# How to Build Powerful Learning Trajectories for Relational Thinking in the Primary School Years

Max Stephens
*The University of Melbourne*
<m.stephens@unimelb.edu.au>

Dian Armanto
*The State University of Medan*
<armanto_dian@yahoo.com>

There are now strong arguments for building closer links between children's understanding of numbers and number operations and the beginning of algebraic (relational) thinking in the primary school years. Rarely, however, do Australian mathematics textbooks give enough guidance for teachers to use good activities in the classroom to promote algebraic thinking. By contrast, Japanese mathematics textbooks introduce students to relational thinking about number sentences, starting from the first grade. The idea of a learning trajectory – or trajectories – seems a fruitful way of looking at how this is achieved and what it might mean for the teaching of Number and algebra in the primary years.

In its scoping paper, *Shape of the Australian Curriculum: Mathematics*, the Australian Curriculum and Reporting Authority (ACARA, 2009) undertook to strengthen the links between the teaching of Number and algebra. Referring especially to the middle and later primary years, the paper argued that "An algebraic perspective can enrich the teaching of number …." (ACARA, 2009, p. 6). Likewise, in its report, Algebra: *Gateway to a Technological Future*, The Mathematical Association of America (2007) argued that "We need a much fuller picture of the essential early algebra ideas, how these ideas are connected to the existing curriculum, how they develop in children's thinking, how to scaffold this development, and what are the critical junctures of this development" (p. 2). A clear learning trajectory, supported by carefully planned activities and good problems, will be important to demonstrate the kind of scaffolding advocated in these two documents.

## Rationale for a Learning Trajectory for Relational Thinking

The development of algebraic thinking depends on the recognition and analysis of patterns that are important components of the young child's mathematical development (Sarama & Clements, 2009, p. 319). Patterns such as subitized patterns, patterns in the number words of counting, "one-more" patterns of counting, numerical patterns, arithmetic patterns, spatial patterns and array structures are typical early childhood practice in patterning (Wu, 2007). Patterning is the search for mathematical regularities and structures, to bring order, cohesion, and predictability to seemingly unorganized situations and facilitate generalizations beyond the information directly available (Sarama & Clements, 2009, p. 319). As a process, a habit of mind in understanding mathematics, Mason, Stephens, and Watson (2009) point out that pattern recognition is based on an assumption that patterns are present and that things that seem to be the case will continue to be the case; emphasising that, for students, investigating patterns sometimes means seeing "the general in the particular" or at other times seeing a "particular case as an instance of a general mathematical truth" (p. 21).

Baroody (1993) mentioned that students learn to find and extend numerical patterns – extending their knowledge of patterns to thinking algebraically about arithmetic. However, we know that this does not happen automatically for children, and that it is heavily dependent on how they are taught. The ability of children to invent, learn, apply, justify, and otherwise reason about arithmetic problems, demonstrates that algebraic thinking can

be an implicit but significant component of primary children's learning of mathematics (Sarama & Clements, 2009). Students also can recognize that equality is preserved if equivalent transformations are made on both 'sides' of a situation, from balance scales to sets of objects, verbal problems, and written equations (Schliemann, Carraher & Brizuela, 2007).

According to Molina, Castro and Mason (2007), when students think relationally, they consider the number sentence as a whole, then analyse and find the structure and important elements or relationship to generate productive solutions. Related research by Carpenter and Franke (2001) and by Stephens (2007, 2008) showed that relational thinking occurs when students see the equals sign as a relational symbol, focus on the structure of expression, and carry out reasonable strategies to solve the number sentence attending to the operations involved.

These key ideas are assumed in most textbook treatments of algebra which typically start at junior secondary school, where it is introduced as generalised arithmetic. Appreciation, understanding, and comprehension of these ideas are necessary for continued development of and success in algebraic thinking throughout secondary school. In this paper we argue that the foundations for this kind of thinking need to be laid in the primary school years. By analysing students' textbooks used in Japan, this paper will illustrate a possible learning trajectory for teaching relational thinking for students and teachers.

## A Concept of Learning Trajectory

In an abstract sense, learning trajectories are descriptions of students' thinking as they learn to achieve specific goals in a mathematical domain. In more concrete terms, a learning trajectory is a conjectured path through a set of instructional tasks designed to produce those mental processes or actions hypothesized to advance students' thinking in a particular domain. It begins with a preliminary design of prototypical instructional activities, followed by teaching experiments, and ends with a retrospective analysis. This iterative process of (re-)designing and testing (Gravemeijer, 1999) aims at building a local instructional theory on how to teach specific topics.

