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Volume 1

Navigating currents and charting directions

MERGA 31

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The 31st annual conference of the Mathematics Education Research Group of Australasia was held at The University of Queensland, Brisbane, Qld, Australia
28 June – 1 July 2008

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Preface

This is a record of the proceedings of the 31st annual conference of the Mathematics Education Research Group of Australasia (MERGA). The theme of this conference is *Navigating currents and charting directions*. The theme reminds us that, although we are constantly pushed to account for the quality and impact of our research, we need to assert some control over our work by making our own research futures.

We are pleased to welcome any conference participants who are attending MERGA for the first time. We hope you will make yourselves known so you can be made welcome and introduced to others who share your research interests. Authors from many countries (e.g., Australia, New Zealand, Singapore, Israel) are represented in these Proceedings, as is nearly every university in Australia and New Zealand with education programs. There are also participants from state and independent school systems and government ministries of education. We look forward to the dialogue that will emerge from the varying perspectives brought by these participants.

All research papers and symposia submitted were blind peer reviewed (without author/s being identified) by one of ten review panels comprising mathematics education researchers with appropriate expertise in the field. Review panels were convened throughout Australia and New Zealand by experienced researchers/reviewers who identified colleagues in their geographic region to join the panel. Panel convenors attended a training day where they reviewed earlybird papers according to clear reviewing guidelines that have been refined over a number of years. They then led their panels through the reviewing of a fixed number of conference papers. Each paper was independently reviewed by two panel members, who then discussed their assessments and produced a single consensus report that provided the author(s) with detailed feedback. For consistency, all reviews recommending that a paper not be accepted were reconsidered by two members of a small panel of highly experienced reviewers. Only those research papers accepted by two reviewers have been included in these conference Proceedings. The abstracts for round table discussions and short communications were also blind peer reviewed (without the authors being identified) by two experienced mathematics education researchers.

The most pleasing aspect of the organisation of this conference has been the spirit of collaboration between MERGA members from all the Brisbane universities: Australian Catholic University, Griffith University, Queensland University of Technology, and The University of Queensland. We thank the institutions that have provided free of charge various meeting facilities and other resources to enhance our planning. We particularly thank the Teaching and Educational Development Institute at The University of Queensland for providing sponsorship in the form of Kylie O'Toole, who undertook most of the administration associated with the reviewing process with grace, efficiency, and good humour.

Merrilyn Goos

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Ray Brown and Katie Makar

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Stars, Compass, and GPS: Navigating Currents and Charting Directions for Mathematics Education Research on Gender Issues

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In the early 1970s, gender differences in mathematics learning outcomes favouring males were identified. Research efforts revealed that learner-related cognitive and affective variables, as well as school-related and societal factors were implicated. Policy changes and funded intervention programs followed and had mixed effects. Both government and research attention have since turned elsewhere. In this paper, I present recent findings on gendered patterns in mathematics achievement and participation rates, and on the effects of technology on mathematics learning outcomes. The data indicate that any narrowing of the gender gap in the past decade now appears to be reversing. While there is a growing tendency to focus on smaller scale, qualitative studies, I argue that there is also the need to continue examining large scale data sources to monitor trends over time. I use three navigational metaphors to challenge thinking on the direction of future Australasian research on gender issues in mathematics education.

Introduction

Members of the mathematics education community share many common goals with respect to research on the learning and teaching of mathematics at all levels. We readily acknowledge the many different facets of the field, all critically contributing to the whole.

At one level, we strive to understand the underlying principles of how people learn mathematics well, identify approaches to the teaching of mathematics that are consistent with this, and then how best to structure and deliver pre-service and professional development programs. At another level, we recognise the great diversity among learners, teachers, schools, learning settings, communities, and societies, and that one size does not necessarily fit all. In aiming to achieve equitable outcomes for all, finding ways to address and overcome disadvantage while simultaneously accepting difference are the guiding principles that provide the major challenges to those conducting research in these areas.

Gender is an "obvious" category of difference and is, I would argue, a variable of disadvantage in mathematics education. The extent of the disadvantage – but rarely the disadvantaged group (females) – varies by location, socio-economic status, ethnicity, societal expectations, socio-political climate, and other factors, with each variable having differential impact in a given context (e.g., McGaw, 2004; OECD, 2007; Teese, Davies, Charlton, & Polesel, 1995). While some may dismiss research on gender as irrelevant to the main concerns within an holistic purview of mathematics education, I contend that gender is a central variable demanding inclusion in all mathematics education research studies.

Why it is Important to Incorporate Gender into Research Studies

In the early work on gender and mathematics education, persistent patterns of gender difference were found in two main spheres: achievement measures and participation rates, particularly in the most challenging mathematics subjects (e.g., Eccles, 1985; Fennema, 1974). Research efforts resulted in a range of contributing factors being identified. In attempts to explain the patterns of gender difference observed, various models were proposed (e.g., Eccles et al., 1985; Fennema & Peterson, 1985; Leder, 1990). Among the learner-related and environment factors in Leder's (1990) explanatory framework (see Figure 1), many of the variables common to the other explanatory models postulated prior to 1990 are found. To this day, Leder's (1990) model continues to provide a useful starting point for research on gender issues, particularly for those identifying with liberal feminist theory and Fennema's (1990) three equity principles: equity with respect to access and opportunity, equity with respect to treatment, and equity with respect to outcomes.

As a result of the early research findings, resources were fairly free flowing during the 1980s and 1990s to support a range of intervention programs (see Leder, Forgasz, & Solar, 1996), with mixed levels of success. Since 1990, there has been an expansion of knowledge in the field; in Australasia, for example, a chapter on gender issues has been found in each of MERGA's four-yearly reviews of the literature since they began in 1984. There have also been challenges to the theoretical viewpoint of liberal feminism which guided the early research. Some have argued that in aiming for women to reach men's levels, liberal feminism was consistent with a deficit view of females. Not all agreed. As a consequence, the field has broadened to incorporate a range of alternative feminist standpoints that underlie research efforts, all aimed at bettering females' mathematics experiences and learning outcomes. The subsequent research findings and their implications for mathematics education more generally have greatly enriched the field.

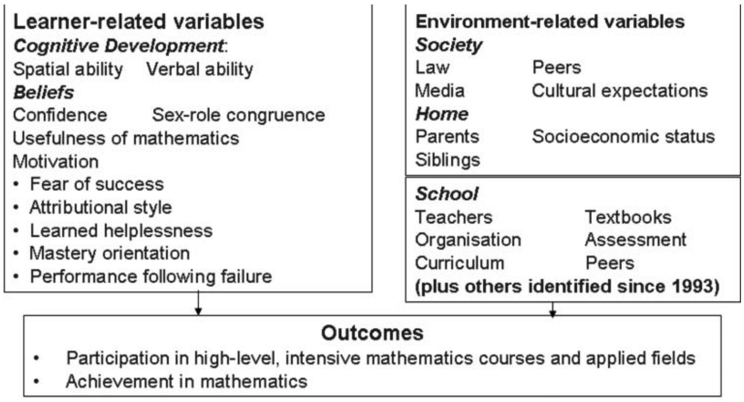

Figure 1. An explanatory framework for gender differences in mathematics learning outcomes. Adapted from Leder (1990).

Methodologically, there have also been changes in the approaches taken to investigate the gendering of mathematics as a discipline, the gender-role stereotyping associated with teachers and learners of mathematics, and the effects of external factors, including technology, on gendered patterns of learning outcomes. In the current evidence-based climate in which we work and conduct research, I believe that an emphasis on small scale, qualitative studies, is likely to have very limited impact beyond the mathematics education research community. It is difficult to convince policy makers, curriculum developers, and teachers of the need for change based on findings from small studies. This is not to say that such studies are worthless or should be abandoned. Rather, my preference has been, and remains, for mixed methods approaches. Large scale data, from which generalisable trends pertinent to a context can be determined, can then be complemented (or supplemented) by focussed, qualitative studies to understand better the identified phenomena and patterns. I consider it regrettable that in some quarters research designs including psychological constructs are increasingly viewed with suspicion, and that in the broader educational research community there is growing disdain for the inclusion of statistical analyses. The consequences include a lack of generalised evidence in many important aspects of education, as well as a loss of expertise in the gathering, analysis, and interpretation of statistical data. It is also evident that there is a lack of sustained effort to monitor trends or evaluate the outcomes of interventions and change. This might be due to poor judgment, or to a level of arrogance for which decisions made are unquestioningly considered correct.

Some of the early advances in redressing female disadvantage in mathematics achievement and participation rates now appear to be reversing. For example, there was no gender difference in the Australian mathematical literacy results for PISA 2003 (OECD, 2004), but in 2006 there was a statistically significant difference favouring males (OECD, 2007). [For New Zealand there were statistically significant differences in favour of males in both the 2003 and 2006 PISA results.]

In some international contexts, attention is only now beginning to focus on gendered patterns of disadvantage in mathematics learning. While it is heartening that researchers in these countries believe it important to understand and overcome the gender differences they have identified, I have been distressed by the lack of awareness of earlier research and of the vast knowledge base in the field that could be drawn upon to inform research agendas.

The Place of Affect in Mathematics Education Research on Gender

Leder's (1990) explanatory model (Figure 1) included a range of learner-related affective factors as contributors to gender differences in mathematics learning outcomes. Among the environment- and school-related variables, several also have affective dimensions, for example, whether teachers and parents hold gender-stereotyped beliefs about boys' and girls' capacities to learn mathematics and about their future career directions. In the field of gender and mathematics learning, affect cannot be ignored.

Various pairings of critical dimensions in mathematics education research are often seen as oppositional. More often, they are complementary and there is a need to establish ‘harmony’. At the Rome conference to celebrate the 100th anniversary of ICMI, Jeremy Kilpatrick used the Chinese concept of Yin (darkness) and Yang (light) (see Figure 2) to highlight the complementary aspects of mathematics and mathematics education. The Yin and Yang concept can also be applied to emphasise the complementary notions of cognition and affect. Both receive attention in contemporary mathematics curricula. Yet, some children’s misconception that there can exist a bigger and a smaller “half” also appears to hold true in this context; invariably more cognitive than affective goals are listed. Within mathematics education research, too, those focussing on affective issues represent a much “smaller half” of the community. In classrooms and the general community there also appears to be greater importance attached to cognition than affect. Yet, people’s reactions to mathematics, particularly their negative reactions, are frequently couched in affective terms: “I hated mathematics at school” is likely to precede comments related to the cognitive such as “I didn’t understand mathematics” or “It was too difficult”. Since it is more often women than men expressing negative sentiments towards mathematics, the need to find greater harmony between affect and cognition in all areas of mathematics education research seems clear.

Figure 2. Symbol for the Chinese concept of Yin and Yang

Research on Gender Issues and Metaphors for Research Pathways

The focus of the rest of this paper is on examining patterns of gender difference in mathematics achievement and participation, two outcomes of mathematics learning central to researchers in the field, and on some of the affective and other factors contributing to them. I draw heavily on findings from my own research. I comment on the relationships of these findings to three navigational metaphors that I have used. The metaphors represent aspects of the types of research and research approaches which I believe have the potential, at different levels, to steer meaningful and effective future Australasian research agendas on gender issues. I conclude the paper with personal reflections on the field of gender and mathematics learning and its future research directions.

Let me begin with brief explanations of the three metaphors – stars, compass, and GPS.

The Stars

As the basis of early navigational instruments, the stars (including the sun) represent what can be learnt from history (Navigation and related instruments, nd). The (mariner’s or sea) astrolabe, said to originate in ancient Greece (Early navigational instruments, nd) and known to have been used by the Persians in C11th (Navigation and related instruments, nd), was used to measure latitude and is claimed to be the first scientific instrument for navigation. The ‘star’ metaphor can be extended. During the day, and when the night stars are obscured by clouds, the boundaries surrounding the usefulness of instruments such as the astrolabe are obvious. Similarly, the limitations of the early research studies, and the data gathering and analytical approaches adopted, need to be recognised. However, there are clear advantages in being able to fall back on earlier knowledge and methods when more contemporary approaches may be inappropriate or fail, or when particular phenomena are being examined for the first time in a new context.

The (Magnetic) Compass

While Christopher Columbus may have said that the compass “always seeks the truth” (Navigation and related instruments, nd), the direction shown on the magnetic compass is not “true north”. Still used today, the magnetic compass is independent of the time of day and the weather, and requires no external energy source. Yet, it, too, has its limitations. Not only does it not point true north, it does not always point towards the “magnetic north” either. Variations are due to distortions in the Earth’s magnetic field, as well as the effects of other localised magnetic fields formed in the presence of iron/steel, electric currents etc. User knowledge and navigational skills can compensate. The magnetic compass represents what is valid, reliable, and relatively stable in our research endeavours – the focus of the research studies conducted; the research instruments, approaches, and methods of analysis used to extend current knowledge and seek new knowledge; and findings that can be compared to those conducted earlier and/or in different locations.

The GPS

A navigational tool for "dummies" is one way of regarding the GPS. A wonder of modern technology, the car version with its human voice can take you wherever you want to go. If a road is blocked or you take a wrong turn there is no problem, as an alternative route will be worked out. However, to use the instrument there is total dependence on being in range of satellites and having access to battery energy. If either fails, users are left floundering. Thus the GPS can signify research efforts dependent on the vagaries and ebb and flow of funding availability and other external factors – human, contextual, technological, and socio-political – that apply pressure to meet immediate perceived need and popular demand. The GPS can also be seen to represent the research context in which we in Australasia have recently found ourselves: beholden to government prerogative, with a focus on product, and consequently starved of funds for "basic" research.

