Two and Two
Make Zero

Two and Two Make Zero

The Counting Numbers,
Their Conceptualization,
Symbolization, and Acquisition

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PREFACE

The nature of the things is perfectly indifferent, of all things it is true that two and two make four.

-Alfred North Whitehead

"Five plus three is really zero," explained mischievous Sarai to her second-grade teacher, " . . . because," she argued, " . . . numbers are *nothing*!" Somewhat bewildered and concerned, her teacher related this incident to me in a parent-teacher conference. I smiled to myself realizing that my little Sarai had discovered that numbers are abstract ideas, not physical *things*.

The charming way she articulated this most fundamental and necessary understanding inspired the title of this work.

I

WHAT'S IN A NUMBER?

I-1. THE IDEA

'One,' 'two,' 'three,' 'ten,' 'hundred,' and 'thousand' are number words with which we count, measure, and calculate. Each of these words communicates a discrete idea of size, as do any of the number words between or beyond them. This size idea is envisioned as, and defined by a fixed sum of units or, if you wish, 'ones.' It is because 'five' comprises more 'ones' than 'three' and fewer 'ones' than 'six' that 'five' is larger than 'three' and smaller than 'six.' What makes 'five' 'five' is the exact sum of its constituent units. Add one unit and it is no longer 'five' but 'six'—take one away and it becomes 'four.'

There are, of course, an infinite number of possible discrete sizes of this kind, many more than there are things to be counted or measured. After all, one may always add units to, multiply, or raise to another power any assembly of units one may wish to consider. In the words of Edward Kasner and James R. Newman, "Mathematics is man's own handiwork, subject only to the limitation imposed by the laws of thought."[1]

I-2. DESCRIBING 'THREE'

Like all other numbers, "three" describes a numerical attribute: *three*. Insofar as the numerical attribute of 'three' itself is 'three,' 'three' is a self-descriptive or self-referential concept. Because numbers are self-referential all attempts to define or describe them result in a tautology. Take for example the mathematician philosopher Bertrand Russell's definition of a number as "the class of all classes that are similar to the given class,"[2] which can be roughly translated into: "three is three." His explanation that "every collection of similar classes has some common

[1] Kasner and Newman, 1989, p.359

[2] Russell, 1952, p. 208; 1996, p. 115

predicate applicable to no entities except the class in question,"[3] does not help to undo the circularity.

But it is because of this self-descriptive, self-referential property that each number constitutes a wholly coherent and complete meaning or mental presentation on its own. A sentence such as: "five plus three equals eight," is entirely intelligible and meaningful, even when 'three,' 'five,' and 'eight' reference nothing but themselves. This conceptually self-referential property of numbers stands in contrast to adjectives, such as 'beautiful' or 'large,' which are also descriptive concepts. 'Beautiful' or 'large,' in and of themselves, do not form definite mental images; they are intelligible only in relation to the objects they describe; hence, their properties are changed and modified according to these objects. For example, a 'beautiful butterfly' and a 'beautiful poem' convey different qualities of the idea of beauty. Similarly, a 'large beetle' and a 'large elephant' define substantially different dimensions of largeness. The notion 'three' on the other hand always conveys exactly the same idea, whether it describes that number of butterflies, poems, beetles or elephants.

Number-concepts' self-sufficiency makes numbers indefinable and at the same time absolute, and definite ideas of size.

I-3. WHY NUMBERS? THREE APPROACHES TO SIZE ASSESSMENT

Numbers are so ingrained in the fabric of our daily life that it is difficult to imagine life without them. Yet empirical evidence shows that early man, and some tribal cultures continuing well into the twentieth century, managed quite well without numbers. Indeed, numbers are not the only size concepts available to the human mind, and are not the only means by which one can objectively and accurately examine quantities.

One path to determining size is through ordinary direct perception. After all, the ability to determine the approximate size of objects—like determining their color, shape, location in space, speed, etc.—is one of the perceptual faculties necessary to the survival of any moving, foraging, preying or preyed upon creature. Sizes conceived in this manner are, of course, a property of perceptible objects. Consequently, size ideas that are established by ordinary perceptual processes are inevitably tied to a distinct phenomenon, namely, the object of that perception. It is through their association with specific objects that size concepts generated through perception become viable, definite, and meaningful. The dimensions of an object, as it is, are stored in memory and can be recalled and imagined whenever

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[3] Russell, 1996, p. 116

the object itself is brought to mind. An elephant brings to mind a specific and definite idea of size, while a beetle brings to mind a different size idea.

The perceptual process judges the size of a phenomenon by its total impression, that is, the amount of space it occupies. Thus, phenomena that become objects of perception are viewed as continuous wholes, and the focus of attention falls on their outline or contour even when the objects to be evaluated are an aggregate such as a flock of birds or a pile of apples.

The cognition that three apples is a larger quantity than two similarly sized apples, or that four apples are more than three, requires no knowledge of numbers. Such distinctions are easily, effectively, and correctly established by means of perceptual criteria without recourse to enumeration. Three apples simply occupy a greater area than two apples and a smaller area than four. However, the global nature of this direct perceptual path to discriminating and evaluating size or quantity does not allow an analytical and objective examination insofar as an analytic determination of size requires division into units and methodical attention to detail. Hence, the direct perceptual approach to size evaluation can only yield a size idea that is subjective, impressionistic, and tentative.

For exactitude and objectivity in examining inventory or ensuring fair trade, pre-number-concept humans used a technique known as *one-to-one correspondence* or ‘exchange.’ This procedure is based on the understanding that when objects in one collection form a one-to-one relationship with the objects in another collection, each group of objects comprises the same numerical size. The primitive one-to-one correspondence is carried out through physically handling objects, which are perceived and treated as constituent units of collections. In contrast to perceptual assessments that view collective phenomena as continuous entities (e.g., a flock of geese, a school of fish, a pile of berries), the one-to-one method breaks down a collectivity into its component parts and considers each separately. Through this division into constituent units and their analysis, the one-to-one examination of a quantity achieves its objectivity and accuracy.

Karl Menninger describes the one-to-one procedure as it is carried out by the Wedda tribe of Ceylon as follows: When a Wedda wants to examine the coconuts in his possession he collects a bunch of sticks, assigns one stick to each coconut, and says, “This is one.” These sticks serve as *auxiliary* or *supplementary* quantities he can later use to examine his coconut inventory as follows: He takes one stick from the ‘supplementary quantity’ and pairs it with one coconut in his collection. He continues to pair sticks with coconuts, one pair at a time, until all the coconuts are exhausted.[4] If one stick is left over, the Wedda ascertains accurately and objectively that one coconut is missing.

4 Menninger, 1992, p.33

The one-to-one technique enables the Wedda to verify losses and gains with an adequate degree of accuracy and assuredness; however, it cannot help him to form a meaningful idea of the total amount of coconuts in his collection. In laboring to pair sticks with coconuts, he directs his attention to a sequential recognition of units, one unit at a time, and is unable to consider his coconut inventory as a totality. The concrete one-to-one correspondence, then, forgoes a global and meaningful conception of size in favor of exactness.

The absence of a comprehensive notion of explicit numerical values in cultures that use the one-to-one procedure is illustrated in Sir Francis Galton's story about a barter made between a shepherd of the Damara tribe and a tobacco trader.[5] The Damara are nomads who move with their herds in small groups. Their number vocabulary contains only three number words. The shepherd, according to Galton, agreed to trade sheep for tobacco at a rate of one sheep per two twists of tobacco, but when the tobacco trader offered him four tobacco twists in exchange for two sheep, the shepherd became confused and declined the offer under that term. Instead, he insisted on breaking the exchange into two separate transactions. Even though he could accurately and effectively extract 'four' via one-to-one exchange, the Damara shepherd was unable to conceive or verify the numerical value of four twists of tobacco—a number for which he had no name or concept.

Without reference to a numerical concept and its symbolic definition, the shepherd could ensure fair trade only by exchanging two twists of tobacco at a time, just as the Wedda tribesman, described by Menninger, could only point to his bunch of sticks in order to indicate how many coconuts were in his possession.

The one-to-one procedure and the direct perceptual impression of size, as described above, seem to be contradictory and irreconcilable approaches to size evaluation, for each establishes its own merit by foregoing the virtue of the other: The perceptual approach produces meaningful ideas of size that are devoid of accuracy and objectivity, and the one-to-one approach produces accuracy and objectivity devoid of a meaningful notion of size.

Yet the two methods share an essential trait: both are inextricably bound to, and helplessly constrained by the concrete objects of their attention: The one-to-one method proceeds through physical manipulation of objects and evaluates the size of a group with respect to another concrete group. The size concepts that are derived from perceptual impressions not only rely on inputs generated by the physical world, but also obtain meaning from, and are defined by the objects of that world. It is this confinement to the concrete and the physical that prevents the analytical and global views from becoming complementary aspects of a single approach to size evaluation and definition.

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5 Galton's 1889 "Narrative of an Explorer in Tropical South Africa," (Cited in Menninger, 1992, p. 34, and Claudia Zaslavsky, 1999, p. 32)

A number, in contrast, is a discrete and abstract model of size. Each number is conceived and imagined as a specific amount of 'units.' Conceptualization of a number requires both the identification of a sum as a whole and the identification of the discrete units that constitute it. Since units must be recognized before being assembled together into a sum, the conceptual identity or mental representation of a sum is built upon prior analysis of units. Thus, a number is an idea of size that is holistic and perceptual at the same time that it is analytical and exact.

It is the abstract nature of that mental construct—the number—that permits the integration of the holistic with the analytical into a singular and distinct new mode of size evaluation and definition.

II

NUMBER APPLICATION

II-1. NUMERICAL EVALUATION OF CONCRETE MAGNITUDES

Numbers are pure abstractions and as such they are neither bound by nor inherent in any physical phenomena; this is the very reason numbers can be used to quantify any magnitude one may wish to consider, be it concrete or abstract. There are two major categories of numerical evaluation of concrete magnitudes: *counting* and *measuring*. The aim of measuring is to answer the question, what is the *physical size* of a given entity, for example, *how big?, how heavy?,* and so on. The aim of counting is to answer the question, *how many units*. These are two different questions, each demands a different interpretation of the numbers that constitute their answers.

Numbers in and of themselves, however, answer only the question, *how many?* Counting is a direct application of numbers since it uniformly pertains only to the sum of the units in a collection (which, in the instance of a concrete collection, are phenomenally discrete objects) and not to a collection's actual physical size or to the properties of the objects it contains. In fact, counting is a reciprocal process in which the units of the quantifying number—the number words—are counted in tandem with the objects of the collections under consideration. In a procedure not unrelated to the Wedda-tribesman's pairing sticks with his coconuts, number words are paired with objects comprising a collection, one pair at a time—"one apple," "two apples," "three apples," etc. There is, however, an important difference between the primitive 'auxiliary quantities' used in the one-to-one procedure, and the number words used in counting. Unlike the Wedda's sticks, each of the number words represents a unique and discretely recognized numerical idea. Without the reference to the abstract numerical concepts that these number words convey, counting can yield no meaningful notion of an exact numerical value of a quantity. For the number-educated counter, every number word defines simultaneously the ordinal value[1] of the object it counts, as well as the cardinal value[2] of the entire group of objects counted thus far (as in the 'one apple,' 'two apples,' etc.). The process of counting concludes when all the objects in the examined group have been exhausted such that the last number word used in the counting procedure fully defines the answer to the question *how many?*

The question *how big?,* on the other hand, concerns the magnitude of a particular physical attribute of a phenomenon, for example, the volume or weight of a watermelon, or group thereof, or the length or width of a piece of lumber. Continuous magnitudes, such as length or weight, must be structured or divided into constituent units because unless a magnitude is viewed and treated as aggregates, it cannot be defined by a numerical value. These units must be of the same nature or description as the attribute they measure; thus, length is measured with length units (centimeters, inches, etc.) and weight with weight units (grams, ounces, etc.). Yet, artificially concocted, the measuring units are arbitrary—different units may be devised to measure the same attribute. For instance, while length cannot be measured with sq. inches, or grams and ounces, it may be measured with centimeters as well as with inches.

Since the actual extent of a concrete size is expressed as the product of a number and a constituent unit (e.g., the length of my calculator is 5 inches, and its weight is 3 ounces), the size of a measurement is in fact determined by two sizes: the size of the measuring unit and the size of the quantifying number. Hence, the same number may express different dimensions depending on the size of the measuring units it counts. For instance, because an inch is a larger unit than a centimeter is, a 4-inch-long pencil is longer than a 4-centimeter-long pencil (see Figure II-1). At the same time, and for the same reason, a numerical definition of the same pencil's length is roughly *10* (10.16 to be exact) when the measuring unit is a centimeter, but *4* when the measuring unit is an inch (see Figure II-2).

As Figure II-1 demonstrates, the same number (4 in our example) defines different pencils' lengths, while different numbers (4 and 10 in our example) define the same pencil's length (Figure II-2). Yet all these numbers express exact and objective measurements of the pencils' lengths. Of course, exact and objective measurement is the whole purpose for the numerical definition of physical magnitudes. After all, a reasonably functional notion of a pencil's length can be obtained merely by looking at it. Our visual perception will tell us that the pencil will fit in a handbag but not in a coin purse.

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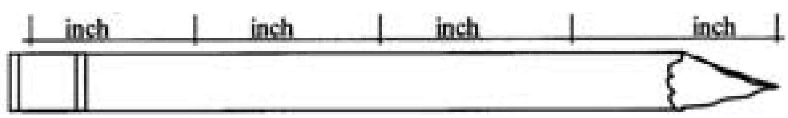
1 Davis, 1961, p. 6, Philip Davis explains that 'ordinal number' answers the question 'How far a long in a line'? The ordinal value of two numbers determines which of them precedes or follows the other.

2 Ibid., Cardinal numbers, according to Davis, express quantity, the cardinal value of a number answers the question: 'How many'? Or 'How much'?

But when we take the trouble to measure an object, the numerical element of that quantification must be understood as an expression of the *ratio* between the measured *magnitude* and the constituent *unit* with which it is measured—the larger the unit of measurement, the smaller the quantifying number, and vice versa. Thus, unlike in counting, in which the resulting number fully and directly defines the answer to the question *how many?*, in measuring, the resulting numbers render a relative value: As such, the numerical element of measurements is only a partial and indirect answer to the question, "how long?" or "how heavy?" etc. Indeed when asked, "How long was your walk?" you would never answer, "Twenty;" such an answer would automatically elicit the question, "Twenty what?—Minutes?—Hours?—Miles?—Kilometers?—Yards?" Whereas, when asked, "How many people came to the party?" the answer, "Twenty," will be adequate.

These differences notwithstanding, in both kinds of applications numbers remain abstract entities. In counting they serve as preconceived abstract models of exact and discretely recognized sizes against which the concrete quantities are evaluated; in measuring, they serve as the ratio between the size of the measured magnitudes and the size of a particular constituent unit.

Pencil length ≈ 4 *inches*



Pencil length ≈ 4 *centimeters*

Figure II-1: Comparative measurements

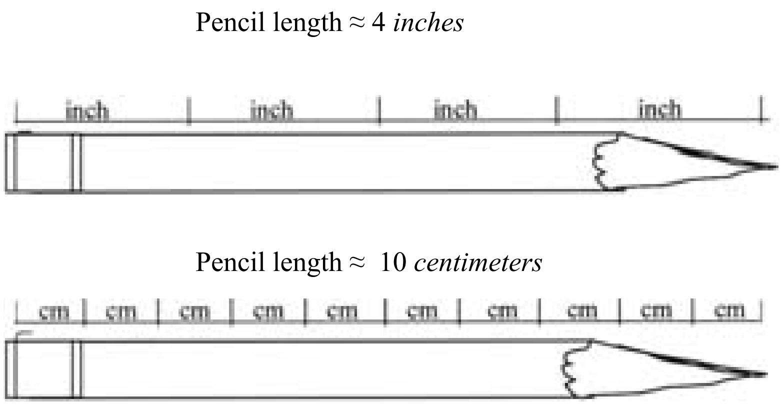


Figure II-2: Comparative measurements

II-2. CALCULATION

"There are some, King Gelon, who think that the number of sand grains is infinite in multitude." So opens the famous treatise, *The Sand Reckoner*, written by the third century BC Sicilian genius, Archimedes. Archimedes took it upon himself to prove that it is possible to find out the number of sand grains, and he meant, "not only the sand which exists in Syracuse and the rest of Sicily, [. . .] but that which is found in every region whether inhabited or uninhabited," in effect, "that of a mass equal in size to the universe." He thought to obtain that number not by counting, of course, but through calculation. In Archimedes' time, the Greeks believed that the universe was contained within a crystal sphere to which the stars were attached.[3] Assuming that grains of sand, like the universe, were spheres, Archimedes calculated the number of sand grains needed to fill the universe by figuring out the ratio of the universe's volume to the volume of a grain of sand. A task he had accomplished by means of geometrical proofs and arithmetic operations, amounting to "a series of calculations that would give a high school boy nightmares," according to Gamow. (Ibid.)

Because the largest number the Greeks could name at that time was "myriad" (10^4), Archimedes had to invent a new number system in order to express the number he pursued. The numerical system he devised was, in essence, a base

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[3] Gamow, 1960, p.17-19

system, in which, much like in our modern ten-base system, ten units of any given 'order' or 'rank' make up one unit of the next higher rank. Archimedes' base was *myriad myriads* (10^8 in our base-ten system). He called this base an *octade* or "a unit of the second class." (Ibid.) With this new number system, Archimedes could define the number of grains of sand in the universe as "not greater than one thousand myriads of units of the eighth class," (Ibid.) which would be ten to the power of sixty-three (10^{63}) in our ten base system.

Archimedes' brilliant treatise is a breathtaking example of how knowledge that is unattainable by direct method of counting or measuring becomes attainable through calculation.

Calculation involves relating numerical or spatial entities to one another in the abstract, the results of which are later applied to the subject of its examination. In Archimedes' "sand-reckoning" example, the subject of examination was a numerical value: the number of sand grains contained within a given magnitude (the universe). This value was obtained by calculating the ratio between a magnitude (the universe) and a unit of measurement (a grain of sand), both of which had to be calculated also beforehand. But calculation can be and is often used for much simpler and mundane purposes. For example, it can be used as an expedient way of counting, say finding the number of almond trees in Ms. Smith's garden by multiplying the number of trees in each row by the number of rows, or as a means of figuring out the measurement of magnitudes that cannot be measured in direct ways, say, figuring out the area of a rectangle by multiplying the measurement of its length by the measurement of its width.

Unlike counting and measuring, which are applied directly to the objects in question, calculation procedures, which are carried out in the abstract, are independent of the objects of their examination and, as it were, need not relate to real world objects in the first place. For instance, the expression, '7x6=42' may represent the area of a rectangular room, the number of almond trees in Ms. Smith's garden, or merely a mathematical truth and nothing else. The difference between the proposition, '7x6=42,' and the propositions, '7 trees x 6 = 42 trees,' or '7 feet x 6 feet = 42 sq. feet' is that the latter two are concrete applications of the purely theoretical former.

Mathematicians call the field of studies that uses mathematical calculations as an instrument for exploration and examination of real world phenomenon *applied mathematics*, and that which uses calculations as a means to explore mathematical ideas for their own sake, *pure mathematics*. The latter can be considered "humanistic mathematics" because it is driven by humans' intellectual and aesthetic needs.[4] Calculation, then, introduces a new way for number concept application, that is, their application in the exploration and development of new mathematical ideas.

4 Wilder, 1968, p. 6

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Legend has it that Archimedes also invented a new written-numeral system, akin to that of our modern Hindu numeral system.[5] But there is no archeological evidence to support that myth. What is clear, though, is that Archimedes' was a "humanistic" mathematical effort par-excellence. Neither the knowledge of how many grains of sand are needed to fill up the entire universe, nor the numerical system that could afford expression of numbers of such magnitude as "one thousand myriads of units of the eighth class (10^{63})" was of any practical use in the Greek civilization of Archimedes' time. Indeed, Archimedes' true intention was not to find the number of grains that could fill up the universe, but rather to demonstrate to king Gelon that there exists no multitude, regardless of its magnitude, that cannot be defined by a number, provided one can establish a method to name numbers as they progress along their infinite sequence.

[5] Ball, 1960, p. 72, According to Ball, some believe that "Archimedes and Apollonius had some symbolism based on the decimal system for their own investigations . . . "

III

NUMBER PERCEPTION

III-1. THE PERCEPTIBLE NUMBER

We tend to take for granted our ability to capture in a glance the exact number of objects in a small group, say three or five, but this seemingly simple and matter-of-fact perception deserves our wonder and consideration. After all, numbers in and of themselves are merely mental constructs, abstract ideas—not physical entities. Lacking texture, color, shape, or smell, they are devoid of perceptible qualities by which they can be discerned, recognized, and distinguished from one another through our sensory system.

That we can perceive the number of a small collection of objects, even though numbers themselves are devoid of physical properties, is because there is much more to human perception than simply responding to environmental stimuli. As is pointed out by psychologist, Ulrich Neisser, "Stimuli themselves cannot possibly have meaning because they are merely patterns of light, sound, or pressure."[1] The function of the perceptual processes is not only to capture environmental stimuli, but also to transform this array of unorganized inputs into coherent and discretely recognized entities.

In his groundbreaking biological theory of consciousness detailed in his 1989 book, *The Remembered Present*, the neuroscientist Gerald Edelman furnishes an important insight into this issue. Edelman distinguishes between two states of human perception: *perceptual categorization* and *perceptual experience*. The more primary process is 'perceptual categorization,' by which sense data are grouped and organized. Perceptual categorization enables humans to discriminate objects and events from their background.[2] It is carried out by increasingly complicated neural connections and organizations, through which sensory inputs are continuously re-categorized and generalized, and occur without the organism's conscious awareness. In contrast to 'perceptual categorization,' Edelman describes

[1] Neisser, 1976, p. 71

[2] Edelman, 1989, p. 49

'perceptual experience' as "consciousness of objects and events."[3] Edelman ties the formation of consciousness to the emergence of concepts and language, and speculates that 'perceptual experience' is attained via a process of contrasting and comparing the conceptual memory with on-going 'perceptual categorization;' "previous memories and current activities of the brain interact to yield primary consciousness as a form of '*remembered present*.'"[4]

The infusion of conceptual references into the human perceptual process, suggested by Edelman, implies that our perceptions of the physical world are as much the outcome of our own conceptual network as they are of the sensory stimuli we pick up from that external world. Thus, our perceptions of the external world may be modified and change over time as a result of experience and education. For instance, what may be perceived as a blotch of color for a newborn is perceived by a toddler as a poor illustration of a lady's boot, and by a first or second grader as the map of Italy. Since humans are capable of forming abstract concepts, these concepts too are destined to be incorporated into our perception of the external world once they have been acquired. As an educated person, I cannot help thinking 'three' when I come across a group of three apples, and do so for the same reason I cannot help thinking 'Italy' when I come across the map of Italy. "Perception," as Ulrich Neisser pointed out, "is where cognition and reality meet."[5]

Because the perception of apples relies upon a phenomenally based concept, it can be triggered by the sensory inputs generated by the physical properties of the apples: their taste, texture, shape, or smell. But the perception of their number, which relies on the abstract and mentally constructed numerical concept, 'three,' can be triggered only by the preconceived idea of that concept. A numerically illiterate person will be blind to the number of apples in a given group even when that number is as small as three. With no conceptual reference to numbers, he is able to respond only to the physical stimuli generated by the apples and, thus, can recognize only the apples, but does not perceive the number their group constitutes.

That the availability of numerical concepts affects the ways we view small aggregates is consistent with anthropological observations made in the early twentieth century. These studies observed that some tribal cultures that did not develop numerical systems beyond two or three displayed an inability to perceive specific numbers in groups that contain more than two or three objects (recall the Damara shepherd in I-3). These findings support the theory that the Indo-European words for 'three—' three, trois, dre, tres, tri, etc.—stem from the same Latin root—'trans—,'which means 'beyond,' suggesting that early humans could not

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3 Ibid., p.155

4 Ibid., p. 105 (emphasis mine)

5 Neisser, 1976, p. 9

perceive the specific number of objects in groups that contained more than two objects.[6]

With regard to very young children, "number blindness" was demonstrated in a dramatic way in one of Descoeudres' 1921 experiments.[7] Descoeudres displayed a small number of objects in front of children of various age groups. She then asked the children to put down the same number of objects as in her display. When only two or three objects were on display, the two and four-year-old children performed this task without error, but when larger quantities were on display they made many errors. These young subjects, who were conversant only with numbers within the three-range, could not replicate numbers beyond that range, even when the concrete examples of those numbers were in plain sight before their eyes. Descoeudres' experiment demonstrates that children cannot recognize (or 'see') specific numbers, and cannot reproduce groups of specific numbers of objects for which they yet have no available conceptual reference.

It should be pointed out that when we think about number education, we have to take into consideration not only Descoeudres' finding, but also that the notion of a 'unit' is an abstract concept in and of itself. In order to relate to an object as 'unit,' one must disregard this object for what it is (an apple, in our example) and extract its abstract significance as a member in a collection. Both the *units* and the *number* they constitute transcend the particularity of the physical reality that makes numbers visible to a numerically literate person.

Hence, there is a pedagogical paradox in the attempt to exemplify numbers to the numerically naive by presentation of concrete aggregates. In showing a child three apples with an intention to teach her the concept 'three,' we actually expect her to ignore the objects' tangible aspects, the elements that she can pick up spontaneously by her sensory system, and to relate, instead, to abstract ideas of 'units' and 'numbers,' which we assume have not yet formed in her mind. Indeed, as Descoeudres' experiment has shown, the concrete examples of aggregates, no matter how small or how carefully displayed, do not enlighten children.

III-2. NUMBER ILLUSIONS

We often fall victim to the illusion that the structures we discern in the real world with the mediation of our thoughts and concepts are imposed on us from outside by our physical surroundings. Number perception is a poignant example of that predicament. Once we have established numerical concepts, not only do we spontaneously perceive the number of a small group of objects in a glance,

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6 Menninger, 1992, Dantzig, 1954, Wilder, 1968

7 Descoeudres, A. 1921 (Cited in Bryant, 1974, p.119)

but we also look at their number as if it were a physical reality in and of itself. Since our own notion of numbers affects the way we teach them to our children, this topic deserves a closer scrutiny. Let us, then, examine the perception of three apples again.

'Three apples' is a percept that involves two distinct and unrelated conceptual references ('apples' and 'three') of which only the apples can generate sensory information. While the proximity of a pile of three apples makes the abstract concept 'three' visible to the numerically literate observer, "There is nothing *in the world* that makes a collection of objects a set other than our choice to so consider them."[8] Because numbers are not physical phenomena, the sensory information generated by a pile of apples is relevant only for the recognition of the apples as 'apples,' not their number. The only relevant reference for the recognition of their number (three) is our pre-existing concept 'three.' Therefore, the recognition that the number of apples we see is *three* (and not two, four, or five) is extracted from our own inner and preexisting conceptual reservoir, not from the sensory information that is generated by the pile of apples.

Owing to the inherently integrative nature of our perceptual system, considerable rational effort is needed to shake off the notion that numbers are independent of our mind. In the case of number perception we tend to blend together the abstract concepts of number with the phenomenally based concepts of objects that make these numbers visible. Because of this perceptual synthesis, a group of objects representing units and the number their collection comprises ('apples' and 'three,' in our example) are perceived as a single integrated entity ('three apples'). As we lose the distinction between an abstract numerical idea and those physical objects that represent its constituent units, the number and the objects the number counts seem to us inseparable from one another. Fooled by this perceptual illusion, we tend to reify or objectify the numerical concept and perceive it as if it was an external phenomenon that is independent of our mind. Mathematicians themselves are not immune to this deception. In the epilogue for their 1940 classic, *Mathematics and the Imagination*, E. Kasner and J. R. Newman write,

> We have overcome the notion that mathematical truths have an existence independent and apart from our own minds. It is even strange to us that such a notion could even have existed. Yet this is what Pythagoras would have thought—and Descartes, along with hundreds of other great mathematicians before the nineteenth century.[9]

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8 Yaseen, 1999, p.7

9 Kasner and Newman, 1989, p. 359

R. Wilder expresses a similar opinion:

> Mathematicians themselves seem prone to ignore or to forget the cultural nature of their work and to become imbued with the feeling that the concepts with which they deal possess a 'reality' outside the cultural milieu.[10]

He quotes a noted twenty-century mathematician, thus:

> I believe that mathematical reality lies outside us, and that our function is to discover or observe it, and that the theorems which we prove, and which we describe grandiloquently as our 'creations' are simply our note of our observations.[11]

In a similar vein, the mathematician, Leopold Kronecker, famously asserted that "The integers were made by God, but everything else is the work of man," and a statement in a book for high school teachers explains,

> The numbers 1,2,3,4,5 . . . . are called *natural numbers* because it is generally felt that they have a natural existence independent of man. The most complicated of the number systems, by way of contrast, are regarded as intellectual constructions of man.[12]

Ironically, it seems that mathematicians may succumb to number illusion without ill effect on their work. "The question, 'what is a number?' is one that a mathematician needs not ask provided he knows enough of the properties of numbers to enable him to deduce his theorems," argued B. Russell.[13] But educators have to be especially cognizant of the transformation and synthesis of what are actually two distinct concepts into a single integrated perceptual entity, such as the example of 'three apples' discussed above. Yielding to the impression that numbers are real world phenomena has led to the common belief that numbers can be deduced intuitively through sensory processes such as manipulating and looking at objects. Unfortunately, and to the detriment of effective teaching, some popular educational theories and didactical methods of teaching basic arithmetic are based on that psychological/epistemological self-deception.

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10 Wilder, 1968, intro. p. 8

11 Ibid., p. 2

12 Ibid., p. 150

13 Russell, 1914, p.191

III-3. SUBITATION

The ability to instantly identify a group of apples specifically as 'four apples' presupposes the possession of the number-concept four. However, the process by which the matching of a number concept with a particular grouping (say, 'four apples' to the exclusion of other groups—'five' or 'six apples') as well as why it is possible to instantly and accurately identify four apples, but not one hundred and thirty-two apples is not explained simply by the possession of appropriate number concepts. The cognitive processes active in the perception of a specific numerical value of a quantity entail more than a simple reference to the appropriate numerical concept.

Human cognition is 'human' as much for its limitations as for its potentialities. By examining the limits of our capacity to accurately perceive the numerical value of quantities at a glance—that is, without resort to mediating mechanisms such as actual counting, grouping or calculating—we may arrive at a deeper insight into the cognitive processes that are active in numerical perception. Moreover, according to Neisser, mental images arise from the same integrative processes that are involved in actual perception.[14] This suggests a relationship between the limits of our ability to perceive a specific number of objects at a glance and our ability to form mental images of specific numbers. If number imagery is fundamental to the formation of numerical concepts, understanding the processes that are involved in the perception of the numerical value of concrete quantities should contribute to our understanding of how number-concepts are created. Fortunately, a number of studies have been conducted that shed significant light on the process of numerical perception.

S. Jevons' 1871 experiment, *The Power of Numerical Discrimination*, is perhaps one of the earliest among the few studies that have examined human perception of the numerical value of groups without counting. To study this subject, Jevons threw a random amount of beans into a box and at the very moment when the beans came to rest he estimated their number "without the least hesitation," and then recorded it together with the number that he obtained by counting. Jevons found that when the amount of beans was less than *five*, his estimation was error free.[15] Thereafter the number of his errors grew in proportion to the number of beans he threw in the box.

Seventy years later E. H Taves conducted a series of experiments aimed to determine how judgment of the numerical value of a quantity is affected by various factors such as the density of items' distribution, the items' configuration, as well as the group's size. Whereas most of the experiments in this series dealt

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[14] Neisser, 1973, p. 209; 1976, p.128, 146-7; 1967, p. 153

[15] Jevons, 1871, p. 281- 282

with comparative judgments, his fifth experiment, *Numerousness and the Direct Estimate of Number*, examined, as did Jevons' before, the absolute judgment of numerical value of quantities. In this study Taves used various fields of dots, ranging from two dots to a hundred and eighty dots. He projected each of these fields on a screen for one fifth of a second and instructed his subjects to report: (1) the number of dots they believed they had seen, and (2) their degree of confidence in the accuracy of their reports. Taves' results showed that between *six* and *eight* dots there was a sharp decline in the accuracy of his subjects' estimations, coinciding with an equally abrupt decline in their confidence. Reviewing the results of his other experiments in that series, as well as the results of others studies (including Jevons'), Taves suggested that these sudden changes indicate that the cognitive mechanism involved in visually estimating fields containing six or fewer dots is different in kind from the cognitive mechanism involved in estimating fields containing more than six dots.

Eight years later, in their 1949 study, *The Discrimination of Visual Number*, E. L. Kaufman, M. W. Lord, T. W. Reese, and J. Volkmann coined the term *subitize* from the Latin "subitus" meaning "sudden," to characterize the process of capturing in a glance an exact number of units in groups containing six or fewer units. They used the term *estimate* to depict the inexact and less assured process of visual enumeration of groups containing more than six items.[16]

As did Jevons, Taves reckoned that the results of his experiment showed that there is a cognitive limit on the number of objects humans can accurately subitize (perceive in a glance). How real this limit is was demonstrated in Saltzman and Garner's 1948 experiment that showed that repetitions did little or nothing to change this limit.[17]

The subitation threshold is by no means the only cognitive threshold known to psychologists. In his 1956 essay, *The Magical Number Seven Plus or Minus Two: Some Limits on Our Capacity for Processing Information*, George Miller examined the results of several studies that investigated the limits of information processing, or people's capacity "to transmit information."[18] Like subitation span, these capacities were measured by comparing the amount of information that was presented to subjects with the amount of information that they could accurately report. The juncture at which subjects were no longer capable of accurately matching their responses with the stimuli presented to them signifies the 'span' of, or 'limit' on information processing.

Some of the experiments, which Miller reviewed, were studies of the span of *immediate memory* (sometimes called *verbal memory*)—the memory for items that

[16] Kaufman et al., 1949, p. 520

[17] Saltzman and Garner, 24, 1948, p. 227- 241

[18] Miller, 1956, p. 81-97

one has perceived only a second or two ago, typically tested by how many discrete words, letters, or digits one can remember and repeat after a brief presentation. He also discussed studies concerning the *span of apprehension* (sometimes called *attention span*)—the number of objects a person can encompass or apprehend in a single glance. Miller pointed out that both the *span of verbal/immediate memory*, and the *span of apprehension* fall within the proximity of seven, as borne out by other studies on the limits on information processing. He pondered the possibility that the fact that all those limitations occur within the same numerical range (give or take two) means that they share common cognitive components. However, he personally doubted that there is something "deep and profound behind all these sevens." (Ibid.)

But George Sperling, four years later, suspected otherwise, for regardless of whether stimuli are presented successively (as in experiments of verbal/immediate memory) or as clusters (as in experiments of apprehension), and of whether they are auditory, tactile, or visual, they must be remembered if they are to be reported by the subjects.[19]

Sperling also hypothesized that in apprehension experiments, the information that is available to subjects during and very shortly after exposure to stimuli is greater than the information that they can remember and report. Thus, the *span of apprehension* is, in fact, determined by the *span of verbal/immediate memory* (that is, the number of identified objects a subject could recall after exposure). In his 1960 study, *The Information Available in Brief Visual Presentation*, which aimed to test this thesis, Sperling projected on a screen various fields of letters and then asked his subjects to report as many letters as they could remember. In order to obtain valid results, he had to circumvent the limitation imposed by *immediate memory*, whereupon he devised a special procedure he called a *partial report*. By using the partial-report technique, Sperling was able to show that "at the time of the exposure, and for a few tenths of a second thereafter, observers have two or three times as much information available as they can later report."[20]

Sperling suggested that the high accuracy of the *partial report* owes to the ability of an observer to read the image that persists in the retinal receptors after a stimulus is removed and before it fades.[21] This residual image, which, according to most experiments in visual memory, may linger up to about one sixth of a second, is called *iconic*[22] or *eidetic*[23] imagery. Since the eidetic image declines so

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[19] Sperling, 1960, p. 20

[20] Ibid., p. 26

[21] Ibid., p. 27

[22] Neisser, 1967, p. 46-48

[23] Sperling, 1960, p. 22

rapidly, Sperling proposed, “Within one second after the exposure the available information no longer exceeds the memory span.”[24]

If, as Neisser claims, “To identify generally means to name, and hence to synthesize not only a visual object but a linguistic—auditory one,”[25] then Sperling’s finding implies that the span of apprehension reflects the limit on what can be synthesized and then verbally stored while the eidetic image lasts. Citing Sperling’s study, Neisser speculated that the subitation span signifies the number of items that can be counted before their image fades from iconic memory. Introspection rarely reveals a clear experience of counting, but this does not prove that none has taken place, just as “Introspection does not reveal the separate and successive impacts of my fingers on the typewriter key board either,” he argued.[26] In other words, *apprehension* (or *attention*) *span*, which on an intuitive level appears to be purely visual, may contain verbal elements as well. Now, if only one object can be acknowledged at a time, subitation may be understood as a sequential process, notwithstanding that the stimuli are presented as a cluster. If true, it should take longer to subitize two dots than to subitize one, and longer still to subitize three.

