# A Reflective Journey through Theory and Research in Mathematical Learning and Development

Shashidhar Belbase

Graduate Student of Mathematics Education
College of Education, University of Wyoming
Laramie, Wyoming

December 20, 2010

A Reflective Journey through Theory and Research in Mathematical Learning and Development

**Abstract**

This paper is an attempt to reflect on class sessions during the fall 2010 in a course '*Theory and Research in Mathematical Learning and Development*'. This reflection as a learning journey portrays discussions based on foundational perspectives (FP), historical highlights (HH), and guiding questions (GQ) related to mathematics learning and development. I think these three key areas: FP, HH, and GQ became the central in all sessions explicitly or implicitly. FP covered the key areas of philosophical and psychological theories of learning and development; HH captured historical developments of psychology and philosophy of mathematics learning and development; and GQ turned critical eye toward theories and research in mathematical learning and development. Complexities of classroom dynamics have been discussed in relation to course contents, lectures, discussions, reviews and presentations as stepping stones in the progress of course with multiplicities of ontologies, epistemologies, and methodologies in actions and operations.

## Prologue

The summer was almost toward the end but there was still greenery with beautiful lawns and trees in and around the Prexy's Pasture at University of Wyoming. The dignity of equality state at the center of the pasture was brighter with clear sunshine. There were some white marks in the blue sky due to plumes of jet engines across the sky of Laramie. A pair of squirrels were jumping around the pine trees near the Wyoming Hall when I was going to attend my class. In the odyssey of academic life with gentle breeze, clean blue sky, flying birds, playing squirrels, group of students going and coming back to and from classes, busy Prexy's pasture…after a long summer holiday, the university was gaining full momentum in the beginning of the fall 2010 by the third week of August.

I was much curious about the nature of the course, course calendar and course works that we had to accomplish. I was waiting the first day of the course session to know about the course in detail. The first session of the course started with course overview and plans. The course instructor distributed us the course plans with calendar for the semester. We discussed plan for each sessions with specific productions and assignments together with course schedule. The plan was well laid out with schedules of activities and discussion topics on different dates.

The course instructor asked us to share our personal theory of mathematics learning and development. He encouraged us to tell what we believed about learning mathematics. We shared the personal theories of mathematics learning and development. Participant 1 shared her personal theory of learning mathematics as cumulative, structural, and sequential. Participant 2 shared personal theory of learning mathematics as influenced by personal and social constructs. For participant 3, her personal theory of learning mathematics was largely dependent on age and mathematical ability of a child which were correlated to each other. Participant 4 opined that integrated learning, constructivism, exploratory learning was his emphasis. Participant 5 mentioned that she just did it as it came and intuition was a way of learning. Her emphasis was on persistence to learn mathematics. For participant 6, mathematics learning process was focused more on the process itself than on the outcomes. She mentioned four integral parts as strategy, curriculum, teacher motivation, and classroom environment as important aspects of mathematics learning. Participant 7 mentioned that receptive learning, learning opportunities to be presented to all students promoting actions that might lead to learning. He further mentioned that learning mathematics was affected by other cognitive process such as reading, writing, listening, and speaking to each other in the classroom environment. Learning was enhanced by physical and psychological maturity of learners. The journey of *epistemic students* (Steffe, 2007) began with sharing of personal theories of learning and development opening avenues for individual and group sharing and discussion, actions and operations, accommodations and building upon our experiences. Such a class in which the subject of discussion, participation, actions and operations are common to all subjects (participants) at the same level of development, whose cognitive structures derive from the most general mechanisms of the co-ordination of actions (Piaget, 1966, p. 308) and, to me, it was epistemic class.

We discussed on participant 1's question, "Why we prefer some theory over another theory of learning mathematics?" We have our biases and preferences about method of knowing and learning that makes us prefer one way or other. We agreed that motivation and readiness as important factors that affect learning. Is there learning by a child if he or she can perform certain way (as expected)? Does the performance show that there is learning? These questions led us to discuss about how learning is manifested through students behavior or performance. Repetition may facilitate learning, knowing basic facts, and memorizing formula and application of formula. Instantiation is the way that can manifest learning. Other ways to observe learning are exemplification, generalization and construction of mathematics. We began discussion on connectionism as learning theory.

Discussion on personal theory of learning was a great beginning. I think every person has his or her own perspective about learning mathematics. Our personal theories of learning mathematics was a blend of ideas such as mathematics learning as cumulative, structural, and sequential; learning is influenced by personal and social constructs; learning mathematics as largely dependent on age and mathematical ability of a child; integrated learning, constructivism, and exploratory learning; learning mathematics as doing it as it comes and intuition as a way of learning; persistence as important factor to learn mathematics; process as important aspect of learning than the outcomes; integral parts of learning as strategy, curriculum, teacher motivation, and classroom environment ; receptive learning, learning opportunities to be presented to all students promoting actions that might lead to learning; mathematics learning as affected by other cognitive process such as reading, writing, listening, and speaking to each other in the classroom environment; and learning to be enhanced by physical and psychological maturity of learners. In developing such personal theories of learning we came to realize that there are analogies between the experiences and the responses of pupils, teachers, and researchers, and analyzing the personal theory of learning of one group of learners can provide implications for the learning of others (Duffin & Simpson, 1995). These personal theories are not haphazard but they have been derived from natural experience, conflicting experience, and alien experience to some extent. These experiences build up our theories through adding new experiences, changing in the existing experiences with new experience, or avoiding some experiences that do not match with our existing experiences. And, this is a continuous process.

A familiarity with student's personal beliefs, values and perspectives about learning mathematics can lead to richer discussions with a higher degree of relevance and also this helps the instructor to understand at what level students are ready to participate in discussions. The starting session motivated me to see learning mathematics differently than what I had understood earlier about it. I thought that learning could be manifested by test results but it was an illusion. Many times I also felt that test results could not show how much I had learnt because of time factor to express the knowledge I had about a problem within a given time. This discussion led me to think that learning is different from what we or our students manifest in test results and these test results are only a part of that and not the whole of what one has learnt; and it cannot be a good tool to judge how much one has learnt and how one has learnt. We should not only focus on end results but also on the process which carries a lot of meaning of learning.

## Taking a Motion into Emergent Ideas

It was the last of August but there was still summer around. The gentle westerly breeze brought the sense of last summer to the fullest. The chirping of birds around the pine trees reminded me of the midsummer. I went to the class and prepared my computer ready to take down important points during the discussion in class. I prepared 'illumination' to be connected to a friend who used to join the class from the North West of Wyoming. The windows were open to the east side of the room and the green lawn in the cemetery was partly visible across the 15$^{th}$ street due to tall pine trees.

The journey of the course moved forward in the second session with brief review of Kilpatrick's (1992) paper on "History of Research in Mathematics Education". Mathematics Education is a relatively new field of study and it does not have long history of research. The history of mathematics education is rooted to the history of mathematics education that has developed over the last two centuries as mathematicians and educators have turned their attention to how and what mathematics is, or might be, taught and learned in school. From the outset, research in mathematics education has also been shaped by forces within the larger arena of educational research. Research in mathematics education has struggled to achieve its own identity. It has tried to formulate its own issues and its own ways of addressing them. It has tried to itself and to develop a cadre of people who identify themselves as researchers in mathematics education. The purpose of research in mathematics education is manifold (Kilpatrick, 1992).

According to Kilpatrick, education was not considered a "discipline" to be studied. Germany (1779) and Sweden (1804) were the first countries to chair departments of education. In the U.S., New York University (1832), Brown University (1850) and University of Michigan (1860) led the way. Mathematics education programs began to develop at the end of the 19$^{th}$

century. Kilpatrick further discusses how research in mathematics education came to be recognized as a university subject. This area of study has been influenced from other disciplines and research in mathematics education was historical and philosophical studies, surveys, and other types of empirical research. Roots of research in mathematics education are traced as they relate, first to mathematics and second, to psychology. Researches in mathematics education were related to mathematical thinking, teaching, learning, testing of teaching and learning, curriculum, child and teacher attitude, motivation and other psychological factors of learning and teaching.

Kilpatrick further states that research in mathematics education started responding to areas of teaching, learning, and other aspect of pedagogy. Many teaching experiments started evaluating teaching, learning and curriculum issues. Educators started evaluating individual differences and whether or not all students would benefit from studying math. Even though its history is not a long history, research in mathematics education is a conversation that began well before today's researchers appeared and that will continue long after they have gone. It is a conversation with thousands of voices speaking on hundreds of topics.

To me Kilpatrick's paper provides a historical foundation of research in mathematics education. It also helps us to build an understanding of prospects and challenges in the research in mathematics education which is deeply rooted in the problems and challenges of mathematics education. Our discussion topics for the day (and future) were: constructing knowledge vs learning, developing mind, experience (external vs internal), mental activity, mental operations and schema (operational), items of knowledge, psychology in action (text and teaching), the child's construction of math, errors and unlearning, and affective dimensions of mathematical experience.

Our further discussion was concentrated on psychology and mathematics education with foundational perspectives, historical highlights, and guiding questions for further research. We also reviewed and discussed our point of views of psychology and mathematics education. Our discussion topics for the day (and future) were: constructing knowledge vs learning, developing mind, experience (external vs internal), mental activity, mental operations and schema (operational), items of knowledge, psychology in action (text and teaching), the child's construction of math, errors and unlearning, and affective dimensions of mathematical experience.

We planned for brief review of history of psychology of mathematics education with focus on Binnet's (1899) scientific pedagogy, mental testing theory of Gall (phrenology), and Galton (inheritance), Wertheimer's Gestalt (perception, productive thinking) and Dunker, Vygotsky (ZPD) and Krutetskii, connectionist assault on transfer (Thorndike after Pavlov, then Skinner's behaviorism) and then Gagne's theory, Judd, Buswell (social utility), Brownell (meaning theory of arithmetic), Dewey, Bruner, Dienes, Davis, Piaget, Montessori, von Glasersfeld, Steffe, and then cognitive science (information processing)---Brown, Greeno, and Skemp.

Then we came to discuss on some guiding questions: What theories of learning and cognitive development have influenced school mathematics? What are key aspects of children's mathematical thinking? How can we use ideas from constructivism to stimulate and guide children's construction of their mathematics? What are some important psychological considerations for teaching mathematical concepts, principles, generalizations, skills, problems, and structures? What are some important psychological considerations for helping children construct their knowledge of numbers, operations, arithmetic, geometry, measurement, algebra, statistics, and probability? What are some important psychological considerations for dealing with children's mathematical errors and unlearning? What are some important psychological considerations related to affective dimensions of children's mathematical experiences?

