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Aspects of a Neoteric Approach to Advance Students' Ability to Conjecture, Prove, or Disprove

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Abstract

Aspects of a Neoteric Approach to Advance Students' Ability to Conjecture, Prove, or Disprove Through Inquiry-Based Learning Using A Modified Moore Method.

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The author of this paper suggests several neoteric, unconventional, idiosyncratic, or unique approaches to beginning Set Theory that he found seems to work well in building students' introductory understanding of the Foundations of Mathematics. This paper offers some ideas on how the author uses certain 'unconventional' definitions and 'standards' to get students to understand the essentials of basic set theory. These approaches continue through the canon and are employed in subsequent courses such as Linear Algebra, Probability and Statistics, Real Analysis, Point-Set Topology, etc.

The author of this paper submits that a mathematics student needs to learn how to conjecture, to hypothesise, *to make mistakes*, and to prove or disprove said ideas; so, the paper's thesis is learning requires *doing* and the point of any mathematics course is to get students to do proofs, produce examples, offer counterarguments, and create counterexamples.

We propose a quintessentially inquiry-based learning (IBL) pedagogical approach to mathematics education that centres on exploration, discovery, conjecture, hypothesis, thesis, and synthesis which yields positive results - - students doing proofs, counterexamples, examples, and counter-arguments. Moreover, these methods seem to assist in getting students to be willing to make mistakes for, we argue, that we learn from making mistakes not from always being correct!

We use a modified Moore method (*MMM*, or M^3) to teach students how to do, critique, or analyse proofs, counterexamples, examples, or counter-arguments. We have found that the neoteric definitions and frame-works described herein seem to encourage students to try, aid students' transition to advanced work, assists in forging long-term undergraduate research, and inspirits students to do rather than witness mathematics. We submit evidence to suggest that such teaching methodology produces *authentically* more adept students, more confident students, and students who are better at adapting to new ideas.

I. Introduction, Background, and the Modified Moore Method.

This paper is one of a sequence of papers ([69], [70], [71], [73], [74], [75], [76], [77]) the author has written over the last decade discussing inquiry-based learning (IBL) and the modified Moore method (M^3) that he employs whilst teaching mathematics, directing research, and doing mathematics himself. We seek to describe the method, the reasons for employing the method, the successes or the lack thereof we have had using the method, how the method has modified and evolved over the years, and the roots of the method which have stayed constant. The purpose of this particular paper is to highlight several neoteric, unconventional, idiosyncratic, or unique approaches to beginning Set Theory that the author has found seems to work well in building students' introductory understanding of logic and naïve set theory. This paper offers some ideas on how the author uses certain 'unconventional' definitions and 'standards' to get students to understand the essentials of basic set theory. These approaches continue through the mathematics canon and are employed in subsequent courses such as Linear Algebra, Probability and Statistics, Real Analysis, Point-Set Topology, etc.

We note that mathematics is built on a foundation which includes axiomatics, intuitionism, formalism, logic, application, and mathematical constructivism.[1] Proof is pivotal to mathematics as reasoning whether it be applied, computational, statistical, or theoretical mathematics. The many branches of mathematics are not mutually exclusive. Oft times applied projects raise questions that form the basis for theory and result in a need for proof. Other times theory develops and later applications are formed or discovered for the theory. Hence, students' education in mathematics ought be centred on the encouragement of individuals thinking for themselves: to conjecture, to analyze, to argue, to critique, to prove or disprove, and to know when an argument is valid or invalid. Perhaps the unique component of mathematics which sets it apart from other disciplines in the academy is proof - - the demand for succinct argument from a logical foundation for the veracity of a claim such that the argument is constituted wholly within a finite assemblage of sentences which force the conclusion of the claim to necessarily follow from a compilation of premises and previously proven results founded upon a consistent collection of axioms.

The author of this paper submits that we humans do have a natural inquisitiveness; hence, we should encourage inquiry for students to learn. We also submit that students must be active in learning. Thus, the student must learn to conjecture and prove or disprove said conjecture. Ergo, the author of this paper submits the thesis that learning requires doing; only through inquiry is learning achieved; and, hence we posit that the experience of creating a mathematical argument is a core reason for an exercise and should be advanced above the goal of generating a polished result.

To place a student in a situation where finished, polished, or elegant

[1] As opposed to educational constructivism which is discussed in this paper. Hence, 'constructivism' when mentioned in this paper means educational constructivism and mathematical constructivism shall be called such.

proofs are presented denies the process of constructing proofs (a "proof appreciation" class). To place a student in a situation where finished proofs are unobtainable - - ever - - denies the student the pleasure in creating an end product. This means that the instructor must construct a carefully crafted set of notes that is an axiomatic introduction to a subject and then builds into more and more complex problems. Further, the instructor ought constantly monitor the progress of individual students and adjust the notes or offer 'hints,' where appropriate at beginning levels. The two, experiential process and final product, cannot be disconnected. Thus, to paraphrase John Dewey, the ends and the means are the same.

Adoption of said philosophy is not enough - - it must be practiced - - hence, the author submits that the method of teaching that he suggests is a modified Moore method (M^3) that he has used successfully in teaching mathematics from the freshman to graduate level. Particular attention will focus on the introductory 'proofs,' 'logic,' or 'transition' courses (herein referred to as a Foundations course) that have entered the mathematics canon in the past forty years and how neoteric, unconventional, idiosyncratic, or unique approaches that are a part of the author's M^3 create authentic learning opportunities for students and how the M^3 is enacted in the classroom.

We opine that it has become rather accepted in mathematics education to include in the canon a course or courses which transition students from the Calculus sequence to upper division course work. R. L. Moore created or adapted a pseudo-Socratic method which bears his name ([24], [40], [41], [42], [43], [60], [119], [120], [121], [124]). He said, "that student is taught the best who is told the least."[2] It is the foundation of his philosophy and it sums up his philosophy of education simply, tersely, and succinctly. Moore believed that the individual teaches himself and the teacher is merely an informed guide who must not trample on the individual's natural curiosity and abilities.[3] The Moore method accentuates the individual and focuses on competition between students. Moore, himself, was highly competitive and felt that the competition among the students was a healthy motivator; the competition among students rarely depreciated into a negative motivator; and, most often it formed an *esprit d' corps* where the students vie for primacy in the class.[4] However, the Moore method is, perhaps, best suited for graduate-level work where there is a rather homogenous set of students who are mature. Moore's philosophy of education is too often considered a method of teaching and, as such, can be adopted and practiced.

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[2] R. L. Moore, *Challenge in the Classroom* (Providence, RI: American Mathematical Society, 1966), videocassette. See, also, Miriam S. Davis, *Creative Mathematics Instruction* Ed.D. dissertation (Auburn, Al: Auburn University, 1970), 25; Benjamin Fitzpatrick, Jr., "The Teaching Methods of R. L. Moore." *Higher Mathematics* 1 (1985): 45; Lucille S. Whyburn, "Student Oriented Teaching - The Moore Method." *American Mathematical Monthly* 77, 4 (1970): 354; and, David Zitarelli, "The Origin and Early Impact of the Moore Method." *American Mathematical Monthly* 111, 6 (2004): 465.

[3] See Davis, *Creative Mathematics Instruction* and Paul R Halmos, How To Teach. In *I Want To Be A Mathematician* (New York: Springer-Verlag, 1985) for a more detailed discussion of Moore's tenets.

[4] See Davis, pages 21 and 119 and D. R. Forbes, *The Texas System: R. L. Moore's Original Edition* Ph.D. dissertation (Madison, WI: University of Wisconsin, 1971), pages 168, 169, 172, and 188 for a detailed psychological analysis of the formation of *esprit d' corps* from competition.