As it were, this very logical assumption was corroborated by the Kaufman’s group study of *the discrimination of visual number* mentioned before. Kaufman et al.’s study was in principle similar to Taves’ 1941 fifth experiment. However, their five subjects’ responses were measured not only for the accuracy of their numerical estimations and their self reported confidence, but also for their ‘reaction time,’ i.e., the elapsed time between presentation of the stimuli and their responses. This measurement documented a steady and consistent increase in reaction time as the fields of dots were increased from one to eight.

As in Taves’ study, the subjects in the Kaufman study were asked to determine the number of dots in fields that were flashed randomly for only 1/5 of a second. Altogether there were thirty-five fields of dots depicting numbers from 1 to 210. The experiment was divided into two parts, one in which subjects were asked to put emphasis on accuracy and one in which the emphasis was on speed. In the part of the experiment that emphasized speed, the results indicated that median reaction times increased by 1/10 of a second between 1 and 2 dots, by 1/5 of a second between 2 and 3 dots, by 1/8 of a second between 3 and 4 dots, by 1/5 of a second between 4 and 5 dots, and by 1/3 of a second between 5 and 6 dots. There was a total increase of 4/5 of a second in reaction time between the subitation of 1 dot and 6 dots—an average of 1/7 of a second increase in median response time with each additional dot.

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24 Ibid., p. 26

25 Neisser, 1967, p. 103

26 Ibid., p. 42-3

Studies of children's subitation show even greater increases in response time. The difference between one and two dots is twice that of adults, and between two and three dots, thrice that of adults.[27] Indeed, Donald G. MacKay observed that children speak at slower rates than adults.[28] MacKay's observation in conjunction with the results of children's subitation studies supports the suggestion of verbal involvement in subitation.

Taken together, the various studies cited above suggest that while subitation is seemingly instantaneous, in actuality it is a temporal process involving latent counting and the immediate memory. Memory is crucial to the perception of a number. The identification of a specific number requires both the recognition of a discrete sum as well as the recognition of discrete units. During counting, units are visually and verbally acknowledged and effectively committed to memory, thereby enabling the observer to reconstruct a vision of their sum. Not less important is the understanding that there is a limit on our ability to grasp an exact numerical value of a quantity as a whole in a glance. And this limit is approximately six.

It is worth noting that the proposition that subitation is a voluntary temporal process involving latent verbal counting and the immediate memory, as described therein, is incongruent with the theory that humans possess a *number sense*. Scholars who believe in the latter imagine an innate number processor or module that enables humans and animals alike to discern the numerical value of phenomenal aggregates instinctively, instantaneously, and without resorting to rational symbolic cognitive functions. The notion of instantaneous and instinctive number recognition clashes with the notion of a sequential process that entails deliberate and conceptual efforts.

A case in point is Stanislas Dehaene—one of the better-known advocates of the number-sense theory—, and his colleague, Laurent Cohen's 1994 study of the subitizing mechanism. Guided by the assumption that the central differences concerning subitation lie in the question of whether subitation is a serial process similar to counting or a "spatially parallel process," Dehaene and Cohen attempted to show that quantification of small sets is based not on a serial process akin to counting, but rather on a spatially parallel process. For this purpose they tested brain-lesion patients who had lost their ability to explore visual inputs serially. The results of their research were inconclusive, for they revealed that four out of these five patients had considerable problems subitizing groups of 3 items and that some had problems with even smaller groups of 2 or 1. Dehaene and Cohen nonetheless continued to uphold their theory that subitation is a spatially parallel rather than sequential process. Yet, they admitted that "subitizing may not necessarily be based on a single procedure," and it is possible that "the fast quantification of sets

27 Chi and Klahr, 1975 (Cited in Gelman and Gallistel, 1978, p. 223)

28 MacKay, 1968 (Cited in Menyuk, 1971, p. 99)

of 3 items might be based on procedures different from those used with sets of 1 or 2 items." Thus, they conclude that the "nature of subitation remains unknown."[29]

Of course, in addition to Dehaene and Cohen's study, there have been numerous other studies on the subject of subitation's accuracy-and-response time with various results that do not always concur with the Kaufman et al.'s study.

But even if the limit on subitation and its process are still being researched and debated in this century and has yet to be fully understood, there is a universal agreement with Jevons' 1871 discovery that there is a limit on our ability to grasp the exact numerical value of a group as a whole in a glance (or to subitize), and this limit is the small numbers within the range of five (plus or minus one).

The assumption that the limit on our ability to accurately perceive a specific number in the concrete reflects the limit on our ability to construct clear and exact mental presentations of specific numbers suggests that only small numbers within the six range may be mentally configured with optimal clarity and authenticity. This conclusion invites two questions: First, what is the mechanism that enables us to enumerate aggregates that exceed the subitation limit? The answer to this question is the subject of the next chapter, which concludes our discussion of number perception. Second, how can we construct accurate numerical concepts for multitudes we can no longer perceive or imagine in an explicit way? This question belongs to the separate issue of number cognition and symbolization, which is the topic of section IV.

III-4. ESTIMATION

You recall that Taves (1941) distinguished between two kinds of cognitive mechanisms involving number perception: one that operates in a rapid and exact estimation of small groups of objects, and the other that operates in a typically imprecise and slower estimation of larger groups. The Kaufman group, which was able to measure the response times for the estimation of the various size fields of dots, documented a consistent increase in response time for each additional dot within the range of *six* after which response times remain within a fixed limit.[30] The absence of increase in response time when subjects estimated larger and larger groups not only indicates a shift in estimation strategy, but also provides a clue as to the nature of the new strategy. The temporal/sequential process of unit analysis,

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[29] Dehaene and Cohen, 1994, p. 958-975

[30] Though, randomly fluctuated, it remained more or less around 1.50 seconds. For instance, the response time for estimating 210 dots at 1.54 seconds, was virtually the same as for estimating 13 dots—also 1.54 seconds (in the part of the experiment that emphasized speed).

or latent counting, characterizes the subitation of small groups, but estimation of the larger groups relies upon a simultaneous, and generally inaccurate, global impression.

In the Kaufman's experiment, the shift to an impressionistic strategy was indicated not only by a substantial increase in errors, but also by the subjects' tendency to define the estimated quantities of dots with a "round number," which indicated that their efforts were aimed from the outset at obtaining general ideas, or categories of numerical sizes, rather than exact numerical values. A striking example of this trend was documented by the Kaufman group in the estimation of the fields of 134, 152, and 170 dots, which were all estimated as—100 (making the highest estimation error close to 60%).[31] The field containing the number closest to 100 (that with 103 dots) was perceived as 75.

As crude as the approximations of larger groups were, the impressionistic methods of numerical estimation are by no means arbitrary. The subjects in these experiments, as suggested above, were guided by a reference to specific numerical concepts, albeit 'round'. Since they were no longer able to accurately analyze and count the units of these larger collections, the subjects had to resort to 'educated guesses,' which matched their non-analytical perceptual impressions with a specific analytical number concept. When their perceptual impressions were subjected to misleading criteria such as the size of the area occupied by objects and the objects' spatial arrangement,[32] the subjects' mistakes were compounded.

The adult subjects in Kaufman et al.'s experiment were higher education students and must have had a solid grasp of numbers such as 135, 152, 170, and 103; nonetheless, they were unable to correctly identify these same numbers presented in the concrete, and committed gross errors in estimation, as was mentioned above. The discrepancy between comprehension and cognitive clarity of large numbers in the abstract, and educated adults' inability to recognize these numbers clearly and accurately in the concrete, suggests that *abstract* and *symbolic* representation of larger numbers is captured in a more meaningful and real way than their concrete representation. This conclusion is consistent with the sharp decline in subjects'

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31 The exact percentage is 58.7%

32 Krueger, L. E. 1972 has shown that adult subjects, just as the children in Binet's and Piaget's experiments, perceived the same number of dots as more numerous when the dots were spread over a larger area (see chapters, VII-1, VII-2, and VII-3). And Norman Ginsburg 1976 and Christopher D. Frith and Uta Frith 1972 demonstrated that a given quantity of dots distributed evenly over a fixed area is perceived as more numerous than the same quantity of dots randomly distributed over that same area.

confidence when reporting estimations of fields containing a large number of dots, as was documented in Taves' and Kaufman's studies.[33]

How the conceptual/symbolic representation of numbers helps us to conceive with certainty and accuracy numbers we can no longer actually perceive or imagine in an explicit and accurate way is the subject of the following section.

[33] To be specific, in Kaufman's experiment, in a scale of 5 to 1 the degree of certainty was very close to the absolute 5 for numbers within the subitation span, but there was considerable erosion in certainty, starting already in number 10—where it went all the way down to 'not certain' (2.85 median points) when subjects were instructed to put emphasis on speed. At the fields of 103 dots the degree of certainty was practically at the 'absolutely uncertain' level (1.40 mean points in speed). At 210 dots confidence was virtually at the lowest level possible (1.08 median points speed).

IV

NUMBER COGNITION AND SYMBOLIZATION

IV-1. THE DICHOTOMY OF NUMBER CONCEPTS

In the context of mathematical thinking, the term, *number* implies an exact and absolute concept of size, conceived and defined by the specific amount of units of which it is comprised. Thus, the number *five* consists of one + one + one + one + one. The *units* element is an essential component of the concept, number, for it is only by its division into units that a magnitude can be analyzed rationally and be defined in an absolute and exact way. The term *number concept,* which speaks about human cognition of numbers, implies a mental representation or image of a *specific* sum that can be recalled and reconstructed at will. The emphasis here is on the word 'specific;' the mere recognition that something is comprised of an unspecified sum of units, a generic notion of plurality or "manyness," is not a concept of number in its mathematical/cognitive sense because each number constitutes a discrete and unique idea of size. Thus, to satisfy the term, 'number concept,' a specific group of units, or *sum*, must be conceived and identified as a singular conceptual entity distinguishable from all other groups or sums: 'five' can never be confused with 'four' or 'six.' The formation of numerical concepts, then, involves both the identification and recognition of discrete *units* and the identification and recognition of discrete *sums*—two discordant and conflicting cognitive processes.

The recognition of units calls for the deconstruction of a sum into its separate constituent units, while the recognition of a sum calls for consideration of these same units as a singular and integrated totality. These two contradictory tasks require conflicting cognitive processes. The analysis of units is essentially a rational/sequential process, for it involves deliberate and methodical attention to one unit at a time. In contrast, the cognition of sums, which demands dealing with units contemporaneously so as to form a holistic or global view of their totality, is essentially a visual, instantaneous and integrative process.

Both approaches to information processing are equally important for the construction of numerical concepts. The function of rational analysis, through which units are identified, is to produce the information that guides and determines the content of imagery or visualization; the function of visual synthesis or imagery, on the other hand, is to establish the unique conceptual identity of a specific sum. Without the former the visualization of numbers has nothing on which to build, and without the latter the identified units are merely indistinguishable aggregates.

Since units must be sorted out or identified prior to being assembled into an explicit image of a sum, number conceptualization is essentially a process of transforming rational/temporal information about units into spatial/visual information about their sum. Conceptualizing numbers, then, is a creative activity, which builds on rational thinking, as do so many other mathematical thought-processes.

Ulrich Neisser hypothesized that the same integrative process that makes ordinary perception possible produces visual images.[1] Indeed, regardless of their information sources, both seeing and imagining require the synthesis of multiple inputs. Our ability to harness the spontaneous perceptual/visual processes to the formulation of the analytical and rational concepts of number owes to the cognitive elements that make imagery distinct from ordinary perception.

Though both perceiving and imagining involve visual synthesis and are based upon the same integrative cognitive mechanism, there is an important difference between them. Seeing is triggered by ongoing and immediate stimuli generated by the phenomenal world. Since the physical inputs that arouse perceptions are independent of the observer's cognitive system, they circumscribe and subject the visual processes to external phenomena over which one has no control. One does not see an orange when one looks at an apple (assuming one knows the difference). In contrast, images that originate in one's own cognitive system are guided by information that can be selected and manipulated voluntarily, making the visualization a self-directed process. Moreover, because the construction of imaginary "objects" can be generated by non-visual stimuli [2] (the words 'one,' 'two' and 'three,' for example), there is nothing to prevent such processes from creating rationally structured images such as numbers.

Of course, as the sizes of numbers increase, the numbers very quickly become too large to visualize in an explicit way. Yet in order for such numbers to continue to be conceptually and numerically meaningful, it is required that one is able to keep both an accurate account of the units they encompass and also comprehend their sums as holistic entities. Thus, the tension between the dual recognition of 'units' vs. 'sums' continues to influence the way we structure numerical concepts throughout their infinite sequence.

The following pages examine how we structure numbers too large to be recognized or imagined in an explicit way, but nonetheless make them both mathematically and cognitively valid.

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1 Neisser, 1967, p. 153; 1968, p. 128; 1976, 146-7, 209

2 Neisser, 1967, p. 97

IV-2. GENERAL PRINCIPLE OF CONCEPTUAL/SYMBOLIC NUMBER—SEQUENCE

Keeping account of all the units comprising a sum without losing sight of the sum as a uniquely recognized entity, and vice versa, is a demanding undertaking, which pulls the mind in two opposite directions; and the larger the number, the more difficult the task. Numerous cognitive studies demonstrate that the mind's capacity to capture the exact number of objects in a group instantaneously (i.e., holistically or globally) is confined to sums within the range of six items—a limitation, which psychologists have called, 'subitation span' (see chapter III-3). Since the 'subitation span' reflects a limit on the ability to form explicit images or mental representations of specific sums, it is reasonable to assume that concepts of numbers larger than six or so do not consist of explicit images of sums. Structuring a numerical image by dividing a sum into smaller groups may be useful, but it can help only up to a point. Hence, for the conceptualization of larger numbers, the engagement of other mental strategies, mainly logical and symbolic thinking, as well as more advanced levels of abstraction, become inevitable.

To appreciate the role of symbolic thinking in the configuration of larger numbers, it is important to understand that the symbolic function is much more than an encoding mechanism; it is an entirely different way of remembering and forming concepts.[3] Symbolization tremendously enhances the mind's ability to retain and retrieve concepts, and thus it significantly expands our capacity to relate concepts to one another, to re-categorize and to re-code them. Most importantly, it enables us to carry out all these mental activities without reference to real world stimuli or perceptions. Symbolization frees our thought processes from dependence on and confinement to perceptual inputs. This ability allows humans to create new and more abstract concepts, concepts that can be created only through abstract symbolic and rational thinking because they have no physical attributes. Such are the mental representations of specific numbers, particularly large numbers, that can be no longer imagined or recognized in the real world.

Still, in the evolution of the human brain, old functions are not discarded to make way for new and more advanced functions. Rather, primitive processes (such as 'perceptual categorization') continue to operate as part of an integrated system along with functions that evolved later. Thus, even purely abstract concepts that have been acquired through logical, symbolic thinking must somehow connect with perceptual content in order to gain meaning.[4] Nothing illustrates this premise more eloquently than the principles that guide the structuring of numbers, as is reflected in their fully developed symbolization.

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[3] Edelman, 1998, p. 92-3

[4] Edelman, 1998, p. 146-8, Deacon, 1997, p. 265

Throughout the ages and across cultures, the mature symbolization of numbers, whether verbal or notational, has been consistently structured in such a way that regardless of the numbers' magnitude, numbers can still be comprehended within the conceptual framework of smaller tangible numbers. Invariably, this effect has been achieved by basing symbolization on two separate and distinct conceptual components; one may be called *sum* value, and the other may be called *unit* value. The *sums* values are quantified by the *base numbers*, 'one' through 'nine' in the base-ten system, and the *units* values, in the same base, are quantified by the numbers 'one,' 'ten,' 'hundred,' 'thousand,' etc.[5] In the instance of the number 'seven hundred,' the numerical value of the sum is seven, and the numerical value of the unit is hundred. The numbers *one*, *ten*, *hundred*, *thousand*, etc., are units because they can be counted as units can. The unit *hundred*, for instance, can be quantified by various sum values as in 'two *hundred,*' 'three *hundred*,' and 'seven *hundred*.' The numbers *one* through *nine* or 'base numbers' are 'sums' because they represent the range of possible sum values of various units. For instance, the sum *three* counts a variety of unit values as in '*three* hundred,' '*three* thousand,' and '*three* million.' The number 'three thousand' is conceived as the product of two kinds of numerical concepts, the *sum* three, and the *unit* thousand; the number 'seven hundred' is conceived as the product of the *sum* seven, and the *unit* hundred.

The 'units' and the 'sums' form the 'base system,' which structures numbers into an inexhaustible gradation system. The term *base* refers to a fixed counting group that indicates when counting should start anew.[6]

Owing to the base-counting scheme, when the amount of any category of units adds up to the designated counting group—the base—, these units form a "new unit [of a higher order]," whereupon the counting of the newly formed unit begins and continues until the accumulation of these "new units" also amounts to the designated base and forms a "new new-unit;" and the counting of these higher order units commences, and so on and so forth.[7] In the base-ten system when all the *one* units add up to the designated base, they form the new unit—*ten*, and when all the *ten* units add up to this base, they form the new new-unit—*hundred*, etc. In his 1872 grammar-school textbook *New Rudiments of Arithmetic*, James S. B. Thomson sums up the base system's infinite gradation process as it plays out

5 Interestingly, while the 'sum' concepts are readily recognized as 'base numbers,' the 'unit' concepts have no agreed-upon term; they have alternately been called 'level,' 'step,' 'degree,' 'rank,' 'denomination,' 'order,' 'class,' 'unit,' and, in the particular case of the base ten system, 'decimal unit.'

6 Ore, 1948, p. 2

7 Menninger, 1992, p. 14-5, & 28

in the decimal system thus: “Universally, ten of any lower order make one of the next higher [order].”[8]

Historically, ten has been by far the most prevalent base, probably because of the immediate accessibility of the fingers of the human hands.[9] And so, although ten exceeds the size of numbers that can be visualized explicitly, which are six or smaller, it is still tangible and comprehensible because of its origin—the human hands. Because all the units that are created via base counting are powers of the base ten (e.g., $1=10^0$, $10=10^1$, $100=10^2$, and $1,000=10^3$), ten serves as a constant reference for the extrapolation of the successively larger units. All the decimal units, then, are related to, or are abstractions of the concept ‘ten.’ Indeed, scholars have noted that the word for ‘hundred’ in some languages means ‘fate’ or ‘strong ten,’ and ‘thousand’ means a ‘great hundred.’[10] Through the association with ten, any concept of decimal unit, be it a thousand or a million, is connected, however indirectly, to a perceptually tangible content, which makes them meaningful.

In all base-counting systems, the largest sum that can be obtained by any given category of unit (e.g., hundred) must always be one unit smaller than the base. Thus, in a base-ten system, the largest sum of any unit’s category can never exceed nine, for the next sum—ten—becomes a unit of a higher power. Ten *tens*, for example, form the unit *hundred*, and ten *hundreds* form the unit *thousand*. Because numbers that quantify sum (as the *six* in *six* hundred, and the *eight* in *eight* hundred) are limited to numbers that are smaller than ten, they too can be conceptualized in a tangible fashion. Consequently, regardless of a given number’s magnitude, the *sum* elements of numbers always remain comprehensible.

And so, the base-counting technique allows us to define numbers of magnitudes we cannot explicitly imagine and do so without losing count of a single unit. These numbers remain conceptually coherent because the two distinctive numerical concepts, which have been created by the mechanism of the base-counting system—the *sums* and the *units*—link any number, regardless of its size, to numerical concepts that are still within the domain of perceptual comprehension and thus meaningful.

The ramification of this conceptual division of labor is the subject of the next chapter.

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8 Thomson, 1872, p. 11

9 Dantzig, 1954, p. 16

10 Gullberg, 1997, p. 27, Ore, 1948, p. 3, Menninger, 1992, p. 47, 132, Hundred derives from the old English ‘hund.’ Thousand is akin to the Old Norse, thushund, that is, great hundred. The prefix, ‘thus’ denoting great is of the same origin as in ‘thumb,’ literally the strong finger.

IV-3. 'SUMS' VERSUS 'UNITS'

In the preceding chapter, I argued that the conceptualization of large numbers builds on two distinct numerical concepts: 'units' and 'sums.' This chapter examines these two concepts in greater detail in order to gain a better understanding of both the concepts of 'units' and the concepts of 'sums' and the processes involved in the conceptualization of large numbers.

The concepts of 'units' and 'sums' are distinguished from each other in two respects: (1) the functions that each fulfills in a number's definition, and (2) the cognitive mechanisms and processes that are required for their comprehension.

The function of *units* is to define a number's category of magnitude (as the alternate terms, 'rank,' 'denomination,' 'class,' 'order,' and 'level' suggest), and the function of *sums* is to tally the exact number of units within a given rank. In any composition of 'units' and 'sums' that expresses a number, the largest unit defines the order of magnitude of that number while the other numerical values just add the detail. For instance in the number 3,456, the 'thousand' unit defines the general size category of that number as a whole, even though the sums of the 'hundreds,' the 'tens,' and the 'ones' are larger than the sum of the 'thousands.' When we are only concerned with gross ideas of size we 'round' numbers 'up' or 'down' to the nearest largest and dominant 'unit' (e.g., we round the number 7,654 to 8,000 and the number 7,123 to 7,000). In some situations, all other elements, including the sum of the largest unit, are dropped entirely, and only the dominant 'unit' is indicated, as in the Biblical saying: "They have ascribed unto David ten thousands, and to me they have ascribed but thousands: and all he lacketh is the kingdom!"[11]

Because 'sums' are contained within the 'base numbers,' they can define only the numerical relationship within a given unit-size category. For example, the sum 'five' expresses a larger number than the sum 'three' only in so far as the five and the three count equivalent units; 500 is larger than 300, but 300 is larger than 50. The function of base numbers becomes important when the emphasis is on exact quantification, and when the objective is to examine numerical relationships and properties, rather than numerical sizes per se. Take for example the numbers 23 and 24: For most practical purposes, the difference in their sizes is insignificant. But from the perspective of their mathematical properties, these two numbers cannot be more different. The number 23 is an odd number and a 'prime' at that; it has no other factors other than one and itself. The number 24, in contrast, is an even number with no less than six factors (2, 3, 4, 6, 8, and 12) in addition to one and itself.

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[11] First Samuel 18:8, attributed to King Saul lamenting the popularity of his petty officer, David.

The cognitive processes that are required for the apprehension of 'sums' and those that are required for the apprehension of 'units' are different from one another as well. The sequence of sum elements (1, 2, 3, 4, etc.), employs a mechanism of addition: 1=0 +1, 2=1+1, 3=2+1, 4=3+1, etc. Ordered along a continuum of increasing numerical value, the 'sums' form *arithmetic progression*, meaning a progression in which the growth rate is constant; starting with '1' the 'sums' or set of base numbers continue to add the same 'one' unit with each additional step along a progression that ends when their accumulation reaches the designated base—'10.' Understanding such a pattern of growth and the terms that constitute it is almost as easy as counting to ten.

The sequence of 'unit' elements (1, 10, 100, 1,000, etc.), on the other hand, employs a mechanism that raises the base to ever growing powers: $1=10^0$, $10=10^1$, $100=10^2$, $1{,}000=10^3$, etc. Thus, appreciating the magnitudes of the decimal units demands a continuous increase in the level of abstraction of ten with each additional step along their sequence. Ordered along a continuum of increasing numerical value, the units form a graduated pattern or a *geometrical progression*, in which not only the 'units' grow with each step, but the rate by which they grow is growing as well, and in the ten base system, it grows by a factor of ten (1, 1x10, 1x10x10, 1x10x10x10, etc.). It is an amazingly rapid growth. Though the procedural logic that propels this growth is relatively simple (raising the base to the next power), matching it with the appropriate numerical concepts becomes more and more difficult as the 'units' continue to ascend and their link to the original perceptual reference '10' is weakened and fades away. As ideas of units become so vague and abstract that they can no longer be conceived in a meaningful way, the comprehension of the units' magnitude relies increasingly on the comprehension of the extremely rapid growth described by their geometrical progression.

Indeed, the comprehension of the units' geometrical progression demands a level of abstraction that even trained scientists find strenuous. When numbers become 'astronomical,' scientists circumvent these difficulties by redefining such huge and difficult to imagine 'units' in a more tangible way. The concept of a *light year*—the distance traveled by light in the course of one year—is an example of such a 'unit.'[12] Another example is the use of *scientific notation,*[13] which expresses numbers as the product of a 'base number' and a power of ten.

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[12] A light-year is a unit of distance used by astronomers. It equals the distance that light travels in one year at a speed of 186,000 miles per second, approximately six trillion miles. (Source: Robert Jastrow, 1977, *Until the Sun Dies*)

[13] Numbers expressed as a product of a base number and any given power of ten, used for writing very large numbers. For instance 9,460,000,000,000,000 meters (the number of meters in a light year) is expressed as $9.46X10^{15}$. (Source. Baron's 1987, *Dictionary of Mathematics Terms*)

The comfort of visualization, however, does not depend only on the number's magnitude, but also on its complexity. Menninger claims that early man acquired the concept for 1,000 before he could grasp the lower number 543, because the former is an extrapolation of 10 and a result of "grouping," whereas 543 relies on counting and on multiple "ascending steps." "Even today," he observes, "if we contemplate the number sequence in our mind, 1,000 seems clearer, more 'available' to us than 543."[14] Indeed, the number 543 involves the visualization of three 'sums' (5, 4, and 3) and three 'units' (1, 10, and 100). Each of these 'units' demands a different level of abstraction. The visualization of 1,000, on the other hand, involves no sum greater than 1, and only one 'unit', which is a single level of abstraction.

Although the base system employs multiple cognitive processes and approaches for the comprehension of numbers such as 543 or even smaller numbers, which can be mentally taxing, the cognitive advantages gained by this scheme make this effort worthwhile. For the base system makes the maximal use of these basic numerical concepts once they have been established. Each of the elemental concepts that have been created by the base system functions as a module. Together they form a modular system in which any *sum*-size may be paired with any *unit*-size. A great quantity of new numbers may be conceived simply by shuffling and recycling a few already existing concepts. For instance, all the ninety-nine unique numerical concepts from one to ninety-nine are various combinations of only ten elemental numerical concepts: the nine base numbers, plus the concept ten, which doubles as both a 'sum' and as a 'unit.' Now, add to this short list the concept 'hundred', and nine hundred and ninety-nine numerical concepts can be figured out. Indeed, all the million discrete numbers from one to one million are constructed with only thirteen basic numerical concepts: the nine base numbers and the units: ten, hundred, thousand, and million.

The next section, A History of Numerical Notations, deals with, among other things, the translation of a number's conceptual/symbolic structure into symbolic language. Both notational and verbal representations of numbers are examined.

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14 Menninger, 1992, p. 46, 127

V

A HISTORY OF NUMERICAL NOTATIONS

V-1. THE GAP

In contemporary Hebrew the numerical dimensions of Noah's ark and the life span of Methuselah are verbally articulated just as they were written in the ancient Hebrew Bible more than two thousand years ago. And no one knows for sure for how many centuries prior to its appearance in the Bible the verbal numeral system on which these numerical descriptions are based had lived in oral tradition, passing from one generation to another. In contrast, the visual representations of these ancient verbal numerals have been long lost. The Hindu numeral system that replaced them, and that is used in modern Israel, appeared only around 800 AD. It took another 800 years before it was fully incorporated into Western cultures (around the 16th century AD).[1] The actual implementation of modern numerical notation occurred, then, approximately a millennium-and-a-half after the Hebrew Bible was compiled. This developmental gap between the maturation of verbal and notational systems, according to Menninger, is not exceptional. Historically, cultures' verbal systems achieve their ultimate efficacy and final forms long before any notational system is perfected.[2]

The relatively late development of a mature notational system cannot be attributed to a lack of effort. On the contrary, in the course of the 20 or 30 millennia that elapsed between man's earliest attempts at numerals and the emergence of the Hindu numeral system, there have been countless attempts to achieve satisfactory and effective visual representations of numbers. Many of these trials were made by highly advanced and sophisticated civilizations around the globe. Numerals, as it were, are probably among the oldest signs used by human kind; as Tobias Dantzig quipped, their history is "as old as private property."[3] Early man began recording numbers long before they started to record words. According to Childe, there

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1 Ore, 1948, p. 21

2 Menninger, 1992, p. 53

3 Dantzig, 1954, p. 20

exist documents that demonstrate that the ancient cultures of Sumer and Egypt were utilizing numeral systems prior to their earliest known writing systems.[4] Moreover, the earliest numerals—those twenty to thirty thousand year old signs which represented numbers by tally marks and were carved on rocks, cave walls, and bones—long predate the emergence of corresponding verbal expressions.[5] If, as has been suggested by Wilder, the very conception of numbers as independent and universal concepts grew out of visual signs,[6] then numerals predate not only number words, but also the very concept of number itself. How then can we account for the very slow crystallization of visual numeral systems?

The differences between the visual and the auditory mediums in which numerical systems are encoded may well have been the reason for the slower development of a mature notational representation of numbers. Let us examine these two kinds of numeral systems—the auditory and visual—from this perspective.

We begin with Menninger's observation that number words reflect the arrangement of groups because they encompass a numerical idea in a single word. The idea, |||, for instance, is rendered by the single word, 'three,' rather than by repeating the word 'one,' three times.[7] Not only is encoding numbers in verbal symbols more economical and effective than encoding them in visual images such as tallies, but it also represents numbers in a more abstract way. It is quite possible that early number words, especially those representing small numbers, were derived from objects with which numbers could be associated (say the word 'hand' with the idea 'five,' or 'eyes' with the idea 'two'). But, according to Menninger, number words lose their original meaning early on.[8] Both the independence of these words from their original meaning and the inherent arbitrary relationship between the auditory pattern of words and their meaning, in general, imposed an abstract property on number words almost at the outset. In other words, the auditory medium of verbal numbers confines the possible numeric expressions to a single method: a discrete and recognizable sound pattern, which is, inevitably, arbitrary and abstract.

The purely abstract form of the verbal numerals was probably a contributing factor to their early maturation. An abstract verbal system that reflects the underlying conceptual structure of large numbers (i.e., the principle of 'base counting' or a unit's gradation, which builds on the concepts of 'sums' and 'units') and employs a practical base size (i.e., a base size that allows a single regrouping criterion) can

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4 Childe, 1948 (Cited in Wilder, 1968, p. 37)

5 Menninger, 1992, p. 39

6 Wilder, 1968, p. 42-3, 65-6

7 Menninger, 1992, p. 45

8 Ibid., p. 89

in no way be further advanced or improved; nor is advancement necessary. It has reached its full and ultimate maturation.

In contrast to verbal systems, which allow only one method of expressing numerical values, notational systems allow several methods. One of the most salient differences between verbal and visual systems is that in verbal systems the *sum* element in numbers (or 'base numbers') and the *unit* element (or rank, order, etc.) are represented in the same way—abstract symbols (words), whereas in most visual systems the ways in which the *sums* are represented differ from the ways the *units* are represented. For example, in our modern Hindu numeral system, sum values (or base numbers) are represented by symbols (or digits) 1, 2, 3, etc., and the unit values by positional means. Positional representation of 'units' means that the magnitude of the 'units' is expressed by the location of symbols that represent their sums. In the Hindu numeral system, the symbol in the far right of a sequence of symbols represents the sum of the unit *one*, the symbol that is second to the right represents the sum or the unit *ten,* etc.[9] Thus, the sequence 123 reads, "one hundred and twenty three." In the ancient Egyptian system (Figures V-1 and V-2), 'units' are represented with symbols or ideograms. For instance, a vertical staff ('|') represents the unit *one*, a horseshoe ('∩') represents the unit *ten*, and a coiled rope ('𓍢') the unit *hundred*, and the base number values by the tallies of these symbols, that is, the number of times a particular unit's symbol is repeated.[10] For instance, a sequence of three horseshoes, thus: '∩∩∩,' represents the number 'thirty.'[11]

In the Babylonian and Mayan systems (Figures V-3 and V-4 respectively), the 'units' are indicated by positional means, much like in the Hindu system, but 'sums' in both systems are indicated by a mixture of tallies and symbols.

That devising a perfect numerical notation system took so long in spite of an early start suggests that the multitude of ways that numerical ideas can be visually represented impeded the development of mature notational systems. It seems that the intuitive impulse to represent numbers pictorially was an obstacle to the development of an abstract and effective visual numeral system. Indeed, notational representation reached its ultimate form with the introduction of the

9 For instance, in the sequence of the numerals, '345,' the digit, '5,' indicates the sum of the unit 'one,' the digit '4' the sum of the unit 'ten', and the digit '3' the sum of the unit 'hundred.'

10 For example, 'one' is represented by the symbol "|", 'ten' is represented by the symbol "∩"; and 'hundred' by the symbol '𓍢,' etc.

11 For example, the symbol "|" representing the unit '1' is repeated four times to depict the number '4' thus, "||||" and the symbol, "∩" representing the unit, '10,' is repeated three times to depict the number, '30,' thus: "∩∩∩". Hence, the sequence "|||| ∩∩∩" read (from right to left) 'thirty four.'

Hindu numeral system which, like the mature verbal systems, is purely abstract. This observation reinforces the conclusion that the abstraction that the auditory medium imposes upon the verbal systems was the root of their early maturation.

But there have been other factors at work. Mathematics is a cultural creation—cultural needs propel it forward, and cultural habits restrict its progress. R. Wilder has termed the major forces that pushed the development of mathematics forward *culture stress*, and the part "tradition" plays in preventing adaptation of obviously more efficient tools or concepts *cultural lag* and *cultural resistance*.[12] The story of the Hindu-numeral-system's slow acceptance in Western civilization—in spite of its obvious superiority over the existing systems—is a quintessential example of *cultural resistance*. The new Hindu numeral system was met with open resentment in Europe, rejected by the learned circles, and adapted only by the "enlightened masses."[13] Although they entered the history of mathematics in 800 AD, the Hindu numerals, complete with zero, had to wait 800 years more to win a "complete victory" and be adopted by Western scholars. A complete exposition of this ingenious numeral system was introduced to the Western world at the very beginning of the 13th century with the publication of Fibonacci's *Liber Abaci* (1,202 A.D.). In 1,299 AD, some 100 years after the publication of Liber Abaci, the Italian merchants of Florence were forbidden to use the Hindu numerals[14] and perhaps used them in secret code.[15] According to Ore, in Nicholas Copernicus's famous work on the solar system, written in the 16th century, "one finds a strange mixture of Roman and Hindu Arabic numerals, and even numbers written fully in words."[16]

One of the greatest advantages of the Hindu numeral system over all the systems that preceded it, and perhaps the major reason Europeans eventually overcame their cultural resistance to it, was its capacity to function as an auxiliary device for numerical calculation. To understand why it was so important to obtain the ability to represent numbers as well as to carry out arithmetic calculation with mediation of visual numerals, we must first understand the import of notational symbols in arithmetic and in mathematical thinking in general.

