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TECHNICAL REPORT

POSSIBLE REASONS FOR STUDENTS' INEFFECTIVE READING OF THEIR FIRST-YEAR UNIVERSITY MATHEMATICS TEXTBOOKS

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Possible Reasons for Students' Ineffective Reading of their First-Year University Mathematics Textbooks

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ABSTRACT
This paper reports the observed behaviors and difficulties that eleven precalculus and calculus students exhibited in reading new passages from their mathematics textbooks. To gauge the effectiveness of these students' reading, we asked them to attempt straightforward mathematical tasks, based directly on what they had just read. These students had high ACT mathematics and high ACT reading comprehension test scores and used many of the helpful metacognitive strategies developed in reading comprehension research. However, they were not effective readers of their mathematics textbooks. In discussing this, we draw on the psychology literature to suggest that cognitive gaps, that is, periods of lapsed or diminished focus, during reading may explain some of the ineffectiveness of the students' reading. Finally, we suggest some implications for teaching and pose questions for future research.

Keywords: reading mathematics, first-year university mathematics textbooks, precalculus, calculus

INTRODUCTION

In a previous technical report, we described a study of students' difficulties in reading their first-year university mathematics textbooks. The current technical report is a reexamination of that data, along with conjectures as to why these students, who were good at reading and good at mathematics, as judged by their ACT reading and mathematics test scores, and their use of many of the metacognitive strategies of good readers, were in fact not effective readers of their textbooks. For completeness, and in order to make this technical report self-contained, we include much of the previous data and examples, but in addition, we provide several conjectures derived from the psychological literature as to why these students had difficulties.

From our own experience and in talking with colleagues, we have come to suspect that many, perhaps most, first-year university students do not read large parts of their mathematics textbooks *effectively*, that is, they cannot work straightforward tasks based on that reading. Whether this is because they cannot read efffectively, or choose not to do so, seems not to have been established. However, there have been a number of calls for mathematics teachers to instruct their students on how to read mathematics (Bratina &

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Lipkin, 2003; Cowen, 1991; Datta, 1993; DeLong & Winter, 2002; Draper, 2002; Fuentes, 1998; Pimm, 1987; Shuard & Rothery, 1988). In addition, textbooks for many first-year university courses, such as college algebra, precalculus, and calculus seem to be written with the assumption that they will be read thoroughly, precisely, and effectively.

For example, the preface of the precalculus book used by the students in this study asserts:

> The following suggestions are made to help you get the most out of this book and your efforts. As you study the text we suggest a five-step approach. For each section,
>
> 1. Read the mathematical development.
> 2. Work through the illustrative examples.
> 3. Work the matched problem.
> 4. Review the main ideas in the section.
> 5. Work the assigned exercises at the end of the section.
>
> All of this should be done with a graphing utility, paper, and pencil at hand. In fact, no mathematics text should be read without pencil and paper in hand; mathematics is not a spectator sport. Just as you cannot learn to swim by watching someone else swim, you cannot learn mathematics by simply reading worked examples—you must work problems, lots of them. (Barnett, Ziegler, & Byleen, 2000, p. xxxi).

Most teachers of beginning undergraduate mathematics would probably agree that the above is good advice. But it is not clear that it is realistic to assume that students will, or even can, adequately carry out the above five steps.

In this partly empirical, partly theoretical, exploratory study we examined what first-year undergraduate mathematics students did when reading their mathematics textbooks both from the perspective of the reading comprehension literature, and from a specifically mathematical perspective, noting the difficulties students encountered. Drawing on those observations, and also on information on our students' reading and mathematical backgrounds, we concluded that our students – and probably many, if not most, first-year mathematics undergraduates – cannot read their mathematics textbooks effectively. This led to an apparent anomaly. Our students were good at both mathematics and reading, applied their reading skills to their textbooks, and did not find the notation or syntax of these textbook passages very burdensome, but still could not read their textbooks effectively. How could this be? We offer a possible psychological explanation for the students' errors, and make some suggestions for teaching and future research.

In the first section, we describe the Constructively Responsive Reading (CRR) framework, a theoretical framework developed in reading comprehension research (Pressley & Afflerbach, 1995), discuss how mathematics textbooks differ from other books, and note the limited amount of research that has been done on how students read their mathematics textbooks. In the next section, we lay out our research questions. After that, in the following section, we describe the students, their courses, and our research methodology. Next there follows a section in which we describe our data and observations concerning students' use of CRR-based strategies, and students' difficulties in working straightforward tasks from their mathematics textbooks. After that, drawing on results from psychology, we provide a possible explanation for such difficulties.

Finally, we summarize our findings and suggest some implications for teaching, as well as propose directions for future research.

BACKGROUND AND LITERATURE REVIEW

Reading Comprehension Research

During the past fifty years, conceptual shifts have led reading researchers to view reading as an active process of meaning-making in which readers use their knowledge of language and the world to construct and negotiate interpretations of texts in light of the particular situations within which they are read. (Borasi, Seigel, Fonzi, & Smith, 1998; Brown, Pressley, Van Meter, & Shuder, 1996; Dewitz & Dewitz, 2003; Flood & Lapp, 1990; Kintsch, 1998; McNamara, 2004; Palincsar & Brown; 1984; Pressley & Afflerbach, 1995; Rosenblatt, 1994; Schuder, 1993; Siegel, Borasi, Fonzi, Sanridge, & Smith, 1996). These conceptual shifts have expanded the notion of reading from that of simply moving one's eyes across a page of written symbols and translating these symbols into verbalized words, into the idea of reading as a mode of thinking and learning (Draper, 2002).

Current discussions of reading focus on how the reader creates meaning as a result of the interaction, or transaction, between the text and the reader (Flood & Lapp, 1990; Pressley & Afflerbach, 1995; Rosenblatt, 1994). Reading researchers have found that competent readers actively construct meaning through a process in which they interact with the words on the page, integrating new information with their preexisting knowledge structures (Flood & Lapp, 1990).

Reading and literacy researchers agree that reading includes both decoding and comprehension. Research on comprehension indicates that there are several strategies that good readers employ before, during, and after they read. These strategies seem to vary from reader to reader and to depend on the material being read and the goals of the reader (Borasi et al., 1998; Brown et al., 1996; Flood & Lapp, 1990; Fuentes, 1998; Palincsar & Brown; 1984; Pressley & Afflerbach, 1995; Siegel et al., 1996).

One of the most comprehensive metastudies of reading research was conducted by Pressley and Afflerbach (1995). They developed a framework called Constructively Responsive Reading (CRR) that combined the frameworks of many previous reading researchers (e.g., Brown, Kintsch, Rosenblatt, and others). They noted about 330 different activities that readers were reported, or were observed, doing while reading. They produced a "Thumbnail Sketch" of the CRR framework that categorized activities of good readers into fifteen constructive responses. We have reduced these fifteen responses to the following eight that we call *CRR-based strategies*.[1] Good readers:

1. Preview the text to be read before reading to gain an overview and to make predictions about it.
2. Pay greater attention to information perceived as the most important.
3. Activate, that is, bring to mind, prior knowledge, integrate reading within the text using prior knowledge to interpret the text, construct meaning and revise/adjust prior knowledge as appropriate.
4. Make inferences about information not explicitly stated.
5. Determine the meanings of new or unfamiliar words.
6. Monitor comprehension and change reading strategies if needed.

7. Evaluate the text, remember the text, and reflect on it after reading.
8. Anticipate the use of the knowledge gained.

In addition to the above CRR-based strategies, we found four slightly different collections of reading strategies. Each included a subset of the eight strategies given above and suggestions on how they might be encouraged or taught. The four strategies are: Reciprocal Teaching (Palincsar & Brown, 1984), Transactional Strategies Instruction (Brown et al., 1996), Transactional Reading Strategies (Borasi et al., 1998) and Self-Explanation Reading Training (McNamara, 2004).

The Writing Style of Mathematics Textbooks

In their own writing, mathematicians appear to prize brevity, conciseness, and precision of meaning. Most first-year university mathematics textbooks currently published in the U.S. contain exposition, definitions, theorems and less formal mathematical assertions, as well as graphs, figures, tables, examples,[2] and end of section exercises. Often the definitions, theorems, and examples are set apart from the expository text by boxes, colors, or spacing. Figures containing graphs and explanatory material often appear in the margins. Typically there is a repeated pattern consisting of first presenting a bit of conceptual knowledge, such as a definition or theorem and perhaps some less formal mathematical assertions, followed by procedural knowledge in the form of a few closely related worked examples (tasks), and finally students are invited to work very similar tasks themselves. In these respects, the textbooks (Barnett, Ziegler, & Byleen, 2000; Larson, Hostetler, & Edwards, 2002) read by the students in this study appear to us to be typical.

Some special features of the style of mathematical writing that can sometimes lead to student difficulties, as indicated by Barton and Heidema (2002) and Shuard and Rothery (1988), include:

1. Reading mathematics often requires reading from right to left, top to bottom, bottom to top, or diagonally.
2. The writing in mathematics textbooks has more concepts per sentence, per word, and per paragraph than other textbooks.
3. Mathematical concepts are often abstract and require effort to visualize.
4. The writing in mathematics textbooks is terse and compact—that is, there is little redundancy to help readers with the meaning.
5. Words have precise meanings which students often do not fully understand. Students' concept images[3] of them may be "thin."
6. Formal logic connects sentences so the ability to understand implications and make inferences across sentences is essential.
7. In addition to words, mathematics textbooks contain numeric and non-numeric symbols.
8. The layout of many mathematics textbooks can make it easy to find and read worked examples while skipping crucial explanatory passages.
9. Mathematics textbooks often contain complex sentences which can be difficult to parse and understand.

In addition, we note that definitions are to be read and used in a special way, and play an especially important role in mathematics. That is, readers of mathematical writing must know how to read a definition as a stipulation of meaning, attending to every part, not adding anything, and ignoring most connotations.[4] Such definitions are unlike dictionary definitions which are often only approximate descriptions extracted from everyday language usage. Edwards and Ward (2004) found that even advanced university mathematics students have difficulty understanding the role and use of mathematical definitions. In our experience, even when students can correctly state and explain a mathematical definition, they may not use it correctly because they do not understand the distinction between mathematical (stipulated) and dictionary (extracted, descriptive) definitions.[5]

When reading mathematical writing, there appears to be little room for an acceptable interpretation of a passage that is different from the one intended by the author. In spite of this, Edwards and Ward (2004) found that formal definitions are *not* used by students as much as their concept images when reasoning about the kind of abstract ideas encountered in a typical upper-level mathematics class such as abstract algebra. This dependence on concept images also occurs for students in lower-division courses (Tall & Vinner, 1981). Furthermore, the concept images of different readers may contain different examples or procedures.

Previous Research on Reading Mathematics Textbooks

Only a little research has been done on how students read their mathematics textbooks. Osterholm (2008) surveyed 199 articles having to do with the reading of word problems, but found little about reading comprehension of more general mathematical text. He has done several studies on secondary and university students' reading of mathematical text (Osterholm, 2005, 2008), but the passages he used were written by him especially for that research.

Recently, Weinberg and Wiesner (2010) presented a framework for considering students' reading of their mathematics textbooks in order to construct meaning. It focused on the concepts of the intended reader (the one the author had in mind), the implied reader (the one with the competencies necessary to make sense of the text), and the empirical reader (the one who actually reads the text). We see Weinberg and Wiesner's views as complementary to ours. Students need both to extract conceptual and procedural knowledge from their mathematics textbooks and to reflect on, interact with, and construct meaning from that knowledge.

In addition, there *has* been an interest in, and some research on, how students read their science textbooks in order to learn science. The April 10, 2010 issue of the journal *Science* had a special section devoted to research on, and to the challenges of, reading the academic language of science. It was noted that, while students have mastered the reading of English texts (mostly narratives), this does not suffice for science texts that are precise and concise, avoid redundancy, use sophisticated words and complex grammatical constructions, and have a high density of information-bearing words (Snow, 2010, p. 450).

RESEARCH QUESTIONS

From the perspective of the reading comprehension literature, we ask what our first-year undergraduates did when reading their mathematics textbooks. In particular, what did they do relative to the eight CRR-based strategies? Did our students exhibit the characteristics generally observed in good readers?

We also ask what our students did from the mathematical perspective. In particular, what mathematical difficulties did they encounter when reading their mathematics textbooks? To what degree were such difficulties traceable to the writing style characteristic of mathematics textbooks, such as any unusual symbolism, syntax, or treatment of definitions?

We go on to ask, could our students read their mathematics textbooks *effectively*? That is, could they carry out straightforward tasks (that mathematics textbooks often label as examples, exercises, or problems) immediately after reading passages explaining how these tasks should be carried out, and with those passages still available to them?