The author opines that this is an error and it is a philosophy of education (see [72] for a detailed argument of said position). Therefore, adoption of the methods that Moore created and practiced would be meaningless and could lead to harm for the students if the practitioner did not subscribe to Moore's philosophy. Whyburn notes that Moore's beliefs "gives one the feeling that mathematics is more than just a way to make a living; it is a way of life, an orderly fashion in which you want to consider all things."[5] Thus, several authors have proposed a modified Moore method for teaching mathematical sciences courses ([12], [13], [34], [42]. [80]) which could be used in a transition to higher mathematics course. These like the Moore method, require that the individual learn without the aid of books, collaboration, subject lectures, and demands (uncompromisingly) talent from the individual.[6]

Our modified Moore method in some classes allows for books. We opine we should not be afraid to direct students to books or use books ourselves; but, we should train our students to use them wisely (for background (pre-college) material mostly) and perhaps sparingly. We should not be inflexible with regard to the use of a computer algebra system (CAS) (such as *Maple* or *Mathematica*) on *rare* occasions to allow the student to investigate computational problems (in Probability and Statistics or Number Theory, for example) where or when such might be helpful to understand material or form a conjecture. We should not be fearful of directing students to a CAS or use a CAS ourselves; but, we should train our students to use them infrequently and wisely. We must be very careful with a CAS for it can become a crutch quickly and there are many examples of students who can push buttons, copy and paste syntax, but not understand why they are pushing the buttons, what is actually the case or not, and are very convinced that crunching 10 quintillion examples proves, for example, a claim let us say a universal claim in $\mathbb{R}$.

A methodology that is not wholly opposed to our M^3 but we believe suffers from sophistic and relativistic pitfalls is constructivism. There are several authors who opine that a constructivist approach to teaching a mathematics course ([52], [88], [105], [114]) is the proper method. The constructivist accentuates the 'community,' focuses on cooperation amongst students - - the group rather than the individual - - and views mathematics as a constructed 'reality' that is relative to each constructor. The constructivist approach includes alternate assessment, group projects, service learning, etc. and closely resembles pedagogically the National Council of Teachers of Mathematics ([83]) standards and Dewey's position ([29], [30], [31]). Our M^3 does not negate or deny objectivism or realism; hence, it is not in the constructivist tradition. Furthermore, the constructivist position focuses, seemingly, more on process and less on content as opposed to this M^3 approach (see [77] for details on this point).

Let us accept, for the sake of argumetn and this paper, that (a) different individuals learn in different ways, (b) there is a basic knowledge base that is necessary for the average student to obtain so that he has a higher likelihood to succeed in upper division courses. It is not as commonly accepted, perhaps, but is argued that (c) proving claims true or false is a skill that can be mastered through brief exposition, much practice, and mostly individual inquiry. Much of the general educational

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[5] Whyburn, page 354.

[6] D. Reginald Traylor, *Creative Teaching: Heritage of R. L. Moore* (Houston, TX: University of Houston Press, 1972): page 171.

research of the twentieth century seemed to be centred on (a), thus we shall not bother wasting paper addressing in detail this point. Much work of professional associations (in particular the Mathematical Association of America (MAA)) research and policy statements of the late twentieth century centred on defining (b) and revising, enhancing, and reviewing (b) ([18], [19], [20], [21], [22], [23], [84]). Proving claims true or false being a skill is grounded in the philosophy of William James and the practice of George Pólya. Just as art schools teach composition techniques, architecture schools teach drafting, etc. schools of mathematics teach theorem proving as a skill that is grounded in logic. There are a finite number of techniques and students are encouraged to learn each one so as to have a basic competency when approaching mathematical claims.

However, the skill of proving or disproving claims is not something that is learnt through reading other people's work or watching a master teacher demonstrate his or her great skill at doing a proof (no errors, elegant, and compleat). The students learns by getting his 'hands dirty' just as an apprentice mechanic must get his hands dirty taking an engine apart and putting it back together. He must explore the parts of the engine, learn how the parts interact, take things apart, put the part or the whole engine together again, make mistakes (leave a part out so the engine does not work or put the parts together incorrectly so the engine does not work), learn from the triumphs and mistakes (**but most especially we learn from our mistakes** (more on this point in section four of this paper). Proving or disproving claims not only is a skill but takes initiative, imagination, creativity, patience, perseverance, and hard work. The nuts-and-bolts of the skill can be taught; but, initiative, discipline, imagination, patience, creativity, perseverance, and hard work can not be taught - - such must be nurtured, encouraged, suggested, and cultivated.

Our modified Moore method (*MMM*)[7] amends several philosophical positions from traditionalism, the true (or original) Moore method, and constructivism. We borrow heavily from Moore's philosophy of education but relax several aspects of the Moore method. Moore's philosophy of education stated is that a person learns alone - without help or interference from others. We hold that a person learns best and most completely alone; *but*, sometimes needs a bit of help, encouragement, or reinforcement. The Moore method assumes the student has a natural inquisitiveness, he must be active in learning, and as a consequent self-confidence and self-directedness is established and builds within the individual.[8] However, the student is not always going to perform at peak efficiency given the constraints of human nature and the diversions of modern society. Therefore, our M^3 assumes the natural inquisitiveness ebbs and flows or intermittently turns on or off much as a distributor cap distributes a charge in an engine or a heart alternates between pulsing and relaxing.[9]

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[7] Herein when referring to an instructor who uses our M^3 in a classroom, sometimes the term '*MMM* or IBL (inquiry-based learning) classroom' will be used to refer to the classroom experience.

[8] See Davis, pages 17, 78, and 173; D. R. Forbes, *The Texas System: R. L. Moore's Original Edition* Ph.D. dissertation (Madison, WI: University of Wisconsin, 1971), page 181; Traylor, page 13; and, Whyburn, page 354.

[9] We are not taking a negative position regarding a student's attention or ability to focus but we argue that it is not perfect. Moore demanded much and it might have been more facile to expect greater focus of a student in 1910 or 1960 as opposed to 2010. This is not a critique of

The Moore method demands that the student not reference any texts, articles, or other materials pertaining to the course save the notes distributed by the instructor and the notes the individual takes during class. Not every student is as mature and dedicated as to be able to follow such a regulation especially in an undergraduate setting and most especially in a transition course. Thus, books are not banished in our M^3 classroom. The student is encouraged to use as many books as he opines is necessary to understand the *background or foundation for* the material; thus, a student may refer to and reference material that is pre-requisite to the course. The student is allowed to read about material that has been discussed, presented, or was tested over *after* such had been discussed, presented, or was tested over.[10] We do not seek maximal coverage of a set amount of material, but standard competency in a given field with some depth and some breathe of understanding of material under consideration. This requires time, flexibility, and precise use of language.

Our M^3 philosophy of education and pedagogical perspective seeks to engage students in a balanced depth and breathe understanding of material with much concern for the precise use of mathematical language, mathematical notation, and proper language. The precise use of language is a key component of our method and a central focus of our paper since some of the language we use is inimitable; original; and, distinctive to the author. We have invented some words, created some definitions, and established some notation over the course of the author's teaching career such that, we claim, said assists students' learning.

The Moore method demands that the students not collaborate. Moore stated this position clearly:

> I don't want any teamwork. Suppose some student goes to the board.
> Some other student starts to make suggestions.
> Suppose some how or another a discussion begins to start.
> One person suggests something, then another suggests
> something else... after all this discussion suppose somebody finally
> gets a theorem... who's is it? He'd [the presenter] want a theorem
> to be his- he'd want a theorem, not a joint product![11]

Our M^3 tempers the position Moore proposed and demands no collaboration on material *before* student presentations and no collaboration on any graded assignment and requests minimal collaboration on material after student presentations. After student presentations, if a student does not understand a part of an argument or nuance of said argument, the students are permitted to discuss the argument as well as devise other arguments. However, they are still encouraged to work alone

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individual students but may be an small indictment of our society. Furthermore, it is a realistic and pragmatic amendment to Moore's demands.

[10] This is 'tricky' to enact and in our M^3 class the instructor does not 'police' the students but has to be vigilant. If one suspects that the work, an idea, or some other a process or procedure was not devised by a student but came from elsewhere, the instructor tries to discuss such with the student in the office privately and explain (re-explain) why such 'ill-gotten' ideas are counter-productive. We assume students are honest and are sincerely attempting to learn; we assume the best in them, hope for the best, and only when such faith is shown to be an error do we then 'deal' with such. Our modified Moore method cannot stand on assuming the worst in the student and expect the worst - - such a stance is fundamentally flawed and overly pessimistic.