We decided that each of us (seven students) would lead a discussion turn by turn one in each session. Participant 1 decided to lead the session in intuition in learning mathematics. Participant 3 chose cognitive science and information processing, participant 5 chose Gestalt psychology, participant 7 preferred to present in Neo-Behaviorism and Gagne's Theory of learning, participant 6 chose psychology of discovery learning and Participant 2 chose phenomenology.

The dynamics of classroom discussion is greatly affected by the instructors lay out of the plan. The plan for discussion on psychology and mathematics education with foundational perspectives, historical highlights, and guiding questions for further research clearly showed that the instructor followed the guided discussion pedagogy with an active learning technique offering many benefits to us. This technique exposed us to a variety of diverse perspectives, helped us recognize and investigate our assumptions, improved listening and conversation skills, fostered connection to a topic, and affirmed us as co-creators of knowledge (Brookfield & Preskill, 2005).

To me the guiding questions were very powerful tool to envision our further discussion in the coming sessions. These questions carry the significance of this course and our participation in the course. The theories of learning and development in relation to mathematics, psychological considerations for teaching and learning mathematical concepts, principles, generalizations, skills, problems, and structures were very important areas that we identified as emergent ideas in the course. This session was totally a motivation session for me that aroused my interest twice it was before this session towards the subject matters laid out in the plan. Division of work for class presentation or leading a session by each participant was very important task. The democratic practice of selection of area of discussion that we would lead in future sessions was very impressive to me. We chose the areas for lead discussion as per our own interest from among many possible topics listed in the guideline.

**Gestalt Psychology and its Application in Mathematics Learning**

As per our schedule, it was turn of the participant 5 to lead a presentation in Gestalt psychology. She distributed power-point slides and brief note to each of us and started presenting on Gestalt psychology. What is Gestalt theory/psychology? What are laws of Gestalt theory? How Gestalt theory can be applied in mathematics education? These were some questions she discussed. She presented the idea of Gestalt psychology with graphics showing relationship of whole and parts…and whole is always greater than sum of all parts (Wertheimer, 1924). It is a large field of psychology and it is also known as phi phenomenon. She also discussed the laws of Gestalt such as: law of perceptual organization, law of pragnanz (law of simplicity), law of proximity, law of continuity, law of closure, and law of symmetry. She finally discussed the possible connection of Gestalt psychology to mathematics education. Parts and whole relation would be very good example in algebra…such as x and 2 are different numbers…and their whole in terms of x^2 or 2^x would be far different from the parts.

From her presentation and literature reviews, I came to know that Gestalt is a German word for '*form*' and it is applied in Gestalt psychology which means '*unified whole*' or '*configuration*'. The word Gestalt has been variously translated as shape, form, figure, and configuration (Westheimer, 1999). This theory is focused on the mind's perceptive. Gestalt was a holistic approach and rejected the mechanistic perspectives of the stimulus-response models. The essential point of Gestalt is that in perception, the whole is different from the sum of its parts or the whole is greater than just the sum of the parts. Gestalt theory influenced thinking and problem solving skills by an appropriate substantive organization, restructuring, and centering of the given in the direction of the desired solution (The International Society for Gestalt Theory and its Application, 2010).

Max Wertheimer developed Gestalt theory and applied this theory to problem solving. According to him, the parts of the problem should not be isolated but instead should be seen as a whole. This way the learner can obtain a new, deeper structural view of the situation. He developed a concept of ‘pragnanz’ which states that when things are grasped as whole, the minimal amount of energy is exerted on thinking. Pragnanz deals with bringing meaning and completeness to our beliefs, values, needs, and attitudes (Alok Jain, 2006).

Phi phenomenon is a perceptual experience that is different from the sum of the sensory elements. Wertheimer explained that we are seeing an effect of the whole event, not contained in the sum of the parts. This phenomenon led to the conclusion that elements sensed are not the only reality. She discussed law of closure which means missing elements are supplied to complete a familiar figure. The mind conceptualizes elements that are not perceived through sensation in order to complete a regular figure.

Law of similarity states that the mind groups similar elements into collective entities or totalities. This similarity might depend on relationships of form, color, size, or brightness. Next law she discussed was law of proximity which states that near objects are grouped together. Spatial or temporal proximity of elements may induce the mind to perceive a collective or totality. Law of symmetry states that symmetrical images are perceived collectively, even in spite of distance. Law of continuity states that piece in smooth continuation are grouped together.

The mind continues visual, auditory, and kinetic patterns. Finally, law of common fate states that elements with the same moving direction are perceived as a collective or a unit (Fultz, 2010).

Later Kohler tested on apes and chickens with detour problems. He found that animals learn with insight. His experiments with animal learning led him to conclude that animals exhibited insights when relations among stimuli and responses were learned, rather than simple stimulus-response connections critical to behaviorist theory. Impact of Gestalt psychology in all field of study was enormous. Psychologists who studied perception had great debate with Gestalt Psychologists. There were a lot of people who rejected Gestalt psychology simply being developed by German Psychologists.

For many psychologists behaviorism became foreground and Gestalt psychology became background. Gestalt psychology might have significance in teaching algebra and geometry. Changing patterns of complex algebraic expressions will lead to different characteristics showing the impact of parts on whole. Gestalt view was also a philosophy, epistemology, psychology and later it developed Gestalt therapy.

After presentation and discussion on Gestalt theory, we discussed Piaget and his theories of learning and development. He clearly spoke about not only cognition but also affect and role of genetics. Piaget was trained as a biologist and his focus of attention became his children and his home was his laboratory. He began developing situations/tasks for his children to explore their cognitive development. Later he founded Geneva Institute to continue his research and he got down to the trenches. We also discussed his theory of cognitive development into four stages: sensory motor stage, pre-operational stage, concrete operational stage and formal operational stage. To apply these theories in teaching and learning mathematics we should have multiple perspectives and requires differentiation and coordination. Matching of child's developmental readiness and what we ask kids to do are very important.

I think Gestalt laws have significance in learning and teaching geometry. "Thinking of a diagram as an object means associating specific figural properties with it, such as position or form. These considerations on what thinking about it as an object means, in Fischbein's way, refer to a mathematical object, this is abstract. The dichotomy between object and concept is related more to a theory need to include non-formalized mathematical aspects, such as position or form, than to the mathematical objects in themselves" (Claudia, 2010, p. 707). Further Claudia states that "A Gestalt-type configuration, as well as the intentionality of solution, should contain a reference to the relation between background and form, that is, Gestalt configuration "adjusts" to the general composition of the diagram" (p. 709). We can agree that the essential point of Gestalt the whole is different from the sum of its parts or the whole is greater than just the sum of the parts carries a powerful meaning in terms of geometrical interpretation of a figure, algebraic structures and functional relationships in various dependent and independent variables in daily life problems that we can bring in classroom as context of teaching and learning.

**Looking back at Piaget's Theory and Behaviorism**

Course calendar was at the center of our sessions and we considered worth reviewing it at the beginning of each session. This time too, we reviewed the course calendar to plan for our activities and discussions. We discussed about paper number two for our assignment with guidelines. Our presentations leading through different topics would constitute our paper number one. Before their presentation we discussed Piaget's first stage in theory of cognitive development that was sensory motor stage that included perception and semiotic functions, role of imitation, symbolic play and gesture, and three levels of transition from action to operation. Sensory motor stage is centered on the basis of schema and the first schemas of infants are to do with movement. A few weeks after its birth, the baby begins to understand some of the information it is receiving from its senses, and learns to use some muscles and limbs for movement. The baby begins to understand how one thing can cause or affect another. By around eight to twelve months, infants begin to look for objects hidden, and this is defined as "*object permanence*" (Edwards et al., 2010).

We further discussed the importance of development in terms of language. Vygotsky has acknowledged the language as a key aspect of development. Piaget's theory of genetic epistemology states that a child has neither innate ideas…nor a nativist theory…and nor is the child a tabula rasa with the real world out there waiting to be discovered. Instead mind is constructed through interaction with environment, what is real depends on how developed one's knowledge is.

According to Piaget, cognitive structures are the means by which experience is interpreted and organized. Early on, cognitive structures are quite basic, and consist of reflexes like sucking and grasping. Piaget referred to these structures as '*schemes*'. These schemes or cognitive structures develop through assimilation and accommodation. Here, assimilation means incorporation of new experiences into existing structures and accommodation means changing of old structures so that new experiences can be processed. According to Piaget, assimilation is *conservative process* whereas accommodation is *progressive process (Block, 1982)*. Why child tries to accommodate? Piaget says that the mind is in a state of equilibrium in normal condition that means existing structures are stable, and assimilation is mostly occurring. A discrepant experience can lead to disequilibrium or cognitive instability and the child is forced to accommodate existing structures. This way development proceeds as the child actively refines his/her knowledge of the world through many small experiments.

A child can make sense of quantities that require measurements. Children's operations are characterized by *generality, reversibility, shared* and *concrete*. Piaget talked about two kinds of reversibility: inverse thinking and reciprocity. According to him, child's conception of space is different from mathematician's idea of space. There is great impact of Piaget's development theory in mathematics teaching and learning at schools. "One contribution of Piagetian theory concerns the developmental stages of children's cognition. His work on children's quantitative development has provided mathematics educators with crucial insights into how children learn mathematical concepts and ideas" (Ojose, 2008, p. 26).

As a matter of usual schedule of lead discussion, the participant 7 presented on early behaviorism focusing on classical conditioning, connectivism, associationism, and operant conditioning. He introduced Ivan Pavlov's classical conditioning as a learning process that occurs through association between an environmental stimulus and a naturally occurring stimulus. He mentioned that classical conditioning can be applied in the class by creating a positive environment to help students overcome anxiety or fear. He further discussed the Thorndike's laws of connectionism as (1) learning requires both practice and rewards, (2) a series of S-R connections can be chained together if they belong to the same action sequence, (3) transfer of learning occurs because of previously encountered situations, and (4) intelligence is a function of the number of connections learned.

He discussed Robert M. Gagne's conditions of learning such as signal learning, SR learning, motor and verbal chain learning, multiple discrimination, concept learning, principle learning, and problem learning. Although Gagne's theoretical framework covers many aspects of learning, "the focus of the theory is on intellectual skills". Gagne's theory is very prescriptive. In its original formulation, special attention was given to military training (Gagne 1962, as cited in Kearsley, 1994a). Gagne proposed five major types of learning levels: verbal information, intellectual skills, cognitive strategies, motor skills, and attitudes. Hierarchy of learning tasks for intellectual skills according to complexity: stimulus recognition, response generation, procedure following, use of terminology, discriminations, concept formation, rule application, and problem solving. The primary significance of this hierarchy is to provide direction for instructors so that they can "identify prerequisites that should be completed to facilitate learning at each level" (Kearsley, 1994a). This learning hierarchy also provides a basis for sequencing instruction.

