Students with Learning Disability in Math Are Left Behind in Multiplicative Reasoning? Number as Abstract Composite Unit is a Likely 'Culprit'

Ron Tzur
University of Colorado – Denver

Yan Ping Xin, Luo Si, Rachael Kenney, Adam Guebert
Purdue University

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Abstract

This study[1] addressed the problem of why students with learning disabilities in mathematics too often fail to develop multiplicative and divisional concepts/operations. We conducted a constructivist teaching experiment with 12 students (nine 5th and three 4th graders). This report focuses on three students' conceptual progress, particularly on Sandy's (most pronounced). Our analysis indicates that, at the outset, those three could only reason additively because they lacked a robust concept of number as an abstract composite unit and were bound to rely on strategies of counting units of one. Once teaching engendered a concept of number in them, they made substantial advances. We argue that lacking the concept of number is a decisive cause for setting students at risk of failing to advance beyond additive reasoning, and that assessing this conceptual cause is vital for providing effective pedagogical interventions.

1. Purpose

This study addressed the problem of why do so many elementary students with learning disabilities (SLDs) in mathematics fail to construct multiplicative concepts/operations. Addressing this problem is important for two interrelated reasons. First, the transition from additive to multiplicative ways of operating is considered a major hurdle in the elementary school, yet a crucial one to promote if students are to ever develop mathematical powers beyond arithmetic (Harel & Confrey, 1994). This transition seems particularly difficult for students with or at risk of developing learning disabilities in mathematics (Xin, in press). We address the conceptual source for this difficulty by using the comprehensive framework proposed by Steffe and his colleagues (Steffe & Cobb, 1988; Steffe & von Glasersfeld, 1985; Steffe, von Glasersfeld, Richards, & Cobb, 1983) for analyzing numerical thinking. This framework emphasizes two fundamental components: (1) the units a learner operates on and (2) the particular nature of the child's operations, including the goal/intention toward which her operations are directed. The specific operations and units that distinguish multiplicative from additive reasoning are presented in the next section. Here, we note that a review of the literature, either within mathematics education (Fuson, 1992; Greer, 1994; Verschaffel, Greer, & Torbeyns, 2006) or special education (Cawley and Parmar, 2003; Cawley, Parmar, and Foley, 2001; Fuchs and Fuchs, 2005; Schmidt and Weiser, 1995), did not reveal an in-depth analysis of *conceptual sources* for difficulties exhibited by SLDs.

The second reason is the need to provide a pedagogical approach that can effectively promote SLDs' progress to multiplicative reasoning. In particular, the NCTM (2000) *Principles and Standards* and the National Educational Goals Panel (1997) documents emphasized the need to promote problem solving and conceptual understandings in *all students*. The students with whom

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we worked are typical examples of the failure of the educational system to do so (see, for example, Carnine et al., 1994). As we shall further report in the Analysis section, after 4-5 years in the public school system, they barely developed mathematical understandings that, according to state standards (IN), are expected of first graders. Because they were 4^{th} or 5^{th} grade students, their school work typically included repetitive exercises of 1- and 2-digit multiplication and division problems and/or the four operations with fractions. Indeed, they were unable to successfully solve such problems and were thus identified as disable or at risk of disability in mathematics. It seemed that not only research, but also typical teaching in schools does not identify the conceptual sources for these inabilities and is not tailored to what SLDs already do know (see more in the Conceptual Framework below). This study reports on our twofold attempt to both identify and rectify conceptual constraints and affordances for their learning.

2. Conceptual Framework

In this section we briefly describe key general and content specific constructs of a conceptual framework that guided our study and pedagogical intervention. The general constructs pertain to learning (conceptual change) and corresponding teaching of mathematical ideas; content specific constructs pertain to the units and operations that distinguish multiplicative from additive reasoning.

2.1 <u>General Constructs</u>

Building on the core constructivist notions of assimilation, reflection, and anticipation (Piaget, 1971, 1985), and on von Glasersfeld's (1995) tripartite model of a scheme (situation/goal, activity, expected result), Tzur et al. articulated the mechanism and stages by which a learner abstracts conceptually advanced mathematical ideas as transformation in her existing schemes/conceptions (Simon & Tzur, 2004; Simon, Tzur, Heinz, & Kinzel, 2004; Tzur, 2007, 2008b, accepted for publication; Tzur & Simon, 2004). The mechanism was coined reflection on activity-effect relationship (*Ref*AER*), and it specifies Piaget's (1985) notion of reflective abstraction for the purpose of guiding mathematics teaching. The *Ref*AER* mechanism refers to two types of comparisons the human brain processes. *Ref*AER* Type-I consists of comparisons between the goal (anticipated effect) set by the learner's scheme and the actual effect(s) of the scheme's activity. The learner's goal regulates her noticing of discrepancies between the two, and may engender adjustments to and re-execution of the activity that are recorded as 'experience instances.' *Ref*AER* Type-II consists of comparisons across those recorded experience instances. It leads to the abstraction of what is anticipated to remain invariant in the use of the novel activity-effect link across different situations, that is, it characterizes the 'sameness' of those situations.