[11] Moore, *Challenge in the Classroom.*

for, we posit, the student is one who must pass tests (not work in groups to pass tests), earn a degree, defend a thesis, earn a job, etc. We opine that too much 'collaboration' can lead to mediocrity; and our M^3 is designed to have student not 'pass' but 'excel.'

The Moore method minimises subject lectures as does our M^3. We include minimal lectures before student presentations over definitions and terminology, an occasional exemplar argument, as well as subsequent lectures after the students discuss the work(s) presented when the instructor finds there is confusion or misunderstanding about the material amongst the students. However, our M^3 is not as 'lecture heavy' as a traditional class - - the instructor does not enter the class begin lecturing and only end recitation at the end of the period. Under our MMM, everything *should* be defined, axiomatised, or proven based on the definitions and axioms whether in class or referenced. In this regard our M^3 is reminiscent of Wilder's axiomatic methods ([116], [117], [118]). Everything cannot be defined, discussed, etc. within class; hence, the allowance for reference material. Indeed, our M^3 avails itself of some of the modern conveniences that technology affords; thus, additional class materials are available for students to download from an instructor created web-site. These handouts have several purposes including delving deeper into a subject; clarifying material in a text (if one is used); correcting a text used in the class; or posing several additional problems and question in the form of additional exercises. Moreover, the handouts present students with material previously discussed, claims which were made during the class (by students or the instructor), and conjectures that were presented but were flawed or were not presented by students in the class along with proposed arguments as to the veracity of the claims. The students critically read the proposed arguments and note whether or not the proposed solution is correct. Thus, our M^3 includes more reading of mathematics materials than the Moore method, though perhaps less than traditional methods. The ability to read mathematics seems to be difficult for some and many students who studied mathematics (this author included) have reported they opine it is challenging to become motivated to read mathematics and it seems that students who studied in traditional settings find the reading of mathematics less cumbersome. Further, we include this reading component in our M^3 since it seems to be the case that today's students (in general) are less enthused about reading (anything) than students 20 or 40 years ago.

A superficial understanding of many subjects is an anathema to a Moore adherent; a Moore adherent craves a deep, full, and compleat (as compleat as possible) understanding of a subject (or subjects),[12] the undergraduate experience is bereft with time constraints; therefore, pace is not determined by the students' progress but is regulated and adjusted by the instructor. Hence, our M^3 shares somewhat a commonality with traditional methods in so far as pacing is concerned. Our method acknowledges that not all questions can be answered and that each time a question is answered a plethora of new questions arise that may not be not answerable at the moment. Therefore, our M^3 seeks to balance the question of 'how to' with the

[12] See Davis, page 70; Fitzpatrick, "The Teaching Methods of R. L. Moore." *Higher Mathematics* 1 (1985): 44; Fitzpatrick, Some Aspects of the Work and Influence of R. L. Moore, *A Handbook of the History of Topology* 1996), page 9; Forbes, page 194; Paul R. Halmos, How To Teach. In *I Want To Be A Mathematician* (New York: Springer-Verlag, 1985), page 262; and, Edwin E. Moise, "Activity and Motivation in Mathematics." *American Mathematical Monthly* 72, 4 (1965): page 409.

question of ‘why.’ A subject that is founded upon axioms and is developed from those axioms concurrently can be addressed with the questions ‘why’ and ‘how to.’ Indeed, our M^3 allows for the concept of minimal competency, that a student needs some skills before attempting more complex material. So, aspects of ‘coverage’ are included in the M^3 classroom. That is to say that there is a set of objectives that we attempt to meet when teaching a class, that one is duty-bound to include that material. However, our M^3 does not attempt to maximise ‘coverage’ of a syllabus. A syllabus designed by an instructor who adheres to our M^3 would include ‘optional’ material and would have a built-in ‘cushion’ so that the set of objectives can be discussed (more than just mentioned), the students have a reasonable amount of time to work with the material, and more than that set of objectives is met each semestre. The goal of education is not, under the M^3 methodology, ‘vertical’ knowledge (knowing one subject extremely well) nor ‘horizontal’ knowledge (knowing many subjects superfluously), but attempts to strike a *balance between depth and breathe; a balance between ’vertical’ and ’horizontal’ knowledge; and seeks to encourage understanding not sophism.*

As with traditional methods, our M^3 includes regularly administered quizzes, tests, and finals. But, a part of each quiz or test (no less than ten percent nor more than ninety percent) is assigned as ‘take home’ so that the student may autonomously compleat the ‘take home’ portion with notes, ancillary materials, etc. Take homes, we opine, mimic how a real mathematician works so the exercise (by working at home under a more generous time-constraint than an in-class test can create) actualises a more authentic experience for the student. Take home questions posed are richer, fuller, and deeper than in-class questions, therefore rendering the take-home portion of a test a more authentic learning experience for the student. Our M^3 does *not include group assignments of any kind.* We are wholly in agreement with Moore’s inferred position that teamwork leads to misunderstandings, incompleat understanding, and inauthentic sophistic arguments, proofs, examples, or counterexamples.

II. A Few Words About the Foundations Class

Much of our discussion of the neoteric methods we use will centre on material students first experience the 'proofs' course at Kutztown University of Pennsylvania (KUP), the Foundations of Mathematics, Math 224, which introduces first order logic, predicate calculus, syllogistic arguments, existentials, universals, elementary set theory, more advanced set theory and its axioms, generalised collections of sets, the real numbers, the field axioms of the reals, the order axioms of the reals, Cartesian product sets, relations, functions, equivalence relations, partial orders, cardinality, and ordinality. The ordinality section of the course is optional and oft we do not delve as deeply into equivalence relations and partial orders as we do functions and cardinality in the course. The depth choice (which subjects to accentuate) is due primarily to the fact that equivalence relations and partial orders are reintroduced to students in Abstract Algebra, Math 311, which is a course required of all majors at KUP whilst functions and cardinality are not as deeply discussed in a course at KUP that is required of all majors and we opine that the theory of functions is content that is seminal to advanced mathematical theory.

The first meeting the course the students are given a syllabus, given a grading policy, told of the expectation of student responsibility, told of the web-site, etc. Definitions are presented and they are sent home with some basic drill exercises on logic. We do not 'go over' the syllabus, grading policies, etc. for students can read (we did not say wish to read, but can read). The second meeting day student presentations begin; and rather than calling on students like the Moore method [121], volunteers are requested (like Cohen's [13] modified Moore method and we have found it not necessary to call on students for many (enough) of the students seem enthused to present and gain credit for attempting to prove or disprove the claims) modified Moore method). We dispense with logic quickly and go on to introduce the basics of the sets of naturals, integers, rationals, and reals along with the Peano axioms, field axioms of $\mathbb{R}$, order axioms of $\mathbb{R}$, the completeness axiom of $\mathbb{R}$, and the Zermelo-Fraenkel set axioms. The fundamentals of sets are discussed with claims of about sets of numbers and students are encouraged to present often. Throughout the first few weeks the class proceeds in this fashion with short talks about new definitions, methods to prove or disprove claims, and introduction to new terminology, notation, etc. Class proceeds in this fashion with short talks about new definitions, brief reviews of methods to prove or disprove claims, comments on additional methods to prove or disprove claims (such as the second principle of mathematical induction, two place quantification, etc.) and introduction to new terminology and notation (where applicable). SAt some point in the semestre a change in the culture of the classroom is enabled or created so that the instructor is able to introduce concepts or definitions in reaction to or in addition to student work rather than spontaneously. This creates a natural flow to the proceedings in the classroom and the students are encouraged to take responsibility for their education (and, hopefully, regard the instructor less as a teacher and more as a conductor). Nonetheless, it must be noted that some days there are no student presentations; so, the instructor must be prepared to lead a class in a discussion over some aspects of the material or be prepared to ask a sequence of questions that motivates the students to conjecture, hypothesise, and outline arguments that can later be rendered rigorous.