Learning trajectories need pedagogical and theoretical constructs (Simon, 1995). Three elements appear to be essential: a set of clear mathematical goals, a clearly marked out pathway through the curriculum along which students can be helped to reach that goal, and a coherent set of instructional activities or tasks, matched to levels of students' mathematical thinking and prior experience, that support these mathematical and pedagogical goals. Teachers need to be particularly careful not to assume that students can see situations, problems, or solutions as adults do. Similarly, when they interact with students, teachers also should consider their actions from the students' point of view (Sarama & Clements, 2009, p. 17). Building good learning trajectories in textbooks helps teachers to think about what objectives should be established, where to start, how to know where to go next, and how to get there (Carpenter, Fennema, Peterson & Carey, 1988).

### An Example of a Japanese Learning Trajectory

The learning trajectory of teaching mathematics in Japanese textbooks is quite clear in developing a close link between calculation and relational thinking. The development of students' relational thinking starts in the first grade of primary school. The examples used in this paper are taken from the most popular textbook series for primary schools in Japan – the Tokyo Shoseki (2006) series. The focus in this paper is on the development of

relational thinking about subtraction and addition in Grades 1 to 3 and the development of relational thinking about division in Grades 4 to 6. A detailed examination of relational thinking about multiplication is not possible here. It is hoped that, by using these chosen areas, several powerful and interconnected learning trajectories can be demonstrated.

First Grade



*Figure 1*. Exploring subtraction relationally in Grade 1

Figure 1 presents three activities that might take place over several lessons or even longer. These are explorations in looking for patterns and relationships, as opposed to "find the answer". Firstly, working in a small group, students look for number sentences, deliberately left in uncalculated form, whose value is equivalent to a number that is named by the student at the head of the table (the answer to each subtraction is written on the back of the cards if students need to check). Here, students see that different subtraction sentences can have the same number as their answer, and become familiar with "what numbers are represented by what sentence". In the first grade, other activities include arrays of cards representing addition. Regular computation methods feature too in all grades.

Initially, those subtractions "with the same answer" are not presented in any systematic form. Subsequently, by working individually (see the middle figure above) students are given a set of cards with the same subtraction questions on the face side, and are asked to place together on the board subtraction sentences that match the various numbers (answers). This more systematic activity focuses on finding (and discussing) different subtraction sentences that produce a given number (answer).

Thirdly, students are presented with the cards arranged by the teacher in an array. This activity builds on the preceding two activities and assumes that students are already comfortable with having sentences in uncalculated form; and that all students know that several subtraction sentences can represent the same number. In the first phase of this activity, students are asked to look at how the cards have been arranged vertically. Teachers may ask students to extend a given vertical column by using a given card or creating a new card. In other cases, the teacher may make gaps in the array which students are expected to fill in. Here, students are asked to find patterns for generalising the relationships that identify the sequence of numbers in a given column. Students might notice, for example, that in each column each card has the same starting number; or that the number of cards in each column grows as one moves from left to right; or that the second number in each column increases as one goes down. These insights are shared with the whole class.

Then, the same questions are applied to how the cards are arranged horizontally. Finally, the teacher or a student may name a given number – shown in the figure as 6 – and

ask students to find all the cards which have answers equivalent to this number. Students need to see that these cards all sit along a diagonal (shown above using a bar which *we* have inserted). Students are expected to describe how the numbers on the cards change as one moves along a diagonal. A related activity might be to look for all the cards in the array that represent the number 5, and so on. This carefully designed activity, spread over several lessons, helps students to understand that several subtraction sentences can represent the same number. Leaving the sentences in uncalculated form is essential to discussing the various relationships between numbers on the cards – horizontally, vertically and diagonally. As Mason, Stephens and Watson (2009) argued, structural (or relational) thinking relies on students learning to apply these multiple perspectives.