Finally, the answers to these questions will uncover an apparent anomaly, which suggests a further, more theoretical, question. Is there some further perspective, different from that of good reading strategies and of the unusual nature of mathematical writing, that might help explain students' difficulty reading their mathematics textbooks? And how can it be that most mathematicians apparently do not have such reading difficulties, despite having had no explicit training in avoiding them or recovering from them?

METHODOLOGY

The Students

The eleven precalculus and calculus students in this study attended a U.S. mid-western comprehensive state university at which they took all their coursework. The university has a student body of 6,500 students of which 5,500 are undergraduates. It has a moderately selective admissions standard. Six were university students. An additional five were students in a mathematics/science magnet secondary school located on the campus of the university. Eight of the students were female, none were minorities. The mathematics courses taken by the eleven students in this study carried normal university credit and were taught by a member of the regular university faculty -- the first author.

Students for the study were selected from a precalculus class of 17 (the secondary magnet school students) and from two sections of Calculus I with 41 students total. In the fourth week, we first identified 33 good readers (12 precalculus, 21 calculus) with a ACT[6] reading scores ranging from 24 (70th percentile) to 36 (99th percentile). Based on the instructor's judgment, nine students (4 precalculus, 5 calculus) were eliminated because it appeared they had no problems reading mathematics and may have seen the material in previous courses. Of the remaining twenty-four students, eleven (5 precalculus, 6 calculus) volunteered to participate in this study. Ten of the students received a small amount of extra credit for participating in the study. The amount of extra credit received did not change any final grades. One calculus student dropped the class before the fourth week of the semester, but agreed to participate anyway. That student was grouped with the precalculus students since that was the passage the student read for the study.

The average reading ACT score for the eleven students was 28.6 (the median, 28, corresponds to the 87th percentile) which compares favorably with the university average reading ACT score of 22.3 for incoming first-year students. For these eleven, the reading ACT scores were further broken down into Social Studies/Science, where their subscores ranged from 12 (68th percentile) to 17 (98th percentile), and Arts/Literature, where their subscores ranged from 12 (63rd percentile) to 18 (99th percentile).

All but two volunteers — both calculus students who were not first-year university students—had ACT mathematics scores ranging from 23 (71st percentile) to 30 (96th percentile). The average ACT mathematics score for the eleven students was 25.2 (median score 27) which compares very favorably with the university average ACT mathematics score of 20.9 for all incoming first-year students at this university. Thus, according to their ACT scores, these students were good students generally, good at mathematics, and good readers in both the Social Studies/Science and the Arts/Literature portions of the ACT.

The Courses

From the beginning, both the precalculus and calculus courses from which the students in this study were chosen had a strong class emphasis placed on reading their mathematics textbooks. The students were given handouts about reading mathematics on the first day of class, and beginning the second class period, students were given reading guides for use with the first several sections of their mathematics textbooks. An example of a reading guide and additional information about the teaching practices of this instructor appeared in Author (2005).

During the first two weeks of the courses, all 58 students from the pre-calculus and calculus classes participated in a diagnostic interview as part of the instructor’s normal teaching practice. This consisted of reading one of four short (one-half to two page) passages on partial fractions, algebraic vectors, absolute value, or symmetry. Students at this level are unlikely to be familiar with readings on these topics, but will normally find them accessible. After reading the short passage, each student was asked to complete a task, based on the passage read. In addition to being used diagnostically in teaching, these interviews served to familiarize the 11 subsequent volunteers with the interview procedures that they would experience later.

The Conduct of the Study

During the sixth and seventh weeks of the courses, the volunteers each selected a 90-minute time slot during which they were asked to read aloud a new section selected by the instructor/researcher from their respective textbooks. These passages were selected because the students would be familiar with the notations and prior definitions used in their respective textbooks and because the students were judged to have the necessary prerequisites for reading them. The five pre-calculus students, and the one calculus student who had dropped the course, read the section entitled “The Wrapping Function” in Barnett, et al. (2000, pp. 336-343). The five calculus students read the section entitled “Extrema on an Interval” in Larson et al. (2002, pp. 160-164). Along with definitions, theorems, examples, figures, and discussions, the precalculus book has “Explore/Discuss” and the calculus book has “Exploration” tasks to encourage students to become active as they read.

The students were stopped at intervals during their reading and asked to try a task based on what they had just read, or asked to try to work a textbook example (task) without first looking at the provided solution. These were the places that the textbook authors would probably have assumed they would independently pause for such activities (see Introduction). The precalculus students were stopped an average of three times (a maximum of four times, a minimum of three times). The calculus students were stopped an average of eight times (a maximum of nine times, a minimum of seven times). The tasks were straightforward ones based directly on the reading and required very little in the way of problem-solving skills. They were what might be called "routine exercises." For instance, after the textbook had defined the wrapping function, *W*, and had explained the calculation of the exact values for $W(0), W(\pi/2), W(\pi)$, and $W(3\pi/2)$, the routine exercise given was: *Find the coordinates of the circular point* $W(-\pi/2)$.

The reading passages along with the interruptions and requested tasks appear in Appendix A for precalculus and in Appendix B for calculus. For example, for the reading from Barnett et al. (2000), the precalculus students were asked to find the coordinates of a circular point, that is, a point such as $W(\pi/2)$ on the unit circle, given by the wrapping function, *W*. From Larson et al. (2002), the calculus students were asked to determine from a graph whether a function had a minimum on a specified open interval. After the entire section had been read and a few final tasks were attempted, the students were questioned about how reading during the interview differed from their normal reading of their mathematics textbooks (Appendix C).

All interviews were audio-recorded and transcribed. The interviewer also made notes during the interviews. The written work produced by the students during the interview was collected. The first author listened to the recordings carefully at least three times, making additional notes. These additional notes, along with the notes taken during the interview and the students' written work, were compared with the transcripts to create Tables 1 and 2 below. The number and kind of CRR-based strategies used (Table 1) and the kinds of mathematical difficulties the students incurred were noted. The number of tasks attempted, done correctly, incorrectly, or not done, by each student was noted (Table 2).

DATA COLLECTION AND OBSERVATIONS

Use of the CRR-Based Strategies

Table 1 indicates that, for the most part, the students were employing the CRR-based strategies characteristic of good readers. It confirms that they were, in general, good readers, as indicated by their ACT reading scores. In Table 1, for each of the eight CRR-based strategies, we provide examples of observed behaviors, together with the number of students exhibiting those behaviors. For example, six students read the title of the section, the introduction, or the caption at the start of their reading and were judged to have employed CRR-based Strategy 1. In another example, Christie[7] made the following comment after reading the definition of the wrapping function. "So, to me it sounds like that they have a circle at […] and it has to have [a] start at this point and there's going to be a line going around it and we're going to find the points on that line."[8] This was coded as a paraphrase in CRR-based Strategy 7. However, some additional good reading strategies might have been present without having been observed.

TABLE 1: Observed CRR-Based Student Strategies

CRR-based strategies (shortened)	Number of students observed	CRR-based strategies, along with examples of observed behaviors
1. Preview text to be read before reading to gain overview and to make predictions about it.	6	• **Preview text to be read.** a. Read the title, the introduction, or the caption at the start of the reading.
2. Pay greater attention to information perceived as most important.	11 3 2	• **Look for and pay attention to material perceived as more important.** a. Reading selectively, slowing down, pausing and rereading sentences. b. Specifically stated something like, "This must be important." c. In questioning at the end, reported only looking at the examples.
3. Allow interaction of prior knowledge and text to interpret text, construct meaning and revise/adjust prior knowledge.	7 11 6 7 5 None	• **Attempt to relate important points in text to one another.** a. Tried to relate a point in the current reading to earlier points. b. Looked at tables or went back in the reading to reread previous parts. • **Activate and use prior knowledge to interpret text.** a. Students did not activate prior knowledge before reading but were observed recalling things learned in previous mathematics courses while reading. • **Relate text content to prior knowledge.** a. Specifically related what they read to something in their prior knowledge. • **Reconsider or revise hypotheses about meaning of text.** a. Showed that they had revised their understanding of the text by the end of the reading. • **Reconsider or revise prior knowledge based on text.** a. There were no overt observations of the changing of prior knowledge, however this does not mean students did, or did not, do this.
4. Make inferences about information not explicitly stated.	11 1	• Infer information not explicitly stated. a. Tried to fill in details and give reasons while reading the examples. b. Filled in a reason incorrectly and subsequently corrected his reasoning.
5. Determine meanings of new/unfamiliar words	11	• **Determine meaning of new words.** a. Recognized when something was not understood and many tried different strategies, such as rereading definitions or paraphrasing, hoping to determine some meaning.
6. Monitor comprehension and change reading strategies if needed.	3	• **Change reading strategies when comprehension is not occurring.** a. Stated they would "go ask for help."
7. Evaluate text, remember it and reflect on it.	11 4 1 3 11 7 11 3 3	• **Use strategies to remember text.** a. Repeated or reread parts of passage. b. Wrote notes or copied important ideas onto paper. c. Seemed to create a concrete visualization of a concept; for instance, comparing the wrapping function to a ribbon. d. Constructed analogies, identifying the u-v coordinate system as the x-y coordinate system. e. Paraphrased, though, not always correctly. • **Evaluate the qualities of text.** a. Several students had specific comments about the text in the debriefing (see Appendix B) related to the appropriateness of examples, the clarity of the author, etc. • **Reflect on text after text has been read.** a. Gave some indication of reflecting on the text while reading. b. Specifically recognized some unresolved understanding at the end of the reading. • **Carry on responsive conversation with the author.** a. Several students commented on "what the book wants" while reading or working examples.
8. Anticipate the use of knowledge gained from the reading.	2	• **Anticipate how the reading would be used in an application.**

Difficulties in Working Tasks

All of the students in our study had considerable difficulty correctly completing some of the straightforward tasks based on their reading. The percent of tasks done correctly by individual students ranged from 13% to 76%. Five of the six students who read the precalculus passage did not find correct values of the wrapping function, *W*, in two or more instances. Also, four of the five students who read the calculus passage containing the definition of extrema of a function on an interval could not determine from its graph whether it had a minimum. Ten of the students stated at some point that they did not understand something, but made no attempt to understand whatever was causing confusion. Five students, three precalculus and two calculus, gave up at some point. They stated that they had no idea what to do either while trying to work a task or when reading through a worked example. When questioned, one calculus student stated she would just move on, the other four stated they would quit and ask for help before continuing. However, they continued to read at the request of the interviewer.

TABLE 2: Correctness of Tasks and Number of CRR-Based Strategies Observed

Student	# tasks attempted	# Correct (% correct)	# Incorrect	# not done (skipped or gave up)	Incomplete	Read/ not worked	Read as worked	"correct" w/wrong reasons	# CRR-based strategies observed
Precalculus									
Alicia *	19	9 (47%)	5	5					7
Bryan	18	9 (50%)	4	2	1	2			8
Christie	21	3 (14%)	7	7	1	2		1	10
Darcy	8	1 (13%)	2	2	1	2			12
Ellis	17	13 (76%)	2	1	1				11
Faye	20	6 (30%)	6	7		1			8
Calculus									
Tara	22	8 (36%)	2	2	4	5		1	9
Vannie	22	12.5 (57%)	2.5	1		2	4		11
Winnie	22	10.5 (48%)	1.5	3	1		6		11
Yates	22	8 (36%)	4	2	2	1	5		10
Zoe	23	8.5 (38%)	6.5	1	1	1	5		8

*All students' names are pseudonyms.

There appears to be little or no relationship between the number of good reading strategies observed and the percent of correctly performed tasks (Table 2).

Understanding and Using Definitions. In mathematical writing, it is intended that everyone who reads the definition of a concept with comprehension will have essentially the same basic understanding of the definition. Different individuals' concept images need not agree, but everyone should be able to agree on whether or not an example satisfies the concept's definition.

For the calculus students, one difficulty appeared to come from an inadequate concept image of the word "function." After reading the definition of extrema (Appendix B), Vannie was asked to look at the graphs of eight functions and determine whether they had minimum values. As she looked at the graph of the first one, 51a, a function with a jump discontinuity, she went back to the definition and tried to compare the graphical information with the definition.

> "You're on the interval *I* as they designate. You're supposed to look at […] Is it *c* or *x* they use? … For all the *x*'s, $f(c)$ is supposed to be your minimum point. Well, $f(c)$ on this portion is your minimum point, is a real number, but on this one it is not because it is open. So, if you look at it from [...] since it's totally two different things coming in. I don't know if you say well this one does have a minimum and this one doesn't or if they go together, then they don't. I don't [...] that part I [...] I'm not clear on."

Vannie came to no resolution. Although she clearly tried to use the definition of extrema, she did not appear to recognize the graph of a function with a jump discontinuity as representing a single function.