Furthermore, the intensity of discussion deepens quickly so that students and instructor have the opportunity to have deep, meaningful dicussin about topics in set theory and the complexity of the discussion increases.[13] The claims included (and proposed by students often) take on a more challenging form (for the most part) as the course progresses.[14] Obviously, the latter material studied in the course sets up more of the foundation for preparation intended for advanced course work by including the axioms of the reals, the axioms of sets, and the theory of functions.[15]

[13]We are purposely not using the word 'sophistication' of the discussion because of it sophist roots. We accentuate this notion to the students overtly and subtly.

[14] The claims included in the foundations of Set Theory are not *necessarily* more challenging. Oft times, students propose claims early in the course which are quite challenging. Indeed, some students offer claims throughout the class which are not answered in the course. I opine this is wonderful for it concretely demonstrates how ideas percolate and how not everything is immediately answerable.

[15]I have held that it seems that if a student grasps well the concept of elements, sets, collections, and functions that student is well prepared for advanced mathematics and has a huge advantage over a student who does not. I hold that only through working on claims can a student fully understand the concept of elements, sets, collections, and functions and no amount of reading about them in books can substitute for the experience of working with these wonderful notions.

III. Our Neoteric Approach

In all these approaches there is a common theme that runs through them, we submit, and that is that a student (and professor) must question, revise, analyse, etc. the work. These approaches require a student (and professor) entertain the possibility he is in error, that there may be flaws in an argument or example presented (hence the need to submit the work to the scrutiny of our peers), or some part of the example or argument needs to be extended ('fleshed out' more so there is no ambiguity or misunderstanding). We maintain the position that mathematics is something which (like anything in the academy) must be open to inquiry, questioning, etc. It is not 'magic'[16] and is reasonable, clear, and logical. We hold the position that mathematics need not be elegant, short, curt, or brief. The M^3 employed seeks to assist students in discovering original proofs, arguments, counterexamples, examples, heuristics, etc. that *they themselves construct for themselves* and that once created is theirs and theirs alone - - it can not be taken away from them and they have done the hard work, reasoned well, and earned the result.[17] The M^3 employed seeks to teach the students that a presenter's work should speak for itself: it needs no promotion, no hyperbole ('hype'), marketing, etc.[18]

Let us assume that students have a fundamental understanding of basic Aristotelian logic and have been allowed to investigate and learn about said logic; the need for a domain of discourse; the basics of atoms and statements; connectives; etc. Having that as a foundation, then a basic notion that is 'drummed in' or 'drilled' in each of our classes (and is central in the Foundations class) is getting students to understand and realise that the establishment of a universe must precede any discussion of a set. We have to take a given universe and then make claims about elements, sets, collections, product sets, etc. after having a defined universe stated or having a universe assumed to exist and making claims therein. It seems esoteric but we opine it is neoteric to insist on the establishment of a universe and accenuate the importance of the existence of a universe. In our classes we need not 'impose' on students Russell's paradox but can produce a line of questioning that helps students to understand the importance of the existence of a universe merely by having them investigate the definition $A^C = \{x|x \notin A\}$ where A is a set and how ridiculous the 'definition of A^C' is without a well defined universe having been defined previous.

We do not purport that this titchy point is not a standard; but, we note that many authors of undergraduate texts do not accentuate this point, many students had not a rigourous discussion of sets previous to college (therefore had not a discussion of a universe), and there is quite a number of students who had pre-college

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[16]Hence the revulsion to any sort of 'mathematical' magic nonsense that some people talk about.

[17]Note how this point somewhat echo's Moore's point about, "his- he'd want a theorem, not a joint product!"

[18]The lesson of this point is to be rather than to seem and do not 'brag.' It is a position we term 'quiet arrogance,' a phrase my sister coined from the lessons our parents taught us. It is an approach to life and deeds our late father and mother taught us– – one should not brag and 'stuff it down' other's throats that we can do something well or very well. We should let our work speak for itself and not 'toot our own horn.' By doing so the actions speak for themselves and we are not trapped in a Sophistic position of sounding as if we are conversant with something; we **are** but do not need to run about announcing it to the world. I added to our family crest, "*Esse quam videri*," so that such was made clear that it is a family tradition to be rather than to seem.

'exposure' to sets that was wrong (since there was no discussion of the need for a well defined universe. Indeed there are quite a cadre of students who do not have an elementary understanding of $\mathbb{R}$, who opine that $\mathbb{I}$ does not exist, that $\mathbb{Q}$ is the reals, and have not even heard of $\mathbb{C}$. When teaching Calculus I, it is noteworthy the number of students who 'view' the Euclidean plane not as $\mathbb{C}$ not as $\mathbb{R} \times \mathbb{R}$ but as $\mathbb{Q} \times \mathbb{Q}$ with some 'fill in the dots.'[19]

Along with stressing the need for the existence of a universe for basic naïve or fundamental set theory, we also stress the need for the axioms of set theory (we do not teach Foundations as an axiomatic set theory course but we have students reference which of the Zermelo-Frankel-Cantor axioms are being referenced or how they are by implication of inference being used).

We turn our attention to the next neoteric method that is used in our M^3 class; that is to incorporate into the class many claims stated in such a manner as to 'prove or disprove.' The following examples from Math 224 illustrate the idea:

<u>Theorem:</u> Let U be a well defined universe. Let A and B be sets.
If $A \subseteq B$, then $B^C \subseteq A^C$. Prove the claim.

<u>Claim:</u> Let U be a well defined universe. Let $A, B, \wedge C$ be sets.
If $A \subseteq B$, then $A \cup C \subseteq B \cup C$. Prove or disprove the claim.

<u>Claim:</u> Let U be a well defined universe. Let $A, B, \wedge C$ be sets.
If $A \nsubseteq B$, then $A \cup C \nsubseteq B \cup C$. Prove or disprove the claim.

<u>Claim:</u> Let $U = \mathbb{R}$. Let x and y be real numbers. Let $x < -1, 1 < y$.
It is the case that $x^2 \cdot y^2 > 1$. Prove or disprove the claim.

<u>Exercise:</u> Construct a well defined universe U and sets A, B, and C
such that $A \nsubseteq B$ and $B \nsubseteq C$, but $A \subseteq C$ or prove such a universe with such sets can not exist.

Note the preponderance of claims to prove or disprove. Fully 50% or more of the work students do in the class is of the form 'prove or disprove' and after something is proven true or false, then it is added into the notes as a lemma, theorem, or corollary, or as an example of something which belies a false claim. We opine that it give a student the opportunity to consider that what is presented may be true or false.[20] Stating claims in a form such that a person shall 'prove or disprove' forces them to think and in our M^3 class, we encourage students to think

[19]Many students report that in high school, $\mathbb{I}$ means 'not real' π is not real and is not a constant since 'it keeps going' (a horrid misunderstanding of decimals). I have had more than a few discussions with students in Calculus I about this 'fill in the dots' and have found much of it is from viewing calculator images rather than constructing a graph and from a high school teacher iterating that the 'fill in the dots' part is not important. Students oft reference terms like 'next' number, 'closest,' 'infinitesimal,' etc. Rather than view such too negatively, I try to remind myself it opens up a plethora of avenues to encourage student learning and exploration; but it seems very difficult to correct that which was learnt incorrectly. It seems it is easier to assist students to learn something a fresh rather than correct mistaken notions.

[20]Indeed the text 'used' for the class has within its pages 'theorems' which are false - - another marvellous opportunity for students to learn!

about the claims and devise devices for exploring the concepts (Venn diagrammes, numeric examples, etc.). Students are challenged to train their minds to think like mathematicians, to discipline their thoughts, and to organise their ideas. There are students who complain about so many claims to prove or disprove, oft protesting that they can not figure out whether or not a claim is true so can not deduce whether to do a proof or counterexample. Such students seem to have had most everything presented to them in a 'nice, neat, orderly, and mimic-ready' fashion in previous courses or do not wish to spend much time on the material. We have had in some classes students who have *demanded* the instructor do something as if the instructor is a dog performing tricks - - such has happened (not often) in class or in the office. A student *demanding* the instructor do X, Y, or Z is nipped in the bud early in the course and some such students drop the course and take it with another instructor, adjust to the demands of the course, or attempt to 'muddle though.' Most choose the first and of the rest a large majority who stay in the course follow the second path.