What are the implications of Gagne's theory in Mathematics Education? His programmed instruction has not gone away from Mathematics Education and many schools still practice this approach for teaching mathematics in schools. Geometer sketchpad and Cabri geometry can be of great help in programmed instruction. "The most pervasive example of an application of Gagne's theories and research to a large-scale curriculum project is Science: A Process Approach (SAPA). These science curriculum materials were influential in schools and colleges during the 1960s and early 1970s and represent a significant large scale curriculum effort utilizing Gagne's theories and research in the area of problem solving and scientific inquiry" (Fields, 1996).

It was nice discussion about Piaget's theory of child's assimilation and accommodation with the environment. The process of taking in new information into our previously existing schemas is known as assimilation. This process is somewhat subjective, because we tend to modify experience or information somewhat to fit in with our pre-existing beliefs.

Accommodation involves changing or altering our existing schemas in light of new information or new experience. New schemas may also be developed during this process and hence it is progressive adaptation. According to Piaget, all children try to strike a balance between assimilation and accommodation, and it is achieved through a mechanism called equilibration. Revision of behaviorist theories such as Pavlov's classical conditioning and Thorndike's operant conditioning were helpful to understand importance of re-enforcement in shaping behaviors. This can be applied in mathematics classroom in order to develop certain positive behavior through positive re-enforcement and may be negative re-enforcement helpful to stop some negative behavior in students that may have hindering impact in learning.

## Transfer Theory and Application in Mathematics Learning

As usual we reviewed the course plan and scheduled for presentation by participant 4 and 1. Participant 4 made his presentation on transfer theory of Thorndike. Transfer of learning occurs when learning in one context or with one set of materials impacts on performance in another context or with other related materials. For example, learning to drive a car helps a person later to learn more quickly to drive a truck; learning mathematics prepares students to study physics. "Transfer is a key concept in education and learning theory because most formal education aspires to transfer" (Perkins & Saloman, 1994, p.6452).

Learning requires a medium of transfer. Transfer learning is almost similar to ordinary learning but transfer learning occurs in a new situation in which learning from other context is applied. Transfer can be positive or negative. A positive transfer occurs when learning in one context improves performance in some other context. Negative transfer occurs when learning in one context impacts negatively on performance in another. Negative transfer typically causes trouble only in the early stages of learning a new domain. There can be near or far transfer of learning. Near transfer refers to transfer between very similar contexts. Far transfer refers to transfer between contexts that, on appearance, seem remote and alien to one another (Perkins & Saloman, 1994).

We further discussed research in transfer theory, E.L. Thorndike's contribution in development of transfer theory, his early research on animals and his law of effect and law of exercise. Thorndike did research in animal behavior before becoming interested in human psychology. His theory of connectionism states that learning is the formation of a connection between stimulus and response. His law of effect states that when a connection between a stimulus and response is positively rewarded it will be strengthened and when it is negatively rewarded it will be weakened. His law of exercise refers to the similarity between a training activity and the actual task one is training for. Actually, Thorndike pioneered the concept of active learning (Mergel, 2010).

Transfer of learning can be discussed as low-road and high-road transfer. Low-road transfer supposedly happens spontaneously and automatically, after a task has been practiced extensively in different environment similar to the expected transfer context. High-road transfer is most likely to happen when students consciously and deliberately abstract ideas or principles and mindfully search for connections and situations in which those principles can be applied. The amount of transfer is determined by the number of identical elements.

Participant 1 presented on intuition and science of learning. She questioned, "What then is mathematics if it is not a unique, rigorous, logical structure?" and she mentioned that, "It is a series of great intuitions carefully shifted, and organized the logic men are willing and apply to any time" (Morris Kline in Dehaen, 2009). She defined intuition as 'ah, hah" of a flash of light or illumination of on the road to the solution of a problem (Feferman, 2000). She further mentioned that the sequence leading to discovery as consisting, to begin with, of conscious preparatory engagement with the problem (Hadamard, 1949). Henry Poincare believed that the intuition constitutes a foundation for all mathematics. He saw inductive reasoning as the main conduit of intuition in mathematics. For Descartes, intuition meant a reliable source of true knowledge that opposed to misleading sense impressions. To Bergson intuition was described as a method which means that a mental strategy to search the essence of phenomena beyond logical argumentation. Piaget interpreted intuition as categories of cognition, directly grasped without a need for justification or re-interpretation (Bergsten, 2004). She further discussed intuition of numbers in infancy and early childhood, role of intuition in geometry education, Godfrey's geometrical eyes, the gap between intuition and formal knowledge, and technology as facilitator that triggers intuition.

Our further discussion continued on the topics: psychology of mathematical ability which is an ability to see generalization in one instance, automation of basic facts, intuition, analytical thinking and logic, some issues of new math, sequential learning, inductive and deductive reasoning, child conception of number, child's understanding of numbers, early number experiences, difference between intuition and perception, primary and secondary intuition, moment of intuition, lived intuition, Zeno's paradox of the tortoise and Achilles, factors influencing intuition and technology as a facilitator that triggers intuition.

Some people may consider the behaviorist and cognitive theories as outdated and not applicable in real context of teaching and learning mathematics. However, I think it was worth reviewing those theories and build a strong foundation of how learning theories emerged with time and contexts. I feel that transfer theory of learning has application in mathematics teaching and learning. I think mathematics learnt in class has great significance in out of classroom behaviors and vice-versa. Kids can experience transfer of graphical interpretation learnt in mathematics equally helpful in social studies class when they interprete the graphs of population. Discussion about intuition as a way of learning and its role in geometry learning, gap between intuition and formal knowledge were important aspects in relation to learning mathematics. Important take home message to me was that the amount of transfer from mathematics to other discipline or other disciplines to mathematics also depends upon extent of common elements across the disciplines.

## Cognitive Science, Information Processing, and Discovery Learning

Participant 3 led the discussion on cognitive science and information processing. She introduced cognitive science as combination of psychology, philosophy, linguistics, anthropology, neuroscience, and computer science (Eckardt, 1995). There is not a single specific contributor in cognitive science. There are many people engaged in this field from psychology, philosophy, linguistics and other fields. Specifically, she defined cognitive science as the science of mind and behavior. This relates to understanding knowledge acquisition and use as the key to understanding mind. Focus is on how we thinking, process information, and create mental models. Cognitive science is the study of the mind with emphasis on the use and acquisition of knowledge and information. The implications of cognitive science can be as interdisciplinary approach to study human mind and its conceptual approach is to explain information processing in terms of neural computations. Cognitive science studies the relation of motivation, intentionality and feelings to the experience of anything as being meaningful through explanation of behavior, perception or communication. For this, we have mathematical information theory at one side phenomenology, hermeneutics and semiotics at other side. Cognitive science is based on the mathematical information processing theory.

Cognitive science deals with information processing through the following steps: perception (acquiring real-time information about the environment), language use (making use of information about syntax, semantics, and phonology), reasoning (combining informations, deriving new information, testing consistency…), action (making use of information in action), and memory (storing and retrieving information). This theory compares our brain with a computer processor. According to this theory, our brain function is analogues to computer processor function. "Cognitive science is concerned with the kind of knowledge that underlies human cognition, the details of human cognitive processes, and the computational modeling of these processes" (Cohen & Lefebvre, 2005, p. 2). Thinking can be best understood in terms of representational structures in the mind and computational procedures that operate on those structures. This raises questions of how do we characterize mental representational structures and computational procedures?

She discussed history of cognitive science as a major shift from behaviorism, trying to objectify mind, trying to create mental processing….analogous to computer processing. Behaviorists removed the focus of the mind and how it works and alter. In 1956, George Miller studied memory (the magic number 7), and this marked a shift in research in the area of cognitive science. According to this theory, we create mental images of objects and process in our brain. Rules, images and connection come together to solve a problem (here mathematical problem). We try to connect the problem situation to our prior understanding as a metaphor of making connection, same as computers connect the information with history (in search engine) that searches and matches. This is the fundamental short coming of viewing brain as part of computer.

Cognitive theory can be used to build a bridge to mathematics education. There are differences in how experts and novices think and approach problems, novices lack abstract solution methods that expert hold, experts are able to identify or articulate key features of solutions but the novices were focused to a few features; and therefore, there are expert and novice way of mathematical problem solving. She raised the questions: Do you believe it is appropriate or useful to compare our minds to computers? Is there a room for affective component about how we think in cognitive science to consider this? The whole idea of cognitive science considers human brain as computer processor. Today's computers can take biofeedback data…like we take biofeedback data from environment. This has made cognitive science a mechanistic way of dealing with human brain as a machine.

Next presentation was led by the participant 6. The presentation was on 'Discovery Learning'. She connected different psychological theories and past experiences as foundation for experiential learning. Experiential learning leads to discovery learning. She introduced teaching methods adopted in discovery learning as inquiry based process, focus on learning through experience, learner building on past experiences, students interacting with environment, discovering facts and relationships on own, and students creating own construct of knowledge through narratives. She further mentioned that discovery learning as a method of learning where the learner does self-exploration, creates his or her own presuppositions mainly using what he or she already knows. The learner is able to learn by himself and therefore, the rate of retention is high, the ability to think and process ideas by oneself is also high. She further explained that the discovery learning is a tool and information for problem solving. Students interact with the process and the actual experimentation.

She mentioned that Dewey, Bruner, Piaget, and Vygotsky were major proponents of this method of teaching and learning. She quoted Veermans (2002) to introduce the five steps in the discovery learning: orientation, hypothesis generation, hypothesis testing, conclusion, and regulative processes. When do these occur? She mentioned that these occur during problem solving and cooperative learning.  She then raised questions: How does discovery learning occur in problem solving? What is discovery teaching and discovery learning? Then she explained that a constructivist approach of learning emphasizes discovery learning in which students rediscover the mathematical ideas. Mathematics has been considered to be a tough subject by many learners but it can be best understood by them if they are allowed to discover concepts for themselves and not to learn by rote.

She discussed advantages of discovery learning as active engagement, promoting motivation, ownership of learning, development of creativity and problem solving skills, and tailored with past learning experiences. She mentioned potential misconceptions about discovery learning as too much information may create cognitive overload, often requires vast resources unavailable in traditional classrooms, lack of teacher control, potential misconceptions, and teacher may fail to recognize misconceptions.