The *Ref*AER* framework suggests that for the majority of learners a new mathematical understanding does not evolve all-at-once. Rather, reflection Type-I brings about the construction of a provisional stage of the new understanding, termed *participatory*. Here, the learner forms a novel mental link between the scheme's activity and effects that were previously unanticipated by the learner. This link is provisional in the sense that the learner can call upon it for problem solving *only* if somehow prompted for the activity to be used. The notorious 'oops' experience is characteristic of the participatory stage; a learner carries out the activity to its effect, then notices she should have known the effect in advance (but did not). By definition, the participatory stage implies that if prompts are *not supplied*, previously established schemes will 'interfere' in the learner's problem solving processes, an implication that is consistent with Siegler's (1995, 2000, 2003) Overlapping Waves Theory (see also Tzur, 2000; Tzur, 2004 for explanation of whole number interferences in fractional situations). Following the construction of a novel, participatory activity-effect anticipation a learner may form the more stable stage termed *anticipatory*. Here, she can independently and spontaneously call upon, use, and justify the novel activity-effect relationship proper to the problem situation. The anticipatory stage is compatible with Sfard's (1991) notion of structural understanding and with Dubinsky's (Breidenbach, Dubinsky, Hawks, & Nichols, 1992; Dubinsky, 1991) notion of object understanding. Conceptually, the anticipatory stage seems a necessary condition for transferring newly acquired knowledge to different contexts/problems. The crucial point for teaching is that, unlike the automatic brain processing of Type-I reflection, Type-II, which is necessary for constructing the anticipatory stage, may not spontaneously occur in the learner's brain (Tzur, accepted for publication).

For pedagogy, the mechanism and stages outlined above imply engaging students in solving tasks/problems that serve 3 principal functions of learning: (a) *fostering students' assimilation* of the tasks into their extant schemes, (b) *orienting students' attention* so they notice new effects intended by the teacher (Type-I reflection), and (c) fostering reflection on and distinction/formation of the new, intended conceptions across problem situations (Type-II reflection). In such a conception-based perspective (Simon, Tzur, Heinz, Kinzel, & Smith, 2000), problem situations are continually tailored to the dynamic windows-of-opportunity (Noss & Hoyles, 1996) that proceed from students' extant conceptions to the intended mathematical understandings. The threefold function of tasks entails a cyclic process of seven principal teaching activities, which further extends Simon's (1995) notion of hypothetical learning trajectory. The cycle always begins by articulating student assimilatory conceptions.

1) <u>Specifying students' current conceptions</u> by inferring into mental processes that might underlie *their* work on previous tasks: goals *they* seemed to set, mental activities *they* likely executed, and effects *they* likely noticed and linked to the activities.
2) <u>Specifying the pedagogical goal for student learning</u> by decomposing intended conceptions into their components—goals, activities, effects, and invariant relationships among them.
3) <u>Identifying a mental activity sequence</u> that, when 'triggered' in students, may generate the intended effects, reflections, and relationship. An activity sequence must be identified relative not only to the intended conceptions but also to the students' extant conceptions.
4) <u>Selecting tasks</u> by deciding on an initial problem situation and follow-up prompts. This is based on explicit hypotheses about (a) how students may assimilate the tasks into and use their extant conceptions, (b) notice intended effects, and (c) reflect on new activity-effect relationship.
5) <u>Engaging students in the tasks</u> while making sure students solve them using *their* goal for regulating *their* activity. This is a delicate challenge; it requires continual negotiation of students' task interpretation so that their goal is compatible with the teacher's without sliding into a futile attempt to govern students' (internal) goals.
6) <u>Monitoring students' progress</u> by continually examining their actual work: which goals seem to regulate their activities, which (mental) activities they seem (inferred) to employ, which effects they seem to notice, and which reflections they seem to undertake.
7) <u>Introducing follow-up questions/prompts</u>, planned or adjusted in real-time. Such follow-ups enable *indirectly* orienting students' attention so they reflect on effects of their goal-regulated activity and relate them with the activity. Activities #6 and #7 feed back into activity #1.