Another approach to the material we take is to advocate that each student presentation, write-up, etc. have a beginning and an end. Therefore, if a student is doing a proof, that student shall begin the proof with the term, 'proof,' begin with a declaration that they are assuming premises for the proof (stated premises, previously proved lemmas, corollaries, theorems, definitions, and axioms). Then the proof is done and the proof shall end with, 'Quod Erat Demonstratum (QED).' We take this position so that there is a standardisation and the students can all read each other's work and note the beginning and the end of the proof. For consideration on a test or quiz, these (*must* be included for we hold it is not the job of the grader or reader to decide where an argument begins or ends; that job is the writer's job to note to the reader of grader where the argument begins or ends).

For example (a proof that no point of a Cantor middle third set is an isolated point):

Claim: $U = \mathbb{R} \quad \forall b \in C, \quad b$ is not an isolated point where C is the Cantor middle third set.

Proof:

Assume the premises.

Suppose $\exists b \in C$ such that b is an isolated point.

Then $\exists \varepsilon_1 > 0$ such that the segment $S = (b - \varepsilon_1, b + \varepsilon_1)$ has the property that $S \cap C = \{b\}$.

Since $\varepsilon_1 > 0$, $\frac{1}{\varepsilon_1}$ exists and is greater than 0.

$\exists m \in \mathbb{N}$ such that $(m-1) \leqslant \frac{1}{\varepsilon_1} < m$.

$\Rightarrow \frac{1}{m} \in (0, \varepsilon_1)$ On the C_m level each interval is $\frac{1}{3^m}$ in length. Further, b is in one of theses intervals.

Since $\frac{1}{3^m} < \frac{1}{m} < \varepsilon_1$, the interval containing b of C_m is contained in $N(b, \varepsilon_1)$, and it contains a point of C distinct from b.

This is a contradiction (#!).

Therefore, b is not an isolated point.

Q. E. D.

Likewise, we take the position and teach the students that each student presentation, write-up, etc. that is a counterexample have the property that it shall begin with the term, 'counterexample,' begin with a declaration of the counterexample; demonstrate that the example fulfils the premises and denies the conclusion of the claim; and, then the counterexample shall end with 'Exemplum Est Factum (EEF).' We take this position so that there is a standardisation and the students can all read each other's work and note the beginning and the end of the counterexample. For consideration on a test or quiz, these (*must* be included for we hold just as with a proof is not the job of the grader or reader to decide where an example begins or ends. The EEF and QED requirements force students to think about where an argument or example is presented and when it ends (if it does). Students who do not end with QED or EEF are indicating to the instructor on a test that they know something is not compleat and oft earn more points than a student who indicated an argument is compleat when it is not.

For example (a ridiculous elementary claim about real numbers):
Claim: Let $U = \mathbb{R}$. Let w, x, y, z be reals such that $w < x$ and $y < z$. Therefore, it is the case that $w \cdot y < x \cdot z$.
Counterexample:
Let $w = -1, x = 2, y = -3, z = 1$ which are real numbers.
Note $-1 < 2$ since $0 < 1$ by Lemma 0 (proven earlier).
Note $-3 < 1$ since $0 < 1$ by Lemma 0 (proven earlier).
It is the case that $-1 \cdot -3 = 3$ from closure of $\mathbb{R}$ under $\times$ and Lemma 4 (proven earlier).
It is the case that $2 \cdot 1 = 2$ from closure of $\mathbb{R}$ under $\times$ and the axiom of the multiplicative identity.
But, $2 < 3$ by Lemma 0 and therefore it can not be that $3 < 2$ (since that would contradict the trichotomy axiom) so the claim is false.

E. E. F.

The norm established in the class that each student presentation, write-up, etc. have a beginning and an end and that there be rigourous arguments and counterexamples, we opine, creates a better learning opportunity for the students and a better preparation for further study in mathematics. The author notes to his students that if a professor, journal, conference, etc. has weaker rules for doing proofs or counterexamples, then the student who learnt to do proofs and counterexamples so 'thoroughly' under such 'nit-picky' norms is well served for he can easily relax the verbiage and methods; but, a student who is not taught to justify every step, to note beginnings and ends to the work, to question, revise, analyse, etc. has difficulty adjusting to more rigorous demands.

A more felicitous neoteric approach to the material in our M^3 class is one that comes directly from the author's experiences in class he was a student in that were taught under the Moore method, and that is, to name claims, lemmas, theorems, or corollaries for the claimant or person who proved it (if it is of such an interesting point, then it may get both). When the author was at Auburn University under the tutelage of Dr. Coke Reed, the author proved:
Theorem: Let $U = \mathbb{R}$. Every bounded sequence of reals has a convergent subsequence.
It was called Mr. McLoughlin's Theorem since the author proved it. Many theorems or examples in classes at Auburn had the claimant or prover (person's name attached) to it such as Mr. Kuperberg's set, Mr. Yu's Theorem, Ms. Lavin's

Corollary, etc. Later (*much later*) we found out what the name of the theorem was if it was a 'big' theorem. However, it was not the case that we, the students, read someone else's proof, summarised it, etc. as is a typical lecture method technique employed but each individual proved it himself, each individual claimed a thing herself, or each individual produced a particular example itself.

We have a plethora of examples and the names do not carry over from year to year (unless with a particular group of students) from classes taught under our M^3. For example there is the Krizan set (named after Mr. Krizan a student) see [64] or [79] for details of the set. We also have had:
From Probability and Statistics II
Theorem 2.13 (Ms. Pulse's Theorem): Let $\{X_i\}_{i=1}^n$ be a collection of independent random variables each from $\Gamma(x_i, \alpha_i, \beta)$ (they are not identical but they are independent) $(X_i \sim \Gamma(x_i, \alpha_i, \beta))$.
It is the case that $Y = \sum_{i=1}^{n}(X_i)$ is distributed $\Gamma(y, \sum_{i=1}^{n}(\alpha_i), \beta)$.
From Foundations of Mathematics
Claim (Bongco's Theorem) A.3: Let U be a well defined universe, A be a set, and B be a set. If A is a subset of a finite set B, then A is finite.
Cornista's Theorem S.11: Let U be a well defined universe, D be a set, C be a set, and f be a well defined function from D to C. Let $A \subseteq D$ and $B \subseteq D$, then: $f[A \cup B] = f[A] \cup f[B]$.
Cornista's Claim: Let U be a well defined universe, D be a set, C be a set, and f be a well defined function from D to C. Let $A \subseteq D$ and $B \subseteq D$, then: $f[A \cap B] = f[A] \cap f[B]$.
Cornista's claim was shown to be false (by Mr. Cornista, which was swell (the sets are not equal in Cornista's claim)). We name true claims and **false claims** for the claimant. False claims provide much insight into the material once the students have investigated the claims (especially in beginning set theory where much of the notation suggests things that are not so (as seeming congruents of arithmetic, for example)).

For each instance of the aforementioned naming of claims, lemmas, theorems, or corollaries for the claimant or person who proved it there have been in our M^3 class dozens more of such. Naming claims after students tends to make them want to claim things in the class - - and we hold that such makes them concentrate on the material more than if they were but reading it or listening to a lecture.[21]

The naming of claims that are false also allows for some levity in the class; we can discuss what we want to be but what is not no matter how much we want it (which is not only a humorous lesson but a worthy lesson). The naming of examples constructed or theorems proven provides a student with an intangible motivator. It provides a boost to the student's self-confidence and internal locus-of-control.

[21]Consider how dull it (for some like the author) to sit and hear about let $U = \mathbb{R}$. Every bounded sequence of reals has a convergent subsequence. It is the Bolzano-Weierstrass Theorem proven in ___ (fill in the blank) and here is the proof. The author has sat in on many lectures that are enjoyable; but few where he is trying to learn something. For the author, trying to learn mathematics means trying to do mathematics. It may be he was acclimated to this but it sincerely seems to be something that 'struck a chord' with him as early as his junior year in high school when he attended a National Science Foundation (NSF) Summer Mathematics Institute (SMI) at Auburn University and was on material taught under a three modified Moore methods by Dr. Jack Rogers, Dr. Stephen Brown, and Dr. Wlad Kuperberg.