We further discussed discovery versus guided discovery. We have to aware of relativistic position of what is meant by discovery teaching and learning. The main feature of discovery learning is asking very provocative question and discover the way they come to understand the notion of mathematical problem (asked in the question). “What are some other key mechanisms of discovery learning?” We talk about discovery teaching but we don’t have much information about discovery learning. The major problem in this approach is how to keep students focused on the problem.

"The potential of discovery learning can be realized by the discovery itself: learning domain knowledge and ideally evoking a certain degree of excitement, and by the discovery process: learning discovery learning skills, thus enhancing the chances of a discovery to occur in another place and another time through transfer of the discovery skills" (Veermans, 2005, p. 1). I like Veermans' stress on chances of discovery to occur in another place and another time. This is beauty of discovery learning that motivates students to think as a discoverer or inventor of mathematics.

To me cognitive science and information processing is a model that deals with mechanical approach of viewing learning as processing in the brain analogous to computer processing. This theory undermines the human nature of thinking, constructing and progressive manipulation of ideas through experience, intuition and perception. There is no room of intuition in this theory but to me mathematics learning is more intuitive than just mechanical processing. Discovery learning may have relation with discovery teaching. I think discovery learning is inductive learning in which learner is posed with problems or situation and then he or she discovers mathematical relationship with minimal or no guidance. Though this approach may take longer time for learning a certain concept or construct but I think it is meaningful learning.

## A Glimpse of Phenomenology

The session started with usual review of course calendar to see where the course and discussion reached. Then it was turn for participant 2 to lead a session for the day. He led the session with a presentation on phenomenology. He discussed phenomenon and phenomenology. Phenomenon is any observable event or occurrence that we can perceive through our sense organs, and some defined it as a rare or exceptional or unusual or significant event. Phenomenology investigates for the very nature of phenomenon, it asks what something is without which it would no longer be what it is (Van Manen, 1990). Phenomenological research is the study of lifeworld (Van Manen, 1990). Phenomenology can be described as the systematic attempt to uncover and describe the internal meaning of structure of lived experience (Van Manen, 1990).

The Stanford Encyclopedia of Phenomenology defines phenomenology as the study of structures of consciousness as experienced from the first person point of view. It further states that the central structure of an experience is its intentionality, its being directed toward something, as it is an experience of or about some object. Phenomenology studies the structures of various types of experiences ranging from perception, thought, memory, imagination, emotion, desire, and volition to bodily awareness, embodied action, and social activity, that includes linguistic activity too. Phenomenology involves the use of thick description and close analysis of lived experience to understand how meaning is created through embodied perception (Sokolowski, 2000). Benz and Shapiro (1998) claimed that the primary tool of the phenomenological study is the in inquirer's own consciousness. According to Trochim (2006) phenomenology is a school of thought that emphasizes a focus on people's subjective experiences and interpretations of the world and the phenomenologist wants to understand how the world appears to others.

He introduced briefly Husserl's (1859 - 1938) logical investigations (1900-01), ideas (1913), Heidegger's (1889 – 1976) being and time (1927), Sartre's (1905 – 1980) nausea (1936), being and nothingness (1943), and Merleau-Ponty's (1908-1961) phenomenology of perception (1945). These historical foundations were important aspects for the development of phenomenology as epistemology, philosophy and psychology. We further discussed about contributions of contemporary phenomenologists such as Max Van Manen's Researching Lived

Experience: Human Science for an Action Sensitive Pedagogy (1990, 1997), Robert Sokolowski's Introduction to Phenomenology (2000), David Woodruff Smith and Amie L. Thomasson's Phenomenology and Philosophy of Mind (2005), and Uriah Kriegel and Kenneth Williford's Self-Representational Approaches to Consciousness (2006).

We then discussed variations of phenomenology. Transcendental phenomenology included Intentionality : all our thinking, feeling, and acting are always about things in the world, epoche or bracketing is suspension of beliefs in order to understand the phenomenon fully, eidetic reduction is to looking into the past through the particularity of lived experience, transcendental reduction suspends the judgment of everything in the world including ego, constitution of meaning: according to Husserl meanings are constituted in and by consciousness. Phenomenology becomes hermeneutic when its method is taken to be interpretive. It claims that there are no such things as un-interpreted phenomena (Van Manen, 1990). The fundamental mode of human existence is not detached knowing but, rather, engaged activity (Heidegger, 1962). Therefore everything one does is interpreted activity. For Heidegger, human being is a 'Dasein' which means being in the world. Being in the world refers to the way human beings exist, act, or are involved in the world. To Husserl everything can be purely described and to Heidegger everything can be interpreted. For Heidegger, meanings are not given directly to us but require interpretation. Merleau-Ponty (1962) says phenomenology is the study of essences. His phenomenology is existential, oriented to lived experience; the embodied human being in the concrete world. Emphasis is on understanding human being in his existential world. It is concerned with being, human existence, and a person's concrete way of thinking.

Phenomenological psychology deals with the meaningful experience as fundamental locus of knowledge (Polkinghorne, 1989). Human behavior is an expression of meaningful experiences rather than a mechanically learned response to a stimulus (Polkinghorne, 1989). Phenomenological psychological research emphasizes on subject's experienced meaning instead of descriptions of their overt actions or behavior. Giorgi (1985 in Ganeson 2006) explores four themes of phenomenology: description, reduction, essences, and intentionality. We discussed phenomenological research which involves a basic methodological structure of a dynamic interplay among six research activities as discussed by Van Manen (1990).

The first step we discussed was *turning to a phenomenon which seriously interests us and commits us to the world.* For this we start from our personal lived mathematical experiences as a starting point. We use written protocols of personal lived mathematical experiences at different moments that we lived as a student and/or teacher. Those protocols help us to connect our lived mathematical experiences to others as they lived in different contexts. One way this makes us conscious of subtleties of personal lifeworlds connecting self with others developing both inside and outside perspective of observing the phenomena of experiences. This perspective requires that we, as researcher, need to engage in the "Epoche processes" of making explicit and setting aside our prejudgments, biases, and preconceived ideas about the phenomena being studied so that we can understand the experiences as lived (Moustakas, 1994 as cited in Hart & Swars, 2009).

Next step in this study is *to investigate experience as we lived it rather than as we conceptualized it* (Van Manen, 1990). In a direct sense, this step involves communication with others to unfold their lived mathematical experiences as they lived. We explore the essences of lived experiences of mathematics educators through different methods such un/semi-structured interviews, reflective journal writing and our experiential protocols. To me, narrative interviews are appropriate method for disclosing the meanings of lived experience (Lindseth & Norberg, 2004). The interviews are one-one informal talk based on the research questions or it can be group discussion. There may be a few opening questions that we may ask participants during the interviews but these questions should be only a starting point for the further discussion. We may ask subsidiary questions with some comments and remarks in depth while staying close to the experience as they lived (Hart & Swars, 2009). The interviews are audio/video recorded for transcribing and analysis.

The third step in this study is *reflecting on the essential themes which characterize the phenomenon* (Van Manen, 1990). The interview records are transcribed into texts and essences of living/lived experiences in different contexts are identified as essential themes. It is the stage of observing our experiential anecdotes and textual transcriptions of interviews of 'mathematical experiences' lived by the participants. The interview transcriptions, reflective journals and our personal protocol are used to identify the essence of experiences (as themes) with respect to time, space, things, the body and others that truly constitute the nature of the specific lived mathematical experiences.

The fourth step is *to describe the phenomena through the art of writing and rewriting* (Van Manen, 1990). Van Manen further states that phenomenology is the application of logos (language and thoughtfulness) to a phenomenon (an aspect of lived experience), to what shows itself precisely as it shows itself. Bringing the essence of lived experiences as the participants lived in different contexts to speech is most commonly a writing activity as suggested by Van Manen. The experiential descriptions from the analysis of interview transcriptions, reflective journals are supported by our experiential descriptions in the protocols through careful review of similarities and differences.

The next step in such study is *maintaining a strong and oriented pedagogical relation to the phenomenon* (Van Manen, 1990). In my understanding the art of writing and rewriting should have the quality of richness, depthness, orientation, and pedagogical wakefulness and thoughtfulness (Denzin & Lincoln, 2005). Phenomenological writing as research is a way to connect the consciousness of self with others and vice-versa and helping the others to be reflexive along their lived experiences as they lived while reading the text. In this process we emphasize constructing a text that is dialogic in nature that connects us with readers advancing the level of consciousness of what we experience beyond what we see or feel at a moment that constituted lived mathematical experience (Van Manen, 1990).

The final step is *balancing the research context by considering parts and whole* (Van Manen, 1990). In phenomenological sense, every part in experiential world has important role for constitution of meaning of a whole and vice versa. There should be a balance between research contexts focusing on the various steps to be taken for the research. For this we should be aware about effects and ethics of the research on research subjects, researcher, people who are associated with research, institution and society as a whole.

Hans Freudenthal (1983) in *Didactical Phenomenology of Mathematical Structures* has both conceptualized and exemplified a new area of critical inquiry in mathematics education. *All the* major concepts of elementary mathematics are dissected, analyzed, explored, viewed mathematically, and viewed phenomenologically (Zalman Usiskin, 1985). Yuichi Handa (2006) in his report on *Variations on the Phenomenology of Knowing and Understanding Mathematics: Through (the Lens of) Relationship* turns to a "hermeneutic phenomenological approach to human science research and writing" (Van Manen, 1990) that pairs the interpretive/hermeneutic tradition with the descriptive/phenomenological orientation in researching pedagogically related phenomenon. Finally, the participant 2 proposes some questions for research: How do students experience about the concrete skills or abilities involved in mathematical creation? Are they engaged in meaningful and soulful mathematical creation in learning mathematics? Is there a reflective interplay between knowledge and action while learning mathematics?

Our further discussion on the day covered foundations of contemporary psychology, John Dewey, Vygotsky, and Bandura's social psychological theory. Also we discussed attribution, the factors of authority, helplessness, social and cultural contexts as critical, major factors of child development, and Russian psychologists. Schools in Russia are still based on Vygotsky's ideas of sociocultural theory in which primary idea is environment and culture. We also discussed in brief about Maria Montessori's theory of learning in which environment is considered as a primary element that plays significant role in learning. Other elements of her theory we discussed were mediation with culture and society, psychology of play, and imaginative child. Children begin to build their own illusive world, rich world of activity and imagination, and in transitional stage – a stick becomes a pivot for serving meaning of horse from a real horse, imagination as very powerful starting point, but at some point meaning of the imagination has to change. We discussed interrelationship between language development and thought, images and words are objects of thought, Vygotsky's interest in self and self regulating process of internalization, emphasis on ZPD, preadolescence and the propositional operations, problem solving were discussed as central piece in child's cognitive development.

