Listening. Learning. Leading.®

Research Report

ETS RR–11-11

Use of Continuous Exponential Families to Link Forms via Anchor Tests

Shelby J. Haberman

Duanli Yan

April 2011

Use of Continuous Exponential Families to Link Forms via Anchor Tests

Shelby J. Haberman and Duanli Yan

ETS, Princeton, NJ

April 2011

Technical Review Editor: Matthias von Davier

Technical Reviewers: Frank Rijmen and Sandip Sinharay

Abstract

Continuous exponential families are applied to linking test forms via an internal anchor. This application combines work on continuous exponential families for single-group designs and work on continuous exponential families for equivalent-group designs. Results are compared to those for kernel and equipercentile equating in the case of chained equating. The conversions produced by all methods are quite similar.

Key words: moments, information theory, nonequivalent groups

Acknowledgments

Alina von Davier and Hexin Chang have assisted in this project.

Application of continuous exponential families to linking has been considered for equivalent-groups designs (Haberman, 2008a) and single-group designs (Haberman, 2008b). The procedure for a single-group design is readily applied to the chained approach to the equating design for nonequivalent groups with anchor tests (NEAT). In this report, the required methodology is described, and application is made to the equating of several forms from several components of a test in which kernel equating is currently used on an operational basis. Results of equating by continuous exponential families are compared to those for kernel equating and to those for equipercentile equating with log-linear smoothing. On the whole, all equating procedures yield quite similar results; however, continuous exponential families have some advantage. As in kernel equating, readily-computed asymptotic standard deviations are available. In addition, unlike in kernel equating, a bandwidth need not be specified or estimated. In addition, continuous exponential families can be applied to continuous score distributions and to score distributions with very large numbers of possible values. This feature may gain increasing significance in the future if scoring begins to include such components as essentially continuous electronically derived features of essays.

Section 1 describes use of continuous exponential families in the NEAT design. In this section, all distributions of random variables and random vectors are assumed known. Section 2 considers the more realistic case in which sample data must be used to determine the appropriate conversions. Section 3 summarizes results of the application to the test data. Section 4 provides some conclusions. Discussion assumes familiarity with kernel and equipercentile equating methods (von Davier, Holland, & Thayer, 2004).

1 Equating for the NEAT Design With Continuous Exponential Families

To equate two test forms with a common anchor test by continuous exponential families is relatively straightforward if the chained approach is employed. Consider two test forms, Form 1 and Form 2, and consider an anchor test A. For $1 \le j \le 2$, let n_j be a positive integer, and let Examinee i, $1 \le i \le n_j$, receive a score X_{ij} on Form j and a score A_{ij} on the anchor test. Assume that the pairs (X_{ij}, A_{ij}), $1 \le i \le n_j$, $1 \le j \le 2$, are mutually independent. For $1 \le j \le 2$, let the joint distribution of (X_{ij}, A_{ij}) be the same for $1 \le i \le n_j$. The examinees who receive Form 1 are not assumed to be from the same population as the examinees who receive Form 2, so that A_{i1} and $A_{i'2}$ do not have the same distributions for Examinee i who received Form 1 and Examinee i'

who received Form 2. For Form j, where j is 1 or 2, possible scores X_{ij} are in the closed interval with finite lower bound c_{Xj} and finite upper bound $d_{Xj} > c_{Xj}$. In addition, the anchor test scores A_{ij} are all in a closed interval with lower bound c_A and upper bound $d_A > c_A$. No requirement is imposed that the scores be integers or rational numbers. Nonetheless, in typical applications, the common distribution function F_{Xj} of X_{ij}, $1 \leq i \leq n_j$, and the common distribution function F_{Aj} of A_{ij}, $1 \leq i \leq n_j$, are not continuous, so that an equipercentile approach to equating of Form 1 and Form 2 based on observed scores normally involves some approximation of the distribution functions F_{Xj} and F_{Aj} by continuous distribution functions G_{Xj} and G_{Aj}, respectively. The distribution function G_{Xj} is strictly increasing on some open interval B_{Xj} that contains both c_{Xj} and d_{Xj}, and the distribution function G_{Aj} is strictly increasing on some open interval B_A that contains c_A and d_A. For each positive real $p < 1$, there are unique continuous and increasing quantile functions R_{Xj} and R_{Aj} such that $G_{Xj}(R_{Xj}(p)) = p$ and $G_{Aj}(R_{Aj}) = p$. With the chained approach, the linking function e_{X1X2} for conversion of a score on Form 1 to a score on Form 2 is then $e_{X1X2}(x) = R_{X2}(G_{A2}(R_{A1}(G_{X1}(x))))$ for x in B_{X1}, while the linking function e_{X2X1} for conversion of a score on Form 2 to a score on Form 1 is $e_{X2X1}(x) = R_{X1}(G_{A1}(R_{A2}(G_{X2}(x))))$ for x in B_{X2}. Both e_{X1X2} and e_{X2X1} are strictly increasing and continuous on their respective ranges, and e_{X1X2} and e_{X2X1} are inverses, so that $e_{X1X2}(e_{X2X1}(x)) = x$ for x in B_{X2} and $e_{X2X1}(e_{X1X2}(x)) = x$ for x in B_{X1} (Haberman, 2008a). If G_{X1} has a continuous derivative g_{X1} at x in B_{X1}, G_{A1} has a positive and continuous derivative g_{A1} at $e_{X1A}(x) = R_{A1}(G_{X1}(x))$, G_{A2} has a continuous derivative g_{A2} at $e_{X1A}(x)$, and G_{X2} has continuous and positive derivative g_{X2} at $e_{X1X2}(x)$, then application of standard results from calculus shows that e_{X1X2} has continuous derivative

$$e'_{X1X2}(x) = \frac{g_{X1}(x)g_{A2}(e_{X1A}(x))}{g_{A1}(e_{X1A}(x))g_{X2}(e_{X1X2}(x))}$$

at x. Similarly, if G_{X2} has a continuous derivative g_{X2} at x in B_{X2}, G_{A2} has a positive and continuous derivative g_{A2} at $e_{X2A}(x) = R_{A2}(G_{X2}(x))$, G_{A1} has a continuous derivative g_{A1} at $e_{X2A2}(x)$, and G_{X1} has continuous and positive derivative g_{X2} at $e_{X2X1}(x)$, then e_{X2X1} has continuous derivative

$$e'_{X2X1}(x) = \frac{g_{X2}(x)g_{A1}(e_{X2A}(x))}{g_{A2}(e_{X2A}(x))g_{X1}(e_{X2X1}(x))}$$

at x.

One method to obtain distribution functions G_{X1}, G_{A1}, G_{X2}, and G_{A2} is to approximate the joint distribution of (X_{ij}, A_{ij}) by use of a bivariate continuous exponential family for both $j = 1$ and $j = 2$ (Haberman, 2008b). For simplicity, let B_{Xj}, $1 \leq j \leq 2$, and B_A be bounded. For $k \geq 0$, let u_{kXj} be a polynomial of degree k on the interval B_{Xj} for $1 \leq j \leq 2$, and let u_{kA} be a polynomial of degree k on B_A. For $1 \leq j \leq 2$ and a pair $\mathbf{k} = (k_{Xj}, k_A)$ of nonnegative integers, let $u_{\mathbf{k}j}$ be the polynomial on the plane such that $u_{\mathbf{k}j}(\mathbf{x}_j) = u_{k_{Xj}}(x_{Xj})u_{k_A}(x_A)$ for real pairs $\mathbf{x}_j = (x_{Xj}, x_A)$. Let $\mathbf{X}_{ij} = (X_{ij}, A_{ij})$. Let $\mu_{\mathbf{k}j}$ be the expectation of $u_{\mathbf{k}j}(\mathbf{X}_{ij})$, so that $\mu_{\mathbf{k}j}$ is a linear combination of the bivariate moments $E(X_{ij}^{h_{Xj}} A_{ij}^{h_A})$ of $\mathbf{X}_{ij}$ for nonnegative integers $h_{Xj} \leq k_{Xj}$ and $h_A \leq k_A$.

Consider a nonempty set K_j of r_j pairs of nonnegative integers $\mathbf{k} = (k_{Xj}, k_A)$ such that k_{Xj} or k_A is positive. Let $\boldsymbol{\mu}_{K_jj}$ be the K_j-array of $\mu_{\mathbf{k}j}$, $\mathbf{k}$ in K_j, and let $\mathbf{u}_{K_jj}(\mathbf{x})$ be the K_j-array of $u_{\mathbf{k}j}(\mathbf{x})$, $\mathbf{k}$ in K_j. If $\mathbf{y}_{K_jj}$ is a real K_j-array of $y_{\mathbf{k}j}$, $\mathbf{k}$ in K_j, and $\mathbf{z}_{K_jj}$ is a real K_j-array of $z_{\mathbf{k}j}$, $\mathbf{k}$ in K_j, then let

$$\mathbf{y}'_{K_jj}\mathbf{z}_{K_jj} = \sum_{\mathbf{k}\in K_j} y_{\mathbf{k}j} z_{\mathbf{k}j}.$$

Assume that, for any real K_j-array $\mathbf{y}_{K_jj}$, the variance of $\mathbf{y}'_{K_jj}\mathbf{u}_{K_jj}(\mathbf{X}_{ij})$ is 0 only if $y_{\mathbf{k}j} = 0$ for each $\mathbf{k}$ in K_j. Let $B_{XjA} = B_{Xj} \times B_A$ be the interval in the plane that consists of pairs (b_{Xj}, b_A) such that b_{Xj} is in B_{Xj} and b_A is in B_A. To treat issues such as internal anchors, let w_j be a bounded and positive real function on B_{XjA}. For numerical work, it is helpful to assume that w_j is infinitely differentiable. Then a unique continuous bivariate distribution with positive density on B_{XjA} has the exponential family density

$$g_{K_jj}(\mathbf{x}) = \gamma_{K_jj}(\boldsymbol{\theta}_{K_jj}) w_j(\mathbf{x}) \exp[\boldsymbol{\theta}'_{K_jj}\mathbf{u}_{K_jj}(\mathbf{x})],$$

$\mathbf{x}$ in B_{XjA}, for a unique K_j-array $\boldsymbol{\theta}_{K_jj}$ with elements $\theta_{\mathbf{k}K_jj}$, $\mathbf{k}$ in K_j, and a unique positive real $\gamma_{K_jj}(\boldsymbol{\theta}_{K_jj})$ such that

$$\int_{B_{XjA}} u_{\mathbf{k}j}(\mathbf{x}) g_{K_jj}(\mathbf{x}) d\mathbf{x} = \mu_{\mathbf{k}j}$$

for $\mathbf{k}$ in K_j and

$$\int_{B_{XjA}} g_{K_jj}(\mathbf{x}) d\mathbf{x} = 1$$

(Gilula & Haberman, 2000; Haberman, 2008b). A random vector $\mathbf{Y}_{K_jj} = (Y_{XjK_jj}, Y_{AK_jj})$ in B_{jA} then exists with density g_{K_jj}. The moment equalities $E(u_{\mathbf{k}j}(\mathbf{Y}_{K_jj})) = E(u_{\mathbf{k}j}(\mathbf{X}_{ij}))$ hold for $\mathbf{k}$ in K_j, so that $\mathbf{Y}_{K_jj}$ has a distribution close to that of $\mathbf{X}_{ij}$ in the sense that the expected log penalty function $I_{K_jj} = E(-\log g_{K_jj}(\mathbf{X}_{ij}))$ is the smallest expected log penalty function $E(-\log g(\mathbf{X}_{ij}))$ for all probability densities g on B_{XjA} such that

$$g(\mathbf{x}) = \gamma_{K_jj}(\boldsymbol{\theta}_{*K_jj})w_j(\mathbf{x})\exp[\boldsymbol{\theta}'_{*K_jj}\mathbf{u}_{K_jj}(\mathbf{x})]$$

for some real K_j-array $\boldsymbol{\theta}_{*K_jj}$, and $E(-\log g(\mathbf{X}_{ij})) = I_{K_jj}$ only if $\boldsymbol{\theta}_{*K_jj} = \boldsymbol{\theta}_{K_jj}$.

If K_j includes the pairs $(1,0)$, $(0,1)$, $(2,0)$, $(0,2)$ and $(1,1)$ and w_j is always 1, then $\log g_{K_jj}(\mathbf{x})$ is a quadratic function

$$\beta_0 + \beta_{Xj}x_{Xj} + \beta_A x_A + \beta_{XjXj}x_{Xj}^2 + 2\beta_{XjA}x_{Xj}x_A + \beta_{AA}x_A^2.$$

If β_{XjXj} and β_{AA} are both negative and if $\beta_{XjA}^2 < \beta_{XjXj}\beta_{AA}$, then g_{K_jj} is the conditional density of a bivariate normal random vector given that the vector is in the interval B_{XjA}. The random vector $\mathbf{Y}_{K_jj}$ with density g_{K_jj} then has the same mean and covariance matrix as (X_{ij}, A_{ij}).