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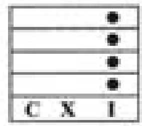
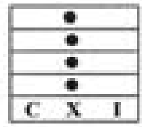
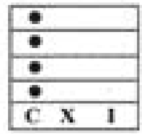
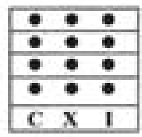
12 Wilder, 1968, p. 175

13 Cajori, 1985, p. 121

14 Ball, 1960, p. 186, Dantzig, p. 33-4

15 Dantzig, 1954, p. 34

16 Ore, 1948, p. 21

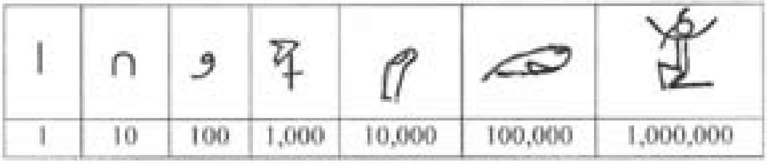
Verbal	*Hindu*	*Egyptian*	*Roman*	*Greek Alphabetical*	*Standard Chinese*	*Abacus*
Four	4 ⇒	|||| ⇐	IV ⇒	Δ ⇒	四 ⇒	
Forty	40	∩∩ ∩∩	XL	M	四十	
Four hundred	400	𓍢𓍢 𓍢𓍢	CD	Y	四百	
Four hundred-and forty four	444	||∩∩𓍢𓍢 ||∩∩𓍢𓍢	CDXLIV	YMΔ	四百 四十四	
Sums Symbolic	*Sums* Symbolic	*Sums* Tallied	*Sums* Mixed	*Sums* Integrated	*Sums* Symbolic	*Sums* Tallied
Units Symbolic	*Units* Positional	*Units* Symbolic	*Units* Symbolic	*Units* Integrated	*Units* Symbolic	*Units* Positional

<u>Reader:</u> Read from left to right: ⇒ Read from right to left: ⇐

Egyptian: |=1, ∩=10, 𓍢=100 Roman: I=1, V=5, X=10, L=50, C=100, D=500

Greek: Δ=4, M=40, Y= 400 Chinese: 四 = 4, 十 =10, 百 =100

Figure V-1: Comparison of Verbal, Hindu, Egyptian, Roman, Greek Alphabetical, standard Chinese, and Abacus numerals



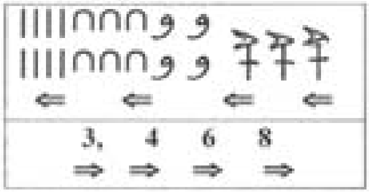
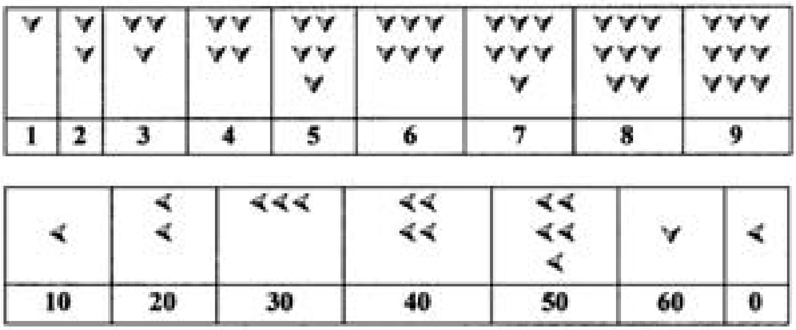
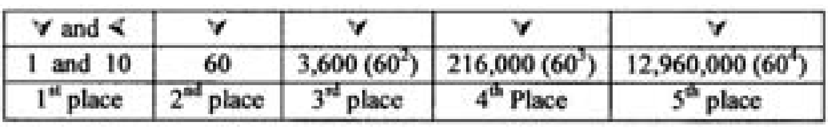


Figure V-2: Egyptian numeral hieroglyphs



Babylonian base numbers



Babylonian base-60-units' progression

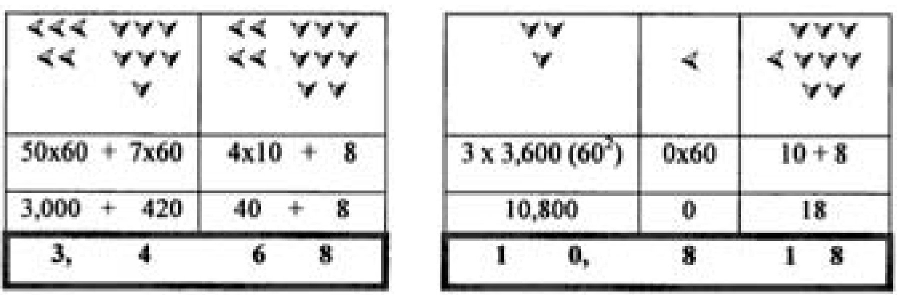
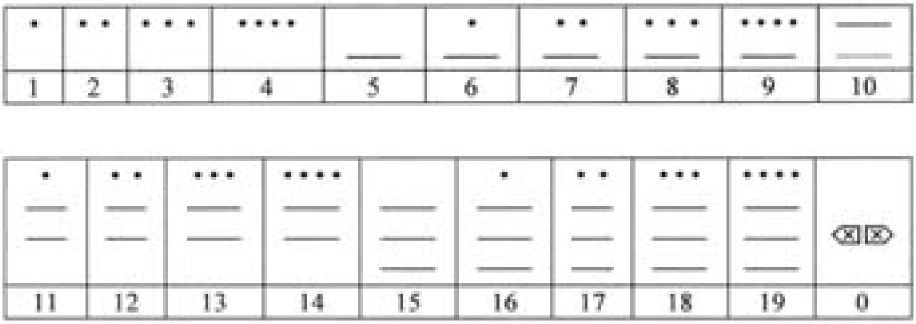


Figure V-3 Babylonian Numeral system



Mayan base numbers

Mayan's base-20 units' progression	1,370,162 In Mayan's numerals	Mayan Hindu's numeral conversion
5th place: $18x20^3$ 144,000	• • • • ——	5th place (x9) 1,296,000
$18x20^2$ 7,200	—— ——	4th place (x10) 72,000
3rd place: 18x20 360	• ——	3rd place (x6) 2,160
2nd place: 20 20		2nd place (x0)
1st place: 1 1	• •	1st place (x2)

Figure V-4: Mayan numeral system

V-2. THE IMPORT OF NOTATIONAL SYMBOLS

Contrary to common lore, which attributes the intellectual challenge embedded in mathematical ideas to the "difficult" and "mysterious" symbols employed by this science, symbols were "invariably" introduced to mathematical thinking "to make things easy," argues the mathematician A. N. Whitehead.[17] Consider the verbal statement with which he illustrates this truth: "*If a second number be added to any*

[17] Whitehead, 1958, p. 40-1

given number the result is the same as if the first given number had been added to the second number." Now compare this verbal rendition of the commutative law of addition, with its symbolic version: '$x + y = y + x$,' and you will no doubt see his point. The former is complex and difficult to follow, while the latter can be grasped effortlessly in a glance. Obviously, the extent of the difficulties in grasping mathematical ideas, of which the commutative law of addition is an example, depends not only on the extent of their inherent complexities but also on the way in which they are rendered. As Whitehead's comparison illustrates, a sequence of mathematical notations representing an idea is deciphered more easily than a sequence of words representing the same idea, even when the words are written down.[18]

Of course, the notational and the verbal symbols share common features. For example, the visual pattern of the figure '*7*' and the sound pattern of the word '*seven*' give away no clues pertaining to their numerical meaning; in both cases, the connection to the concept they represent must be established through deliberate training, and in both, once that association is established, it occurs automatically and subconsciously.[19] Yet the different mediums in which the visual signs and the auditory signs are encoded require different cognitive processes in their decoding. To understand the cognitive benefit of notational symbols for mathematical thinking let us analyze these processes and how they differ from one another.

In spoken words the speech elements, or phonemes, are produced in particular sequences. Since the phonemes and the combinations in which they occur are pertinent to the meaning of the utterance, both must be retained in the memory long enough to allow the entire phonemic sequence of a single word or a series thereof to be considered and synthesized as a whole. Identifying, remembering, and synthesizing the elements of the spoken words employs various mental processes and considerable cognitive effort; this is true even when words are recognized automatically and subconsciously. The transitory and fleeting nature of the spoken sequence makes the mnemonic effort all the more essential for grasping its meaning. This effort is required not only for the immediate decoding of the verbal signals, but also for committing them into longer-term memory for future recall and manipulation.

In contrast, visual notations such as '7,' '8,' or '3' convey an entire idea with a single symbol, thus their perception requires none of the analytical, mnemonic, and synthesis efforts that are needed for decoding a single spoken word. Moreover, a group of visual signs may be processed simultaneously because visual processes are adapted to deal with multiple spatial stimuli contemporaneously.

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18 Decoding the written words involves auditory processes similar to those involved in decoding spoken words even though the former are presented visually.

19 Edelman, 1989, p. 201

But the efficiency and directness of symbols' decoding is not the only advantage of the visual signs for mathematical thinking. Carved, crafted, or written, a visual sign has a lasting physical presence that is independent of the human mind. Its physical presence liberates the mind from the mnemonic effort required to decode and store the temporal, vanishing spoken words. Thus, one can fully focus on the examination of the subject at hand. The Russian psychologist, L. S. Vygotsky, noted that human ability to actively remember with the help of physical signs changes the dynamic of the memory process. "Even such comparatively simple operations as tying a knot or marking a stick as a reminder change the psychological structure of the memory process," he explained. "They extend the operation of memory beyond the biological dimensions of the human nervous system and permit it to incorporate artificial, or self-generated stimuli, which we call signs."[20]

Because thinking is about forming relationships between remembered ideas, the use of visual signs or "external objects" (Ibid.) to aid the mnemonic element of thinking is tantamount to the incorporation of signs in the thought process itself. No other discipline is in greater need of extending memory and thinking capacity beyond the biological limitation of human mind than mathematics. Indeed, among other things that make mathematical thinking challenging is that it typically involves contemplation of various relationships between multiple numerical and spatial concepts simultaneously. Even when dealing with basic arithmetic, we are often required to hold in our memory some 'figures' while we actively consider others. Just try finding the sum of the numbers 'two-hundred-and-seventy-four' and 'eight-hundred-and-ninety-six' without jotting down some figures on a piece of paper.

Still, depending on the kinds of principles that underlie their constructions, written symbols may be perceived, processed, and utilized in different fashions, and not all are equally helpful for mathematical thinking. Friedrich Waismann distinguishes between two kinds of written symbols: symbols that represent sound, like 'r' or 'o,' and symbols that represent concepts, like '3' or '+.'[21] He calls the former "phonetic symbols" and the latter "ideographs." Decoding symbols that represent sound (i.e. letters) is primarily an auditory/verbal task, although the medium in which the stimuli are presented is visual.[22] Since verbal symbols or words are typically constructed as a sequence of several phonemes, decoding written words is akin to decoding speech in a few respects: it occurs over time and involves mnemonic effort, as well as auditory analysis and synthesis. Decoding ideographs such as '3' or '+', on the other hand, because they do not involve auditory/verbal processing, creates an instant, almost automatic connection

[20] Vygotsky, 1978, p. 39, a similar idea expressed in p. 51

[21] Waismann, 1966, p. 51

[22] Neisser, 1967, p. 105-137

between a visual cue and a whole concept.[23] By transforming complex conceptual configurations into a visual image, the abstract notational expression allows us to grasp a complex set of ideas almost in a glance. And so, whereas a combination of the phonetic symbols, say, '*m*,' '*e*,' '*l*,' '*o*,' '*d*,' '*y*' (or for that matter the numeral '||||||'), conveys a single concept, an expression written with a combination of abstract 'ideographs,' such as '*5-2=3*' or '*x+y=y+x*,' conveys not only several concepts simultaneously, but their relationship to one another as well.

Analyzing the mathematical progress made by Vieta, Descartes, and Leibniz in the 17th century, the mathematician R. Wilder was struck "by how much it actually consisted of the invention of a new, and powerful symbolic apparatus."[24] But to appreciate the importance of good notation, one need not resort to advanced mathematics. As Wilder observed, "Even the elementary concept of number could not have advanced very far until a suitable symbolic apparatus—a numeral system—was set up."[25] Moreover, as history showed, the advanced mathematical symbols of Vieta, Descartes, and Leibniz could not have evolved before an appropriate method of representing numbers was in place.

But devising a perfectly appropriate notation system for the simplest and most primitive of all mathematical concepts—the counting numbers—proved to be a great challenge to mankind.

The Hindu numerical system was introduced to mathematical thinking more than a millennium after the completion of Euclid's 300 BC masterpiece, *The Elements*, which laid the foundation for the modern method of scientific investigation. Yet, it took another 700 to 800 years "of blind stumbling and chance discovery, of groping in the dark and refusing to admit the light," before the Hindu system was finally adopted in daily and scholarly discourse in the west.[26] Its victory was nonetheless inevitable. The Hindu numeral system is indisputably the ultimate method for writing numbers, a method that leaves nothing to be improved upon and an "admirable illustration for the importance of good notation."[27] Indeed, it owes directly to the Hindu numeral system that virtually all grammar school

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23 Holender and Peereman, 1987, p. 77 maintained that letters depict phonological units, i.e., the meaning of letters is a property that the right hemisphere is unable to process. Word recognition is mediated by letter or syllable recognition. "By contrast, Arabic numerals have a meaning in a symbolic system that has nothing to do with phonology." Therefore, they explained, "there is no reason why the right hemisphere could not generate a semantic representation of the digit and transfer it to the other side, a task it seems able to perform with objects as well."

24 Wilder, 1968, p. 171

25 Ibid., p. 4

26 Dantzig, 1954, p. 20

27 Whitehead, 1958, p. 39 & 42

children today routinely perform arithmetic calculations that a few centuries ago only a small circle of the most educated people of the time could manage.[28]

The next chapters examine the greatness of the Hindu numerical system by comparing this system with those preceding it, and by studying how it allows ordinary people to handle complex calculations.

V-3. THE THREE METHODS OF VISUAL REPRESENTATION

The visual medium affords the representation of numerical ideas by three methods:

1. *Concrete images*, namely *tally*—the representation of a number by repeating a sign as many times as there are units in that number.[29] The Roman's numerals *I*, *II*, and *III* representing the numbers *one*, *two*, *three* respectively, are an example.
2. *Abstract symbolic signs*—the representation of a number by a unique symbol. The Hindu numerals *1*, *2*, *3*, *4* representing the numbers *one*, *two*, *three*, *four* respectively, are an example.
3. *Location*, or relative position in space—the representation of number by using the position of a symbol in relation to other symbols. For example, in the sequence, *333*, each of the identical symbols represents a different numerical value according to its location in the sequence; reading from left to right, these numerical values are: *three-hundred*, *thirty*, and *three*.

To understand the great achievement of the Hindu numeral system, we must analyze these three options of denoting numerical values from their cognitive aspects and consider the ways each of these methods was used by pre-Hindu numeral systems.

Concrete images: Because they are the most direct and intuitive method of representing numbers, *tally* numerals were the earliest and most prevalent numerals prior to the introduction of the contemporary Hindu numerals. As seen in Figures V-1-through-V-7, except for the Chinese and the Alphabetic systems, all pre-Hindu systems combine some form or other of tally numerals. Tally, notwithstanding its intuitive qualities, is an inefficient method of number

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28 Dantzig, 1954, p. 193-4

29 Ibid., p. 7, The term 'tally' comes from the Latin word ' talea' meaning 'cut'—a reference to the most prevalent 'tallying' method which was carving lines on various surfaces, such as bon, wood, cave walls, and clay.

representation as much for its decoding as its encoding processes. Compare for instance the Egyptian's tally numeral '||||||||' with the Hindu's '8;' both represent the same number: Writing '||||||||' entails stroking a surface eight times, and its decoding requires counting eight symbols. Not only is it laborious to write 'eight' this way, but the counting process that its decoding necessitates interferes with the formation of an automatic symbol-to-concept link, thereby neutralizing the advantage gained by symbolization. In contrast, the Hindu's '8,' which is comprised of only one symbol that is captured in a glance, connects to the concept eight in an immediate and direct way, and in addition is encoded with a single motion. Furthermore, the concrete quality of the tally is incompatible with the arithmetic's modus operandi, which—like mathematics in general—is abstract thinking.

Abstract symbolic signs: The alternative to tallying is *cipherization*,[30] that is, the use of a single symbol to represent a group of units. *Cipherization* arose out of the necessities of evolving cultures to use larger numbers and maintain written records.[31] Historically, the earliest cipherization was that of numerals denoting 'units' (or ranks) rather than that of numerals denoting 'sums' (or base-numbers) as in our familiar Hindu system. The 'base-numbers' in these older systems were represented in tally fashion. The Egyptian hieroglyphic numerals (Figures V-1 and V-2), tracing back to about 3500 BC, are the purest and perhaps most ancient model of such a numeral system. The Egyptian's hieroglyphic was written mostly from right to left, with the larger 'units' indicated first.[32] An example is the sequence '||||∩∩∩𓍢𓍢,' which reads from right to left "two-hundred and thirty four." A coiled rope ('𓍢') represents the unit hundred; a horseshoe ('∩') represents the unit ten, and a vertical staff ('|') the unit one.

Most of the pre-Hindu numeration methods were, by and large, variations on the Egyptian prototype meaning that they incorporate tally elements to depict *sum* (or base-numbers).[33] The derivations from the Egyptian method were short cuts aimed at economizing with the number of symbols. The difference between the Egyptian and the other systems, then, was one of convenience, not principle. For

30 Dantzig, 1954, p. 32, The English 'cipher' is the derivative of 'zephirum'—a "Latinization" of the Arabic's 'sifr'—in itself a translation of the Hindu—'sunya.' The word 'sifr' underwent several changes that ended with its Italian version, "zero." Because the zero serves an important function in the Hindu numeral system, both in writing numbers and in arithmetic calculation, the word 'cipher,' originally meaning 'zero' in popular use, came to denote Hindu numerals in general.

31 Wilder, 1968, p. 54

32 Ifrah, 2000, p. 170-1, *The New Encyclopaedia Britannica* (1986-15th edition). Vol. 29 p. 1001-2

33 Dantzig, 1954, p. 22

instance the Roman system (Figures V-1 and V-5) represents each ‘decimal unit’ with a single sign (I=1, X=10, C=100, and M=1,000). Each ‘half unit’ is similarly represented (V=5, or 10/2; L=50, or 100/2; and D= 500, or 1,000/2). The Greek Acrophonic, or Attic system, from which the Roman system was probably derived, worked on the same principle.[34] Thus, the Romans represented the number ‘678’ by 9 signs—‘DCLXXVIII—’ compared to the Egyptians,’ ‘||||||||∩∩∩∩∩∩∩999999’ that employed no less than 21 signs to represent the same number. It is noteworthy to mention, however, that numerals that combine symbols and tallies require adding and subtracting in addition to subitizing and decoding symbols as seen in the case of the Roman’s ‘XLVII,’ which is read, ‘50-10+5+2.’[35] These requirements, much like the counting involved in decoding the Egyptian’s ‘|||||||,’ interfere with the formation of instant automatic and subconscious connections to whole concepts and undermine the advantage of symbolization.

That cipherization of the *units* preceded by cipherization of the *sums* makes perfect sense. Ciphered numerals not only are a more sensible method of expressing the large numbers that characterize ‘units,’ but are also consistent with the abstract mode of thinking that guides a unit’s conceptualization. ‘Sums,’ in contrast, not only are small enough to be tallied with relative ease, but can be visualized explicitly, such that the tallied sums, much like the ciphered units, are consistent with their conceptual property. The combination of enciphered ‘units’ and tallied ‘sums,’ therefore, is compatible with the functions and the conceptual properties of these two distinctive concepts with which larger numbers are configured; hence, the resiliency and durability of these cumbersome numeral systems.

There were other encipherment strategies, such as the use of the alphabet letters, as employed by the Phoenicians, Ionian Greeks, Jews, and Arabs around 300 BC. For example, the Ionian Greek (Figures V-1 and V-6) represented the nine base numbers with the first nine letters of their alphabet, such that ‘A’ (alpha) represented ‘1,’ ‘B’ (beta), ‘2,’ ‘Γ’ (gamma), ‘3,’ etc. The next nine letters denoted multiples of 10 (‘10,’ ‘20,’ ‘30,’ etc.). To have enough letters to represent the series of the nine multiples of 100 (‘100,’ ‘200,’ ‘300,’ etc.), the Greeks had to add 3 letters to their alphabet, whose active letters numbered only 24. They represented the 1,000s series by repeating the first nine letters—preceded by a stroke, as in the examples of ‘A, ‘B, and ‘Γ for ‘1,000,’ ‘2,000,’ and ‘3,000.’ And represented the 10,000s series by the same first nine letters atop the letter M (the first letter of the word ‘myriad’), as in the examples of ‘M^{α},’ ‘M^{β},’ and ‘M^{γ},’ for ‘10,000,’ ‘20,000,’ and ‘30,000.’[36] At

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[34] Ifrah, 2000, p. 182, Found in Attic inscriptions from 500 BC.

[35] L= 50, X=10, V=5, and I=1

[36] Ifrah, 1990, p. 274

that point they exhausted all the possible numbers that could be represented by their system, and herein lies one of the disadvantages of that strategy.[37]

The Chinese numeral system presents another encipherment strategy. In the Chinese numeral system (Figures V-1 and V-7), both, 'units' and 'sums' are symbolized; it is the only visual system that actually recapitulates the verbal number sequence.

To their detriment, the alphabetic and Chinese systems, which are purely symbolic, do not restrict symbolization to 'sums' (base numbers). As a system that is bound to the verbal representation of numbers, the Chinese system offers no significant operational advantage over verbal systems. While the alphabetic methods neutralize the advantage of the base-counting system by representing the same 'sum' values by different letters in synchronization with the units they represent (Figure V-6). For example, in the Greek system, '2' is represented by the letter 'B' (beta), '20' by the letter 'K' (kappa), and '200' by the letter 'Σ' (sigma).[38] In changing the symbols of the 'sum values' according to the units they tally, the alphabetic methods obscure the modularity principle of the concepts 'sums' and 'units' and consequently do not reflect the base-systems conceptual structure. The Attic Greek and Roman methods, which are hybrids of an abbreviated Egyptian and the Alphabetic methods, shared the deficiencies of both.

Location: Perhaps the most obvious difference between spoken numerals and 'enciphered' written numerals—both of which denote numerical ideas by a single abstract symbol—is that, unlike the spoken numerals, which could be recognized only by their sound patterns, the written numerals could be recognized by their position relative to other symbols as well as by their pattern. This quality allows each symbol to convey two distinct numerical values at the same time, as in the case of '333,' mentioned previously. The contemporary Hindu numeral system is an example of this positional strategy but is by no means the only system that employs this strategy.

In all the 'place-value' or 'positional' systems of the present and the past, the shapes of symbols represent a 'sum' value, and their locations represent a 'unit' value. In these systems each position or 'place' in the sequence of symbols represents a certain power of the base in question, such that any symbol is understood as a product of a 'sum' or a 'base number' and a certain power of the relevant base. For instance, the symbols '5,' '4,' '3,' and '2' represent the 'sum' element of numbers in the Hindu numeral system. In this base-ten system, which is read from left to right, larger units first, the sequence '5,432' means '$5x10^3 + 4x10^2 + 3x10^1 + 2x10^0$.'

Though the ancient Babylonian and the Mayan systems, like the Hindu system, were 'place-value systems,' their huge bases proved to be a disadvantage.

37 Wilder, 1968, p. 56

38 Ifrah, 1990, p. 272

Cognitively, the Mayan base of 20 (Figure V-4) and the Babylonian base of 60 (Figure V-3) were too large to effectively serve as visual references for the conceptualization of ‘units’. The large base sizes had adverse effects for symbolic representation, as well; they meant either huge tallies for the representation of base numbers, or too many ‘ciphered’ numerals (i.e., discrete symbols)—20 in the instance of the Mayans, and 60 in the instance of the Babylonians. Indeed, both systems used numerals that combined tally and cipherization to represent the sum values. The combination of cipherization and tally actually creates a secondary grouping criterion that competes with the grouping criteria set by the primary base of those systems. The conceptual utility of the base system, which depends on the ability to use a single standard group as a reference for further abstraction of ‘units’ (as in 100=10x10, and 1,000=10x10x10, etc.), is inadvertently compromised if not neutralized altogether by this kind of sub-grouping. Moreover, systems that have used mixed methods for representing base numbers as a means of abbreviation merely substitute the cumbersome process of counting by equally cumbersome addition and subtraction operations. Take for example the Roman ‘IIX’ and ‘VIII,’ or the Mayan ‘° ° °,’ which present the number eight: the Roman numeral IIX is suggestive of arithmetic operation, ‘10-2,’ while the numeral ‘VIII’ suggests the arithmetic operation ‘5+3,’ as does the Mayan ‘° ° °.’ The Hindu numeral ‘8,’ on the other hand, connects to the concept ‘eight’ without tying it to specific operations or structural images. The Hindu symbolism, therefore, is more effective both for decoding the numeral sequence and for the utilization of its symbols in calculation.

The Hindu method combines the best qualities of all the other alternative systems. By clearly distinguishing between the sum’s element and the unit’s element, it effectively uses the modular system created by the *sum* and *unit* conceptual elements. Its uniform base—10—is a conceptually practical size. And it is a place-value system. But as this description of other historic notational methods for representing numbers clearly shows, none of these important qualities—base-ten system, positional system, use of ‘0,’ and ciphered numerals—, for which the Hindu numerical system is known, were unique to the Hindu system.

What then was the distinctive and novel quality that set the Hindu system apart and high above all the systems that preceded it? The following chapter deals with that question.

M	D	C	L	X	V	I	MMMCDLXVIII ⇒ ⇒
1000	500	100	50	10	5	1	3, 4 6 8

Figure V-5: Roman numeral system

Alpha A =1	Beta B =2	Gamma Γ=3	Delta Δ =4	Epsilon E =5	Digmma* [=6	Zeta Z =7	Eta H =8	Theta Θ =9
Iota I=10	Kappa K=20	Lambda Λ =30	Mu M =40	Nu N =50	Ksi Ξ =60	Omicron O =70	Pi Π =80	Koppa* C =90
Rho P=100	Zigma Σ=200	Tau T =300	Upsilon Y =400	Phi Φ =500	Chi X =600	Psi Ψ =700	Omega Ω=800	San* m=900

‟Γ Y Ξ H = 3, 4 6 8

Figure V-6: Classical Greek alphabet numerals according to Ifrah, 2000 (p.220): Asterisked letters—Digmma, Koppa, and San—are letters of the ancient Greek alphabet that were already obsolete in the classical period.

一	二	三	四	五	六	七	八	九	十	百	千
1	2	3	4	5	6	7	8	9	10	100	1,000

三 千 四 百 六 十 八
3, 4 6 8

Figure V-7: Chinese numeral system

V-4. THE HINDU NUMERALS BREAKTHROUGH

In most textbooks the Hindu numeral system is described as a base-ten-place-value system. But this definition does not reveal the secret innovative element that made the Hindu system different from all other systems. Base-ten systems have been around from time immemorial, right from the point that humans began using their fingers as auxiliary devices in counting. The place-value principle also cannot account for the fundamental changes in the science of numbers that followed the introduction of the Hindu system. That same place value principle depicting the units' category was used by the Babylonians as early as 1800 BC and by the Mayans by 600 AD. Both systems also used a sign equivalent to the 'zero' to indicate the place of a missing ‘unit’ category for which the Hindu system is so famous (Figures V-3 and V-4). What set the Hindu numerical system apart from all other methods was that it alone opted to cipher ‘base numbers—‘the ‘sum’ element of numbers—and only base-numbers. The Hindu numerals ‘1,’ ‘2,’ ‘3,’ ‘4,’ ‘5,’ ‘6,’ ‘7,’ ‘8,’ and ‘9’ depict the *sum*-or- base-number element by a single symbol, without resorting to any other method.

As varied as the pre-Hindu numeral methods were, it is easy to recognize a salient thread common to all. With the exception of the Chinese and the alphabetic systems, all feature some form of tally in their depiction of base-numbers. The element of ‘tallying’ can be recognized not only on the 20 and 30 thousand year old inscribed wolf’s bones and on prehistoric cave walls, but also in the numerals used by the highly developed cultures of Sumer, Egypt, Greece, and Rome. Even the Babylonian and the Mayan place-value systems, both of which utilize signs that are equivalent to the Hindu zero, used a combination of tallied and ciphered numerals to denote base-number values.[39] For example, the Roman numeral ‘VIII’, like the Mayan numeral ‘$\underline{\circ\ \circ\ \circ}$’ (both representing 8) uses a ciphered ‘5’ and tallied ‘3.’

This is not to say that cipherization was a new idea. The method of cipherization (or ‘encipherization’) has been around for a very long time, as far back as 3300 BC. However, cipherization was mostly used to denote the ‘units’ categories (the Egyptian ‘∩’ and the Roman ‘X’ for the unit *ten* for example). The Alphabetic and the Chinese systems ciphered both the *units* and the *sums*. Among the systems that used cipherization, the Hindu numeral system alone opted to restrict cipherization to *sums*, which are the smallest and the most concrete component of numerical concepts—the numerical values that can be most easily represented by tally. Some historians maintain that the *cipherization* of the base-numbers-or-sums, not the invention of the *zero*, was the greatest achievement of the Hindu system.[40] The zero, in spite of its poetical reputation, has a prosaic history. It was an inevitable byproduct of the invention of the place-value system. All the positional systems known to us (the Babylonian, Mayan, and Hindu) eventually invented a symbol akin to zero; it was a necessary device to mark a missing units’ category so as to avoid ambiguity.[41]

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39 Menninger, 1992, p. 236, “Tally sticks” have been especially popular and long lasting. They were used by Swiss diary-farmers until as late as the 19th century, not to mention the ‘Exchequer Tallies’—the notched sticks that served the British Royal Treasury as the official tax payments records. English bureaucrats used them until 1826 A.D.

40 Wilder, 1968, p. 55, Menninger, p. 398

41 Wilder, 1968, p. 152, Even the Babylonian system, which rarely had to indicate a missing denomination, used an equivalent sign for zero, though it took the Babylonians

Unlike decoding combinations of tallied and ciphered numerals, decoding the Hindu numerals does not involve processes of analysis or counting. Thus the Hindu numerals form an immediate and direct connection between a symbol and an established concept. And unlike the tally numerals '||,' '|||,' or '||||,' the 'ciphered' Hindu numerals '2,' '3,' or '4,' bear no descriptive reference to the image of the number they encode; they are entirely arbitrary. Thus their decoding must rely on an automatic connection between a visual sign and its meaning. The automatization and directness of decoding the Hindu numerals makes these symbols a part of the perceiver's thought processes and optimizes their effectiveness. A sequence of signs, say '534,' which can be grasped in a glance, evokes a vivid representation of the compound number, 'five hundred, and thirty-four,' notwithstanding its three 'sums' and three 'units' categories. This abstract visual directness is one of the most important cognitive advantages of the Hindu numerals.

In addition, the combination of ciphered numerals and the positional method of depicting 'units' allows minimal and effective use of symbols. Leonardo of Pisa, better known by the name Fibonacci, introduced the Hindu numerals in his seminal computation book, "Liber Abaci" of 1202 AD, thus: "The nine numerals of the Indians are these: 9 8 7 6 5 4 3 2 1. With them and with this sign 0, which in Arabic is called 'cephirum' [cipher], any desired number can be written."[42] In this memorable and dramatic statement Fibonacci captured the essence of the magic power of the Hindu numerals system, that is, the ability to write any number, regardless of its size or complexity by mean of only ten signs.

And not least, the symbolized *sums*, and the localized *units* make the Hindu system both purely visual and yet purely abstract. As such the Hindu system is entirely compatible with the cognitive mode of mathematical and arithmetic thinking. Moreover, with its symbolized *sums* that are simple enough to be written rapidly, its positional strategy to denote *units*' categories, and with a special sign to indicate empty positions—the *zero*—the Hindu system is not only a superior method for recording and decoding numbers, but is also an effective tool for mediating arithmetic calculation. Indeed, it is its utility in numerical calculation that has established the Hindu-numeral-system's unchallenged universality and sovereignty in contemporary civilizations.

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almost a millennium to create the symbol. Cajori, 1985, p. 5, noted that the "number with a missing sexagesimal place are rare, fewer than 1.7% of the numbers from 1 to 216,000 are of this kind, as compared with roughly 40% that require a zero (or several zeroes) in the decimal system." In addition, because of their big base, the disparity between numbers of different denomination is so big that they are less likely to be confused (see Figure V-3).

42 Menninger, 1992, p. 425

V-5. THE NEW ARITHMETIC

Scholars unanimously view the Hindu numeral system as the ultimate method of number representation, a method that leaves nothing to be improved upon, a system that signifies the end of a long, winding search for an appropriate expression for numbers. George Ifrah considers the invention of the Hindu numerals "as important as the invention of agriculture, the wheel, writing, or the steam engine."[43] No doubt, the introduction of the Hindu numeral system to mathematical thinking brought with it significant cultural progress: For the first time in history mankind could use visual signs as an auxiliary device to aid thinking.

Until the appearance of the Hindu system, written numerals were used exclusively to record numbers, while arithmetic calculations were made with the aid of an *abacus* or *counting board* (Figure V-8). The origin of the term 'abacus' is not clear. Some trace it to the Semitic 'abac' meaning dust; others believe that its origin is the Greek 'abux' meaning slab. The early abacus was a board or a table with lines representing unit categories, on which pebbles were placed to represent base number values in a tally fashion. The word, *calculate* derives from the Latin *calculus* meaning pebble and alludes to this kind of calculation practice.[44] Pebbles and lines drawn in dust or clay slabs were later replaced by beads, which were strung on wires or strings that were attached to a frame.[45]

The Hindu positional decimal system is a symbolic version of the abacus according to Dantzig. He speculated that the Hindus might have used dust boards and erasable marks instead of counters and pebbles for arithmetic calculation. (Ibid.) This is consistent with Ifrah's hypothesis that Indian scholars, who also used counting boards to execute arithmetic operations, began at some point to replace the concrete counters (beads, pebbles, etc.) with the first nine signs of numeration—the digits *1,2, 3 . . . 9*.[46] The digit in the first column on the right indicated the sum of the unit *one*, the numeral inscribed in the second column from the right the sum of the unit *ten*, and so on. Later, when the custom of considering the values of units in a particular order had become "ingrained," the columns' marks were eliminated. (Ibid.) Once the marking of columns was no longer used, a special sign to indicate an absent unit (which was previously indicated simply by leaving the column empty) had to be introduced in order to avoid ambiguity. Ifrah surmised that the *zero*, which originally was termed *sunya* meaning void,

43 Ifrah, 1985, p. 437

44 Dantzig, 1954, p. 7

45 Ibid., p. 28-9

46 Ifrah, 1985, p. 457-8

was a mere substitute for the empty column of a counting board. Only later on did it acquire the "meaning of 'nothing' as in '10 minus 10' [equal 0]." [47]

It is likely that the transformation of the "sunya" to a numerical concept occurred as a result of using the sign for zero in an arithmetic operation. Ifrah noted that in his 628 AD, *Brahmas Iddhanta*, the mathematician and astronomer Brahmagupta explained the rules for the six basic arithmetic operations—addition, subtraction, multiplication, division, raising to a power, and extraction of roots—on positive numbers, negative numbers, and the *null number*.[48] His successor, Bhaskara, born 1114 AD, outlined the "rules of cipher" in the first chapter of his astronomy book, *Lilavati*, as follows: $a+0=a$, $a-0=a$, $0^2=0$, $\sqrt{0}=0$, $a\div 0=\infty$ (the sign ∞ indicates infinity).[49] Indeed, according to Wilder, the use of symbols in formal mathematical operations often forces the introduction of new concepts, and the zero, like many other signs, achieved a "conceptual status" only after many centuries of such use.[50] And so, as the use of the Hindu numerals for calculation transformed piles of pebbles into abstract numerical concepts, it also transformed the sign '0,' which marks the place of an empty column, into a new mathematical concept: a number that counts nothing.