Research Studies on Gender and Mathematics Education

Participation

Findings from large scale studies, with their concomitant limitations, provide broad brush overviews of phenomena that invite explanations to be sought through additional studies. The three Johns – Dekkers, de Laeter and Malone – began monitoring enrolment patterns in grade 12 mathematics and science subjects across Australia quite early, and repeated their investigations regularly (Dekkers et al., 1986, 1991, 2000). In 2006, I conducted a study for ICE-EM on enrolment numbers for grade 12 mathematics subjects across Australia (Forgasz, 2006) that built on this earlier work. I was interested in determining patterns of enrolment over the years 2000-2004 in the various mathematics options available and, of course, whether there were gender differences. Of interest, too, was how these enrolment patterns would compare with those reported earlier.

I generally followed the methodology of Dekkers, de Laeter, and Malone, but made some variations. Their work only looked at enrolment numbers. I gathered data on overall grade 12 cohort sizes and calculated the percentages of grade 12 students enrolled in each category of mathematics subject. Dekkers et al. used three categories – high, intermediate, and low – that were related to tertiary entry requirements and career pathways. Barrington and Brown (2005) classified the subjects offered in 2004 by content demands into three levels – advanced, intermediate, and elementary. Strict comparisons with earlier results were tricky because the bases for the subject categorisations differed, subject names had changed, and there had been curricular changes. I included many caveats to my report.

Across Australia in 2000-2004, females comprised about 53% of the grade 12 cohorts each year. To be representative, M:F ratios in all grade 12 subjects should have been about .89. Focussing only on data for intermediate level mathematics, the most common pre-requisite for tertiary study, Australia-wide enrolment data by gender are shown in Figure 3 for the years 1990-2004. In Figure 4, the same enrolment data are shown but expressed as percentages of grade 12 cohort numbers. Slightly different stories about intermediate level mathematics enrolments can be inferred from the data in Figures 3 and 4.

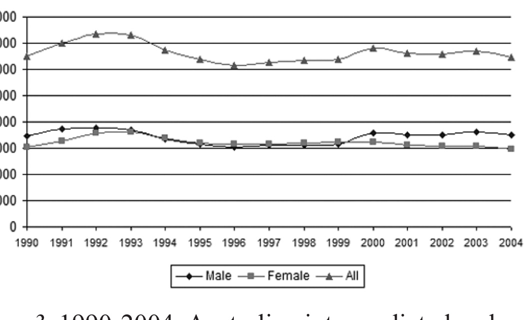

Figure 3. 1990-2004: Australian intermediate level year 12 mathematics enrolments, by gender.

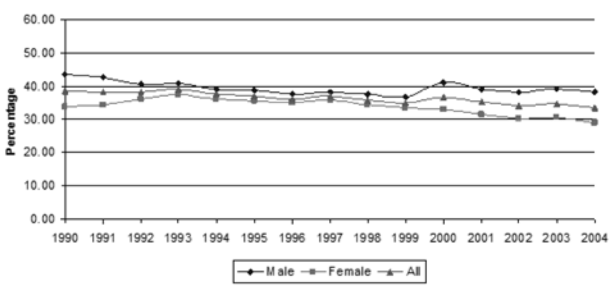

Figure 4. 1990–2004: Australian intermediate level mathematics enrolments as percentages of year 12 cohorts, by gender

In Figure 3 it can be seen that from 1995-1999 about equal numbers of males and females were enrolled, female enrolments were fairly constant from 1995-2004, and male enrolments increased suddenly in 2000 and then stabilised.

The data in Figure 4 show that from 1995-1999 there was a small but steady decline in the proportions of male and female students, in 2000 there was a jump in the proportion of males, and from 2000 the steady decline in the proportions of males and females continued.

So which story should be told? Which story is correct? Why are the stories different? I believe that speaking in terms of enrolments alone, without considering grade 12 cohort sizes, can be misleading. However, it could also be argued that the proportions of cohort data can be misleading, particularly if the composition of the cohorts is very different (as would be the case for 1990 compared to 2004). Both sets of data require additional information. We are dealing here with methodological and epistemological issues. An appreciation of the socio-political context affecting the composition of grade 12 cohorts over time is important. For the years 2000-2004, both sets of data tell virtually the same story since the economic and educational conditions for students across Australia were fairly stable with respect to mathematics curricula, tertiary pre-requisites, and employment.

In Table 1, male to female ratios for enrolment numbers in 2000-2004 are shown for advanced, intermediate and elementary levels of mathematics subjects, with state-wide data for the intermediate level subjects only.

Table 1

M:F ratios for grade 12 mathematics enrolments across Australia in advanced, intermediate, and elementary level courses, 2000-2004

	2000	2001	2002	2003	2004
Advanced	1.70	1.71	1.63	1.60	1.58
Intermediate	1.11	1.12	1.14	1.17	1.19
ACT	*.81*	*.89*	*.74*	*.83*	*1.00*
NSW	*1.04*	*1.05*	*1.09*	*1.13*	*1.14*
NT	*.79*	*1.38*	*1.08*	*1.22*	*1.56*
Queensland	*1.09*	*1.14*	*1.16*	*1.15*	*1.15*
SA	*1.21*	*1.29*	*1.29*	*1.40*	*1.44*
Tasmania	*1.13*	*1.29*	*1.18*	*1.29*	*1.46*
Victoria	*1.19*	*1.14*	*1.15*	*1.18*	*1.20*
WA	*1.29*	*1.26*	*1.33*	*1.38*	*1.43*
Elementary	.85	.89	.90	.93	.91

The data in Table 1 reveal that:

– males were over-represented in advanced and intermediate level mathematics subjects (M:F >>.89)
– there was a small, but steady decrease in the M:F ratio for advanced level subjects
– there was a small, but steady increase in M:F ratios for intermediate level subjects overall, with large variations in the M:F ratios by state/territory and different patterns of change over time from one state/territory to another
– there was a small overall increase in the M:F ratio for elementary level subjects; at the elementary level, the M:F ratios are fairly representative of grade 12 cohorts.

The trends over time suggest that compared to females, males were less likely to choose advanced level mathematics subjects, and more likely to opt for intermediate and elementary level subjects. Also, while there was an overall decline with respect to intermediate level mathematics subjects, the decline was slightly greater among females.

These data are open to multiple interpretations. Advanced level mathematics courses are very rarely stipulated today as pre-requisites for tertiary study. Thus, in the competitive tertiary entry environment, it could be argued that by moving away from advanced level mathematics courses, some males are being pragmatic. Perhaps, too, males are more certain than females of their future career trajectories, less swayed by the idea of keeping options open, and more aware of pre-requisites, hence opting for elementary and intermediate rather than advanced level courses. A potential positive from these data is that with relatively more females studying advanced level mathematics subjects, this may, over time, be reflected in the field of mathematics itself.

Missing from this study was a qualitative dimension in which the effects of socio-political factors on mathematics enrolment patterns could be explored, a better understanding of why some boys may be opting out of advanced mathematics sought, and reasons for the large variations in male and female enrolments across the states examined.

This research study demonstrates how the "stars and compass" can be used together to advance knowledge in the field. [A GPS energy source would have been welcome.]

Performance (of High Achievers)

I undertook an analysis of the 2007 Victorian Certificate of Education [VCE] results in the three mathematics subjects offered, by gender, for the highest achievers. In the Victorian newspaper, *The Age*, the names and schools attended by students obtaining *study scores*[1] of 50 (highest) down to 40 are listed for each VCE subject. Table 2 includes the 2007 VCE cohort size by gender, and enrolments in each mathematics subject by gender.

Table 2

VCE enrolments by gender

	All	**M**	**%**	**F**	**%**
VCE cohort	48840	22588	46.2%	26252	53.8%
Specialist mathematics	4804	3012	62.7%	1792	37.3%
Mathematical methods & CAS[2]	15427	8600	55.7%	6827	44.3%
Further mathematics	24787	11623	46.9%	13164	53.1%

The figures in Table 2 indicate that females comprised 53.8% of all grade 12 VCE students, and that for all mathematics subjects, including Further mathematics, males were over-represented in relation to their VCE participation. The analysis of the gender break-up of the highest achieving students revealed that the gender gaps favouring males were even wider. In 2007, students with study scores of 46-50 represented about 1.3% of the cohorts in each mathematics subject. The numbers of students scoring 46-50 in each subject, and the numbers and percentages by gender are shown in Table 3. [NB. When students' genders could not be determined from given names, they were classified 'unknown' (?).]

1 Standardised scores such that for each VCE subject: mean ≈ 30, sd ≈ 7

2 Combined enrolments for the two parallel subjects, Mathematical Methods and Mathematical Methods CAS

Table 3

Male and female achievements by study score (50-46) in VCE mathematics subjects, 2007. [in bold: gender group over-represented]

	Specialist mathematics				**Mathematical methods (& CAS)**				**Further mathematics**			
Score	All	M	F	?	All	M	F	?	All	M	F	?
50	14	**12**	1	1	36	**24**	9	3	60	**43**	13	4
		86%	7%	7%		**67%**	25%	8%		**72%**	22%	7%
49	5	**5**	-	-	2	9 **24**	5	-	36	**19**	14	3
		100%				**83%**	17%			**53%**	39%	8%
48	12	6	6	-	27	**18**	6	3	48	**30**	17	1
		50%	50%			**67%**	22%	11%		**63%**	35%	2%
47	13	**11**	2	-	41	**29**	11	1	60	**37**	22	1
		85%	15%			**71%**	27%	2%		**62%**	37%	2%
46	21	**15**	6	-	66	**38**	19	9	108	**58**	48	2
		71%	29%			**58%**	29%	14%		**54%**	44%	2%

When the data in Table 3 are compared to the proportions of male and female students enrolled in each of the three VCE mathematics subject (Table 2), it is very clear that males dominate over females at the highest levels of performance – study scores of 50-46.

This study is again representative of the 'stars' and 'compass' working together.

Attitudes Towards Computers for Mathematics Learning

In a three year ARC-funded project, I was able to conduct a mixed methods study examining grade 7-10 students', and their teachers', beliefs about the effects of computers on mathematics learning, and to identify factors contributing to the patterns of belief found. Among a range of findings, the following, I believe, were most noteworthy.

In 2001 and 2003, large samples of students were asked if computers helped their mathematics learning. Their responses, and the results of chi-square tests by gender, are shown in Table 4.

Table 4

Students' beliefs about computers helping mathematics learning, 2001 & 2003

2001: N=2140						**2003: N=1613**					
F (n=1015)			***M (n=1112)***			***F (n=810)***			***M (n=796)***		
Yes	*Unsure*	*No*	*Yes*	*Unsure*	*No*	*Yes*	*Unsure*	*No*	*Yes*	*Unsure*	*No*
194	334	429	326	331	384	185	269	300	239	237	256
20%	35%	45%	31%	32%	37%	25%	36%	40%	33%	32%	35%
Gender difference: $\chi^2 = 32.5$, $p<.001$						Gender difference: $\chi^2 = 12.1$, $p<.01$					

The data in Table 4 reveal that males believed more strongly than females that computers helped their mathematics learning. The students (2001: 26% and 2003: 29%) were much less convinced than their teachers (61% in both years) that computers assisted mathematics learning. This study also included a qualitative component in which six grade 10 mathematics classrooms were observed when the students used computers, a number of self-report instruments were completed, and teachers and four students from each class were interviewed. The teachers were asked to rate, on 5-point scales (excellent – poor), each student's mathematics achievement, level of co-operation, persistence with and confidence in using computers. T-tests conducted on the pooled ratings by gender revealed statistically significant differences favouring males for confidence (M=3.93, F=3.32, p<.001) and persistence (M=3.69, F=3.3, p<.05). On the large scale teacher surveys (2001: N=96; 2003: N=76) and at the interviews with those whose classes were observed, teachers were asked if there were differences in the ways boys and girls worked with computers. Their comments indicated that they believed that it was students who were competent with computers, rather than necessarily mathematically strong, who gained most from computer use for mathematics learning; boys were considered to be more computer savvy.

Gender stereotyped comments were also evident. For example:

> … my observation [is] that girls naturally are not… as good in mathematics as boys are… [T]hey are better in language skills and they have different strengths than the boys… [It] doesn't apply to everyone, but it's the general trend… [B]ecause they're…not good in maths as naturally boys are, so I suggest to them to have a bit more practice so the concepts are… more consolidated and they could use it when they need. So I think they need a bit more practice than boys.

Methodologically, this study again reflects the "stars" and the "compass". However, the topic, and the funding support, suggest that the "GPS" had a role to play.

Effects of Technology on Performance

In another study of VCE results, I looked at students' performance, by gender, on the two parallel running VCE courses – Mathematical Methods and Mathematical Methods CAS – during the three year trial of the CAS subject. The numbers taking the CAS option were small and only the data for the final year of the trial 2004 are presented here. The within gender percentages and M:F ratios of percentages for students gaining A+ for the three assessment tasks for the two subjects are shown on Table 5.