The moment equations expressed in terms of $u_{\mathbf{k}j}$ can be interpreted in terms of conventional moments if the set K_j satisfies the hierarchy rule that (k_{Xj}, k_A) is in K_j whenever (h_{Xj}, h_A) is in K_j, $k_{Xj} \leq h_{Xj}$, $k_A \leq h_A$, k_{Xj} and k_A are nonnegative integers, and k_{Xj} or k_A is positive. The equations $E(u_{\mathbf{k}j}(\mathbf{Y}_{K_jj})) = E(u_{\mathbf{k}j}(\mathbf{X}_{ij}))$ for $\mathbf{k}$ in K_j then hold if, and only if, $E(Y_{XjK_jj}^{k_{Xj}} Y_{AK_jj}^{k_A}) = E(X_{ij}^{k_{Xj}} A_{ij}^{k_A})$ for all $\mathbf{k}$ in K_j.

For $1 \leq j \leq 2$, the distribution function G_{XjK_jj} of Y_{XjK_jj} and the distribution function G_{AK_jj} of Y_{AK_jj} are strictly increasing and continuously differentiable on their respective ranges B_{Xj} and B_A. If B_{XjyA}, y in B_{Xj}, consists of all pairs (y_{Xj}, y_A) such that y_{Xj} is in B_{Xj}, y_A is in B_A, and $y_{Xj} \leq y$, then

$$G_{XjK_jj}(y) = \int_{B_{XjyA}} g_{K_jj}(\mathbf{x})d\mathbf{x}.$$

If B_{XjAy}, y in B_A, consists of all pairs (y_{Xj}, y_A) such that y_{Xj} is in B_{Xj}, y_A is in B_A, and $y_A \leq y$, then

$$G_{AK_jj}(y) = \int_{B_{XjAy}} g_{AK_jj}(\mathbf{x})d\mathbf{x}.$$

The inverse R_{XjK_jj} defined by $G_{XjK_jj}(R_{XjK_jj}(p)) = p$ for $0 < p < 1$ and the inverse R_{AK_jj} defined by $G_{AK_jj}(R_{AK_jj}(p) = p$ for $0 < p < 1$ are also continuously differentiable and strictly increasing, so that the conversion functions

$$e_{X1X2K_1K_2} = R_{X2K_22}(G_{AK_22}(R_{AK_11}(G_{X1K_11})))$$

and

$$e_{X2X1K_1K_2} = R_{X1K_11}(G_{AK_11}(R_{AK_22}(G_{X2K_22})))$$

are also continuously differentiable and strictly increasing. Note that $e_{X1X2K_1K_2} = e_{AX2K_2}(e_{X1AK_1})$, where $e_{X1AK_1} = R_{AK_11}(G_{X1K_11})$ provides a conversion from Form 1 to the anchor test and $e_{AX2K_2} = R_{X2K_22}(G_{AK_22})$ provides a conversion from the anchor test to Form 2, while $e_{X2X1K_2K_1} = e_{AX1K_1}(e_{X2AK_2})$, where $e_{X2AK_2} = R_{AK_22}(G_{X2K_22})$ provides a conversion from Form 2 to the anchor test and $e_{AX1K_1} = R_{X1K_11}(G_{AK_11})$ provides a conversion from the anchor test to Form 1.

As in other cases of continuous exponential families (Haberman, 2008a, 2008b), numerical work is simplified if computations employ the Legendre polynomials P_k for $k \geq 0$ (Abramowitz & Stegun, 1965, chapters 8, 22). These polynomials are determined by the equations $P_0(x) = 1$, $P_1(x) = x$, and

$$P_{k+1}(x) = (k+1)^{-1}[(2k+1)xP_k(x) - kP_{k-1}(x)],$$

$k \geq 1$. If $\inf(B_{Xj})$ is the infimum of B_{Xj} and $\sup(B_{Xj})$ is the supremum of B_{Xj} for $1 \leq j \leq 2$, $\inf(B_A)$ is the infimum of B_A, and $\sup(B_A)$ is the supremum of B_A, then it is relatively efficient for numerical work to let $\beta_{Xj} = [\inf(B_{Xj}) + \sup(B_{Xj})]/2$ be the midpoint of B_{Xj} for $1 \leq j \leq 2$, to let $\beta_A = [\inf(B_A) + \sup(B_A)]/2$ be the midpoint of B_A, to let $\eta_{Xj} = [\sup(B_{Xj}) - \inf(B_{Xj})]/2$ be half the range of B_{Xj} for $1 \leq j \leq 2$, to let $\eta_A = [\sup(B_A) - \inf(B_A)]/2$ be half the range of B_A, to let

$$u_{kXj}(x) = P_k((x - \beta_{Xj})/\eta_{Xj})$$

for $1 \leq j \leq 2$, and to let

$$u_{kA}(x) = P_k((x - \beta_A)/\eta_A).$$

In applications considered in this report, for integers $r_{Xj} > 1$ and $r_{Aj} > 0$, $1 \leq j \leq 2$, the set K_j consists of the $r_{Xj} + r_{Aj} + 1$ elements $(k_{Xj}, 0)$, $1 \leq k_{Xj} \leq r_{Xj}$, $(0, k_A)$, $1 \leq k_A \leq r_{Aj}$, and $(1, 1)$, so that the hierarchy principle holds and, for $1 \leq j \leq 2$, Y_{XjK_jj} and X_{ij} have the same r_{Xj} initial moments, Y_{AK_jj} and A_{ij} have the same r_{Aj} initial moments,, and Y_{XjK_jj} and Y_{AK_jj} have the same correlation as X_{ij} and A_{ij}. Thus Y_{XjK_jj} and X_{ij} have the same mean and variance for each j, and Y_{AK_jj} and A_{ij} have the same mean and variance for each j. If $r_{Xj} > 2$, then Y_{XjK_jj} and X_{ij} have the same skewness coefficient. If $r_{Xj} > 3$, then Y_{XjK_jj} and X_{ij} have the same kurtosis coefficient. Similarly, if $r_{Aj} > 2$, then Y_{AK_jj} and A_{ij} have the same skewness coefficient.

If $r_{Aj} > 3$, then Y_{AK_jj} and A_{ij} have the same kurtosis coefficient. In the case of $r_{Xj} = r_{Aj} = 2$ in which Legendre polynomials are used, if $\theta_{\mathbf{k}}$ is negative for $\mathbf{k}$ equal to $(2, 0)$ or $(0, 2)$ and $\theta^2_{(1,1)}$ is less than $36\theta^2_{(2,0)}\theta^2_{(0,2)}$, then $\mathbf{Y}_{K_jj}$ corresponds to a bivariate normal random variable $\mathbf{Z} = (Z_{Xj}, Z_A)$ (Haberman, 2008b). The distribution of $\mathbf{Y}_{K_jj}$ is the same as the conditional distribution of $\mathbf{Z}$ conditional on Z_{Xj} being in B_{Xj} and Z_A being to B_A (Haberman, 2008b). One alternative choice of K_j (Wang, 2008) has K_j contain all pairs (k_{Xj}, k_A) of nonnegative integers such that k_{Xj} or k_A is positive, $k_{Xj} \leq r_{Xj}$, and $k_A \leq r_A$.

In typical cases, w_j is just the constant 1; however, in some cases with internal anchors $A_{ij} \leq X_{ij}$, $\inf(B_{Xj}) = \inf(B_A)$ and $\sup(B_A) < \sup(B_{Xj})$. In such a case, it may be reasonable to let

$$w_j(\mathbf{x}) = \frac{\exp[z_j(x_{Xj} - x_A)]}{1 + \exp[z_j(x_{Xj} - x_A)]}$$

for $\mathbf{x} = (x_{Xj}, x_A)$ in B_{XjA}, where z_j is a positive real constant. As z_j becomes large, $w_j(\mathbf{x})$ goes to 1 for $x_{Xj} > x_A$ and to 0 for $x_{Xj} < x_A$. In applications in this report, $z_j = 2$. This choice of w_j and z_j facilitates use of 20-point Gauss-Legendre integration (Haberman, 2008b).

2 Estimation of Parameters

The parameters $\boldsymbol{\theta}_{K_jj}$, the information criterion I_{K_jj}, the distribution functions G_{XjK_jj} and G_{AK_jj}, and the conversion functions $e_{X1X2K_1K_2}$ and $e_{X2X1K_1K_2}$ are readily estimated (Gilula & Haberman, 2000; Haberman, 2008a, 2008b). For $\mathbf{k}$ in K_j, let $m_{\mathbf{k}j}$ be the sample mean $n_j^{-1}\sum_{i=1}^{n_j} u_{\mathbf{k}}j(\mathbf{X}_{ij})$, and let $\mathbf{m}_{K_jj}$ be the K_j-array with elements $m_{\mathbf{k}j}$, $\mathbf{k}$ in K_j. If the covariance matrix of $\mathbf{m}_{K_jj}$ is positive definite, then $\boldsymbol{\theta}_{K_jj}$ is estimated by the unique K_j-array $\hat{\boldsymbol{\theta}}_{K_jj}$ such that

$$\int_{B_{XjA}} \mathbf{u}_{K_jj}(\mathbf{x})\hat{g}_{K_jj}(\mathbf{x})d\mathbf{x} = \mathbf{m}_{K_jj},$$

$$\int_{B_{XjA}} \hat{g}_{K(j)j}(\mathbf{x})d\mathbf{x} = 1,$$

and

$$\hat{g}_{K_jj}(\mathbf{x}) = \gamma_{K_jj}(\hat{\boldsymbol{\theta}}_{K_jj})w_j(\mathbf{x})\exp[\hat{\boldsymbol{\theta}}'_{K_jj}\mathbf{u}_{K_jj}(\mathbf{x})]$$

for $\mathbf{x}$ in B_{XjA}.

For $1 \leq j \leq 2$, as the sample size n_j approaches ∞, $\hat{\boldsymbol{\theta}}_{K_jj}$ converges to $\boldsymbol{\theta}_{K_jj}$ with probability 1, and $n_j^{1/2}(\hat{\boldsymbol{\theta}}_{K_jj} - \boldsymbol{\theta}_{K_jj})$ converges in distribution to a multivariate normal random variable with zero mean and with covariance matrix $\mathbf{V}_{K_jj} = \mathbf{C}^{-1}_{K_jj}\mathbf{D}_{K_jj}\mathbf{C}^{-1}_{K_jj}$ (Gilula & Haberman, 2000). Here

$\mathbf{D}_{K_j j}$ is the covariance matrix of $\mathbf{u}_{K_j j}(\mathbf{X}_{ij})$ and $\mathbf{C}_{K_j j}$ is the covariance matrix of the K_j-array $\mathbf{u}_{K_j j}(\mathbf{Y}_{K_j j})$. Thus

$$\mathbf{C}_{K_j j} = \int_{B_{XjA})} [\mathbf{u}_{K_j j}(\mathbf{x}) - \boldsymbol{\mu}_{K_j j}][\mathbf{u}_{K_j j}(\mathbf{x}) - \boldsymbol{\mu}_{K_j j}]' g_{K_j j}(\mathbf{x}) d\mathbf{x}.$$

The estimate of $\mathbf{C}_{K_j j}$ is

$$\hat{\mathbf{C}}_{K_j j} = \int_{B_{XjA}} [\mathbf{u}_{K_j j}(\mathbf{x}) - \mathbf{m}_{K_j j}][\mathbf{u}_{K_j j}(\mathbf{x}) - \mathbf{m}_{K_j j}]' \hat{g}_{K_j}(\mathbf{x}) d\mathbf{x}.$$

The estimate of $\mathbf{D}_{K_j j}$ is

$$\hat{\mathbf{D}}_{K_j j} = n_j^{-1} \sum_{i=1}^{n_j} [\mathbf{u}_{K_j j}(\mathbf{X}_i) - \mathbf{m}_{K_j j}][\mathbf{u}_{K_j j}(\mathbf{X}_i) - \mathbf{m}_{K_j j}]'.$$

Thus $\mathbf{V}_{K_j j}$ has estimate

$$\hat{\mathbf{V}}_{K_j j} = \hat{\mathbf{C}}_{K_j j}^{-1} \hat{\mathbf{D}}_{K_j j} \hat{\mathbf{C}}_{K_j j}^{-1}.$$

For any nonzero constant K_j-array $\mathbf{z}_{K_j}$, the estimated asymptotic standard deviation (EASD) of $\mathbf{z}'_{K_j} \hat{\boldsymbol{\theta}}_{K_j j}$ is

$$\hat{\sigma}(\mathbf{z}'_{K_j} \hat{\boldsymbol{\theta}}_{K_j j}) = n_j^{-1/2} (\mathbf{z}'_{K_j} \hat{\mathbf{V}}_{K_j j} \mathbf{z}_{K_j})^{1/2},$$

so that

$$(\mathbf{z}'_{K_j} \hat{\boldsymbol{\theta}}_{K_j j} - \mathbf{z}'_{K_j} \boldsymbol{\theta}_{K_j j}) / \hat{\sigma}(\mathbf{z}'_{K_j} \hat{\boldsymbol{\theta}}_{K_j j})$$

converges in distribution to a standard normal random variable.