Content-Specific Constructs

Steffe et al. (1985; 1983) suggested that children's conceptualization of number is abstracted via reflection on and coordination of mental operations on objects, particularly counting. Once numerosity of a collection of objects is anticipatory (the child anticipates and uses the *result* of counting without carrying it out), she can progress to solving addition and subtraction problems via *Counting-All*. Given two collections and having counted each (say, 4 & 3), and asked how many items are in both, the child would begin recounting the first collection from 1, then continue to the second: 1-2-3-4; 5-6-7. Through *Ref*AER* on the invariant *starting point* for the second collection (the *number after* the anticipated numerosity of the first collection, n+1), she abstracts the scheme of *Counting-On*. This scheme coordinates anticipation of (a) where to start and (b) the need to keep track of how many items of the second collection were counted (e.g., Four; five-is-1, six-is-2, seven-is-3; so 4+3=7). The latter has been termed double counting, because the child simultaneously counts the 1's of the second addend as belonging to two number sequences. Such a counting scheme indicates that the child's meaning (at least) for the first collection is of an abstract composite unit. It's taken for granted as a single, numerical quantity that symbolizes for the child the potential effect of counting and is thus available as input for further operations. Steffe et al. attributed the concept of number to a child who anticipates the combination of where to start and how to monitor where to stop counting. Lambert and Tzur (2009) have recently demonstrated that each of these invariants (know where to start, know where to stop via counting one's own counting acts) includes a participatory stage, at which the child cannot yet call upon counting-on without a prompt and is likely to 'revert' to using counting-all, thus at risk of being left behind.

Counting-on provides the conceptual basis for further additive operations on abstract composite units, such as counting-up to solve a missing addend problem (e.g., for 3+?=7 the child would count, "3; 4-5-6-7" while keeping track of how many numbers were said for the second addend). It also provides the basis for using strategies such as 'Through-10' (e.g., 8+5=? is solved by decomposing the second number (composite unit of 5) into two sub-units (e.g., 2+3) and then adding 8+2=10 and 10+3=13). Eventually, these strategies may lead to a child's anticipation and retrieval of invariant addition facts (e.g., '6+6=12', as well as '6+5=11 because it's one less than 6+6'). What marks all *additive* operations on units is that the nature of units upon which one operates does not change (e.g., 4 apples + 4 apples + 4 apples = 12 apples, which essentially means figuring out that '4 units of one + 4 units of one + 4 units of one = 12 units of one').

On the other hand, *multiplicative reasoning*—a central mathematical theme in grades 3-5 (NCTM, 2000)—involves not merely computing multiplication and division with numbers but rather making sense of activities of coordinating quantities in problem situations. A key difference between additive and multiplicative reasoning is that in the latter the items of one composite unit are distributed over the items of a second composite unit to produce a third composite unit (e.g., 17 *dogs* X 4 *legs-per-dog* = 68 *legs*). Schwartz (1991) characterized this difference as *referent preserving composition* (additive) and *referent transforming composition* (multiplicative). There is a wide range of analyses regarding multiplicative structures and operations (Confrey & Harel, 1994). Schwartz's distinction grew out of focus on relationships among the three terms in multiplicative problem situations (factor-factor-product) and distinguishing between the factors: an intensive quantity (unit rate, UR) and an extensive quantity (CUs). Vergnaud (1994) introduced the notion of multiplicative conceptual field, while sorting multiplicative problem structures into three types, *isomorphism of measures*, *product of measures*, and *multiple proportions*. He pointed out difficulties in representing the intensive quantity because it is about quantification of an abstract relation between other quantities.

Our work, and this study in particular, drew on the *developmental analysis* provided by Steffe (1994) and his colleagues (Steffe and Cobb, 1988, 1998), because it articulates how children construct pre-multiplicative, multiplicative, and part-to-whole unit-coordinating schemes (conceptions) as transformation in their additive, iteration-based, number schemes. For example, on the basis of the construct of a child's anticipation, Steffe and Cobb (1998) distinguished between two children's conceptualization as indicated by their solution for the problem of "4 x 5." The more conceptually advanced child, Jason, anticipated the operation of double counting—producing and iterating a composite unit of "5" four times. That is, Jason constructed "5" as an iterable composite unit for which he could anticipate the need to coordinate (distribute across) two counting schemes—counting by ones and counting by an iterable composite unit (e.g., 1-is-5, 2-is-10, 3-is-15, 4-is-20). The less advanced child, Peter, could only implement his concept of multiplication by first producing composite units of five. For Peter, each "5" he produced via circling five units of one constituted a different entity. He was able to then count the total of items in all four circles, but he did not seem to regard all "fives" as essentially the 'same' entity. Although the distinction between the participatory and anticipatory stages was not available at the time, using it for re-analyzing Steffe and Cobb's data indicates to us that Peter could notice the relationship between his activity of creating one composite unit after another and the effect of possibly counting those units *only after* he actually produced and operated on them—a typical manifestation of the participatory stage.

SLDs' capacity (or lack thereof) to double count (multiplicatively) in anticipation as did Jason served as a major conceptual benchmark in our study. At issue is that such double counting requires operation on number as an abstract (numerical) composite unit. Once a child constructs multiplicative double counting at the anticipatory stage, further distinction, selection, and operation on units of one vs. composite units can be promoted (e.g., solving how many bags of 6 cookies in each would there be if one has 5 bags and 18 more cookies). Tzur et al. (2009) have reported on the construction of an anticipatory stage of such a mixed-unit coordination scheme, which includes a divisional operation on the 18 units of one. This scheme provides a necessary conceptual 'bridge' between multiplication and division, and for the later construction of algebraic ideas, and was thus considered as the intended goal for our SLDs' learning.