I think the third question proposed by the presenter 2 was very important in relation to application of phenomenology in mathematics teaching and learning. The reflective interplay between knowledge and action while learning mathematics should be a powerful connection with phenomenological implication of mathematics. The lived mathematical experience of teachers and students can reveal some important characteristics of mathematical experience that can be helpful to develop a more just, equitable, student friendly, contextual and meaningful mathematics curriculum and pedagogy. The discussion on Maria Montessori's theory of learning by play was important take home message to me. According to this theory, man's natural tendency is toward learning and growth. When this tendency is nurtured and cultivated in childhood, a young person can direct himself into a happier, healthier, and more productive life. The basic aim is to free the individual child's potential for self development in a prepared environment.

## Personal Beliefs about Learning

Our discussion began with the question: How do we learn mathematics? The course instructor encouraged us to come up with our personal theories of learning and identify what factors might have facilitated in our learning mathematics and what factors might not have been helpful to us in learning. He asked us to draw our attention as a teacher or leaner and identify aspects of learning and share our experiences about how we learned certain mathematical ideas. Participant 5 mentioned that she struggled a little bit…about how we learn but not about how others teach us. She liked to sit down and work on something like ax^2+bx+c. Participant 3 assumed a bigger picture and specific picture…time of the day…could learn best early in the morning. Participant 1 mentioned that as a student when classroom environment and motivation was a key for learning, e.g. she preferred to learn math in environment with visual impacts, posters, computer, a round table ….and the physical environment. Participant 2 mentioned that he preferred night time for learning at the loneliness. He further mentioned that he could learn better by teaching others (peers), also in group practice, learning and understanding from basic ideas. Participant 7 mentioned that he was very visual and he had to see things while learning, needed active involvement. Participant 4 said that for him a visual was not major factor, but a focused attention. The participants showed a diversity of their beliefs and practices about what they thought of learning and how they could learn best.

To me these views clearly indicated toward the juxtaposition of two opinions: the view of teaching as a 'direct transmission' with learning seen as an 'acquisition' of knowledge and a 'cognitive construction' view of learning (Schlichter, Watermann, and Nueckles, 2010). The participant's views were not explicitly clear about the teaching and learning patterns: 'transmission' vs. 'construction' views. Though we can see 'acquisition' vs. 'participation' metaphors (Sfard, 1998), and 'content-centered' vs. 'learning-centered' teaching (Kember & Kwan, 2000). We further discussed the questions: What an observer is doing during observation? Whether there is processing, or active participation? If we don't do something during observation, then there is no learning. Show-and-tell method of teaching is dominant these days which does not engage kids in learning but it encourages memorization of facts and information. The current demand is dynamic discourse (interactive dialogue) in which a teacher may be *asking-good-questions* and *listening-to-students*, trying to use what they are trying to say. In fact, questioning should exhibit thinking and processing and there are processes of learning and ways of thinking mathematics such as making errors, recognizing, correcting, trial and error ( you have to perceive the error and you have to make decision what to try…and think of next step). We discussed intellective or cognitive development.

Further discussion continued with the question: Does problem solving involve learning? Lots of people know how to solve habituated problems but we should encourage non-routine problem solving. Learning itself is *a-problematic-situation* and learning gradient may not be the same all the time. If a student does not grasp the responsibility of learning as their business, learning does not occur. National Council of Teachers of Mathematics (NCTM) in Principles and Standards for School Mathematics (2000) states problem solving standards as: build new mathematical knowledge through problem solving, solve problems that arise in mathematics and other contexts, apply and adapt a variety of appropriate strategies to solve problems, and monitor and reflect on the process of mathematical problem solving. These standards are guidelines for teaching and learning mathematics in schools but they are not mandatory for the schools to follow. According to Resnick (1987) a problem solving approach contributes to the practical use of mathematics by helping people to develop the facility to be adaptable when, for instance, technology breaks down. Cockcroft (1982) also advocated “problem solving as a means of developing mathematical thinking as a tool for daily living, saying that problem-solving ability lies 'at the heart of mathematics” (p.73) because it is the means by which mathematics can be applied to a variety of unfamiliar situations (Taplin, 2010).

Next question was: What’s your orientation of your learning? Some of us opined that orientation of learner is the major factor in learning, and we have to re-orient learner for learning, parents support is key factor for students learning (not only teachers and schools). I think some of participants in our class had learning orientation as *interactive learner*. They preferred subjects that dealt with people rather subjects focused on technical details. They liked to create, speculate, write, and interpret, and they tended to be benefited from study groups. I think some participants in our class matched with *spontaneous learner*. They like action and tend to be easily distracted when passive learning situations arise. Some of us were *organized learners* who preferred structure, details, and advanced notice. They seemed to manage time effectively and plan ahead. They needed detailed instructions and with clear expectations. Some of us were *conceptual learners* who thrived on the opportunity to work independently. These participants enjoyed asking questions, seeking answers, and solving complex problems. Also, they were keen to understand how information was important to them by examining it analytically and objectively.

Further discussion question was: Is the student self determined? Or the student is dependent? We need to pay attention on time to reflect (take a lot time to process), practicing and reflecting, questioning (how widely applicable), reflecting can lead to progress (incubation), reflection as “haunting”. We discussed strategy of looking-back and looking-ahead (looking at whole and then looking at parts). Next question we discussed was: What could we say about factors about learning fractions and decimals? What are other learning factors? How we learn Euclidean algorithm for greatest common factor of two natural numbers? Learning may be affected by a felt need. Learning algorithm should be preceded by prior understanding of what it is.

Then finally we discussed on paper (assignment) # 3. We were suggested to select a learning session of 60-90 minutes, who participated (subjects), would not be a problem, we (EMAT students), could serve as participants or we could invite anybody suitable to us (even family members or friends). The course instructor encouraged us not to be worried about generalization and we could limit our thinking through very specific and very limited activities. We had to stay on mathematics for the learning session. We needed to focus on: What are the tasks, situations, modeling….make decision about what you were going to do as activity that showed that the participant learnt? Then we needed to assess learning (being careful that person learned...but the person should not have that idea before). This way we wanted to create evidence of learning but assessing what participant already learnt would be an issue. We could document and describe: What did the learner do? Could we study process of learning or we could study only results of learning? How did we know the participant learnt? There were all kinds of ways to show that they learnt. We should pay attention to process of learning not the results of learning.

For the learning session (paper # 3) we discussed if there could be some teaching activities such as posing situations, asking questions, talking to each other, etc. Yes, but it should not be an object of analysis and the sole focus of the study should be learning. Procedure might not have to be in algorithm and it could be solving problem without algorithm, it might involve teaching but focus should be on learning. Once we chose an activity, we would seek a few literatures if anybody had done anything about what we were going to do and the main focus was asking good questions. We had to identify some prospects of this study and see that what some possibilities were. The instructor advised us that the descriptions of teaching should be done as conditions of learning…but focus should be on learning.

I think the idea of *asking-good-questions* and *listening-to-students* were very important things that we discussed in this session. Our personal views about how we learnt were interesting to share in the class and know each other's way of learning. The diversity of learning methods or approaches was interesting characteristics among us and I think these differences were outcome of our cultural and social contexts. The way I learnt might not be necessarily same with learning of my colleague from Turkey or Saudi Arabia or the U.S. Discussion on developing paper number 3 was very helpful to decide about the structure and content of the paper more or less.

**Again on Our Personal Beliefs about Learning**

We continued our discussion about personal theory of learning. Our focus was on differentiation of learning and learnt. We discussed that learning theories drawn heavily from psychology and philosophy and often our thinking is focused on teaching, people describe about teaching for particular types of learning. In terms of discussion on how students learn, what actually they do for their learning…and this part is much more missing in literature. The instructor posed the discussion questions: “Is learning harder to know about than teaching? What do you think about learning? What is learning for you? Do we know moments of learning? How do we know we have learnt?”

May be by indicators that shows the results of learning but we should not get confused about our teaching with learning. Motivation matters, attention matters, some level of intensity matters in learning and there is more to be learned …and learning is very illusive and it is hard to describe learning. Psychologists study the behavioral path of learning…they have not discussed much about learning. Processes of learning are not an easy way…they are too complex and learning is in fact a very complex matter…and we are so far away from adequate description of learning. We might not be able to look at construction in our mind during learning.

Participant 1 shared her experience of learning binomial distribution theory in statistics and later she expressed that she reconstructed her learning. Then we could take the word reconstruction off and again think of learning and what was there as learning. Some people put on some lenses and view learning through those lenses. One of the conditions for learning was connections to prior learning (prior experiences). We could say learning as transforming. We saw that a fish may not be able to explain sea because it is in it. Our learning looks like as a path or trajectory …in a stream of consciousness. Neuroscience describes the structure of brain but still far to describe learning. Most important thing for us is to create conditions of learning….and lets guard about evidences of learning. In this way the instructor was drawing our attention toward very meaning of learning.

"What does it mean to learn? Usually what we are doing is describing conditions of learning. We talk about teaching and impact of teaching on learning but we cannot observe learning. We observe either the conditions of learning or consequence of learning. We are still in the darkness about the process of learning and understand about what is learning. There is gap between learning and consequences of learning or conditions of learning." The instructor continued explaining difference in learning and learnt. Teachers are agents of change in behavior and learning is permanent change in behavior (for some people). What is classroom learning? Learning in classroom may not occur in a way as expected…much learning occurs outside classroom. We should be careful when we talk about learning and consequences of learning.

Revisiting our personal beliefs about learning and our own ways of learning provided an insight about viewing learning through our personal epistemologies. Emphasis on 'how we learn' helped me to understand the process of learning as important aspect that we should strive to promote. The lack of literatures to describe learning can be that it is more elusive and it is too personal. The process of learning differs from person to person and situation to situation and there might not be any grand theory of this process but there are many theories of learning which mention the conditions of learning. Whether we study behaviorism, cognitivism, constructivism or any other theories of learning, they describe conditions of learning. To me learning can be discussed as a trajectory and its nature is different for different persons.

## Epistemology of Learning Mathematics

We reviewed course calendar to see what we had to do further. We then were reviewing and trying to think about history of psychological theories that possibly speak to mathematics learning and development. Next we were discussing about human development. Our discussion was followed by the distinction between learning and what is learnt (discussed last week). There is research on what is learnt…but rarely do they talk about learning. Our new focus of discussion was epistemological frameworks of learning and development.