The minimum expected penalty $I_{K_j j}$ may be estimated by

$$\hat{I}_{K_j j} = -\log \gamma_{K_j j}(\hat{\boldsymbol{\theta}}_{K_j j}) - \hat{\boldsymbol{\theta}}'_{K_j j} \mathbf{m}_{K_j j}.$$

As the sample size n_j increases, $\hat{I}_{K_j j}$ converges to $I_{K_j j}$ with probability 1 and $n_j^{1/2}(\hat{I}_{K_j j} - I_{K_j j})$ converges in distribution to a normal random variable with mean 0 and variance

$$\sigma^2(-\log g_{K_j j}(\mathbf{X}_{ij})) = \boldsymbol{\theta}'_{K_j j} \mathbf{V}'_{K_j j} \boldsymbol{\theta}_{K_j j}.$$

The EASD of $\hat{I}_{K_j j}$ is then

$$\hat{\sigma}(\hat{I}_{K_j j}) = n_j^{-1/2} (\hat{\boldsymbol{\theta}}'_{K_j j} \hat{\mathbf{V}}'_{K_j j} \hat{\boldsymbol{\theta}}_{K_j j})^{1/2}$$

(Haberman, 2008b).

For $1 \leq j \leq 2$, the distribution function G_{XjK_jj} has estimate $\hat{G}_{XjK_jj}$ defined by

$$\hat{G}_{XjK_jj}(y) = \int_{B_{XjyA}} \hat{g}_{K_jj}(\mathbf{x})d\mathbf{x}$$

for y in B_{Xj}, and the quantile function R_{XjK_jj} has estimate $\hat{R}_{XjK_jj}$ defined by

$$\hat{G}_{XjK_jj}(\hat{R}_{XjK_jj}(p)) = p$$

for $0 < p < 1$. The distribution function G_{AK_jj} has estimate $\hat{G}_{AK_jj}$ defined by

$$\hat{G}_{AK_jj}(y) = \int_{B_{XjAy}} \hat{g}_{K_jj}(\mathbf{x})d\mathbf{x}$$

for y in B_A, and the quantile function R_{AK_jj} has estimate $\hat{R}_{AK_jj}$ defined by

$$\hat{G}_{AK_jj}(\hat{R}_{AK_jj}(p)) = p$$

for $0 < p < 1$. Let

$$\mathbf{T}_{XjK_jj}(y) = \int_{B_{XjyA}} [\mathbf{u}_{K_jj}(\mathbf{x}) - \boldsymbol{\mu}_{K_jj}]g_{K_jj}(\mathbf{x})d\mathbf{x}$$

and

$$\mathbf{T}_{AK_jj}(y) = \int_{B_{XjAy}} [\mathbf{u}_{K_jj}(\mathbf{x}) - \boldsymbol{\mu}_{K_jj}]g_{K_jj}(\mathbf{x})d\mathbf{x}.$$

As the sample sizes n_1 and n_2 approach ∞, $\hat{G}_{XjK_jj}(y)$ converges to $G_{XjK_jj}(y)$ with probability 1 for y in B_{Xj}, so that $|\hat{G}_{XjK_jj} - G_{XjK_jj}|$, the supremum of $|\hat{G}_{XjK_jj}(y) - G_{XjK_jj}(y)|$ for y in B_{Xj}, converges to 0 with probability 1. Similarly, $\hat{G}_{AK_jj}(y)$ converges to $G_{AK_jj}(y)$ with probability 1 for y in B_A, so that $|\hat{G}_{AK_jj} - G_{AK_jj}|$, the supremum of $|\hat{G}_{AK_jj}(y) - G_{AK_jj}(y)|$ for y in B_A, converges to 0 with probability 1 (Haberman, 2008b). In addition, $[\hat{G}_{XjK_jj}(y) - G_{XjK_jj}(y)]/\sigma(\hat{G}_{XjK_jj}(y))$ converges in distribution to a normal random variable with mean 0 and variance 1 if

$$\sigma(\hat{G}_{XjK_jj}(y)) = n_j^{-1/2}\{[\mathbf{T}_{XjK_jj}(y)]'\mathbf{V}_{K_jj}\mathbf{T}_{XjK_jj}(y)\}^{1/2},$$

and $[\hat{G}_{AK_jj}(y) - G_{AK_jj}(y)]/\sigma(\hat{G}_{AK_jj}(y))$ converges in distribution to a normal random variable with mean 0 and variance 1 if

$$\sigma(\hat{G}_{AK_jj}(y)) = n_j^{-1/2}\{[\mathbf{T}_{AK_jj}(y)]'\mathbf{V}_{K_jj}\mathbf{T}_{AK_jj}(y)\}^{1/2},$$

Similarly, $\hat{R}_{XjK_jj}(p)$ converges to $R_{XjK_jj}(p)$ with probability 1, and $[\hat{R}_{XjK_jj}(p) - R_{XjK_jj}(p)]/\sigma(\hat{R}_{XjK_jj}(p))$ converges in distribution to a normal random variable with mean 0 and variance 1 if

$$\sigma(\hat{R}_{XjK_jj}(p)) = [g_{XjK_jj}(R_{XjK_jj}(p))]^{-1}\sigma(\hat{G}_{XjK_jj}(R_{XjK_jj}(p)))$$

and $g_{XjK_jj}(y)$ is the marginal density corresponding to G_{XjK_jj}. Thus $g_{XjK_jj}(y)$ is the integral of $g_{K_jj}((y, x_A))$ over x_A in B_A.

The estimate $\hat{R}_{AK_jj}(p)$ converges to $R_{AK_jj}(p)$ with probability 1, and $[\hat{R}_{AK_jj}(p) - R_{AK_jj}(p)]/\sigma(\hat{R}_{AK_jj}(p))$ converges in distribution to a normal random variable with mean 0 and variance 1 if

$$\sigma(\hat{R}_{AK_jj}(p)) = [g_{AK_jj}(R_{AK_jj}(p))]^{-1}\sigma(\hat{G}_{AK_jj}(R_{AK_jj}(p)))$$

and $g_{AK_jj}(y)$ is the marginal density corresponding to G_{AK_jj}. Thus $g_{AK_jj}(y)$ is the integral of $g_{K_jj}((x_{Xj}, y))$ over x_{Xj} in B_{Xj}.

Estimated asymptotic standard deviations may be derived by use of obvious substitutions of estimated parameters for actual parameters. Thus

$$\hat{\sigma}(\hat{G}_{XjK_jj}(y)) = n_j^{-1/2}\{[\hat{\mathbf{T}}_{XjK_jj}(y)]'\hat{\mathbf{V}}_{K_jj}\hat{\mathbf{T}}_{XjK_jj}(y)\}^{1/2},$$

where

$$\hat{\mathbf{T}}_{XjK_jj}(y) = \int_{B_{XjyA}} [\mathbf{u}_{K_jj}(\mathbf{x}) - \mathbf{m}_{K_jj}]\hat{g}_{K_jj}(\mathbf{x})d\mathbf{x},$$
$$\hat{\sigma}(\hat{R}_{XjK_jj}(p)) = [\hat{g}_{XjK_jj}(\hat{R}_{XjK_jj}(p))]^{-1}\hat{\sigma}(\hat{G}_{XjK_jj}(\hat{R}_{XjK_jj}(p)),$$

and $\hat{g}_{XjK_jj}(y)$ is the marginal density corresponding to $\hat{G}_{XjK_jj}$. In like manner,

$$\hat{\sigma}(\hat{G}_{AK_jj}(y)) = n_j^{-1/2}\{[\hat{\mathbf{T}}_{AK_jj}(y)]'\hat{\mathbf{V}}_{K_jj}\hat{\mathbf{T}}_{AK_jj}(y)\}^{1/2},$$

where

$$\hat{\mathbf{T}}_{AK_jj}(y) = \int_{B_{XjAy}} [\mathbf{u}_{K_jj}(\mathbf{x}) - \mathbf{m}_{K_jj}]\hat{g}_{K_jj}(\mathbf{x})d\mathbf{x},$$
$$\hat{\sigma}(\hat{R}_{AK_jj}(p)) = [\hat{g}_{AK_jj}(\hat{R}_{AK_jj}(p))]^{-1}\hat{\sigma}(\hat{G}_{AK_jj}(\hat{R}_{AK_jj}(p)),$$

and $\hat{g}_{AK_jj}(y)$ is the marginal density corresponding to $\hat{G}_{AK_jj}$.

The estimate $\hat{e}_{X1X2K_1K_2}$ of the conversion function $e_{X1X2K_1K_2}$ from Form 1 to Form 2 satisfies

$$\hat{e}_{X1X2K_1K_2} = \hat{e}_{AX2K_2}(\hat{e}_{X1AK_1}),$$

where

$$\hat{e}_{AX2K_2} = \hat{R}_{X2K_22}(\hat{G}_{AK_22})$$

and

$$\hat{e}_{X1AK_1} = \hat{R}_{AK_11}(\hat{G}_{X1K_11}).$$

The corresponding estimate $\hat{e}_{X2X1K_1K_2}$ of $e_{X2X1K_1K_2}$ satisfies

$$\hat{e}_{X2X1K_1K_2} = \hat{e}_{AX1K_1}(\hat{e}_{X2AK_2}),$$

where

$$\hat{e}_{AX1K_1} = \hat{R}_{X1K_11}(\hat{G}_{AK_11})$$

and

$$\hat{e}_{X2AK_2} = \hat{R}_{AK_22}(\hat{G}_{X2K_22}).$$

As the sample sizes n_1 and n_2 become large, $\hat{e}_{X1X2K_1K_2}(y)$ converges with probability 1 to $e_{X1X2K_1K_2}(y)$ for y in B_{X1}, and $\hat{e}_{X2X1K_1K_2}(y)$ converges with probability 1 to $e_{X2X1K_1K_2}(y)$ for y in B_{X2}. In addition, $[\hat{e}_{X1X2K_1K_2}(y) - e_{X1X2K_1K_2}(y)]/\sigma(\hat{e}_{X1X2K_1K_2}(y))$ converges in distribution to a standard normal random variable if

$$\begin{aligned}
&\sigma^2(\hat{e}_{X1X2K_1K_2}(y)) \\
&= n_1^{-1}[\mathbf{T}_{X1K_11}(y) - \mathbf{T}_{AK_11}(e_{X1AK_1}(y))]'\mathbf{V}_{K_11}[\mathbf{T}_{X1K_11}(y) - \mathbf{T}_{AK_11}(e_{X1X2K_1K_2}(y))] \\
&\quad \{[g_{AK_22}(e_{X1AK_1}(y))]/[g_{AK_11}(e_{X1AK_1}(y))g_{X2K_22}(e_{X1X2K_1K_2}(y))]\}^2 \\
&\quad +n_2^{-1}[\mathbf{T}_{AK_22}(e_{X1AK_1}(y)) - \mathbf{T}_{X2K_22}(e_{X1X2K_1K_2}(y))]'\mathbf{V}_{K_22} \\
&\quad [\mathbf{T}_{AK_22}(y) - \mathbf{T}_{X2K_22}(e_{X1X2K_1K_2}(y))]/[g_{X2K_22}(e_{X1X2K_1K_2}(y))]^2.
\end{aligned}$$

In like manner, $[\hat{e}_{X2X1K_1K_2}(y) - e_{X2X1K_1K_2}(y)]/\sigma(\hat{e}_{X2X1K_1K_2}(y))$ converges in distribution to a standard normal random variable if

$$\begin{aligned}
&\sigma^2(\hat{e}_{X1X2K_1K_2}(y)) \\
&= n_2^{-1}[\mathbf{T}_{X2K_22}(y) - \mathbf{T}_{AK_22}(e_{X2AK_2}(y))]'\mathbf{V}_{K_22}[\mathbf{T}_{X2K_22}(y) - \mathbf{T}_{AK_22}(e_{X2X1K_1K_2}(y))] \\
&\quad \{[g_{AK_11}(e_{X2AK_2}(y))]/[g_{AK_22}(e_{X2AK_2}(y))g_{X1K_11}(e_{X2X1K_1K_1}(y))]\}^2 \\
&\quad +n_1^{-1}[\mathbf{T}_{AK_11}(e_{X1AK_2}(y)) - \mathbf{T}_{X1K_11}(e_{X2X1K_1K_2}(y))]'\mathbf{V}_{K_11} \\
&\quad [\mathbf{T}_{AK_11}(y) - \mathbf{T}_{X1K_11}(e_{X2X1K_1K_2}(y))]/[g_{X1K_11}(e_{X2X1K_1K_2}(y))]^2.
\end{aligned}$$