A second difficulty apparently arose from not rereading a definition. Christie read about the wrapping function and how to calculate its values for integer multiples of $\pi/2$, orally answered two worked examples incorrectly (with the work hidden), and then read their solutions. Next she tried to answer the first matched problem, *Find the coordinates of* $W(-\pi)$. She said, "It's going to be (1,0) because you're going . . . up π every time, every quarter of a circle…. So if we just start at the top [(0,1)] and then go down one π, I think we'd be at (1,0)." Not only did Christie start wrapping at the wrong point, but she also did not know that the measure of a quarter circle is $\pi/2$ and that positive angles are measured in the counterclockwise direction. She had not gone back to the definition to check her starting point. Somewhat later in her reading, she did discover that the starting point was (1,0) instead of (0,1). However, at the end of the interview, when asked if there was any notation that had bothered her, she was still confused about the starting point. She said, "And I still don't [...] I mean they still start you at the *v*-axis sometimes, and they start you at the *u*-axis sometimes, I think. So, I'm not real sure on that aspect of it."

A third difficulty related to definitions apparently came from not distinguishing between definitions with similar wording, such as relative extrema versus absolute extrema. One of the tasks given the calculus readers included the directions, *Determine whether the function has a relative maximum, relative minimum, absolute maximum, absolute minimum, or none of these on the interval shown.* (Larson, et al., 2002, p. 165) Zoe worked through the exercise, looked up the definition of extrema on an interval which included absolute extrema, but not relative extrema (Appendix B). In the debriefing (Appendix C), she was asked if there were any words that had bothered her. From her comments, one can see that Zoe had not distinguished between definitions of related concepts.

> "It said to find any relative minimum, relative maximum, absolute minimum, and absolute maximum. But in the first of it [definition of extrema], they said that those are the same things. So I wasn't quite sure why they were asking me to find possibly four different things if they're supposed to be just the same thing, but synonyms. … Since I didn't know, I just went under the assumption that they're the same thing."

Using Theorems. The students in this study also had difficulties related to theorems encountered in the textbook. Some students could not assign the correct authority to a theorem and some had difficulties understanding the implications of a theorem. The calculus students read the Extreme Value Theorem, followed by an Exploration (Appendix B). They were then asked to answer a true/false question: *If a function is continuous on a closed interval, then it must have a minimum on the interval.* One student, Tara, answered the true/false question correctly and correctly gave the Extreme Value Theorem as the reason.

Vannie tried to use the definition of Extrema on an Interval, but could not use it to answer the true/false question. She consulted the definition of Extrema on an Interval, and then "pled the fifth" because she did not know the answer. Vannie seemed to understand that the definition was not enough, but did not know where else to look, even though she had just read the Extreme Value Theorem.

Two errors combined when Zoe seemed to rely on a visual part of her concept image that incorrectly indicated that a horizontal line had no extrema, and also confused an implication with its converse. Zoe justified her incorrect answer with an example that seemed to show visual reasoning instead of using the Extreme Value Theorem. She said, "That's false, … it's not continuous. [...] *If a function's continuous on a closed interval*, [...] Well, … I can't think of an example. If it were a [horizontal] line it wouldn't necessarily have a minimum. I guess that will be my example." She seemed unaware that her answer was incorrect.

Consideration of Examples. In the interviews, students were asked to solve each worked example (task) with the textbook solution covered with a Post-it© note. Most of the precalculus readers could work through most of these examples without looking at the solutions. These worked examples required only one or two reasoning steps. After working the example themselves, some students read the textbook's solution thoroughly, while others only skimmed it. One precalculus student, Christie, did not write down her answers so upon checking the solution, was unaware of errors she had made. Two precalculus students, Faye and Darcy, had difficulties with fractions, such as recognizing that $6\pi/2 = 3\pi$.

None of the calculus readers was able to complete the final three worked examples (tasks) in their textbook passage, without looking at the solutions provided or comparing their work with that of the book. These worked examples concerned finding the extrema of a trigonometric function and estimating extrema for graphically presented functions. The calculus textbook provides procedural knowledge in the form of a list of steps to follow to find extrema on a closed interval. Although the calculus students tried to follow these steps, three of them had difficulty with algebraic concepts (negative exponents, factoring, trigonometric identities), and all of them gave up trying to figure out the trigonometric example.

Although they continued to read for the interview, two calculus students stated they would normally give up before reaching the final example. Vannie indicated she would ask her group for help before continuing, and Tara indicated she would ask the teacher about the example in the next class period. Another difficulty occurred when students did seem to not pay close attention to relevant definitions as they worked examples.

Another difficulty occurred when one precalculus student, Faye, focused on the development of the wrapping function and tried to derive its values for multiples of $\pi/4$

directly, rather than using symmetry as suggested by the textbook. Of the five precalculus students who read this passage, she was the only one who did not use symmetry. Faye seemed very interested in showing she could derive the values directly. Faye first read the algebraic development of the coordinates of $W(\pi/4)$. Then, just before an example to work, she read: "*Using the symmetry properties of a circle, the unit circle is symmetric with respect to both axes* [She repeated this phrase.] *and the origin, we can easily find the coordinates of any circular point that is reflected across the vertical axis, horizontal axis, or origin from* $W(\pi/4)$*.*" Faye then read the directions to Example 2:

> "*Find the coordinates of the circular points* A. $W(5\pi/4)$ *and* B. $W(-\pi/4)$. …
> Let's see, one, two, three, four, five […] I don't think so […]There's nothing for me to count. [...] There's no axis there […] I don't know if … my counting would be equal … I didn't know, if I would … count like, in one ...
> I don't know, just because there's nothing to count on."

She started to rederive by writing $a^2 + b^2 = 1$, then read the solution to part A, mostly silently. She did some deriving, but the answer she wrote was $(-1/\sqrt{2}, 1/\sqrt{2})$ which is incorrect. She read the solution to part B concerning $W(-\pi/4)$.

> "*… (1,0) we proceed one-eighth the way around the unit circle in a clockwise direction…the fourth quadrant*…wait … that's right…*on the circle halfway between (0,-1) and (1,0) as indicated in Figure 6* [in the textbook]"

followed by silence and low whispering. She rederived the values during this silence. "That works. OK." She had written the answer to Example 2B as $(-1/\sqrt{2}, -1/\sqrt{2})$, which is incorrect. When Faye next tried the matched exercise 2A, which was her third attempt to calculate one of these values, she apparently *did* use symmetry to find her answer. At the end of the interview, when Faye tried to find the value of the wrapping function at $\pi/6$, she correctly rederived the wrapping function at $\pi/4$, instead of finding W($\pi/6$).

The students, particularly those in the calculus group, seemed to find it difficult to work the examples and reconcile their work with that shown in the textbook. At the end of the interview, Winnie said, "A lot of the times their examples are the easier problems and then the ones you see in the lesson are […] (shrugging)."

Perhaps not surprisingly, the two students with the lowest ACT mathematics scores 16 and 20 (23^{rd} and 55^{th} percentile, respectively), Tara and Vannie, had great difficulty completing the required algebra and in explaining the solutions given in the calculus textbook. Their incomplete prior knowledge of algebra caused them difficulties.

In particular, Vannie's incomplete prior knowledge of negative fractional exponents caused her to become frustrated and give up attempting to understand a calculation. She tried to work Example 4 that asked the reader to find the extrema of $f(x) = 2x - 3x^{2/3}$ on the interval [-1,3]. Vannie attempted to take the derivative and set it equal to zero. She incorrectly wrote $f'(x) = 2x - 2x^{-1/3} = 0$. At this point she checked the solution to confirm her derivative and said,

> "They did something crazy. Ok. What did they do? […] I'm confused. . . . I don't understand their math or their […] what they did. … I figured it was just a basic [...] you did the derivatives in the subtraction."

She eventually fixed her derivative but still could not get the form of the derivative shown in the textbook, which was $f'(x) = 2 - \frac{2}{x^{1/3}} = 2\left(\frac{x^{1/3} - 1}{x^{1/3}}\right)$. The negative exponent confused her, even though she had tutored college algebra in the past. Her final comment after reading through the entire solution was, "At this point, if I was really reading this I would be frustrated and quit and then I would go ask somebody." Vannie did not seem to realize that incomplete prior knowledge was a component of this difficulty and did not attempt to do anything about it herself.

Explorations. A feature of both textbooks in this study is a more open-ended type of task, called an "Explore/Discuss" or an "Exploration" task, where no guidance is given and an explanation may be required. The precalculus students were given the option of doing the two Explore/Discuss tasks (Appendix A), and the calculus students were given the option of attempting the Exploration which had two parts (Appendix B). Most of the precalculus students chose not to do the Explore/Discuss tasks either with comments such as, "I don't understand what they want me to do," from Christie, or "They might think that's an effective memory aid but that's confusing me so I'm moving on" from Ellis.

Four of the five calculus students chose to do the first part of the Exploration, and one, Tara, also did the second part. Tara came to a wrong conclusion on the first part, saying that there was no maximum for a quadratic function on a closed interval. She said, "I think it's infinity because the graph keeps going and I can't see any point." On the second part with a cubic on the same closed interval, she said, "I think the minimum and maximum of both of these is infinity since I can't find an ending point on either one of them." The fact that she had read the Extreme Value Theorem just prior to this did not lead her to see a conflict between the theorem and her answers. However, as noted above, Tara answered correctly, with a correct reason, the true/false question posed immediately after this, *If a function is continuous on a closed interval, then it must have a minimum on the interval.* Zoe chose not to do the Exploration because "…that's not going to help me."

Reading the Exposition. All students read the expository parts of the textbook since that was part of the interview, but upon questioning at the end, some students viewed exposition as of minor importance -- something often to be skipped or skimmed. Students wanted to concentrate on problems and find worked examples similar to the exercises given in the text, and often ignored the exposition that tied together conceptual and procedural knowledge.

Some of the student comments included: "I learn by examples."—Winnie. "Sometimes it's just jibberish. But stuff that they mean to attempt to stand out, then I read that. But usually, at the beginning of the chapter, I try not to read. I just read the definition because otherwise it's just confusing."—Zoe. "It takes quite a while to read through [the section] like that, too, maybe an hour, hour and a half."—Yates.

Students' Difficulties and the Unusual Style of Mathematical Writing

We pointed out a number of ways that reading mathematical writing can differ from reading other text, and such differences might in some situations contribute to ineffective reading. However, most of these differences did not occur in the passages our students read, and what differences were there did not often cause our students to stumble in reading. For example, they could read equations and the notations for functions, intervals, and points without difficulty.

Summary of Observations

According to our eight CRR-based strategies, our students exhibited the characteristics of good readers, in agreement with their relatively high ACT reading scores. Also their relatively high ACT mathematics scores indicated they were good at mathematics.

However, all of our students had considerable difficulties completing many of the straightforward tasks based on the readings. The difficulties occurred across a wide spectrum and were associated with tasks involving definitions, theorems, examples, and explorations. Only the expository passages did not produce difficulties, although many students regarded them as of little importance. There appeared to be little relationship between the percent of students' correctly performed tasks and the number of good reading strategies they employed (Table 2). There also appeared to be little connection between the students' difficulties and the writing style of mathematical textbooks such as unusual symbolism, syntax, or treatment of definitions.

It is perhaps a main purpose of this kind of textbook that readers should be able to reliably work such tasks, or similar tasks, to demonstrate their understanding, and in support of later understanding of more complex tasks. However, only three of our eleven students (Bryan, Ellis, and Vannie) could work at least half of the tasks, and only one of these, Ellis, could work three-fourths of them (Table 2). Furthermore, our judgment agrees with the students' own views, that is, they believe they do not benefit from reading major parts of their mathematics textbooks, and often avoid doing so.

In short, our students were good at mathematics and reading, applied their reading skills to these texts, and did not find the unusual style of mathematical writing very burdensome, but still could not read their textbooks effectively. This apparent anomaly suggests another anomaly. It is common knowledge among mathematicians that some of them occasionally teach a course in order to learn the topic, which requires the very reading skills our students lacked. Of course, it should not be surprising that mathematicians can benefit from reading a textbook more than students. What is remarkable is that mathematicians appear to have developed what can be seen as a major, complex skill without having noticed much, if any, of their learning processes. Thus, to more fully understand our observations, it might be helpful to look for an additional perspective from which to view them. In the next section, we consult the psychological literature.

ANALYSIS AND AN ADDITIONAL PERSPECTIVE

We have examined our students' reading from the perspective of the writing style of their mathematics textbooks; however, there is another significant aspect of the textbooks to consider.

The Integration of Conceptual and Procedural Knowledge

Much of the content in our students' textbooks is a close integration of conceptual knowledge with corresponding procedural knowledge, introduced through worked examples (tasks) immediately following the conceptual knowledge. Both the conceptual knowledge and the tasks often call on students' assumed prior knowledge, and often the procedures needed to work the tasks are explicitly described.