With the introduction of the Hindu numerical system into mathematical thinking, arithmetic operations that for centuries had been executed by the manual manipulation of objects could now be executed with the mediation of symbols that communicate abstract ideas by abstract means. In doing so, the Hindu numeral system paved the way for the use of symbols as auxiliary devices in other mathematical topics and helped to usher in a new language of mathematical discourse.

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47 Ibid., p. 459, It was also called "hah"—sky, "ambara"—atmosphere, and "agana"—space, which convey similar ideas for void or emptiness.

48 Ibid., (emphasis is mine)

49 Ball, 1960, p. 150-1

50 Wilder, 1968, p. 61

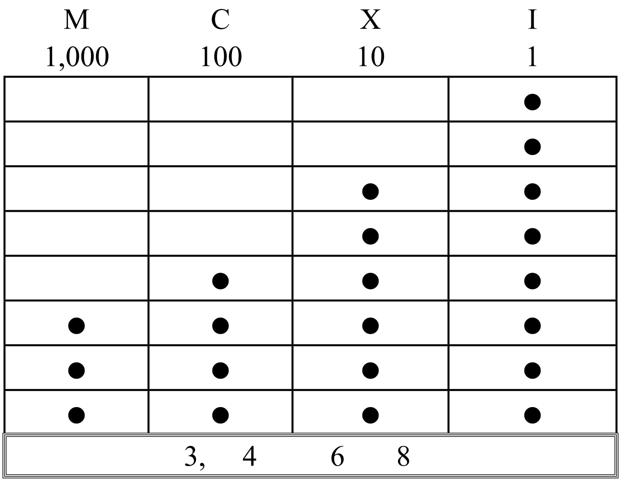

Figure V-8: Abacus

V-6. NEW ARITHMETIC VERSUS OLD ARITHMETIC

In his book, *A Short Account of the History of Mathematics*, W. W. Rouse Ball dates the creation of modern mathematics to the introduction of the "Arab text books" into Europe,[51] that is, from the time Europeans became familiar with the Hindu numerals. Ball's opinion that the Hindu numeral system was of central importance to the development of modern mathematics is well justified.

Much of European medieval Arithmetic, prior to the introduction of the Hindu numeral system, was based on the books of Boethius (475-528 AD), "the last Roman of note who demonstrated interest in Greek literature."[52] Apparently, Boethius' arithmetic was not new either. In fact, it was a translation of Nicomachus (100 AD), who, according to Ball, remained the authority on the subject for more than a thousand years. Boethius's arithmetic, like that of Nicomachus's, reiterates the Greek's distinction between *arithmetic*—'the science of number—' and *logistica*—'the art of calculation,' a distinction first made in 600 BC by the Pythagorus school. The Greek's *arithmetic* focused on the theoretical

[51] Ball, 1960, p. 263

[52] Ibid., p. 33

aspect of numerical relationships and proceeded either by rhetorical means or by geometrical demonstrations. *Logistica*, on the other hand, dealt with everyday practical calculation and thus centered on the use of the abacus. The former was considered worthy for philosophers, while the latter an art that should be relegated to slaves.

The new "logistica," which carried out calculation by means of symbols instead of the abacus' beads, was known in Western Europe as *algorithm*, or the "art of Al-Karism."[53] Al-Karism was a librarian in the court of caliph Al-Mamun in Baghdad. His seminal 830 AD book, *Al-gebr we' L Mukabala*,[54] in which he explained his celebrated method of balancing algebraic equations as a device for their simplification, was the first arithmetic text book in which the Hindu numerical system was fully expounded.[55] It is believed that the Arabs, who had extensive commercial contacts with India and by this time expressed numbers both as written words and in alphabetic numerals, became acquainted with the Hindu numeral system through Brahamgupta's (600 AD) astronomy book, *Siddhanta*, or other astronomical texts and tables.

The Hindus and Arabs' rules of numerical operations were not founded on deductive methods for they had no interest in logical arguments and deductive "proofs."[56] Being practically oriented, they were interested in arithmetic and algebra because both yield quantified results. Perhaps because algebra is in principle an abstraction of arithmetic, the Indians and the Arabs made no distinctions between them treating arithmetic as part of algebra.[57] In any event, many of the symbols that are used today in arithmetic were used first in algebra to replace the rhetorical methods by which it proceeded earlier. For instance, in his 1557 algebra the English mathematician Robert Recorde introduced the sign, '=' (equal to) with the following explanation: " . . . to auoide [avoid] the tediouse repetition of these woordes: is equalle to: I will sette as I doe often in woorke vse, a pair of paralleles,

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[53] Ibid., p. 156, 'Al-karism' is a corruption of the Arabic 'Al-Khwarizmi' meaning, 'from Khwarizmi,' a province in Persia. His full name was Mohammed Iben Musa Abu Djefar Al-Khwarismi.

[54] Dantzig, 1954, p. 79, 'Mukabala Al-gebr', which is the origin of the term *algebra*, means 'restoration,' 'Mukabala' means simplification. 'Al- gebr we' L Mukabala' means "on restitute and adjustment"—a reference to a method of simplifying algebraic equations by performing the same operations on both sides of the equation. For instance, the algebraic expression $3x+8=20$ can be simplified with Al-Khwarismi's method by subtracting eight from both sides of the equation and then dividing both by three, thus, $(3x+8-8)\div 3= (20 -8) \div 3$, to get the ultimate simplification: $x = 4$.

[55] Ball, p. 1960, 156, Dantzig, 1954, p. 79

[56] Kline, 1980, p. 111-113

[57] Ball, p. 1960, 158,183

or gemowe [twin] lines of one lingth, thus: =, bicause noe .2. thynges, can be moare equalle."[58]

The growing cultural need for more advanced applied mathematics favored the Hindu/Arab mathematics, which is concerned with practical quantitative topics, over the Greek's, which adheres to theoretical purity in mathematical thinking.

With the mediation of the Hindu numerals and the operational signs, arithmetic calculation could be carried out by means of conceptual thinking and was no longer restricted to the mechanical operations of the abacus. That development created a fusion (or confusion) between what theretofore were two distinct studies, one—*arithmetic*—devoted to theoretical quests, and the other—*logistica*—devoted to practical purposes. Consequently, contemporary arithmetic combines both aspects of numerical reckoning, and it is up to our educational approach whether we make arithmetic calculation worthy for philosophers or a slave's chore.

V-7. INSTRUMENTAL VERSUS CONCEPTUAL USE OF THE HINDU NUMERALS

The history of the Hindu numeral system indicates that the familiar vertical algorithm can be traced to the use of the abacus. Just as on an abacus each column represents a particular unit's category, so too in the Hindu system, numbers are written one beneath the other in a way that aligns the same categories of decimal units in the same column. For example:

$$\begin{array}{r} 1\ 2\ 3 \\ +\underline{4\ 0\ 5} \\ 5\ 2\ 8 \end{array}$$

The link to the abacus implies that this kind of algorithm incorporates an instrumental as well as conceptual element.

The early vertical algorithm that was practiced by the Hindus, the inventors of this system, and the Arabs, who were the first to adopt it, operated on the largest units first and moved toward the smaller units in a left-to-right motion.[59] The left-to-right direction of the Hindu numerals corresponds to the direction of the Sanskrit script, which Indian scholars used at the time they invented their numerals. That early use of Hindu numerals in calculation proceeded from left to right attests to their conceptual element, because the conceptual preference is to consider the larger units, which render the general size category of a number, first.

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[58] *The New Encyclopaedia Britannica* (1986-15th edition). Vol. 23, p. 612

[59] Cajori, 1985, p. 91; Ball, 1960, p. 88

This conceptual preference manifests itself in the receding order in which numbers are typically written and verbally articulated, and in mental calculation, which also tends to start with the largest units and proceeds toward the smaller units.

However, the left-to-right direction necessitated making corrections as numbers were added. For instance, in adding the numbers 648 and 275 (as seen in the example below) the numeral 8, which is the sum of 6 and 2, in the *hundred*-column, had to be erased and change into 9 after the numerals 4 and 7 in the *ten*-column had been added producing an additional hundred-unit. And the numeral 1 (the right numeral of eleven, which is the sum of 4 an 7), had to be erased and change into 2 after the numerals 8 and 5 in the *one*-column had been added producing an additional *ten*-unit, thus:

648+275 = 6 4 8
+ 2 7 5
→ ~~8~~
9 ~~1~~
9 2 3

Tables or boards dusted with sand or flour were intended to make erasing and rewriting easy. The contemporary approach to vertical addition and subtraction, which involves moving from right to left, starting with the smallest unit and working up toward the largest unit, was invented in sixteen hundred A.D. by an Englishman about whom we know nothing except his name—Garth.[60] The date places this invention some eight hundred years after the emergence of the Hindu system, and four hundred years after its introduction to Europe. It seems that Garth's algorithm, which deals with the smaller units first and works up toward the largest units, calls for viewing numbers in a counterintuitive manner. The intuitive origin of the left to right direction was probably the reason that the inconvenient Hindu/Arab algorithm survived eight hundred years before it gave way to the much more convenient right to left direction (which does not require erasing and rewriting).

Yet, even with its left-to-right direction, which reflects the conceptual dictate of considering larger terms first, the Hindu/Arab method, much like Garth's, favors instrumental utility of numerals over mental and conceptual efforts in computation. Like Garth's, it positions written numbers vertically—an addend underneath an addend in addition, and a subtrahend underneath a minuend in subtraction such that the same units categories are aligned in a single column. Both methods achieve their utilitarian effect by breaking numbers into disjointed decimal-units and dealing with each category of unit separately.

[60] Ball, 1960, p. 188

The consequence of the vertical algorithm is that, regardless of the number of columns representing the constituent ranks and sizes of the numbers, arithmetic operations invariably involve no more than two columns at a time. Thus, the entire procedure of addition or subtraction in vertical algorithms remains in the cognitive framework of numbers within the 20s range. Take for example the subtraction of 1,245 from 2,763. Because 5 is written underneath 3, one must extract a ten from the 63 (of the minuend 2,763), and add it to the 3 in the *one*-column as to allow the subtraction of 5 (of the subtrahend, 1,245) thus:

$$2{,}763\text{-}1{,}245 = \begin{array}{rrrrr} & 2, & 7 & {}^{5}\cancel{6}^{(1)} & 3 \\ - & 1, & 2 & 4 & 5 \\ \hline & 1, & 5 & 1 & 8 \end{array}$$

The operation seen above involves five steps: (a) extract 10 and add it to 3 in the one-column; (b) subtract 5 from 13; (c) subtract 4 [tens] from 5 [tens]; (d) subtract 2 [hundreds] from 7 [hundreds]; and (e) subtract 1 [thousand] from 2[thousands].

The first and second operations (*extracting 10* and *adding it to 3 in the one-column,* and then *subtracting 5 from 13*) will be the same regardless of what the minuends or the subtrahends are as a whole so long that the digits that represent the sum of the unit—*one* in the minuend is 3 and that of the subtrahend is 5. Moreover, it makes no difference what the sizes of the units involved are. So long as a 5 of any unit-size has to be subtracted from a 3 of an equivalent unit, the same operation will be repeated as a higher unit must be extracted from the next number on the left as to allow the subtraction of 5, thus:

(N = any number)

$$\begin{array}{rrrr} {}^{N-1}\cancel{N}, & (1)\,3 & N & N \\ -\,N, & 5 & N & N \\ \hline N, & 8 & N & N \end{array}$$

The consequence of this itemization and atomization of numbers is that the child's ability to form concepts of numbers larger than 20, not to speak of numbers beyond 100, is compromised. Indeed, even when they can easily subtract one from one hundred mentally, some students are perplexed and lost when they have to subtract one from one hundred written vertically; they do not know how on earth to subtract one from a zero, let alone, from two zeros! The vertical arithmetic operation prevents them from recognizing the numerals, 1, 0, and 0 as constituent elements of the same number (100).

Whereas *vertical* algorithm places the numbers one beneath another and aligns same decimal units in a single column so as to allow calculation in the fashion of abacus operation, *horizontal* algorithm places numbers one next to the other, as in 50+34=84, such that an abacus-like operation is not possible. The horizontal algorithm, then, forgoes the instrumental value of the Hindu-numerals and instead uses them primarily for their conceptual value. In this capacity the numerals are used as mnemonic signs for numerical concepts. And as mnemonic signs these numerals become an auxiliary for promoting arithmetic thinking, rather than an auxiliary for promoting expediency and ease in the execution of arithmetic calculation. Although not as efficient as the vertical plan, the horizontal algorithm has its important pedagogical advantage over the latter, because by removing the instrumental aspect from the Hindu numerals (i.e., the potential for operating in an abacus-like fashion), the horizontal presentation helps children to relate to numbers as integrated whole. The consideration of numbers as an integrated totality enhances conceptual understanding of numbers and their relationships to one another, and promotes better understanding of multi-unit numbers. The method of solving arithmetic problems horizontally is practically the same method as solving them mentally; that is, operating with the largest unit first and then moving on to the smaller units, as the conceptual approach intuitively proceeds in mental calculation. It seems then, that the horizontal algorithm could mediate and aid mental computation; as such, they are invaluable for basic arithmetic education.[61] After all, in mental calculation the students must keep in their working-memory (the short-memory span) the totality of all the numbers that are related to a given arithmetic problem; this effort, in turn, enhances their understanding of the properties of the numbers and the numerical relationships involved in the task at hand.

In fact, neither the Hindu/Arab vertical algorithm nor the currently used vertical algorithm devised by Garth were devised for instructing grammar school students in their first attempts to grapple with numerical ideas and their arithmetic; instead, they were aimed to meet the needs for increasing expediency in calculations of seasoned professionals who already had a firm grasp of the counting numbers and beyond. Vertical algorithms—which are necessary only when the arithmetic involves large and complex numbers, and in long-multiplication and long-division operations—can and should be introduced after children have established solid understanding of the Hindu numerals' positional system, and have demonstrated proficiency in mental calculation and horizontal algorithm in the arithmetic of smaller or less complex numbers.

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[61] English and Halford, 1955, P. 168-71, Katz, 1981, Booklet A, p. 19

VI

THE ORIGIN OF NUMBER

VI-1. THE SEARCH FOR NUMBER SENSE

Numbers, in one or another form, are indispensable to all known civilizations, both ancient and modern. Therefore, it is not surprising that scholars from many disciplines are interested in understanding the origin of numerical concepts, not the least the question of whether humans are endowed with a biologically based *number sense*. There are various categories of numerical concepts. For example, the category mathematicians call the *counting numbers* includes all positive whole numbers from 'one' on, exclusively. Because the counting numbers are the most primal and elementary of all numerical categories, they are the foundation upon which all of the other numerical concepts are constructed; understandably, this class of numbers is at the center of the search for 'number sense.' It is important to bear in mind, that as a pivotal element of scientific and mathematical contemplations, numerical concepts—even the most rudimentary, such as the counting numbers—must be available to voluntary and rational thinking or they will be deemed irrelevant. It is the purpose of this chapter to explore whether there is any merit to the proposition that there exists a biologically based apparatus that is capable of forming mathematically valid counting-numbers. This question is examined from three perspectives: (1) Is there reasonable biological evidence of a number-specific apparatus? (2) Is the proposed mechanism of such an apparatus consistent with neuroscientific knowledge? (3) Is there historical or anthropological evidence that provides support for its existence? Let us begin with a brief review of the conceptual properties of the counting numbers:

The 'counting numbers' are discrete ideas of specific sizes. Each of these numerical sizes is envisioned as, and defined by a fixed sum of nonspecific abstract units: Three is larger than two because it has one unit more than two, and it is smaller than four because it has one unit less than four. By basing definition of size on analysis of generic units, these concepts define sizes in an objective, exact, and absolute, albeit purely abstract, way. The significance of number concepts is that they institute the cognitive tools for replacing the spontaneous sensory quantification—which is subjective, inaccurate, and relative—with a quantification that is objective, exact, and absolute. Numerical evaluation is typically used only in situations that call for accurate definition of size. Tasks that can be accomplished by ordinary sensory impressions, such as choosing the larger pile of berries or a space large enough to park one's car, require no numerical reference.

It seems, then, that the term "number sense" harbors conceptual conflict on two levels: First, numbers in and of themselves are devoid of any physical attribute and, therefore, possess no sensory information that can be perceived by our senses. Second, it is precisely because number quantification is based on rational, analytical, and voluntary processes rather than on spontaneous sensory impressions that it achieves its exact and objective results. Thus, from a cognitive standpoint, the term "number sense" is an oxymoron. It is antithetical to the cognitive properties and to the cognitive significance of numbers, the ways numbers function, and the very rationale for using numbers.

It is worth noting, however, that in everyday language the term *number* can be a synonym for such words as *many*, *few*, *some*, and *several*. Also, as is pointed out by Dehaene,[1] *number* may indicate written or spoken numerical symbols. In addition there is a tendency to confuse the decidedly abstract concept of *number* with quantities of phenomenal units (e.g., groups of toys, candies, counters, etc.). The latter confusion stems from the human tendency to objectify the abstract counting number, especially the small numbers within the subitation range, and perceive them as if they were physical phenomena imposed on us from without (see Chapter III-2).

The term *sense* itself has a few interpretations as well. In its literal understanding, *sense* means a faculty with which we perceive attributes of the physical world such as sound, temperature, light, smell, etc.; but in its figurative interpretation, it could also mean skill, knowledge, understanding, and the like, or it can mean instinct, intuition, or feeling. The conjunction of the terms *sense* and *number* has two major interpretations depending whether the term *sense* connotes instinct, or whether it connotes knowledge and skill. In pedagogical literature the term *number-sense* often means number competency that is acquired in schools, specifically, the understanding and mastering of numerical relationships, numerical operations and their algorithms, and the ability to apply this knowledge in solving problems. Although the term *sense* in the conjunction 'number sense' is not interpreted literally in an educational context, it is still puzzling; why are all other scholastic subjects taught for achieving skill, proficiency, or knowledge, while arithmetic alone is taught for acquisition of sense? But the "number sense" that must be taught in school cannot be considered the origin of number in human cultures. In contrast, in the natural-sciences literature where the term *sense* connotes instinct, the conjunction 'number sense' speaks about an innate

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1 Dehaene, 1997, p. 35

or intuitive number receptor that is biologically founded. This interpretation of number-sense has a direct implication for the discussion of the origin of numbers in human cultures.

The principle scientific basis for a biologically based number sense is a hypothetical inborn brain-module programmed specifically for processing numbers. Proponents of various innate-number theories recognize this apparatus by the term, *accumulator*. The concept, *accumulator,* was introduced by Meck and Church in their 1983 research paper: *A Mode Control Model of Counting and Timing Processes*, in which they described their study of rats' response to time and number. Because the aforementioned study and the concept of 'accumulator' to which it gave rise have a prominent place in theories of the origin and acquisition of number concepts, we ought to examine this research to learn what the accumulator is all about. Let us begin with the description of Meck and Church's 1983 experiment and their proposed counting and timing processor:

In the initial phase of their experiment, Meck and Church trained rats to press the far-most-left *or* far-most-right lever in a 10-lever box in response to white-noise signals. The noise signals were constructed by various series of *cycles*. Each cycle was made up of a period of noise and an equal period of no noise: | Noise | No Noise. The rats are rewarded when they press the left lever in response to a 2-second signal consisting of 2 noise cycles and when they press the right lever in response to an 8-second signal consisting of 8 noise cycles. The duration of a single cycle of both the 2-second and 8-second signals was 1 second. As seen in Figure-VI-1, in this training phase both of the signals' attributes—namely, (a) the *total duration*, assumed by Meck and Church to test the rats for *time* perception; and (b) the *number* of noise-cycles, assumed to test the rats for *number* perception—were synchronous, or in their words "confounded."

In the actual test Meck and Church introduced two additional sets of signals, each set consisting of six new signals. In one of these sets the *total duration* of signals was held constant at 4 seconds, while the *number* of cycles varied between 2, 3, 4, 5, 6, or 8 cycles. In the other set the *number* of cycles was held constant at 4 cycles, while the *total duration* of the signals varied between 2, 3, 4, 5, 6, or 8 seconds. Their rational for holding one of the signals' attributes (example, duration of signal) constant at the intermediate value (4 seconds in this example) while varying the other attribute (number of cycles in this example) was to obtain a controlled measurement of the latter (response to the number of cycles in this example).

As seen in Figure-VI-2, the results showed that *both 4 cycles with a total duration of 2 seconds* (each cycle lasting ½ a second) and *2 cycles with a total duration of 4 seconds* (each cycle lasting 2 seconds) elicited *left*-choice responses. The former was deemed, by Meck and Church, to convey *time* discrimination, while the latter was deemed to convey *number* discrimination. On the other hand, both the four 2-second cycles totaling 8 seconds and the eight ½ a second cycles totaling 4 seconds elicited right-choices responses. Meck and Church deemed the former an indicator of time discrimination and the latter an indicator of number discrimination.

This experiment and Meck and Church's interpretation of its data gave rise to the idea that *time* discrimination and *number* discrimination use the same mechanism. Meck and Church proposed an internal mechanism that measures time duration and numerical value of quantities simultaneously. They further speculated that this mechanism emits pulses that can be controlled in three "modes:"[2] (1) the *run* mode, in which the signal generates a continuous process that ends when the signal ends; (2) the *stop* mode, in which the signal and process are synchronized throughout the administration of the signal; and (3) the *event* mode, in which "each onset of the stimulus produces a relatively fixed duration of the process regardless of stimulus duration," seen in Figure VI-3. (Ibid.) Put simply, in the *run* mode the number and the duration of discrete noise-cycles are ignored, and only the total duration of signals is registered, while in the *event* mode the total duration of signals and the durations of discrete noise-cycles are ignored, and only the number of noise-cycles are registered. Only in the *stop* mode is the stimulus registered as is.

The mechanism for timing and counting comprises three functions, as illustrated in *Figure-VI-4*: The first function is a *pacemaker-switch-accumulator*, which consists of three components: (a) a *pacemaker*, which emits pulses with fixed intervals between pulses; (b) a *switch*, which controls these pulses in three modes: run, stop, and event, as explained above; and (c) an *accumulator*, which accumulates the resulting information as time or as number. The *pacemaker-switch-accumulator* is used as a "clock" (for estimating time-duration) when the "switch" operates in *run* or *stop* mode, and as a "counter" (for estimating the number of events in a cycle) when the "switch" operates in *event* mode; thus, it "may be called either a clock or counter."[3] The accumulator values (of both time- and number-outputs) are passed on to the second function, that is, the *memory*: first to working memory and from there to reference memory. The accumulator's current status is passed through both kinds of memory to the third function of this mechanism, that is, the *comparator* from which the left-or-right-choices ensue, and in which current values are compared with the rewarded values that were stored in the reference memory.

This model of a counting and timing mechanism is at the center of the accumulator concept and of the major innate-number theories.

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[2] Meck and Church, 1983, p. 323, In Meck and Church's actual words: "Perhaps there is an internal mechanism that puts out pulses that can be controlled in several modes […]."

[3] Ibid., p. 323-4

One way to verify the scientific validity of Meck and Church's hypothesis is to examine whether or not it predicts animal behaviors in other experiments. In this respect, some biologists beg to differ with Meck and Church, in particular with Gibbon's *scalar-expectancy theory* (SET) on which their *pacemaker-switch-accumulator* model is based. Gibbon proposed a mechanism that delivers pulses at a more or less even pace and measures time in a direct way. But Machado[4] and Joana Arantes[5] argue that the SET model, which implies evenly measured and unchangeable timing, failed to predict the behavior of the pigeons they studied. Their experiments suggest that the context in which stimuli were presented affected the pigeons' choices. Machado and Pata proposed an alternative model they called the *learning-to-time* (LeT) model. In fact more and more experiments substantiate that the same period of time may be perceived as having different durations. Several factors that affect perception of time emerged in these studies; one of them is the rapidity or speed at which the stimuli are administrated. For example, Khoshnoodi et al. used tactile vibrations with different frequencies on their human subjects. Their experiment indicates a direct correlation between the increase in the frequency of the vibrations and the subjects' *overestimation* of duration of time.[6] Other studies suggest that heightened agitation, excitement, or fear results in overestimation of time. Perhaps the most dramatic example of these studies is David Eagleman's highly publicized experiments involving free falls from a height of 150 feet into a net. The students who volunteered for these experiments estimated that their fall took much longer than it actually did—36% longer than their compatriots on the ground estimated these same falls to be.[7]

However the most pertinent question for the issue of number sense is whether or not the Meck and Church's experiment succeeded in demonstrating that rats can perceive numbers. As it were, the focus of Meck and Church's research was to establish a connection between rats' counting and timing processes, not to study rats' counting or numerical abilities per se. In the initial phase of their experiment, Meck and Church trained rats to discriminate between two periods of noise, one lasting for 2 seconds and encompassing 2 cycles, and the other lasting for 8 seconds and encompassing 8 cycles. Consequently, the time-variables (the total duration of signals) and the number-variables (the number of cycles in signals) were completely synchronized such that the rats could use either one of the signals' attributes in making their choices. Under the conditions of the training phase, then, one cannot ascertain whether the rats were using a number cue, or a time cue. Hence, no valid evidence for either a counting—or a number—discrimination

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4 Machado et al., 2005, p. 111-122

5 Arantes, 2007, p. 269-278

6 Khoshnoodi, 2008, p. 623-633

7 Eagleman, 2008, on line

process was demonstrated at this stage. On the other hand, the results of the actual experiment give an impression that rats can discriminate number and time for they pressed the *right* lever in response to the *longest-duration* as well as to the *most-numerous*-cycles signals, and the *left* lever in response to the *shortest-duration* as well as the *fewest-cycles* signals. Alas, by holding one of the variables constant (in order to observe the changes in the other variable) in their actual test, Meck and Church inadvertently introduced a third variable—the duration of single cycles. For example, in the "test for *time*," in which the number of cycles was held at 4, the *2-second* signal could *accommodate* 4 cycles of only ½ *a second* each while the eight-*second* signal could *accommodate* 4 cycles lasting as long as 2 *seconds* each. In the "test for *number*," in which the duration of signals was held at 4 seconds, the *8* cycles could last only ½ *a second* each, while the 2 *cycles* could last as long as 2 *seconds* each. But Meck et al., who theorized that in the *event* mode all noise cycles are registered as having equal duration regardless of their actual duration or the total duration of the signal, did not include this additional variable in their calculations and analysis of the experiment's results, notwithstanding that the ratio of ½ a second to 2 seconds is 1:4—the same as the ratio that they believed was necessary to allow the rats in their experiment to distinguish between many vs. few, and long vs. short.

Even if justified by their hypothetical '*event* mode,' Meck and Church's disregard for the differences between the durations of a discrete events (or noise-cycles) is puzzling: Why cannot an animal, able to distinguish differences in total durations of a string of noise-cycles when these differences are of 1:4 ratio, distinguish differences in durations of *events* (single noise cycles) when these differences are of the same 1:4 ratio? After all according to their model, "in the 'stop' mode the process occurs whenever the stimulus occurs,"[8] suggesting that the rats can recognize these two kinds of duration: the total duration of a signal and the duration of each segment in the stimulus (otherwise this mode could not be possible). Moreover, Meck and Church's earlier study demonstrated "that a rat can time two signals simultaneously and independently," (Ibid.) suggesting that rats can attend to two different aspects of the same auditory signal simultaneously. And not less important, why can an animal discriminate between different *numbers* of noise-events better than between the *rapidity* (or speed) with which these noise-events are delivered? After all, noise and the noise pattern, which are vibrations of air, are decisively physical phenomena that can be recognized through the senses, while numbers per se are devoid of physical manifestation such that they cannot be recognized through the senses. That is to say that the fact that a signal of *4-sec. duration* with *faster* administration of noise cycles elicited the *same* response as a signal of *8-sec. duration* with *slower* administration of

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8 Meck and Church, 1983, p. 323

noise cycles could be attributed to the rats' ability to use two distinct properties of the noise signals, namely, the total duration signal and the rapidity at which the noise-cycles of the signal proceed. This suggests in turn that the rats did not have to rely on number cues in making their choices.

Recognizing scientists' penchant for attributing human qualities to animal behaviors, the zoologist and psychologist C. Lloyd Morgan postulated,

> *In no case may we interpret an action as the outcome of the exercise of a higher psychical faculty, if it can be interpreted as the outcome of the exercise of one which stands lower in the psychological scale.*[9]

According to the above principle, known as *Morgan's Canon*, it is more plausible that the rats in Meck and Church's experiment responded to the total duration of noise, and the speed or rapidity with which increments of noise were administered; those are the two variables in the signals that, unlike number discrimination, could be described in terms of an animal's auditory function that need not resort to higher psychological processes such as labeling and counting. In fact, Meck and Church themselves acknowledged that the term *counting*, which is associated with labeling, may not be appropriate for describing the rats' behavior in their research and that, "Whether or not animals can apply symbolic labels to the numerical attributes of stimuli remains uncertain."[10] It seems, then, that the scientific foundations of the accumulator are not sound enough to provide an unequivocal support for theories of number-sense.

Nonetheless, the accumulator concept became a fertile ground for various number-related innateness theories. By 1997 when Dehaene published his book, *The Number Sense,* accumulator became commonly used jargon.

> Animals possess a mental module, traditionally called the 'accumulator,' that can hold a reregister of various quantities. [This mechanism] opens up a new dimension of sensory perception through which the cardinal [value] of a set of objects can be perceived as easily as their color, shape, or position. [. . .] This number sense provides animals and humans alike with direct intuition of what number means.[11]

In their theory of extracting the *integers* from the set of *real numbers,* Gallistel and Gelman propose that number representation in the accumulator encompasses

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9 Morgan, 1894, p. 53

10 Meck and Church, 1983, p. 333

11 Dehaene, 1997, p. 4-5

the entire range of real numbers in a nonverbal form.[12] Children extract the whole numbers from the 'real' by mapping the accumulator's "preverbal numerical magnitudes to the verbal and written numerical symbols and the inverse mappings from these symbols to the preverbal magnitudes."[13]

Gallistel and Gelman's explanation is at odds, however, with most proponents of innate-number faculty. Stephen Laurence and Eric Margolis, claim that the "standard" version of number representation in the accumulator is that it represents numerical value of quantities approximately. "Instead of picking out 17 (and just 17), an Accumulator-based representation indeterminately represents a range of numbers in the general vicinity of 17."[14] And Laurence and Margolis explain:

> Accumulator represents numerosity via a system of mental magnitudes. [. . .] Instead of using discrete symbols, the Accumulator employs representation couched in terms of a continuous variable. [. . .] Imagine water being poured into a beaker one cupful at a time and one cupful per item to be enumerated. The resulting water level (continuous variable) would provide a representation of the numerosity of the set: the higher the water level, the more numerous the set.[15]

Karen Wynn agrees that the way the accumulator represents numbers is different from the way verbal counting does. "It is the entire fullness of the accumulator, not the final increment alone" that represents the numerical value of a quantity of items, she explains.[16]

The proposition that numbers are represented by the "entire fullness of the accumulator" (or continuous estimate) suggests that the accumulator represents numbers that have no units. The question is then, what qualifies these presentations to be considered numbers? After all numerical sizes are distinguished from one another by the specific sum of their units. Without the separation into units, numbers

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12 The set of *real numbers* consists of, in addition to whole-positive-and-negative numbers, the set of *rational numbers*, which are all the numbers that can be defined as a ratio between two whole numbers (e.g., 5/1, 1/2, or 23/47), and the set of *irrational numbers*, which are the numbers that cannot be defined by relationships between whole numbers. For example the ratio between a circle's circumference and its diameter—known as pi—, and the ratio between a diagonal of a square and its two sides—the square root of two—represent *real* geometrical relationships that have numerical values, yet they cannot be defined as a ratio between two whole numbers.

13 Gallistel and Gelman, 1992, p. 43-74

14 Laurence and Margolis, 2005, p. 221

15 Ibid., p. 218-9

16 Wynn, 1992, p. 228

cannot answer the question, “How many?” It is possible that for the proponents of number-sense, the theory that the accumulator assembles its information in successive *increment*s (that is, the separate inputs of the pacemaker) grants sufficient evidence for the presence of numbers. If true, the accumulator must be a mechanism through which physical aggregates are transformed into a mental continuum.

Undeniably, as Siegler and Laski’s and Booth and Siegler’s 2006 experiments demonstrated, numerical magnitudes can be visualized as continuous physical sizes such as length of lines. In the aforementioned experiments Siegler at el. represented the escalating sequences of the counting numbers as linear continuums in which only the endpoints of the line indicated a numeral. For example, if the sequence of numbers was 0 through 100, the numeral ‘0’ was indicated on the left end of the line and the numeral ‘100’ on the right end; the same line could represent the sequence 0 through 1,000 by substituting the 100 with the latter. Both child and adult subjects had to find the location of various numbers on this line. For instance, on a line representing the numbers 0 through 100, the location of 50 would be exactly in the middle of the line; but on a line representing the numbers 0 through 1,000, the location of 50 would be exactly at the end of the first twentieth portion of this line. The Siegler et al. studies showed a positive correlation between subjects’ arithmetic aptitude tests and overall mathematical knowledge and experience, and their success in correctly locating numbers on the lines. These results indicate that the ability to conceive number as continuous size relies on pre-existing numerical concepts, which means that numerical definitions of continuum are a product rather than the source of numerical concepts. In other words, concepts of numerical sizes could be converted into concepts of continuum sizes, but continuums could not be converted into numerical concepts spontaneously and without an existing grasp of numbers.

In addition to being a continuum, the accumulator’s size estimation is also an approximation that is tied to concrete phenomena. The merging of these properties raises the question of what distinguishes number sense from the ordinary perceptual estimation of phenomena’s sizes. The proponents of number-sense do not dwell on this question; on the other hand they do try to provide solutions to a related problem that is raised by the accumulator’s properties: Given that the accumulator presents a numerical value of quantity as a continuous and approximate entity, what is the process through which this continuous approximation is transformed into a numerical concept that is an exact and discrete sum of units?

Gallistel and Gelman offer that the only thing that children have to do in order to transform the accumulator’s information into specific numbers is to find the appropriate matching “cells” of the preverbal magnitudes (i.e., the ‘real numbers’), which are represented in the accumulator “in tabular arrangements” of “answers.”[17] For Gallistel and Gelman (and as Carruthers, Laurence, and Stich pointed out)

[17] Gallistel and Gelman, 1992, p. 43-74

integers occur "only against the background of representational resources that most others take to be a *far greater* psychological achievement."[18] By suggesting that the *real* are more basic than *integers*, they turn on its head the conventional wisdom that considers the set of the counting numbers the basic system out of which other numerical systems have developed. But this is not the only problem with their solution: Gallistel and Gelman's image of "tabular arrangements" of "answers" to which the child may attach matching symbols suggests a simple labeling process that limits the applications of these symbols—as will be detailed later. Moreover, their postulation, which speaks of the "retrieval" of "number facts" by means of matching verbal and written numerals with the "preverbal magnitudes," which in turn are used for finding "appropriate cells in tabular arrangements of the answers," (Ibid.) conflicts with neuroscientific research into human memory. The growing consensus among scientists is that humans' memories are malleable and unstable, and not orderly indexed in secured and reliable tables, as proposed by Gallistel et al.

Dehaene offers a different solution: The transformation of the accumulator's registered continuum into a specific number is achieved by human symbolic capacity. He argues, "Language allows [humans] to label infinitely many different numbers;" these labels "symbolize and discretize any continuous quantity."[19] A child acquires number ideas by learning to label quantities; he accomplishes this task by correlating numerical symbols with his accumulator's response to their corresponding quantity. When the child realizes that the word 'three' is very often mentioned when his mental accumulator is in a particular state (in a response to the presence of three items), he eventually will understand the meaning of the word 'three.'[20] But like Gallistel et al.'s theory of attaching numerals to the "appropriate cells in tabular arrangements," Dehaene's labeling theory also amounts to a simple matching or mapping of symbols, albeit, his is the matching of linguistic symbols with the accumulator's fullness, instead of matching linguistic symbols with brain "cells" and vice versa. Both theories are inconsistent with the way neuroscientists view human symbolic function.