The EASD of $\hat{e}_{X1X2K_1K_2}(y)$ satisfies

$$
\begin{aligned}
&\hat{\sigma}^2(\hat{e}_{X1X2K_1K_2}(y)) \\
= \;& n_1^{-1}[\hat{\mathbf{T}}_{X1K_11}(y) - \hat{\mathbf{T}}_{AK_11}(\hat{e}_{X1AK_1}(y))]'\hat{\mathbf{V}}_{K_11}[\hat{\mathbf{T}}_{X1K_11}(y) - \hat{\mathbf{T}}_{AK_11}(\hat{e}_{X1X2}(y))] \\
& \{[\hat{g}_{AK_22}(\hat{e}_{X1AK_1}(y))]/[\hat{g}_{AK_11}(\hat{e}_{X1AK_1}(y))\hat{g}_{X2K_22}(\hat{e}_{X1X2K_1K_2}(y))]\}^2 \\
& + n_2^{-1}[\hat{\mathbf{T}}_{AK_22}(\hat{e}_{X1AK_1}(y)) - \hat{\mathbf{T}}_{X2K_22}(\hat{e}_{X1X2K_1K_2}(y))]'\hat{\mathbf{V}}_{K_22} \\
& [\hat{\mathbf{T}}_{AK_22}(y) - \hat{\mathbf{T}}_{X2K_22}(\hat{e}_{X1X2K_1K_2}(y))]/[\hat{g}_{X2K_22}(\hat{e}_{X1X2K_1K_2}(y))]^2,
\end{aligned}
$$

and the EASD of $\hat{e}_{X2X1K_1K_2}(y)$ satisfies

$$
\begin{aligned}
&\hat{\sigma}^2(\hat{e}_{X2X1K_1K_2}(y)) \\
= \;& n_2^{-1}[\hat{\mathbf{T}}_{X2K_22}(y) - \hat{\mathbf{T}}_{AK_22}(\hat{e}_{X2AK_2}(y))]'\hat{\mathbf{V}}_{K_22}[\hat{\mathbf{T}}_{X2K_22}(y) - \hat{\mathbf{T}}_{AK_22}(\hat{e}_{X2X1}(y))] \\
& \{[\hat{g}_{AK_11}(\hat{e}_{X1AK_2}(y))]/[\hat{g}_{AK_22}(\hat{e}_{X2AK_2}(y))\hat{g}_{X1K_11}(\hat{e}_{X2X1K_1K_2}(y))]\}^2 \\
& + n_1^{-1}[\hat{\mathbf{T}}_{AK_11}(\hat{e}_{X1AK_1}(y)) - \hat{\mathbf{T}}_{X1K_11}(\hat{e}_{X2X1K_1K_2}(y))]'\hat{\mathbf{V}}_{K_11} \\
& [\hat{\mathbf{T}}_{AK_11}(y) - \hat{\mathbf{T}}_{X1K_11}(\hat{e}_{X2X1K_1K_2}(y))]/[\hat{g}_{X1K_11}(\hat{e}_{X2X1K_1K_2}(y))]^2.
\end{aligned}
$$

3 Application

Equating was considered for the verbal, quantitative, writing, and English tests for two administrations. In each case, results are based on 1,414 examinees for the new form and 1,271 examinees for the old form. To avoid identification of the assessment, details concerning the test are omitted. Kernel equating with log-linear smoothing, equipercentile equating with log-linear smoothing, and equating by exponential families were compared. To facilitate comparison, current practices were followed in the following ways. Log-linear models used linear, quadratic, cubic, and quartic terms for main effects, and a linear-by-linear interaction. In continuous exponential families, the corresponding model was used, so that each K_j included the pair $(1, 1)$ and the pairs $(k, 0)$ and $(0, k)$ for $1 \le k \le 4$. Ranges of tests used in kernel equating or equipercentile equating were used to specify c_A, d_A, c_{X1}, d_{X2}, c_{X2}, and d_{X2}. The sets B_{X1}, B_{X2}, and B_A were selected to have $\inf B_{Xj} = c_{Xj} - 0.5$ and $\sup(B_{Xj}) = d_{Xj} + 0.5$ for $1 \le j \le 2$, $\inf(B_A) = c_A - 0.5$, and $\sup(B_A) = d_A + 0.5$. Anchors were internal. Bandwidth selection in kernel equating was based on the criterion in von Davier et al. (2004, p. 63) with $K = 1$. Bandwidths used are found in Table 3. Results for conversion of the new form to the base form are summarized in Tables 1–5 and in Figures 1–8. Note that conversions are not provided outside of the observed range of raw scores.

Table 1
Bandwidths Used in Kernel Equating

	Verbal	Quantitative	Writing	English
New form	0.7	1.9	0.6	1.6
New anchor	0.6	2.5	2.1	0.6
Old form	1.8	2.1	0.6	1.6
Old anchor	1.5	0.7	0.6	1.6

Table 2
Equating Results for Verbal Test

	Exponential		Kernel		Equipercentile	
Score	Conversion	EASD	Conversion	EASD	Conversion	EASD
24	20.26	2.25	19.53	0.91	21.69	2.18
25	23.41	2.52	21.14	1.12	22.93	2.20
26	25.45	2.42	23.03	1.30	24.10	1.95
27	27.08	2.25	25.06	1.36	25.26	1.86
28	28.50	2.08	27.03	1.31	26.47	1.80
29	29.81	1.91	28.82	1.22	27.64	1.33
30	31.02	1.74	30.42	1.12	28.75	1.24
31	32.18	1.58	31.84	1.02	29.89	1.17
32	33.29	1.44	33.13	0.93	31.05	1.11
33	34.36	1.31	34.30	0.84	32.16	0.98
34	35.40	1.19	35.40	0.77	33.27	0.91
35	36.41	1.08	36.44	0.70	34.39	0.85
36	37.41	0.98	37.44	0.64	35.50	0.77
37	38.38	0.90	38.40	0.59	36.57	0.63
38	39.34	0.82	39.33	0.55	37.64	0.59
39	40.28	0.76	40.25	0.51	38.71	0.56
40	41.20	0.70	41.15	0.48	39.76	0.53
41	42.11	0.66	42.03	0.45	40.81	0.49
42	43.01	0.61	42.90	0.43	41.85	0.50
43	43.90	0.58	43.77	0.41	42.89	0.47
44	44.78	0.55	44.63	0.40	43.91	0.45
45	45.65	0.52	45.47	0.38	44.94	0.44
46	46.50	0.49	46.32	0.37	45.95	0.43
47	47.36	0.47	47.16	0.36	46.96	0.41
48	48.20	0.45	48.00	0.35	47.96	0.39
49	49.03	0.42	48.83	0.33	48.96	0.38
50	49.86	0.40	49.66	0.32	49.95	0.36
51	50.69	0.38	50.48	0.31	50.94	0.35
52	51.51	0.36	51.31	0.30	51.93	0.33

	Exponential		Kernel		Equipercentile	
Score	Conversion	EASD	Conversion	EASD	Conversion	EASD
53	52.32	0.34	52.13	0.29	52.91	0.32
54	53.13	0.32	52.96	0.28	53.89	0.31
55	53.94	0.31	53.78	0.27	54.87	0.30
56	54.75	0.29	54.60	0.26	55.84	0.29
57	55.55	0.28	55.42	0.26	56.82	0.29
58	56.35	0.26	56.24	0.25	57.79	0.28
59	57.15	0.25	57.07	0.25	58.76	0.29
60	57.95	0.25	57.89	0.25	59.74	0.28
61	58.75	0.24	58.71	0.25	60.71	0.29
62	59.56	0.24	59.54	0.25	61.69	0.29
63	60.36	0.24	60.37	0.25	62.67	0.29
64	61.17	0.25	61.20	0.25	63.64	0.30
65	61.97	0.25	62.03	0.26	64.61	0.30
66	62.79	0.26	62.87	0.26	65.60	0.30
67	63.60	0.26	63.71	0.26	66.58	0.31
68	64.42	0.27	64.56	0.27	67.57	0.31
69	65.25	0.27	65.41	0.27	68.56	0.31
70	66.08	0.28	66.27	0.27	69.55	0.32
71	66.91	0.28	67.14	0.28	70.54	0.33
72	67.76	0.29	68.01	0.29	71.54	0.35
73	68.61	0.30	68.94	0.30	72.54	0.37
74	69.47	0.32	69.81	0.32	73.54	0.40
75	70.33	0.33	70.74	0.34	74.55	0.44
76	71.21	0.36	71.69	0.36	75.56	0.50
77	72.10	0.39	72.69	0.39	76.58	0.55
78	73.00	0.42	73.73	0.43	77.60	0.61
79	73.92	0.47	74.84	0.46	78.62	0.70
80	74.86	0.52	76.02	0.50	79.66	0.74
81	75.82	0.58	77.30	0.54	80.70	0.80
82	76.83	0.64	78.68	0.56	81.73	0.88
83	77.91	0.72	80.14	0.57	82.80	0.86
84	79.11	0.79	81.64	0.54	83.86	0.83
85	80.58	0.87	83.07	0.48	84.93	0.78
86	82.96	0.93	84.36	0.41	85.99	0.70

Note. EASD = estimated asymptotic standard deviation.

Table 3
Equating Results for Quantitative Test

	Exponential		Kernel		Equipercentile	
Score	Conversion	EASD	Conversion	EASD	Conversion	EASD
8	8.12	0.28	3.09	0.46	6.55	0.64
9	9.15	0.48	5.45	0.45	7.59	0.61
10	10.10	0.53	7.45	0.42	8.63	0.58
11	11.05	0.53	9.11	0.40	9.68	0.57
12	12.01	0.50	10.56	0.38	10.75	0.51
13	12.99	0.46	11.91	0.37	11.82	0.47
14	13.98	0.41	13.21	0.35	12.90	0.42
15	14.98	0.37	14.48	0.32	13.99	0.38
16	16.00	0.33	15.72	0.30	15.08	0.34
17	17.04	0.30	16.96	0.28	16.18	0.30
18	18.09	0.28	18.18	0.27	17.28	0.28
19	19.15	0.26	19.38	0.25	18.38	0.26
20	20.23	0.26	20.58	0.25	19.49	0.25
21	21.33	0.26	21.76	0.24	20.59	0.23
22	22.44	0.26	22.93	0.24	21.71	0.23
23	23.56	0.27	24.09	0.24	22.81	0.23
24	24.70	0.28	25.23	0.23	23.93	0.24
25	25.85	0.29	26.36	0.23	25.04	0.23
26	27.00	0.29	27.48	0.23	26.16	0.23
27	28.16	0.30	28.58	0.23	27.27	0.23
28	29.33	0.32	29.66	0.23	28.38	0.23
29	30.50	0.33	30.74	0.23	29.49	0.23
30	31.66	0.34	31.80	0.24	30.60	0.24
31	32.82	0.36	32.85	0.24	31.71	0.25
32	33.97	0.38	33.89	0.25	32.81	0.25
33	35.11	0.40	34.93	0.26	33.91	0.26
34	36.24	0.41	35.95	0.27	35.00	0.27
35	37.35	0.43	36.96	0.28	36.08	0.28
36	38.44	0.44	37.97	0.29	37.16	0.29
37	39.50	0.45	38.97	0.30	38.22	0.31
38	40.55	0.46	39.96	0.32	39.29	0.31
39	41.58	0.47	40.95	0.33	40.33	0.32
40	42.58	0.47	41.93	0.34	41.37	0.33
41	43.56	0.48	42.91	0.35	42.40	0.34
42	44.52	0.48	43.89	0.36	43.41	0.35
43	45.46	0.49	44.86	0.37	44.43	0.36
44	46.38	0.50	45.82	0.39	45.42	0.37
45	47.27	0.51	46.77	0.40	46.40	0.39

	Exponential		Kernel		Equipercentile	
Score	Conversion	EASD	Conversion	EASD	Conversion	EASD
46	48.14	0.53	47.71	0.42	47.37	0.41
47	48.99	0.55	48.63	0.44	48.33	0.44
48	49.82	0.58	49.54	0.46	49.28	0.47
49	50.63	0.62	50.44	0.48	50.21	0.51
50	51.42	0.66	51.31	0.51	51.13	0.56
51	52.19	0.70	52.17	0.53	52.05	0.63
52	52.94	0.73	53.01	0.56	52.94	0.67
53	53.67	0.77	53.83	0.59	53.80	0.75
54	54.37	0.80	54.65	0.62	54.66	0.85
55	55.05	0.82	55.47	0.66	55.54	0.98
56	55.71	0.82	56.31	0.70	56.18	0.77
57	56.33	0.81	57.18	0.76	57.18	0.85
58	56.91	0.78	58.12	0.83	57.98	0.96
59	57.45	0.72	59.16	0.91	58.78	1.08
60	57.95	0.63	60.35	1.01	59.58	1.21
61	58.40	0.52	61.73	1.10	60.43	0.82
62	58.79	0.38	63.31	1.16	61.36	0.80
63	59.12	0.23	65.11	1.18	62.36	0.66
64	59.39	0.08	67.12	1.17	63.47	0.35

Note. EASD = estimated asymptotic standard deviation.