Table 5

A+ results, by gender, for each assessment component for Mathematical Methods and Mathematical Methods (CAS), 2004

	Mathematical Methods					**Mathematical Methods (CAS)**				
	Male		**Female**		**M:F**	**Male**		**Female**		**M:F**
	N	%	N	%		N	%	N	%	
School-based	1747	18	1261	15	**1.20**	42	17	20	13	***1.31***
Examination 1	1102	12	793	10	**1.20**	35	14	16	11	***1.27***
Examination 2	1013	11	593	7	**1.57**	33	14	12	8	***1.75***

In Table 5 it can be seen that while the percentage of male students gaining A+ in all three assessments in both subjects was consistently higher than for females (M:F ratios > 1), it is clear that the gender gap was greater for each assessment task in the CAS version of the subject (bold italics on Table 5). Forgasz and Griffith (2006) reported that with very few exceptions this pattern was repeated across the three assessment tasks for the two subjects for each year of the trial, 2002-2004, and for each of three levels of achievement: A+, A, and B+. The reasons behind the apparent increased gender gap in the CAS results need to be explored. Considering that small sample sizes associated with the enrolments in the CAS option, the trends that were evident in this study demand future monitoring. Interestingly, Thomson and De Bortoli (2008) noted that Australia's significant gender difference for mathematical literacy in PISA 2006 was higher than the OECD average and:

> appears to have come from the higher end of achievement; 18 per cent of females achieved at Level 5 or 6 in 2003, compared to 13 per cent in 2006. For males the corresponding proportions were 22 per cent and 20 per cent respectively. (p. 245).

Is there a link between the Thomson and De Bortoli (2008) finding and my results that suggest that CAS use increases the gender gap in mathematics performance? Are the latest PISA results associated with an increased emphasis, Australia-wide, on technology use for mathematics learning across all grade levels? These questions invite further investigation.

The effects of technology on learning outcomes, the topic of my research study, aligns with the "GPS", but only a basic model with very low energy demands and restricted map coverage. Methodologically the study was indicative of the steady, reliable "compass".

Other Studies

There are two other studies I would have liked to tell you about but am unable to do so. Both were DEST projects for which confidentiality was required. What I can tell you is that the findings would have added to what is known on gender issues. Metaphorically speaking, total seduction by access to the latest model GPS has resulted in the findings from these two studies being lost to the mathematics education research community. The GPS, it seems, was only on short term loan!

Concluding Comments

The research findings I have presented show that gendered patterns from the past are still evident in the context of contemporary mathematics education in Australia. While I have no hesitation in acknowledging that there are inequities in the system disadvantaging boys as a group of learners, particularly in literacy and related areas, the same cannot be said for mathematics education. It is clear that equity, at least from Fenemma's (1990) standpoint, has not yet been achieved. Despite interventions, curricular change, and a better understanding of the relationship between assessment types and male-female performance outcomes (e.g., Cox, Leder, & Forgasz, 2004), females remain under-represented in higher level mathematics subjects, their performance levels are below those of males and there are signs that the gender gap may be increasing. Less functional belief patterns with respect to the impact of technology on females' mathematics learning are also evident.

In this presentation, I did not discuss findings on the more traditional affective variables associated with gender issues and mathematics learning. Recent research on beliefs about the stereotyping of mathematics as a male domain included many traditional affective variables – enjoyment, interest, confidence, attributions for success and failure, sex-role congruity, perceived usefulness, and perceptions of parents' and teachers' views. Findings defying established stereotyped patterns were reported for students in grades 7-10 (Forgasz, Leder, & Kloosterman, 2004), but not among pre-service teachers (Forgasz, 2005). International comparisons have revealed differences in the extent to which the gender stereotyped views have been challenged, with Australian students faring well in this regard (e.g., Barkatsas, Forgasz, & Leder 2002; Forgasz et al., 2004; Forgasz & Mittelberg, 2007).

In the real world, I love my GPS and the stress-free liberty it provides when travelling to unknown destinations, although I still cope pretty well with maps. At the same time, I greatly admire those who can skilfully navigate by the sun and stars, and those for whom a magnetic compass is all that is needed in open terrain. Metaphorically, the "stars" (the past) and "compass" (reliable but with limitations) are my preferred navigational tools as they reflect the rationale for my research and the methodological approaches I adopt. Post-positivism is the epistemological perspective informing my world view – the knower cannot be separated from the known and there is no single truth, although it is the ultimate goal. Since the socio-political context, the times, the location, and the setting all impact on the particular "truth" uncovered by the research I conduct, it is the limitations of the "stars" and the "compass" that sit comfortably with my perspective of the research enterprise. I am not fazed by research results that vary from one time and place to another; in fact, knowing about them can be used to inform the directions of future investigations. However, persistent patterns in findings present a dilemma and demand continuous monitoring.

I believe that the metaphors for the stars, compass, and GPS that I have used are equally applicable in all research areas within mathematics education. Experienced and novice researchers must be cognisant of the past in order to build upon and extend the knowledge base. While the precision of the GPS can only be admired, the instrument has a more devastating inherent weakness than the stars and compass in that it has the potential to fail us completely. Governments come and go, one topical issue is replaced by another, money flows and then is cut off. Worthwhile research is fundamental, aimed at expanding knowledge to better the human condition, and must be shared with the community. Despite the obstacles and temptations put before us, working this way is my suggested path forward if we are to remain true to our moral obligations as mathematics education researchers.

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Praxis and Practice Architectures in Mathematics Education

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This paper discusses the way "right conduct" in education – *praxis* – is embedded in social practices – *practice architectures* – which enable and constrain conduct in three dimensions: cultural-discursive, material-economic, and social-political, or "sayings", "doings", and "relatings". It is argued that changing professional practices like mathematics education is not just a matter of changing the understandings (cf. sayings), skills, and capabilities (cf. doings) or values and norms (cf. relatings) of practitioners, but also changing the practice architectures that enable and constrain practitioners' actions and interactions. The practice architectures of mathematics education are constructed not only by the knowledge, capabilities, and values internal to traditions in mathematics education but also by meta-practices external to those traditions – particularly the meta-practices of educational administration and policy making, initial and continuing teacher education, and educational research and evaluation. Today, the elaborateness, the rigidity and the compulsions associated with these meta-practices threaten the vitality of education in general, including mathematics education. Educators are thus confronted by an invidious choice. Should they conduct their practice as *praxis*, oriented by tradition and by considerations of the good for each person and the good for humankind as these are expressed in the conduct of education as a practice, as *agents* of education? Or should they conduct themselves as the *operatives* of the education systems in which they find themselves, following the rules and procedures that constitute the functional rationality of those systems?

Praxis and Practice Architectures

This paper discusses the way "right conduct" in education – *praxis* – is embedded in social practices – *practice architectures* – which enable and constrain conduct in three dimensions: cultural-discursive, material-economic, and social-political, or "sayings", "doings", and "relatings".

In this first section of the paper, I will briefly outline the working theory of practice that is guiding some of the research of our "Pedagogy, Education and Praxis" project at Charles Sturt University and in collaboration with fellow-researchers from the Netherlands, the Nordic countries, and the United Kingdom. Before outlining our view of how the special form of action called "praxis" constructs and is constructed in "practice architectures", however, I would like to make some remarks about "education". This is "the big picture" for those of us concerned with praxis and practice in education as a discipline and as a profession. Although saying so is by no means original, I want to recall explicitly that education is always and necessarily a moral and political endeavour.

Education

In an unpublished monograph prepared for students in the Bachelor of Education (Primary) course at Charles Sturt University, Wagga Wagga (Kemmis, 2006), I asserted that

> Education has a double purpose. On the one hand, it aims to form and develop individuals with the knowledge, capabilities and character to live good lives – that is, lives committed to the good for humankind. On the other hand, education aims to form and develop good societies, in which the good for humankind is the principal value. (p. 5)

This view of the double purpose of education follows a long tradition in the discipline of Education, echoed by the centuries-long European tradition of Pedagogy (in Dutch, *Pedagogiek*; in Swedish, *Pedagogik*; in German, *Pädagogik*; see Ponte, 2007; Ponte & Ax, forthcoming 2008). Yet today there are many (even graduates of teacher education courses) who appear unable to make a distinction between "education" and "schooling" – between "education" with this double, individual and social, purpose and "schooling" which is the institutional formation of learners to attain (usually state-) approved learning outcomes which might or might not be in the interests of the students themselves or the good for humankind (though of course it is usually intended to be). People unable to make this distinction are thus unable to conceive of schooling which is non-educational or even anti-educational, as might occur in a totalitarian regime. People unable to make this distinction might thus be prevented from forming a critical view about whether schooling, as we have it here and now, in our schools and our states, is "educational", properly speaking. And people who are educators might also be expected to know other, related distinctions like those between teaching and education, between learning and education, between training and education, between socialisation and education, and between indoctrination and education.

It might reasonably be expected that teachers should be educators, and that their practices as teachers might be not only "teaching practices"[2] but "educational practices". That is, on the one side, it might reasonably be expected that when a teacher acts as an educator, it is in the interests of the particular learners present: in the interests of their self-development, their self-expression and their self-determination (Young, 1990) as persons. On the other side, the teacher acting as an educator acts in the interests of the good for humankind: in the interests of the "self"-development, "self"-expression and "self"-determination of the various social and political collectivities in which we live our lives. On this view, education has the moral, social, and political purpose that aims to develop not only good people but good societies.

To have a good society – a democratic society, for example – we must have not just individuals each of whom is personally committed to the good for humankind; we must also have *the kinds of structures, practices and relationships which foster, express and, in the end, constitute the collective good for humankind*. These structures, practices, and relationships will value truth in some sense above falsehood, self-deception, and fraud in the cultural-discursive dimension. They will be the kinds of structures, practices, and relationships which value human and global well-being in some sense above harm, suffering, and waste in the material-economic dimension. And they will be structures, practices, and relationships which value goodness, care, equity, legal and political equality, recognition and respect for persons, and justice for all in some sense above evil, exclusion, tyranny, and injustice in the social-political dimension. These are the kinds of social structures, practices, and relationships we most hope for in societies that claim to be democratic.

In democratic societies, teachers who are educators are necessarily concerned with education for democracy. A democratic society aims to constitute, through its formal and informal structures, practices and relationships, the conditions in which each can thrive and all can thrive, and in which each and all are protected from the worst – including the harm we can do to each other. A teacher who is also an educator is thus concerned as much with the well-being of the social whole, the collectives of the classroom, the school, the community, the state, and the world, as with the well-being of each learner. An educator thus stands for something more than the interests of individuals and sees societies as something more than undifferentiated aggregates of individuals, each of whom must make their own way in our flawed, frail, and fragile world.

Such a teacher sees a society as a social space constructed in an organic tissue of structures, practices, and relationships that enable and constrain people so each and all have equal and achievable opportunities to self-development, self-expression, and self-determination. Such a teacher, through her or his educational practice, conscientiously and resiliently assists in creating this organic tissue of connections in her or his classroom, school, and community. Such an educator aims, necessarily, to be wise and prudent – to envisage the possible long-term consequences of action in the here and now, and to envisage possible harm and suffering that could be avoided. Such an educator is also necessarily critical – able to be critical about present and existing structures, practices, and relationships that, by their immediate effects or their long-term consequences, cause or permit harm and suffering.

The kind of educator I am describing necessarily aims to act in ways that are morally and ethically appropriate, and in ways that are politically appropriate – that is, to act in ways that avoid harm and thus make a positive contribution to our shared fates as co-habitants of the planet and our solidarity with one another as persons and citizens.

Two Views of Praxis

In the *Ethics* (2003), Aristotle (384 BCE – 322 BCE) drew distinctions between different kinds of actions, guided by different kinds of dispositions. These are presented in Table 1 below (excerpted from Kemmis & Smith, 2008), supplemented by another kind of action and disposition which I have adapted from Habermas's (1972) *Knowledge and Human Interests* (in which Habermas suggests replacing the ancient idealist notion of *epistēmē* with a critical-emancipatory disposition):

2 In a special issue of the *Journal of Philosophy of Education* concerned with whether teaching is a practice, Nel Noddings (2003) concludes that it is, describing teaching as a caring practice.

Table 1

Four Perspectives on Dispositions and Action

	Theoretical perspective	**Technical perspective**	**Practical perspective**	**Critical-emancipatory perspective**
Telos (Aim)	The attainment of knowledge or truth	The production of something	Wise and prudent judgment; acting rightly in the world	Overcoming irrationality, injustice, suffering, felt dissatisfactions
Disposition	*Epistēmē:* The disposition to seek the truth for its own sake	*Technē:* The disposition to act in a true and reasoned way according to the rules of a craft	*Phronēsis:* The moral disposition to act wisely, truly and justly; with goals and means both always open to review	*Critical:* The disposition towards emancipation from irrationality, injustice, suffering, felt dissatisfactions
Action	*Theoria:* Contemplation, involving theoretical reasoning about the nature of things)	*Poiēsis:* “Making” action, involving means-ends or instrumental reasoning to achieve a known objective or outcome	*Praxis:* “Doing” action, involving practical reasoning about what it is wise, right and proper to do in a given situation	*Emancipatory:* Collective critical reflection and action to overcome irrationality, injustice, suffering, harm, unproductiveness or unsustainability

Here, I want to highlight the concept of “praxis”. In Table 1, “praxis” is used in an Aristotelian sense, to be understood in terms of “right conduct” or, as Kemmis and Smith (2008) put it,

> *Praxis* is a particular kind of action. It is action that is *morally-committed, and oriented and informed by traditions in a field*. It is the kind of action people are engaged in when they think about what their action will mean *in the world*. *Praxis* is what people do when they take into account all the circumstances and exigencies that confront them at a particular moment and then, taking the broadest view they can of what it is *best* to do, they *act*. (p. 4; emphases in original)

It is important to note that “praxis” is *the action*. Of course, one’s thinking may be part of one’s doing, but praxis is not the thinking that may precede action (cf. Aristotle’s notion of the guiding disposition of *phronēsis*, and the practical deliberation which may also come before the action; Schwab, 1969; Reid, 1978). Praxis is the action itself, in all its materiality and with all its effects on and consequences for the cultural-discursive, material-economic, and social-political dimensions of our world in its being and becoming. Praxis emerges in “sayings”, “doings”, and “relatings” that are usually more or less coherently “bundled together” (Schatzki, 2002). As praxis “enters” the world, guided by good intentions for individuals and humankind, and shaped by traditions of thought about a particular field of practice, it begins to change the world around it (as do all actions, whether praxis or not). It aims to be wise and prudent, but as it happens, it immediately begins to affect the uncertain world in uncertain and indeterminate ways. Consequences begin to flow, whether for good or for ill. Now the ones who act begin to learn the measure of their wisdom and prudence: do things turn out as they had hoped, anticipated, and intended? So: praxis changes the world, with consequences that ripple “out” from the location in which the action took place, and cascade “down” through time.