3. Methods

The qualitative, constructivist teaching experiment (Cobb & Steffe, 1983; Steffe, Thompson, & von Glasersfeld, 2000) method was used with twelve SLDs (nine 5th graders and three 4th graders) in two elementary public schools in the US Midwest. This experiment was conducted within the larger context of the NSF-funded project, *Nurturing Multiplicative Reasoning in Students with Learning Disabilities in a Computerized Conceptual-Modeling Environment* (NMRSD-CCME). About 26 weekly teaching episodes of 25-35 minutes were conducted with each child (individually or in pairs) over the course of 8 months. The participants were those students identified by their school as requiring special pedagogical attention in mathematics. This sample reflected the shift in identification models in the state of Indiana, away from the old discrepancy model (1-1.5 STD between the student's IQ and math performance) to a Response-to-Intervention (RTI) model. The latter identifies students as having learning disability in math even if, for example, their IQ score is at the 85th percentile and their math score is at the 80th. The twelve SLDs included all students identified by the schools as SLDs in mathematics, that is, an exhaustive sample.

An initial assessment of every student's assimilatory conceptions was conducted via an individual diagnostic interview that used the instrument found in Appendix A. It revealed nine SLDs who have already constructed the concept of number and were thus placed in the 'Multiplicative Reasoning Group'. This research report focuses on the other three SLDs, who we initially placed in the "Additive Reasoning Group'. In particular, those three used *counting-all* to solve problem #1, and were unable to solve problems #2a (missing 1st addend), #2b (missing 2nd addend), #4 (procedural skip-counting, except for 'Twos' and 'Fives'), #5 (missing both addends for composing a number in the second decade), and #6, #8a, #8b, #10, #13, and #14 (multiplication/division problems, which students in the 'Multiplication Group' also failed initially). We hoped to support those three students' construction of abstract composite units to an extent that would allow advancing them to multiplicative reasoning—first to multiplicative double-counting (mDC) and then to multiplicative mixed-unit coordination (mMUC).

The pedagogical intervention for promoting the three students' construction of number proceeded from addition and subtraction realistic problems with one-digit numbers. They were first oriented to use Unifix cubes and/or beads on a two-row abacus. Later, they were asked to produce the two addends, which were immediately covered by the researcher (one collection and later both), and encouraged to substitute the concrete objects with figurative items, such as counting on their fingers or tapping motions (see Steffe, et al., 1983). To solve 'covered item' problems, the students brought forth, used, and justified the mathematical appropriateness of counting-on, counting-up, counting-back-from, and through-10 strategies. Eventually, additive problem situations were presented to those three students in an abstract form, asking them to pretend quantities were produced and orienting them to use their numbers for the solution. Due to these students' rather limited short-term memory, we did not stress memorization of additive facts, though some improvement could be detected particularly for small numbers. When solutions to 'for pretend' problems consistently indicated that an individual student among the three from the 'Additive Reasoning Group' had constructed an anticipatory stage of additive operations (or at least high participatory indicated by internal self-prompting soon after attempting a lower-level strategy), we re-placed her into the 'Multiplicative Reasoning Group'.

The pedagogical intervention for promoting all students' multiplicative reasoning engaged them in a few variations of a turn-taking, 'platform' game that the first author had created and termed "Please Go and Bring for Me …" (PGBM). Its base-format involves one player sending the other player to a box with Unifix Cubes located away from both of them to produce, and then bring back, a tower composed of a given number of cubes. After some 'trips' (2-9) for bringing the *same-size tower each time*, the 'bringer' was asked how many towers (i.e., Composite Unit, denoted CU) she brought, how many cubes are in each tower (i.e., unit rate, denoted UR), and how many cubes (denoted 1's) there are in all (hereafter, *N* towers of *M* cubes each are symbolized as NT_M; e.g., $5T_4$ = 5 towers of 4 cubes each). The key feature of this game is the child's repeated-but-clearly-separated activity sequence of producing an anticipated composite unit out of so-many-1's (hence, UR), followed by explicitly distinguishing between CU and 1's-within-UR, and coordinating both into an anticipatory, single quantity (total 1's). To promote multiplicative double counting at the anticipatory stage, this basic variation was first used with concrete cubes and towers, then with actual-but-covered items, and finally with abstract items (e.g., 'Pretend I asked you to PGBM $4T_7$; how many cubes would you have in all?'). Obviously, the latter form of the game necessitates availability to the child of an abstract composite unit within her mental system. When a child seemed to operate consistently with mDC in the cubes/towers context, realistic word problems were also used, including orienting prompts to explicitly 'translate' units from those problems into cubes and towers or vice versa. Then, two higher-level variations of PGBM were used to promote construction of two, conceptually more advanced understandings: (a) multiplicative same unit coordination (mSUC) and (b) multiplicative mixed unit coordination (mMUC). The former might ask: "I placed $6T_4$ under the cover. How many *towers* will we have *if* you brought additional $3T_4$?" The latter, that Steffe and Cobb (1988) considered as a strong indication of an Explicitly Nested Number Sequence (scheme), might ask: "I covered $6T_4$ here and 12 cubes there. *If* you put all 12 cubes into T_4 and moved them under the other cover, how many *towers* will you have in all?"