Table 4
Equating Results for Writing Test

	Exponential		Kernel		Equipercentile	
Score	Conversion	EASD	Conversion	EASD	Conversion	EASD
0	1.02	0.93	-0.65	1.28	2.67	0.79
1	2.20	1.10	0.80	1.38	4.40	1.93
2	3.25	1.22	2.11	1.46	5.80	1.11
3	4.26	1.28	3.33	1.46	7.21	1.67
4	5.23	1.29	4.46	1.40	8.46	1.18
5	6.19	1.25	5.53	1.31	8.94	0.90
6	7.12	1.18	6.56	1.20	9.73	0.84
7	8.04	1.09	7.57	1.09	10.47	0.90
8	8.96	0.99	8.57	0.98	11.03	0.71
9	9.89	0.88	9.57	0.87	11.80	0.63
10	10.81	0.79	10.57	0.76	12.50	0.44
11	11.75	0.69	11.57	0.67	13.17	0.53
12	12.70	0.61	12.58	0.58	13.95	0.42
13	13.67	0.53	13.59	0.50	14.70	0.35
14	14.64	0.46	14.61	0.44	15.52	0.31

	Exponential		Kernel		Equipercentile	
Score	Conversion	EASD	Conversion	EASD	Conversion	EASD
15	15.63	0.41	15.64	0.39	16.33	0.34
16	16.64	0.37	16.67	0.35	17.17	0.30
17	17.65	0.34	17.71	0.32	18.03	0.27
18	18.68	0.32	18.75	0.30	18.95	0.26
19	19.72	0.30	19.80	0.29	19.87	0.25
20	20.77	0.29	20.86	0.28	20.82	0.25
21	21.83	0.28	21.91	0.27	21.80	0.24
22	22.90	0.28	22.97	0.26	22.77	0.23
23	23.97	0.27	24.03	0.24	23.79	0.23
24	25.04	9.26	25.08	0.23	24.80	0.22
25	26.11	0.25	26.14	0.22	25.81	0.22
26	27.18	0.24	27.18	0.22	26.85	0.21
27	28.24	0.24	28.22	0.21	27.87	0.20
28	29.29	0.24	29.26	0.21	28.89	0.20
29	30.32	0.24	30.28	0.21	29.90	0.20
30	31.34	0.24	31.28	0.22	30.89	0.21
31	32.35	0.24	32.28	0.22	31.87	0.21
32	33.33	0.24	33.26	0.22	32.83	0.21
33	34.30	0.24	34.23	0.23	33.77	0.22
34	35.25	0.24	35.18	0.23	34.69	0.22
35	36.18	0.24	36.12	0.23	35.58	0.22
36	37.09	0.23	37.04	0.24	36.45	0.21
37	37.99	0.23	37.95	0.24	37.31	0.21
38	38.87	0.24	38.84	0.25	38.15	0.22
39	39.73	0.25	39.74	0.27	38.95	0.24
40	40.58	0.26	40.62	0.29	39.75	0.27
41	41.41	0.29	41.50	0.32	40.52	0.31
42	42.24	0.32	42.38	0.34	41.29	0.29
43	43.06	0.36	43.29	0.37	42.05	0.36
44	43.88	0.39	44.24	0.39	42.85	0.41
45	44.72	0.43	45.25	0.40	43.59	0.48
46	45.58	0.45	46.38	0.39	44.32	0.38
47	46.53	0.44	47.65	0.36	45.29	0.44
48	47.68	0.29	49.12	0.31	46.33	0.38

Note. EASD = estimated asymptotic standard deviation.

Table 5
Equating Results for English Test

	Exponential		Kernel		Equipercentile	
Score	Conversion	EASD	Conversion	EASD	Conversion	EASD
1	1.15	0.33	1.75	0.36	0.62	0.35
2	2.13	0.46	2.48	0.38	1.66	0.46
3	3.02	0.49	3.22	0.39	2.74	0.54
4	3.86	0.49	3.99	0.39	3.83	0.59
5	4.67	0.47	4.78	0.38	4.94	0.62
6	5.47	0.44	5.58	0.36	6.06	0.65
7	6.27	0.41	6.40	0.33	6.87	0.39
8	7.08	0.36	7.25	0.30	7.77	0.33
9	7.91	0.32	8.11	0.28	8.67	0.26
10	8.77	0.29	8.99	0.25	9.54	0.22
11	9.65	0.26	9.89	0.22	10.40	0.26
12	10.57	0.23	10.81	0.20	11.26	0.21
13	11.52	0.22	11.75	0.19	12.12	0.18
14	12.50	0.21	12.72	0.18	13.02	0.17
15	13.53	0.21	13.71	0.17	13.91	0.17
16	14.59	0.21	14.73	0.17	14.81	0.17
17	15.68	0.21	15.78	0.17	15.72	0.16
18	16.82	0.21	16.86	0.17	16.63	0.17
19	17.99	0.21	17.97	0.18	17.56	0.17
20	19.19	0.21	19.11	0.18	18.49	0.16
21	20.42	0.21	20.27	0.19	19.43	0.17
22	21.68	0.21	21.47	0.19	20.39	0.17
23	22.95	0.21	22.70	0.20	21.35	0.18
24	24.22	0.22	23.95	0.21	22.35	0.18
25	25.50	0.24	25.21	0.23	23.33	0.19
26	26.76	0.25	26.49	0.25	24.37	0.20
27	27.99	0.27	27.77	0.26	25.41	0.21
28	29.20	0.28	29.03	0.28	26.49	0.22
29	30.38	0.29	30.28	0.29	27.60	0.26
30	31.51	0.29	31.50	0.31	28.76	0.27
31	32.59	0.29	32.68	0.31	29.94	0.30
32	33.63	0.29	33.82	0.32	31.18	0.31
33	34.62	0.29	34.92	0.33	32.42	0.33
34	35.57	0.29	35.97	0.34	33.74	0.40
35	36.48	0.30	36.98	0.36	35.09	0.42
36	37.35	0.32	37.95	0.38	36.44	0.45
37	38.19	0.34	38.88	0.41	37.82	0.56
38	38.99	0.38	39.78	0.45	39.23	0.61
39	39.78	0.42	40.66	0.49	40.58	0.74
40	40.55	0.47	41.51	0.53	41.80	0.80

	Exponential		Kernel		Equipercentile	
Score	Conversion	EASD	Conversion	EASD	Conversion	EASD
41	41.31	0.53	42.33	0.57	43.07	0.89
42	42.07	0.59	43.14	0.60	44.28	0.68
43	42.85	0.66	43.94	0.63	44.93	0.73
44	43.64	0.73	44.73	0.65	45.50	0.57
45	44.47	0.79	45.50	0.64	46.12	0.61
46	45.35	0.83	46.26	0.61	46.67	0.72
47	46.32	0.82	46.99	0.55	47.29	0.39
48	47.55	0.59	47.68	0.45	48.06	0.33

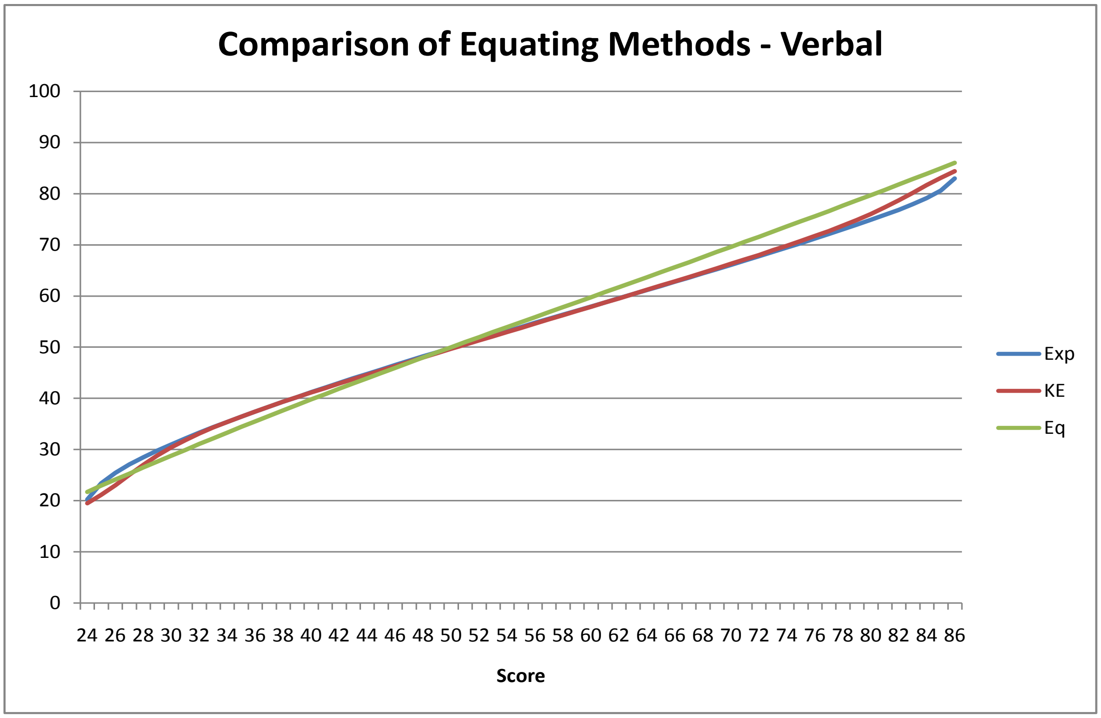
4 Conclusions

On the whole, results for all methods are quite similar. Differences are most noticeable for the highest and lowest scores. The results do illustrate an occasional difficulty with kernel equating based on the normal density. The equated score can be somewhat beyond the range of possible scores. This issue does not arise with continuous exponential families or equipercentile equating. It can also be avoided by use of alternate density functions (Lee & von Davier, 2008). The asymptotic standard deviations for the kernel method do not consider the effects of selection of bandwidth on the basis of data. In the equipercentile case, the discontinuities in the fitted density function are not considered. These issues do not arise with continuous exponential families.

The data do not provide a compelling case in favor of or against any of the alternative equating methods. Current implementations of kernel equating with log-linear smoothing and equipercentile equating with log-linear smoothing assume that the scores to be equated are integers, as is the case with the operational test examined. Continuous exponential families can be applied to scores that are arbitrary real numbers; however, this feature does not have direct impact in this example. Although both approaches require selection of a polynomial, equating by continuous exponential families does have the advantage over kernel equating because a bandwidth need not be selected.

The exact method of adjustment in continuous exponential families for internal rather than external anchors had negligible impact for the data examined. Virtually the same results are obtained if the weight function is simply set to 1.

Results here are for chained equating rather than for post-stratified equating. The authors plan to consider the latter approach in a separate report.