There are different formulations of psychologies in the classroom such as item of knowledge perspective, epistemology, role of epistemology etc. What are some characteristics of epistemological framework? The way we approach that we reach to mathematics teaching and learning. Epistemology is theory of knowledge. "What are epistemological foundations of learning mathematics…might there be other side too? How we know it…why we know it….? When we know about it?" The instructor raised these questions for discussion in the session. Knowledge is a phenomenon that is extremely complex and difficult to describe. We need to focus on what we exist for. We as mathematics educators/teachers exist for helping construct mathematical knowledge. Purpose of education is probably for knowledge. "What we mean by knowledge? What do you mean by gaining knowledge, extending or increasing knowledge?" He asked us. Politics related to mathematics education is enormous. How you view the knower and how you view the knowledge? Knowledge has been heavily viewed as what we can produce in a test such as glory, power, beauty and significance about knowledge. We need to distinguish a performance and performer. Knowledge is equivalent to performance (some theories say). Engaging in performing can enhance knowledge.

The participant 4 brought a point that a person if has enough practice in computer can have knowledge about computer without formal training or learning opportunities. It is possible to become highly knowledgeable even without entering into a classroom (through practice and study). "What is difference between skill and knowledge?" Participant 1 asked. How skillful behavior fits in knowledge, it is still not much known. For proficiency both knowledge and skill are needed. Knowledge of equilateral triangle is not about the triangle that we see on the board but it is our construction in mind, the figure on the board might be only a representation of our knowledge of equilateral triangle. Knowledge is dynamic and goes on changing and adapting with situations that we experience. We only can talk about metaphors; we don't have anything to see it.

There is a different philosophy which assumes that the tree is out there. We understand tree better today than in last 50 years. Important distinction is how about who knows it. We can transfer representation of knowledge across the generation. Prenatal knowledge has been a point of discussion. There is perspective that knowledge is actively constructed but not transferred from one person to another except just passing some representation of knowledge. Knowledge cannot be transmitted. Knowledge is dynamic, continue learning, and processed all the time. Learning is so ill defined….and we only know little about that. "One set of beliefs that influence teaching and learning fall within the domain of personal epistemology (PE). The research on PE demonstrates that teachers' beliefs about knowledge and how one comes to know influence their practice" (Bondy et al, 2007, p.68).

We should begin process of construction and reconstruction of knowledge. Knowledge is too complex and it is difficult (almost impossible) to define in a few words. Whole enterprise of Mathematics Education is to foster child's development of mind and understanding mathematics. Standardized tests are extremely poor tool to measure knowledge. As species (human being) we have a lot potential to develop what people have today (in relation to mathematics). Goal of education is and should be developing knowledge. When we talk about epistemological framework…we have more to say about this. Thompson and Saldanha (2000) mentioned epistemological analysis (EA) when they were investigating the question, "What does it mean to understand x, and how might people develop such an understanding?" They intended to produce an epistemic subject, a framework for creating models of individual persons' mathematical thinking and knowing that explain why they behave as they do, both individually and in interaction with others.

"Let's think about a question", the instructor oriented us to next situation. "In your opinion or based on readings, what are some epistemological ideas that seem to be appearing in mathematics education literatures?" We were encouraged to think about how the way we think about knowledge (mathematical knowledge). What may be things that shape ways are we think about knowledge? How do you think of knowledge of novice in relation to skill they have? We are barely in the position to distinguish mind and brain; we don't have enough information about knowledge in mind or brain. When we talk about epistemology of learning mathematics and research, two predominant perspectives on approaches to knowing have influenced research in mathematics education: *the cognitive constructivist perspective* and t*he sociocultural perspective* (Cobb, 1996 as cited in Muis, 2004). According to Cobb, *the cognitive constructivist perspective* views students as actively constructing individual ways of knowing by establishing coherence across various personal experiences. *The sociocultural perspective* emphasizes the socially and culturally situated nature of activity (Cobb, 1996). The idea of the cognitive constructivist perspective was inspired by Piaget's (1971, 1977) genetic epistemology and sociocultural perspective was based on Vygotsky's work on cultural transmission (Cobb, 1996).

Rescher (2003) states that the mission of epistemology, the theory of knowledge, is to clarify what the conception of knowledge involve, how it is applied, and to explain why it has the features it does. Rescher further states that knowledge claims can be regarded from two points of view: *internally-commitally* and *externally-detachedly*. The first view is subject to an acceptance thereof as correct and authentic. The second view is subject to epistemic distance without the commitment of actual acceptance and seen as merely representing purported knowledge. There is, I think, epistemic war (many people call it paradigm war) due to these two view points to see mathematics knowledge or way of knowing mathematics.

Though we started discussion from epistemology of learning but we engaged more on distinguishing knowledge and knowing. The implicit discussion on positivist epistemology (knowledge is out there and we have to discover it) and constructivist epistemology (knowledge is constructed by our mind based on interactions with environment) was very interesting. Review of literatures on epistemology of learning mathematics helped me to understand two perspectives: the cognitive constructivist perspective and the sociocultural perspective, as recent and more dominant epistemological foundations of mathematics learning. I think, this certainly needs further discussion to widen our understanding epistemological foundations of learning and development.

## Further Discussion on Epistemology

We started the session by looking at the course calendar to see where we were. We began our discussion by giving testimony to role of epistemology in mathematics learning and developing. We were to discuss different epistemological frameworks. How does a theory of knowledge seem to influence our classroom teaching and learning? “Any question about role of epistemology in mathematics education?” the instructor asked us. Epistemology influences the way we teach and the way our students learn. He encouraged us to think on these questions: How do kids think about mathematical knowledge? How do you know something is true about mathematics? How you come to know something? How does this impact lives in classroom?

Epistemology of mathematics teaching and learning has strong root in Piaget's genetic epistemology (1971, 1977) which was an interdisciplinary approach to understanding human intellectual, moral, and social development (Thompson & Saldanha, 2000). Thompson and Saldanha further mentioned that genetic epistemology made deep connections among biology, philosophy, psychology, and logic, and used both structural and functional approaches to understanding what might constitute human knowledge. "Vygotsky (1978) stressed the Zone of Proximal Development (ZPD), whereby a learner is engaged in social interaction with another more knowledgeable person, and culturally developed symbols and signs as psychological tools for thinking" (Muis, 2004, p. 322).

What is genetic epistemology? How does this theory contribute in knowledge? Why this theory is important? These are some questions that I wanted to know further. I used online Wikipedia to understand the basics of genetic epistemology. Wikipedia defines genetic epistemology as a study of the origins of knowledge. The goal of genetic epistemology is to link the validity of knowledge to the model of its construction. According to Jean Piaget (1971), genetic epistemology attempts to explain knowledge, and in particular scientific knowledge, the basis of its history, its sociogenesis, and especially the psychological origins of the notions and operations upon which it is based (www.marxist.org). In genetic epistemology, as in developmental psychology, too, there is never an absolute beginning. We can never get back to the point where we can say, "Here is the very beginning of logical structures."

What is sociocultural theory? How sociocultural theory contributes in learning mathematics? How this theory is different from genetic epistemology? Learning Theories dot com site lists the major themes of sociocultural learning theory of Vygotsky. The first theme states that social interaction plays a fundamental role in the process of cognitive development. In contrast to Piaget's understanding of child development, Vygotsky felt social learning precedes development. Vygotsky (1978) states that every function in the child's cultural development appears twice: first, on the social level, and later, on the individual level. The second theme assumes more knowledgeable other. The more knowledgeable other refers to anyone who has better understanding or a higher ability level than the learner, with respect to a particular task, process, or concept. The more knowledgeable other is thought of as a teacher, coach, or older adult, it also can be a peer, or even younger. The third theme is about the zone of proximal development (ZPD). The ZPD is the distance between a student's ability to perform a task under adult guidance and/or with peer collaboration and the student's ability to solve problem independently. According to Vygotsky, learning occurs in this zone. Vygotsky's sociocultural theory of human learning describes learning as a social process and the origination of human intelligence in society or culture.

Next we discussed a little about item of knowledge perspective. It is not a recent perspective; it has been around for long time. It can be found in theories of Dewey, in behaviorism, and recent theories of learning. The place where it has been most crystallized is in Robert Gagne's theory. Gagne and many others call it concepts, principles, generalizations, rules, properties, algorithms, forms of representation. We can even see it down to the level of proof, definitions and principles…and found in each piece or items. A big part of theory of knowing is the knowing of pieces. We know the cat, behavior of cat, variations of cat….cat as in item of knowledge.

We then discussed structuralist view of knowledge which assumes that knowledge is structured. We might refer to grammar as structure of language; poetry might refer to structure of certain idea. If A: B = C: D then, AD = BC is a structure that represents an item of knowledge.

Can we think of a fraction as a ratio? Yes, we can. Structuralism was a dominant domain of epistemology in 70s and 80s. Structuralism focuses on concept mapping and learning hierarchy. Is knowledge a finished product? What is role of process of development or generation? Learning trajectories suffer from this problem because probably it is over simplification. Parsimony is a notion of reduction (tries to reduce complexity through simplification). One of the beauties of human knowledge is unpredictability. Mathematics being built by knower is missing part of mathematics knowledge. There is conflict in intent in different epistemological grounds.

Participant 5 opined that we can never finish understanding mathematics….it is then silly to consider mathematics as a finished good. Structuralist perspectives…knowledge are representable in a form as finished product. Let's consider the concept of trapezoid. What about concept learning? What we do with concepts? Actually, definitions are proved (they are not theorems). What is a concept? What makes something about an object makes a concept of object? Personal pedagogical epistemology guides us how we teach and learn.

We discussed about neo behaviorism which was the second phase of behaviorism. Neo behaviorism was associated with Tolman, Hull, and Skinner. Like Thorndike, Watson, and Pavlov, the neobehaviorists believed that the study of learning and a focus on rigorously objective observational methods were the keys to a scientific psychology. The instructor encouraged us to think about the role of epistemology. Presenting mathematics theory as finished items may not work for many of our learners. Mathematical models have certain qualities or features. Students need to deconstruct and reconstruct a model; they have to think about that. In today's technological world, models are important ideas. "Can a problem be model?" asked participant 1. In what sense a statement of a problematic situation becomes a model?

Instantiation, many instantiations that fit to an idea create a model. Clearly a deductive system is arbitrary and we can have axiomatic systems…do such systems create knowledge? People think about knowledge of mathematics that impacts in their teaching and learning of mathematics. In mathematics one of the most basic process ideas that we advocate is the idea of abstracting. Example: What is a five? The concept of five is idea that is bound up with numerocity. Five exists only in a place that is in one's mind. What is triangle? If you draw a figure with three sides bounding an area, then it is representation of triangle. The concept of triangle is not in the picture but in the mind.