It is an implied, "good job," declaration; it is an expression of, "wow," and, it accentuates that much of the work is wonderful, but executable. In a lecture with polished presentation, exquisitely written notes on a board, where a student sits and hears about some famous Theorem that the lecturer then presents polished, elegant, and succinct. Where is the excitement for the student? Where is the adventure? Where is the hard work and failed attempts before 'a light goes off,' inspiration hits, and a product is produced *by the student who then later presents it to his peers and instructor.* The student is robbed of all the agony and ecstasy of the search for a solution to a claim, to the solution for a problem, or for an iota of truth in a confused and non-logical life. The example produced or theorem proved is the property of the student who did it and no one can take that away - - that it was proven before, by someone, at another time is fine but is of little import compared to the overwhelming sense of confidence the student gains by doing it himself. The confidence so obtain is *authentic* and *not vapid and shallow*. Some of today's 'edu-babble' highlights and advocates sophistic praise and vapid acknowledgement (as exceptional) of the routine. Lost in this 'edu-babble-ist's' sophistic praise of nonsense is the appreciation for the exceptional, encouragement to drive for betterment or the best, and the *healthy* competition that can take place between students. The Moore method allows for and encourages competition; the naming of an example produced or theorem proved by a student provides (for some students) an unmatched tool to direct the student toward a career in the mathematical sciences.

Now, let us turn out attention to, what the author opines, is one of the two most useful and most pedagogically helpful neoteric approaches we have developed. We introduce to students (or encourage students to create) definitions or terms which are non-standard to the canon but aid in understanding material, concepts, etc. Such definitions or terms (inventive, descriptive, etc.) have been invented in the author's classes. The particular neoteric definition we will highlight herein makes for a more meaningful and fuller understanding of function is used in our M^3 classes; but, it makes its debut in the Foundations of Mathematics class (Math 224) and was invented in the analogue to the Foundations class at Morehouse College, Introduction to Set Theory (Math 255) in the mid-1990s. We note that the standard discussion of functions has associated with it the domain, codomain, and range of a function. Students were confused by the standard definition of a well defined function from a domain to a codomain and were frustrated by many authors of texts introducing function *before* relations. The supposed reason for such is that 'students are more familiar with function from high school or earlier college work;' but, it is our experience that students are more familiar with the *wrong definition of function and wrong understanding of that which it is*.

We follow a scheme with our material so that after students study naïve set theory, the definition of a product set is introduced and students prove or disprove, create examples or counterexamples, to claims about product sets. Following the study of product sets, the definition of a relations from a set to a set or a relation on a set is introduced and students prove or disprove, create examples or counterexamples, to claims about relations and types of relations. Students 'lightly' study reflexivity, antisymmetry, symmetry, or transitivity of relations (or lack thereof) on a set, the use Haas diagrammes to assist in visualising partial orders, graphs to assist in visualising equivalence relations; examples of (or not of) equivalence relations, partial orders, linear orders, maximal or minimal elements in an order, bounds of sets under an order, image sets, inverse image sets, composition of relations, inverse relations, etc. But, all are discussed in more detail in a (subsequent to the Foundations course) course (and rightly so) in our programme so a student is able to return to the concept later and delve in the ideas more deeply at some later date. Recall, such is not the case in our programme with the sets associated with a relation: the domain, codomain, range, and our corange along with image sets and inverse image sets. These assist in building a meaningful and alternate definition of function so in our M^3 Foundations class we concentrate on the concept.

In our M^3 classes we emphasise the definition of function and create a non-standard concept: that of the corange of a function, and then have students prove or disprove, create examples or counterexamples, about functions. We have them apply such throughout the remainder of the course and (hopefully) help them understand the use of such trough the discussion of cardinality and ordinality. The corange of a function comes from our creation of the definition of the corange of a relation. The following example comes from our Foundations notes:

Def. 13.01 : Let the universe $W = U \times V$ be defined from the well defined universes U and V such that $A \subseteq U$ whilst $B \subseteq V$.
The set $A \times B$ is the set $\{(a,b) : \forall a \in A, \forall b \in B\}$.
Let R be a subset of $A \times B$, $R \subseteq A \times B, R$ is called a **relation**.[22]
Also:
$dom(R) = A,\ cod(R) = B,$
$cor(R) = \{x \in A : \exists b \in B \ni (x,b) \in R\}.$
$ran(R) = \{x \in B : \exists a \in A \ni (a,x) \in R\}.$

Claim 13.01R : Let the universe $W = U \times V$ be defined from the well defined universes U and V such that $A \subseteq U$ whilst $B \subseteq V$. Let R be a subset of $A \times B$, it is the case that $ran(R) \subseteq B$
Claim 13.01C : Let the universe $W = U \times V$ be defined from the well defined universes U and V such that $A \subseteq U$ whilst $B \subseteq V$. Let R be a subset of $A \times B$, it is the case that $cor(R) \subseteq A$

Def. 14.01: Let the universe $W = U \times V$ be defined from the well defined universes U and V such that $A \subseteq U$ whilst $B \subseteq V, A \neq \emptyset$, and $B \neq \emptyset$.
Consider the relation f from A to B.
Let the relation f have the following properties:
(1) $\forall a \in A \ \exists b \in B \ \ni \ (a,b) \in f$; and,
(2) $(a,x) \in f \wedge (a,y) \in f \implies x = y,$
then it is the case that f is called a **function** from A to B (or more precisely a well-defined function from A to B) and we symbolise it as

$$f : A \longrightarrow B \qquad \text{or} \qquad A \xrightarrow{f} B$$

Def. 14.01 (Alternate): Let the universe $W = U \times V$ be defined from the well defined universes U and V such that $A \subseteq U$ whilst $B \subseteq V, A \neq \emptyset$, and $B \neq \emptyset$.
Consider the relation f from A to B.

[22]Big whoop!

f is a function from A to B if and only if: (1) $dom(f) = cor(f)$; and, (2) $(a, x) \in f \wedge (a, y) \in f \implies x = y$.

The definition of the corange (so instead of a 'hole' in the idea of relations and functions - - students complained about a domain, codomain, and range (that a codomain needed to be defined, that why was the domain so hard to work with, etc.)) yielded positive results: the creation of a corange complements the range, defining a corange accentuates that the full domain needs to be 'used' in order to possibly have a well defined function, defining a corange accentuates how the range and codomain differ (as the domain and corange do), defining a corange yielded a symmetry for students, and seems to assist their later understanding of injections and surjections:

Def. 14.02: Let the universe $W = U \times V$ be defined from the well defined universes U and V such that $A \subseteq U$ whilst $B \subseteq V, A \neq \emptyset$, and $B \neq \emptyset$.
Consider the well defined function f from A to B.
f is an injection (an injective function) from A to B if and only if $(x, a) \in f \wedge (x, b) \in f \implies a = b$.

Def. 14.03: Let the universe $W = U \times V$ be defined from the well defined universes U and V such that $A \subseteq U$ whilst $B \subseteq V, A \neq \emptyset$, and $B \neq \emptyset$.
Consider the well defined function f from A to B.
f is a surjection (a surjective function) from A onto B if and only if $cod(f) = ran(f)$.