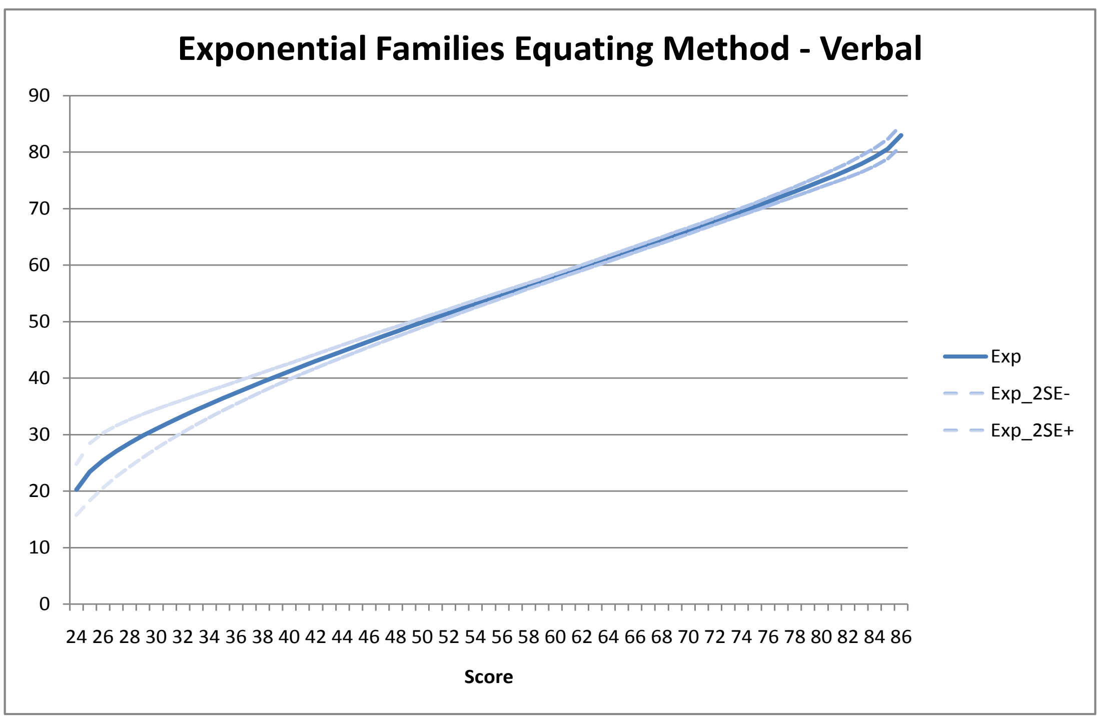
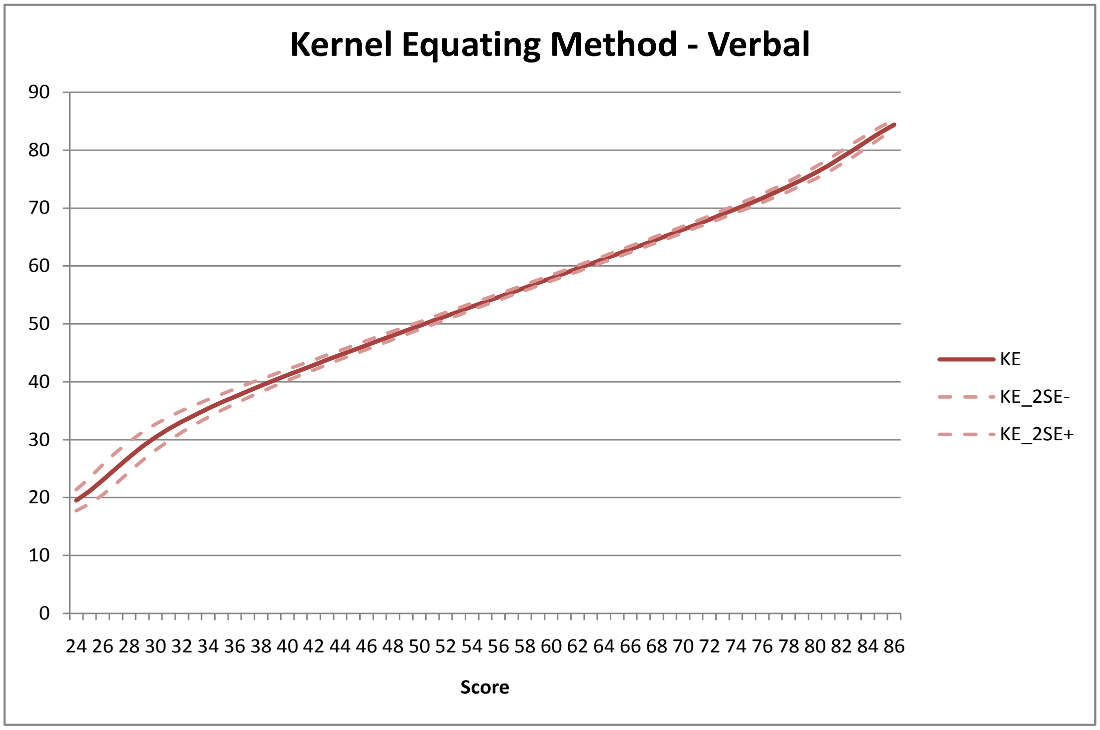


Figure **1 Verbal Results: Continuous Exponential Case**



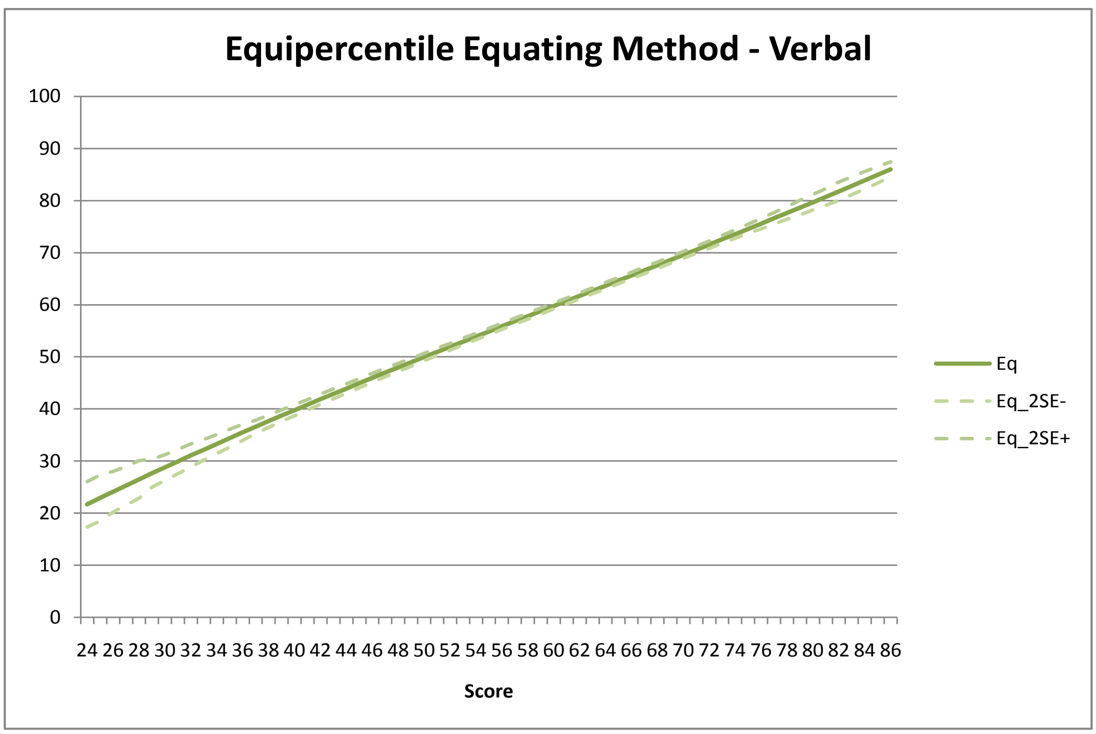
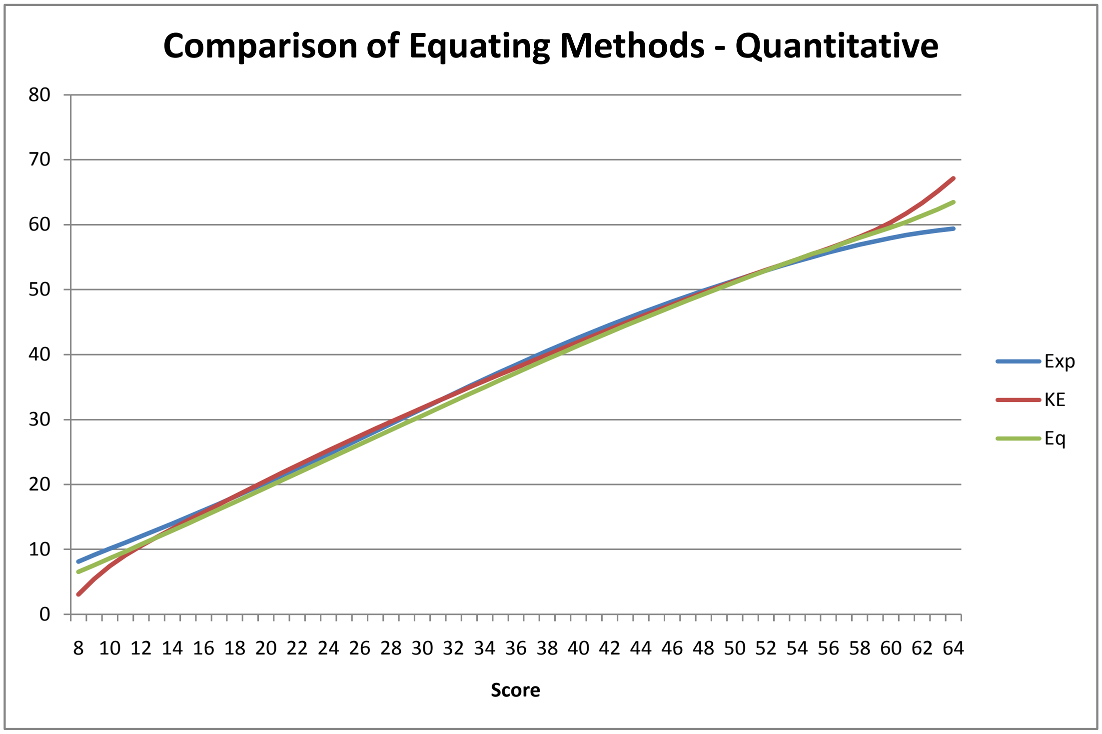


***Figure* 2 Verbal Results: Other Methods**



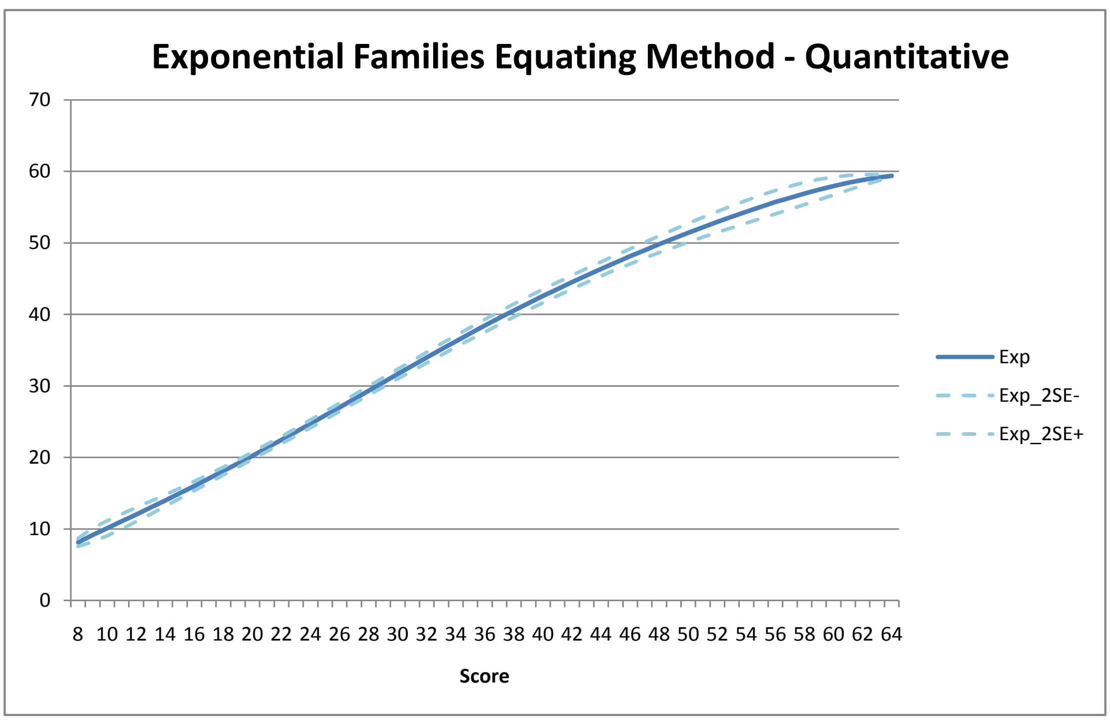
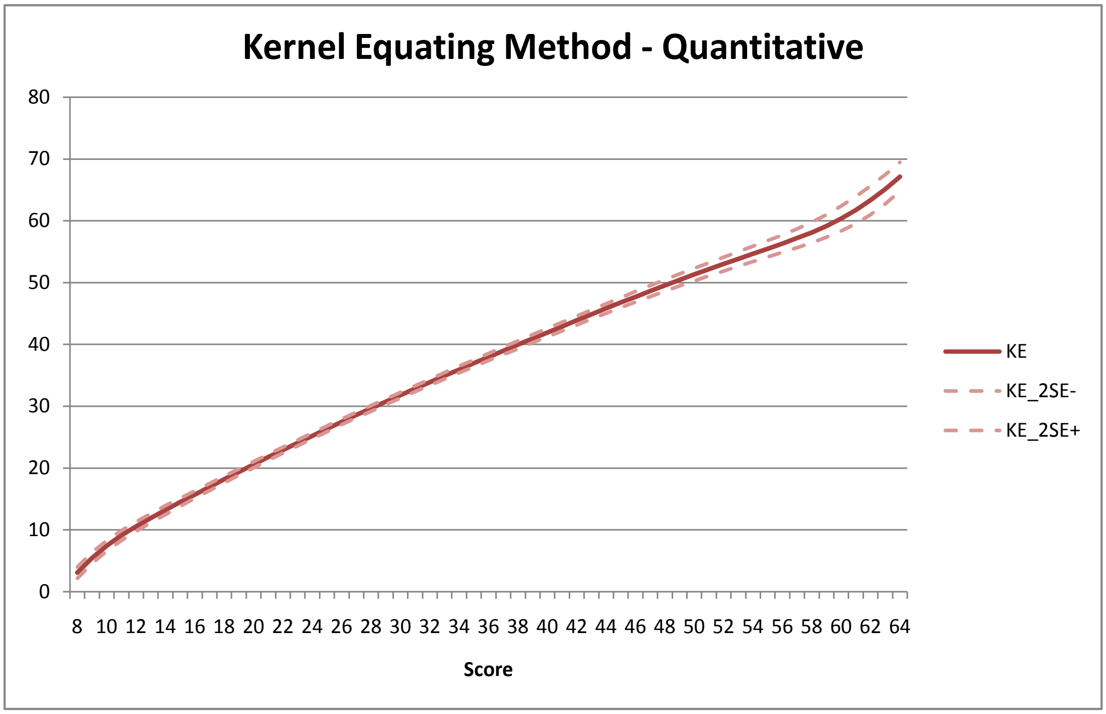