For neuroscientists, humans' symbolic function is much more than a labeling mechanism. It is a multileveled system that is constructed upon a rich and intricate web of conceptual and symbolic connections. According to Edelman, the symbolic function is an entirely different way of remembering and forming concepts.[21] Immensely efficient in retaining and retrieving concepts, symbolization significantly expands humans' capacity to relate concepts to one another, to

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18 Laurence and Margolis, 2005, p. 221

19 Dehaene, 1997, p. 5

20 Ibid., p. 106-7

21 Edelman, 1998, p. 92-3

re-categorize and to re-code them. Deacon describes the effect of our symbolic function thus:

> We are not just a species that uses symbols. The symbolic universe has ensnared us in an inescapable web. Like a 'mind virus,' the symbolic adaptation had infected us, and now by virtue of the irresistible urge it has instilled in us to turn everything we encounter and everyone we meet into symbols,[…].[22]

Labeling reduces the complex-multileveled relationships that characterize human symbolic function into "a simple mapping relationship."[23] It does not recognize the difference "between the rote understanding of words that my dog possesses and the semantic understanding of them that a normal human speaker exhibits." It is precisely because, in humans, symbols rely on a rich "web of associative relationships," which can create references "to impossible things," that the direct correspondence between words and objects is "secondary" and "subordinate" to the wider, more abstract conceptual applications of words. (Ibid.)

Deacon distinguishes between two kinds of symbolic processes: the *symbolic reference*, which is the complex system described hitherto, and the *indexical association*.[24] The "indexical association" is formed when one learns to connect a pattern of sound with "something else in the world." This kind of association can be acquired through operant conditioning.[25] Although, words can serve indexical functions, he warns, their "symbolic content" in this function is "minimal." (Ibid.) That is to say that the meaning of the words, used in the indexical function, is immaterial to animal response to those words. This observation implies that forming an indexical association between the verbal pattern of the word 'three,' and a group of 3 cherries does not necessarily initiate or indicate grasping the meaning of the word 'three.' Moreover, in human interactions the probability that the occurrence of a word and its corresponding referent is frequent enough for forming an indexical association is extremely low, for it is acquired through operant conditioning, which requires frequent and incredibly numerous repetitions.[26] Charles Ferster and Clifford Hammer's early-1960s experiments, in which they trained two chimpanzees to count, drive this point home. Trained by

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22 Deacon, 1997, p. 436

23 Ibid., p. 69-70

24 Ibid., p. 79, 412

25 Ibid., p. 80, Operant conditioning modifies the spontaneous responses to stimuli through reinforcing or inhibiting behavior depending upon their desirability.

26 Ibid., p. 70

the operant-conditioning technique, the chimpanzees could successfully match the binary numerals 1 through 7 (1, 10, 11, 100, 101, 110, 111) with the corresponding aggregates. In their comprehensive review of a wide range of experiments in animal counting, Hank Davis and John Memmott pointed out that although the chimpanzees in Ferster et al.'s experiment "became highly proficient at identifying each of the seven different binary numbers," they needed approximately 500,000 trials in 200 sessions (that is, 2,500 trials per each of the 200 sessions) to develop this skill.[27] Durkin, Shire, Riem, Crowther, and Rutter's study of the spontaneous use of number words of mothers and children aged 9 to 36 months further weakens the validity of Dehaene's theory, as it revealed "considerable scope for possible confusion/contradiction" in parental inputs related to number words.[28] For example, the word *one* in English can be used as a "deictic pronoun," as in the case of one mother talking to her 9-month-old baby, "Put two in, that's one, that's one."[29] There are also homophones for some number words, such as *to* and *too* vs. two, or *for* vs. four as is the case of the utterance: "One for Mummy [. . .] that one too, one to me." (Ibid.) Add to this the use of number words both in and out of the number-string context as in "there are five buttons" vs. counting to five,[30] or a lexical-"mismatch" string such as "one, two, three, tickley vs. One, two three four,"[31] and so on and so forth.

These findings imply that the likelihood that humans learn to associate number words with the matching concrete aggregates, not to mention acquiring the words' meanings by operant conditioning, is slim if not nil. In contrast, the efficiency of the humans' rich conceptual and symbolic network enables them to form associations between words and any objects without the need of thousands of repetitions. Better yet, the same network is capable of forming concepts that are independent of external input, and to employ these concepts in abstract thinking. Indeed, Durkin et al., Beatrice, Riem, Crowther, and Rutter speculate that it is the conflicting information wrought by social interactions that promotes development. They explain,

> If the input data are problematic, then the child must acquire competence either independently of the input or because of the progress due to resolving conflicts and contradictions in the strategies he or she develops to cope with the input.[32]

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[27] Davis and Memmott, 1982, p. 553

[28] Durkin et al. 1986, p. 269

[29] Ibid., p. 283

[30] Ibid., p. 271

[31] Ibid, p. 279

[32] Ibid., p. 284

In addition, the question of how children accomplish this can be partially answered by considering "what children *do* with numbers, especially as they become more active contributors to number-oriented dialogues." (Ibid.) Kevin Durkin et al.'s conclusion implies that children acquire numerical concepts through the employment of their conceptual/symbolic system in an active process of thinking.

One of the obstacles in searching for instinctive number is that studies and experiments relevant to this subject must exclusively involve animal and preverbal human infants as to avoid reliance on meaningful numerical symbols; otherwise what will be revealed in them would not be an instinct or instinctive process. The methods of these studies, therefore, are inevitably restricted to comparisons between two concrete aggregates in which one magnitude serves as the reference (in lieu of meaningful numerical symbols) for the quantification and definition of the other magnitude. But this method raises a dilemma, because one of the properties that distinguishes number from other approaches to size estimations is that numbers define sizes in absolute terms such as *three* or *four*, whereas the measurement of one group's size by another group yields relative terms such as *larger*, and *more* (than the other). Moreover, the recognition that one group of objects is larger than another group of objects of the same size can be effectively achieved on the basis of ordinary perceptual processes by relying on the overall area occupied by these two groups. Consequently the results of these studies cannot serve as evidence for the presence of numerical concepts, or as an indication of an enumeration process.

Of course, with regard to animals, there is also the question of the evolutionary import of number capability. In their aforementioned survey of animal-enumeration studies, Davis and Memmott commented, "There may be relatively few items that need to be counted in infrahuman realm. [. . .] There may always be a simpler, more effective way of coping with salient stimuli." Davis et al. speculated that maternal care for offspring might be the only natural situation in which animals might benefit from counting. Albeit, they found out that studies of rats' maternal behavior "show no evidence that number per se plays a role. Rather, the successful retrieval of pups seems to depend on a host of non-number-related factors, most notably ultrasound." They pointed out that animal counting behavior is exclusively related to laboratory experiments, which they described as "restricted to relatively unnatural and extreme conditions."[33] A case in a point are the 15 days and 45 hours required to train rats to recognize differences between quantities at 1:4 ratios in Meck and Church's study, and the 500,000 number of trials that were needed to train chimpanzees to recognize seven numbers in the Ferster et al. experiment.

But even if one accepts the assumption of biological number-processor or number-sense one must come to grips with a difficult issue: How an instinctive

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33 Davis and Memmott, 1982, p. 567

quantification, which is an involuntary response to sensory input generated by the physical world, becomes a rational, analytical concept that is independent of external stimuli and is available for a voluntary conscious examination as all mathematically valid number concepts must be? After all, pre-wired instincts and abilities are restricted to the particular task to which they are dedicated, and therefore, in addition to being inherently involuntary, they cannot be applied in tasks other than the one for which they were designated: Bees, for instance, are known as masterful builders of hexagons. This skill, of course, is guided by their pre-wired instincts over which they have no control. Therefore, they cannot employ this skill in tasks other than building honeycombs. It is precisely because humans, unlike bees, cannot rely on pre-wired instincts and must rely instead on their capacity for thinking, planning, and using measuring tools when constructing hexagons, that their architectural skills extend far beyond honeycomb constructions.

Moreover, the instinctive responses are tightly bound to the sensory inputs of the physical world, which of course do not include numbers, for numbers in and of themselves are devoid of physical attributes. Indeed, even the most ardent supporter of instinctive numbers would admit that instinctive numbers are quite different from the numbers we recognize as the numbers with which we count and do arithmetic. Recognizing this truth, Dehaene explained in his book about number sense that although the accumulator is capable of registering only continuous estimates,[34] this "primitive" processor "*prefigures*, *without quite matching it,* the arithmetic that is taught in our schools [italics mine]."[35] No wonder that only when imprecise descriptions of counting and numbers are used is it possible to attribute to animals and preverbal humans number-related capacities.

Take for example Davis and Memmott's conclusion of their aforementioned critical survey of experiments that deal with animal-number abilities: "Some definitions of counting behavior preclude the possibility that animals can count,"[36] they write. They then proceed to redefine counting in "a way that is not only consistent with its occurrence in humans but also allows for its demonstration in other species."[37] They accomplish it by substituting the use of meaningful numerals in counting with an unspecified "cardinal chain," and the "application of that chain in one-to-one correspondence to the external world."[38] But in exchanging meaningful numerical symbols with a generic chain of tags in counting, Davis and Memmott inadvertently omitted a pivotal requirement for a meaningful counting, that is, the conceptual references, which are embedded in these symbols. The term

34 Dehaene, 1997, p. 5

35 Ibid., p. 4

36 Davis and Memmott, 1982, p. 547

37 Ibid., p. 565

38 Ibid., p. 549

number, as mentioned before, has various interpretations, and few, if any, are clearly defined.

The failure to satisfactorily explain the link between 'number sense' and mathematically valid numerical concepts is of no surprise. It is inconceivable that an abstract number concept may be created without a conscious coordinated effort merely on the basis of sensory experience of the physical world or the association of such experience with a mental module, no matter how relevant the innate and the external inputs seem to be in the mind of the observer. Ultimately, the conception of mathematically valid numerical concepts must involve deliberate effort, as is required for all higher-order thinking processes. If either the spoken word 'three' or the visual sign '3' is to be employed in conscious thought processes, it must become available to human attention through the mind's own conscious and voluntary activities. This prerequisite calls for the association of the signal 'three' with a concept, an image or a mental representation of the idea 'three' that is independent of external cues. The development of numerical concepts, then, involves changes in one's conceptual network, which cannot materialize without conscious and voluntary processes.

The theory of instinctive number seems to rest on the assumption that subconscious intuitive knowledge could be automatically transformed into conscious rational knowledge. This assumption is inconsistent with neuroscientists' view on this issue. For example, Edelman maintains that conscious attention plays a key role in the initial learning tasks of various motor or cognitive routines such as speaking, writing, riding a bicycle, playing a musical instrument, or carrying out calculations. "Successful learning leads to automatization" of these routines. That is, after a skill has been learned, "conscious attention is often not required for performance and is only called up if novelty appears or if a goal is not reached."[39] The traffic of knowledge and skills such as enumeration and calculation, then, seems to flow from the conscience to the sub-conscience in the opposite direction of that suggested by innate-number theory. Neuroscientist and evolutionary anthropologist Terrence Deacon echoed Edelman's view by asserting that the tendency of human cognitive processes is to relegate the task of information processing to unconscious and automatic operations, for these operations are a lot more efficient than conscious processes. To put it in Deacon's words, "consciousness is messy," and thus, the cognitive processes aim to become automated, unconscious, and mechanical, a simple "input-output matching."[40] It seems, then, that the experience of intuitive and spontaneous recognition of a number in response to a specific physical aggregate is actually a 'learned instinct;' that is to say, it is an automated enumeration that is the product of a consciously acquired numerical concept.

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39 Edelman, 1989, p. 201

40 Deacon, 1997, p. 456

If there were any merit to the notion that enumeration is a natural disposition of humankind, one would expect that the history of human culture would reflect this propensity for number. But to the contrary, history indicates that humans' facility with the universal concepts of exact and specific sums is actually a relatively recent phenomenon. The word for 'three' in many Indo-European languages can be traced to a time when three was beyond the human ability to count. Some tribal cultures did not develop number words beyond two or three up to the beginning of the 20th century. It seems that numerical concepts are an imposition on the human mind rather than an innate faculty. Indeed, time and again, the history of the development of numerical concepts has shown that mankind employed alternatives to numerical evaluations and to analyses of quantities whenever they could satisfactorily accomplish these tasks without reference to numerical concepts. That humans have problems in dealing with numerical ideas is also suggested by the countless devices that were invented over generations and across cultures to aid them in numerical calculations.

Historians and mathematicians alike have long considered number a cultural phenomenon rather than a biological one and have pointed out that historically the extent of a numerical system's development reflects cultural needs. This view is substantiated by the fact that cultures in which social structure, economics, and technologies remain so simple as to render numerical systems unnecessary managed quite well for thousands of years without developing numbers beyond the most elementary level.

If numbers are indeed a cultural phenomenon, the quest for the origin of numbers is more likely to be answered by studying human cultures rather than human biology. The next two chapters examine the cultural habits of quantification and how they advanced or, as the case may be, delayed the evolution of universal and abstract notions of rational numbers.

Number of Cycles	Duration of Total Signal		Reinforced Response
↓	↓		↓
2 cycles	2 sec.	|__|--|__|--	Right
↓	↓		↓
8 cycles	8 sec.	|__|--|__|--|__|--|__|--|__|--|__|--|__|--|__|--	Left

Figure VI-1: The training-phase stimuli adapted from Meck and Church's data in Table-1, depicting "Design of Experiment 1: testing" (p. 322) and Figure-4: "Diagram of signal types for Experiment 2" (p. 327)

Signal duration: 2 seconds, Number of Cycles:
at ½ - sec. per cycle *4 Cycles*
|_|--|_|--|_|--|_|--
Un-reinforced-responses when *number* of cycles in held at 4: Considered rats' *time* perception. *Left*

Number of cycles: 2, at 2-sec. Signal Duration:
per cycle *4 Seconds*
|________|-------|________|-------
Un-reinforced-responses when signal *duration* is held at 4 seconds: Considered rats' *number* perception. *Left*

Signal duration: 8 Seconds, Number of Cycles:
at 2-sec. per cycle *4 Cycles*
|_______|--------|_______|--------|_______|--------|_______|--------
Un-reinforced-responses when *number* of cycles is held at 4: Considered rats' *time* perception. *Right*

Number of cycles: 8, at ½-seconds Signal Duration:
per cycle *4 Seconds*
|_|--|_|--|_|--|_|--|_|--|_|--|_|--|_|--
Un-reinforced-responses when signal *duration* is held at 4 seconds: Considered rats' *number* perception. *Right*

Figure VI-2: The Meck and Church's experiment results adapted from data cited by Meck and Church's Table-1: "Design of Experiment 1: testing" (p. 322) and Figure 1 (p.323) and Figure 4: "Diagram of signal types for Experiment 2" (p. 327)

Stimulus: --|___|------|________|-----------
1. The *Run* mode: --|___________________________
2. The *Stop* mode: --|___|------|________|------------
3. The *Event* mode: --|_|----------|_|----------------------

Figure VI-3: The three modes of operations adapted from Meck and Church's Figure 2: "Diagram of three modes of operation of the accumulation process" (p. 323)

PACEMAKER-SWITCH-ACCUMULATOR MECHANISM

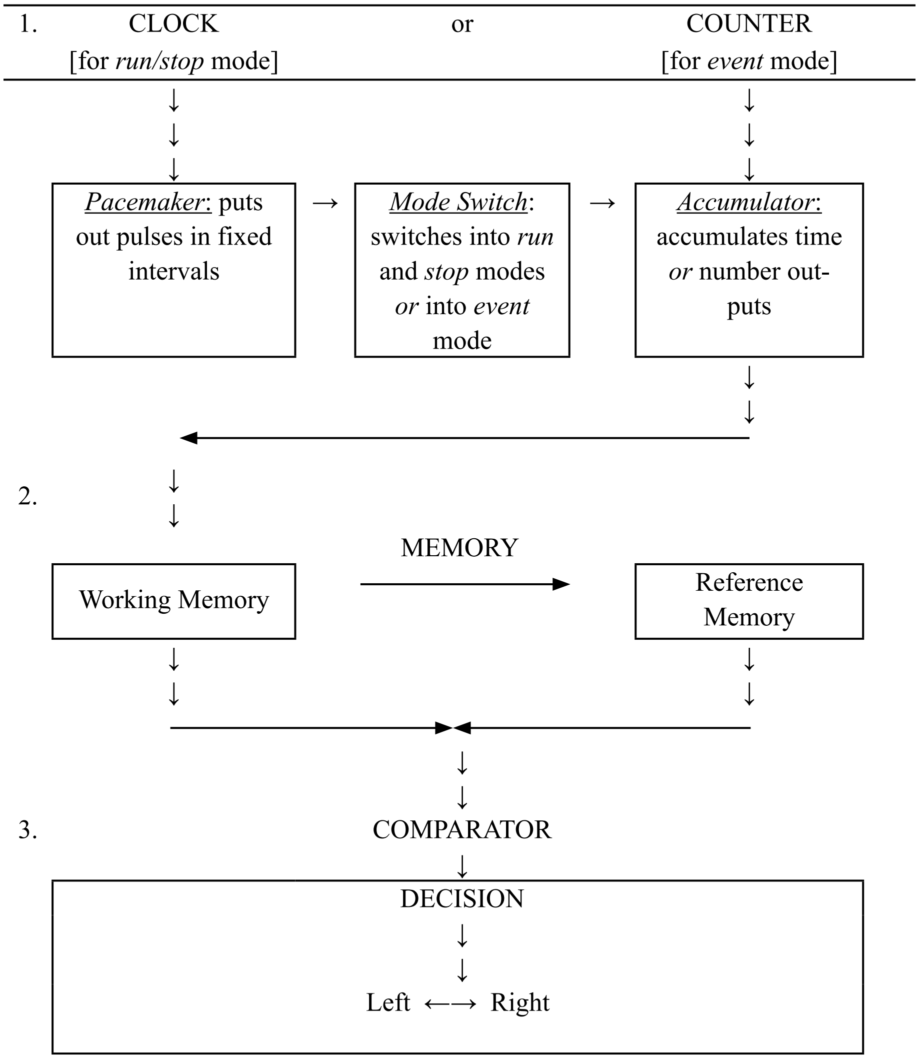

Figure VI-4: The model of the counting-and-timing information processor adapted from Meck and Church's Figure 3: "Functional units of an information-processing model of counting and timing" (p. 324)

VI-2. NUMBER AS A PROCESS

Many historical accounts of the development of numbers begin with a description of a procedure known as *one-to-one exchange*, a process that supposedly enabled early man and some tribal societies into the 20th century to accurately analyze numerical value of quantities without resorting to numbers. But historical facts do not support this assumption. This chapter examines the failure of the 'one-to-one' to generate universal concepts of number.

The principle of 'one-to-one,' when used in trading situations, involved exchanging goods by swapping one object, or a group thereof, with objects of the exchange group; a transaction that repeated itself as many times as was needed to obtain the desired amount of goods. In situations that called for checking inventory, the one-to-one scheme utilized *supplementary* or *auxiliary quantities* (i.e., groups of a fixed number of objects such as sticks or pebbles that served as a quantification instrument). It proceeded by matching the objects of the auxiliary groups with the objects of the examined quantities, one to one (see detailed description in I-3).

There is an obvious procedural similarity between actual counting and the primitive one-to-one practice. During the process of counting, number words are paired with objects comprising a collection, one to one, in the same manner with which early man used exchanged or auxiliary objects in their one-to-one procedure. Moreover, both methods are based on the understanding that sets or collections are numerically equal when the members of one set form a one-to-one correspondence with the members of the other set. The use of auxiliary quantities as an evaluation tool brings the process of the one-to-one procedure even closer to counting, for in both procedures magnitudes are measured with respect to predetermined models of quantities, notwithstanding that in the primitive one-to-one procedure this model is a collection of sticks or shells, whereas in counting this model is a collection of verbally articulated concepts. But perhaps the most important justification for viewing the primitive one-to-one procedure as a harbinger of number concepts is that this procedure—with or without the use of 'auxiliary quantities'—signals a cognitive shift from a *perceptual* mode of magnitude evaluation, which is spontaneous and impressionistic, to a *rational* analytical mode of magnitude evaluation akin to the process that is employed in counting. In both counting and the early-man one-to-one procedure, magnitudes are perceived and treated as sums and their evaluation proceeds through the analysis of their constituent units.

The rational analytical mindset and the principles on which the one-to-one method is based seem to be but a small cognitive step away from a true conception of numbers. It is natural to assume that concepts of number grew out of that practice. But as surprising as it may be, numerous cultural groups in the distant and not so distant past have been practicing the one-to-one procedure for hundreds if not thousands of years, both in trading situations and in inventory evaluations, without ever attaining a well-developed numerical system. The Fiji Islanders, for example, were able to amass the exact quantity of a thousand coconuts, which they called *saloro,* but according to Menninger, they had "no number sequence, at least not an extensive one."[41]

In a closer examination of the primitive one-to-one procedure, the reasons for its failure to produce true numerical concepts in their operators' minds become apparent. Although they are similar in their rational analytical approach to quantification and the method by which they proceed to obtain their analysis, counting and the concrete one-to-one exchange are dissimilar in their purpose as well as in their conceptual referents: The objective of counting is to answer the question, 'how many units are there in a single group?' whereas the objective of the concrete one-to-one correspondence, which involves pairing objects of two quantities one pair at a time, is to answer the question, 'which of the two groups under examination is larger, or has more units?' Hence, counting establishes the *absolute*-size value of the single quantity it counts up, while the primitive *one to one* establishes the *relative*-size values of the two quantities it compares. Moreover, as mentioned above, these two methods of quantification are also dissimilar in their conceptual referents. Counting uses verbal symbols in its one-to-one matching—the number words. Each of these words represents a distinct numerical value. The last word used in counting—that that is assigned to the last object counted—identifies the numerical value of the entire group. Without a prior knowledge of the conceptual content that each of these number words indicates, counting is a meaningless operation. In contrast, the primitive one-to-one practice proceeds and achieves its full meaning by using in its matching operation objects that have no numerical implication. This procedure satisfies its quantification goal without resorting to numerical concepts because its measuring tool is not conceptual; it is another concrete aggregate.

The explicit conceptual referents of counting allow it to define a quantity in an absolute term, whereas the concrete referents of early man's one-to-one practice allow it to define quantities only in a relative term such as "more" or "larger." In fact, the method of matching the units of two sets—one pair at a time—is used in George Cantor's 19th-century "arithmetic of infinity"[42] to avoid a definite articulation of numbers.[43] It seems that the absence of exact and absolute concepts of numbers is at the root of Cantor's ideas about the method of comparing infinities

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41 Menninger, 1992, p. 12

42 Gamow, 1960, p. 25-34

43 Kasner and Newman, 1989, p. 43-4 noted that according to Cantor, "An infinite class has the unique property that the whole is not greater than some of its parts." This statement is incompatible with "finite arithmetic," in which "the whole is always greater than any of its parts."

as much as it is the root of the primitive one-to-one exchange of earlier times. The 'counting-numbers' are, of course, absolute ideas of size; it is inconceivable that they could be originated by a process that aims to establish a relative size value.

Not less important, the procedure of matching objects one-to-one ties one's cognitive process exclusively to the actual perception and manipulation of physical entities. Without the ability to detach quantification activities from concrete entities, early man could not form abstract concepts of number. On the contrary, in many instances, the focus on the immediate and the perceptible married the idea of number to particular objects. Indeed, even though the Fiji's *saloro* (thousand) defines a specific numerical value of a quantity, this word cannot be considered an indication for the concept 'thousand.' Groups of specific size can be technically obtained and even named without reference to specific numerical concepts. The meaning of the Fijian *saloro* could not be envisioned as independent of the notion of coconuts.[44] Bound to particular objects, the word *saloro* did not convey a clear and true universal notion of the number 'thousand,' that is, a number that counts anything. Even the idea 'ten' figured vaguely in their vocabulary; ten boats were *bola*, but ten coconuts, *koro*. The object-specific *bola* and *koro* indicate that the idea 'ten' was indistinguishable from the objects it counted.

And so, in spite of its important similarity to counting, the one-to-one practice failed to generate true numerical concepts. It seems that the creation of numerical concepts could not have originated by the process of units' analysis alone.

VI-3. FROM ADJECTIVE TO NOUN

In order to be mathematically valid, numbers must be perceived as universal, absolute, and independent conceptual entities. But cross cultural etymologies of words that express ideas of numbers or pluralities attest to a time when humans had difficulty disassociating the notion of numbers and pluralities from objects and as a result perceived numbers as the objects' attributes. The Fiji's *bola* (ten boats), *koro* (ten coconuts), and *saloro* (one-thousand coconuts), mentioned in the preceding chapter, indicate that some cultures completely meshed together number and objects into a single concept, according to Menninger. The Fijian's number words are by no means the only example of this trend. The English language assigns the word 'school' to denote a group of fish, but not a group of wolves or deer; the word 'pack' denotes a group of wolves, but not a group of fish or sticks; the words 'yoke,' 'pair,' 'duet,' and 'twin' convey the idea 'two,' but their use is confined to a specific category of objects: It is a 'yoke of oxen',

44 Menninger,1992, p. 11

and a ‘pair of shoes,’ but not a ‘yoke of children’ or ‘duet of shoes.’[45] Menninger maintained that some pairs are so strongly felt like a single whole that one item of the pair is indicated as half. He gives as example the Irish ‘di-suil,’ which denotes two eyes, and ‘leth-suil’ denoting half eye, that is, one eye.[46] Wilder called these object-specific number words *adjectival numerals*,[47] Menninger called them *number as attribute*.[48] Perhaps the most astonishing example of adjectival number words is that of the Tsimshian tribe. In the Tsimshian’s number system, each of the numbers between one and ten has five, and sometimes even seven, different names. Each name is determined by the number’s function or the category of the objects it counts. The number ten, to give an example, has six different names. It is called ‘gyap’ for flat objects, ‘kpeel’ for round objects, ‘kpeentvam’ for long objects, ‘kpal’ for men, and ‘gyapsk’ for canoes; it is ‘gyap’ again when used in counting, but ‘kpeont’ when used in measuring.[49]

Dantzig observed that the object-specific words ‘flock,’ ‘herd,’ ‘set,’ ‘lot,’ and ‘bunch’ are native to the English language while the generic words ‘aggregate’ and ‘collection,’ which express the same idea in an abstract way, are of foreign import.[50] This observation implies that the adjectival ‘many’ are remnants of older and more primitive times. Indeed, Wilder considers the formation of the generic ‘many’ as a step forward in the evolution of mathematically valid number concepts and regards the explicit ‘duet,’ ‘twin,’ ‘koro,’ or ‘gyap’ a “dead end” and hindrance to that development.[51]

From whichever perspective Dantzig, Menninger, and Wilder examine the aforementioned etymologies; none considers the adjectival numbers true number concepts. As Menninger put it: “Number as adjective is number not.”[52] The “object imprint on number,” to use his expression,[53] was a mental obstacle that mankind had to overcome in order to develop a number concept that is congruent with mathematical thinking. A true number concept must be an abstract and universal idea, a number that is independent of the things it counts, and thus, may count anything.

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45 Ibid., p. 30

46 Ibid., p. 12

47 Wilder, 1968, p. 40

48 Menninger, 1992, p. 11

49 Wilder, 1968, p. 41

50 Dantzig, 1954, p.6

51 Wilder, 1968, p. 40

52 Menninger, 1992, p. 11

53 Ibid., p. 30

Insofar as numbers are understood as a cultural phenomenon, the question is, "what was the cultural development that brought about or aided the separation of numerical ideas from the counted objects?"

Wilder has speculated that numbers have never been viewed as entities in and of themselves, or in his words, "as a noun," until some kind of ideographs such as '2,' or more likely '||,' had been used for some time.[54] It is highly plausible that the universal notion of numbers derived from written symbols rather than from the perception of objects' aggregates or from the one-to-one practice. Numerical symbols convey the numerical ideas without committing them to anything in particular. Signs, such as '2' or '||', for example, may represent two cows as much as they represent two coconuts, two trees, or two anything. Better yet, at the same time that '2' and '||' liberate the idea 'two' from the domination of particular objects, they are themselves objects in the sense that they have a physical presence that can actually be perceived and pointed at. As things that have a physical existence of their own, yet, are free of any particular reference other than the numerical concepts that they represent, written numerical symbols mediated and facilitated the process of disassociating the ideas of numbers from the things they counted and establishing them on an abstract foundation. Wilder's hypothesis suggests that number's visual symbols were instrumental in the evolution of numbers as independent entities, and consequently, to the development of genuine numerical concepts. And of course the use of symbolic representation, much like the use of enumeration, grew out of cultural necessities.

The understanding of numbers as a cultural phenomenon and their evolution as a primarily cultural process, as well as the important function number symbolization has played in that evolution, merit a re-examination of our assumptions regarding children's acquisition of numerical concepts.

The next section reviews leading studies and theories of children's acquisition of numbers.

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[54] Wilder, 1968, p. 42, 66

VII

THEORIES OF CHILDREN'S ACQUISITION OF NUMBER CONCEPTS

VII-1. ALFRED BINET'S PIONEERING STUDIES

"That which is called intelligence, consists of two principal things: first, perceiving the exterior world, and second, reconsidering these perceptions as memories, altering them and pondering them," asserted Alfred Binet.[1] Known chiefly for the 1905 'Binet-Simon Scale,' which laid down the principles for standardized psychometric tests, Binet was a man of diverse interests. He wrote about microorganisms and insects, co-authored several plays with Arde de Lore, and not least, he was interested in the nature of intelligence.[2] In the concise and memorable statement quoted above, Binet articulated effectively the age-old premise that underlies many, if not all, theories of children's cognitive development, namely, that our thoughts and knowledge derive from, and are founded on our perceptions of the physical world. According to this view, intelligence has two components. Binet identified one as "perception" and the other, which builds on the former, as "ideation." In ideation he included reasoning, judgment, memory, and power of abstraction.[3]

Binet's original and innovative 1890 experiments in children's perceptions, specifically his studies of their perceptions of length, number, and color (published together in the 1890 volume of *Revue Philosophique*), are seminal in that variations of their basic methodological principles were repeatedly utilized in many of the studies that followed his own.

In his studies of number perception, Binet sought to answer the question, "how the perception of numbers occurs in children who do not know how to count." (Ibid.) The formulation of this question alludes to an assumption that number perception is an instinctive ability rather than a deliberate and rational one as is counting.

1 Binet in Pollack and Brenner, 1969, p. 93

2 Ibid., P. ix

3 Ibid., P. 85-6

Conceiving children's perception of numbers as an intuitive process, Binet believed it required neither conceptual prerequisites nor learned skills. "Fortunately" (so he thought) his little four-year-old daughter, whom he studied, did not know how to read or count. That is, she could count only up to three, and "beyond the number three," he wrote, "she said the numbers entirely by chance and the spoken enumeration was no longer of any help to her." (Ibid.)

Believing that his daughter could not count and did not possess rational numbers, Binet opted to circumvent the requirements for counting and the use of number words, or number symbols, by presenting two groups of objects side by side and instructing the child to indicate the larger or the smaller of the two. It was the same method he had used in his studies of child's perception of lengths and angles. By means of this method both the process of numerical size evaluation as well as the definition of this size could proceed without reference to number concepts and symbols, for each of the two magnitudes (just as in the comparison of two continuums) served as a perceptual reference for the quantification of the other magnitude. Thus, counting and a reference to concepts of number were unnecessary. As for the definition of the quantities in question, words such as 'more,' 'less,' 'smaller,' and so forth sufficed.

Binet found out very quickly that there was a great difference in his daughter's ability to make accurate evaluations of continuum sizes and her ability to do the same with regard to numerical sizes. He made the discovery of children's "deficiency" in the perception of number in an experiment that involved a comparison of two rows of counters (counting pieces, typically in the form of beads or other round objects). One row contained 16 counters, each 4 cm in diameter, and the other row contained 18 counters of 2.5 cm diameter. The fewer but larger counters formed a longer line than did the more numerous but smaller counters; and his daughter judged the larger (but fewer) counters as more numerous. Binet continued this experiment by removing some of the larger counters one by one. His daughter continued to judge the fewer but larger counters as more numerous until only 9 of them were left to be compared with the 18 smaller counters. He concluded that the child perceived the group of objects as a continuum and judged the number of objects by the place their group occupied on the paper, rather than by how many of them were in the group. In other words, the child answered the question 'how large,' instead of the question 'how many.' Hers is not a genuine enumeration, he concluded.

In another study, however, his daughter consistently made the correct choice when the differences between the number of the larger beads and the number of the smaller beads were as follows: 1 large counter versus 2 small; 2 large versus 3 small; 3 large versus 4 small; and 4 large versus 5 small. But when presented with 5 large beads versus 6 small ones, she judged the fewer, but larger beads as more numerous, again. These results implied that within the range of 4 or 5 items, the child was able to ignore the global impression of groups and to recognize correctly that some groups that are smaller globally nonetheless contain more counters. This indicated that her evaluation utilized numerical criterion. She shifted her estimation strategy back to perceptual impression only when she had to compare larger groups of objects.

Given the close relationship between the range of numbers she could count effectively (which was a definite 3) and the range of numbers for which she was able to arrive at correct enumeration (which was 4 and perhaps 5), it seemed that the child's difficulties lie not in recognizing the larger aggregates for what they were, but rather in her inability to enumerate without the possession of the specific conceptual referents that matched the quantities on display.

Indeed, even though "she substituted the perception of the whole for that of its discrete elements" when she quantified larger amounts of beads, as Binet alleged, "she continued to indicate that she was trying to perceive numbers." "'There,' she said, pointing toward one of the groups, 'there are more of them here.'"[4]

Binet, who was "astonished" to have discovered the great discrepancy between children's ability to perceive phenomenological sizes such as length and their ability to perceive the number of items in groups, asked himself "whether there was a real difference between these two modes of perception." (Ibid.) The term "two modes" alludes to a qualitative distinction between these two processes. The results of his experiments should have led him to answer this question in the affirmative, that is, that the perception of length and the perception of number are two different modes of perception. Yet, Binet confined himself to the conclusion that "[children's] perception of the whole occurs more easily and more correctly than the perception of number." (Ibid.) By defining this difference with the quantitative term *more*, rather than the qualitative term *what* (as in, 'what are the properties of'), Binet apparently thought that the perception of numbers and the perception of length share the same cognitive properties, and the difference between them lies only in the relative difficulty of their application. Thus, he remained convinced that the child perceives numbers without employing what he called, "ideation," that is, conceptual symbolic thinking. "However limited her instinctive enumeration," he writes, "it goes much beyond learned verbal enumeration which, in the child examined, did not exceed the number three." He identified the number six as the limit of that accurate "instinctive" perception. (Ibid.) Having identified his child's enumeration experiments as studies of ordinary perceptions, and concluding that children are able to correctly estimate numbers in the small range of 1-6 without reference to verbal enumeration, Binet remained convinced that numbers within that small range are perceived intuitively.

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[4] Ibid., p. 88-9

Binet's comparative numerical evaluation technique, which was based on his view that "perception" gives rise to "ideation"[5] and that enumeration is essentially an intuitive perceptual task, became the signature method for studies of children's acquisition of number concepts until late into the 20th century. Some of these later studies even proceeded with experimentation of children's perceptions of length.

VII-2. PIAGET AND THE ORIGIN OF NUMBER IN CHILDREN

Among the psychologists who studied children's number conceptions, Piaget is the most well known. His theory of *The Origin of Number in Children*, formulated in 1941, continues even today to influence the subject of children's acquisition of number concepts.[6] The theory is best understood in the context of his general views concerning children's cognitive development and particularly his concept of developmental stages. We begin, then, with a brief summary of this aspect of his theory.

Piaget imagined the development of intelligence as a process in which children progress along a fixed hierarchy of cognitive modes or 'stages.' Each stage is characterized by a more advanced level of cognitive operations than its predecessor. Overall, the development of cognitive operations leads "from intuitive and egocentric pre-logic to rational co-ordination that is both deductive and inductive."[7]

The first two stages of children's intelligence development are the *sensory-motor* stage—from birth to age two, and the *intuitive* stage—from age two to age seven. In these stages the child's cognitive processes are governed by perceptual impressions. This perceptual domination binds the child's cognitive system to a fixed perceptual 'schema' that in turn prevents him from considering diverse perceptual information simultaneously. The child, therefore, is not capable of relating different pieces of perceptual information to one another and of incorporating them into a "dynamic whole or system of relationships."[8] The failure to examine information logically from various points of view impairs the child's ability to form coherent, logical, or rationally coordinated thought. According to

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5 Pollack et al. 1969, p, 85, Binet believed that "intellectual development begins with those lower functions which attain a very high degree and almost end their evolution at a moment when the higher functions are in a rudimentary state."