The other of the two most useful and most pedagogically helpful neoteric approaches is to **encourage mistakes**. Perhaps the most controversial of the neoteric approaches that the author uses, we believe it reaps the most rewards for the students: because it assists them in taking chances, in opining, in acting not receiving, in hypothesising - - in thinking. Mathematics is a science of thinking, one can not *do* mathematics with no thought, mathematics is an epistemological endeavour so how can one do matehamtics without thought? We should allow and encourage exploration, discovery, and **inquiry**; hence, allow and encourage mistakes. We do not propose that the encouragement of mistakes is to lead students astray or set them up for failure - - however we do learn from failures, we learn from erring, we learn from making mistakes; so, why deny the student the experience of making an error then correcting said error and learning from it?[23]

To allow stuents to make mistakes is to give up some control in the classroom and allow the dialectic to occur without interference (too much interference). This approach means that we let students go down wrong paths of inquiry (not *too far asunder*) and attempt to reason why something is or is not the case, how to prove or disprove a claim, to claim something, to challenge claims made, etc. Some students are pathologically trained or conditioned to be 'right' - - not to speak or think unless 'perfect' by an educational system that is far from 'right' or 'perfect' but which sophistically sets itself and its members up as such. Note how oft there are examples of instruction where all the examples are presented and 'right,' the

[23]We take as a given that a mathematician, a mathematics educator, or a professor reading this tome (if honest with oneself) learnt from failures, learnt from erring, learnt more from making mistakes than from successes and had more blunders than triumphs (such was so for the author).

notes are 'perfect,' there are seemingly no errors, there are no flaws to examine, and the instructor's presentation is flawless (oral and written). What exactly does a student infer from such, is it any wonder students are unwilling to opine (or challenge)? What do such sterile instruction produce?[24]

Under our M^3, we further promote errors and we bfreward student errors. We reward students for presentations that are flawed, incompleat, or flat-out wrong by awarding 'board points' no matter the outcome. 'Board points' are added into the total points for the semester (and though minimal) students react well to this neoteric aspect of our M^3 classroom. We note that students will learn not only from their individual mistakes but from other's mistakes[25] (including mistakes made by the author) so it is emphasised in our modified Moore method to try to maintain a collegial atmoshphere in the classroom and encourage different students to present (even more than one student to present a claim or at a time (not groups by individually)).[26] Under our M^3, a goal is to engage students in a dialogue in class, promote errors so that students **are not frozen** in panic and **fear of being wrong - -** we opine that the fear of being wrong causes students to be unwilling to speak, present, or engage and is worse than making a mistake so we dedicate the time in class to presenting this idea to the students in both words and deeds (mostly deeds so as to avoid hypocrisy and sophistry). We have encountered many students who are afraid of being wrong, who panic at the thought of losing control, who dread being called upon to posit, answer, or converse. Many (if not most) also are obsessed with making (not earning) an 'A" rather than in truly learning and in being 'right' even if 'wrong' (the ends matter to such people rather than the process or means and such people are of the sort who argue for 'getting' something rather than earning it. Our M^3 tends to be not to such person's liking and we have found students who display such behaviour or attitude either change and adapt (oft well) to a modified Moore method or drop the class nd take the calss with an instructor who does not instruct in a modified Moore method. Nonetheless, we claim that if we are successful in enabling a student to realise that an 'A' (or a high grade and low understanding of the content ofd a course) is not the true measure of success in learning and that genuine learning matters, such persons will be more successful than those who cling to perverse notions of 'perfection.'

[24]We opine sterile, thoughtless instruction produces sterile, thoughtless automatonic people. The American educational system sems to have done very well, very well indeed, if that is what the point of scooling is.

[25]This is a direct result of the author's Moore method experiences as a student. Oft I would learn to approach an idea, to construct an example or counterexample, or to prove something based on one of my fellow student's attempts. More often than not I recall it was Joseph Yi-chung Yu's work that would afford such because Joseph loved to 'wing it' which means to go to the board without having prepared a proof or an example. It is important to note this was not *refining his errors or using his work to then correct what he did* but to note the raw thought process on display and seek insight from said. Once again, what our modified Moore method seeks to instill in students is a willingness to explore, discover, and **inquire** not sit, receive, and witness. Also, let us not forget that the thrill of the competition was a part of the classroom setting - - I wished to learn so I could go up and present.

[26]More than one person presenting a claim concurrently - - they write their work on different boards and then each is presented individually. The class compares and contrasts the work, the ideas, the techniques, and the errors or lack therof in each presentation.

IV. Discussion

A focal point of the discussion of the methods of proof under our M^3 is the uncompromising demand for justification. The instructor must insist that his students (and he himself) justify every claim, every step of a proof (early on in the Foundations course (indeed early in each course where students study beginning ideas of an area of mathematics)), and explain to the students the rationale for such a policy.

We hold that for an authentic modified Moore method class to be run well we must assist students to imbibe in the the love of knowing and knowing does not necessarily mean believing. Note that if one happens upon a fact but really does not know why the fact is indeed so, does he really know the thing he claims to know? In classical philosophy, epistemologically in order for person A to know X: (a) X must exist; (b) A must believe X; and, (c) A must justify why X is. A Moore method instructor or a modified Moore method instructor allows for (a), does not request the students adopt (b), but must insist on (c). This is because there are enough examples of truths in mathematical systems such that (a) and (c) are the case but (b) certainly is not for the majority. One can over time come to accept (b) because of the irrefutability of the argument that establishes the certainty of the claim. This is a key lesson for students. Another key lesson for students is that after one assumes an axiom system, definitions are pliable; but, methods of proof and the fundamental methods of reasoning are not. So, the neoteric definitions that we employ (or that other instructors employ) are fine so long as they enable authentic learning and understanding.[27] Neoteric approaches to material should also assist students to desire to demand understanding of what is and why it is, what is not know and an understanding of why it is not known, recognition of the difference between the two, and a confidence that if enough effort is exerted, then a solution can be reasoned. In this way, our M^3 is perhaps most similar to the Moore method. Consider:

> Suppose someone were in a forest and he noticed some interesting things in that forest. In looking around, he sees some animals over here, some birds over there, and so forth. Suppose someone takes his hand and says, 'Let me show you the way,' and leads him through the forest. Don't you think he has the feeling that someone took his hand and led him through there? I would rather take my time and find my own way.[28]

The confidence must be *authentic* and not forced whilst tempered with humility and realism.

We posit that this pseudo-Socratic method should be considered by more instructors of mathematics. A methodology can only be recommended if there is some evidence of success. We submit that there exists evidence for the success of our M^3; and, we offer it in three parts.

The author has tracked students who completed a M^3 Foundations class and

[27] It is ridiculous to allow for definitions which are self-contradictory; which contradict an axiom system that one is working in; that are not consistent; or, that cause unnecessary or unjustifiable confusion.

[28] Moore, *Challenge in the Classroom.*

notes that there is a *higher success rate* for students who completed the M^3 Foundations class in subsequent classes (Abstract Algebra I and Real Analysis I) than students who completed a non-M^3 Foundations class. The students also earn approximately half a letter grade better on average in Abstract Algebra I than in the M^3 Foundations class and an approximately equal letter grade in Real Analysis I. Ergo, we claim that this seems to provide concrete evidence to posit that in the M^3 Foundations class it is not the case that grades are 'given away' and that preparation for classes subsequent in acceptable or above acceptable when compared to a non-M^3 Foundations class.[29]

The last time the author complied rigorously quantitative data about the grades earned in the Foundations course and grades earned in Abstract Algebra I; and, grades earned in Real Analysis I for a sample of students who took the author's Foundations class from the Fall of 1999 to the Spring of 2001. We considered grades earned in Abstract Algebra I or Real Analysis I for the period of Spring 2000 through Fall 2001.

Let g be grade earned in Abstract Algebra I, a be grade earned in Real Analysis I, b be grade earned in Foundations of Mathematics (FoM). Of the 61 students sampled, 34 had earned grades in FoM and Abstract Algebra I; with correlation estimate and regression equation estimates of $\hat{r} \doteq 0.554, p < 0.01$ and $\hat{g} \doteq 0.989 + 0.519 \cdot b$. Of the 61 students sampled, 31 had earned grades in FoM and Real Analysis I; with correlation estimate and regression equation estimates of $\hat{r} \doteq 0.461, p < 0.01$ and $\hat{a} \doteq 1.233 + 0.415 \cdot b$.

Furthermore, it was interesting to note that of the 61 students enrolled, 30 earned a grade which was less than a 'C' and therefore had to repeat the FoM course before moving on to either Abstract Algebra I or Real Analysis I. Moreover, 22 had not completed Abstract Algebra I by the next academic year and 22 had not completed Real Analysis I by the next academic year (which were not the same 8 students [6 completed both successfully]).[30]

It should be emphasised quite strongly and emphatically that no inferences should be derived from these statistics; they are only descriptive and illuminating insofar as it seems to suggest that for this particular group of students in the particular time-frame the grade in the FoM class was a positive predictor of the grade earned in either Abstract Algebra I or Real Analysis I.