Of course, the action that is praxis does not have its origins solely in the actor, either. She is always already pre-formed by the “sayings”, “doings”, and “relatings” that have made her who she is today. In the past and present, she has been changed by “sayings”, “doing”, and “relatings” – her own or others’ – and she has inherited a world partly made by her history of action in the world. She is a “subject” who has been formed as well as a person who forms herself and others.

And the action that is praxis comes into a world always already pre-formed for this particular possibility of action, for this praxis. At the moment of action, it is a world ready for this. It is a world in which such an action as this is possible, and usually relevant and appropriate. It may be a moment in which it is time to speak out. It may be a moment when the most important thing is to show care. It may be a moment when one must listen. It may be a moment in which it is necessary to tell the patient his illness is terminal, or the student that he must leave the room.

I want to emphasise that praxis is action in which the practitioner is aware of acting in history, that it is history-making action, that it has, for the one acting, some world-historical significance (even if a small thing in itself). And this is a second sense of the term "praxis" that was taken up and developed by Marx from Hegel. If the Aristotelian sense of praxis finds its locus in the one who acts – the actor – the post-Hegelian, post-Marxian sense of praxis finds its locus in the world acted within and upon, and in the unceasing flow of history made by human, social action. This is the sense of "praxis" discussed by Bernstein in his (1971) book *Praxis and Action*.

So: praxis comes into the world through the actions (sayings, doings, and relatings) of people, individually and collectively, changing the world through the immediate effects and long-term consequences of their actions, and these effects and consequences become conditions that in turn shape actors and the media in which they can act in the world – the medium of language in the cultural-discursive dimension of human sociality, the medium of work in the material-economic dimension, and the medium of power in the social-political dimension.

Educational praxis is action that is consciously directed not only by the intention or purpose (*telos*) of aiming towards the good for students and the good for humankind; educational praxis is action consciously directed towards *forming* good individuals and good societies. Education consists in this process of formation – educational praxis is *doing* this forming. Praxis is the action, not the intention.

Practice Architectures

Under some circumstances, cultural-discursive, material-economic, and social-political conditions produced by human beings acting in the world become sedimented rather than ephemeral, institutionalised rather than fluid or contested. They thus come to be among the conditions that frame, and enable and constrain, subsequent action-possibilities. They may be institutionalised (and become contested):

- in the cultural-discursive structures of language; in discursive practices; and in the form of relationships between speakers and hearers, authors and readers;
- in the material-economic structures of work and life in the natural world; in work practices; and in the form of material and economic relationships between producers and consumers, or owners and workers; and
- in the social-political structures that give form to power; in administrative and regulatory practices; and in the form of social and political relationships between people and groups like the hierarchical relationship between managers and the people they manage, the complementary relationship between professional practitioners and their clients, or the reciprocal relationships of solidarity and legitimacy that form among people in interest groups or public spheres.

When these structures, practices, and relationships become sedimented and institutionalised, they then function as *mediating preconditions* for subsequent practice and praxis: preconditions that pre-form what kinds of practice or praxis will be possible in particular kinds of circumstances – like the circumstances of a professional practice, for example, like the practice of mathematics education.

Table 2 shows these dialectical relationships of mutual formation and transformation between actors and action possibilities – action possibilities that can become sedimented into "practice architectures" that enable and constrain future action.

Table 2

The Dialectic (Mutual Constitution) of Action/Praxis and Practice Architectures

Action and praxis	Dimension/medium	Practice architectures (mediating preconditions)
"Sayings" (and thinking)	The cultural-discursive dimension (in the medium of language)	Cultural-discursive structures, practices and relationships
"Doings" (and 'set-ups')	The material-economic dimension (in the medium of work)	Material-economic structures, practices and relationships
"Relatings"	The social-political dimension (in the medium of power)	Social-political structures, practices and relationships

At the beginning of this paper, I argued that education has a double purpose: the formation of individuals and the formation of societies. I then argued that praxis has two meanings, one referring to the "right conduct" of individuals and another referring to the "history-making action" of individuals and people collectively. "Educational praxis", then, is that particular kind of human and collective action – educational action – by which good individuals and good societies are formed and transformed. Finally, I argued that the consequences of our actions construct and frame future possibilities for action, in the form of "practice architectures" that enable and constrain action, including not only the actions of educators, but also those they educate, as well as all those others whose work and practices function as "meta-practices" that shape the practices of others – including, in education, teacher educators, educational policy-makers and administrators, and educational researchers and evaluators. To express it as Karl Marx did (1852/1999),

> [People] make their own history, but they do not make it as they please; they do not make it under self-selected circumstances, but under circumstances existing already, given and transmitted from the past. The tradition of all dead generations weighs like a nightmare on the brains of the living. (p. 1)

Changing Practices

We should note that not only action possibilities, but also actors and their self-understandings – their identities and their subjectivities – are also framed and constructed by these practice architectures. By their engagement with, in, and through these practice architectures, people construct their self-understandings and their understandings of the world, their modes of activity and their skills and capabilities, and their roles and patterns of relating to others. They construct themselves in the terms made available to them by the practice architectures they inhabit. Educators and teachers are made the educators and teachers they come to be by complying with and also by resisting the particular practice architectures in which they live and work.

Some practice architectures favour the development of educators; some permit only the development of teachers. Particular forms of initial teacher education and continuing professional development, particular kinds of educational policies and educational administration, particular forms of educational research and evaluation, favour the development of educators; others permit only the development of teachers.

If we want schools and kinds of education different from the schools and schooling we have today, then we must change not only the educators and the teachers; we must change the practice architectures that construct their action possibilities, their self-understandings and their understandings of the world, who they are and who they can become.

It follows, then, that changing professional practices like mathematics education is not just a matter of changing practitioners' own particular understandings and self-understandings (cf. sayings), skills and capabilities (cf. doings) or values and norms (cf. relatings), but also changing the practice architectures that enable and constrain what practitioners can do.

How System Meta-Practices Endanger Traditions: A Crisis for Education?

The practice architectures of education and mathematics education are constructed not only by the knowledge, capabilities, and values internal to traditions in mathematics education but also by meta-practices external to those traditions – particularly the meta-practices of educational administration and policy making, initial and continuing teacher education, and educational research and evaluation. These meta-practices function as practice architectures that enable and constrain possibilities for action in education and mathematics education. They may enable and constrain action to such an extent that they may even make *education* – or mathematics education – almost impossible to enact.

When the Board of Studies school mathematics syllabus imposes too many demands on what teachers should cover, when the National Assessment Program – Literacy and Numeracy (NAPLAN) or the Organization for Economic Cooperation and Development (OECD) Program for International Student Achievement (PISA) testing programs place too great an emphasis on what will count as "numeracy", when a teacher education program places too great an emphasis on a particular approach to teaching and learning in mathematics, or when the research program of the International Council for Mathematics Education is dominated by particular approaches to research on mathematics education, dangers emerge. Some of the advice mathematics teachers receive from those bodies will assist them, help them to see the difficulties of students more clearly, help them to make school mathematics more engaging and success in school mathematics more attainable. But attending to these multiple sources of advice may not come without costs – they contain contested and contradictory messages about what good mathematics education – and good education – consists in. If teachers are obliged to follow all the available advice too slavishly, if they take their eyes off the students in front of them because they are obliged to listen too closely to the voices of the advisors and administrators behind them, they may find themselves working on what the state intends – schooling – rather than for the good of their students and the society. The syllabus, instead of being a source of guidance and inspiration for teachers and students, may become a litany of imposed tasks to which teachers and students cannot do justice. It is possible for a state or an education system to ask too much. In the ever-more-prescribed, ever-more-regulated circumstances of education and mathematics education in Australia, the mania for extending such advice threatens to overwhelm even the most capable and committed teacher. For many teachers, there is just too much to do to follow all the guidelines, meet all the objectives, deploy all the suggested teaching strategies and curriculum resources, and attain all the targets. Thus, they choose. They interpret. They do their best to meet the needs and interests of their students and communities. But they may have to do so by teaching against the grain of the advice they receive. And that can be risky.

As Australian Ministers of Education once again consider a national curriculum, and national testing, the spectre emerges of a curriculum for all that will be a curriculum for none. Will a national curriculum better serve the needs and interests of students in remote Indigenous communities in central Australia? Will it better serve the interests of students in those communities with exceptional needs on the margins of our major cities? How will objectives and strategies be specified so that teachers have the capacity and the opportunity to respond to differences in the needs and circumstances of students and communities? How will the state, curriculum bodies, teacher education accreditation agencies, teacher registration bodies, and agencies conducting national testing restrain themselves from prescribing too much? Many already prescribe too much. Why, in an ever-more-diverse Australian society, should we expect greater success from ever-more-standardised solutions to the problems of teaching and learning? (The standardisation that works for DVD players, and might work better for mobile telephone chargers might not work for human beings.) In *The Farewell Party*, Milan Kundera (1993) wrote of the "longing for order",

> a desire to turn the human world into an inorganic one, where everything would function perfectly and work on schedule. The longing for order is at the same time a longing for death, because life is an incessant disruption of order. Or to put it the other way around: the desire for order is a virtuous pretext, an excuse for virulent misanthropy. (p .85)

If it is allowed to run unchecked in educational policy and administration, teacher education, and educational research and evaluation, the longing for order will endanger education and mathematics education. There is not space here to explore all of these arenas, but an example – concerning educational theory – might show the kind of way in which the practice architectures of a certain kind of educational science and educational theory might endanger education.

In his (1947) book *The Eclipse of Reason*, Max Horkheimer, one of the founders of the Frankfurt School of critical social theory, feared that science had been overcome by scientism – science's belief in itself – and that the power of critical thought was in jeopardy. People had become so captivated by the possibility of progress through science that all realms of thought were being colonised by scientistic, functional and instrumental rationality.

In the 1960s, Hans-Georg Gadamer (1975) argued that technical rationality and the idea of "scientific method" was displacing practical rationality even in the realms where the latter was the only appropriate form of rationality, and even in fields like theology, art, and the humanities.

In three powerful books, *After Virtue* (1983), *Whose Justice? Which Rationality?* (1988) and *Three Rival Versions of Moral Inquiry* (1990), Alasdair MacIntyre argued that the tradition of the virtues – what we might here roughly describe as the tradition of moral action in civil societies – has already been threatened and had already been overwhelmed in our bureaucratic, technocratic societies of modern times. In MacIntyre's view, even the universities have been transformed into places where the tradition of the intellectual virtues is under threat – and, as inhabitants of the postmodern university, we might agree.

In the 1980s, Jürgen Habermas published his *Theory of Communicative Action* (1987, 1989). It included his theory of lifeworld and system, including the thesis that the person-to-person connections of the lifeworld, in which people meet one another as human subjects in an intersubjective space, was increasingly being colonised by the functional rationality of economic and administrative systems, debilitating lifeworld relations between people. Our capacities for self-understanding, for example, are increasingly given in terms imposed by the institutions we inhabit – the university, for example. Many Australian universities provide institutional definitions of what constitutes an "active researcher", thus imposing particular kinds of understandings and self-understandings on their staff. The Australian Government defines what counts as "quality" in research publications, and thus imposes particular kinds of understandings and self-understandings on Australian researchers. Increasingly, we are obliged to understand ourselves – even if we resist the categorisations – in institutional or systemic terms. In the process, we may be put at risk of becoming less human than we are.

Authors like these are not alone in identifying the threats posed by "the technologisation of reason" (Kemmis, 1980).

On the view of reason as "method" (Carr, 2006b), it is thought that all problems can be solved, given sufficient time and resources: all systems can be better run; all employees can work more efficiently and effectively; all states can better manage their internal and external affairs with better policies that meet or exceed measurable targets and outcomes. On this view of reason as "method", well-elaborated systems of schooling and training, systems of teacher education and deployment, systems of educational policy and administration, and systems of technical research and evaluation – all these functional systems can achieve all that can reasonably be achieved in terms of the development of people as citizens and clients, and as producers and consumers.

The view of reason as "method" expresses itself in technical, instrumental or functionalist rationality according to which goals must first be established and defined, and the means to reach those goals identified and implemented. Expanded economic and administrative systems must be organised and administered in the service of the defined goals and methods of attaining them. Increasingly, the setting of goals (or "targets"), the identification and organisation of means, and the evaluation of goal-achievement are conducted in a rationality that emerged from Max Weber's (1948/1920) ideas about bureaucracy as the most rational of organisations. Powered by reflexive modernity (Giddens, 1991), public administration and modern management theory have further developed Weberian ideas of bureaucracy into ever-more flexible forms of management and control. Giddens describes reflexive modernity in this way:

> Modernity's reflexivity refers to the susceptibility of most aspects of social activity, and material relations with nature, to chronic revision in the light of new information or knowledge. …
>
> In respect of both social and natural scientific knowledge, the reflexivity of modernity turns out to confound the expectations of Enlightenment thought – although it is the very product of that thought. The original progenitors of modern science and philosophy believed themselves to be preparing the way for securely founded knowledge of the social and natural worlds: the claims of reason were to overcome the dogmas of tradition, offering a sense of certitude in place of the arbitrary character of habit and custom. But the reflexivity of modernity actually undermines the certainty of knowledge, even in the core domains of natural

> science. … The integral relation between modernity and radical doubt is an issue which, once exposed to view, is not only disturbing to philosophers but is *existentially troubling* for ordinary individuals. (pp. 20-21)

Instrumental, technical, functional rationality has its place – just as technique, expertise, and control have a place. But there are dangers when notions of technique, expertise and control furnish the principal models of excellence in reason (or science), or when they dominate the landscape of social life, pushing other forms of reason and rationality to the margins (or excluding them altogether). Most especially they become problematic when they are applied to the kinds of matters that are beyond their competence and control – matters concerning the moral and political actions and lives of people, and the often unexpected and unanticipated consequences that cascade from them in the course of human and natural history. Science and administration make this error when they treat people as means to ends and not, as Immanuel Kant famously put it (1998/1785), as "ends in themselves" (p. 38).