Without attending learning might not take place. Mathematics is a coding system, knowing similarity depends in part in our coding, a process of particularizing, and a process of instantiating (important process). To understand fractions, one has to coordinate two dimensions and in such situation varying is a tremendous process. Idea of reasoning a variation is important process. Process of quantification and quantifying is a powerful concept in mathematics. These are all mental processes that we conceptualize in our mind. Interface between mental action and physical action both are involved in creation of knowledge. We need to pay attention to epistemological processes that connects our learning. There are broader psychological meanings in these processes while building abstraction. This is nature of knowledge that we continue processing. Learning how to solve a problem is more important than solving many problems. Some epistemological contrasts are: objects of knowledge, items of knowledge and process of knowledge. These contrasts help us to understand nature of mathematics and mathematics learning. Finally, it is worth mentioning here "there is always an epistemic gap between subjective experience and objective fact" (Rescher, 2003, p. 62). We can only minimize this gap through established or emerging epistemic methods addressing the epistemic issues through epistemic valor.

**Items of Knowledge and Readiness and Mathematical Thinking**

We looked at the course calendar and saw where were on the day and what we needed to discuss. The instructor mentioned about a journal published by graduate students of UGA. This online journal "The Mathematics Educator (TME) is a student produced journal. It is published by Mathematics Education Student Association (MESA) in the Department of Mathematics and Science Education at the University of Georgia. Also, MESA is an affiliate of the NCTM. He encouraged us to write papers and submit to this journal and they might publish it if the paper meets its standard.

Then we discussed about paper (assignment) # 2, if time allows working through draft would be a good step. The instructor encouraged us to try to complete our work on paper # 2. Then we planned for informal meeting on the Tuesday before the Thanksgiving holiday in order to complete our paper # 2 and 3. Then we planned for the last session during the exam week of December 6-10. He suggested us to practice for presenting paper # 3 in the class on December 7th. The presentation could be supported by a hand out or power point slides. We also fixed our class session earlier than usual class time (from 12:15 to 3:15).

We began discussing the subject matter with a question: What do we find prevailing in terms of focal points of this course Learning and Development? It is always risky to generalize about these matters. If we look at the literatures of mathematics education (journal articles…), what we may find about what frames the learning and development. Mostly we find items of knowledge such as: concepts, skills, generalization, proofs. We itemize and list all kinds of items of knowledge in mathematics. It is Neo-behaviorism kind of perspective which emphasizes on objects of knowing…and there is nothing about process of knowing. We can read what teachers talk about in their writing (in papers). Mostly they talk about teaching, strategies of teaching and rarely people talk about students' learning, their strategies of learning, and processes of learning.

Understanding a graph is a process and graphing is process of plotting in which there is one-to-one correspondence. An experience of creating a new coordinate system (a student invented an oblique coordinate). We can see items of knowledge which is one perspective and items of processes is another perspective. Then we should strive to promote the second perspective. For this, readiness is an important factor in learning. Readiness is quite general term and literatures also speak to developmental readiness. Developmental readiness connects with some developmental potential. We have holes in our knowledge…that prevents us in succeeding in new learning situation. A seven year child is not ready for dealing with fractions (in general). We have lack of readiness …both developmental and background knowledge. Literature is not rich mentioning about these kinds of readiness for learning. There are limited research programs on readiness factors such as Vygotsky's ZPD.

We discussed that kids in Wyoming are not ready for Algebra II, we could ask questions such as: Why they are not ready? Do we know they are not ready? How do we know they are not ready? College Algebra may be partly High School Algebra with some higher degree of abstraction. Many students are still not good in Algebra….might be the reason that they were not ready, they didn't learn….the question is why didn't they learn? Participant 4 opined that maybe there was no connection between what they learn in school and outside of school. Readiness is more important than motivation. Is the mathematics relevant to the learner?

Our further discussion were about motivation, intrinsic and extrinsic motivation, delayed gratification and instant gratification, disconnection as a social problem, and grouping students based on their readiness background factors for differentiation in instruction. When knowledge is counted as commodity then the concept of items of knowledge is validated. The items of knowledge are ways to map knowledge to accomplish certain task. That is to say that to accomplish B, one has to first accomplish A. There are series of knowledge items with required time to acquire those items. This itemization of knowledge has commoditized knowledge in economic value in terms of how long one needs to impart/acquire that knowledge. The aim of knowledge valuation is to identify these knowledge items that are considered to be more valuable (Kingston, 2001). The idea of knowledge valuation in terms of items of knowledge introduced the concept of knowledge auditing and calculation of opportunity cost of knowledge. Mathematics knowledge also became the subject of auditing through items of knowledge and items of process. To me this perspective is related quantifying knowledge in terms of items and processes. There are not sufficient literatures in this area.

A paper by Tall (1994) was worth reviewing which discusses issues of developing advanced mathematical thinking, the comparison of knowledge in mathematics, visualizing mathematical concepts, using symbolism to mathematical compress process into concept, sequential and procedural compression, transition to formal mathematics, can we teach students to think mathematically, and reflections on mathematical thinking. Tall further discusses cognitive principles in order to focus on the value of cognitive development of student. These cognitive principles should be helpful to understand the cognitive structures and foster mathematical thinking in different situations. Cognitive principle I states that for survival the individual must maximize the use of his or her cognitive structure by focusing on concepts and methods that work, discarding earlier intermediate stages that no longer have value. The cognitive principle II states that the brain has a small focus of attention and a huge space for storage and therefore cognitive growth needs to develop. His third cognitive principle states that a powerful agent in learning with understanding is by going through mathematical constructions for oneself and then reflecting on one's own knowledge- thinking about thinking.

He mentions various methods of comparison of knowledge in mathematics, including: representing information visually, using symbols to represent information compactly, and if a process is too long to fit in the focus of attention, practice can make it routine so that it no longer requires much conscious thought. These processes lead a student to the transition to formal mathematics. Students usually find formal mathematics in conflict with their experience. It no longer about precept-symbols representing a process to be computed or manipulated to give a result. The concept in formal mathematics is no longer related so directly to objects in the real world. This may create conflict of understanding mathematics from context or valuing mathematics in a context but it may also be an opportunity to view mathematics within cultures in a tacit way.

The discussion session and review of papers helped me to understand difference between items of knowledge and items of process. Also, there should be more emphasis on items of processes to promote meaningful learning. Though we have frames of constructivist classrooms and teaching…and still we lack constructivist frame of learning or how students go through processes of learning. To me it is easy to talk about end rather than processes in between two cognitive stages or learning situations. This has been a hindrance for meaningful learning. A lot of literatures on teaching but a very few in processes of learning indicates that we are still far to understand processes of learning mathematics. Items of knowledge in mathematics are, narrowly, only a part of items of information in relation to mathematics contents. Existence of items of knowledge reminds us the existence of items of ignorance in foreground and errors of commission and errors of omission in the background (Rescher, 2003). Authentic mathematical experiences can bring this gap closer with avoidance of epistemic ignorance.

## Epilogue

There was beautiful summer at the start of the semester. The weather was fine and I could walk around with a shirt and pant without putting on a jacket. There was high enthusiasm for the courses and classes. But slowly the weather changed and the flora changed around the Prexy's Pasture. I like changes and I think it is inevitable of nature that brings changes in the weather with seasons and accordingly changes can be seen in the floral community around the Pasture. The squirrels continue playing around the pine trees though we cannot see birds around due to waning temperature that invites snowflakes instead of rain drops and makes the land full of white cover everywhere till the end of the semester. Then I put on warm clothes and snow-boot to be safe from the wind chill and walk on the snow. Slowly the festive mood grows in the small town with Halloween and Thanks Giving that continues with the Christmas trees and colorful lights signaling the coming of Christmas. The enthusiasm reaches to the pick during the finals week when I was busy to complete papers, reports, and journals and preparing for tests in the first week of the December.

Every session in the course '*Theory and Research in Mathematics Learning and Development*' went very well with great discussions on diverse topics related to theory and research in mathematical learning and development. Session one was an opportunity to know each other's personal theory of learning and development. Our diversity of personal theories made the class discussions even more lively and interesting. We formed questions which, in most of the cases, reflected our personal epistemology. We discussed foundational perspectives (FP), historical highlights (HH), and guiding questions (GQ) related to mathematics learning and development. I think these three key areas: FP, HH, and GQ became the central in all of our sessions explicitly or implicitly. Certainly, this attempt to bring the voices of the participants is never complete in such personal reflective writing, however, I tried to reflect upon the major ideas that contributed in my understanding of foundational issues of mathematical learning and development.

I learnt from the sessions and review of literature about Jean Piaget's cognitive development theory which assumes that intelligence constructs the structures that it needs to function; knowledge is an interactive process between the learner and the environment. Lev Vygotsky's sociohistorical theory of psychological development assumes that the processes of human mental development are part of the process of historical development. Albert Bandura's social-cognitive theory assumes that learning is a three-way interaction among the environment, personal factors and behavior. Bernard Weiner's attribution theory assumes that the search for understanding is a primary motivator. Skinner's operant conditioning assumes that learning is behavior, behavioral change, represented by response frequency, is a function of environment and conditions. In Robert Gagne's conditions of learning, human learning in all its variety is the focus of study. Learning is more than a single process, and these distinct processes cannot be reduced one to other. Next theory we discussed was information processing theory which assumes that human memory is a complex and active processor and organizer of information that transforms learning into new cognitive structures. These assumptions were the key player in the epistemology and philosophical foundations of the respective theories. These assumptions were the basic characteristics of these theories which guided the practices of teaching, learning and research in education (and mathematics education).

We had opportunity to lead presentations in the class. I think, participant 5's Gestalts, participant 3's cognitive and information processing, participant 7's neo-behaviorism with a focus on R. Gagne's theory, participant 6's discovery learning, participant 1's intuition and science, participant 2's phenomenology, and participant 4's transfer theory provided great opportunity to develop and understating of the major psychological theories. The detailed discussion of theories of Piaget and Vygotsky were very impressive. We discussed epistemological foundations of mathematical learning and development. I think we discussed all major theories of learning and development in the thirteen sessions. Review of literatures to develop understanding of epistemic distance, epistemic gap, epistemic circumstances, epistemic issues, epistemic valor, epistemic justification, and epistemic policy in relation to pedagogy and research in mathematics education. I can claim that the learning journey contributed in our ontologies, epistemologies, and methodologies of developing an understanding of *praxis* in the field of mathematics education and our journey certainly progressed toward *epistemic students* through the support of *epistemic subject* and *epistemic teacher (professor)*.