Second Grade — Here, subtraction sentence cards use larger numbers, as shown in Figure 2. Here, as in Grade One, the focus is on subtraction sentences with the same answer. However, larger numbers are used and the shift to a more systematic array takes place more quickly. Once again, it is expected that students are comfortable working with sentences in uncalculated form. In Figure 2 on the left, students are asked to line up the cards in order beginning with the smallest minuend and ending with the largest. Here students are asked to find the relationship among six subtraction sentences, all of which are equivalent to 28. A student might think that when the minuend (first number) increases by 1 the subtrahend (the number being subtracted) has to increase by one in order for the difference to remain the same. Students can also see which of the six cards represents the easiest calculation for them to make. The student in the frame on the right asks what will happen to the subtrahend if 84 is increased to 88 – increasing by one is not always necessary! Decreasing is also possible. Molina et al. (2007) emphasise this flexible aspect of relational thinking.
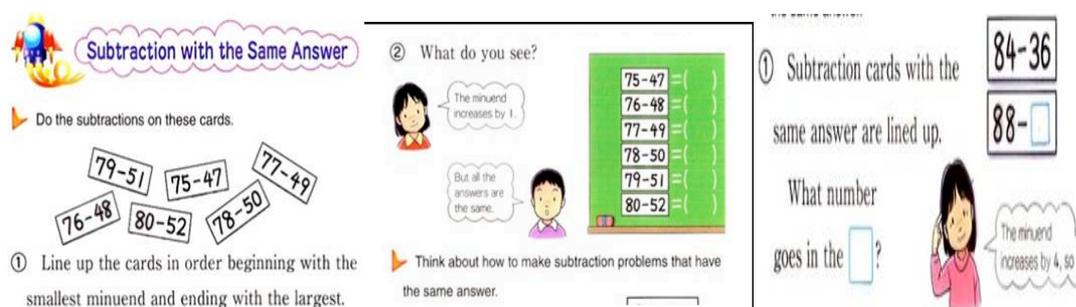


*Figure 2*. Exploring subtraction relationally in Grade 2

The pairing of 84 – 36 with 88 – □ is quite deliberate, as is the use of an empty box to denote a missing number. Clearly, 88 – 40 is easier. So a link between relational thinking and calculation is maintained in the Japanese textbook. Working with equivalent number sentences using subtraction is essential if one is to calculate intelligently, for example, in thinking about 234 – 99, or 87 – 39. It is a vital component of this learning trajectory.

Third Grade — In the third grade, the link between relational thinking and calculation is very clearly illustrated by the following two activities involving addition and subtraction (see Figure 3). Number sentences involving multiplication are also introduced in this grade.

*Figure 3*. Exploring addition and subtraction relationally in Grade 3

Working with number sentences in uncalculated form, students are invited to transform a given expression into an "easier-to-calculate" expression, involving the same operation. The original addition question 298 + 120 is transformed to 300 + 120, for which the answer is 420. Having obtained this answer, students have to think whether this answer is too large or too small in relation to the original question. Here, since a larger number has been added, the answer obtained is larger (by two) than the answer to the original question. Therefore, two has to be subtracted from the answer. In the case of subtraction, taking away a larger number (in this case 200) means that one has taken away too much and the answer obtained is smaller (in this case, smaller by 2) than the answer to the original question. Therefore, two has to be added to the result of 500 – 200 to get the answer to the original question. This method is distinct from, but clearly complements, the activity involving equivalent subtraction sentences in Grade 2. Students can now also reason that 500 – 198 is equivalent to 502 – 200. Teachers are expected to discuss these two methods. Relational thinking tends to branch out in different directions, as Schliemann et al. (2007) emphasise.

Fourth Grade — In the fourth grade, Japanese students learn to think relationally about division. Once again, students are expected to focus on the left-hand side of the expression. Using sheets of origami paper, students are invited to think what will happen if the individual sheets are replaced by bundles of ten and later by bundles of five.

The link to calculation is again clear, as the three examples shown at the bottom right of Figure 4 show. Students are expected to think independently before transforming each expression. The first example may be simplified by dividing both the dividend and divisor either by three or by ten, but equally both could be divided by 30. In the third example, students have several options: dividing both the dividend and divisor by 5 or 25, or multiplying both by 4. Relational approaches to division are returned to again in Grade 6.