Wilfred Carr has recently written several important papers (2004, 2006a, 2006b, 2007) challenging the idea and consequences of the twentieth century view of "method", and, following Gadamer, has called for the abandonment of the hundred-year-long intellectual project of *educational* science and theory unleashed by that idea of method. In his (2006a) "Education without Theory" paper, he concludes that

> … all that now remains to be done is to accept, without regret or nostalgia, that the educational theory project has run its course and that the time has now come for us to bring it to a dignified end. (p. 150)

Instead, Carr (2006a, 2006b, 2007) calls for development of forms of educational philosophy and educational research as "a practical science" or as "practical philosophy".

In the traditions of educational studies and educational philosophy and theory, and the tradition of *Pedagogik* in Europe, education (and *Pedagogik*) was understood to have a double purpose of the kind I described at the beginning of this paper. The questions of what it meant to be a "good person" or an "educated person", and the question of what was in the interests of any one person or in the interests of humankind, were regarded as permanently open questions, to be re-considered and re-thought for every changing society and for different historical moments. The questions could not and cannot be closed once and for all by the answers given in any particular time or place. The task of the "Education tradition", if I might call it that, was – and is – to continually review and revise past answers to those questions in the light of changed historical times and changed social circumstances. It was – and is – necessarily an *interpretive* task of attempting to understand the present with the resources given by the past, and in the light of changed present circumstances to rise above the tradition and to revise its resources, and, by doing so, to reinvigorate and renew the tradition by critical development of the concepts and practices internal to the tradition. The Education tradition did not and does not consist in an accumulating set of answers to an unfolding series of questions put by history, but in an enduring argument about those questions. The tradition is *the argument*. As Alasdair MacIntyre (1988) put it,

> A tradition is an argument extended through time in which certain fundamental agreements are defined and redefined in terms of two kinds of conflict: those with critics and enemies external to the tradition who reject all or at least key parts of those fundamental agreements, and those internal, interpretative debates through which the meaning and rationale of the fundamental agreements come to be expressed and by whose progress a tradition is constituted. Such internal debates may on occasion destroy what had been the basis of common fundamental agreement, so that either a tradition divides into two or more warring components, whose adherents are transformed into external critics of each other's positions, or else the tradition loses all coherence and fails to survive. It can also happen that two traditions, hitherto independent and even antagonistic, can come to recognize certain possibilities of fundamental agreement and reconstitute themselves as a single, more complex debate. (p. 12)

Moreover, the interpretive task is just the first stage of what is needed in, or asked of, the "Education tradition"; beyond the interpretive task, the Education tradition also had a *practical* task: not merely informing educators but also providing a framework for making *practical judgements* about what to do in any particular situation, whether considering the whole curriculum for a school, or what to do at this or that moment in the hurly-burly of teaching a class, or in thinking about how a school should relate to its community. What Carr calls "practical science" or "practical philosophy" is not only the work of general intellectual, moral, and social *preparation* for educational judgement and educational practice, but also the work of *deliberation* about *what* *to do* in the practical circumstances in which we find ourselves, on various scales of time (whether thinking about the next minute, the next class, the next year, or the next generation), place (on various levels from local to global), and social space (whether in relation to this student, this class, this community, or this world).

Perhaps, by now, some of you will have heard the lines too often, but it seems to me that Aristotle was right, in the *Ethics*, to quote the ancient Athenian poet Agathon (448-402 BCE), friend of Plato, who wrote:

> For one thing is denied even to God:
>
> To make what has been done undone again.

This ineluctable materiality of actions and their consequences, something that always escapes our categories, our interpretations and our grasp, is what a practical science of education and practical philosophy aim to address. They aim to help us to swim in the sea of history, not just to get this or that job done, good though it may be to get those jobs done and to do them well. They acknowledge at the outset that we always see only partially, even with the aid of our most powerful theories, or the wisdom borne of long and thoughtful experience. The tide of history rolls on despite us, despite our best intentions, our expertise, and our intelligence; practical wisdom and practical reason can only help us to swim in that tide; they cannot reduce the uncertainty to certainty, they cannot make the unknowable known. The focus of practical reason is on unfolding history – on what happens in all of its "happening-ness". Its concern with "what is to be done?" does not stop with the question or even the answer, but with what happens and what happens after that. Its concern is with what *is* done, not the contemplation about it. As Marx (2002/1845) said, "philosophers have only interpreted the world, in various ways; the point is to change it."

In Carr's view, the Educational Theory tradition (as distinct from the Education tradition which it reinterpreted into a particular kind of philosophising) became introverted and self-absorbed, practised within practice architectures that directed the attention of the philosophers of education operating within that tradition principally to one another's voices and ideas. Absorbed by arguments within the Education Theory tradition, they attended decreasingly to the concerns of the practitioners of education in the field and the profession of education. In this way, one might say, they became dislodged from the field of practice and absorbed with another field – the field they made for each other by their specific practices of philosophical communication, production, and organisation. Not surprisingly, perhaps, their preoccupations and their work became largely irrelevant to the day-to-day work of teachers – although the theorists in the tradition continued their efforts in the sincere belief that their work remained relevant to problems of contemporary education and educational practice. The field of philosophy of education they constituted by their practice was a field of (a certain kind of) philosophical practice (a practice of philosophical discourse; Hadot, 1995, p. 269), not educational practice. It was decreasingly a resource for practice, but rather a commentary on what it construed that practice to be.

Instead, Carr (2006b, 2007) calls for forms of "practical science" and practical philosophy that involve educators in investigating their own practices, for example, through forms of educational action research aimed at critically reconstructing their practices, by understanding more deeply the character and consequences of their actions. Such a practical science would seek to provide local and current answers to the enduring educational questions about the extent to which teachers' practices foster the good for each student and the good for humankind by avoiding harmful consequences for students and their communities.

This case of a certain kind of educational theorising – the Educational Theory – tradition shows how even the most determined efforts to understand the nature and conduct of educational practice may lead away from practice rather than into it. According to the French historian of Hellenistic and Roman philosophy Pierre Hadot (1995),

> … modern philosophy is first and foremost a discourse developed in the classroom, and then consigned to books. It is a text which requires exegesis. (p. 271)

He contrasts this form of modern philosophy with ancient philosophy which was what he calls "a way of life" – in which philosophical discourse only oriented people towards the living of good lives, not just having noble thoughts about how lives should be lived.

There is not space here to do more than suggest that this separation between theory and practice, based in a division of labour between theorists and practitioners, has become endemic to other realms also, in which relationships with practitioners emerge as a relationship between managers and those they manage, those who know and those who (are told what to) do.

I put it to you that there is today a relationship between managers and those to be managed in the realms of educational policy and administration, the control of initial and continuing teacher education, and the control of educational research and evaluation. In each field, professional practitioners are being brought more closely under the control of educational and administrative systems and authorities.

Under such circumstances, educators are thus confronted by an invidious choice: to conduct their practice as *praxis*, oriented by tradition and by considerations of the good for each person and the good for humankind as these are expressed in education, as *agents* of education, or to conduct themselves as *operatives* of the education systems in which they find themselves, following the rules and procedures that constitute the functional rationality of those systems. Although not without precedent, this situation has emerged with new force in the performativity required of teachers in the last two decades. A new kind of tragedy has become possible right at the heart of educational practice, in the encounter between teachers and learners – a choice between behaving professionally or compliantly. It is a tragedy with which we shall all become more familiar as the juggernaut of the national curriculum rolls through Australia in the coming years.

Teachers will increasingly be called to make a choice between, on the one hand, acting as professionally and acting educationally, for the good of each student and the good of humankind, and, on the other, doing what they are told.

Conclusion: Recovering Praxis Traditions

In the chapter "Conclusions and Challenges" in *Enabling Praxis*, Tracey Smith and I concluded that there are five challenges for the contemporary education profession and for educators:

> (1) the re-moralisation of practice, (2) the re-invigoration of professionals, (3) the revitalisation of professional associations, (4) the renewal of educational institutions, and (5) a recovery and revitalisation of educational traditions. (p. 274)

The fifth may seem the most innocuous and indirect, but I think it may also provide a reconnection with education and its purposes. What is most needed, it seems to me, is a relocation of every educator's, every teacher's, understandings of education as a practice in terms of the enduring question posed by the Education Tradition (as distinct from the Educational Theory tradition I also discussed). What is most needed is to have every educator enter and participate in the enduring debate about the extent to which our practice as educators *in fact* forms and develops good persons and the good for humankind.

We can only judge our success in these things in terms of history – a longer timescale than is conventional for the weekly test or annual waves of state-wide testing or Year 12 examinations – or the post-test in our empirical research study. As educators, did we produce good people? Have we produced a good society?

It is obvious, of course, that as educators, we do not have sole responsibility for the formation and development of good people and a good society. (Nor, as mathematics educators, are you solely responsible for producing good people and a good society.) We are not alone in forming students, society, the state, the economy, or the state of the planet. In addition to parents and caregivers, others – in the conduct of business, the productions of the media, the workings of the state, warfare, and global terror – have also created practice architectures that shape how human practice has produced the world in which we live today.

We, however, have our particular task, and I believe that it is one that will require greater courage to pursue than in earlier years. Our task is to do what people in the Education Tradition have always tried to do: to produce good people and a good society. It is our job to attend to this first, and to be able to debate with one another, and with other people in the communities and societies we serve, whether we are indeed doing this job. Are these good people? Is this a good community and a good society?

I believe education is under threat in the West today, and that our times will require educators to speak in defence of education. Education is being replaced by a species of administration – the administration of personhood and the administration of the state, in which persons appear only in the faded roles of consumers and clients, emptied of their personhood and what makes them ends-in-themselves. Emptied of what is most precious – their being and becoming as persons and as fellow-citizens in what should be a good society.

Here, in this conference, it falls to you to debate whether and how the *practice* – not just the theory – of mathematics education contributes to the formation of good persons and a good society. Perhaps a Mathematics *Learning* Research Group of Australasia would have a different task.

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Facilitating Communities of Mathematical Inquiry

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In the current shifts in mathematics classrooms teachers are challenged to use effective pedagogy to develop inquiry communities in which all participants are offered opportunities to engage in the reasoning discourse of proficient mathematical practices. The challenge for teachers is to know what pedagogical actions support the development and use of effective mathematical practices. This paper examines how a group of teachers used a purposely designed communication and participation framework as a tool to scaffold development of inquiry communities and the use of progressively more proficient mathematical practices within their classrooms..

The past decade has seen ambitious calls by national and international policy makers and researchers for significant changes to be made to the teaching and learning of mathematics (e.g., Franke, Kazemi, & Battey, 2007; Ministry of Education, 2007; RAND, 2003). Increasingly, reform efforts have centred on the potential benefits of giving specific attention to the teaching and learning of mathematical practices; practices which encompass the mathematical know-how which constitutes expertise in learning and using mathematics (Boaler, 2003b; RAND). The RAND group maintain that for students to develop robust mathematical thinking and reasoning processes students need opportunities not only to construct a broad base of conceptual knowledge; they also require ways to build their understanding of mathematical practices. A common theme of literature related to development of mathematical practices (Ball & Bass, 2003; Boaler, 2003b; Franke et al., 2007) is the need for students to participate in reasoned collective discourse so that they learn to construct and communicate powerful, connected, and well reasoned mathematical understanding. In keeping with this notion, Ball and Bass explain that reasoning in this form "comprises a set of practices and norms that are collective … and rooted in the discipline" (p. 29).

The advocated changes necessitate radically different roles and responsibilities for the teachers and students, including how they are to relate to each other, to the classroom power and authority base, and to the discipline of mathematics itself (Boaler, 2003a; Sowder, 2007). A central hallmark of the changes is a vision of students and teachers actively engaged in the shared dialogue of inquiry and argumentation, using the mathematical practices of able problem solvers, within classrooms which resemble learning communities (Goos, 2004). For example, in the new national curriculum document for New Zealand, teachers are charged with the responsibility of putting into effect pedagogy which facilitates inquiry climates where "everyone, including the teacher, is a learner; learning conversations and learning partnerships are encouraged; and challenge, support, and feedback are always available" (Ministry of Education, 2007, p. 34). Here the mathematical practices include "justifying claims, using symbolic notation efficiently, defining terms precisely, and making generalisations [or] the way in which mathematics users are able to model a situation to make it easier to understand and to solve problems related to it" (RAND, 2003, p. xviii).