4. Data Sources and Analysis

Data collection consisted of videotaped teaching episodes of about 25-35 minutes each with individual or pairs of students, as well as the written plans for each episode (about 26 per student). As implied by the teaching experiment methodology, SLDs' learning was concurrently promoted and studied. Immediately following each episode, ongoing analysis (3-5 team members) focused on every child's anticipation development. This ongoing analysis resulted in a tentative plan for the next episode(s), which the first author later revised into a final plan a day before the next episode. Retrospective analysis was conducted later, where conjectures about changes in a child's anticipation were noted, and those noted claims were held in check against confirming and/or disconfirming evidence in the data.

5. Results

The critical point of departure in our results is the distinct contrast between the 'Multiplicative' and 'Additive' groups. Our central argument, that the lack of a concept of number as an abstract composite unit is a plausible conceptual cause for inability to progress to multiplicative reasoning, proceeds from this contrast. All 12 SLDs in our exhaustive sample failed to independently initiate a solution to (let alone fully solve) multiplicative problems in the diagnostic interview (particularly #6, #8a, #8b, #10, #13, and #14). However, the diagnosis indicated that they did construct the prerequisite concept of number and could operate on it. They almost never used counting-all; and they properly used and justified counting-on, counting-up, and counting-back strategies. Particularly telling was these SLDs' capacity to solve problem #5 by independently generating a number for the first addend, then figure out the second addend via one of these strategies. A few were also capable of reflecting on their first two solutions (e.g., 10+7=17, 9+8=17) and noticed an invariant between the twofold activity of reducing an addend by one while increasing the other by one, which they then anticipated to yield the same effect (sum, 17), so they could continue with 11+6=17, 12+5=17, etc.). Further, during the diagnostic interview, when prompted to solve problem #8a (Miguel's 4 bags of 3 marbles each) via modeling the situation with cubes, they did not rely on counting-all visible items, but rather on some rudimentary form of counting groups of 3 (e.g., 3; 6; 7-8-9; 10-11-

12). Our consequent work with them via the PGBM game and more realistic problems indicated their conceptual progress, which suggested to us that they previously failed to move beyond additive reasoning due mainly to inadequate instruction.

In contrast, successful solutions of the three SLDs placed in the 'Additive' group were obtained primarily via counting all (i.e., operating solely on units of 1). Quite often, they were unable to operate on invisible units. For example, upon being prompted they would solve problem #8a by placing 3 cubes on the desk or drawing four groups of 3 circles ('marbles') each. Then, they would count each and every cube/circle (e.g., 1-2-3-4-5-6-7-8-9-10-11-12) without any indication that the grouping was considered. As we therefore predicted, when we attempted to teach them the base-format of the PGBM game they could only solve the problem of how many total cubes there were if all cubes were visible and by counting-all. The report below begins after 6-8 sessions in which our pedagogical intervention for promoting additive reasoning indicated their evolving construction of number as an abstract composite unit (we do *not* claim that these sessions alone produced that conceptual leap). This report demonstrates how we could then capitalize on the conceptual prerequisite of number and use instruction tailored to their particular characteristics to promote construction of multiplicative reasoning and problem solving.

<u>Lia</u>: After Lia's return to the 'Multiplicative' group, it took us 12 additional episodes to nurture her ability to use multiplicative double counting (mDC) not only for solving, but also for posing equal-group multiplication problems. To this end, we had to address two major individual issues. We realized that, for Lia, the PGBM context did *not* support her ability to solve problems. Guided by the notion of prompt, we therefore conducted a mini-interview with her to find what contexts in her real life were possibly providing supportive prompts for her distinction among and coordination of CU, UR, and 1's. It turned out that buying shirts with buttons was such a context. Thus, we posed to her problems in this context to promote independent and willing engagement in the activity of double-counting composite units. For example, to solve, "Every shirt needs 6 buttons; you have 3 shirts; How many buttons do you need?" without any concrete object available, Lia would count 6 and raise her finger for one shirt, then repeat raising a finger for the 6-button groups in the two remaining shirts (e.g., 6; 12; 13-14-15-16-17-18). The second issue was Lia's difficulty to pose a problem. She experienced serious difficulties in coordinating all 3 quantities involved into a single, coherent multiplicative situation (unit rate, number of units, total items). Her struggle to 'simply' utter problems such as, "If you bring 4 towers/shirts with 7 cubes/buttons in each, how many cubes/buttons you'd have in all?" pointed to this coordination as a conceptual stumbling block on the path to constructing a robust (beyond problem solving) anticipatory concept of multiplication.