For example, for the precalculus students the description of the concept of the wrapping function, *W*, calls on assumed prior knowledge including an understanding of the real number line, the rectangular coordinate system, the unit circle, and the concept of function. Then calculation of *W*(*x*), for various real numbers *x*, is illustrated by giving steps sufficient to determine that $W(\pi/2) = (0,1)$. These steps include recalling that the circumference of the unit circle is 2π, that $\pi/2$ is one-fourth of 2π, and that starting at (1,0) and moving counterclockwise one-fourth of the way around the circle arrives at (0,1). Then the student is asked to calculate other values, such as $W(-\frac{\pi}{2})$. (See Appendix A.)

For the calculus students, the definition of the concepts of minimum and maximum of a function on an interval calls on assumed prior conceptual knowledge including an understanding of the concepts of function, interval of real numbers, and the usual order relation on the real numbers. Finding minimums and maximums is immediately illustrated in Figure 3.1. Shortly thereafter students were invited to work similar tasks in Exercises 51-54. (See Appendix B.) These tasks present graphs of functions in the first quadrant and ask for minimums, if any. Procedural methods are not explicitly provided for these tasks, but some students may have sufficient experience to just "see" the minimum or note that there is none. A student without such experience would need to recall that *f*(*x*) is the length of the vertical line segment from the *x*-axis to the graph of *f*, after which trial and error might be used to identify a minimum or suggest that there is none.

From a constructivist perspective, when a concept is introduced in a textbook as above, a reader's own subsequent development of procedural knowledge associated with tasks should help that reader construct a corresponding (inner) conception.[9] Indeed, after some time, the reader's conception may owe as much to the reader's induction from working tasks, as it does to the original textual representation of the concept. Also, calling on his or her own prior knowledge while working tasks should help the reader integrate the new conception into his or her existing knowledge base. Finally, the results of attempting to work the tasks should provide the reader with evidence for, or against, the developing conception's viability (von Glasersfeld, 1995, pp. 68-69), as well as evidence as to whether that conception can be taken-as-shared (Cobb, Yackel, & Wood, 1990), relative to the textbook author's conception.

Thus, working tasks, along with the associated procedural knowledge, is useful, perhaps even necessary, in the construction of mathematical conceptions. The textbooks' close integration of conceptual with corresponding procedural knowledge, and the textbooks' authors expectation that students will work tasks as they proceed in their reading make it reasonable for us to use students' working of tasks to judge their reading effectiveness.

Reading Effectiveness

We have also examined our students' reading from the perspective of their reading comprehension strategies (Table 1). However, reading comprehension appears to differ from what we are calling *reading effectiveness*.

In undergraduate mathematics teaching, it is common to assess students' knowledge (including their understanding and comprehension) through their demonstrated ability to use that knowledge – typically in working tasks. The working of tasks is guided by a student's related (inner) procedural knowledge and often involves activities such as solving an equation or proving a theorem, rather than, say, merely stating a definition. Perhaps tasks are used in assessments because it is widely recognized that the ability to state a theorem or definition, or even describe a procedure, does not imply the ability to actually use it properly. In this study, Tara could correctly answer a true-false question about the Extreme Value Theorem, but could not use the theorem in an exploration.

In judging the *effectiveness* of our students' reading, we followed the usual practice of teachers of first-year university courses, and examined students' ability to work straightforward tasks. That is, we asked them to work tasks that had typically just been explained or illustrated, but that did not involve many steps or long sequences of inferences or much logic beyond common sense. The working of the tasks should have been facilitated by our students' textbooks that link conceptual knowledge to procedural knowledge that is intended for use in working the subsequent tasks.

A Comparison of what the ACT Tests Measure with Effective Reading

Our students were good readers in the sense that they did well on the ACT general reading comprehension test, as well as on the Social Studies/Science and the Arts/Literature parts of the test. The test calls on students' reasoning skills to: determine main ideas, find significant details, understand sequences, make comparisons, understand cause-effect relationships, understand context-dependent words, draw generalizations, and understand an author's voice or method (ACT Reading Test Description, 2010). However, the test does *not* emphasize students' working tasks that are dependent on newly read procedural knowledge, which in turn depends on newly read conceptual knowledge. Thus, while the skills indicated by a good score on the ACT reading comprehension test might be helpful in effective reading of mathematics textbooks, there seems to be no reason to suppose they are sufficient.

Our students were also good at mathematics in the sense that they did well on the ACT mathematics test. This test is a 60-question, 60-minute test designed to measure the mathematical skills students have typically acquired in courses taken by the end of 11th grade (ACT Mathematics Test Description, 2010). The test *does* ask students to work tasks. However, those tasks call on well integrated prior knowledge and practiced procedures gained through instruction, *not* on recently read conceptual knowledge and procedures. Thus, while the knowledge and skills indicated by a good score on the ACT mathematics test are likely to be helpful in effective reading of mathematics textbooks, again there seems to be no reason to suppose they are sufficient.

Now we are in a position to explain the apparent anomaly mentioned earlier. The ACT tests of reading and mathematics do *not* test the kinds of skills sufficient to yield effective reading of mathematics textbooks. However, this alone does not explain why our students made many errors in working recently explained, straightforward tasks. An understanding of the sources of such errors might be useful to a teacher or a researcher in helping students become more effective readers of their mathematics textbooks.

Possible Sources of Error

We have described our students' difficulties in working tasks, including a number of errors. Some errors arose unavoidably from a student's inadequate or incorrect prior knowledge. For example, this was the case for Vannie, whose inadequate conception of function apparently prevented her from recognizing the graph in Problem 51*a* as that of a single function with one jump discontinuity. But there were other kinds of errors, where the source is less clear. For example, in attempting to find $W(-\pi)$, Christie started the wrapping function at the wrong point, (0,1). This error could not have been a matter of incorrect prior knowledge, as *W* had just been introduced.

We described difficulties that we observed in terms of mathematics, but their origins might also be described in psychological terms. In order to work a task, our students must have combined an inner model,[10] that they had just constructed from externally presented procedural knowledge, with their own prior knowledge, to guide their actions. We suspect it is not uncommon for such inner models to have incorrect or incomplete parts for psychological reasons. What are some possible psychological sources of student errors? A definitive answer to this question is beyond the scope of this study. While our observations are adequate for detecting errors in working tasks, they indicate little of the psychological sources of such errors. We will, however, suggest a perspective through which some psychological sources of error might be understood and investigated.

Cognitive gaps. The external representation of mathematical knowledge, such as a definition or a description of how a task can be worked, often consists of a number of smaller parts that fit together in a specific way. In constructing an inner model of the external representation, a student needs to maintain a sustained focus on that representation long enough to notice and comprehend all of its smaller parts and their interrelations. Apparently maintaining such a sustained focus is more difficult than might be thought, and *cognitive gaps,* that is, periods of lapsed or diminished focus can occur. A period of lapsed focus may result in a part of an external representation being missing from a student's inner model. However, where coherence demands it, that missing part may be replaced with some plausible, but possibly incorrect, information.[11] Also, a period of diminished focus may correspond to a part of an external representation containing fine distinctions, and some of those fine distinctions may be missing from a student's inner model.

The idea of cognitive gaps and their consequences for inner knowledge construction from reading, and for working tasks, appears to be consistent with our students' ineffective textbook reading. Below we give three examples from our students' reading to illustrate this. We emphasize that we have not observed such gaps, but only that they are consistent with some of our students' difficulties.

First, Christie seems to have missed the fact that one starts the wrapping function at (1,0), and from there, measures around the circle in the counterclockwise direction. This is consistent with having a lapse in focus while she was reading that part of the passage on the winding number. Since she must start wrapping somewhere on the circle, it would be plausible for her to follow the convention of clocks, and start at the "top," that is, at (1,0) and move clockwise, which is what she did. She was acting as if her inner model, constructed from the reading, included this incorrect information.

Second, from Zoe's debriefing comments, it is clear that she did not distinguish between relative and absolute extrema, as she incorrectly asserted, that the textbook said they were the same. But she had read the definition of absolute extrema, which does not include relative extrema. Somewhat later she read the definition of relative extrema (in terms of absolute extrema). Then she consulted the definition of absolute extrema again while working a task asking for any relative minimum, relative maximum, absolute minimum, and absolute maximum, which she did incorrectly. This suggests she had diminished or lapsed focus at least while reading the definition of relative extrema.

Third, Vannie, having recently read the Extreme Value Theorem, could not answer the true/false task: *If a function is continuous on a closed interval, then it must have a minimum on the interval*. Except for omitting the "*f*" and "[*a,b*]" and substituting the words "a minimum" for "a minimum and a maximum," this task is very close to the textbook's statement of the theorem. Since the logic required for this task is not much beyond common sense, and since readers do not normally forget the existence of what they have read (and comprehended) only one paragraph ago, we think that Vannie may have had a lapse in focus when reading the theorem.

Probably cognitive gaps can occur for a number of reasons, including tiredness or insufficient sleep, and may not always be observable. In addition, there is research from psychology that supports the existence of cognitive gaps for other reasons. Both the work on mind wandering during reading (Schooler, Reichle, & Halpern, 2004; Smallwood, Fishman, & Schooler, 2007; Smallwood & Schooler, 2006) and on sensory awareness (Hurlburt, Heavey, & Bensaheb, 2009) suggest that there are likely to be occasional gaps in many people's sustained focus. Much of that research has been described in terms of the content of the gaps, that is, in terms of what *is* being focused upon (during the time of the gaps), rather than in terms of the gaps' existence, that is, on what is *not* being focused upon, and the resulting consequences.

Mind wandering during reading. Psychological research related to mind wandering has been conducted under several different constructs, such as task-unrelated thought, task-unrelated images and thoughts (TUITs), stimulus-independent thought, mind pops, and zone-outs (Smallwood & Schooler, 2006). While many of the experiments have considered mind wandering when subjects were given dull tasks, such as signal detection, to perform, a few have considered mind wandering (or zoning out) during reading. Schooler, Reichle, and Halpern (2004) report two such experiments that specifically checked participants' zoning out during reading.

In the first experiment, 45 participants were to self report zoning out, and were also probed randomly every 2- 4 minutes. Participants were familiar with the concept of zoning out, described to them as having "no idea what [they had] just read" and "not really thinking about the text, but … of something else altogether." They read parts of the opening chapters of *War and Peace* for 45 minutes. On average, participants caught themselves zoning out 5.4 times. On approximately 67% of the zone-out responses, participants indicated that they believed they had not been aware that they had been zoning out. When asked, they reported thinking about such things as school-related topics (27% of the time), fantasies (19% of the time), nothing at all (18% of the time) and themselves (11% of the time). Though participants were often unaware that they were zoning out, their minds were occupied “with rich thoughts that were completely unrelated to what they were reading” (Schooler, Reichle, & Halpern, 2004, p. 210).

The second experiment replicated those of the first, but in addition, when compared on text recognition performance, participants who were zoning out had “lower comprehension levels than the baseline performance of those participants who were randomly given text recognition probes. … This finding provides behavioral evidence consistent with the claim that zoning-out episodes are associated with particularly low levels of attention to the [meaning of the] text” (Schooler, Reichle, & Halpern, 2004, p. 212).[12]

A third experiment reported by Smallwood, Fishman and Schooler (2007) used two types of probes to determine the effect of mind wandering on reading comprehension: inference-critical probes and random probes. Because readers need to build a coherent narrative, or situational model, of what they read, the experimenters had participants read Sherlock Holmes’ detective story, “The Red-Headed League,” to determine whether they could identify the villain. The story has four inference-critical events and some participants received probes at junctures in the text that reveal a fact critical to subsequently identifying the villain. Participants whose minds were wandering when they received these inference-critical probes were less able to identify the name of the villain than those who received random probes, suggesting that “even brief failures in attention, if they occur at critical junctures in the narrative, can impair an individual’s ability to create a model of the discourse, leading to downstream failures … ” (Smallwood, Fishman & Schooler, 2007, p. 233). Similarly, since every step in executing a procedure is critical, a lapse in focus when learning, or executing, one step is likely to lead to failure to get a correct answer. In addition, “Successful learning requires that individuals integrate information from the external environment with their own internal representations. … Because mind wandering is a state of decoupled attention, it represents a fundamental breakdown in the individual’s ability to attend (and therefore integrate) information from the external environment.” (Smallwood, Fishman, & Schooler, 2007, p. 230).

Sensory awareness. This is the direct, primary focus on some sensory aspect of the body or inner or outer environment, without regard to its instrumental purpose. It is not some aspect of perception, but a central feature of one’s experience, and a completed phenomenon in its own right (Hurlburt, Heavey, & Bensaheb, 2009, pp. 239 & 245).

The prevalence of sensory awareness (Hurlburt, Heavy, & Bensaheb, 2009) was detected using the Descriptive Experience Sampling (DES) method in order to investigate the momentary conscious, inner contents of a person’s mind in natural environments (Hurlburt, 1997; Hurlburt & Schwitzgebel, 2007). In this method, a subject carries a beeper that sounds randomly, through an earphone, about six times per day. Subjects are to attend to their experiences at the last undisturbed moment before the beep and to immediately jot down notes on it. Within 24 hours, the subjects are interviewed, and this sampling and interview procedure is typically repeated on two more days. According to Hurlburt, Heavey, and Bensaheb (2009), sensory awareness is just one of the five most common inner experiences; the others are inner speech, inner seeing, feeling, and unsymbolized thinking.