As the author stated in [73] and there is a strong case for and a need for a dispassionate, objective, and quantitative study to be designed and executed that could delve into the question of whether or not a particular teaching method results in more students pursuing advanced degrees or more students having success in subsequent course-work in a mathematics programme. Such a study might prove extremely difficult if not impossible to create and might be controversial since there are more than a few faculty in mathematics departments in the U. S. A. who opine that 'subsequent course-work' is a misnomer, that pre-requisites should be minimised or done away with (see [78], [86], and [8]), and that much (if not most) of the positions forwarded in this paper are contrary to what said faculty believe

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[29] It should be noted that there could be a problem of self-selection. If the student had the self-confidence necessary to do the work in the Foundations class, then he may have selected the author's M^3 class (and other classes) because they were reputed to be 'hard;' therefore, of worth in preparation for subsequent courses.

[30] The correlation between the grade in Abstract Algebra I and Real Analysis I was $\hat{r} \doteq 0.796, n = 31$.

should constitute a mathematics programme.

Students have reported that this M^3 was challenging and students have returned to thank the author for the preparation gained from the course (both students still matriculating and students who graduated and were either in the work-force or graduate school). Nineteen students who took the author's FoM course later did research with the author or wrote an undergraduate thesis under the direction of the author in the past decade. Such anecdotal evidence seems to suggest that for the particular group of students that the author taught there were more than a handful who seemed affected by the method and who then wished to do more work in a modified Moore way and under the direction of the author. Student feed-back that is not included in this paper is the Student Rating of Instruction (SRI) and open ended questions (free response) that are administered at Morehouse College and Kutztown University of Pennsylvania.[31]

We note another point to support the idea that this methodology creates a standard of success for students is that there have been a number of students who took the M^3 Foundations class and then chose later to further their mathematical education by pursuing an advanced degree. Many students who were in the author's Foundations course (and usually at least two or more courses beyond the Foundations course with the author) went on the graduate school. In the last decade, 21 students who were in the author's Foundations class pursued post-baccalaureate work in the mathematical sciences. The author does not claim it was him but the *the modified Moore method* which was key in encouraging the students to pursue post-baccalaureate work. In fact, there is a possible explanation for the number of students who were in the author's Foundations class who pursued post-baccalaureate work - - it may have been due to student self-selection. If the student in the back of his mind thought of the possibility of graduate school or subliminally had the self-confidence necessary to do such work, then he may have selected the author's Foundations class (and other classes) because they were reputed to be 'hard.' Further, another possible explanation for the number of students who were in the author's Foundations course who pursued post-baccalaureate work - - the author's own bias toward 'smart' students![32] Therefore, there is a strong caveat in inducing any 'success' at all the author seems to have had from the number of students who pursued further study in mathematics. However, we note that the trend toward students going on to graduate school who worked under the author is now also occurring at Kutztown University of Pennsylvania - - which seems to provisionally create a credible case that Foundations of Mathematics taught in a modified Moore method way yields positive results and can be deemed 'successful.'

We view mathematics as a great puzzle to be studied and understood, full of structure, beauty, and ideas. I view the task of understanding mathematics as putting the pieces together so they support each other and lead toward a better, deeper, and fuller understanding of mathematics.[33] The work must be singular

[31]The author opines these are meaningless and drivel. The rating seem to correlate to grades *given* not earned and the free response tomes are oft tosh.

[32]There has always been claims by some that the Moore method favours the 'already mathematically inclined.' Such a view seems to assume there is a latent mathematical ability, not everyone possesses it or possesses as strong an ability, and that adherents to the Moore method subliminally favour 'better' students.

[33]It is like a large Lego sculpture each piece (idea) locking together with other pieces to create a structure that build 'mathematics' in one's mind. I opine that the M^3 provides a 'user's manual' or

and not group-work. That which I learnt the best was that which I *did* myself, rather than be told about, lectured to, or even read about. I must *do* in order to *understand*. That I can not explain something does not mean it does not exist, it simply means that it is unknowable (at this point or perhaps it is never knowable).

Our M^3 is based on the idea that learning is a never-ending process rather than a commodity or entity that can be given like the metaphor of an instructor cracking open the head of a student then pouring the knowledge into said head. In that regard it is very much reminiscent of reform methods and the philosophy of John Dewey. Dewey stated, "the traditional scheme is, in essence, one of imposition from above and from outside,"[34] and "understanding, like apprehension, is never final."[35] So, our modified Moore method does not create a setting for imposition from above but attempts to engage each student as a unique person and create multiple dialectics in a classroom full of individual scholars. This dynamic is crucial for the creation of a meaningful M^3 class. P. J. Halmos recalled a conversation with R. L. Moore where Moore quoted a Chinese proverb. That proverb provides a summation of the justification of our M^3 employed in teaching Foundations. It states, "I see, I forget; I hear, I remember; I *do*, I *understand*." It is in that spirit that a core point of the argument presented in the paper is that each student should be engaged as an individual, each student should be given encouragement, and each student should be allowed to create his own argument, example, counter-argument, or counterexample. By so doing, the student is enabled to reach his own potential (rather than meet so pre-determined level of mediocrity that exists seemingly in the educational system we presently have) and thereby have an educationally meaningful experience and in order to properly transition the student from an elementary understanding to a more refined understanding of mathematics.

If there was ***one*** way to teach mathematics, perhaps this paper would not exist. The object of a lesson in an authentic inquiry-based learning (IBL) classroom and the object of furthering an undergraduate's progress toward authentic understanding of mathematics seems to be in harmony: encourage thought, encourage deliberation, encourage contemplation, and encourage a healthy dose of scepticism so that one does not wander too far into a position of subservience, 'give-me-the-answer'-ism, or a position of arrogance, 'know-it-all'-ism. I become more convinced each day that R. L. Moore was right - in the competition between Sophistry and Socraticism, Socraticism is correct, authentic, and should be preferred. Unfortunately, Sophistry is ascendant in the 21^{st} century from elementary through post-graduate study. It prevails in many a classroom because:
1) it is easier for the instructor–no arguments with students, parents, or administrators; complaints of things being 'hard' are almost non-existent if one employs sophistry and the instructor does not have to "think as hard;"
2) it is easier for the student–he does not have to "think as hard" (or think at all), she can "feel good," it can have its self-esteem 'boosted;'
3) it is easier for the institution– standardisation can be employed (which seems to be a goal at many institutions); students retained ('retention' and 'assessment'

'instruction book' to assist the student who is attempting to build the sculpture. I also mean this metaphor to be about Lego blocks not the modern Lego kits that are pre-arranged and therefore yield not a work of creation that the child imagines but something that is on the box cover (such as a helicopter, a ship, etc.).

[34] John Dewey, *Experience and Education* (New York: Macmillan, 1938), page 18.

[35] John Dewey, *Logic: The Theory of Inquiry* (New York: Holt and Company, 1938), page 154.

seem to be a 21^{st} 'buzz words'), graduation rates increase,[36] and accrediting agencies are mollified (such as the Middle States Association of Colleges and Schools ('Middle States'), the Southern Association of Colleges and Schools ('SACS'), or National Council for the Accreditation of Teacher Education ('NCATE')).

So, a core point of the argument presented in the paper is that inquiry-based learning (IBL) as actualised by our modified Moore method (M^3) is a pedagogical position that deviates from the 'norm' insofar as it argues one must *do* in order to *understand*. The method centres on exploration, discovery, conjecture, hypothesis, thesis, and synthesis which yields positive results - - students doing proofs, counterexamples, examples, and counter-arguments. We have found that the neoteric definitions and frame-works described herein seem to encourage students to try, aid students' transition to advanced work, assists in forging long-term undergraduate research, and helps encourage students to do rather than witness mathematics. We submit that we have provided evidence to suggest that such teaching methodology produces *authentically* more prepared students, more confident students, and students who are better at adapting to new ideas. Moreover, these methods seem to assist in getting students to be willing to make mistakes for, we argue, that we learn from making mistakes not from 'always' being correct!

[36]Not because of higher achievement; therefore, a sophist success.

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