Of central importance in shifting the discourse towards inquiry are the enacted sociocultural and mathematical norms (Sullivan, Zevenbergen, & Mousley, 2002); the negotiated variables constituted within discursive interaction. Sociocultural norms relate to the stable patterns of behaviour or practices, organisational routines, and forms of communication valued in classroom communities. Mathematical norms support higher level cognitive activity and relate directly to mathematics. They evolve within mathematical activity and are the "principles, generalisations, processes and products that form the basis of the mathematics curriculum and serve as the tools for the teaching and learning of mathematics itself" (Sullivan et al., p. 650). Theorising that mathematical norms and mathematical practices are interrelated offers a way to explain how mathematical practices are transformed as they are negotiated in the discursive dialogue and emphasises the importance of teachers attending to the discourse used in mathematical activity. A number of studies (e.g., Franke et al., 2007; Hufferd-Ackles, Fuson, & Sherin, 2004; Wood & McNeal, 2003) have illustrated deeper student engagement in mathematical practices when teachers explicitly foster communicative patterns which shift the collective discourse from inquiry to argumentation, challenge, and debate.

Successful implementation of such discourse is a challenging task. The literature provides ample evidence (e.g., Franke et al., 2007; Hufferd-Ackles et al., 2004) of the considerable complexities involved in establishing collective discourse which provides students with space to engage in disciplined ways of reasoning and inquiry. For many teachers their fundamental beliefs about teaching and learning are challenged as they rethink their roles and responsibilities and those of their students within the classroom discourse patterns. At the same time, the changed communication and participation patterns also create challenges for students. Not only is their notion of the teacher's role as unquestionable authority in dispute; changes reflecting the wider diversity of their roles, task demands, and novel interactional scripts also add to the demand (Forman, 1996).

Whilst readily acknowledged that the pedagogical actions used to guide and negotiate the mathematical and sociocultural norms are pivotal to facilitating communities of mathematical inquiry, it is less clear how teachers might effect such a change. This is particularly an issue for those teachers currently in the classroom who too often lack experience of learning in inquiry environments or using effective mathematical practices (Hufferd-Ackles et al., 2004; RAND, 2003). Seldom do curriculum documents clarify a teacher's role in such learning environments, nor provide guidance on how to constitute the sociocultural and mathematical norms (Sullivan et al., 2002). In addressing this concern, this paper reports on how a group of teachers used a purposefully designed communication and participation framework to map out the establishment of an inquiry community. The central focus of the paper is on how the teachers adapted and used the framework as a tool to constitute the sociocultural and mathematical norms of mathematical inquiry communities. Exemplars of how the teachers scaffolded student engagement in reasoned discourse that supported the use of more proficient mathematical practices capture the ever present "dynamic process of interpretation and mutual adjustment that shapes students learning" (Ball & Forzani, 2007, p. 531) within inquiry communities.

The theoretical standpoint of this study is derived from a sociocultural perspective on learning. Within this perspective, mathematics learning is viewed as contextualised, which is to view learning-in-activity. The social, cultural, and institutionalised contexts are not considered merely as factors which may aid or impede learning; rather, these social organisational processes are integral features of the learning itself and are mutually constitutive (Forman, 1996). As Lerman (2001) explains, when the social practices of classroom communities are discursively constituted "people become part of practices as practices become part of them" (p. 88). Thus, within the sociocultural lens of this study, the learning and use of mathematical practices is matched by an "increasing participation in communities of practice" (Lave & Wenger, 1991, p. 41); a dynamic process of change which involves shifts in positioning of all members of the community.

Research Design

The reported research is from a larger classroom-based design study (Hunter, 2007). Conducted at a New Zealand urban primary school, the study involved four teachers and 120 Year 4-8 students. The majority of students came from low socio-economic home environments and were of Pasifika or New Zealand Maori ethnic groupings.

A year-long partnership between researcher and teachers supported the design and use of a participation and communication framework (see Table 1). Adapted from the theoretical framework proposed by Wood and McNeal (2003), the framework drew on a wide range of research findings related to those communication and participation patterns that had been found to be effective in supporting student engagement in a variety of mathematical practices. These were positioned in the framework as conjectures of possible actions teachers could scaffold students to use, to provide them with opportunities to learn and use mathematical practices within the collective inquiry discourse.

As an organising tool to assist teachers to scaffold students' use of proficient mathematical practices within reasoned inquiry and argumentation, the framework was structured around two components: communication patterns and participation patterns. Vertically, the framework outlined a set of collective reasoning practices matched with conjectures relating to the communicative and performative actions teachers might require of their students to scaffold their participation in learning and using mathematical practices. Likewise, conjectures of a set of participatory actions teachers may expect of their students to promote their individual and collaborative responsibility in the collective activity were included. The horizontal flow over three phases sketched out a possible sequence of communicative (and performative) and participatory actions teachers could scaffold their students to use as they went about establishing communities of mathematical inquiry.

Table 1

The Communication and Participation Framework

	Phase One	**Phase Two**	**Phase Three**
Making conceptual explanations	Use problem context to make explanation experientially real.	Provide alternative ways to explain solution strategies.	Revise, extend, or elaborate on sections of explanations.
Making explanatory justification	Indicate agreement or disagreement with an explanation.	Provide mathematical reasons for agreeing or disagreeing with solution strategy. Justify using other explanations.	Validate reasoning using own means. Resolve disagreement by discussing viability of various solution strategies.
Making generalisations	Look for patterns and connections. Compare and contrast own reasoning with that used by others.	Make comparisons and explain the differences and similarities between solution strategies. Explain number properties, relationships.	Analyse and make comparisons between explanations that are different, efficient, sophisticated. Provide further examples for number patterns, number relations and number properties.
Using represent-ations	Discuss and use a range of representations to support explanations.	Describe inscriptions used, to explain and justify conceptually as actions on quantities, not manipulation of symbols.	Interpret inscriptions used by others and contrast with own. Translate across representations to clarify and justify reasoning.
Using mathematical language and definitions	Use mathematical words to describe actions.	Use correct mathematical terms. Ask questions to clarify terms and actions.	Use mathematical words to describe actions. Reword or re-explain mathematical terms and solution strategies. Use other examples to illustrate.

Participatory actions	Active listening and questioning for more information. Collaborative support and responsibility for reasoning of all group members. Discuss, interpret and reinterpret problems. Agree on the construction of one solution strategy that all members can explain. Indicate need to question during large group sharing. Use questions which clarify specific sections of explanations or gain more information about an explanation.	Prepare a group explanation and justification collaboratively. Prepare ways to re-explain or justify the selected group explanation. Provide support for group members when explaining and justifying to the large group or when responding to questions and challenges. Use wait-time as a think-time before answering or asking questions. Indicate need to question and challenge. Use questions which challenge an explanation mathematically and which draw justification. Ask clarifying questions if representation and inscriptions or mathematical terms are not clear	Indicate need to question during and after explanations. Ask a range of questions including those which draw justification and generalised models of problem situations, number patterns and properties. Work together collaboratively in small groups examining and exploring all group members reasoning. Compare and contrast and select most proficient (that all members can understand, explain and justify).

Data collection over one year included three teacher interviews, twice weekly video captured lessons, field notes, classroom artefacts, written and recorded teacher reflective statements, and teacher recorded reflective analysis of video excerpts. On-going data collection and analysis maintained focus and flexible revision of the emerging communication and participation patterns which supported development of proficient mathematical practices. Data analysis occurred chronologically using a grounded approach to create codes, and to identify coherent patterns and themes. Trustworthiness was maintained through use of constant comparative methods which involved interplay between the data and theory and sustained engagement with participants in the field by the researcher.

Results and Discussion

The communication and participation framework was used by the teachers as a flexible and adaptive tool to map out development of an inquiry environment in which the students were offered opportunities to participate in learning and using progressively more proficient mathematical practices. In using the framework the teachers all reached similar endpoints at the conclusion of the study, but the individual pathway they each mapped out and traversed was unique. How the teachers positioned themselves in the mathematical discourse was an important initial consideration; those who positioned themselves in a more traditionally oriented role encountered more challenges in effecting change and their journey to construct a community of inquiry was lengthened.

Constituting Intellectual Partnerships which Supported Mathematical Arguments

To initiate change, all of the teachers initially addressed the sociocultural norms in their classrooms. They repositioned themselves as participants in the discourse and they emphasised student responsibility for active listening and sense-making. Their immediate focus aimed to establish safe, supportive learning environments that promoted social and intellectual risk-taking. They employed a range of strategies to attend to students' affective needs, including direct discussion of the need for collegiality, inclusion, and intellectual and social risk-taking. Each of the teachers also implemented specific strategies to re-mediate situations in their existing classroom culture, including closely engineering learning partnerships (e.g., placing Maori, Pasifika and female students in supportive pairs initially), and the use of specific talk-formats that valued assertive communication, construction of multiple perspectives, and affirmation of effort over ability. For example, in the following teacher comment the social and cultural background of the students were drawn on and linked to the expectations and obligations of the developing community—referred to as a whanau—a family and collective concept in which the more knowledgeable are positioned as valued knowledge sources within the collective:

> Teacher: Remember you are a member of our whanau so you need to be loud and proud and confident … we are all ready to think and listen.

A focus on communal construction and examination of mathematical explanations occurred in partnership with development of the sociocultural norms. Guided by the framework, the teachers addressed how students in small heterogeneous groups were to discuss, negotiate, and construct a collective solution strategy. However, each teacher adapted the communication and participation framework to guide the specific needs of their classroom context. For example, Table 2 illustrates an extended adaptation one teacher made to the trajectory she used within her class to help her students develop ways of managing their initial discussions.

Table 2

An Adapted Section: Phase 1 Making Mathematical Explanations

Think of a strategy solution and then explain it to the group. Listen carefully and make sense of each explanation step by step. Make a step by step explanation together. Make sure that everyone understands. Keep checking that they do. Take turns explaining the solution strategy using a representation. Use equipment, the story in the problem, a drawing or diagram or/and numbers to provide another way or backing for the explanation.	Keep asking questions until every section of the explanation is understood. Be ready to state a lack of understanding and ask for the explanation to be explained in another way. Ask questions (what did you…) of sections of the explanation. Discuss the explanation and explore the bits which are more difficult to understand. Discuss the questions the listeners might ask about the explanation

In the sharing sessions which followed small group activity, student presentation and sense-making of conceptual explanations were closely structured. The teachers provided models of questions to elicit further clarification of the reasoning, and they prompted explainers to make the explanations experientially real for the listeners. They also directly interceded and structured the discourse to allow space and time for sense-making. As a result, within each classroom, within differing timeframes, the students realised their responsibility for reasoned sense-making. At the same time, the communal construction and examination of explanations as mathematical arguments provided an important foundation for the teachers to press towards explanatory justification and generalisation.

Maintaining Intellectual Partnerships in Collective Justification and Generalisation

The participation structure the teachers made available to the students operated as a scaffold for the development of argumentation. However, the shift to consider mathematical explanations as a form of argument caused conflict for the teachers. Not only did they acknowledge their own novice status in an inquiry environment, they also expressed concern at what they considered to be a lack of fit between the students' cultural and social norms and the requirement that they engage in the discourse of inquiry and argumentation.

Using the framework as a reflective tool the teachers critically analysed video excerpts for student engagement in interrelated mathematical practices. They examined student use of the questions and prompts which supported emergence of mathematical practices. They re-mapped their pathways and planned their next foci. Student attention was focused beyond the development of mathematical knowledge to rich ways to use and extend the reasoning mathematically. With their attention directed to discussing and modelling mathematical argumentation, the teachers required that the students construct multiple explanations. The teachers used rich tasks and problems they had collaboratively designed in accord with their next goals. To strengthen their ability to encounter challenge students were required to examine their arguments closely and rehearse possible responses to questions or challenge. The teachers scaffolded and probed the students to use the questions, and prompts which drew justification and generalisations. They promoted the use of "thinking time" as a pause in the dialogue to provide the students with opportunities to analyse explanations, frame questions, and reconsider and restructure arguments. They also explicitly positioned students to voice agreement or disagreement backed by mathematical reasoning.

The use of these practices provided the students with a predictable framework for strategy/solution reporting, inquiry and argument and resulted in extended reasoned dialogue. A consistent pattern occurred in each classroom; as the discourse of inquiry and argumentation increased the teachers began to explicitly focus on, attend to, and build on, the students' observations of patterns and relationships. This is illustrated in the following episode in which a teacher asks the students to analyse a strategy in which a student had justified his group's collective explanation for a decimal problem using fractions:

Sally: There's three different ways to basically explain a fraction, the fraction way, a decimal point way, and a decimal way. That's why he has picked one of them. Instead of just doing the fraction or percentage, he's picked the decimal point way because he may think that that is actually his easier point of doing the fraction way.

Teacher: But can you do that?

Sonny: Yes because they are equivalent like just the same.

The increased student agency in the discourse led to repositioning of all participants in the classrooms. Within the negotiated and extended dialogue the teachers assumed facilitative roles, stepping in and out of the dialogue as they guided development of a shared perspective from which all community members drew on a range of different mathematical practices as integrated tools for *using* and *doing* mathematics. The following excerpt illustrates how a teacher supported her students to autonomously use their mathematical knowledge and practices to analyse a solution strategy for a problem which involved multiplying forty by twenty-four:

Kuini: [Examining the representation] Hang on. So if you are saying you got the two from the twenty then do you mean that ten times twenty-four equals two hundred and twenty-four times another two equals four hundred and eighty is the same as twenty times twenty-four? But why start with two? You need to convince us.

Kuini has closely examined the representation and uses her interpretation to question further. The teacher without speaking turns to Akeriri and nods to affirm his need to provide explanatory justification.

Akeriri: Because two is easier than four timesing. It's sort of like what Saawan showed us yesterday. Yeah and then I go times two again and it's nine hundred and sixty because that is the same as four times ten times twenty-four or forty times twenty-four. Are you all convinced?