*Figure 4*. Exploring division relationally in Grade 4

527

Sixth Grade — Here, students are invited to think again about division by providing different division sentences which have the answer 4 (see Figure 5 below). In the figure on the right, these different answers have been re-arranged starting with 4 and finishing with the highest number 40 so that students are able to investigate the structure of equivalent division sentences.
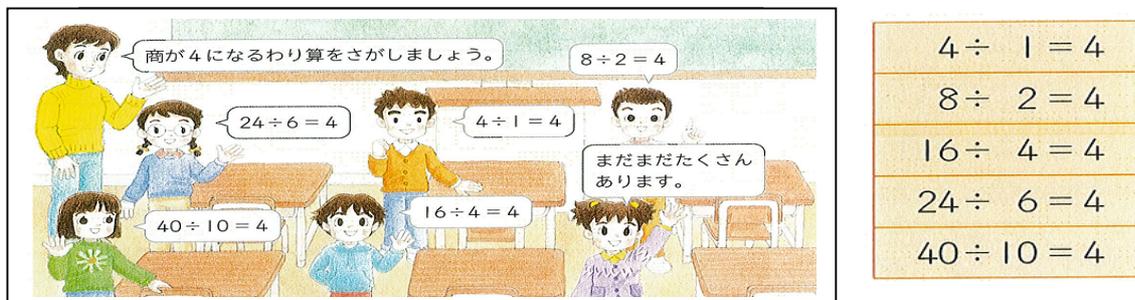


*Figure 5*. Teaching division relationally in Grade 6

The teacher uses the various division sentences and their repeated answer of 4 (see the right of Figure 5 above) to promote understanding the relationships between the numbers involved in these sentences. Students apply multiple perspectives to explain the relations: top-to-bottom, bottom-to-top, pair-wise comparisons, as Mason et al. (2009) point out.

When the relational properties of division were introduced in Grade 4, the dividend and the divisor were both reduced by a factor of 10, and later by 5 (see Figure 4). Here dividends and divisors are arranged with the smallest numbers at the top, and so dividends and divisors may be multiplied by the same factors without changing the result. A more systematic and formal treatment (shown in Figure 6 below) shows students that when the dividend is multiplied by 6, the divisor should be multiplied by 6 as well. But, as the frame on the right shows, dividends and divisors can be divided by the same factor as well. Being able to think relationally about division has clear pay-offs, as Wu (2007) shows, when students encounter division by decimals, and convert a decimal divisor into a whole number; and this same thinking is very helpful for understanding division by a fraction.
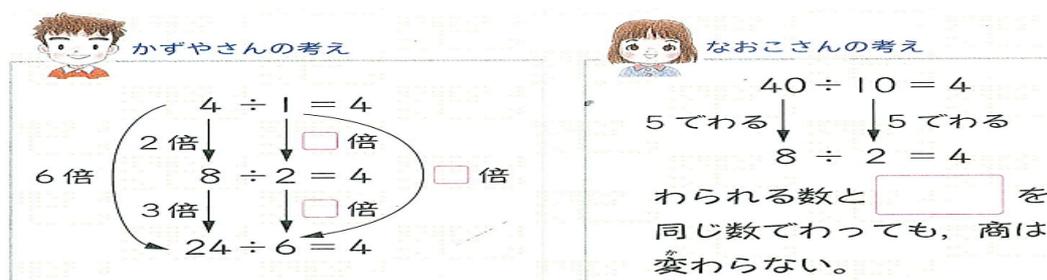


*Figure 6*. More relational thinking using division in Grade 6

By the end Grade 6, clear links have been built between calculation and relational thinking across the four operations – multiplication and division, and addition and subtraction – and how they behave differently. In these activities, teachers and students focus on the relationship of the numbers involved rather than looking first to calculation. Computation is used for reassurance and validation.

# Building a Stronger Learning Trajectory

It can be seen that this mainstream Japanese textbook focuses on creating learning trajectories where students think deeply about the relationships in number sentences using each of the four operations. As Jacobs, Franke, Carpenter, Levi and Battey (2007) show, relational thinking has the added benefit of enhancing and supporting fluency of and efficiency in calculation. These learning trajectories are designed so that all students can engage with the problems presented to them. Of course, some will learn more than others, but the textbook authors ensure that key ideas are visited again at different grades of the primary school. While this is intended to provide a strong foundation for starting algebra in junior high school, in a sense, algebraic thinking has already started in the primary school. The learning trajectories shown here are not simple and linear, but complex and "branchy". Students are expected to choose transformations that make sense for them – not to follow a lock-step method. The activities chosen have to be rich enough for every student to engage and conceptually strong to develop key ideas and to promote deep mathematical thinking. Even from the examples above, some students will notice that the direction of compensation in equivalent division sentences is not unlike the direction of compensation for equivalent subtraction sentences.