6 Piaget's 1941 book "La Gense do Nombre Chez L'enfant" (The Origin of Number in Children) was transliterated to English in 1952, under the title, "The Child's Conception of Number."

7 Piaget, 1952, p. vii

8 Ibid., p. 87

this view, the child's thoughts at these first-two stages are immobile, irreversible, or not "operational." Thus, Piaget considered these stages in children's cognitive development the *pre-operational* stage. Only when the child reaches the age of 7 is he capable of entertaining diverse perceptual data simultaneously, thereby engaging in an *operational* mode of thinking. Piaget divided the operational stage into two stages, the *concrete operational* stage of the 7 to 11 year olds, in which the child's mode of thinking is more flexible but still concrete, and the adolescent's *formal-operational* stage, in which the child's thinking begins to resemble that of adults.[9] Piaget's experiments in the subject of number-concept development involved children between the ages of 4 and 7, that is, children in the 'pre-operational' (more specifically, 'intuitive') stage and the beginning of 'concrete-operational' stages.

Piaget's understanding of children's number-concept acquisition and of the cognitive process he deemed necessary for number comprehension was also affected by his own conception of number. Let us, then, briefly summarize Piaget's notion of number as well.

According to Piaget, number arises from the fusion of the *logic of class*, which is the thinking that underlies *classification*, and the logic of *asymmetrical relations*, which is the thinking that underlies the activity of ordering elements and forming sequences. Hence he also called the latter *seriation*. The 'logic of class' concerns the 'qualitative equivalent' of the elements counted—the classification (or categorization) of entities as equivalent units, so as to allow their inclusion into a single system. For instance, "Red and blue counters are counters irrespective of their color" and, therefore, can be included in the same class, the "counters class" (as in 2 blue counters + 3 red counters = 5 counters). The 'logic of asymmetrical relations,' on the other hand, is about recognition of the differences between units, which are "differences only of order." For instance, "when counter B is on the right of counter A, and a certain distance away, A and B are conceived as different."[10] Since the difference between 'classification' and 'seriation' concerns only the way in which the same units are considered (that is, they are classified and assembled in the former, and distinguished and ordered in the latter) every unit is both equivalent and asymmetrical at the same time.

But so long as a set is classified by the concrete properties (or "identities") of the elements it encompasses, the logic of class and logic of asymmetrical relations stand apart. It is only when the same units can be subjected both to classification and seriation, without regard for their properties, that the synthesis of "inclusion and seriation of the elements into a single operational totality take place." Piaget identified this "operational totality" as the sequence of whole numbers

9 Ginsburg and Opper, 1969, P.133

10 Piaget, 1952, p. 94-5

that—consistent with their formation process—are "indissociably cardinal and ordinal."[11] This is to say that each concept of number is produced by the synthesis of its cardinal value (the amount of units it encompasses) with its ordinal value (its position on the sequence of the counting numbers), such that the number's cardinal value and its ordinal value become indistinguishable. This view suggests that for Piaget, *numbers* mean not only the discrete numbers such as 'three' or 'four,' but also the counting sequence in which they are ordered—a rather complex description of an otherwise irreducibly plain concept. Indeed, according to Ginsburg and Opper, *number*, as Piaget envisioned it, does not pertain only to the numbers that can be used in arithmetic or computation—tasks, that in his opinion, require no understanding and can be carried out simply by rote memorization. Instead, it pertains also to the fundamental concepts that underlie numerical ideas, for instance, one-to-one correspondence and 'conservation,' which is the understanding of the invariance of numbers.[12] Likewise, the development of numerical concepts entails not only acquisition of numbers as such or the arithmetic relationships between numbers, but also obtaining the generic ideas on which numbers are founded.

Believing that a conception of 'number' requires the coordination of two different kinds of logic, Piaget proposed that the "construction of number goes hand-in-hand with the development of logic and that the pre-numerical period corresponds to the pre-logical level." (Ibid.) This understanding, together with his ideas concerning children's cognitive development, led him to the conviction that children start making progress toward true understanding of number only when they reach the age of 6 or 7. At this age children are in their 'operational stage' and are finally able to unite 'class' and 'asymmetrical relations' into a single operational system.

Another pivotal element in Piaget's theory of children's acquisition of number concepts is the assumption that children's own activities in relation to their physical environment have a greater role in their cognitive development and the formation of their thoughts than social input including language.[13] Thus, the promotion from one stage to another is propelled by the child's own activities in the physical world. Guided by this belief, he studied cognitive issues by orchestrating experiments in which children act and reason in relation to an immediate and deliberately staged physical surrounding. Piaget's conclusions and some of his unique terminology are closely tied to the methodologies of these experiments.

Of the many terms he coined, the term *conservation* is particularly important in Piaget's influential theory of children's acquisition of number concepts, and it became a widely used jargon. The notion of 'conservation' grew out of his

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[11] Ibid., p. VIII

[12] Ginsburg and Opper, 1969, p. 142

[13] Ibid., p. 171

experiments concerning quantification tasks in which he discovered that 4, 5, and 6 year olds tended to offer different estimations for the same quantity after it was subjected to changes in its spatial appearance. For example, he found that children judged the volume of liquid as greater, or 'more,' once it was poured from a shallow and wide vessel to a taller but narrower one, or considered as more numerous the same number of counters once they were spread out over a larger area or over a longer line. The most astonishing aspect of these results was that children evaluated the same quantities differently even though they were observing the changes that were being made.

The concept of *conservation* with regard to quantification, whether it refers to continuous or to discontinuous quantities, implies the understanding that unless something is added to or subtracted from a magnitude, the magnitude remains the same regardless of the changes that may occur in its appearance. 'Number conserved' is described by Piaget as a number that "remains identical with itself, whatever the distribution of the units of which it is comprised."[14]

Piaget believed that children's cognitive development is independent of language. Consequently, many of his experiments in the topic of children's acquisition of number concepts concerned comparisons of two aggregates by way of one-to-one correspondence, which does not require the use of number words. The following experiment is an example:

> Fu (age, 5:9) poured the content of 6 bottles into 6 glasses and put the glasses in front of the empty bottles. 'Is there the same number of bottles and glasses?—*Yes*.—(the bottles were grouped closer together in front of glasses—Are they the same?—*No*.—Where are there more?—*There are more glasses*.—(The reverse process then took place.)—And now?—*There are more bottles*.—What must we do to have the same number?—*We must spread out the glasses like, this, no, we'll need some more glasses.'*[15]

Even counting will not persuade the child that the number of bottles and glasses remain the same when one of the set forms a longer line or is spread over a larger area as in the following example: When Mul (age 5:3) responded in a way similar to Fu (as described above), Piaget asked him to count. After the child correctly counted 6 bottles that were still spread apart and 6 glasses that were close together, Piaget asked again if there are as many glasses as there are bottles. "*There are more where it's bigger*," Mul insisted. (Ibid.)

[14] Piaget, 1952, p. 3

[15] Ibid., p. 45

In light of his experiments in one-to-one correspondence, Piaget thought that conservation occurs when the child forgoes his reliance on global impressions of the groups and instead relies exclusively on one-one-correspondence, or as he worded it: when "correspondence triumphs over perceptions."[16] This, in turn, enables the child to decompose entireties and coordinate the relationships of their parts, an achievement that permits him to construct a "reversible system" and still retain the set as a constant whole.[17] Viewing the concept of conservation through the prism of his experiments with one-to-one-correspondence, Piaget associated conservation with "lasting equivalence,"[18] meaning, a dependable ability to recognize that two quantities remain equivalent even if a different distribution of items in one of them has changed the size of the area it occupies. Lasting equivalence and conservation are achieved when the child is able to consider length and density simultaneously, and to relate these two perceptual inputs to one another so as to understand that the decrease in length is exactly compensated by an increase in density.[19] This understanding enables him to realize that the numerical value of an aggregate does not change even if the objects of which it is comprised are spread over a larger area. It follows that the concepts of equivalency and conservation depend on the ability to concurrently entertain different perceptions and ideas, to which the 'pre-operational' child, whose cognitive system is confined to fixed perceptual schemes, is incapable. The child can 'conserve' only when he reaches the 'operational stage,' at around the age of 7.

Piaget admitted that when the "sets" are small, say up to 4 or 5 objects, the 'pre-operational' child is already capable of "simultaneous perception of the whole and of the elements." But he deemed those numbers *intuitive*, that is, that they are based on perceptual rather than operational processes.[20] At this stage, the classification process is not yet detached from actual objects; consequently, it cannot merge with 'seriation' into a coherent and dynamic whole, and therefore there is no true union of class and asymmetrical relations, or the "intermingling of cardinal and ordinal processes that constitutes number."[21]

Piaget expanded the discussion about children's number cognition beyond the strictly perceptual and the intuitive of his predecessors. The many new terms he coined, and the new ideas, methods of experimentation, and thinking he originated still occupy an important place in the study of children's number cognition. The thoughts and experimentations of other scholars in relation to Piaget's theory are examined in the following chapter.

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16 Ibid., p. 37,55

17 Ibid., p. 89-90

18 Ibid., p. 85

19 Ibid., p. 94

20 Ibid., p. 199-200

21 Ibid., p. 154

VII-3. PIAGET'S THEORY REVIEWED

Piaget introduced into the discussion of children's number cognition an array of new ideas and concepts and wove them together into a logically coherent vision of children's number-concept development. Yet, the strength of his theory must stand not only on its internal logical consistency. Equally important is the question of whether the basic assumptions that underlie his theory, his experimental methodology, and his conclusions are supported by other studies relevant to that topic. This chapter examines Piaget's theory in light of the thoughts, the studies, and the findings of other scholars. It focuses mainly (1) on Piaget's proposition that children in their *pre-operational/pre-logical* phases of cognitive development are not capable of forming genuine numerical concepts and (2) on his concepts of *equivalency*, *one-to-one correspondence*, and *conservation* and the part they play in children's number-concepts acquisition.

Let us begin with Piaget's presumption that children, approximately 7 and under, lack the cognitive capacity for genuine deductive reasoning.[22] This pivotal tenet in Piaget's theories was strongly called into question by the results of Bryant and Trabosso's series of experiments in "deductive transitive inferences" with 4-, 5-, and 6-year-olds, which employed rods distinguished by two criteria: length and color.[23] Their experiments demonstrated that children as young as four years old are capable of applying diverse perceptual information logically, provided they can remember this information. In a 1973 experiment, Bryant specifically targeted the relationship between memory and inference by using only verbal stimuli, for example, Joe is taller than Susan, Susan is taller than Tom, etc. Once again the results indicated a tight correlation between memory and inference ability. Bryant concluded that when errors in inference occur they are most probably the result of memory failure rather than a failure to reason logically. Bryant et al.'s demonstration of transitive-reasoning ability in children as young as four years old (Piaget's 'pre-operational' period) undermines one of the essential premises on which Piaget's theory of children's acquisition of numerical concepts is founded along with his explanation of children's conservation errors, which he attributed to their 'pre-logical' or 'pre-operational' stage.

His conclusion, notwithstanding, Bryant did not attribute children's conservation errors to mnemonic factors.[24] Instead, he argued that because of the comparative

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22 Ginsburg and Opper, 1969, p. 83

23 Bryant, 1974, p. 43-50, (Y.) 'Transitive inference' is a logical rule pertaining to the relations between members of a given sequence, for example: if a>b, and b>c, then a>c. According to Ginsburg and Opper, 1969, p.112, Piaget called this logic, 'ordinal relationship.'

24 Ibid., p. 130-1

context of Piaget's conservation experiment the children tended to judge the size of the group that was spatially changed by the criteria they normally use in comparative tasks and were using in their first comparison. A more suitable test of the child's understanding of invariance of quantities would have been to look at her response to a change in only one quantity.[25] To substantiate this argument, Bryant had recourse to Elkind and Schoenfeld's 1972 experiment in which the length of only a single row of objects was changed. Their experiment demonstrated that children, as young as four-years-old, did not view the change in the *length* of a single row of objects as if it was a change in the *number* of these objects, and they made almost no conservation errors. As it were, Elkind and Schoenfeld's intention in this experiment was to confirm their supposition that Piaget's classic conservation experiments do not test for the understanding of the invariance of a quantity as such, which would be a "conservation of *identity*," but for the understanding of the invariance of equivalence, which would be a "conservation of *equivalence*."[26] Eysenck pointed out that in Elkind and Schoenfeld's experiment the 'conservation' task required only recognition that the single quantity in question remains the same in spite of the changes to its spatial distribution, whereas in Piaget's experiments the 'conservation' task required three steps: first, to establish that the two compared quantities are equal; second, to recognize that the quantity that was spatially changed remained the same quantitatively; and third, to recognize that the spatially changed quantity maintained its quantitative equivalence with the quantity with which it was previously compared. The difference between these two kinds of conservations, then, is that the 'conservation of identity,' which involves a single quantity, is a direct or "pure" indication of understanding of the invariance of quantities, whereas 'conservation of equivalence,' which involves a comparison of two quantities, requires, in addition, transitive inference.[27]

That said, one should keep in mind that Piaget's association of 'conservation' with 'equivalence' did not derive only from his assumption that 'pre-operational' children are incapable of logical thinking such as transitive inference, but also from his belief that children's logical thinking draws primarily from actions that do not engage language; which means that children do not rely on verbal or symbolic inputs for their logical thinking. Just as Binet used perceptual comparisons of two concrete aggregates in order to circumvent reference to verbal symbols and counting in his number-discrimination experiments, so Piaget used one-to-one correspondence for the same end. The association of 'conservation' with 'equivalence,' then, grew out of Piaget's experiments with one-to-one-correspondence. These experiments led him to believe that 'conservation' occurs when "correspondence triumphs

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25 Ibid., p. 175-6

26 Elkind at al., 1972, (Cited in Bryant, 1974, p. 130-2, 148-50, emphasis mine)

27 Eysenck, 2004, p. 528-9

over perceptions."[28] He assumed that such a "triumph" occurs when the child can co-ordinate diverse perceptual inputs into a dynamic whole.

Piaget's dismissal of the contribution of children's symbolic function to their cognitive development is perhaps one of the weakest elements of his theory. After all, the conceptual/symbolic function does not depend on physical stimuli; therefore, it enables children to retrieve remembered perceptions and concepts and relate them to one another without input from the external world. Such ability is imperative for the creation of new concepts, in particular, abstract concepts that have no physical referent. Numbers in-and-of-themselves are devoid of any recognizable physical attributes—they are pure abstractions—and as such, can have no existence except as concepts and symbols. It is inconceivable, then, that numerical concepts can be constructed without conceptual/symbolic thinking.

Piaget's conviction that children's thoughts are guided by actions rather than by language is equally incredible. To examine that proposition let us turn to L. S. Vygotsky, whose views about manipulation of objects are particularly instructive. Ever mindful of the influence of the social/cultural environment and language upon children's learning and cognitive development, Vygotsky analyzed the effect of symbolic thinking on their interactions with their physical surroundings, or as he saw it, the effect of speech on human behavior and tool use. According to Vygotsky, speech enables the child to include stimuli that are not part of his visual field; this inclusion enables him "to ignore the direct line between actor and goal."[29] It is then, through the mediation of the child's inner "stimuli," that is to say, his conceptual/symbolic network, that he is capable of creating "a time field," which in turn enables him to "view changes in his immediate situation from the point of view of the past activities" and to "act in the present from the viewpoint of the future."[30] That ability to separate actions from perceptions and to postpone responses, frees the child from the constraint of the immediate-sensory stimuli and the dictates of the concrete world, enabling him instead to direct his actions by means of his own thinking and intentions. Though formulated at the beginning of the 20th century, Vygotsky's view is consistent with contemporary thinking of neuroscientists. Gerald Edelman, for example, raises the possibility that conscious attention to a given task depends upon what he called, "*negative* influences," such as the suppression of sensory and perceptual inputs by means of stimulation of concepts and imagery.[31] Terrence Deacon concurs, and adds that

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[28] Piaget, 1952, p. 37, 55

[29] Vygotsky, 1978, p. 26

[30] Ibid., p. 36

[31] Edleman, 1989, p. 200-1, In Edelman own words: "other than that related to the carrying out of a given motor plan."

instinctive responses to immediate and concrete stimuli conflict with symbolically mediated actions.[32] These views imply that children's interactions with their physical surrounding are guided by their conceptual/symbolic function, and not the other way around, as Piaget believed.

A similar emphasis on the actual perceptual processes rather than on conceptual/symbolic processes is implicated by Piaget's understanding of *conservation*. Piaget maintains that a permanent conservation, and ultimately a true conception of number, depends on operational thinking as demonstrated by a child's recognition that the density of a row of objects exactly compensates for the row's length.[33] Built on the perception of two concrete entities (length and density), Piaget's explanation places the process of conservation within a strictly perceptual framework. In the same vein, Norman Ginsburg defines 'conservation' as "the capacity of the organism to maintain an invariant response to an invariant property of the world in spite of changes in irrelevant but distracting properties."[34] 'Conservation,' as recognition of an "invariant property of the world," pertains to an aspect of a physical thing. It seems, then, that the kind of logical reasoning involved in 'conservation,' as Piaget and Ginsburg imagine it, relates to physical phenomena. If so, perhaps the most pertinent question regarding the concept of 'conservation' is whether or not the invariance of physical quantities is at all relevant to the acquisition of numbers, which themselves lack any physical qualities. The invariance of concrete quantities can be established by considering their various spatial/perceptual variables. For example, the *density* and *length* in the instance of a group of bottles that are positioned closer together or farther apart, and the *height* and *width* in the instance of water that is poured from a tall, thin vessel into a short, wide one. But the invariance of the numerical value of a collection, which is devoid of any perceptual elements of its own, cannot be established by consideration of spatial/perceptual data. On the contrary, spatial/perceptual information must be ignored to allow attention to the group's numerical value. The presence of a specific numerical concept helps the child to ignore irrelevant perceptual information and focus her attention only upon a group's numerical value. It is no surprise, then, that that the smallest numbers are the first numbers to be "conserved."

Extensive evidence suggests that 'conserving' the numerical value of a group depends upon a referent to a specific numerical concept, not on logical consideration of perceptual stimuli. Starting with the early experiments of Binet with rows of beads, studies consistently show that once a child acquires a specific numerical concept, she will not be confused by changes to an arrangement of

32 Deacon, 1997, p. 435

33 Piaget, 1952, p. 94

34 Ginsburg, 1976, p. 667

objects within that number. Indeed Piaget's own experiments show that when the groups in question are small, children of 'pre-operational' ages can 'conserve.'

It is reasonable to assume that younger children who do not yet possess the concepts of larger numbers opt for a perceptual instead of enumerative strategy when they compare the size of larger groups. When numbers exceed subitation range, an absence of reference to specific numerical concepts may also be attributed to lack of an opportunity to count. In this case, determining the numerical value of a group by a global impression of the area occupied by that group is a legitimate alternative to true enumeration, albeit prone to mistakes. In fact, impressionistic estimation is susceptible to erroneous judgment not only in children but also in adult subjects, as has been documented in a few studies. For example, Krouger's 1972 experiments demonstrated that adults perceived the same number of objects as more numerous when they were spread over a larger area. In 1976, Norman Ginsburg conducted three experiments concerning the question of how random as opposed to regular arrangements of dots effect estimation. The 53 college students who participated perceived regular patterns as "significantly more numerous" than random arrays. Mindful of Piaget's theory, Ginsburg called this phenomenon a "breakdown of conservation of number"[35] and claimed that his adult subjects "regressed to a stage of non-conservation."[36] In their 1972 study involving adults and 8-year-old children, Christopher and Uta Frithe explored the "solitaire illusion—" the illusion that a single large cluster of dots appears to contain more elements than several small clusters—and found no significant difference between adults' and children's performance.[37] All these studies demonstrate that adults err similarly to children when they use perceptual strategies in number-estimation tasks.

Piaget's emphasis of the important contribution of 'one-to-one-correspondence' activity to the formation of numerical concepts was also questioned. For example, Gelman and Gallistel observed that Piaget's one-to-one-correspondence method of experimentation detracts from the children's ability to 'conserve.' They explain that the pairing of objects directs the child's attention to the structure and pattern of the groups under consideration instead of to their numerical values. When the child's attention is directed to a specific group she is more successful in conservation tasks, as Gelman's experiments have demonstrated.[38] It seems that when dealing with a single group children are better able to tie their numerical evaluation to a specific number.

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[35] Ibid., p. 663

[36] Ibid., p. 667

[37] Frith and Frith, 1972, p. 410

[38] Gelman and Gallistel, 1978, p. 230, 233

Beside interfering with the process of number recognition, the ‘one-to-one’ process also hinders the very formation of numerical ideas. Not that units’ recognition, which characterizes one-to-one correspondence, is not a necessary prerequisite for the formation of numerical concepts, but when the one-to-one process focuses on comparing two concrete aggregates it inevitably leads to a relative, not an absolute evaluation of the considered quantities. Bound exclusively to the examination of relative values, the one-to-one process cannot yield numerical concepts, which are inherently definite. The history of human cultures attests to this conclusion: Thousands of years of one-to-one-comparison practiced in a variety of social and commercial contexts failed to advance the development of mature numerical systems in those societies in which it was practiced.

But then, it is possible that Piaget perceived numbers as relative concepts in the first place as is implied by his belief that each number must be understood in the context of its ordinal placement in the sequence of the counting-numbers, and therefore numbers are “disassociably cardinal and ordinal.” Let us note that insofar as its cardinal value defines a number in an absolute term, and the ordinal value defines a number in a relative term, there is a certain self-contradiction in this postulation. Yet some researchers agree that children understand numbers by connecting their cardinal with their ordinal values. Take for example Wynn’s assertion, “[linguistic] symbols obtain their numerical meaning by virtue of their positional relationships with each other.”[39] This idea is at odds with the fact that the conceptual identity of a number is established solely by its cardinal value, that is, its sum size. Of course, the sum size that is denoted by a given number word does not change, no matter where this word is located in a sequence or how it is used. Not that there is no association between the cardinal and ordinal meaning of number words, but it is always the cardinal meaning of a number word that determines where this number word is placed in relation to a sequence of other number words. For example, *three* is the third number in the counting-number sequence, but in the prime-numbers sequence, *three* is the second number, and in the multiples-of-three-sequence it is the first number. The meaning of the word *three* determines its ordinal location in those different sequences, and in spite of *three*’s different ordinal locations in these sequences, its meaning does not change.

The emphasis on the association between the ordinal and the cardinal meanings of number words is unfortunate for it confuses the application of number words in counting, in which their ordinal value is important, with their wider application, namely, the representation or numerical concepts, where the cardinal value these words convey is important. Wagner and Walters observed that the difference between the semantic and the procedural aspects of number words

[39] Wynn, 1992, p. 228

is one of the causes for toddlers' errors and confusions in various enumeration tasks. They lamented, "it is a linguistic misfortune that we do not count by ordinal lexemes ("first, second, third") so as to distinguish between the cardinal value of a collection and an ordinal positioning [in] a sequence."[40]

Finally, Piaget thought that children's ability to correctly identify small numbers, combined with their inability to 'conserve' larger numbers, indicated an "intuitive" perception of small numbers. He proposed that the perception of numbers within the range of 5 by the 'pre-operational' children is merely a result of reproduction of a fixed perceptual "schema" rather than the result of logical operation. Although there are some contemporary scholars who claim that infants can discriminate between small numbers intuitively, the range of the numbers they consider 'innate' is within the range of 3, not within the range of 5 as Piaget thought. But even at this more limited range, the idea of innate number is typically accompanied by a disclaimer. Karen Wynn maintains that the innate 'cardinal tags,' that is, the naturally-occurring-nonverbal numbers, "[are] not available to conscious inspection and so cannot inform linguistic, culturally supported counting activity."[41] Wynn's description excludes these nonverbal "numbers" from the realm of numbers that are applicable in thinking in general, and in arithmetic thinking in particular. Similarly, Dehaene describes the 'innate number' as a process that "prefigures, *without quite matching*" the numbers used in school arithmetic.[42] Yet, this notion of 'intuitive number' was altogether discredited by studies of other psychologists even prior to Piaget's own foray into this subject. The most often cited research concerning this topic is H. Beckmann's 1924 study of two-and-a-half and three-year-old children's perception of number. Beckmann observed that his subjects could recognize the number of objects in small groups only after they had learned to count up to that number, no matter how small this number was. Moreover, the younger they were the more they were inclined to count aloud when asked about the number of objects in a group.[43] Beckmann's finding, according to Bryant, precludes the possibility that numbers can be "subitized" purely on the basis of perception.[44] Beckmann/Bryant's view gained a lot of muscle from Gelman's 1978 studies and many others who were inspired by her research concerning counting. In her research, Gelman noticed that whenever children were asked to judge the equivalency of two sets of objects, they spontaneously resorted to counting. Even two-and-a-half-year-olds preferred

[40] Wagner, and Walters, 1982, p. 149-50

[41] Wynn, 1992, p. 223

[42] Dehaene, 1994, p. 4 (emphasis mine)

[43] Gelman and Gallistel, 1978, p.69

[44] Bryant, 1974, p. 120

to base their estimation on counting rather than on direct perception.[45] In fact, the younger the children were, the more emphatic and deliberate was their counting. For example, the younger subjects tended more than the older to point to the object counted and count it aloud, while estimation without overt counting was more typical to the older children.[46] Evidence that younger children rely on actual counting more than older children in number perception tasks implies that they depend more than their seniors on deliberate analytical processes for numerical evaluations; this conclusion contradicts the idea that children's perception of number in their 'pre-operational' age is a mere reproduction of a fixed 'schema.' As it is, the only evidence Piaget offers in support of his assertion about intuitive numbers is his own theory.

Although, as shown above, many dispute this or that aspect of Piaget's theory and methods of experimentation, no one ignores his work. His studies and thoughts touch many subject matters and reveal the richness and complexity of children's number cognition. Perhaps, one of the most enduring contributions of Piaget to the understanding of *the origin of number in children* is the great interest in children's acquisition of number that his thought-provoking work imbued in other scholars.

The following two chapters deal with Rochel Gelman and C.R Gallistel's counting theory and related issues of their influential work, "The Child's Understanding of Number".

VII-4. GELMAN AND GALLISTEL: THE CHILD'S UNDERSTANDING OF NUMBER

Gelman and Gallistel's 1978 *The Child's Understanding of Number* was influential mainly due to its analysis of children's counting behavior, its elucidation of the principles necessary for a successful counting, and the volume of new data pertaining to children's counting and number reasoning that the study generated.

Gelman and Gallistel speculated that children's perception and abstraction of numbers involves a rapid sub-vocal counting and that counting has a role in number-concept acquisition.[47] They based their proposition on the studies of Beckmann (1924) and Descoeudres (1921) which demonstrate that a number must first be counted before it can be perceived, even if it is as small as 2 or 3, and that the younger the child the greater the tendency to count. Gelman's observation of children's spontaneous counting in her own research further strengthened

45 Gelman and Gallistle,1978, p. 163, 228

46 Ibid., p.78

47 Ibid., p.70

this hypothesis.[48] Abandoning the notion of intuitive perception of numbers, and observing children's inclination to count, Gelman and Gallistel viewed children's counting as intuitive behavior.

Much of their theory and analysis of preschoolers' conceptions of numbers grew out of research by Gelman and her students into the cognitive abilities of preschool children,[49] in particular "The Magic Experiments"[50] and "The Videotape Counting Study."[51] Let us briefly review these two studies:

Though originally designed to study children's reasoning about numbers, the 'magic experiments' proved useful for examining and analyzing children's counting, since in this experiment the children's counting was spontaneous and self-motivated. Children's inclination to count in the 'magic experiments' was particularly remarkable because in order to avoid any reference to number they were instructed to identify the "winner group" instead of the more numerous group. The experiments, which involved 2-, 3-, and 4-years-olds, were designed as a game with two phases: The first phase, which was primarily an identification game, was intended to create an expectation for a specific number. It proceeded by displaying two plates, each containing a different number of small toys (for example, one plate with two mice and the other with three). The experimenter pointed to the "winner" plate, and then covered the toys in both plates. Next, the plates were shuffled around until the children lost track of the "winner" (or the "loser") plate. The children then had to guess which of the two covered plates is the "winner" plate. Once the children made their guess they were asked to lift off the cover and check whether or not they had guessed correctly. The second phase involved making secret changes to the "winner" plates by replacing, removing, or adding objects, or by changing the objects' distribution (hence the term "magic" in the experiments' title). This secret move was followed by a series of questions, such as, "Has anything happened? If so, what? How many objects are now on the plates? How many objects used to be on the plates?" and so on. [52] The children's responses to the secret changes made to the "winner" array and their answers to the experimenter's questions were recorded.

The approach of the videotaped-counting study was more direct than that of the 'magic experiments' in the sense that there was no attempt to conceal that its subject matter is 'number'. The 2-, 3-, 4-, and 5-years-olds that were involved in this experiment were presented with arrays of objects and with the direct question,

48 Ibid., p. 69-70, 222-3

49 Ibid., p. vii

50 Ibid., p. 83-104,161, 249. (Citing: Gelman, 1972a, 1972b, 1977, and Gelman and Tucker, 1975)

51 Ibid., p. 83-104,161, 249 (Citing, Gelman, 1977)

52 Ibid., p. 85

"How many?" or with the instruction to "count them." In addition, the objects in the videotaped-counting study were presented nonlinearly as well as linearly, while in the 'magic experiments' objects were presented only linearly. The children's task was to choose one set for themselves and one set for a puppet. The sets' sizes were, 2, 3, 4, 5, 7, 9, 11, and 19, and in order to keep the children engaged, the sets were presented in pairs, say, 2 and 3, 7 and 9, etc.

The hypothesis that guided Gelman's experiments was that children have "more capacity than meets the eye."[53] Because she set out to find what children can do instead of what they cannot do, Gelman was sensitive to children's states of mind and gave their "proclivities" precedence over the design of the experiment.[54] She also made a special effort to ensure that the children were familiar with the experimenter at least one week prior to the experiment and that the experiments were short and playful. The game-like quality of Gelman et al.'s experiments captured the children's interest and helped in eliciting their spontaneous behavior, yielding rich data and, sometimes, unexpected results. Moreover, this strategy proved to be effective in drawing out children's cognitive skills including 'conservation,' thereby revealing the depth of their understanding of numbers even before they reach school age. Gelman regarded the children's spontaneous behavior as important data. She used this data in designing subsequent research and in formulating her and Gallistel's theory. Gelman noticed that counting was "a salient behavior whenever the experiment permitted." "Indeed," she commented, "it was the prevalence of spontaneous counting behavior that alerted her [Gelman] to the role counting might play in the way children think about number."[55]

She and Gallistel identified five counting principles. They called the first three rules that guide the actual counting procedure, "how to count principles." The first rule in this category is the *one-to-one correspondence*, the understanding that one and only one word should be assigned to each item counted. It includes the ability to distinguish between objects that have already been counted and objects that are not yet counted. Second is the *stable order principal*, the requirement that there must be a fixed word order in counting (as in 'one', 'two', 'three', etc.). Third is the *cardinal principle*, the understanding that the last word used in the counting process indicates the cardinal value of the set. The acquisition of the *cardinal principle* may be expressed either by emphasizing the last word in the series or in repeating it. The remaining two rules concern the abstract nature of counting. They called the fourth rule, which deals with the decision "what to count," the *abstract principle*. Observance of this principle indicates the understanding that the particularities of units (color, size, shape, function, etc.) do not change the

53 Ibid., p. 242

54 Ibid., p. 105

55 Ibid., P.68

result of counting. The child's mastery of the *abstract principle* is manifested by his willingness to include a diversity of objects in the group he counts. The fifth rule, the *order irrelevance principle*, is the understanding that the order in which units are counted does not affect the result of counting. The child's mastery of the *order irrelevance principle* is indicated by his willingness to start counting the same group from a different place or from a different object.[56]

Gelman's studies show that with the exception of the *cardinal principle* children as young as two-and-a-half years old demonstrated a surprising mastery of the counting principles, provided that the sets involved were within numerical sizes with which they were already conversant, that is, smaller than 4 or 5.[57] Children of all ages were highly aware of and paid close attention to the *one-to-one principle*, as they were careful to point to and even touch each of the objects counted. In fact, errors in 'one-to-one' correspondence seldom occurred, even in the case of the two-and-a-half-year-old; and when they did occur, errors were likely to be technical, namely, due to lack of coordination rather than cognitive lapses (for instance, not counting the last object or continuing to recite a number-word after the last object was counted).

Evidence that children of an even younger age, the two-year-olds, followed the stable order rule was, in Gelman's words, "striking."[58] Indeed, when children were able to adhere to only one of the 'how to count' rules, it was usually the stable order principle. Even within the limited range of number-words that they could master, their ability to follow the stable order rule surpassed, by far, their ability to accurately judge number, as in the following example: The experimenter presents 2 items. "How many on the plate?" D.S. age two-and-a-half: "um-m, one, two." Experimenter displays 3 items. "How many on this plate?" D.S.: "one, two, six." Experimenter, "do you want to do this again?" D.S.: "ya, one, two, six."[59]

Children typically used conventional number-words in their ordinal sequence. The tendency to use idiosyncratic words appeared only in the youngest, the two-year-olds. But even the few children who used idiosyncratic words were using them in a fixed order and together with number-words. Usually, the first two or three counting words that were used by the children were the conventional 'one', 'two', 'three.' By age three, their list consisted exclusively of number-words in their ordinal counting sequence.

Whereas the principle of 'stable order' was the first to be acquired, the *cardinal principle*—the understanding that the last number word used in counting expresses the numerical value of the group—was the hardest for children to follow

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56 Ibid., p. 77-82

57 Ibid., p. 101

58 Ibid., p. 131

59 Ibid., p. 91

and the last to be acquired.[60] In contrast, the principle of *order irrelevance*, as far as Gelman could tell, was a non-issue to begin with; children were practically "oblivious" to the order in which they "tagged" objects, she commented.[61] Likewise, the *abstraction principle* posed no problems to children. To Gelman's surprise, the heterogeneity of items in the group had no detectible effect on the children's proficiency in numerical estimation. Even the youngest children did not hesitate to include in a counted group a diversity of objects, and in the rare instances in which the youngest children used idiosyncratic words, they never "tagged" objects in a way that bore reference to their physical properties such as 'blue' or 'mouse.'[62] All of these observations indicate that children understand that the physical properties of the objects being counted are immaterial to the results of the counting.

Their insight into the complexity of the counting task and their hunch that counting plays a role in children's acquisition of number notwithstanding, Gelman and Gallistel claimed that the process of counting is merely a "serial tagging" of objects belonging to a set. The children need not use conventional number words in their counting. Indeed, the "tags need not even be verbal" so long as (1) they are used in marking or in ticking off objects, (2) they are used in a fixed order, and (3) they are arbitrary in the sense that they bear no descriptive reference to the objects that are being counted.[63] Moreover, the children remember a list of their own making better than the conventional list of number words. They shift to conventional number words only in order to make others recognize their knowledge.[64] Gelman and Gallistel expanded the ideas of innate 'tags' even further in their work, "Non-Verbal Numerical Cognition: From the Reals to the Integers," where they hypothesized that the entire range of *real numbers* are stored in a nonverbal form by means of an innate apparatus known as the *accumulator* (See VI-1). Gelman and Gallistel coined the term *numerons* to depict all the possible *tags*, and called the commonly used counting words, which they deemed a subset of *numerons*, *numerlogs*.[65] The proposition that children (perhaps animals as well) use *numerons* as *tags* for "ticking off" objects implies that the meaning of the words used in the counting is immaterial to this process. The principle of *stable order*, for instance, "is neutral with respect to the type of tag; it simply requires that the tags used be drawn from a stably ordered list."[66]

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60 Ibid., p. 126

61 Ibid., p. 217

62 Ibid., p. 116-7

63 Ibid., p. 76

64 Ibid., p. 207

65 Ibid., p. 77

66 Ibid., p. 206

The aforementioned requirement, however, is not trivial; "a significant part of the development of numerical abilities centers around the need to solve the practical difficulties posed by the stable-order principle."[67] This is so, Gelman and Gallistel explained, because the counting procedure requires the memorization of a very long "stably recallable list of distinct numerlogs—" a list that is too extensive for the limited capacity of the human memory.[68] To explain how children deal with this requirement, Gelman et al. proposed that rules must be established for generating words to fill the higher position in the number-word sequence.[69] Once these rules are understood, the list of words required to be learned by rote consists of only the first 12 or 13 number-words,[70] for "all other count words can be derived from the application of the generative rules embodied in the already mastered count words."[71] In other words, the principles for generating higher-order count words are inherent in the 12 or 13 previously acquired counting words.