Figure **3 Quantitative Results: Continuous Exponential Case**



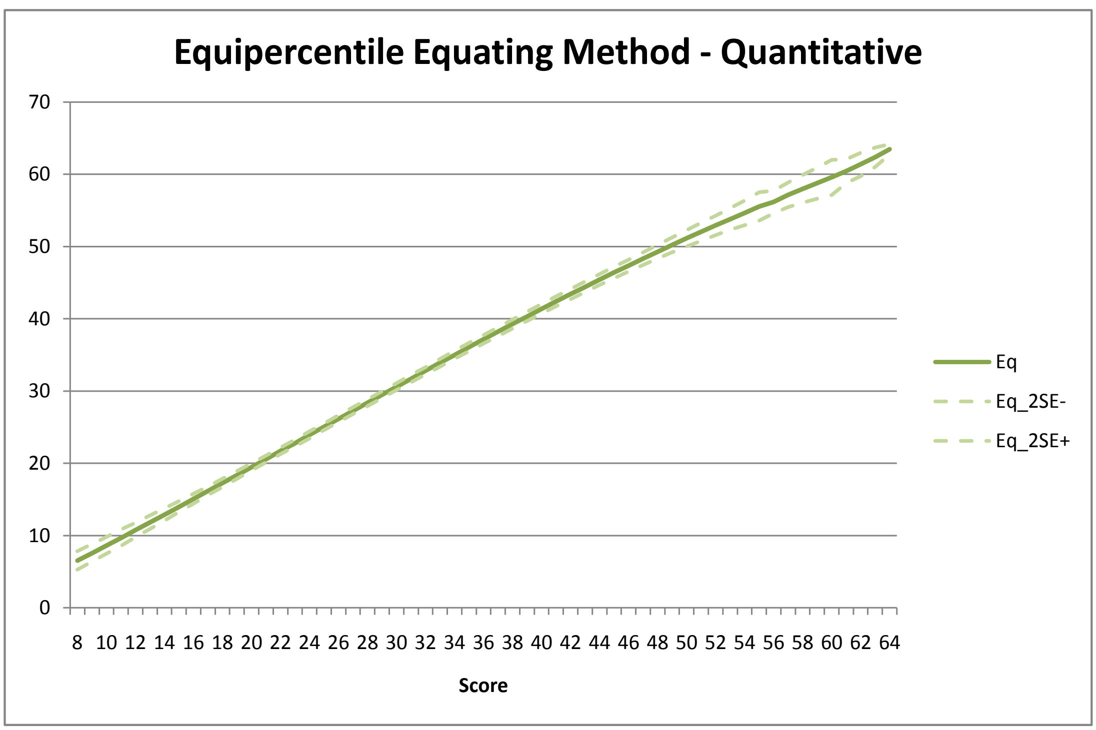
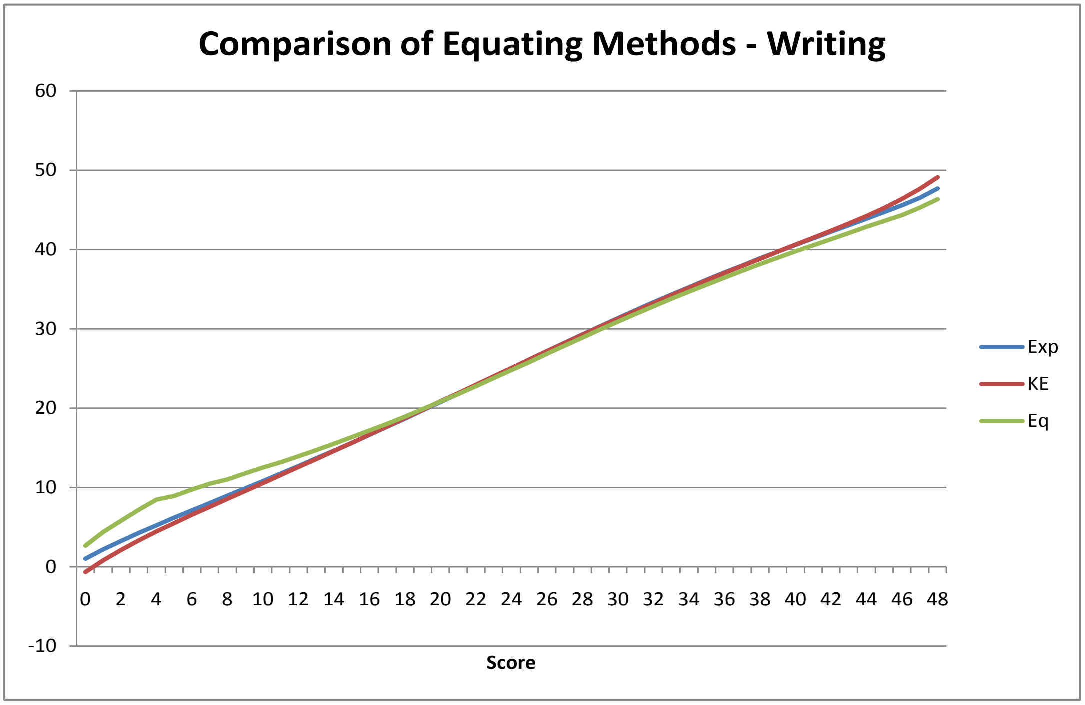


Figure **4 Quantitative Results: Other Methods**



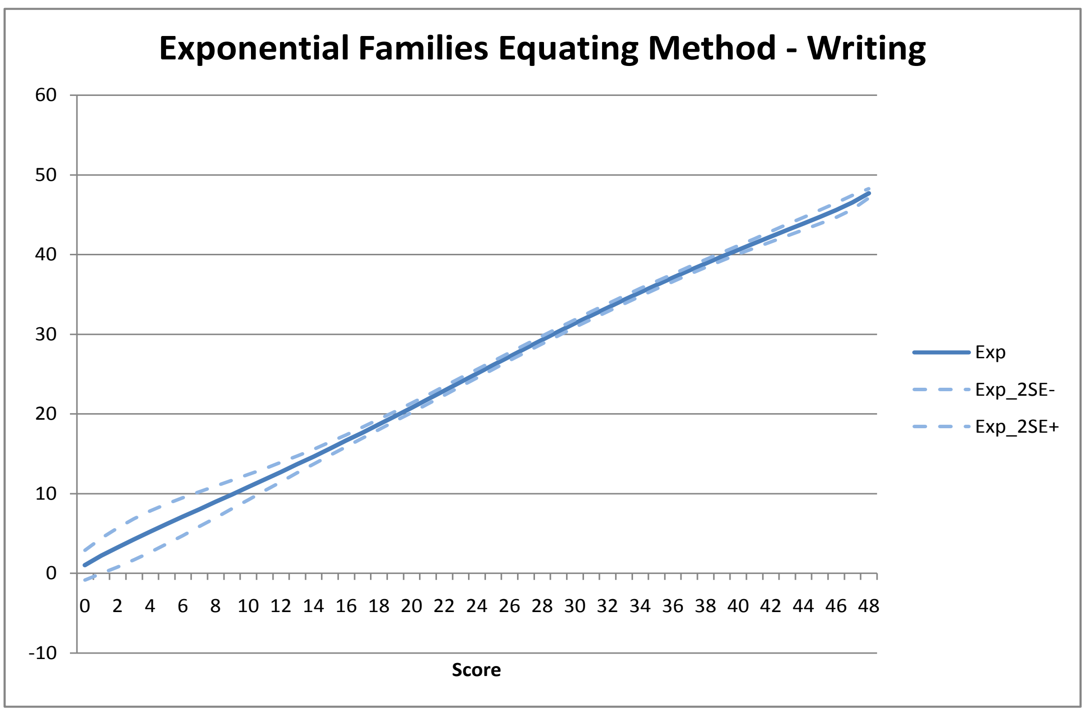
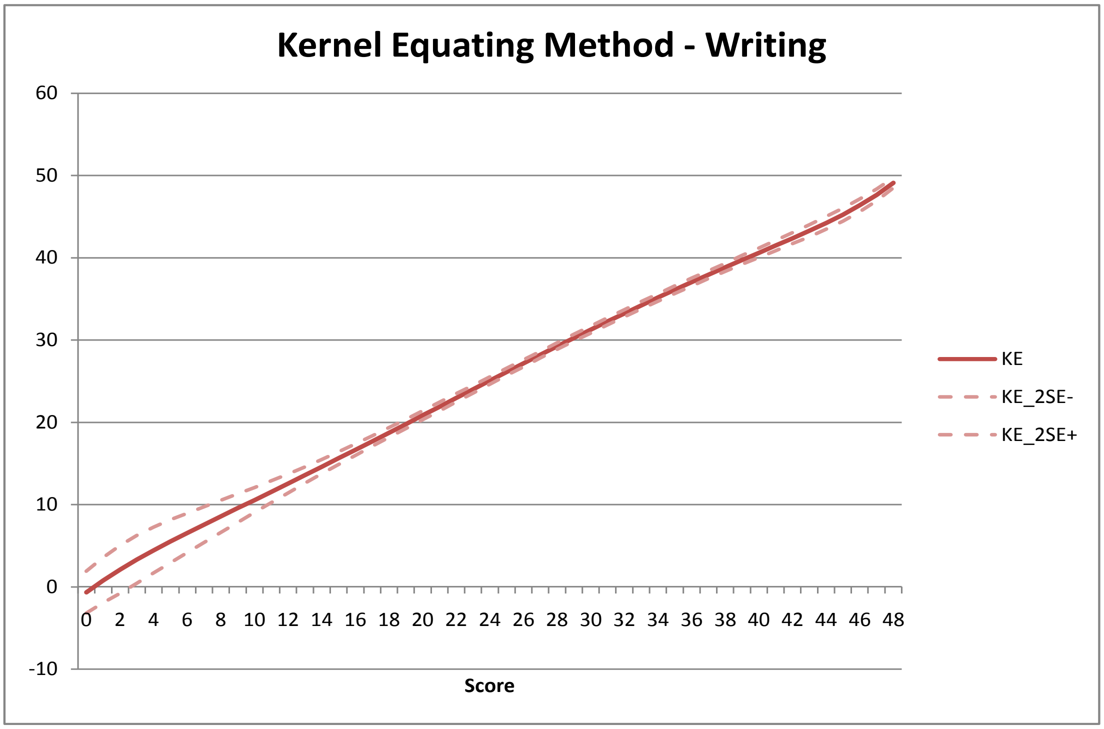