It appears that during an experience of sensory awareness, a person can carry on an unrelated activity, such as engaging in a conversation or dialing a phone, especially if that activity has been more-or-less automated. However, that person's primary focus is upon the prevailing experience of sensory awareness. The phenomenon of sensory awareness is perhaps best understood through two examples given by Hurlburt, Heavey, and Bensaheb (2009, pp. 231-232).

The first example shows how one can "tune out" to what another person is saying. Betty was in a conversation with her friend Wendy. At the moment of the beep, she was taking a sip of the drink, Dr. Pepper. She was drawn to the coldness of the liquid as it moved down her throat. Wendy continued to talk, but Wendy's voice was not part of Betty's (main) conscious experience. She was focused on the coldness in her throat. The second example shows that one can carry on an unrelated activity, perhaps automatically, while having an experience of sensory awareness. Andrew was dialing his cell phone. At the moment of the beep, he was "zeroed in" on the shiny blueness of the aluminum phone case. He was not paying attention to the number he was dialing. His conscious experience had momentarily left the task, which continued as if on autopilot.

According to Hurlburt (1997), sensory awareness is a common occurrence and most subjects exhibit or report embarrassment when describing such experiences. Also, without the stimulus of the beeper and the note taking, experiences of sensory awareness tend be forgotten immediately. Indeed, some subjects who reported experiences of sensory awareness using the DES method were previously completely unaware that they had such experiences. Among 30 students, representative of entering students in a large U.S. university, Heavey & Hurlburt (2008) found that sensory awareness occurred in 22% of sampled experiences.

We suggest that experiences of sensory awareness while reading might have implications for students' reading difficulties. While attempting to construct an inner model of an outer multi-part representation of mathematical knowledge, experiences of sensory awareness are likely to lead to cognitive gaps, and hence, to errors in working associated mathematical tasks. Furthermore, the contents of sensory awareness and mind wandering tend to be rather different, suggesting that taken together the rate of occurrence of gaps in sustained focus that are due to these two phenomena is likely to be more than either of their separate rates of occurrence.

Cautious Reading

The existence of mind wandering and experiences of sensory awareness, especially where unrecognized, suggest that in an initial reading some students may not be able to entirely avoid errors and omissions in the inner models they construct of the externally represented knowledge in their mathematics textbooks. Thus they could not entirely avoid errors in associated tasks. But this applies to mathematicians' reading of a similar kind of text on an unfamiliar topic, so it is reasonable to expect that they, too, would occasionally construct imperfect inner models of what they have just read, and hence make errors in working associated tasks. However, mathematicians are generally very effective readers of mathematics, so much so that they sometimes successfully teach undergraduate courses that they have never studied. Furthermore, many mathematicians seem to have acquired this ability without noticing much about how they acquired it.

We suggest that this apparent anomaly can be resolved not so much by comparing the errors and confusions in learning from reading made by mathematicians with those made by students, but by comparing their respective reactions to those errors and confusions. Most of our students appeared to be strikingly unconcerned about their errors or lack of understanding and did not seem to believe they could have independently done anything about it. Ten of the students stated they did not understand something during the interviews, but made no attempt to determine what was causing their confusion, and five gave up at some point.

In contrast, our experience as mathematicians suggests that most mathematicians read the above kind of material in what might be called a very cautious way. They tend to be sensitive to, and to look for, hints of their own misunderstanding. They work, and evaluate their performance on, tasks provided by the author, and even occasionally invent and work additional tasks. When they find an error or confusion, mathematicians are likely to reread the associated passage and rework the task until the error is corrected or the confusion is resolved. This suggests that, in contrast to our students, mathematicians have come to believe they can (autonomously) succeed in understanding mathematical texts through such reworking. Finally, we further suspect that this kind of careful, usually slow, reading is based on the (perhaps tacit) meta-mathematical knowledge that in mathematics neglected small errors are likely to lead to significant later errors, and that one's own reading can occasionally generate such small errors.

The above combination of sensitivity, belief, and meta-mathematical knowledge that contributes to cautious reading was probably learned tacitly and inductively by most mathematicians from their experiences working tasks, and conjecturing and proving theorems. That is, we suggest that mathematicians learn cautious (and effective) reading because they experience it as a necessary part of doing advanced mathematics. The characteristics of cautious reading suggested here are similar to those of the CRR-based strategies in that both are meant to be characteristics of good readers. However, their emphasis is different. The CRR-based strategies are about constructing as much knowledge as possible from text, while cautious reading is about detecting and correcting errors, misunderstandings, and confusions. We suspect that good readers, in the sense of both comprehension *and* effectiveness, will show both kinds of characteristics.

Although a more thorough examination of the nature and genesis of cautious reading is beyond the scope of this study, we can point to evidence of its early development in one of our students. Ellis was the most effective reader, as indicated by the highest percent of correct tasks (Table 2). He was the only student who, when he did not find the correct coordinates for $W(-7\pi/6)$, created his own example to make sure he had understood the calculation. Thus, some of Ellis' actions were consistent with cautious reading.

DISCUSSION AND CONCLUSIONS

After first reviewing the literature, we posed several research questions. Did our students exhibit the characteristics of good readers? What mathematical difficulties did they encounter when reading their mathematics textbooks and was the writing style burdensome? Could our students read their mathematics textbooks effectively? Is there some further perspective, different from that of the good reading strategies and of the unusual writing style of mathematical textbooks that might help explain our, and other, students' reading difficulties? And how can it be that most mathematicians apparently do not have such reading difficulties, despite having had no explicit training in avoiding, or repairing, them?

We found that our students *did* seem to exhibit many of the characteristics of good readers as given by the CRR-based strategies (Table 1). They were also not very disturbed by the writing style. However, we observed our students having considerable difficulties in working tasks over a wide spectrum of definitions, theorems, examples, and explorations. Consequently, we judged that our students could not read their mathematics textbooks effectively, as indicated by their inability to properly carry out many straightforward tasks. We then suggested an additional perspective based on their textbook's integration of conceptual and procedural knowledge. This allowed us to suggest possible sources of error, in addition to inadequate prior knowledge, namely, the existence of cognitive gaps due partly to mind wandering or sensory awareness during reading. We went on to discuss the cautious reading of mathematicians, who seem to be able to repair imperfect knowledge arising from cognitive gaps during reading.

Refinements of the CRR-Based Strategies

Our students were ineffective readers of mathematics, but called on many of the CRR-based strategies (Table 1). Two kinds of observed difficulties suggest refinements in what effective readers of mathematics might be expected to do. Both move the CRR-based strategies toward cautious reading.

Strategies 1 and 2 indicate that good readers often preview parts of the text and decide which parts are worthy of the most attention. Our students clearly *did* that, but decided where to focus their attention badly. Some reported normally reading mainly examples[13] and said that exposition was confusing and of minor importance. They read the expositions only because they were part of the interview. But in reading mathematics little can be safely omitted entirely, except possibly historical notes. We expect effective readers of mathematics would understand this and attend to much more of the text.

Strategies 4 and 5 indicate that good readers infer the meaning of implicit information, and in particular, of new or unfamiliar words. But how this is done in mathematics differs considerably from how it is done in most other reading.
In most nonmathematical reading inferring information, including the meanings of words, is very common, and it is even one of the reading comprehension skills tested by the ACT reading test. Different readers can infer different, even conflicting, information because the words on which the inferences are based can sometimes be interpreted in more than one way. However, for the stipulated definitions of mathematics, it is necessary for readers to construct inner models whose outer applications are logically equivalent to the author's. Furthermore, for previously introduced words, authors of mathematics textbooks assume their readers' meanings will yield the same results as their own in working tasks.

Implications for Teaching

This study suggests that many first-year university students, including those good at reading and mathematics, could benefit from some instruction in reading their mathematics textbooks. The CRR-based strategies might be one useful guide for such instruction, especially for students less well-prepared than ours. However, this study also suggests that for students to become effective readers, something more should guide teaching, perhaps the refinements of the CRR-based strategies mentioned above. Instructors may wish to encourage their students to become more active in the way they read. This might include getting students to do a better job of activating their prior knowledge, teaching students strategies to help them integrate what they are reading and learning with prior knowledge, getting students to approach definitions as stipulative rather than descriptive, and teaching students how to construct their own examples and nonexamples by carefully consulting the formal mathematical definitions of concepts. Also, students need to actively engage in working from their concept images to the actual definitions, and vice versa, in order that they come to a reasonable semblance of the meaning intended by textbook authors (Pinto & Tall, 2002).

Readers need to know how to "look up" definitions, and that it is important, sometimes even necessary, to go back to definitions when reading mathematical writing. Also, readers need to learn to pay attention to each word in a definition since changing even one word can signal a difference between two concepts. The students in this study did sometimes look up definitions, but frequently did not appear to, or could not, use that information correctly when attempting to carry out a mathematical task.

One approach might be to try to help beginning university students both with the tendency to, and ability to, work through and understand the mathematical tasks that typically follow immediately after the introduction of new ideas or techniques. Perhaps it would be helpful to assign, in each class, a brief passage of new material to be read as homework. A small portion of the next class could then be devoted to helping students having difficulties with that passage. In that way, some of the difficulties we observed might be identified and dealt with. Students might come to understand that it is appropriate to look back to previous definitions and theorems, and to be very careful about the meanings of words. Indeed, they might develop the habit of doing these things, and thus move towards cautious reading. For a more nuanced, but still practical, approach to helping students with reading their mathematics textbooks, see Author (2005).

Future Research Questions

Turning to future research, here are a few questions that have come up in this study. It would be interesting to know how the CRR-based strategies could be further developed and used as a guide for teaching. What additional specialized strategies are critical for understanding mathematics textbooks? In what ways would using such extended strategies reduce the kinds of student reading difficulties we observed?

It would also be very helpful to investigate student attitudes and beliefs about reading mathematics textbooks. Do many students believe that they cannot usefully read a textbook without help? Do many students believe that they will benefit most by reading mainly, or only, the worked examples? Do they feel it is worth attempting a task that is already worked out in the textbook?

This study looked at good readers, as indicated by their high ACT reading scores and their use of many of the CRR-based strategies, and found many could not read their mathematics textbooks effectively, despite being in a class that emphasized reading their textbooks. What are the actual reading practices of more typical students? Are some of them cautious readers? For example, do they look back to the details of a definition while attempting a task? Students' actual practices may differ from what they report doing.

It would be good to better understand the genesis of flaws in the inner knowledge, or mental models, that students construct directly from reading, or for that matter, from any kind of direct communication or teaching of procedural mathematical knowledge. One might expect both incorrect prior knowledge and cognitive gaps due to periods of mind wandering or sensory awareness to play roles in causing errors in immediately thereafter worked tasks. The genesis of flaws in the construction of mental models seems not to have been well investigated in mathematics education research.

Cautious reading appears to be a fairly unusual skill that mathematicians have acquired and that leads to reading effectiveness. One could describe and analyze the cautious reading of mathematicians by observing them read. In this regard, it would also be good to investigate: (a) whether, and to what extent, mathematicians believe that small errors generated in reading mathematical text can lead to significant later errors; (b) the extent of mathematicians' sensitivity to, and search for, their own misunderstandings; and (c) whether, and to what extent, mathematicians believe that they can autonomously eventually succeed in understanding most mathematical text. Without these three traits, most students may be unwilling to invest the time required for cautious reading. An investigation of cautious reading might go some way towards explaining the centuries old folk belief that "mathematics trains the mind.

END NOTES

1. Here we use "strategy" to indicate that the reader tends to carry out certain activities in certain situations, not that the activities necessarily result from conscious intention or in response to advice.
2. The word "example," as used here, normally refers to a mathematical task, sometimes with a solution provided in the textbook, sometimes not. This contrasts with some other mathematical writing where "example" refers to an object, such as in saying "6 is an example of an even integer."
3. One's concept image (Tall & Vinner, 1981) is a mental construct including such knowledge as relevant examples, non-examples, facts, properties, relationships, diagrams, images, and visualizations, that one associates with the concept.
4. In normal writing, "but" means "and" with a negative connotation. However, in the logic of mathematical reasoning, "but" simply means "and".
5. In a stipulated, also called an analytic, definition one must use all parts of the definition and not infer additional conditions. Such a definition can bring a concept or mathematical entity into existence. In contrast an extracted, also called a synthetic or a descriptive, definition is a description of an already existing entity. One need not use all parts of such a synthetic definition and may even appropriately infer additional conditions (Edwards & Ward, 2004). Synthetic definitions can be right or wrong; analytic definitions cannot.
6. The ACT (American College Test, 2010) is a university admissions examination that includes a mathematics portion and a general reading portion. More information is provided in the section, "A Comparison of what the ACT Tests Measure with Effective Reading".
7. All student names are pseudonyms. Names starting with letters at the beginning of the alphabet are precalculus students and names starting with letters at the end of the alphabet are calculus students.
8. When students are speaking, their comments are shown in regular typeface; when they are reading from their textbooks, this is shown in italics; pauses are shown as […] and … indicate omissions.
9. We will use *concept* for an external, textual description or a definition and *conception* for an inner mental structure corresponding to that concept.