In the continued exploratory discussion within the discourse of mathematical inquiry and argumentation the students connect previous reasoning and explore new directions.

Pania: Hey. Would that strategy work with other numbers? Hey what about this? You could do eighty times over twenty-four hours.

Guided by the teacher's facilitative stance the students continue to explore and analyse other numbers and patterns, using agreement and disagreement validated by mathematical evidence.

Kuini: I agree but those are all even numbers. So does it work with only even numbers because you can't half an odd number?

Akeriri: Saawan did it yesterday when he did nineteen but that wasn't the same strategy, that's changing it.

Pania: I disagree. It's just changing the numbers. I think you could take one lot off and then multiply it and then add the one lot back and it's the same. Or use Akeriri's way put the other one back. It works so you can do odd.

Through the extended examination of the argument the reasoning is validated through use of explanatory justification and generalised reasoning.

Conclusions and Implications

Although the teachers and their students all began as novices in the discourse of inquiry and argumentation the participation and communication framework provided a flexible tool which over the duration of the study supported the renegotiation of contexts and the development of detailed pathways for individual classroom communities. The attention placed on the sociocultural and mathematical norms was of significance in developing communal dialogue and individual and collective responsibility to sense-make. Success with scaffolding the students' participation in mathematical reasoning at higher intellectual levels in turn affirmed the teachers' continued press for inquiry and argumentation. In accord with current literature (e.g., Franke et al., 2007; RAND, 2003; Wood & McNeal, 2003) the teachers' increased expectations provided the students with a platform to learn and use explanatory justification, generalised reasoning, the construction of a range of inscriptions to validate the reasoning, and a more defined use of mathematical language.

The participation and communication framework was an effective tool which focused teachers' attention on specific communicative and performative actions they might require students to use to scaffold their engagement in the interrelated mathematical practices. Importantly in this study, the teachers, working within a supportive community of teacher learners including the researcher, were able to adopt and adapt the framework to meet their precise needs. Further research with different groupings of teachers and students to explore the adaptations that particular teachers make to the framework is needed. A greater understanding, both of the framework tool and the associated professional development, is needed to enable such a tool to be more widely used to support teacher learning and change.

Practical Implications

The challenge of creating and sustaining change in teachers' pedagogical practices is a well-documented and on-going issue. Extant beliefs and attitudes teachers hold about their role in the mathematical discourse and activity of mathematics classrooms which shape how they position themselves can prove a significant barrier to change, as can teachers' prior experiences. Importantly many teachers have not experienced learning (or teaching) in classrooms which promote mathematical dialogue, inquiry and argumentation—nor in those which explicitly focus student learning beyond acquiring mathematical knowledge towards learning and using proficient mathematical practices to *do* and *use* the mathematical knowledge. Findings reported in this paper show how teachers can successfully be scaffolded to reflectively adopt and adapt new pedagogical practices that represent a shift from the traditional foci on rote learning of computational rules and procedures to one in which all members of the community are active participants in collective analysis and validation of mathematical reasoning.

In recent times most teachers in New Zealand primary schools have had opportunities to participate in national numeracy professional development programmes. However, how to sustain the initial teacher change and support generative teacher change remains challenging. Importantly, this study found the communication and participation framework (CPF) gave those teachers who had prior involvement in the Numeracy Project an effective tool to support substantive professional learning. The framework not only scaffolded teachers to critically examine their pedagogy and classroom practices in terms of inquiry but prompted ongoing generative change towards creating a more effective community of mathematical inquiry as evidenced by students' use of increasing sophisticated mathematical practices,

To enact substantive professional learning requires that the preconceptions, prior experiences, and practical routines—the tacitly held personal theories of action teachers hold—are interrupted. For change to current practices to occur teachers require space to consider, reflect on, and as appropriate experience dissonance in these routines. This paper suggests that within a study group setting with spaces for individual and collective reflection—the CPF potentially provides a useful and practical tool teachers can use to analyse and understand the tacit theories of action which underpin their current practices.

Critical to using the CPF was the teachers' involvement in study groups comprised of teachers/teachers and/or a researcher, and access to, and discussion of, research literature which provide models of mathematical practices in inquiry environments. By making direct links to the communicative and performative actions outlined within the first phase the teachers were able to critique and evaluate the adequacy of their currently enacted classroom sociocultural and mathematical norms and in turn map out possible change scenarios and pathways. The descriptive detail of communicative and participatory actions teachers may require students to use provided an overt way to interrogate current practice while also acting as a mentoring tool for independent or collegial planning. This was especially effective when applied to the video replay of the teachers' lessons. Such analysis, when guided by the CPF, provided key points for discussion and pressed teachers to develop their own sets of questions and prompts and rich mathematical tasks for engaging students in mathematical inquiry practices. Through collegial discussion, examination, exploration and trialling of the CPF the teachers were provided with opportunities to learn in the act of developing new ways to orchestrate classroom interaction patterns.

A key feature of the CPF design was the horizontal organisation of phases. This signalled to teachers that shifting between and across the phases and the mathematical practices to match the shifting needs of the classroom context was something that took time. As they themselves through their professional growth gradually shifted from novice to expert facilitators of the mathematical discourse and practices, so too would their students need time to acquire competence and confidence with a range of mathematical practices associated with mathematical inquiry. In practice, the immediate and continued focus on the development of sociocultural norms caused gradual shifts in the roles and responsibilities of all members of community. Within these important interactional shifts both the teachers and students had time and space to practise and explore using the developing discourse. Moreover, the space within the phases provided the teachers with opportunities to examine and explore appropriate ways they could draw on their students' home contexts to develop mathematical argumentation in socially and culturally responsive ways.

For change to be sustained beyond a professional development programme teachers need to both adopt and adapt new knowledge into their own conceptual framework about teaching and learning. Integral to the use of the CPF in this study was the opportunities for teachers to apply the information and skills within their own situated practice. Learning in the act of teaching is influenced by the space and time provided to them and the depth of their professional growth is related to their interaction with the new learning. For example, at the beginning of the study the participating teachers, influenced by their recent involvement in the Numeracy Project, included student reporting of solution strategies in their current practices. However, closer analysis highlighted that such reports took the form of "show and tell". The combined approaches described in this paper challenged their personal theories in action, made these problematic, and pressed them to rethink the role of reasoned discourse in mathematics classrooms. It was the practical set of pedagogical actions outlined on the CPF which scaffolded the teachers, and in turn their students, to deep engagement with both the discourse of inquiry and argumentation and use of a range of rich interrelated mathematical practices.

In summary, this paper highlights the importance of teachers developing a coherent conceptual framework of pedagogical strategies that can support the development of proficient mathematical practices within reasoned mathematical dialogue. The CPF provided a practical but flexible and adaptive tool which supported the establishment of the discourse of mathematical inquiry communities. The provision of time and space through flexible scheduling of three phases of change and movement towards inquiry communities and practices acknowledged the complexity of teaching and teacher change, and provided manageable steps for teachers to individually and collectively enact when changing the discourse practices in classrooms.

Too often professional development initiatives and action research type programmes are evaluated by teacher change–with the assumption that teacher change in the advocated direction, with adoption of advocated practices–is the measure of success or an end point. This study however, gave teachers the opportunity to assess the effectiveness of pedagogical change by providing a framework that directly addressed changes in student mathematical behaviour and thinking. The changes the teachers made were generative; they had learnt the skills of reflecting-in-action as they responsively attended to growing both student mathematical knowledge and its use in increasingly proficient ways.

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Assessing Primary Preservice Teachers' Mathematical Competence

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The diagnostic test results from a second cohort of preservice teachers in a longitudinal project are presented. Data analysis using the Rasch Model showed consecutive intakes' performances improved. However, diploma-education students consistently performed lower than foundation-education students. Item analyses indicate problem solving, fractions, and interpreting complex diagrams are most difficult. Findings suggest the university needs to address the mathematical needs of at-risk preservice teachers and to develop policies to regulate entry requirements and enhance course content and delivery.

Professional Teaching Standards mandate teachers demonstrate excellence in their knowledge of the subject content and how to teach that content to students (NSWIT, 2007) otherwise they struggle to effectively "design mathematically accurate explanations that are comprehensible and useful for students … and interpret and make mathematical and pedagogical judgements about students' questions, solutions, problems and insights (both predictable and unusual)" (Ball, Bass, & Hill, 2004, p. 59). Also, Hill, Rowan, and Ball (2005) found in their empirical study, elementary mathematics teachers' knowledge had a significant positive impact on student achievement gains at first and third grades.

Students and teachers in Samoan schools are predominantly Samoan. Unlike the cultural identity crisis Samoan students experience when studying overseas, those in Samoa undergo a seamless practice of Samoan customs and traditions including the infusion of Samoan values in their formal schooling and daily home practices. Consequently, problems students face when learning mathematics, is not primarily attributed to social cultural factors as the situation would be for Samoans studying in New Zealand or Australian schools where the dominant culture of the particular country is privileged. For example, Anthony and Walshaw (2007) present evidence from intervention studies conducted to specifically address the needs of Pasifika students in New Zealand schools against a backdrop of continuously low mathematics achievement in numeracy and international tests. In contrast, problems learning mathematics more meaningfully in Samoa may be the result of untrained teachers who are not competent to teach mathematics, poor pedagogical practices, and/or the mismatch between the prescribed, taught and examined curriculum. To address a small part of these problems, the Samoan study reported here, focussed only on the identification of Samoan primary PS teachers' knowledge of the mathematics content they are expected to teach. Because of the continuously poor performance of primary students during national numeracy tests and later on, as secondary students in their national mathematics examinations (Afamasaga-Fuata'i, 2002), it became increasingly important for the country's only teacher education provider, that mathematical competence levels and needs of incoming PS teachers be identified initially at the point of entry so that timely remediation could be provided for at-risk students, and longitudinally during the program to monitor student progress and before exit to ensure graduate primary teachers are certified mathematically competent (Afamasaga-Fuata'i, Meyer, Falo, & Sufia, 2007). The presented data focuses only on the diagnosis of two consecutive cohorts of PS teachers' mathematics content knowledge as measured by a written test. The paper's focus questions are: (1) What are the levels of mathematical competence of the 2006 intakes into the entry foundation programs? (2) What are the trends of mathematical performances of the two consecutive intakes for the foundation education and diploma education programs? (3) What are some of the concerns emerging from these preservice comparisons for the university?

Theoretical Framework

Shulman's (1986) theory of teacher knowledge originally contained three categories of subject matter knowledge for teaching, namely content knowledge, pedagogical content knowledge (PCK) and curriculum knowledge. Content knowledge includes both facts and concepts in a domain and also how knowledge is validated, produced and structured in the discipline while PCK includes both the knowledge of the subject matter and knowledge of the subject matter for teaching. Of relevance here is the mathematics content knowledge of PS teachers. Ball (1990), based on analyses of classrooms, distinguished further teachers' content knowledge for teaching into knowledge about mathematics (knowledge of concepts, ideas and procedures and how they work) and knowledge about "doing" mathematics (how one decides that a claim is true, a solution is complete or a representation is accurate). In general, these perspectives theorise that teacher effects on student achievement are driven by teachers' ability to understand and use subject matter knowledge to carry out the tasks of teaching. Empirically identifying the effects of teachers' knowledge on student learning and the kinds of teacher knowledge that matter most in producing student learning led Hill, Schilling, and Ball (2004) to develop an instrument to measure teachers' mathematical knowledge for teaching elementary school mathematics. The instrument not only captured the actual content teachers taught – for example, decimals, area measurement – but also the specialised knowledge of mathematics needed for the work of teaching such as knowing how to represent ¼ in diagrams or how to appraise multiple solutions for 35 x 25. This instrument was subsequently used in an empirical study to determine the effects of teachers' mathematical knowledge for teaching on student achievement. "Mathematics knowledge for teaching" is the mathematical knowledge used to carry out the *work of teaching*, which includes explaining terms and concepts to students, interpreting students' statements and solutions, using representations accurately in the classroom, and providing students with examples of mathematical concepts, algorithms, or proofs (Hill, Rowan, & Ball, 2005). Hill et al. found teachers' mathematical knowledge plays a significant role even in the teaching of very elementary mathematics content. These findings further inform that teachers' content knowledge should be at least content-specific and even better specific to the knowledge used in teaching children. Unlike the Hill, Rowan, and Ball (2005) study, the study reported here focussed only on assessing primary preservice (PPS) teachers' content-specific knowledge and ability to solve items that centred primarily on content areas (whole numbers, fractions, decimals, percentage, operations, multi-digit subtraction, rate, ratio, proportion, area and perimeter) which comprised a significant portion of primary mathematics and including items that a first year secondary student is capable of solving (geometry, probability, and basic algebra). Therefore, PS teachers' *mathematical competence* is conceptualised as the ability to solve problems based on the relevant syllabus's content, requiring PS teachers *have* the appropriate mathematical knowledge and understanding of the specific content areas being examined and *ability to link* mathematics to experiences and *to ask* questions about the application of particular mathematical knowledge (Hogan, 2000).