***Figure* 5 Writing Results: Continuous Exponential Case**



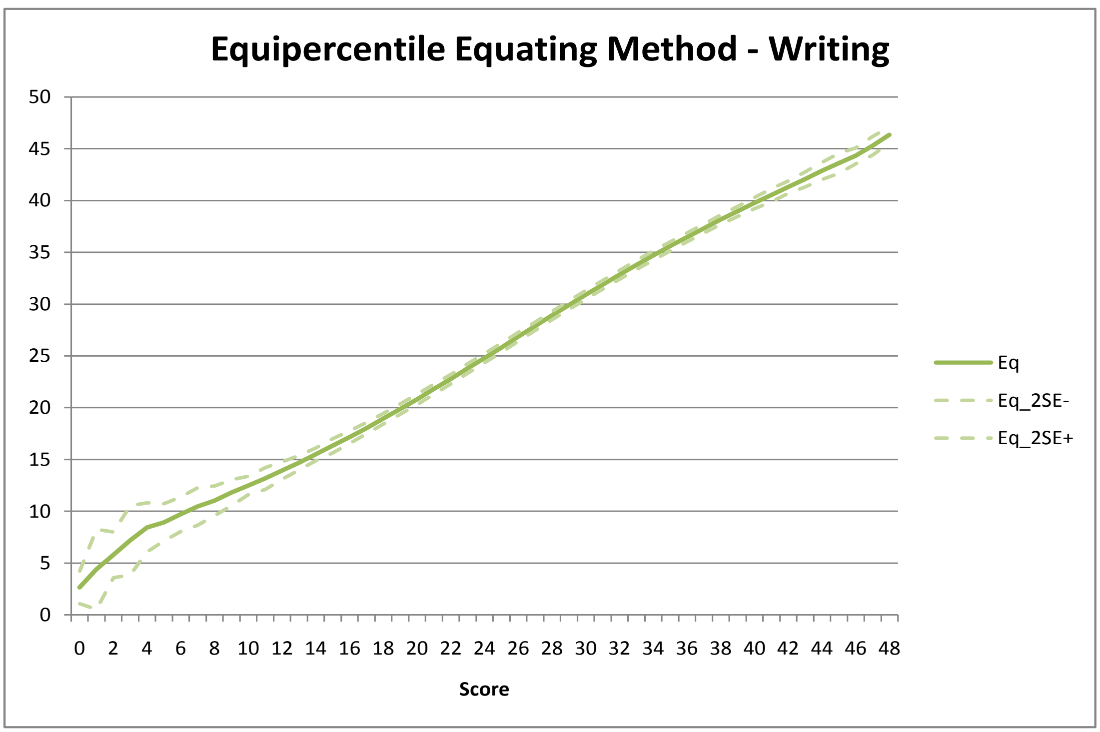
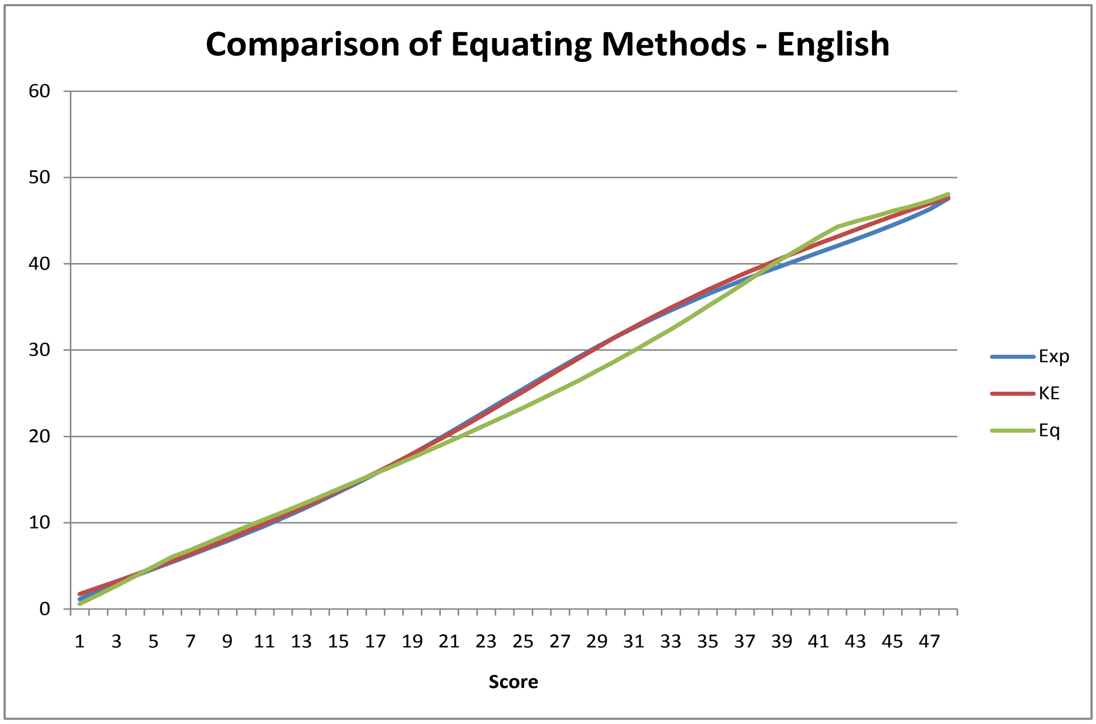


***Figure* 6 Writing Results: Other Methods**



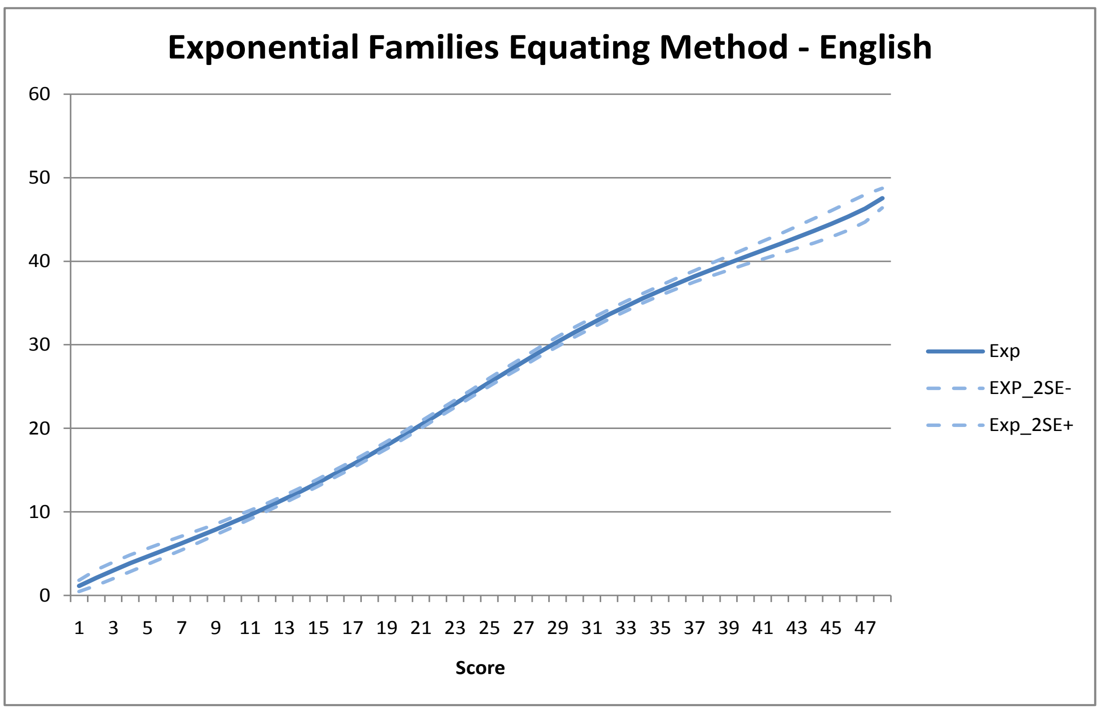
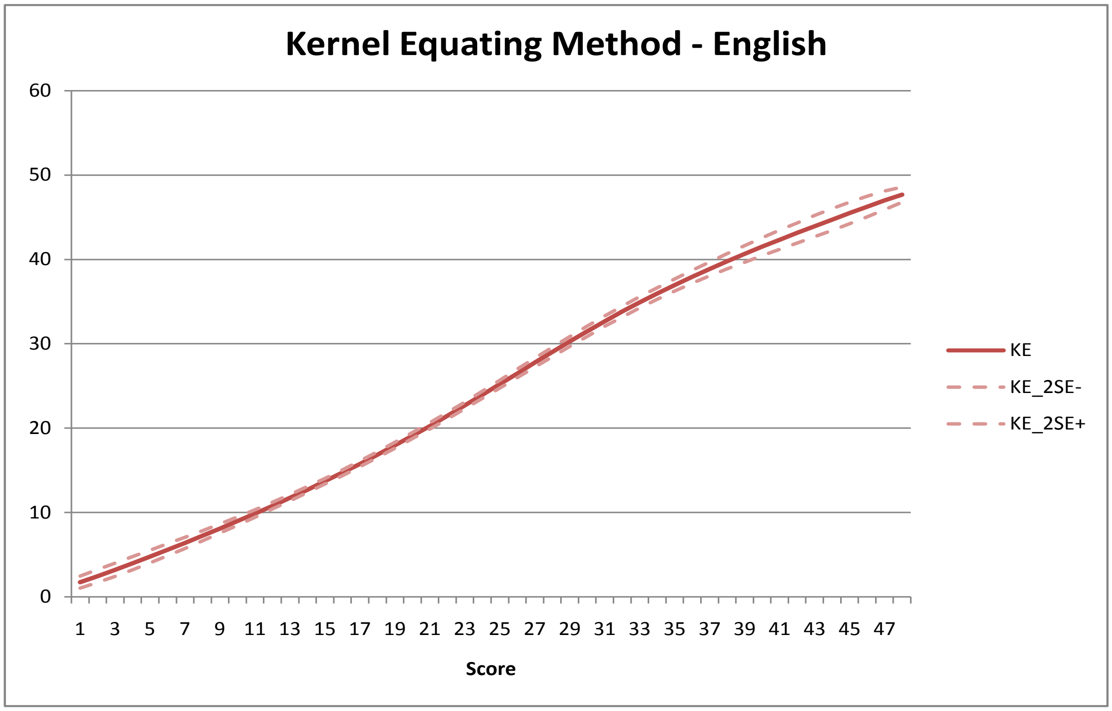


***Figure* 7 English Results: Continuous Exponential Case**



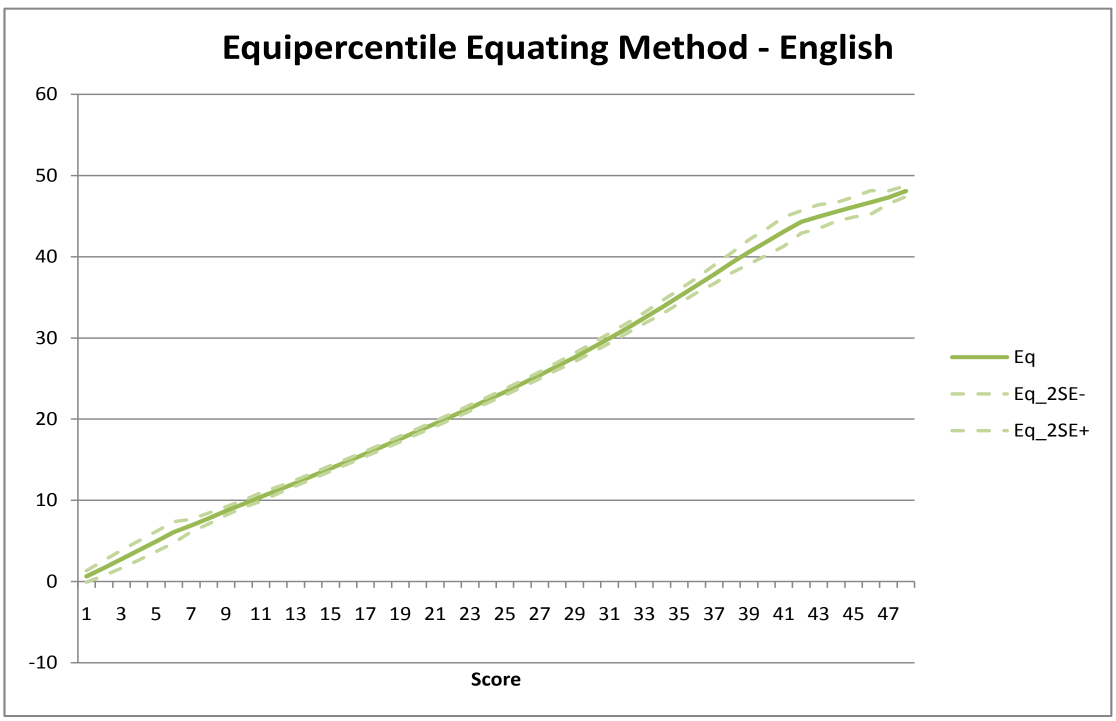


Figure **8 English Results: Other Methods**

In general, it appears that continuous exponential families can be applied to nonequivalent groups with anchor tests. This approach is competitive with kernel approaches and approaches with equipercentile equating. The principal potential gain from use of continuous exponential families is achieved when the number of possible combinations of scores is very large.

References

Abramowitz, M., & Stegun, I. A. (1965). *Handbook of mathematical functions.* New York, NY: Dover.

Gilula, Z., & Haberman, S. J. (2000). Density approximation by summary statistics: An information-theoretic approach. *Scandinavian Journal of Statistics*, *27*, 521–534.

Haberman, S. J. (2008a). *Continuous exponential families: An equating tool* (ETS Research Report No. RR-08-05). Princeton, NJ: ETS.

Haberman, S. J. (2008b). *Linking with continuous exponential families: Single-group designs* (ETS Research Report No. RR-08-61). Princeton, NJ: ETS.

Lee, Y.-H., & von Davier, A. (2008). *Comparing alternative kernels for the kernel method of test equating: Gaussian, logistic, and uniform kernels* (ETS Research Report No. RR-08-12). Princeton, NJ: ETS.

von Davier, A. A., Holland, P. W., & Thayer, D. T. (2004). *The kernel method of test equating.* New York, NY: Springer.

Wang, T. (2008). The continuized log-linear method: An alternative to the kernel method of continuization in test equation. *Applied Psychological Measurement*, *32*, 527–542.