I feel that we were lucky to get the professor as our course instructor who himself is a witness of major changes in the history of theory and research in mathematical learning and development in the United States in the last fifty years. In his every speech in the class, I found history of theory and research in mathematical learning and development. In every question he asked or raised, I found a voice for change in the current teaching and learning practices which have deprived students of being active learner, creator, and inventor of their mathematics. Group dynamics of the class was very interesting phenomena going on throughout the semester. Though we did not work in group for any presentation or paper writing but every session itself was a dynamic group. The active participation of each class member with questions, discussions and presentations made the class live and enthusiastic. Finally, I would like to conclude this journey with contemplation of a few research questions: How do the dominant theories of learning and development interplay among knowledge, action and behavior of a child in relation to mathematics? How do these theories address the issues of complexity and chaos of classroom teaching and learning? How do the theories of mathematics learning and development inform and guide us in the practices of teaching, learning and research through techne, poesis, praxis, dialogos, phronesis, polis and theoria?

**Acknowledgements**: I would like to acknowledge the voices of all participants (anonymous in this paper) in the class which helped me to develop this reflective journey.

**References**

Benz, V.M., & Shapiro, J.J. (1998). *Mindful inquiry in social research.* Thousand Oaks, London, New Delhi: SAGE.

Bergsten, C. (2004). *Exploring the gap between intuitive and formal knowledge in mathematics*. Regular talk at ICME 10, Copenhagen, (published in the ICME 10 proceedings). Online retrieved on November 20, 2010 from: http://www.icme10.dk/proceedings/

Block, J. (1982). Assimilation, accommodation, and the dynamics of personality development. *Child Development*, 53, 281-295.

Bondy, E., Ross, D., Adams, A., Nowak, R., Brownell, M., Hoppey, D., Kuhel, K., McCallum, C., & Stafford, L. (2007). Personal epistemologies and learning to teach. *Teacher Education and Special Education*, 30, 2, 67-82.

Brookfield, S., & Preskill, S. (2005). *Discussion as a way of teaching: Tools and techniques for democratic classrooms*. (2nd ed.). San Francisco, CA: Jossey-Bass.

Claudia, A. (2009). Gestalt configurations in geometry learning. *Proceedings of CERME 6 (pp. 706-715)*, January 28-February 1, Lyon France.

Cobb, P. (1996). Where is the mind? A coordination of sociocultural and cognitive constructivist perspectives. In C. T. Fosnot (Ed.), *Constructivism: Theory, perspectives, and practice* (pp. 34–52). New York: Teachers College Press.

Cockcroft, W.H. (Ed.). (1982). *Mathematics counts.* Report of the Committee of Inquiry into the Teaching of Mathematics in Schools, London: Her Majesty's Stationery Office.

Cohen, H., & Lefebvre, C. (2005). Bridging the category divide. In H. Cohen & C Lefebvre (Eds.), *Handbook of categorization in cognitive science* (pp. 2-15). New York: Elsevier.

Creswell, J. W. (1998). *Qualitative inquiry and research design: choosing among five traditions*. London, New Delhi: SAGE

Dehaene, S. (2009). Origins of mathematical intuitions: The case of arithmetic. Ann NY Academy of Science, 1156, 232-259. Online retrieved on December 10, 2010 from the World-Wide-Web: http://www.unicog.org/biblio/Year/2009.html

Denzin, N.K., & Lincoln Y.S. (Eds.). (2005). *The discipline and practice of qualitative research. In Handbook of qualitative research* (3rd ed., pp. 01-32). Thousand Oaks, CA: SAGE.

Duffin, J. M., & Simpson, A. P. (1995). A theory, a story, its analysis, and some implications. *Journal of Mathematical Behavior*, 14, 237-250.

Edwards, L., Hopgood, J., Rosenberg, K., & Rush, K. (2010). *Piaget's stages of cognitive development*. Online retrieved on December 20, 2010 from the World-Wide-Web: http://ehlt.flinders.edu.au/education/DLiT/2000/Piaget/stages.htm

Feferman, S. (2000). Mathematical intuition vs mathematical monsters. *Synthese*, 125, 317-332. Retrieved on November 22, 2010 from the World-Wide-Web: http://www.springerlink.com/content/m7782730151vr164/fulltext.pdf

Fields, D. (1996). *The impact of Gagne's theories of practice*. Paper presented as a part of the proceedings of Selected Research and Development at the 1996 National Convention of the Association for Educational Communications and Technology, Indianapolis, IN. ERIC No. ED397794 retrieved from: http://www.eric.ed.gov/PDFS/ED397794.pdf

Fultz, J. (2010). *Gestalt psychology theory*. Online retrieved on December 23, 2010 from the World-Wide-Web: http://homepages.ius.edu/rallman/gestalt.html

Gagne, R. (1962). Military training and principles of learning. *American Psychologist*, 17, 263-276.

Ganeson, K. (2006). *Students' lived experience of transition into high school: A phenomenological study.* A doctoral dissertation, Queensland University.

Giorgi, A. (1985). (Ed). P*henomenology and psychological research.* Pittsburgh, PA: Duquesne University Press.

Gubrium, J., & Holstein, J. (2000). *Analyzing interpretive research* (2nd ed.). Thousand Oaks, CA: SAGE.

Handa, Y. (2006). *Variations of phenomenology of knowing and understanding mathematics: Through (the lens of) relationship.* A doctoral dissertation, University of Delaware.

Hart, L.C., & Swars, S.L. (2009). The lived experiences of elementary prospective teachers in mathematics content coursework. *Teacher Development, 13, 2, 159 — 172.*

Heidegger, M. (1962). *Being and time.* Oxford: Basil Blackwell.

Jain, A. (2006). *Gestalt theory*. Online retrieved on December 23, 2010 from the World-Wide-Web: http://alokjain.pbworks.com/w/page/12300948/GestaltTheory

Kearsley, G. (1994a). Conditions of learning (R. Gagne). Online retrieved on December 18, 2010 from World-Wide-Web: http://tip.psychology.org/gagne.html

Kember, D., & Kwan, K. (2000). Lecturers' approaches to teaching and their relationship to conceptions of good teaching. *Instructional Science*, 28, 469-490.

Kilpatrick, J. (1992). A history of research in mathematics education. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 3-38). New York: Macmillan.

Kingston, J. (2001). Ontologies, multi-perspective modeling and knowledge auditing. *Proceeding of the second German/Austrian Conference on Artificial Intelligence*, September 2001.

Mergel, B. (2010). *The basics of behaviorism*. Online retrieved on December 23, 2010 from http://faculty.mdc.edu/jmcnair/Joe14pages/basics_of_behaviorism.htm

Merleau-Ponty, M. (1962). *Phenomenology of perception*. London: Routledge and Kegan Paul.

Moustakasa, C. (1994). *Phenomenological research methods*. Thousand Oaks, CA: Sage Publications.

Muis, K. R. (2004). Personal epistemology and mathematics: A critical review and synthesis of research. *Review of Educational Research*, 74, 3, 317-377.

National Council for Teachers of Mathematics. (2000). *Principles and standards for school mathematics*. Reston, VA: The Author.

Perkins, D., Saloman, G. (1994). Transfer of learning. In T. Husen and T. Postlethwaite (Eds.), *The international encyclopedia of education* (2nd ed., pp. 6452-6457), 11. Oxford: Elsevier.

Piaget, J. (1971). *Genetic epistemology*. New York: W.W. Norton.

Piaget, J. (1977). *Psychology and epistemology: Towards a theory of knowledge.* New York: Penguin.

Piaget, J., & Beth, E. W. (1966). *Mathematical epistemology and psychology*. Boston: D. Reidel Publishing Company.

Polkinghorne, D.E. (1989). Phenomenological research methods. In R. S. Valle & S. Halling (Eds.), *Existential phenomenological perspectives in psychology* (pp. 41-60). New York: Plenum Press.

Rescher, N. (2003). *Epistemology: An introduction to the theory of knowledge*. Albany, NY: State University of New York.

Resnick, L. B. (1987). Learning in school and out. *Educational Researcher,* 16, 13-20.

Schlichter, N., Watermann, R., Nueckles, M. (2010). Personal beliefs about learning and teaching: Comparison of student teachers in the science and humanities at different stages of their studies. *Proceeds of the 9th International Conference of the Learning Sciences*, 1, 564-571.

Sfard, A. (1998). On two metaphors for learning and the dangers of choosing just one. *Educational Researcher*, 27, 2, 4-13.

Smith, D.W. (2003). Phenomenology. In *Stanford encyclopedia of philosophy*. Online retrieved from http://plato.stanford.edu/entries/phenomenology/

Sokolowski, R. (2000). *Introduction to phenomenology*. Cambridge: Cambridge University Press.

Sokolowski, R. (2000). *Introduction to phenomenology*. Cambridge: Cambridge University Press.

Steffe, L. P. (2007). *Basic problems in mathematics education research*. Powerpoint presentation retrieved from www.sigrme.org/Steffe_slides_AERA2007.ppt.

Tall, D. (1994). *Understanding the processes of advanced mathematical thinking*. An invited ICMI lecture at the International Congress of Mathematicians, Zurich.

Taplin, M. (2010). *Mathematics through problem solving*. Retrieved on December 10, 2010, from www.mathgoodies.com/articles/problem_solving.html.

The International Society for Gestalt Theory and its Application. (2010). *What is Gestalt theory*? Online retrieved from http://www.ipria.org/publications/wp/2004/IPRIAWP06.2004.pdf

Thompson, P., & Saldanha, L. (2000). Epistemological analyses of mathematical ideas: A research methodology. *Proceedings of the Twenty-second Annual Meeting of the International Group for the Psychology of Mathematics Education*. Tuscon, Arizona.

Trochim, W.M.K. (2006). *Research methods knowledge base. In web center for social research methods*. Online retrieval from http://www.socialresearchmethods.net/kb/qualapp.php

Van Manen, M. (1990). *Researching lived experience: Human science for an action sensitive pedagogy*. Ontario: Althouse Press.

Van Manen, M. (1990, 1997). *Researching lived experience: Human science for an action sensitive pedagogy.* Ontario, Canada: The Althouse Press.

Veermans, K. (2002). *Intelligent support for discovery learning*. Doctoral dissertation, University of Twente. Enschede, The Netherlands: Twente University Press.

Vygotsky, L. S. (1978). Mind and society: The development of higher mental processes. Cambridge, MA: Harvard University Press.

Wertheimer, M. (1924). Ueber Gestalt theorie. Lecture before the Kant Gesellschaft [reprinted in translation in W. D. Ellis (Ed.), (1938), *A source book of Gestalt psychology* (pp. 1-11). New York: Harcourt, Brace.

Westheimer, G. (1999). Gestalt theory reconfigured: Max Wertheimer's anticipation of recent developments in visual neuroscience. *Perception*, 28, 5-15.