Devoid of numerical meaning, the number words that children typically use in counting provide them no guidance in this procedure. Instead, the children's counting behavior is propelled and guided by an innate structure—the *counting scheme*. Children count for the sake of realizing this scheme's rules and principles. The *counting scheme*, then, is the means to its own end. But the *counting principles* or *counting scheme* not only motivates the child to practice counting, it also constitutes a "reference against which the child can evaluate and refine his counting."[72] And so, "their scheme plays the role of a personal tutor who both goads and guides." (Ibid.) Guided by the counting principles, the meaning of number-words or *numerlogs* is not pertinent for the process of counting. As it were, Gelman and Gallistel argued that counting in and of itself does not promote number development; it is merely the mechanism that provides the "representations of reality upon which the reasoning principle operates."[73] They explained that young children treat counting "as an algorithm that creates the representation of numerosity employed in reasoning."[74]

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[67] Ibid., p. 79

[68] Ibid., p. 237

[69] Ibid., p. 211, 237

[70] Ibid., p. 211-2, In other part of their book Gelman and Gallistel identify these number words in the English speaking cultures thus: the base numbers to ten, eleven to sixteen, the first few instances of multiples of 10 (twenty, thirty etc.), and the 2nd, 3rd, 6th and so on power of ten.

[71] Ibid., p. 212

[72] Ibid., p. 208-9

[73] Ibid., p. 161

[74] Ibid., p. viii

To complete Gelman and Gallistel's theory of children's number concept development, let us examine the *reasoning principle,* which they consider an important indication of children's understanding of number.

Gelman and Gallistel identified three main components of the *reasoning principle*: (1) *Relation*—the cognition that the numerical equivalence, or the lack thereof, between quantities is based exclusively on the number of units in these groups. All other characteristics (units' properties, place occupied, etc.) do not affect the numerical values or the numerical 'identities' of groups. (2) *Operation*—the understanding that the only thing that can change the numerical value of a group is the addition or subtraction of units. (3) *Solvability* or *reversibility*—the recognition that addition can undo or cancel subtraction and vice versa. That is, addition and subtraction are reverse operations, such that any number can be obtained, or "fixed," by adding or subtracting units from another number.

Except for the limitations that children manifested in the *solvability principle*, Gelman's studies have shown that within the numbers 4 or 5, with which they were already familiar, the children demonstrated remarkable mastery of number reasoning principles. As for the *solvability principle*, Gelman's studies revealed that pre-school children had difficulties in mentally solving how many units to add or subtract when the difference between the target number and the number presented was more than one. But they were able to indicate what direction to take, that is, whether to add units or subtract units.

Similar to Piaget, who suggested that true understanding of the counting numbers must rely on a fusion of the 'logic of *class*' and the 'logic of *seriation*,' they too tied number comprehension to a more advanced form of thinking. Theirs, though, was advanced mathematical thinking such as algebra. The child is able to deal "with that ethereal abstraction called number," only when she reaches the stage in which "arithmetic reasoning is no longer limited to dealing with representations of numerosities," that is, the stage she can reason algebraically. [75]

Gelman and Gallistel's theory raises many questions, which will be discussed in the next chapter. Still, their focus on children's counting behavior places the process of number-concept acquisition in the suitable psychological framework, that is, one that is rational/analytical even if for them counting is only a preparatory step for true number-concept acquisition.

VII-5. GELMAN AND GALLISTEL'S THEORY REVIEWED

Reviewing many scholastic responses to Gelman and Gallistel's work, "The Child's Understanding of Number," one cannot escape noticing that to the extent

[75] Ibid., p. 236-7

that scholars disagree with them, these disagreements are not with their methodology and experimental findings, but with several of their theoretical proposals. Although Gelman and Gallistel's "counting principles" have been well received and are widely used as a tool for analyzing and evaluating children's enumeration activities, their view that these principles represent an inborn scheme is far less accepted, as is their closely related proposal that children make their own counting words and shift to conventional words only in order to make others recognize their knowledge. Some also question the usefulness or the validity of giving primacy to principles over concepts and skills. This chapter examines Gelman and Gallistel's theory with respect to the thoughts and studies of other scholars. Its main focus is on Gelman and Gallistel's propositions that children's counting is generated and guided by an innate scheme rather than by conceptual or social/cultural inputs, and that the words children prefer to use in counting are both self-generated and immaterial to this process.

Let us begin with Gelman and Gallistel's supposition that children's counting behavior is originated and guided by an innate scheme of counting principles: One of the scholars who questioned the aforementioned idea is Catherine Sophian.[76] She maintains that because the three *how-to-count principles* (the stable order, the one-to-one, and the cardinal) are logically interdependent they must be applied simultaneously in order to fulfill their function. But there is evidence that children often violate one or two of these principles in order to comply with another. For instance, in order to finish counting a group without violating the one-to-one principle, children repeat or "recycle" number words (e.g., "1, 2, 1, 2, 1, 2, 3) thereby disregarding the stable-order principle. On the other hand, when children are more interested in using up all the counting words at their disposal they readily abandon the one-to-one principle. Wagner and Walters noticed that 2- and 3-year-olds tend to either count small sets again, or to count more than once some objects until they run out of all the number words they know. They call these tendencies the "list exhaustion scheme."[77] Children were also observed to break the cardinality principle in order to meet the researcher's expectations. For example in her 1992 study of children's acquisition of the number words and counting, Karen Wynn asked children to give her five objects. One girl gave her three, and upon Wynn's request to count these objects, the child counted: "one, two, five."[78] Sophian questions,

> [. . .] what the conceptual significance of the counting principles could be for children who violate one principle to satisfy another. What, for instance, could children who make recycling errors understand about why it is important to tag each and every object? [79]

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76 Sophian, 1998, p. 34-5

77 Wagner and Walters, 1982, p. 143-4

78 Wynn, 1992, p. 225

79 Sophian, 1998, p. 36

It is more plausible, she argues, that children's incomplete use of the counting principles reflects their not entirely successful attempts to reproduce the mechanics of counting that they observed others perform, rather than "*a priori* knowledge" of counting principles. (Ibid.) Sophian, who studied children's counting from a developmental perspective, found that children's understanding of the utility of counting develops over time. For example, her experiment, which involved 3- and 4-year-old children, demonstrated that while all children counted when questioned "how many," one quarter—mostly 3-year-olds—did not count when they were requested to "put 'n' [objects]" somewhere or "make [two groups] equal;" 4-year-olds, on the other hand, did count in all of these instances.[80] Karen C. Fuson's observation that children's counting ability involves gradual improvement, rather than "sudden insights" followed by flawless counting,[81] is consistent with Sophian's view that counting is an acquired skill, not an innate ability. Likewise, Nunes and Bryant, who generally consider Gelman and Gallistel's counting principles very "useful," nonetheless comment that Gelman et al.'s model of "principle before skills" implies that children already know the 'how-to-count' rules and need not learn them. But there is no evidence to support that assumption.[82]

Not less problematic is Gelman and Gallistel's claim that children generate their own 'tags' for counting. These self-generated tags are an important component in their innate-counting-scheme theory, and for Gelman et al., they constitute evidence for a tagging impulse. Empirically, however, this proposition has been decisively refuted. For example, Wagner and Walters write, "In 5 years of transcripts and hundreds of counting instances, we found no evidence whatsoever for the strong form of the stable-order principle assuming a nonstandard string."[83] Similarly, in her extensive 1988 work, which details voluminous research data including her own, Fuson commented, "In all the counting done by the several hundred children aged 2 ½ to 6 reported in [Fuson's] book, we almost never had children use anything except number words to count."[84]

Another postulation of Gelman and Gallistel that does not stand up to scrutiny is that counting is merely "ticking off objects" for which any "tags" will do as long as they are arbitrary and their application follows a fixed order—a premise that suggests that the meaning of the counting words is immaterial to the counting process. While it is true that during the actual procedure of counting the number

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80 Ibid., p. 39

81 Fuson, 1988, p. 193

82 Nunes and Bryant, 1996, p. 23-5

83 Wagner and Walters, 1982, p.151

84 Fuson, 1988, p. 389

words may be used mechanically just as sticks in the primitive one-to-one procedure, potentially each of these words may define the numerical value of the entire group counted. For this reason an informed counting requires that the counting words have numerical meanings. By claiming that the meanings of the number words are immaterial to the counting process, Gelman and Gallistel omit from their analysis the conceptual aspect of counting. Stripped of its conceptual component, the complex process of counting is reduced into a meaningless modus operandi. Furthermore, there is no empirical evidence for the proposal that counting words convey no meanings. Although there is no intrinsic connection between a number word and the concept it represents (as it is of course true of all words), there is considerable evidence that children use counting words selectively, not arbitrarily. Fuson, for example, observed that children differentiate number words from non-number words very early, and use only number words when reciting the counting-words sequence. Even when they make recitation mistakes, children persist in using only number words.[85] According to Wynn 2-½ year olds almost always respond with a counting word to the question, "how many?"[86] This indicates that they understand that counting words constitute an answer to that question. As it were, Gelman's own experiments generated strong evidence that children's choices of counting words are in fact discriminatory. These studies showed that children, as young as two years old, preferred to use number words in their counting and rarely used idiosyncratic words; by age three they use exclusively conventional number words in their commonly used sequence.[87] Moreover, even when the three-year-old children's accuracy in identifying the number of items faltered when group sizes were greater than three or five items, their estimations were not "undifferentiated *beaucoup* [many]."[88] In their 1975 experiment, Gelman and Tucker demonstrated that children as young as three years old tended to represent larger sets of objects with number words that come later in the number word sequence. Only a quarter of the participants displayed Descardeu's pattern ("one, two, three, beaucoup"),[89] suggesting that even when the groups' sizes exceeded the size of numbers with which children were familiar, they nonetheless understood that words that come later in the number words' sequence signify larger groups. In other words, children were aware that the escalating order of the

85 Ibid., p. 58

86 Wynn, 1992, p. 249

87 Gelman and Gallistel, 1978, p. 90, 131

88 A reference for Descoeudre's "un, deux, trois, beaucoup" (French for 'one, two, three, many'), coined by Descoeudre to depict the discrepancy between two-and four-year-olds' abilities to grasp small numbers and their ability to grasp large numbers (Descoeudre, 1921, cited in Bryant, 1974, p.119).

89 Gelman and Gallistel, 1978, p. 58-9

counting words corresponds to the escalating cardinal value of the sums to which these words were applied.[90] This finding indicates that far from being arbitrary tags, each number word, including those children do not fully understand yet, conveys a discriminative idea of size. It seems, then, that the meanings of the words children use in counting is important to this process even when their exact connotations have not yet been acquired.

However, the characterization of counting as a process devoid of numerical meaning complements Gelman and Gallistel's belief that the sole purpose of counting is to assemble "representations of reality" upon which a second set of rules, 'the numerical reasoning principles,' operate.[91] These 'reasoning principles,' they hypothesized, provide the basis for generating genuine numerical concepts.

Perhaps by describing number words as *tags*, which have no relevance to the principles of counting or numerical concepts, and proposing an additional set of principles—number reasoning—, Gelman and Gallistel intended to distinguish between the process of counting and the idea of numbers. True, numbers are complete and self-contained conceptual entities; they are independent of the ways in which they are applied, whether it is perception, calculation, or counting. This truth, however, does not mean that the rules of counting are independent of the concepts of number. On the contrary, insofar as the purpose of counting is to determine the numerical value of aggregates, numbers and counting share the same conceptual plan (or, if you will, scheme), namely, to answer the question "how many" and nothing else; hence, the logic of the counting rules and the ideas of number are inseparable. And because all numbers, regardless of their size, answer the question "how many" and nothing else, every discrete concept of number encompasses both the properties of a generalized theoretical number as well as the counting and reasoning principles that Gelman and Gallistel consider necessary for genuine number.

Indeed, scrutiny of Gelman and Gallistel's counting principles makes it apparent that they are derivative of the very concepts of numbers. Consider the following: If each number is a discrete concept of an exact amount of units, then verbal counting requires that one and only one word be assigned to each object counted so that the number of words will exactly match the number of objects in the counted set, thus fulfilling the *one-to-one principle*. Since the purpose of the counting process is to establish *how many* units are in a set, the tacit presumption that drives the whole process is that the word that concludes that process will answer that question; it will convey the number of the things counted, thus fulfilling the *cardinal principle*. But, if the last word is to indicate

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90 Ibid., p. 62

91 Ibid., p. 161

the total sum reliably, counting words have to be applied in such a way that each word will convey an idea of number that is larger by one and only one unit than the number conveyed by its preceding word, forcing a fixed or *stable order* of number words, starting with the word 'one.' Since counting, like number concepts, answers the question 'how many' and nothing else, anything that is not related to this question is immaterial to the results of counting. It follows that things such as the particularities of the objects counted, the area they cover, or the order in which they are counted may be ignored, hence the *abstraction principle* and the *order irrelevance principle*. Here you have it, all the five counting principles. An analysis of the 'reasoning principles' would result in a similar conclusion. Indeed, Fuson commented that Gelman and Gallistel's reasoning principles (relations, operations, and reversibility) are in fact cognitive elements of numerical concepts, and questions the use of the term *principles* for depicting what is essentially a range of concepts and their interrelationships.[92]

Since the range of numbers with which Gelman's subjects dealt were mostly within the lower end of the subitation range, they were also within the children's capacity to imagine and to conceptualize. It is unlikely that these toddlers were using the complex, highly generalized and abstract counting and reasoning principles, as Gelman and Gallistel proposed, when the application of already existing numerical concepts sufficed.

It is far more reasonable to assume that the numerical expertise that the children exhibited in Gelman's studies indicates that they possessed a modest collection of specific abstract number concepts, which they applied in the counting tasks. That conclusion is consistent with Fuson's opinion that conceptual, procedural, and "utilizational" abilities are more helpful terms than principles.[93] Children's reliable adherence to counting and reasoning principles whenever they enumerate quantities within the 3 or 4 range demonstrates, then, that they grasp the essence of what numbers are truly about. That children's application of these principles is deficient when they attempt to enumerate larger quantities, for which they do not yet possess matching concepts, does not invalidate their existing achievements; instead, it confirms that children's ability to adhere to the various sets of principles depends upon established numerical concepts.

Finally, Gelman and Gallistel's theory does not consider the contribution of the cultural milieu in which children are growing as a factor in their number-concept acquisition. In their view children's counting behavior and use of conventional number-words in their fixed (conventional) order has nothing to do with social/cultural inputs. On the contrary, they maintain that children's counting behavior is triggered by an innate scheme, which is practiced for its own sake. They further

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[92] Fuson, 1988, p. 398

[93] Ibid., p. 402

separate children's counting from their cultural milieu by asserting that children generate their own list of counting words. However, Gelman and Gallistel's theory of children's development of number concepts is not an exception in this regard. Many psychologists habitually omit from their discussion about preschooler's acquisition of number the contributing factors of the children's social, cultural, and linguistic environment. Contemplating this trend, Durkin et al. comment that in the studies of children's number cognition the emphasis on a child's own activities promotes "a metaphor of the child as a detached mini-scientist, to the neglect of the child as a social participant, in contexts where numbers are used and counting is developed in interactions with others." Durkin et al. further note that while many studies focused on preschoolers' production of the number-words' sequence, "These investigations leave unexamined the very early use of number words and they have not been concerned with the interpersonal context in which numbers are first encountered and used."[94]

The next section revisits the topic of children's acquisition of number concepts in an attempt to examine it from all the necessary angles. In addition to children's cognitive development and their enumeration activities in various laboratory experiments, it investigates number acquisition in light of numbers' conceptual properties, the functions of number symbolic representation, and the children's social milieu.

[94] Durkin et al., 1986, p. 270

VIII

CHILDREN'S NUMBER-CONCEPT ACQUISITION REVISITED

VIII-1. SYMBOLS FIRST

The narrative of children's number-concept acquisition overwhelmingly centers on children's relations to concrete aggregates by means of subitizing, counting, and comparison activities. And although the list of number words children know and the way in which children use this list has been thoroughly studied, little thought has been given to how number words in-and-of themselves contribute to the acquisition of numerical concepts. Instead, number-words are typically viewed as a tool for tagging objects in counting, and as labels for subitized aggregates or locations in the counting sequence. This chapter explores the possibility that the numerical symbols in and of themselves—be they verbal, notational, or signed—furnish the launching pad for the development of number concepts and the base upon which they are constructed.

Let us begin with the fact that numerical ideas are purely abstract and as such are devoid of recognizable physical attributes, whereas the spoken numerical symbols and their written or signed counterparts do have recognizable physical attributes; hence, they can be perceived by sensory means. Of course numerical ideas can be represented by objects, which, like the spoken, written, and signed symbols can be accessed by sensory means. Two oranges, two apples, or two giraffes, for instance, may represent the concept 'two.' This fact gives rise to the assumption that children are introduced to number ideas through displays of specific amounts of objects. But the idea of any specific number, such as the idea 'two' in our example, is independent of the objects that may represent it; what is physical and accessible to sensory perceptions in the arrays of two apples, two oranges, and two giraffes (i.e., their shape, color, smell, etc.) are not germane to the concept 'two.' These objects may represent a number only insofar as the observer already possesses a corresponding numerical concept (See chapters III-1, and III-2). Although number symbols, a.k.a., *numerals*, are also arbitrary and not intrinsic to the numerical concepts that they represent, each symbol is uniquely assigned to a specific number and, hence, identifies its associated number as a unique entity. For instance, the word 'three' sounds different from the word 'two' and each is exclusively associated with a specific number. Similarly the signs '2' and '3,' which represent two discrete numbers, are discretely recognized on the basis of their unique visual patterns. In comparison, apples and giraffes are not exclusively associated with any particular number. It seems, then, that only the spoken, written, or signed numerical symbols can create the signifying physical attributes by which a specific number may be identified and discriminated from other numbers on the basis of sensory information alone. This conclusion implies that infants become aware of the numerical symbols prior to the acquisition of the numerical concepts that these symbols represent.

The prospect that the development of number concepts is triggered by symbols does not rest solely on the logical argument presented above; it is also supported by the simple fact that humans are predisposed to symbolic thinking, and particularly by humans' propensity for perceiving and producing speech. Studies reveal that the spoken words are phenomena in which human infants are intensely interested and to which they pay special attention. From birth, infants display a clear preference for human sounds over all other sounds. A four-week-old baby stops crying when his mother talks to him and remains quiet and alert as long as she continues talking.[1] By four weeks, he can already produce his own non-crying sounds, such as grunts, cries, gasps, and so on.[2] At four months, the baby actively searches for the speech source and, when spoken to, reacts by kicking in excitement or freezing in attention. In addition, he shows his caregiver that he enjoys her verbal outputs by smiling and answering with his own mixture of conversational sounds. At four and five months, a baby's babble already consists of syllables that resemble words.[3]

At this early stage of language acquisition, babies are able to identify any phoneme used in any language, not only those that are specific to their linguistic environment.[4] But gradually they display increasing preference for the phonemes used in their linguistic milieu and start to lose their ability to discriminate foreign phonemes. This "loss" is beneficial, however, as it allows the babies to develop their ability to discriminate the phonemes specific to their native language with an astonishing accuracy and to do so in a short time. Ten-month-old babies who grow up in an English-speaking culture are able to distinguish between the sounds L and R, and even improve that ability while their Japanese counterparts, growing up in a culture in which these sounds (L and R) are indistinguishable, no longer can.[5] Perhaps the most enchanting aspect of babies' language development is that

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1 Leach, 1983, p. 190

2 Menyuk, 1971, p. 3

3 Leach, 1983, p. 61-2

4 Gopnik, Meltzoff, and Kuhl, 2001, p. 105-6

5 Ibid., p. 107

this great craving for speech patterns in its perception and its production occurs at a time when they have no practical use for speech, for at this period in their development babies cannot communicate their needs through speech, relying instead on crying and gesturing.

That babies are interested in and are able to perceive and emulate the phonemes of their mother tongue before they can apply them in meaningful speech implies that the acquisition of the words' sound patterns and the acquisition of the words' conceptual contents are two distinct cognitive processes, and one is not necessarily dependent on the other. It follows that association between a word's sound pattern and its conceptual content could be a process that occurs either by attaching a newly acquired conventional verbal pattern to an already established concept or by figuring out the conceptual content or meaning of an already acquired conventional verbal pattern or a sequence thereof ('one, two, three' for instance). Indeed, empirical evidence shows that at least in its early stages, language acquisition progresses in two directions, that is, from verbal pattern to concept formation, and vice versa. It has been observed that one-year-old babies are able to associate some conventional words with the particular objects or events they encode and to communicate with these words in an appropriate way.[6] At the same time, some of the words babies produce seem to be entirely original in that they are not even approximations of familiar words, yet babies use them consistently in the presence of particular objects.[7] Since numbers do not have physical manifestations other than the symbols that represent them and since it is improbable that infants form numerical concepts spontaneously without any perceptual input, the development of number concepts must begin with the acquisition of symbols followed by the configuration of their conceptual contents. In other words, symbolic recognition is the fulcrum for the development of number concepts and the starting point from which it commences.

Reflecting upon the many studies described in the previous chapters, it becomes quite obvious that the numbers children accurately identify via perception, count correctly, and 'conserve' are fewer than the number words they can recall or recite. Moreover, there are indications that children value their ability to recite number words even when they do not fully understand their meaning. For example, Wagner and Walters noticed that children tend to repeat the counting of small sets or tag objects multiple times until they run out of all the number words they know. They coined the term, "list exhaustion scheme," to depict both tendencies.[8] Fuson's observation that children's counting errors are sometimes caused by their temptation to count objects more than once in order to complete number-words

6 Leach, 1983, p. 274

7 Menyuk, 1971, p. 168

8 Wagner and Walters, 1982, p. 143-4

recitation to ten, point to the same phenomenon.[9] The observable fact of "list exhaustion" indicates unequivocally that the inventory of conventional number words children can recite is larger than that of the numbers of which they are cognizant, and that children are interested in the number words and their sequence for their own sake. Both are compelling evidence that the pattern of the spoken number words is acquired prior to the numerical concepts they embody. The same is true with regard to number-words' place in their conventional counting sequence.

Learning the symbolic representation of numbers alone does not constitute knowledge of numbers, as the child must still figure out their numerical meaning. Yet, the introductory and fundamental role numerical symbols have in the development of number concepts permanently ties these concepts to their formal symbolic representation, making the symbols of numbers an integral element of numerical concepts. Numerical symbols are, then, inseparable from the concepts they represent from the outset, and are not a later acquisition as commonly believed.

Finally, considering that children are born into a numerate culture that is saturated with numerical symbols, it is quite probable that their great interest in number words and the counting activity associated with them stems from social/cultural-adaptation motives. The development of number concepts, then, should be understood as the child's attempts to explore and comprehend her cultural environment through playful emulation of verbal utterances and their related behaviors. As such, number-concepts acquisition should be viewed as a process of social/cultural adaptation rather than exploration of the physical environment. Indeed, from the children's perspective, the need to understand the meaning of symbols, which seem to be of great importance for the adults in their lives, is unquestionably much more pressing than the need to quantify magnitudes in an exact and objective way. Only the necessity to adapt to social environment can explain why children as young as two and three years old seem to be so interested in a topic that has no utilitarian value for them.

VIII-2. PARENTAL INPUTS INTO CHILDREN'S-NUMBER DEVELOPMENT

The path of number development that was delineated in the previous chapter suggests that number education begins almost at birth. However, most experimental studies of children's number acquisition involve children between 2 ½ and 6 years old—the ages at which they are mature enough to understand instructions and

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[9] Fuson, 1988, p. 170

be cooperative participants in such research. Moreover, this research tends to concentrate on children's relations to concrete aggregates. Consequently, it misses both the earliest period in which children cannot yet verbally express and act upon their knowledge, as well as the social and linguistic context that characterizes the introductory phase of children's number experiences.

Fortunately a group of five scholars, Kevin Durkin, Beatrice Shire, Roland Riem, Robert S. Crowther and D. R. Rutter, who were interested in the interpersonal and linguistic circumstances in which children first experience numbers, studied the use of number words and counting by 9- to 36-month-old children and their mothers. Durkin et al.'s was a longitudinal study consisting of 15-minute observational sessions involving 10 infants (3 females and 7 males) and their mothers. Since the mothers were told that the objective of the experiment was to study the development of verbal communication in children, all references to number during this study were spontaneous. Before the children's second birthday the intervals between sessions were one-month long, and thereafter three months long. The small studio room in which the study took place was minimally equipped. Besides four cameras, recording equipment, light fixtures, and curtains, it contained a plant, a coffee table placed on a rug, and a wastebasket.

The results of the study indicate that besides reciting the number-word sequence and quantification activities, number words were used in a variety of ways. For instance, throughout the experiment there was considerable spontaneous, or "incidental" use of number words such as "Two sugar please." They listed also the playful sequences such as: 'one, two, three, go,' 'ready, one, two,' or 'one step, two step and a tickly under there,'" which they called, "sequential complements," and to a lesser degree, nursery rhymes, stories, and songs. The latter two kinds of inputs declined noticeably during the children's third year.[10] Not surprisingly, children's use of number words increased with time, whereas the mothers' use remained about the same.[11] The most often used number words were the first four numbers.[12] In fact, words for larger numbers were rare even after the child's second birthday.[13] As could be expected, there was no evidence of children's independent recitation of the number-words sequence before 21 months, but once they started to recite the number words on their own, the frequency of independent recitation increased steadily. One of the most significant findings in Durkin et al.'s study was the mothers' unmistakable instructional nature in their use of number words and counting. Starting when the children were about 15 months old, mothers seemed to be engaged in an active pedagogical endeavor; up

[10] Durkin et al., 1986, p. 277

[11] Ibid., p. 272, 280

[12] Ibid., p. 279

[13] Ibid., p. 275

to sixty per cent of the total number references used by mothers in this period can be characterized as instructional or pedagogical discourse, because they require the child's attention, participation, or imitation.[14] Three different pedagogical discourses were observed:

(1) The *recitation of number words* in their counting order. Recitations of this kind were often prompted by the mother with a specific reference to counting and involved actions such as pointing to, or touching objects, for example:

"M: Count them. Look, one, two, three, four."

(2) The *repetition and clarification of cardinality*, for example:

"M: Look. *Four*. 'One, two, three, four.'
M: Let's count the cameras. 'One, two, three, four. *Four* cameras.'"

(3) The *alternating strings* or "joint 'dialogic' number string" with a clear teaching intention, for example:

"M: Count with me. One . . .
C: One
M: Two
C: Two." (Ibid.)

Of these three kinds of number-word experiences, it was during the "alternating strings" dialogue that the18-month- to 3-year-old children produced most of their number words.[15]

Not less significant in Durkin et al.'s study was their observation of an abundance of linguistic ambiguities in the parental inputs. In the sentence 'take that one,' for instance, the word *one* is used as a pronoun. The use of 'one' as pronoun was so prevalent that, in the table in which they listed the mean frequencies of number-word usage, two separate columns were devoted to 'one,' the first for 'one' as a pronoun and the second for 'one' as a number word. Other sources of linguistic ambiguities were homophones of number words. For instance the preposition 'to' and the adverb 'too,' as in "give this one *to* Dan," and 'that one *too*,' could be confused with *two*. Similarly, the preposition *for* as in 'one *for* you

[14] Ibid., p. 277

[15] Ibid., p. 279

and one *for* mom' can be confused with the number word, *four*.[16] The playful use of number words in the *sequential-complements* routine, such as 'one, two, three, go,' mixes the number-words sequence with words that have nothing to do with numbers or enumeration activity, thereby obscuring the function of the number words that are used in these sequences. The imitative *alternating strings* routine in which the child is asked to repeat the number word used by her mother (e.g., "M: 'one;' C: 'one,' M: 'two,' C: 'two'") is equally confusing, for it distorts the proper 'one-two-three-' counting sequence.[17]

But Durkin et al. did not consider these contradictions as hindrances to number acquisition. On the contrary, they maintained that the semantic ambiguities inherent in parental inputs actually contribute to children's number development as much as the overall pedagogical nature of this input and the information it provides.[18] They argued that when the available information is inconsistent and confusing, the child must acquire competence either through an alternative source of information or because of the cognitive gain that was achieved as a result of the strategies she developed to resolve these inconsistencies. Only when children "become more active contributors to number-oriented dialogues," can we learn how children solve these contradictions.[19]

The way children deal with the ambiguities involving numbers and number-word usage, however, is related to their cognitive development in general as much as to the social/linguistic context in which they are first introduced to numbers. In this respect, the period between their first and third year of age is of a particular interest. During this period children's mastery of their mobility, dexterity, bodily functions, and speech articulation significantly increases. As these capabilities develop children's sense of autonomy and self-reliance grows, and not always in a way that makes their parents happy.[20] This spirit of independence is also reflected in 2-year-old children's speech. Mussen, Conger, and Kagan, for example, comment that at 30 months a child's words and grammar are seldom a faithful repetition of adult speech.[21] It has long been observed that young children use words in a peculiar way, which, though erroneous, indicates that they are making intelligent hypotheses about grammar. Alison Gopnik, one of the coauthors of *The Scientist in the Crib*, provided a charming example of this prevalent phenomenon by citing her sister's description of their rather large family: "All of we's is childs."[22] Gopnik's

16 Ibid., p. 271

17 Ibid., p. 283

18 Ibid., p. 270, 286

19 Ibid., p. 284

20 Mussen, et al., 1969, p. 261

21 Ibid., p. 191

22 Gopnik, Meltzoff, and Kuhl, 2001, p. 119

sister's speech suggests that she generalized the underlying grammatical rules of English and applied them according to her own independent thinking.

A similar growth in linguistic independence could be seen in Durkin et al.'s data that show that the frequency of children's use of the words 'one' and 'two' stands in a sharp contrast to that of their mothers.' Whereas the mean frequencies for the mothers' use of the word 'one' exceeded that of all other number words throughout the study, the children's use of the word 'one—'both as a number word and as a pronoun—lagged behind their use of the word 'two' until they were 24 months. But when the use of 'one' as number word alone was considered, children's use of 'one' continued to be less than that of 'two' until they were 33 months old. At 15 months, which was the earliest age at which children produced number-words, the word 'one' was completely absent in their vocabulary compared to a mean frequency of 0.89 in using the word 'two.' In contrast, their mothers' mean frequency of using the word 'one' was more than twice than that of the word 'two' (9.86 for both kinds of 'one' compared to only 4.00 for 'two'). By 18 months the gap between mothers' and children's use of 'one' and 'two' was even greater: While the mothers' use of 'one' was more than four times the frequency of their use of 'two' (11.44 for both kinds of 'one' against a meager 2.55 for 'two), the children's combined use of 'one' was one third of their use of the word 'two' (at 0.33 for both kinds of 'one' against 1.00 for 'two').

The fact that children's outputs do not match parental inputs suggests that in number-word acquisition, as in language acquisition in general, children develop their own ideas about words and become active contributors to the process of learning. Likewise children's interpretation and usage of words might be in conflict with that of their parents.' The excerpt below, which was taken from Durkin et al.'s study, is an example of such friction arising from the child's resistance to using the word 'one' in counting:

M (to Ben, 24 months): Tell Mummy, how many eyes has Mummy got? How many eyes?
C: Mm . . . er. (C walking towards M, looks at her face)
M: One, (C crawls under table)
C: Doo, fwee. (C remains under table)
M: Two, three? Oh, poor Mum! [23]

This recorded exchange hints that the child is unwilling to use the word 'one;' but knowing that this is a word his mother wants him to include in his counting, he crawls under the table where he probably feels safer to hold on to his own way of counting, namely, "Doo, fwee [two, three]" instead of "one, two."

[23] Durkin et al., 1986, p. 284

The phenomenon of delayed use of the word 'one' in a toddler population was observed four years prior to Durkin et al.'s study by Wagner and Walters in their 1982 longitudinal study of early number concept. In that research, Wagener and Walters discovered an intriguing feature in the 1- to 2-year old children's use of the first number words, that is, that they included *two* in their vocabulary before *one*.[24] Besides indicating that children feel sufficiently autonomous to use their preferable words, children's hesitation to use the word 'one' must also signify what they understand about number words. The notion that the delayed uses of the word 'one' is linked to the way children conceive the concepts of number did not escape Wagner and Walters. They speculated that children deem a singular item first as an object rather than a number. Possibly, *one object* "is first of all 'the object.'" Only after a child understands what "two" of the same things is, can she understand what "one" thing is. (Ibid.)

By measuring the frequencies of number-word use, Durkin et al. explicated Wagener and Walters's observation of the delayed use of the word 'one,' yet they did not offer a hypothesis of their own to explain this delay. Still, their theory of learning through resolving contradictory inputs may point to an additional explanation for that phenomenon. To understand why, we have to take into consideration what children of that age already know about numbers. As some of the previously cited research demonstrates, children uniformly responded to the question, 'how many' either by counting or with a number word (see VII-5). These responses indicate that children associate number words with the idea of 'many,' that is, pluralities, and with the counting thereof. 'One' does not fit into this framework, for it is not only incongruent with the ideas of plurality or 'many,' but also, when there is only one object there is no need to count. Hence, the conjunction of 'one' with the question, 'how *many*,' or the activity of counting could be confusing for toddlers. Greek philosophers, who undoubtedly were numerically savvy (and most probably employed the word 'one' in their counting), did not regard 'one' as a number either. 'One' was inconsistent with their definition of number as "a multitude composed of units."[25] It is possible that children feel as the Greek philosophers did that there is a logical contradiction between the idea of *number*, which connotes multitude and the idea of *one*, which connotes singleness. Some children, like 24-month-old Ben in our example, resolve this contradiction by simply ignoring 'one,' and starting to count from 'two.' In an indirect way, the delayed use of the word 'one' reaffirms that children as young as two understand the basic intent of number words.

Though confused by ambiguities and often erroneous, children's independent contribution to the usage of number words is an important development for

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24 Wagner and Walters, 1982, p. 147

25 Euclid's Elements Book VII

it points to their spontaneous interest in numbers without which the mentally challenging acquisition of more and larger numerical concepts cannot proceed. This genuine interest in numbers is also documented statistically in Durkin et al.'s study. Their data showed that by age 3 children's independent recitation of number strings constituted close to half (46.24%) of their number-word usage, even though their mother's recitations dropped to zero.[26] Indeed, it was this spontaneous counting that caught Gelman's attention and made her wonder about the role of counting in children's development of numerical concepts[27]—to be discussed in the next chapter.

But children's interest by no means implies a propensity for numbers. As Durkin et al.'s study shows, parental inputs with their distinctive pedagogical nature start early on: By 9 months babies are already exposed to the 'sequential complements,' by 12 months nursery rhymes, stories, and songs are added to this exposure, and by 15 months babies are already exposed to unmistakable pedagogical inputs such as the 'repetition and clarification of cardinality,' 'number string recitation,' and 'alternating strings.' That parents initiate number familiarity so early in their children's lives, suggests that it takes considerable effort and time on the part of parents to awake their children's interest in numbers. However independent and self-directed children's interest in numbers eventually becomes, it is still the invaluable linguistic and social inputs and the active coaching by parents or other social agents combined with the children's own motivation to adapt to their social and cultural milieu that sets in motion the whole process of number-concept acquisition.