As the Japanese examples show, computational knowledge is essential to lead students to relational thinking, and relational thinking has the potential to help students understand computation better and to perform it more fluently. As students move along this trajectory, they are expected to grow in confidence in using a variety of relational words, such as "equivalent to" or "means the same as" or "has the same value as". Other relational words need to be used, such as "difference between", "product of", "quotient" and so on. As they move through the primary school, they will use relational words in more complex and more varied sentences, such as "The answer will be the same (in a subtraction) if each number is increased by the same amount", or "The difference remains the same", or "We can make this addition (or subtraction, multiplication or division) easier to calculate by changing the numbers in the following way", or simply saying "We know the answers are equivalent". We should expect that all students are able to think this way – some will be more fluent than others, of course, but learning to think relationally should be something that all students can develop to some degree and use in calculating more efficiently.

Japanese textbooks give just as much attention to computation and correct calculation, but they have a clear learning trajectory, which progressively develops relational ways of thinking about the four number operations. This learning trajectory is continuous and systematic throughout the primary years. We should assume that not all Japanese students are ready to understand relational ideas the first time they are introduced, but they will have many other occasions to think relationally about number sentences. Japanese textbooks develop strong patterns of relational thinking by the end of primary school. They do this not by having longer and heavier textbooks but by carefully chosen examples and a well developed learning and teaching trajectory. This is an area that many other countries could well consider for their teachers and students.

# References

Australian Curriculum and Reporting Authority (2009). *Shape of the Australian Curriculum: Mathematics*. Canberra: Author.

Baroody, A. (1993). *Problem solving, reasoning, and communicating (K-8): Helping children think mathematically*. New York: Merrill/Macmillan.

Carpenter, T. P., Fennema, E., Peterson, P., & Carey, D. (1988). Teachers' pedagogical content knowledge of students' problem solving in elementary arithmetic. *Journal for Research in Mathematics Education*, *19*, 385 – 401.

Carpenter, T. P., & Franke, M. L. (2001). "Developing algebraic reasoning in the elementary school: Generalization and proof". In H. Chick, K. Stacey, J. Vincent, & J. Vincent (Eds.), *The Future of the Teaching and Learning of Algebra*, (Proceedings of the 12th ICMI Study Conference, pp. 155-162). Melbourne: University of Melbourne.

Gravemeijer, K. (1999). Developmental Research: Fostering a Dialectic Relation between Theory and Practice. In J. Anghileri (Ed.). *Principles and Practice in Arithmetic Teaching*. London: Open University Press.

Jacobs, V. R., Franke, M. L., Carpenter, T. P., Levi, L., and Battey, D. (2007). Developing children's algebraic reasoning. *Journal for Research in Mathematics Education*, *38*(3), 258-288.

Mason, J., Stephens, M. & Watson, A. (2009). Appreciating Mathematical Structure for All. *Mathematics Education Research Journal* (MERJ), *21*(2), 10-32.

Molina, M., Castro, E., & Mason, J. (2008). Elementary students' approaches to solving true/false number sentences. *Revista de Investigacion en Didactica de la Matematica* PNA *2*(2), 75-86.

Sarama, J., & Clements, D. (2009). *Early Childhood Mathematics Education Research: Learning Trajectories for Young Children*. New York: Routledge.

Schliemann, A., Carraher, D., & Brizuela, B. (2007). *Bringing out the algebraic character of arithmetic*. Mahwah, NJ: Lawrence Erlbaum Associates.

Simon, M. (1995). Reconstructing mathematics pedagogy from a constructivist perspective. *Journal for Research in Mathematics Education*, *26*(2), 114-145.

Stephens, M. (2007). Students' emerging algebraic thinking in the middle school years. In J. Watson & K. Beswick (Eds.), *Mathematics: Essential Research, Essential Practice*, (Proceedings of the 30th annual conference of the Mathematics Education Research Group of Australasia, pp. 678-687). Adelaide: MERGA.

Stephens, M. (2008). Some key junctures in relational thinking. In M. Goos, R. Brown & K. Makar (Eds.), *Navigating currents and charting directions* (Proceedings of the 31st annual conference of the Mathematics Education Group of Australasia, pp. 491-498). Brisbane: MERGA.

Tokyo Shoseki (2006). *Mathematics for Elementary School* (Textbook Series translated in to English – almost all). Tokyo: Tokyo Shoseki Publishing.

The Mathematical Association of America (2007). *Algebra: Gateway to Technological Future*, edited by V. J. Katz. MAA: Author.

Wu, H. (2007). *Whole numbers, fractions and rational numbers*. Berkeley, CA: University of California.