<u>Sandy and Jen</u>: Both girls progressed much faster than Lia. To illustrate the argument of number as a conceptual cause, we focus on Sandy's most pronounced progress. It should be noted that Sandy was a fourth grader whose school identified as needing special support in mathematics, but *not* as a SLD in mathematics according to the traditional model. Within 11 episodes past her return to the 'Multiplicative' group, Sandy could independently use mDC to *both* solve and pose multiplication problems in different contexts. Therefore, we advanced our teaching to the conceptually challenging, multiplicative mixed-unit coordination (mMUC) understanding in which both 1's and composite units (CU) are operated upon. For example, she would solve (and later pose to the researcher) problems such as: Grandma baked cookies and placed them in 6 bags with 4 cookies in each; if she baked 12 more cookies and put those in bags of 4, how many bags would she have in

all? To solve such a problem, she would spontaneously first organize in her mind and use her fingers to count how many bags were needed for additional 12 cookies (e.g., 4-is-1, 8-is-2, 12-is 3). She then proceeded to add the 3 bags to the initial quantity of 6 bags and answer, 'Nine bags'. Her explanation of this two-step solution indicated that her overarching goal for the activities she carried out was to operate additively on the CU (bags). To this end, she established for herself a sub-goal of figuring out how many bags were needed for the 1's (twelve cookies) and accomplished that sub-goal via calling up her anticipated mDC for a part of the problem that was essentially divisional (which we were yet to teach her).

Again, a much more robust anticipation of the units and operations/activities needed on them was required for Sandy to pose such mMUC problems, as she gradually learned to do after a few episodes of only solving such problems. To do so, she must have anticipated an entire activity sequence as well as the effect to be asked about (how many *bags*). That is, we inferred she first differentiated the 1's from the CUs, then operated on the 1's via mDC to compose a total of 1's (e.g., 12 cookies) for which a whole number of CUs (e.g., 4 bags) could yield a whole number of UR (e.g., 3 cookies per bag), then adding the number of both given and composed CUs (6 bags + 3 bags = 9 bags) to figure out the solution and be able to 'check' the researcher's answer. Most importantly, she could carry out this entire complex activity sequence without shifting to operating on 1's, as she responded in the beginning her problem solving (e.g., 18 cookies; 19-20-21; 22-23-24; 25-26-27 cookies). Tzur et al.'s (2009) recent case study of another child who started with the "Multiplicative Reasoning Group" sheds light on how SLDs might make the transition from a participatory to an anticipatory stage of this highly sophisticated scheme.

In closing of this section, we would like to make a cautionary note about Sandy's (and Jen's, not reported here) fast-pace progress in *solving and posing* mDC tasks, through mSUC tasks, all the way to mMUC tasks. Although we are convinced that the PGBM game in its varied formats is highly supportive of learning to distinguish among and multiplicatively coordinate CU, UR, and 1's, we do not claim that it could alone explain the rather rapid progress these two students made. Indeed, students from the 'Multiplicative' group, as well as Lia, made slower progress and encountered some serious conceptual stumbling blocks. What we do emphasize is the critical role that these girls' consolidation of an anticipatory concept of number played in that fast-pace progress. We speculate that much of the needed conceptual prerequisites for such progress have already been in place prior to their return into the 'Multiplicative' group (e.g., distinguishing 1's from groups, or anticipation of additive effects such as combining makes larger and subtracting makes smaller). However, the missing piece for applying those prerequisites lacked the 'object' upon which to operate multiplicatively, namely, number as an abstract composite unit. It seemed as if the construction of anticipatory number released a 'conceptual spring' in both Sandy and Jen, which brought forth their application of established operations to it (e.g., counting CUs and 1's simultaneously—as in mDC, or organizing 1's into and then adding CUs—as in mMUC).

6. Scientific Significance

At the beginning of our work with 12 SLDs, two 5th graders (Lia and Jen) and one 4th grader (Sandy) in our exhaustive sample (25%!) were reasoning additively at or below the 1st grade level. This study found that the reason for their disturbingly inadequate mathematical understandings was the lack of a robust concept of number to operate on/with. Unlike their nine SLD peers, whose inability to reason multiplicatively seemed to be rooted in instruction, these three girls were yet to construct the fundamental concept upon which coordination of equal-group units (CUs X UR) is used to produce a total number of 1's they can figure out. Such coordination is needed for solving (and more importantly posing) realistic word problems in which unit items of one quantity (4 buttons per shirt, 6 cookies per bag, 13 cubes per tower) is distributed across items of a composite unit with specific numerosity (5 shirts, 4 bags, 2 towers) to produce a third unit (total number of buttons, cookies, or cubes)(Schwartz, 1991). Promoting the girls' establishment of the concept of number at an anticipatory stage opened the way for advancing their thinking to solving and posing multiplication problems via multiplicative double-counting (mDC, Lia), and to successfully doing this also for multiplicative mixed-unit coordination problems (mMUC, Jen and Sandy).