10. We are not suggesting anything about the psychological or neurological nature of such an inner model, except what might be inferred from the students' subsequent actions.
11. This is similar to the way one's experience of a large relatively high resolution visual field is constructed, outside of consciousness, from a large low resolution field, plus a number of small high resolution areas, with the gaps filled in plausibly.
12. Schooler, Reichle, & Halpern (2004) speak of "attention" instead of "focus" or "focus of attention."
13. Mathematics textbook authors may inadvertently encourage this by visually setting apart information such as definitions. This is probably to encourage readers to look back to that information when reading later passages. Such looking back to check details is a common practice among skilled readers of mathematics.

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Appendices A and B include copies of the textbook pages that the students were asked to read. Permission to include these pages has been received from the publisher. The pages have been cut apart so that comments and tasks could be inserted to indicate when the students were asked to perform a task.

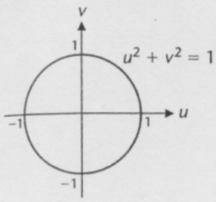
Appendix A contains the passage read by the precalculus students. All tasks that the precalculus students were asked to perform were contained within the selected pages.

Appendix B contains the passage read by the calculus students. In addition to the selected pages and indicated tasks, it also includes copies of the exercises that the students were asked to attempt.

Appendix A (Part I) —Precalculus reading passages with interruptions:

Definition of the Wrapping Function

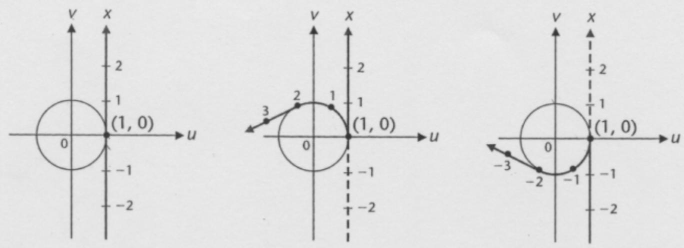
FIGURE 1
Unit circle.



The important *circular function* definitions presented in the next section are based on a function *W*, called the *wrapping function,* whose domain is the set of real numbers and whose range is the set of points on the *unit circle*. By the **unit circle** we mean the circle of radius 1 with center at the origin of a rectangular coordinate system. Its equation is $u^2 + v^2 = 1$ (see Fig. 1).*

In defining the **wrapping function,** we "wrap" a real number line with origin at (1, 0) around the unit circle—the positive real axis is wrapped counterclockwise, and the negative real axis is wrapped clockwise. In this way, each real number on the real line is paired with a unique point, called a **circular point,** on the unit circle, as shown in Figure 2.

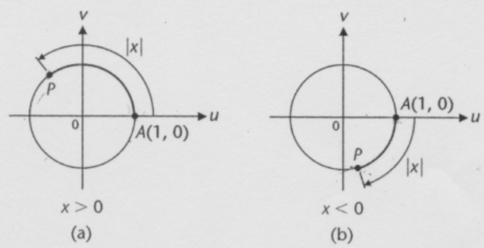
FIGURE 2
The wrapping function.



To locate the circular point associated with a number such as 37 or −105, the number line is wrapped many times around the circle.

An equivalent way of pairing real numbers with points on the unit circle is to think in terms of *arc length,* assuming we know what arc length is. To find the circular point *P* associated with the real number *x,* we start at *A*(1, 0) and move $|x|$ units along the unit circle, counterclockwise if *x* is positive and clockwise if *x* is negative. The length of arc *AP* is $|x|$ (see Fig. 3).

FIGURE 3
The wrapping function and arc length.



It is important to be able to find the coordinates (*a*, *b*) of the circular point *P* associated with a given real number *x* so that we can write $W(x) = (a, b)$. In general, this is difficult and requires the use of a calculator. However, for certain real numbers, integer multiples of $\pi/6$, $\pi/4$, $\pi/3$, and $\pi/2$, we can find the exact coordinates of the corresponding circular points using simple geometric properties of a circle.

*We use the variables *u* and *v* instead of *x* and *y* so that *x* can be used without ambiguity as an independent variable in defining the wrapping function in this section and the circular functions in the next section.

338 | 5 TRIGONOMETRIC FUNCTIONS

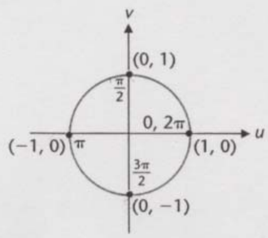
Exact Values for Particular Real Numbers

We start our investigation by finding the circumference of the unit circle. Since radius $r = 1$, the circumference is

$$2\pi r = 2\pi(1) = 2\pi \quad \text{Circumference of the unit circle}$$

One-fourth, one-half, and three-fourths of the circumference are, respectively, $\pi/2$, π, and $3\pi/2$. The circular points corresponding to these real numbers are on the coordinate axes, and hence, their coordinates are easily determined (see Fig. 4).

FIGURE 4
Circular points on the coordinate axes.



$$W(0) = (1, 0)$$

$$W\left(\frac{\pi}{2}\right) = (0, 1)$$

$$W(\pi) = (-1, 0)$$

$$W\left(\frac{3\pi}{2}\right) = (0, -1)$$

$$W(2\pi) = (1, 0)$$

Following the same procedure, we can find the coordinates of *any* circular point on a coordinate axis—that is, for any circular point corresponding to a real number that is an integer multiple of $\pi/2$.

EXAMPLE 1 Finding the Coordinates of Circular Points

Find the coordinates of the circular points.

(A) $W(-\pi/2)$ (B) $W(5\pi/2)$

The student readers were interrupted at this point and asked to work this example without looking at the solution, which was covered with Post-it© notes. Then each student was asked to read through the solution and work the Matched Problem.

Solutions

(A) Starting at (1, 0), we go one-fourth the way around the unit circle in a clockwise direction (see Fig. 4). Thus,

$$W\left(\frac{-\pi}{2}\right) = (0, -1)$$

(B) Starting at (1, 0) and proceeding counterclockwise, we count quarter-circle steps, $\pi/2$, $2\pi/2$, $3\pi/2$, $4\pi/2$, and ending at $5\pi/2$. Thus, the circular point is on the positive vertical axis, and we have

$$W\left(\frac{5\pi}{2}\right) = (0, 1)$$

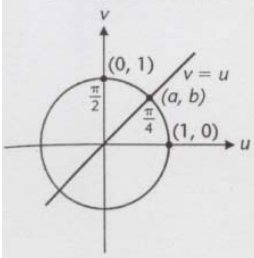
MATCHED PROBLEM 1

Find the coordinates of the circular points.

(A) $W(-\pi)$ (B) $W(3\pi)$

The student was then asked to continue reading:

FIGURE 5
Circular point $W(\pi/4)$.



We now find the coordinates of the circular point $W(\pi/4)$. Since $\pi/4$ is one-half the arc joining (1, 0) and (0, 1), the circular point $W(\pi/4)$ must lie on the line $v = u$, as shown in Figure 5. Since $W(\pi/4)$ is on the line $v = u$ and on the circle $u^2 + v^2 = 1$, its coordinates (a, b) must satisfy both equations. That is,

$$a = b \quad \text{and} \quad a^2 + b^2 = 1$$

Substituting a for b in the second equation, we have

$$a^2 + a^2 = 1$$

$$2a^2 = 1$$

$$a^2 = \frac{1}{2}$$

$$a = \pm\frac{1}{\sqrt{2}}$$

$$a = \frac{1}{\sqrt{2}}$$ $a = -1/\sqrt{2}$ must be discarded, since $W(\pi/4)$ is in the first quadrant.

Using the first equation, we see that

$$b = a = \frac{1}{\sqrt{2}}$$

Therefore,

$$W\left(\frac{\pi}{4}\right) = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

Using symmetry properties of a circle—the unit circle is symmetric with respect to both axes and the origin—we can easily find the coordinates of any circular point that is reflected across the vertical axis, horizontal axis, or origin from $W(\pi/4)$.

EXAMPLE 2

Finding the Coordinates of Circular Points

Find the coordinates of the circular points.

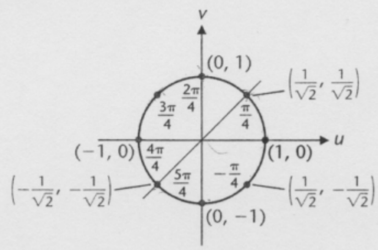
(A) $W(5\pi/4)$ (B) $W(-\pi/4)$

The student was interrupted at this point and asked to work this example without looking at the solution which was covered with a Post-it© note. The student was then asked to read the solution and work Matched Problem 2.

Solutions (A) Starting at (1, 0) and counting in one-eighth circle steps counterclockwise ($\pi/4$, $2\pi/4$, $3\pi/4$, $4\pi/4$, $5\pi/4$), we find ourselves in the third quadrant on the circle halfway between $(-1, 0)$ and $(0, -1)$, as indicated in Figure 6 on the next page. Using symmetry with respect to the origin, we have

$$W\left(\frac{5\pi}{4}\right) = \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$$

FIGURE 6



(B) Starting at (1, 0), we proceed one-eighth the way around the unit circle in a clockwise direction and end up in the fourth quadrant on the circle halfway between $(0, -1)$ and $(1, 0)$, as indicated in Figure 6. Using symmetry with respect to the horizontal axis, we see that

$$W\left(-\frac{\pi}{4}\right) = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$$

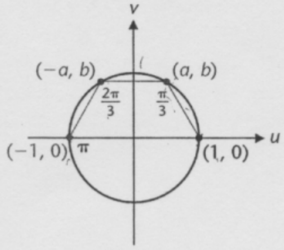
MATCHED PROBLEM 2

Find the coordinates of the circular points.

(A) $W(3\pi/4)$ (B) $W(-7\pi/4)$

After checking answers, the student was asked to continue reading:

FIGURE 7
Circular point $W(\pi/3)$.



We continue our investigation by finding the coordinates of the circular point $W(\pi/3)$. Referring to Figure 7, we divide the upper semicircle from (1, 0) to (−1, 0) into thirds. The circular points $W(\pi/3)$ and $W(2\pi/3)$ are symmetric with respect to the v axis; hence, if $W(\pi/3)$ is given coordinates (a, b), then $W(2\pi/3)$ must have coordinates $(-a, b)$. The chord joining $W(2\pi/3)$ and $W(\pi/3)$ is thus $2a$ units long. Using the distance formula (see Section 1-1), we find the length of the chord joining $W(0)$ and $W(\pi/3)$ to be given by $\sqrt{(a-1)^2 + b^2}$. The two chords are equal in length, since congruent arcs are opposite congruent chords on the same circle. Thus,

$$\sqrt{(a-1)^2 + b^2} = 2a$$

Squaring both sides, we obtain

$$
\begin{aligned}
(a-1)^2 + b^2 &= 4a^2 \\
a^2 - 2a + 1 + b^2 &= 4a^2 \\
a^2 + b^2 - 2a + 1 &= 4a^2 \\
1 - 2a + 1 &= 4a^2 \qquad a^2 + b^2 = 1 \text{ (Why?)} \\
4a^2 + 2a - 2 &= 0 \\
2a^2 + a - 1 &= 0 \\
(2a-1)(a+1) &= 0 \\
a = \tfrac{1}{2} \qquad \text{or} \qquad a &= -1 \\
a &= \tfrac{1}{2} \qquad a = -1 \text{ must be discarded. (Why?)}
\end{aligned}
$$

Substitute $a = \frac{1}{2}$ into $a^2 + b^2 = 1$ and solve for b.

Appendix A (Part II) —Precalculus reading passages (continued)

$$\left(\frac{1}{2}\right)^2 + b^2 = 1$$

$$b^2 = \frac{3}{4}$$

$$b = \pm\frac{\sqrt{3}}{2}$$

$$b = \frac{\sqrt{3}}{2} \qquad b = -\sqrt{3}/2 \text{ must be discarded. (Why?)}$$

Thus,

$$W\left(\frac{\pi}{3}\right) = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$$

Proceeding in a similar manner, or using symmetry with respect to the line $v = u$, we can obtain

$$W\left(\frac{\pi}{6}\right) = \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$$

The key results from the preceding discussion for the first quadrant are summarized in Figure 8.