VIII-3. THE ROLE OF COUNTING IN THE ACQUISITION OF PRIMAL NUMERICAL CONCEPTS

It has long been held that children learn numbers by means of counting, and psychological studies have provided scientific weight to this intuitive belief. However, the ways in which counting helps children to understand numbers are still not entirely clear. This chapter discusses the studies and theories that deal with the relationship between counting and the acquisition of numerical concepts.

The earliest controlled study to establish a connection between a child's perception of number and counting was Beckmann's 1924 study of the development of number competence in 2-6-year-old children. In this study, Beckmann observed that children's ability to accurately estimate the numerical value of a group depended on whether or not they could count up to that number. Psychologist Peter

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26 Durkin et al., 1986, p. 278

27 Gelman and Gallistel, 1978, p.68

Bryant regards Beckman's findings as persuasive evidence that the recognition of absolute numerical values of small groups cannot rely on perception alone; "counting precedes subitizing [and] not the other way around," he argued.[28] Wynn's 1992 research demonstrating that children have the ability to correctly count groups of objects long before they are able to correctly retrieve a requested number of objects[29] is consistent with Beckmann's and Bryant's conclusion that counting precedes number recognition. Gelman's observation (in her studies that involved a comparison between two aggregates) that children started to count without ever being asked to do so reinforces this conclusion. Likewise Kaufman et al.'s 1949 study of number discrimination, which demonstrated an increase in subjects' response time as the number of dots presented to them increased, implies that what its adult participants may have experienced as capturing the exact number of objects in small groups at a glance is, in fact, a process of very rapid counting (see detail in III-3). Indications of a covert counting process in subitation are even more striking in a child population. Gelman and Gallistel pointed out that adults require about 46 milliseconds more to identify two items than to identify one item. But five-year-old kindergarteners require 120 milliseconds, approximately 2.6 times longer than adults. And while the difference in response time between estimation of two items and three items remains 46 milliseconds for adults, it jumps all the way to 280 milliseconds for children, that is, six times more than that of adults.[30]

Still, the idea that counting gives rise to number concepts contains, in its simplistic interpretation, a paradox, and empirical evidence of such exists. The logical contradiction is plain: If counting is to inform the numerical value of a given group, each counting word must convey a numerical meaning, for potentially it may be the one that defines the requested numerical value. Without their attached numerical signification, the words used in counting are as helpful for forming a meaningful idea about the total number of objects in a collection as are the sticks used by the Wedda tribesmen in the primitive one-to-one procedure described in chapter I-3. "We cannot be said to be discovering the number of objects counted unless we attach some meaning to the words one, two, three," warned Bertrand Russell.[31] That truth implies that the prerequisite for effective counting is the presence of already existing numerical concepts. It is illogical to assume that the counting procedure may originate the very concepts without which its results are devoid of meaning. Questions regarding the role of counting in the acquisition of number concepts also emerged from Piaget's studies (described in VII-2), which

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28 Bryant, 1974, p. 120

29 Wynn, 1992, p. 234

30 Gelman and Gallistel, 1978, p. 64-71, 222

31 Russell, 1952, p.193

showed that counting had no effect on the child's ability to 'conserve' (the ability to base estimation of a group's size on purely numerical criteria regardless of its spatial distribution). For instance, when a group of objects that had just been counted was spread out, children, nonetheless, judged that same group to be more numerous. Piaget concluded that even when a child can count he still prefers to quantify by relying on 'global' impressions of the groups.[32]

That said, it is important to recognize that the counting that aims to establish the numerical value of a group is different in kind from the counting that is interpreted as the child's way to grasp her social environment. The former, as pointed out by Russell, is meaningless without pre-existing numerical concepts, while the latter is an activity that may serve as a tool to instill meanings into the child's observation of counting behavior and the words used therein. What, then, does the number-naïve child learn by counting, and how?

Wynn simply maintains that the children learn about the cardinal meaning of number words by their position in the counting sequence.[33] If this claim were true, English readers would associate each letter of the alphabet with a specific numerical content as a matter of fact. After all, they have recited these letters in their fixed order throughout their lives since early childhood. Apparently the ordinal position of these letters in their standard sequence does not inform numerical values. Why, then, should the order of number words be more informative? Much of other scholars' explanations regarding the connection between the counting procedure and the formation of numerical concepts are influenced by Piaget's dictum that numbers are "disassociably cardinal and ordinal."[34] This thesis implies that each number concept encodes simultaneously the specific quantity of units it encompasses as well as its ordinal position in the counting sequence. Accordingly, the processes of number-concept formation involve the synthesis of numbers' cardinal and ordinal values, or in Piaget's words, the "intermingling of cardinal and ordinal processes."[35] In a similar vein, Fuson called the ability to connect between the cardinal meaning of the number words and the counting process "count to cardinal transition."[36] As does Piaget, she believes that the desirable level of understanding is that in which it is "difficult to disentangle" the cardinal meaning of the number words from their counting sequence—they become a "cardinalized number-word sequence."[37] Wagner and Walters echoed

[32] Piaget, 1965, p. 28, 29

[33] Wynn, 1992, p. 228

[34] Piaget, 1965, p. VIII

[35] Ibid., p. 154

[36] Fuson, 1988, p. 250, 363

[37] Ibid., p. 363

this idea with their belief that numerical concepts are inherently a synthesis of their ordinal and cardinal values.[38]

The problem with these assumptions is that a number's cardinal value is an absolute property that does not change, whereas a number's ordinal value is a relative property that changes in relation the particular sequence in which they were ordered. For example, 'three,' which is the cardinal value of the *third* number in the counting sequence, is the *second* number in both the odd number and the prime number sequences, and is the *first* number in the three-multiples number sequence. Only in the counting sequence does a number's ordinal value match its cardinal value. In order to fulfill the purpose of counting, 'three' is ordered after 'two' and before 'four' because its sum is exactly one unit larger than the former, and one unit smaller than the latter. As is the case with the counting sequence, in all the other sequences the ordinal values of 'three' are determined by virtue of its cardinal value. Since the ordinal value of a number is a product of its cardinal value, the relationship between them is that of a cause and effect, or a fact and result. For the same reasons that effects cannot generate or transform their causes, the order in which number words are used in counting cannot instill in these words cardinal values or transform them in any way or fashion. Owing to the dependence of the ordinal value of a number upon its cardinal value, it becomes clear that the evolution of numerical meaning is not simply the result of an interactive process between the cardinal and ordinal aspects of numbers. The relation of counting to the acquisition of numerical concepts cannot be found within the framework of the counting procedure alone; it must be based on an additional process that is independent of counting, as Russell had advised a long time ago: "[Counting] has no meaning unless the numbers reached in counting have some significance *independent* of the process by which they are reached."[39]

To better understand how counting helps pre-number concept children to establish numerical concepts, let us first reestablish that each number is a discrete concept of an explicit and definite amount of units. This property implies that the formation of any given numerical concept involves the identification of constituent *units* as well as the identification of their *sum* as a discretely recognized entity. Since units must be recognized individually before they are viewed together as a specific sum, the formation of a specific concept of number begins with the analysis and acknowledgment of the units. This initial analysis and recognition of units is well served by establishing a visual contact with the objects counted while pointing to them in a temporal/sequential manner. In addition to acknowledging units, pointing also helps to control and regulate the separation of units already counted from those yet to be counted. But however carefully and accurately it is

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38 Wagner and Walters, 1982, p. 144

39 Russell, 1952, p.192 (emphasis mine)

executed, unit recognition alone will not result in the formation of numerical ideas. It must be followed by the visualization of these units as a uniquely recognized sum. For this purpose, the units, which have been identified through the process of counting, must be retained in memory long enough so that they can be viewed or imagined simultaneously. By assigning number words (or other symbols) to units, especially when the units are visually acknowledged and words are actually articulated, the child helps herself to register and commit these units to active memory.[40]

Because imagery draws on the same integrative and visual processes on which actual perception relies,[41] number imagery (as imagery in general) demands the detachment of the cognitive system from the individual's external stimuli.[42] As Neisser explained, "Visualizing one thing and looking at another is just as difficult as looking at two things simultaneously."[43] Visualization, then, must be a process that is separated from the actual counting activity, which requires the acknowledgement of external inputs—the objects to be counted. As a process that is detached from the counting procedure per se, the visualization of sums introduces an additional element that, though originated by the counting procedure, is still autonomous and separated from it. In other words, visualization gives meaning to number words by a process that is *independent* of the counting procedure, fulfilling Russell's requirement for effective counting.

The a priori assumption that spatial/visual processes have a role in the conceptualization of numbers is consistent with recent neuroscientists' studies that with the aid of new imaging technologies have shown that the same regions in the brain that represent numerical values also represent perceptual or spatial magnitudes.[44] The familiar tendency of children in their early phases of number acquisition to denote numbers by means of showing fingers instead of by verbally naming them adds another intimation that spatial/visual processes are involved in the conceptualization of numbers. It is highly plausible that children prefer to indicate numbers in this manner because displaying fingers allows them to remain within the visual-spatial mode by which their initial and rudimentary numerical concepts have been created, whereas naming numbers forces them to give up this

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40 It worthwhile to mention in this regard the counting method used by native-signers of American Sign Language (ASL). Secada, 1984, cited by Fuson 1988, p. 116, The hearing impaired signers count either by pointing with one hand while signing with the other, or by using only one hand, pointing and then signing at each object counted.

41 Neisser, 1976, 128-134

42 Ibid., p. 85

43 Ibid., p. 146, Neisser's proposition is supported by several studies: Brooks (1967 and 1968), Segal and Fusella, (1970), Byrne (1974), and Salthouse (1974 and1975)

44 Sarama and Clement, 2009, p. 42

congruous modus operandi and shift to the verbal sequential mode with which number words are generated. What is more, the fingers provide children with a perceptible feedback for these abstract concepts, which at this point are in their earliest stages of formation. The behavior of finger showing is so prevalent among preschoolers that in an experiment conducted by Fuson and Hall children were asked to identify the number of stars on cards that were presented to them by showing the appropriate number of fingers.[45] To produce the particular number of fingers that were needed to represent a given number, the child had to direct her attention away from the objects she counted (stars in our example) to her fingers. The child, as it were, counted twice: first when she was counting the objects at hand, and second, when she was counting fingers intended for the presentation and verification of the freshly constructed numerical image. In this context the fingers serve as a perceptible metaphor for the abstract concept of unit, whereas the counted stars continue to be the objects that they are—stars. As an indicator of units the fingers should be viewed as an *ideogram*, or a symbolic representation of a number rather than a presentation of the particular objects counted. Thus, Fuson and Hall were correct to consider finger showing a suitable substitute for number words.

This analysis suggests the presence of symbolic thinking even at the earliest stages of number-concept acquisition. Of course, the involvement of the symbolic function is essential for the formation of mathematically viable numerical concepts. Consider this: the words children employ in counting various groups of objects constitute a standard list, and by assigning each object counted a word from this list (instead of tagging, pointing, or naming objects) the counted objects are no longer identified as what they are, but as abstract units. In other words, the use of the numeric symbols is instrumental in releasing the abstract idea of *units* from their particular phenomenological manifestations and transforms them into workable building blocks for the construction of genuine numerical concepts. Number imagery that is based on such abstraction of units creates numerical concepts that are not restricted to particular objects and hence forms universal rather than adjectival ideas of numbers (see VI-3).

Besides their role in disassociating units from the objects that represent them, symbols are also essential for the process of unit-analysis and number-imagery. Unlike the sticks that the Wedda tribesmen used in the one-to-one procedure, each number word has a unique sound pattern. Consequently assigning one of these verbal symbols to each object counted helps the child to recognize units as discrete entities and keep them as such. Moreover, preschool children, as shown in the studies mentioned earlier, are already familiar with the first counting words even if they do not yet know their precise numerical meanings. The mnemonic support that these rehearsed verbal symbols afford is invaluable for successful

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45 Fuson, 1988, p. 213

visualization of universal numbers, which requires the child to hold the identified units in working memory in order to view them together as unique sums. Because the symbolic function significantly increases the mind's ability to retain, retrieve, and develop concepts,[46] their association with the familiar symbols enhances these evolving numerical concepts.

The verbal counting performed by the pre-number-concept child is a cognitive operation that makes use of symbolic thinking upon which mathematical thoughts depend. Hence it is valuable not only for the future development of numerical concepts but also for the development of mathematical thinking in general. Not less important, the efforts children invest in the serial matching of objects with number words during counting is a deliberate and mindful behavior; in so doing the pre-number-concept counting couches the process of number conceptualization in the appropriate mental framework, that is, a conscious and rational thinking mode. Conscious attention is believed to play a key role in the acquisition of both cognitive and motor skills.[47]

VIII-4. THE LIMITATIONS OF COUNTING AS A TOOL OF ARITHMETIC EDUCATION

The understanding that counting is a process through which units are marked and mentally registered for subsequent visualization of their sum suggests that counting can be useful only for the conceptualization of numbers that are small enough to be imagined globally, in an explicit way, namely, numbers within the subitation range. As construction of larger numbers can no longer be based on explicit mental representation of sums, it must rely upon conceptual/symbolic thinking that uses previously acquired concepts of smaller numbers. For example, the number 7, which exceeds the subitation range, could be perceived as 4+3 or other combinations of smaller numbers; numbers beyond the base range, which require involvement of decimal units, create even more complex structures and higher levels of abstraction (see IV-2, and IV-3). As was argued earlier, children are introduced to the topic of number via the visual and spoken numerical symbols; consequently, even concepts of numbers that could be subitized are already tied to and are embodied in their symbolic representation. This enables children to structure numbers on a conceptual, symbolic level early on.

The test of a valid acquisition of a specific number concept is the child's ability to mentally take that number apart and put it together again. This task is an inherently contemplative activity that relies on abstract conceptual thinking.

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46 Edelman, 1989, p.92-3, 104, Deacon, 1997, p. 434

47 Edelman, 1989, p. 201

Counting is an activity in which the child must visually focus on the object counted. Since it relies on the same visual processes that guide imagery[48] and abstract conceptual thinking, counting interferes with the internal processes of conceptual/symbolic thinking on which the envisioning of numbers that exceed the subitation range depend.

Another issue to consider is that children learn about numerical relationships and their mathematical logic by means of mental manipulation of numbers. These arithmetic operations entail both the ability to treat numbers as a whole, as well as to decompose and recombine them according to the arithmetic task at hand. For instance, the addition, '7+5' calls for decomposing 5 into '3+2,' and then to add the 3 to the 7 as to complete it to the unit 10, thus: (7+3)+2= 10+2=12. Moreover, the further development of arithmetic skills and understanding necessitates increasing the speed of manipulating numbers mentally. The more numerical operations one is able to accommodate in one's working memory span, the more complex are the numerical relations one is able to form and to grasp. Since counting replaces the rapid and instantaneous mental processes of conceptual thinking with a temporal and time-consuming 'one-to-one' process, it prevents the child from fitting into her working memory the amount of operations and numerical concepts needed to form more complex numerical relationships. Experience shows that children who depend on counting for basic subtraction and addition operations become perplexed and lost when they are required to solve more advanced problems. Therefore, the adverse results of excessive use of counting in basic arithmetic education are felt only when children move on to study more advanced arithmetic. Unfortunately, at this point learning not only must address the subject at hand but must also undo distracting habits.

Besides the short-term benefits of using conceptual/symbolic functions in arithmetic calculation, there are also long-term benefits: In arithmetic, as in mathematics in general, simpler and more primitive concepts serve as the raw material and referent for the generation, abstraction, and development of more advanced concepts. Counting as a means to obtain answers to arithmetic problems bypasses the employment of existing conceptual networks in arithmetic thinking and thus precludes the formation of new and more advanced arithmetical concepts that can only be developed by utilizing the already established conceptual network.

Counting is pedagogically justified as long as it is used as an instrument to familiarize children with the numerical symbols and their sequence, the concept of unit, the principle of 'one-to-one correspondence,' and the recognition of numbers within the subitation range. Requiring or encouraging children to count after they have already outgrown the developmental phases in which counting is helpful and

48 Neisser, 1973, p. 209-210; Neisser, 1976, p.128, 146-7; Neisser, 1967, p. 153

relevant for number-concept learning is counter-productive and a waste of time. Most of all, it denies children the opportunity to engage in true and constructive mathematical thinking.

VIII-5. THE NATURE OF NUMBER-CONCEPT DEVELOPMENT

The number proficiency of preschool children is known to be restricted to the first three or four small numbers. This restriction, combined with preschoolers' tendency to use phenomenological criteria for ascertaining the numerical values of larger groups, is attributed by some psychologists to an absence of the cognitive maturity requisite for true number comprehension. Piaget proposed that a true understanding of the counting numbers must rely on a fusion of the logic of 'class' and the logic of 'seriation,' and it is only when they reach the 'operational stage' (at around seven years) that children can achieve that 'fusion.' Gelman and Gallistel claimed that children's true understanding of numbers commences only when "the child's reasoning moves from a dependence on specific representations to an algebraic stage in which representations of numerosity are no longer required."[49] According to these theories a genuine understanding of numbers demands either employment of two kinds of logical processes or advanced mathematical thinking.

But because each number is a concept of size that is defined by, and conceived as, a fixed sum of units, each and every number concept, however small, embodies the essence of the abstract idea of *number*. Consequently, mastery of even the smallest numbers satisfies the requisites for a true number comprehension. The fact that preschoolers' number proficiency is limited to small numbers does not make their acquisition mathematically invalid, immature, or less genuine.

Another issue that should be considered is that insofar as each number is envisioned as a distinct sum of units, each number is a unique concept. The amount of units each number encompasses determines the extent of sophistication and level of abstraction needed for its conceptualization. The level of erudition and abstraction required for conceptualization of numbers grows in tandem with their sizes. For example, the acquisition of numbers within the base does not require comprehension of the base system, not even the consideration of the number 10 as a standard unit. Within the base there is a further distinction between the subitizable numbers and the numbers that can no longer be subitized. The former can be envisioned more or less explicitly, while the latter, which are constructed from concepts of smaller numbers, must resort to conceptual/symbolic processes. Numbers beyond 10 require the ability to regard certain sums as standard units

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[49] Gelman and Gallistel, 1978, p. 245

known as *decimal units* (1, 10, 100, etc.). Each of these units requires its particular level of abstraction. Numbers may employ several different decimal units, each accumulating different sums, such that larger numbers may involve several levels of abstraction and base-number concepts as in the instance of the number 764, for which the child has to successfully deal with three levels of abstraction to figure the units, 1, 10, and 100, and three different sum values, to figure the base numbers, 7, 6, and 4. The view that number-concept acquisition hinges upon a cognitive epiphany whereby all numbers may be conceptualized once and for all ignores the different levels of abstraction and sophistication presented by the great variety of numerical sizes and complexities.

The systematic increase in sophistication and level of abstraction needed for the conceptualization of the counting numbers as they progress along their continuum raises another consideration, namely, children's cognitive readiness. The proposition that numerical-concepts acquisition is based upon a fixed set of principles or processes separates number acquisition from children's overall cognitive development, their knowledge, experience, and training. As it were, some of the limitations cited by scholars as evidence of a lack of a genuine understanding of numbers can be explained by the children's other developmental agendas. As previously argued, the visualization of numbers involves verbal and short-term memory tasks. Hence numerical competence must correlate with the level of skill at these tasks, which in two-to-four-year-old children is not yet fully developed. The boundary of the numerical competence of preschool children, therefore, should be viewed as a reflection of their verbal and short-term memory proficiency, rather than an indication of their immature conception of number, or their lack of this or that principle and logic. The preschoolers' state of development of the aforementioned skills is also likely to be the reason for what Descoeudres[50] famously called "un, deux, trois, beaucoup" (French for 'one, two, three, many'), that is, the well-known phenomenon that there is a great discrepancy between two- and four-year-olds' abilities to grasp small numbers and their ability to grasp larger numbers.

Since the level of abstraction required for a particular number's comprehension is determined by the amount of units it encompasses, the increasing abstraction levels required for the conceptualization of larger numbers is used not for better comprehension of *what numbers are*, but for better comprehension of their *sizes*. In other words, once the child attains the fundamental principles of number, the development of number concepts is about the construction of discrete concepts one by one and with accordance to the child's cognitive development and educational experiences. Taken as a whole, then, the development of numerical concepts does not progress from a lesser understanding of numbers to a more

[50] Descoeudre, 1921, cited by Bryant, 1974, p.119

correct understanding, but rather from smaller and less intellectually demanding numbers to larger and more challenging ones.

There is an understandable skepticism regarding the proposal that the generic idea of number is acquired prior the development of specific numerical concepts for it implies that two- and three-year-olds have the ability to form abstract concepts, whereas the conventional wisdom in child psychology is that such abilities only appear much later in children's cognitive development. What is more, the human tendency to perceive small numbers as if they are physical phenomena (as described in III-2) deems such abstraction unnecessary in the first place. Yet, the evidence that two- and four-year-olds' concepts of number are abstract early on is manifested by just about everything children at these ages do and say—or as the case may be, do not do and say—whenever they deal with enumeration tasks.

Let us, then, examine the validity of the proposal that children are aware early on that each number is an idea of size that is described exclusively by an explicit amount of units, and that both numbers and units have universal meanings in light of the research data described thus far:

That children know early on that numbers are ideas of size had already been discovered in Binet's pioneering explorations of this topic. Binet asked his 2 ½- and 4-year-old daughters to compare two rows of beads, each row assembled from beads of a different size, such that the fewer but larger beads formed a longer line than that of the more numerous but smaller beads (see VII-1). His experiments led him to the conclusion that children evaluated the number of beads by the length of the row rather than by how many beads it contained. The numerous experiments, which followed Binet's, confirmed his finding that, when children encounter groups for which they have not yet established matching number concepts, they will judge the same number of objects spread over a larger area or forming a longer line as more numerous; in these instances, children use perceptual criteria for size evaluation, which unequivocally demonstrates that they conceive numbers as size ideas. Discussing his discovery, Binet was quick to point out that while the child substituted the size of an area occupied by a group for true enumeration, her choice of words indicated that she saw a group of objects—"there are more of them here"—, he quoted her.[51] Binet's observation suggests that even when they use perceptual criteria in their evaluation of numbers, children are aware that these size ideas are composed of discrete units. More indications for the recognition of the involvement of units were demonstrated by later research that showed that children associate number words with counting, and that when they count, children as young as two years old pay close attention to units identification by stating aloud one number word while pointing to and touching each object counted. The phenomenon of omitting the word 'one' in early counting (discussed in VIII-2) is

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51 Binet in Pollack and Brenner, 1969, p., 88

another indication that children understand that each numerical size consists of a *group* of units. Indeed, according to Wynn children learn at a very early stage of counting that number words refer to specific sums of units.[52] That children also understand that these units represent universal concepts of equivalent members belonging to the same group is demonstrated by children's tendency to indicate the number of items counted by finger showing (discussed in VIII-3).

Perhaps the most compelling evidence that children understand numbers, and the units that construct them, as universal concepts is their approach to number words, including those whose meanings they have not yet acquired: All 3-year-old and most 2-year-old children begin counting with the conventional sequence of number words. From age 2½ on children seldom use words that are not number words in counting. By age three, children's counting words consist exclusively of the conventional number words, and when counting, use only number words.[53] Children's willingness to repeat the same list of words to count diverse groups of objects is an unmistakable indication that they understand that the content of these words bear a universal meaning that transcends the particular objects to which the words were assigned, even if they point and touch each object while stating aloud one of these words.[54] This a priori conclusion is consistent with Wynn's claim that the empirical data show that at no stage in their development of numerical concepts do children believe that the number words, which they use in counting, refer to the objects counted.[55] Gelman's observation that even in the rare instances in which children did use non-number words in their counting, the words they chose did not refer to the objects' properties (e.g., green) or identities (e.g., frog), confirms that indeed children do understand the abstract or universal nature of number as well as the units by which they are constructed.

Gelman's experiments show that when dealing with a number with which they are familiar, children seemed to be oblivious to any perceptual information that was not relevant to enumeration, such as density, length or area covered by the aggregate as a whole, the kinds of objects or their physical attributes;[56] they disregarded the order in which objects were counted, and they followed all the other counting and reasoning rules, including the 'one-to-one' procedure specified by Gelman and Gallistel (see VII-4). In Fuson's meticulous and methodical 1988 *Children's Counting and Concepts of Number*, in which she refers to voluminous

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52 Wynn, 1992, p. 220 and p. 244. For the record in Wynn's terminology: 'numerosities.'

53 Fuson, 1988, p. 58, 190, 387, 389; Gelman and Gallistel, 1978, p. 90, Wagner and Walters, 1982, p. 151

54 Gelman and Gallistel, 1978, p. 51, 54, 55, and 205

55 Wynn, 1992, p. 224

56 Gelman, and Gallistel, 1978, p. 54-55

research data including her own, she comments about children's counting, thus, "the single most striking attribute of the data about children's counting and concepts of number reported in this book is the really amazing level of competence young children display."[57] Fuson's conclusion is consistent with Gelman and Gallistel's observation that when the preschool children dealt with numerical quantities for which they possess a matching numerical concept,[58] they were able to "bring to bear reasoning principles of surprising sophistication."[59] Since even the smallest numbers embody the abstract principle that underlies all numbers, it appears that the errors children make when dealing with larger numbers are not an indication that the children's understanding of number is lacking or mathematically invalid, but rather, that they have not yet formed a specific and definite conceptual reference for these larger numbers.

The analysis of both the cognitive requirements for the acquisition of numerical concepts and the empirical research data mentioned here suggests that recognizing numbers as abstract ideas is an early and elemental step, and not necessarily one that poses the greatest challenge among the many steps taken in a process that may continue into adulthood. Grasping numbers as universal concepts of size is the beginning, not the ultimate completion, of number concept development. For the development of numerical concepts is not about a progress in the ability to understand what numbers are; it is about a progress in the ability to grasp increasingly larger numbers.

VIII-6. STEPS IN THE ACQUISITION OF NUMBER CONCEPTS

This concluding chapter presents a comprehensive outline of number development by describing the cognitive tasks involved in the acquisition of the counting numbers from the smallest and most tangible to the very large and elusive, step by step. Some of the issues were discussed at length in previous chapters; they are briefly reconsidered along with other subjects in order to offer a cohesive and complete view of a development that begins in infancy and continues into adulthood.

Introduction to numbers: Children's number development is triggered by contacts with numerical symbols, which are a salient element in their cultural and social milieu. Research reveals that at the preliminary stage of number-symbol

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57 Fuson, 1988, p. 402

58 Gelman, and Gallistel, 1978, p. 51, For the record, Gelman's exact words were, "when the young child encounters numerosities that he can represent numerically."

59 Ibid., p. 51

acquisition children's involvement with numbers is initiated by parents. By 18 months parents' actions assume a clear pedagogical characteristic. Children start to become independently interested in numbers around the age of two. At this time they spontaneously initiate recitation of the number-words sequence and their employment in counting, and develop their own hypothesis about the meaning of these words. Data show that toddlers know to associate number words with the question 'how many' and with counting, indicating that they understand that number words signify groups and that they are aware of the relevancy of units and unit analysis even if they have not yet developed definite ideas of numbers.

Comprehension of numbers within subitation range: The first numbers to be acquired are numbers that can be still envisioned explicitly, that is, numbers within the subitation range (up to 4 or 5). These are the smallest numbers and, as pointed out by Dehaene, are also the most frequently used in enumerated cultures.[60] In this initial phase counting plays an important role for it provides the mechanism for committing units to memory, which, in turn, enables the child to visualize them simultaneously so as to form a mental representation of a specific number. Since perceiving numbers within the subitation range is as much a verbal task as it is visual,[61] the size of numbers preschool children can visualize and conceptualize must be commensurable with their verbal skills, particularly the length of the word sequence that they are able to retain in their working memory. The major cognitive processes involved in number conceptualization in this phase are the analysis of units and the rationally guided visualization of their sums as unique entities, while the children's verbal and short-term memory capacity sets the boundaries of the sum sizes of this visualization. Even if children's numerical proficiency is limited to the first two or three numbers, those numerical concepts are genuine in the sense that they are established by units' analysis, and are not associated with specific objects, and hence are mathematically valid. In enumerated cultures most children acquire the concepts of subitizable numbers before they start their formal education.

Comprehension of the first ten numbers: Number proficiency rests upon the mastery of the first ten numbers, that is, the *base-numbers* 1 through 9, and the *base* 10. These numbers provide the conceptual infrastructure and referents for the development of all other numerical ideas. The first ten numbers include both numbers that can be subitized and numbers too large for subitation. As was discussed above, the conceptualization of the former, which relies on counting, is suitable only for the smallest numbers, those that may be perceived and imagined

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60 Dehaene, 1997, p. 110

61 Neisser, 1967, p. 42, 43

explicitly. Construction of larger numbers, which can no longer rely on distinct mental images, must build upon the combination of previously acquired numerical concepts. The number 6, for example, may be constructed by smaller numerical concepts, say 5 and 1, 4 and 2, or, most likely, by 3 and 3. The conceptualization of larger numbers, then, employs conceptual/symbolic thinking, which requires more advanced cognitive operations and levels of abstraction than the visualization of explicit sums. Consequently, in the development of these larger numerical sizes, counting stops to be helpful and symbolic thinking takes over. At this juncture number-concept development becomes a mediated endeavor, that is, a learning that is achieved with adult guidance or with collaboration with more capable peers.[62] The learning mode at this level is arithmetic calculations that aim toward gaining command of the numerical relationships of numbers within the unit 10. Experience demonstrates that mental computation (a process of solving problems 'in the head,' so to speak) is the most helpful method to achieve this goal. Hence, mental/oral arithmetic should be practiced before moving on to written-arithmetic exercises.[63] It is worthwhile to mention that in the early school years the very task of writing numerals demands from children an investment of considerable cognitive effort. When children are still struggling with the technical aspects of writing, written assignments distract them from the arithmetic task at hand, whereas oral exercises allow them to fully concentrate on those tasks. The rapid numerical operation afforded by oral exercises is necessary for establishing fluency and automatization of addition and subtraction operations within ten.[64]

Comprehension of numbers up to one hundred: The comprehension of numbers larger than the 'base', such as 13, 24, or 56, introduces into the process of number configuration the idea that a standard group of units can be considered and counted as a unit much like the unit one. For example, the digit 5 in the number 56 represents the 'sum' of the unit 10, and the digit 6 represents the 'sum' of the unit 1. Numbers such as 20, 30, 40, etc., can be conceived as products of 10 multiplied by a base number—making multiplication operations part of number-concept formation. Hence the construction of numbers beyond 10 involves multiple levels of abstraction and cognitive operations. Whereas in the first ten numbers the focus is on fluency in addition and subtraction of the base numbers, the focus in numbers within the 100 range is the preliminary understanding of the base system's structure, which is achieved by inductive generalization of the concept of unit.[65] Acquisition of numbers within the hundred, then, requires a

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[62] Vygotsky, 1978, p. 86

[63] Hope, Reys and Reys, 1988, p. V; Katz, 1981, Booklet A, p. 19

[64] Katz, 1981, Booklet A, p. 24

[65] Ibid., p. 35-36

cognitive shift to a more advanced level abstraction. This requirement can prove mentally demanding, and many educators are tempted to introduce concrete objects known as manipulatives in order to provide sensory illustration so as to lessen the mental demand. Alas, the manipulatives and the activities associated with them lock the children's thinking in a concrete mode, which impairs their ability to make the necessary leap to a higher abstraction level. Indeed, in her, 1992 article, *Magical Hopes: Manipulatives and the Reform of Math Education*, Deborah Loewenberg Ball has discussed a few of the problems that arise from the use of manipulatives to illustrate mathematical ideas. With regard to the counting numbers in the 100 range, Ball described teachers' frustration with the fact that many students use 'base-ten blocks', or the "relatively" more flexible bundles of Popsicle sticks grouped by tens, as instructed, but cannot transform these concrete activities to the appropriate symbolic operations.[66] Whereas some use of concrete demonstration could be useful, ultimately the acquisition of what many consider one of mankind's greatest intellectual achievements cannot be reached without a considerable mental investment in abstract, conceptual, and symbolic thinking.

Yet, even if the conceptualization of 100 and the two-digit numbers that lead up to 100 may become quite complex and abstract, numbers within 100 do not require comprehension of the graduated pattern of the base system. The acquisition of these numbers can still be contained within the conceptual realm of *arithmetic progression* (in which the rate of growth remains constant), that is, within the framework of addition, subtraction, and multiplication. The number 34, for example, may be conceived as (10+10+10)+(1+1+1+1), or as (10x3)+4, or as 10+10+10+4. These operations and the numerical concepts they employ are within first and second grader's ability to grasp.

Comprehension of numbers beyond one hundred: As numbers grow, conceptualization depends more and more on the comprehension of their largest decimal unit. The sequence of decimal units is formed by raising the base 10 to ever growing powers: $1=10^0$, $10=10^1$, $100=10^2$, $1{,}000=10^3$, etc. The mechanism by which units grow is simple, yet, as the decimal units continue to ascend, the link to the original perceptual reference 10 by which they were abstracted weakens, and as a result the appreciation of their magnitudes become increasingly challenging. The level of abstraction necessary for conceptualization increases along with the decimal-unit sizes. When the decimal units can no longer be matched with an appropriate concept, the comprehension of units depends on the conceptualization of the extremely rapid growth of their *geometrical progression*, in which not only the terms grow, but the rate of their growth is growing as well.

66 Ball, 1992, p. 29-30

Indeed, in a series of experiments aimed to evaluate subjects' ability to locate the position of a number on a number line in which only its endpoints were indicated by a numeral (detailed in VI-1), Siegler, Laski; and Booth, Siegler effectively demonstrated that estimating correctly numbers in the 1,000 range calls for a different approach from that employed for estimating numbers in the 100 range. They described two major approaches for locating a number on the number line: *linear* and *logarithmic*. A linear estimation is proportionally accurate. For example, in a number line representing the numbers between 0 and 1,000, the *linear* estimator is able to locate the number 100 at the end of the first tenth of the line, and the number 20 at the end of the first fiftieth portion of the line. One who estimates *logarithmically*, on the other hand, tends to overestimate the magnitude of the smaller numbers and underestimate that of the larger numbers, such that she may disproportionately locate the numbers 100 and 20 closer to the 1,000 end of the number line than they should be. Their research data show that when second graders were asked to locate numbers between 0 and 100 on the number line the majority operated on a linear level, but when they were asked to do the same on the number line that presented numbers between 0 and 1,000, the majority produced logarithmic estimates.[67] It is significant that at this range even adults had an "impulse" to take a logarithmic approach for locating numbers on the number line.[68] It is safe to assume that the mental challenge presented by the requirement to move from the arithmetic-progression framework to that of the more intellectually demanding geometric progression contributed to the errors in locating numbers between 0 and 1,000 on a number line.

Astronomers, who must measure the huge distances that lie between celestial bodies, circumvent the difficulties of dealing with 'astronomical' and difficult-to-imagine 'units' by redefining units in a way that makes them more tangible. To this end they invented the unit called the 'light year,' which defines the distance traveled by light in the course of one year (approximately six trillion miles). The unit of a 'light year,' which uses a concept of time to define a unit of length, helps astronomers to 'bring down to earth' (so to speak) celestial distances. Another example of scientists' way to circumvent the difficulties in comprehending very large units is the use of *scientific notation*. The 'scientific notation' expresses numbers as the product of a 'base number' and a power of ten. For example, the expression '$9.46x10^{15}$' communicates the number 9,460,000,000,000,000, which is the number of meters in a light year. [69]

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67 Booth Siegler, 2006, p. 189-190

68 Ibid., p. 200

69 Source. Baron's 1987, Dictionary of Mathematics Terms

Concluding notes: Number-concept development is a process of comprehending increasingly larger numbers, and thus increasingly complex and abstract numbers, as they progress along their infinite sequence. In the initial stages of number development the child's progress is linked to verbal and short-term memory capabilities. Progress in the more advanced stages is tied to the scope of the child's arithmetic experiences and concomitant expansion of his/her numerical-operations skills and powers of abstraction. In enumerated cultures such as ours, children already understand the generic idea of number before they start grade school. Hence educators need not ask how children acquire the primary concept of number as a special category of size ideas, but rather how children build upon this acquisition the understanding of new and more advanced numerical concepts; and how, in their role as teachers, they can most effectively facilitate the child's learning.

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