Consequently, this study contributes to the knowledge base by identifying two interrelated plausible conceptual sources of SLDs' mighty difficulties to reason multiplicatively. One source, indicated in the case of Lia, Jen, and Sandy, is the failure of school teaching to engender in SLDs a concept of number as an abstract composite unit—a thing in and of itself (Steffe, 1994; Steffe & Cobb, 1998; Steffe & von Glasersfeld, 1985). In such a case, SLDs' are most likely to be left behind when their class progresses to multiplicative reasoning because they literally lack the mathematical object upon which one operates multiplicatively. A second source, indicated by the other nine participants in our sample, is the failure of school teaching to engender a transition to multiplicative reasoning in SLDs who do use numerical structures for strategic additive operations. Our work with both groups of children, mostly awaiting publication (but see Tzur, et al., 2009), demonstrated a useful set of pedagogical tasks designed on the basis of the reflection on activity-effect relationship framework (Simon, et al., 2004; Tzur, 2007, 2008a; Tzur & Simon, 2004). The teaching cycle embedded within this framework stresses tailoring teaching-learning interactions (e.g., PGBM game with its varied formats) to students' assimilatory conceptions (Tzur, 2008b). Such tailoring includes finding contexts (e.g., shirts and buttons) that improve students' ability to interpret and solve/pose the tasks. Not the least, in our work with the three SLDs their progress was made possible because of the particular attention to the two distinct stages of knowing (participatory and anticipatory) and two related types of reflection (anticipated vs. actual effect, across activity-effect records) through which number and multiplicative operations on/with it are conceptualized. All in all, this study demonstrated that when teaching does promote construction of and capitalizing on number as the *conceptual* prerequisite—substantial progress into multiplication and division can be nurtured.

7. References

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Appendix A: Diagnostic Clinical Interview Instrument

(Do not use without explicit permission from authors)
(In the real instrument, problems were presented one per page, 16-point font)

Math Riddles

<u>Introduction</u>:
I will give you several problems to solve.
Some may be easy, others more difficult.
Solve each problem the best you can.
There are several ways to solve each problem (not just a single right answer).
I am interested in your way of solving.
To better understand your thinking, try to talk out loud when you work on a problem.
I will ask you questions to clarify how you think; if I ask a question it does not say anything about whether your solution is right or wrong.
It's perfectly fine if you don't know how to solve a problem; just let me know and we shall move on.
Use any of the materials here whenever they are useful to you (point to the paper & pencil, 100-Chart, Base-10 Blocks, Unifix Cubes, and Counters). It's also perfectly fine to use your fingers.
This is not a test. Your work will help us know how to program the software that the project is producing and how to teach you better in the weeks to come.

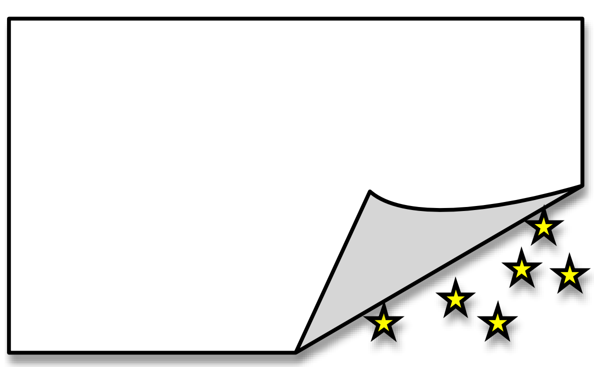
Are you ready to start?