COORDINATES OF KEY CIRCULAR POINTS

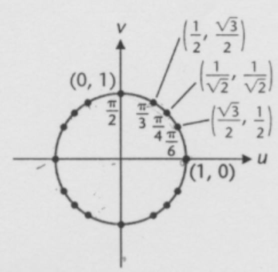


FIGURE 8

It is important that you memorize these first quadrant relationships.

Explore/Discuss 1

An effective **memory aid** for recalling the coordinates of the key circular points in Figure 8 can be created by writing the coordinates of the circular points $W(0)$, $W(\pi/6)$, $W(\pi/4)$, $W(\pi/3)$, and $W(\pi/2)$, keeping this order, in a form where each numerator is the square root of an appropriate number and each denominator is 2. For example, $W(0) = (1, 0) = (\sqrt{4}/2, \sqrt{0}/2)$. Describe the pattern that results.

The student was stopped and was asked if he/she would do the Explore/Discuss. None did. The student was then asked to continue reading:

342 | 5 TRIGONOMETRIC FUNCTIONS

The reason for memorizing the coordinates of key circular points in the first quadrant is that by using these, along with the symmetry of the unit circle, we can find the coordinates of *any* circular point that corresponds to *any* integer multiple of $\pi/6$, $\pi/4$, $\pi/3$, and $\pi/2$.

asked to try without looking at solution

EXAMPLE 3 **Finding Coordinates of Circular Points**

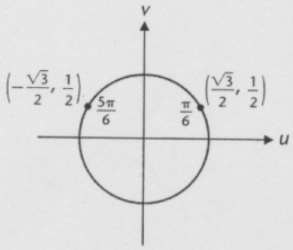
Find the coordinates of the circular points.

(A) $W(5\pi/6)$ (B) $W(-2\pi/3)$

Again, the student was asked to stop and to try to work this example with the solution covered. Then the student read the solution, worked Matched Problem 3, and continued reading.

Solutions

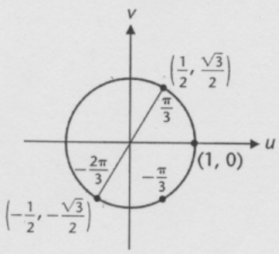
FIGURE 9



(A) Note that $5\pi/6$ is $\pi/6$ less than $\pi = 6\pi/6$. Locate $5\pi/6$ in the second quadrant and use Figure 8 and symmetry with respect to the vertical axis to find $W(5\pi/6)$ (see Fig. 9).

$$W\left(\frac{5\pi}{6}\right) = \left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$$

FIGURE 10



(B) Locate $-2\pi/3$ in the third quadrant and use Figure 8 and symmetry with respect to the origin to find $W(-2\pi/3)$ (see Fig. 10).

$$W\left(-\frac{2\pi}{3}\right) = \left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$$

MATCHED PROBLEM 3

Find the coordinates of the circular points.

(A) $W(5\pi/3)$ (B) $W(-7\pi/6)$

The Wrapping Function Is Not One-to-One

It is easy to see that the wrapping function is not a one-to-one function. Each domain value, a real number, corresponds to exactly one range value, a point on the unit circle. However, each range value, a point on the unit circle, corresponds to infinitely many domain values, real numbers. For example, we see that

$$W\left(\frac{\pi}{2}\right) = (0, 1)$$

That is, exactly one range value corresponds to the domain value $\pi/2$. But how many domain values correspond to the range value (0, 1)? Every time we go around the unit circle 2π units in either direction from (0, 1), we return to the same point. Thus, if we are asked to solve

$$W(x) = (0, 1)$$

we have to write

$$x = \frac{\pi}{2} + 2k\pi \qquad k \text{ any integer}$$

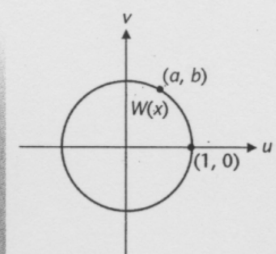
and there are infinitely many domain values of W that correspond to the range value (0, 1). In general, the following theorem applies.

THEOREM 1

A WRAPPING FUNCTION PROPERTY

For all real numbers x,

$$W(x) = W(x + 2k\pi) \qquad k \text{ any integer*}$$



We will have more to say about the implications of this important property of the wrapping function in subsequent sections.

Explore/Discuss 2

(A) Solve the circular point equation $W(x) = (0, -1)$, $-2\pi \le x \le 2\pi$.

(B) Write an expression that would represent all solutions to $W(x) =$ $(0, -1)$.

If the interview got this far, the student was asked to try this Explore/Discuss.

Appendix B Calculus reading passages with interruptions:

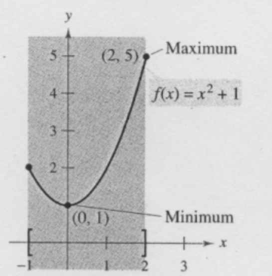
160   CHAPTER 3   Applications of Differentiation

Section 3.1   Extrema on an Interval

- Understand the definition of extrema of a function on an interval.
- Understand the definition of relative extrema of a function on an open interval.
- Find extrema on a closed interval.

Extrema of a Function

In calculus, much effort is devoted to determining the behavior of a function f on an interval I. Does f have a maximum value on I? Does it have a minimum value? Where is the function increasing? Where is it decreasing? In this chapter you will learn how derivatives can be used to answer these questions. You will also see why these questions are important in real-life applications.



(a) f is continuous, $[-1, 2]$ is closed.

Definition of Extrema

Let f be defined on an interval I containing c.

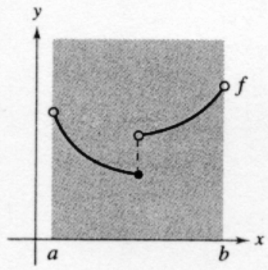
1. $f(c)$ is the **minimum of f on I** if $f(c) \leq f(x)$ for all x in I.
2. $f(c)$ is the **maximum of f on I** if $f(c) \geq f(x)$ for all x in I.

The minimum and maximum of a function on an interval are the **extreme values,** or **extrema,** of the function on the interval. The minimum and maximum of a function on an interval are also called the **absolute minimum** and **absolute maximum** on the interval.

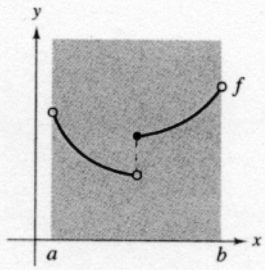
The student was stopped at this point and was asked to try Exercises 51-54 below.

In Exercises 51–54, determine from the graph whether f has a minimum in the open interval (a, b).

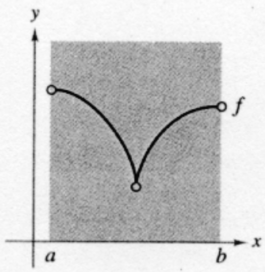
51. (a)



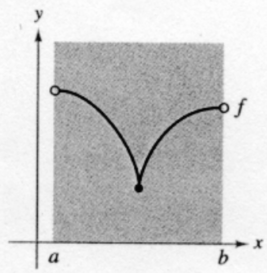
(b)



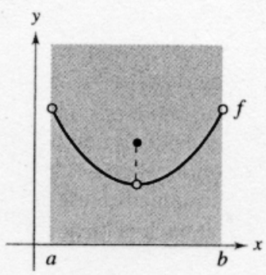
52. (a)



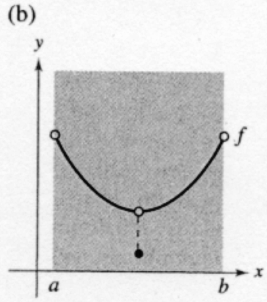
(b)



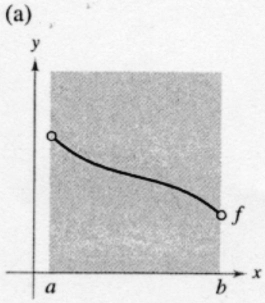
53. (a)



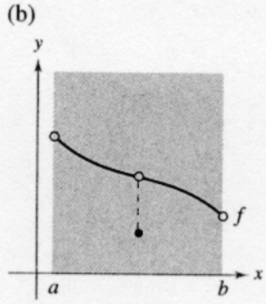
(b)



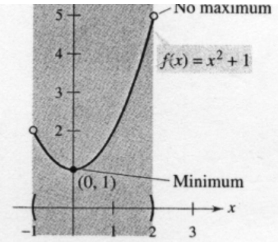
54. (a)



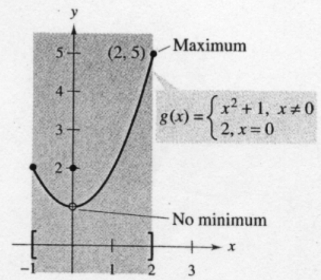
(b)



The student was asked to continue reading. (Note some of the figures are on the left side of the first portion of the reading which is on the previous page.)



(b) f is continuous, $(-1, 2)$ is open.



(c) g is not continuous, $[-1, 2]$ is closed.

Extrema can occur at interior points or endpoints of an interval. Extrema that occur at the endpoints are called **endpoint extrema.**

Figure 3.1

A function need not have a minimum or a maximum on an interval. For instance, in Figure 3.1(a) and (b), you can see that the function $f(x) = x^2 + 1$ has both a minimum and a maximum on the closed interval $[-1, 2]$, but does not have a maximum on the open interval $(-1, 2)$. Moreover, in Figure 3.1(c), you can see that continuity (or the lack of it) can affect the existence of an extremum on the interval. This suggests the following theorem. (Although the Extreme Value Theorem is intuitively plausible, a proof of this theorem is not within the scope of this text.)

THEOREM 3.1 The Extreme Value Theorem

If f is continuous on a closed interval $[a, b]$, then f has both a minimum and a maximum on the interval.

EXPLORATION

Finding Minimum and Maximum Values The Extreme Value Theorem (like the Intermediate Value Theorem) is an *existence theorem* because it tells of the existence of minimum and maximum values but does not show how to find these values. Use the extreme-value capability of a graphing utility to find the minimum and maximum values of each of the following. In each case, do you think the x-values are exact or approximate? Explain your reasoning.

a. $f(x) = x^2 - 4x + 5$ on the closed interval $[-1, 3]$

b. $f(x) = x^3 - 2x^2 - 3x - 2$ on the closed interval $[-1, 3]$

The student was stopped and was asked if he/she would try the Exploration. Upon completion of the Exploration, student asked to try the two true/false questions below from Calculus textbook, page 167.

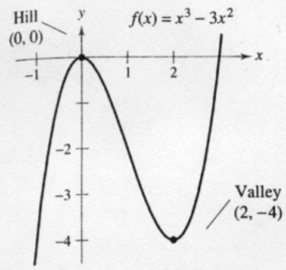
True or False? **In Exercises 61-64, determine whether the statement is true or false.**

If it is false, explain why or give an example that shows it is false.

61. The maximum of a function that is continuous on a closed interval can occur at two different values in the interval.

62. If a function is continuous on a closed interval, then it must have a minimum on the interval.

Then the student was asked to continue reading:



f has a relative maximum at (0, 0) and a relative minimum at (2, − 4).
Figure 3.2

Relative Extrema and Critical Numbers

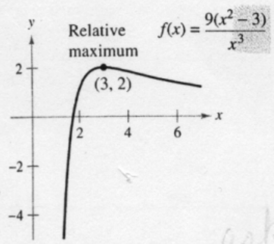
In Figure 3.2, the graph of $f(x) = x^3 - 3x^2$ has a **relative maximum** at the point (0, 0) and a **relative minimum** at the point (2, −4). Informally, you can think of a relative maximum as occurring on a "hill" on the graph, and a relative minimum as occurring in a "valley" on the graph. Such a hill and valley can occur in two ways. If the hill (or valley) is smooth and rounded, the graph has a horizontal tangent line at the high point (or low point). If the hill (or valley) is sharp and peaked, the graph represents a function that is not differentiable at the high point (or low point).

Definition of Relative Extrema

1. If there is an open interval containing c on which $f(c)$ is a maximum, then $f(c)$ is called a **relative maximum** of f.
2. If there is an open interval containing c on which $f(c)$ is a minimum, then $f(c)$ is called a **relative minimum** of f.

The plural of relative maximum is relative maxima, and the plural of relative minimum is relative minima.

Example 1 examines the derivatives of functions at *given* relative extrema. (Much more is said about *finding* the relative extrema of a function in Section 3.3.)

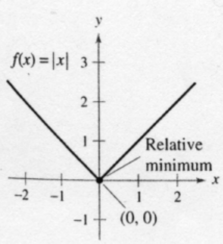


(a) $f'(3) = 0$

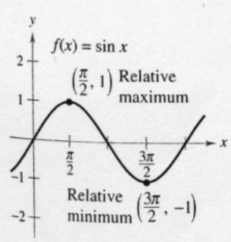
Example 1 **The Value of the Derivative at Relative Extrema**

Find the value of the derivative at each of the relative extrema shown in Figure 3.3.