Problem #1

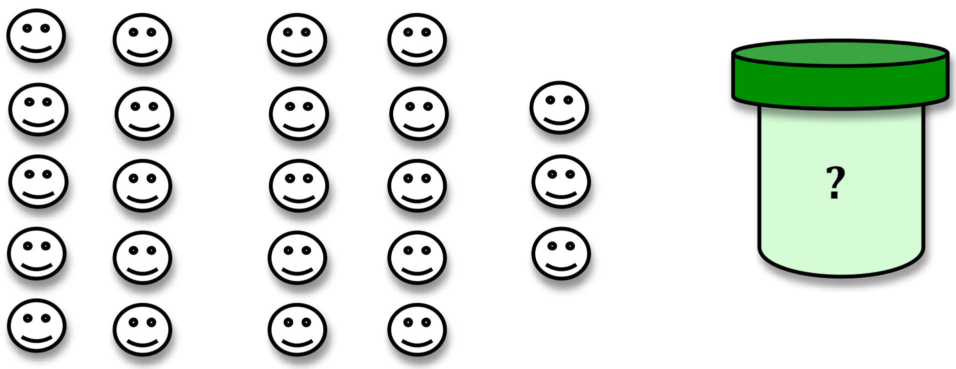
I will ask you to add two numbers using this chart. For example, if the problem was to hop 6 spaces (do this) and then 5 more spaces (do it), I'll then ask you how far the cube is from the start.
a. Please hop over 13 spaces. How far is the cube from the start?
   Use the cube again to hop 6 more spaces.
   How far from the start is it now?
b. Let's do another one. First hop over 47 spaces.
   Now hop over 14 more spaces.
   How far from the start is it now?
c. If you will hop backward 7 spaces, how far from the start will the cube be?

Problem #2:

a. JoAnn marked 17 stars on the paper.
   She then covered some stars and left the rest uncovered.
   How many stars did she cover?



b. A baker baked 31 Smiley cookies.
He packed some in the box and put the rest on the table.
How many cookies did he pack in the box?



Problem #3:

Please complete the following sequences of numbers and let me know how do you know which number to write in each place:

a. 3, 6, 9, 12, _____, _____, _____

b. 25, 33, 41, 49, _____, _____, _____

c. 76, 83, 90, _____, _____, _____

d. 223, 218, 213, 208, _____, _____, _____

Problem #4:

I know how to skip count by 4s (researcher counts out loud):
4, 8, 12, 16, 20, 24, 28, 32, 36, 40.

Can you skip-count by 10?
Can you skip-count by 5?
Can you skip-count by 2?
Can you skip-count by 3?
Can you skip-count by other numbers?

Problem #5:

(Researcher gives an example of how ‘5’ can be produced by adding 4+1 or by 3+2, then asks):
Suggest three different ways in which you can add two numbers to get 17.

Problem #6:

Miguel put all his marbles into 4 bags.
In each bag he has 3 marbles.
How many marbles does Miguel have?

Problem #7:

Write a number that is made of: 6 Tens, 3 Ones, and 5 Hundreds.

Problem #8:

a. Tara collects car toys.
Tara’s cars have 4 different colors (blue, green, red, yellow).
Before last Christmas, in each color Tara collected 6 cars.
How many cars did she collect before last Christmas?

b. For last Christmas, Tara's parents gave her more cars, 6 were black and 6 were white.
How many cars did she have after Christmas?

Problem #9:

Superwoman spotted a burning building with some living creatures in it.
Of course, she immediately set out to save them.
She saved: 2 girls, 7 cats, 4 boys, 5 goldfish, and 6 dogs.
How many pets did she save?

Problem #10:

A new computer game requires the player to organize spaceships in platoons (groups).
A platoon must have exactly 7 spaceships.
The player received 21 spaceships to begin the first game.
How many full platoons can be made?

Problem #11:

a. Selina had several comic books.
Then, her brother Andy gave her 42 comic books.
Then, Selina had 67 comic books.
How many comic books did Selina have before receiving the books from Andy?

b. Luis had 73 candy bars.
Then, Dina gave him some candy bars.
With the candy bars from Dina - Luis has 122 candy bars.
How many candy bars did Dina give Luis?

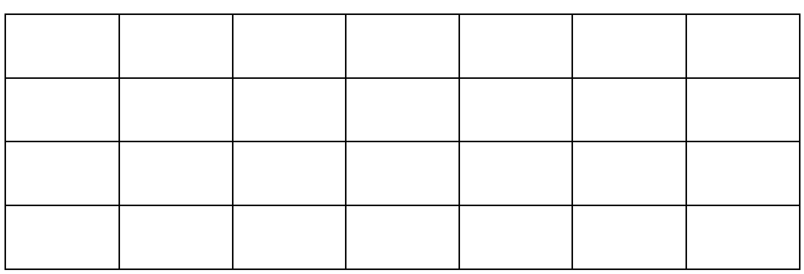
Problem #12:

a. One teacher had 61 flashcards for his students.
Another teacher had 27 flashcards.
To have the same number of flashcards, how many does the second teacher need to obtain?

b. Tiffany collected 23 bouncy balls.
Tiffany has 11 more balls than her friend, Elise.
How many balls does Elise have?

Problem #13:

Look at the chart below. I will cover it now (cover leaves one row and one column visible).
Please find out and tell me, without lifting the cover, how many small squares there are.



Problem #14:

Nora took her five (5) children to the grocery stores.
She promised to give each of them exactly four (4) brownies.
Before shopping Nora does not have any brownies.
She wants to buy exactly the number of brownies needed.
How many brownies does Nora need to buy?