The student was stopped and was asked to work Example 1 with the solution covered with Post-It© Notes. Note that the remaining two figures for Figure 3.3 are beside the solution given below. After the student had worked the example, he/she was asked to read the solution and then continue reading.



(b) $f'(0)$ does not exist.



(c) $f'\left(\frac{\pi}{2}\right) = 0$; $f'\left(\frac{3\pi}{2}\right) = 0$
Figure 3.3

Solution

a. The derivative of $f(x) = \dfrac{9(x^2-3)}{x^3}$ is

$$f'(x) = \frac{x^3(18x) - (9)(x^2-3)(3x^2)}{(x^3)^2} \qquad \text{Differentiate using Quotient Rule.}$$

$$= \frac{9(9-x^2)}{x^4}. \qquad \text{Simplify.}$$

At the point (3, 2), the value of the derivative is $f'(3) = 0$ (see Figure 3.3a).

b. At $x = 0$, the derivative of $f(x) = |x|$ *does not exist* because the following one-sided limits differ (see Figure 3.3b).

$$\lim_{x\to 0^-} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0^-} \frac{|x|}{x} = -1 \qquad \text{Limit from the left}$$

$$\lim_{x\to 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0^+} \frac{|x|}{x} = 1 \qquad \text{Limit from the right}$$

c. The derivative of $f(x) = \sin x$ is

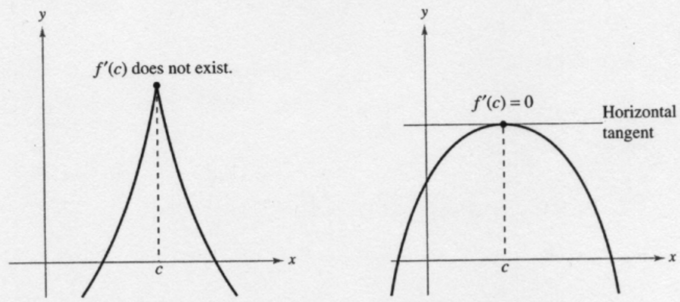
$$f'(x) = \cos x.$$

At the point $(\pi/2, 1)$, the value of the derivative is $f'(\pi/2) = \cos(\pi/2) = 0$. At the point $(3\pi/2, -1)$, the value of the derivative is $f'(3\pi/2) = \cos(3\pi/2) = 0$ (see Figure 3.3c).

Note in Example 1 that at the relative extrema, the derivative is either zero or does not exist. The x-values at these special points are called **critical numbers.** Figure 3.4 illustrates the two types of critical numbers.

Definition of a Critical Number

Let f be defined at c. If $f'(c) = 0$ or if f is not differentiable at c, then c is a **critical number** of f.



c is a critical number of f.
Figure 3.4

THEOREM 3.2 Relative Extrema Occur Only at Critical Numbers

If f has a relative minimum or relative maximum at $x = c$, then c is a critical number of f.

PIERRE DE FERMAT (1601–1665)
For Fermat, who was trained as a lawyer, mathematics was more of a hobby than a profession. Nevertheless, Fermat made many contributions to analytic geometry, number theory, calculus, and probability. In letters to friends, he wrote of many of the fundamental ideas of calculus, long before Newton or Leibniz. For instance, the theorem at the right is sometimes attributed to Fermat.

Proof

Case 1: If f is *not* differentiable at $x = c$, then, by definition, c is a critical number of f and the theorem is valid.

Case 2: If f is differentiable at $x = c$, then $f'(c)$ must be positive, negative, or 0. Suppose $f'(c)$ is positive. Then

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} > 0$$

which implies that there exists an interval (a, b) containing c such that

$$\frac{f(x) - f(c)}{x - c} > 0, \text{ for all } x \neq c \text{ in } (a, b). \qquad \text{(See Exercise 58, Section 1.2)}$$

Because this quotient is positive, the signs of the denominator and numerator must agree. This produces the following inequalities for x-values in the interval (a, b).

Left of c: $x < c$ and $f(x) < f(c)$ ⇨ $f(c)$ is not a relative minimum

Right of c: $x > c$ and $f(x) > f(c)$ ⇨ $f(c)$ is not a relative maximum

So, the assumption that $f'(c) > 0$ contradicts the hypothesis that $f(c)$ is a relative extremum. Assuming that $f'(c) < 0$ produces a similar contradiction, you are left with only one possibility—namely, $f'(c) = 0$. So, by definition, c is a critical number of f and the theorem is valid.

At the end of the proof, the student was stopped and was asked what the proof meant to him/her. The student was then asked to continue reading:

Finding Extrema on a Closed Interval

Theorem 3.2 states that the relative extrema of a function can occur *only* at the critical numbers of the function. Knowing this, you can use the following guidelines to find extrema on a closed interval.

Guidelines for Finding Extrema on a Closed Interval

To find the extrema of a continuous function f on a closed interval $[a, b]$, use the following steps.

1. Find the critical numbers of f in (a, b).
2. Evaluate f at each critical number in (a, b).
3. Evaluate f at each endpoint of $[a, b]$.
4. The least of these values is the minimum. The greatest is the maximum.

The next three examples show how to apply these guidelines. Be sure you see that finding the critical numbers of the function is only part of the procedure. Evaluating the function at the critical numbers *and* the endpoints is the other part.

Example 2 **Finding Extrema on a Closed Interval**

Find the extrema of $f(x) = 3x^4 - 4x^3$ on the interval $[-1, 2]$.

The student was stopped and was asked to try the example without looking at the solution which was covered. Then the student was asked to read the solution and continue reading.

Solution Begin by differentiating the function.

$f(x) = 3x^4 - 4x^3$ Write original function.

$f'(x) = 12x^3 - 12x^2$ Differentiate.

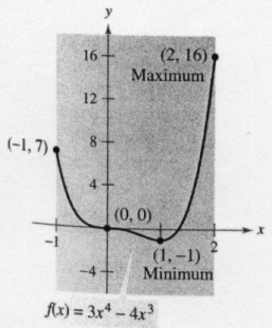
To find the critical numbers of f, you must find all x-values for which $f'(x) = 0$ and all x-values for which $f'(x)$ does not exist.

$f'(x) = 12x^3 - 12x^2 = 0$ Set $f'(x)$ equal to 0.

$12x^2(x - 1) = 0$ Factor.

$x = 0, 1$ Critical numbers

Because f' is defined for all x, you can conclude that these are the only critical numbers of f. By evaluating f at these two critical numbers and at the endpoints of $[-1, 2]$, you can determine that the maximum is $f(2) = 16$ and the minimum is $f(1) = -1$, as indicated in the table. The graph of f is shown in Figure 3.5.

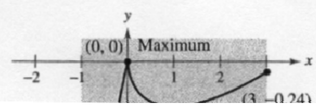


On the closed interval $[-1, 2]$, f has a minimum at $(1, -1)$ and a maximum at $(2, 16)$.
Figure 3.5

Left Endpoint	Critical Number	Critical Number	Right Endpoint
$f(-1) = 7$	$f(0) = 0$	$f(1) = -1$ Minimum	$f(2) = 16$ Maximum

In Figure 3.5, note that the critical number $x = 0$ does not yield a relative minimum or a relative maximum. This tells you that the converse of Theorem 3.2 is not true. In other words, *the critical numbers of a function need not produce relative extrema.*

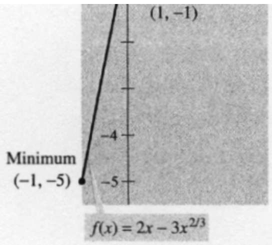
164   CHAPTER 3   Applications of Differentiation



Example 3 **Finding Extrema on a Closed Interval**

Find the extrema of $f(x) = 2x - 3x^{2/3}$ on the interval $[-1, 3]$.

The student was stopped and was asked to try the example without looking at the solution which was covered, and then to read the solution.



On the closed interval $[-1, 3]$, f has a minimum at $(-1, -5)$ and a maximum at $(0, 0)$.
Figure 3.6

Solution Differentiating produces the following.

$$f(x) = 2x - 3x^{2/3} \qquad \text{Write original function.}$$

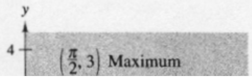
$$f'(x) = 2 - \frac{2}{x^{1/3}} = 2\left(\frac{x^{1/3} - 1}{x^{1/3}}\right) \qquad \text{Differentiate.}$$

From this derivative, you can see that the function has two critical numbers in the interval $[-1, 3]$. The number 1 is a critical number because $f'(1) = 0$, and the number 0 is a critical number because $f'(0)$ does not exist. By evaluating f at these two numbers and at the endpoints of the interval, you can conclude that the minimum is $f(-1) = -5$ and the maximum is $f(0) = 0$, as indicated in the table. The graph of f is shown in Figure 3.6.

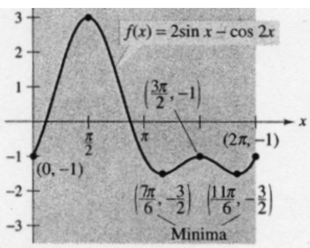
Left Endpoint	Critical Number	Critical Number	Right Endpoint
$f(-1) = -5$ Minimum	$f(0) = 0$ Maximum	$f(1) = -1$	$f(3) = 6 - 3\sqrt[3]{9} \approx -0.24$

Example 4 **Finding Extrema on a Closed Interval**

Find the extrema of $f(x) = 2 \sin x - \cos 2x$ on the interval $[0, 2\pi]$.



The student was stopped and was asked to try the example without looking at the solution which was covered. The student was then asked to read the solution.



On the closed interval $[0, 2\pi]$, f has two minima at $(7\pi/6, -3/2)$ and $(11\pi/6, -3/2)$ and a maximum at $(\pi/2, 3)$.
Figure 3.7

Solution This function is differentiable for all real x, so you can find all critical numbers by differentiating the function and setting $f'(x)$ equal to zero, as follows.

$$f(x) = 2 \sin x - \cos 2x \qquad \text{Write original function.}$$

$$f'(x) = 2 \cos x + 2 \sin 2x = 0 \qquad \text{Set } f'(x) \text{ equal to 0.}$$

$$2 \cos x + 4 \cos x \sin x = 0 \qquad \sin 2x = 2 \cos x \sin x$$

$$2(\cos x)(1 + 2 \sin x) = 0 \qquad \text{Factor.}$$

In the interval $[0, 2\pi]$, the factor $\cos x$ is zero when $x = \pi/2$ and when $x = 3\pi/2$. The factor $(1 + 2 \sin x)$ is zero when $x = 7\pi/6$ and when $x = 11\pi/6$. By evaluating f at these four critical numbers and at the endpoints of the interval, you can conclude that the maximum is $f(\pi/2) = 3$ and the minimum occurs at *two* points, $f(7\pi/6) = -3/2$ and $f(11\pi/6) = -3/2$, as indicated in the table. The graph is shown in Figure 3.7.

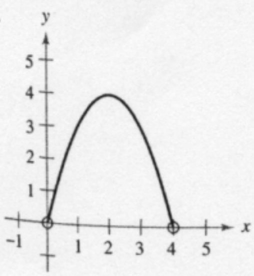
Left Endpoint	Critical Number	Critical Number	Critical Number	Critical Number	Right Endpoint
$f(0) = -1$	$f\left(\frac{\pi}{2}\right) = 3$ Maximum	$f\left(\frac{7\pi}{6}\right) = -\frac{3}{2}$ Minimum	$f\left(\frac{3\pi}{2}\right) = -1$	$f\left(\frac{11\pi}{6}\right) = -\frac{3}{2}$ Minimum	$f(2\pi) = -1$

indicates that in the Interactive 3.0 *CD-ROM and* Internet 3.0 *versions of this text (available at* college.hmco.com) *you will find an Open Exploration, which further explores this example using the computer algebra systems* Maple, Mathcad, Mathematica, *and* Derive.

The student was then stopped (this was the end of the reading section) and asked to try Exercises 8 and 12.

In Exercises 7–10, approximate the critical numbers of the function shown in the graph. Determine whether the function has a relative maximum, relative minimum, absolute maximum, absolute minimum, or none of these at each critical number on the interval shown.

7.



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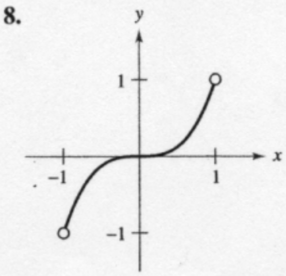

In Exercises 11-16, find any critical numbers of the function.

12. $g(x) = x^2(x^2 - 4)$

\

Appendix C—Debriefing questions:

1. Were there any words or terms that bothered you as you read?
2. Were there any symbols or notation that bothered you as you read?
3. Are there any other ways this passage was difficult for you to read and/or understand?
4. What things do you do when you read the textbook?
5. Have you seen the material this passage covered anywhere before? (If so, where?)
6. Did the reading help you do the task? In what way?
7. Is there anything else you would like to say?