Mathematics Diagnostic Test, Methodology and Analysis

A written Mathematics Diagnostic Test (MDT), developed at an Australian regional university, assesses the mathematical needs of different cohorts of PPS teachers (Mays, 2005; Afamasaga-Fuata'i, 2007). The same test and the content areas it examined was validated with the content of the Samoan Ministry of Education, Sports and Culture's (SMESC) primary and early secondary mathematics (PESM) syllabi (SMESC, 2003) after consultations with experienced Samoan educators to ensure the content areas' appropriateness to the local syllabi and item contexts' relevance to the Samoan cultural setting. The MDT1 test (see Afamasaga-Fuata'i et al., 2006) consisted of thirty items selected from the TIMSS-R 1999 mathematics paper (Mullis, Martin, Gonzalez, Gregory, Garden, O'Connor, Chrostowski, & Smith, 2000) and five mental computations. TIMSS-R 1999 items were selected as these have available reliability and validity data, been used already to measure the mathematics achievement of eighth grade students (ages 13 and 14 years) from 38 countries and been trialled previously in many international classrooms. Mental computation items included those on fractions (*Item 4:* $\frac{1}{2}+\frac{1}{3}$), percentage (*Item 2: What is 30% of 50?*), decimals (*Item 5: 0.3 x 0.3*) and multi-digit subtraction (*Item 3: 8006 – 2993*), which primary students commonly have difficulties solving (Callingham & Watson, 2004). The student:professor problem (Item 14) was included to test PS teachers' problem solving and algebraic thinking skills. The Samoan validation exercise ensured all items directly mapped to the prescribed content areas of Samoa's PESM syllabi and sampled the PESM content with a range of items first year secondary students are most likely to solve successfully. "(I)f teaching involves helping others to learn then understanding the subject content to be taught is a fundamental requirement of teaching" (Aubrey, 1997, p. 93). Because PS teachers had just completed secondary schooling, they should be capable of solving the items since the mathematics content is mainly primary mathematics with some basic algebraic computations typically encountered in early secondary (*e.g., Item 28: Write in simplest form n x n x n* and *Item 19: If x=3, what is the value of* $\frac{5x+3}{4x-3}$*?*). This paper reports the first diagnostic test (MDT1) data from two consecutive groups entering the foundation-education and primary diploma-education programs at the national university.

MDT1 was administered early in Semester 1. One hundred and forty (140) foundation- and primary diploma-education students took MDT1 in March 2005 while 263 completed it in March 2006. Cohort 2006 included other foundation students (i.e., science, arts and commerce) to provide a comprehensive snapshot of school leavers' mathematical competence across the university's foundation programs. PS teachers (secondary and primary) undertake the foundation-education year first before splitting up into their major teaching areas the following year. Students undertake the 2-year primary-diploma-education program only after satisfactorily completing foundation-education. With the latter, PPS teachers without a pass in the school certificate mathematics examination (at Year 12) should take the bridging mathematics elective to upgrade their content knowledge. Included in primary diploma-education are two compulsory mathematics methods courses. For this paper, only data based on whether students' responses were Correct (1), Incorrect (0) or Blank (B) are presented to identify general trends of cohort and group performances in order to inform policies and reform practices at the university. Although analysing students' error responses for specific misconceptions is not included in this paper, item analyses data is provided for content specific to primary mathematics. The three categories of student responses are analysed using the Dichotomous Rasch Measurement Model (Rasch, 1980) and QUEST software (Adams & Khoo, 1993). Rasch Model examines only one theoretical construct at a time on a hierarchical "more than/less than" logit scale (unidimensionality). Rasch parameters, item difficulty and person ability, are estimated from the natural logarithm of the pass-versus-fail proportion (*calibration of difficulties and abilities*). Both tests had exactly the same items; hence the 2006 MDT1 data was anchored on 2005 item estimates, hence the baseline cohort is the 2005 one. This test equating would enable meaningful and valid comparisons of ability estimates between the two cohorts (see Bond & Fox, 2001, for more details on test equating).

Results and Discussion

Rasch statistics from QUEST for item difficulty and case ability estimates (Table 1) indicate cohort 2006 had a higher mean ability than cohort 2005. This was predictable given the composition of cohort 2006. Both case standard deviation values (1.18 and 1.27) indicate a similar spread of cases around ability means.

Table 1

Cohorts' Rasch Estimates

	2005 Item	Cohort Case	2006 Item	Cohort Case
N	38	140	38	263
mean	0.00	-0.93*	0.00	-0.10*
Standard deviation	1.79	1.18	1.77	1.27
Reliability R	0.97	0.83	1.00	0.86
Zero scores	1	1	0	0
Perfect scores	0	0	0	0
*Significance	*t=6.547*	*p<0.00*	*df=302*	

Table 2

2006 Foundation Group Comparisons

	Foundation FED06 item	Education FED06 case	Foundation FSC06 item	Science FSC06 case	Foundation FOT06 item	Other (Arts & Commerce) FOT06 case
N	38	51	38	61	38	13
Mean	0.02	-0.25*	0.00	0.93*	0.06	0.99*
Standard deviation	1.71	0.92	1.77	1.08	1.76	1.60
Reliability R	1.00	0.74	1.00	0.81	0.99	0.90
Zero scores	2	0	0	0	0	0
Perfect scores	1	0	0	0	1	0
*significance tests	*t=6.2437*	*p<0.00,*	*df=93,*	*d=1.17*	*t=2.6835,*	*p<0.025, df=14, d=1.17*

One item (*Item 27: The length of the rectangle is twice as long as its width. What is the ratio of the width to the perimeter?*) in 2005 had zero score but 5.3% (14/263) of cohort 2006 cohort were successful. Also one student in 2005 (diploma-education) got a zero score. The high reliability (*R*) of the measures from the MDT1 instrument was established previously (Afamasaga-Fuata'i et al., 2007) but is again evident in 2006 ($R_{items}=1.00$ and $R_{cases}=0.86$). Statistical testing indicated a significant difference between the cohorts' mathematical performances ($t=6.5457$, $p<0.00$, $df=302$) and a moderate practical significance (*effect size=0.67*).

2006 Foundation Group Comparisons

Intakes for the foundation programs are primarily school leavers who completed the Year 13 Pacific Senior School Certificate (PSSC) Examinations. Minimum entry requirements to the various foundation programs are differential and are based on a cut-off aggregate score as regulated by the university. A comparison (Table 2) between foundation-education (FED06) and foundation-science (FSC06) groups shows a statistically significant difference (t=6.2437, p<0.000, df=93) with a large effect size (d=1.17) implying a strong practical significance. With the foundation-other (FOT06) group, a statistically significant better performance was noted (t=2.6835, p<0.02, df=14) compared to FED06. Practical significance was also strong (d=1.17) indicating FOT06 students were significantly more competent mathematically (>1 s.d.) than FED06. Collectively, these findings establish that incoming students to foundation programs are characterised differently on mathematical competence. Those who are mathematically competent are not choosing teacher education implying the university needs to review minimum entry requirements to foundation-education, to identify at-risk PS teachers and to provide adequate support for their mathematical needs, and for the education ministry to consider a systemic approach to make teaching a more attractive career choice.

2006 Preservice: Foundation Education and Diploma Education Comparisons

The gap between 2006 FED06 and diploma-education (DIP06) groups' performances is significantly different ($t = 2.5123, p<0.02, df = 97, d=0.40$) but the practical significance is small. Nevertheless, unlike the FED06 students who have just entered the university and the PS program for the first time, those in DIP06 have already in 2005, completed their foundation-education year and in 2006, progressed onto year one of their diploma program. It follows then that the statistical and practical significances suggest the DIP06 group consisted mainly of mathematically weaker students from foundation-education 2005 (FED05) who may not have taken the bridging mathematics elective or barely passed (at 50%). The rest of the FED05 group were secondary PS teachers, who in 2006, have progressed onto their secondary programs. Most importantly, the evidence suggests the need to urgently remediate the identified misconceptions of at-risk DIP06 PS teachers (guided by their individual kidmaps from the Rasch analysis, not presented here) to ensure achievement of mastery mathematical competence level before exit the following year. Whilst group difference (FED06 and DIP06) is a small effect size ($d=0.40$), that between FED06 and other 2006 foundation groups ($d=1.17$) is more than doubled that of the PS groups. This finding accentuates, yet again, the need for the university to review minimum entry requirements to teacher education (foundation and diploma levels), and for the education ministry to consider providing financial support to the university to specifically redress the PPS teachers' mathematical needs admitted below minimum entry requirements at foundation level.

2005 and 2006: Consecutive Preservice Program Comparisons

As clearly illustrated in Figure 1, there is a consistent gap in the groups' mean abilities each year. Foundation-education (FED) intakes performed consistently higher than diploma-education (DIP) intakes. Table 3 illustrates a statistically significant difference ($t=2.835, p<0.01, df=118$) between consecutive FED intakes' performances (FED05 and FED06). However, the practical significance ($d=0.48$) is only moderate. Similarly, for the consecutive DIP intakes, DIP06 performed significantly better than DIP05 ($t=3.0018, p<0.01, df=75, d=0.56$) but the practical significance is also only moderate. Collectively, these findings indicate improvement with the consecutive PS intakes' mathematical performances; however, it is still much lower than that for the 'other' foundation groups. In spite of having 'satisfactorily' completed foundation-education, DIP groups consistently performed below FED groups. These findings raise two main concerns for the university. First, the FED program structure should be flexibly designed to encourage at-risk PS teachers to take the bridging mathematics elective to redress their mathematical needs. If at-risk PPS teachers did select the mathematics elective, then the university needs to examine ways in which this support could be more effectively implemented by reviewing the course's content and delivery and raising the "passing grade" higher to mastery level (>80%) instead of the existing at least 50%. Second, in retrospect given the demonstrated 2005 and 2006 trends (Figure 1), it would be more cost-effective for the university to level minimum entry requirements with those of the other foundation programs. Doing so could attract more mathematically capable students to teacher education, who would then most likely filter through to primary diploma-education. However, there are also pitfalls such as potentially low student enrolments in the education faculty and subsequently acute teacher shortage nation wide in the near future.

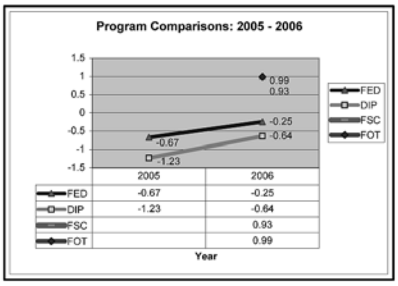

Figure 1. Consecutive Preservice Comparisons

Table 3

Rasch Estimates - Preservice Group Comparisons

	Foundation FED05 item	Education FED05 case	Foundation FED06 item	Education FED06 case	Diploma DIP05 item	Education DIP05 case	Diploma DIP06 item	Education DIP06 case
N	38	89	38	51	38	51	38	128
Mean	0.00	-0.67*	0.02	-0.25*	0.00	-1.23*	0.00	-0.64*
Standard deviation	1.75	0.93	1.71	0.92	1.77	1.86	1.77	0.98
Reliability R	0.95	0.80	1.00	0.74	0.91	0.85	1.00	0.81
Zero scores	1	0	2	0	3	1	0	0
Perfect scores	0	0	1	0	0	0	0	0
*significance tests	$t=2.835$	$p<0.01,$	$df=118$	$d=0.48$	$t=3.002$	$p<0.01,$	$df=75$	$d=0.56$

2005 Foundation Education Transition to 2006 Diploma Education

A comparison of FED05 to DIP06 mean abilities showed no statistically significant difference ($t=0.2286, p<,$ $df=197$) and a trivial effect size ($d=0.03$), implying that the year spent at foundation-education level appeared not to have had any substantive impact on DIP06 students' mathematical competence either because the PPS teachers did not select the bridging mathematics elective or alternatively, they may have barely passed it (≈ 50%). This raises a concern, which the university needs to investigate more fully to rectify the situation such as possibly, requiring a mastery pass score (>80%) instead of at least 50% and pedagogically enhancing the course's delivery.

Identified Content Areas for Remediation

It was noted from item person maps (not shown here), that all students found solving word problems involving multiplicative thinking and multi-step procedures most difficult. Item analyses data (Table 4) of the content specific to primary mathematics (i.e., fractions, decimals, percentage, operations, number sense and measurement) provide further evidence that the PS teachers found solving word problems difficult (Items 24, 18 and 31). Furthermore, ordering fractions (Item 8) was quite difficult (<14% success rate). Many students attempted the items but were unsuccessful (>78%) with the rest ignoring the items. Ordering decimals (Item 6) was difficult with less than one-third of the PS teachers correctly solving it. With the exception of the FED06 (39%) group, the rest illustrated less than 29% success in shading an area to model an equivalent fraction (Item 26). Also between 25% and 54% of the PS teachers successfully generated equivalent fractions (Item 22) whilst between 27% and 59% were successful in determining percentage of a whole number (Item 2). Decimal multiplication was successfully completed by between 33% and 57%. Less than 22% success was noted for the measurement items (#s 15 and 34) involving the interpretation of complex diagrams.

Table 4

Item Analyses Data for the Primary Content-Specific Areas

Number of students, n	Foundation 89 FED05 Correct % / Number	Education 89 FED05 Incorrect / Blank	Foundation 51 FED06 Correct % / Number	Education 51 FED06 Incorrect/ Blank	Diploma 51 DIP05 Correct % / Number	Education 51 DIP05 Incorrect / Blank	Diploma 128 DIP06 Correct % / Number	Education 128 DIP06 Incorrect / Blank
Fractions								
Item 24- operations fractions	4.5 (4)	71.9 / 23.6	5.9 (3)	84.3 / 9.8	7.8 (4)	74.5 / 17.6	2.3 (3)	71.9 / 25.8
Item 8 – Ascending fractions	9.0 (8)	85.4 / 5.6	13.7 (8)	78.4 / 7.8	5.9 (3)	92.2 / 2.0	7.8 (10)	88.3 / 3.9
Item 6 – Ascending decimals	32.6 (29)	65.2 / 2.2	21.6 (11)	74.5 / 3.9	13.7 (7)	82.4 / 3.9	23.4 (30)	74.2 / 2.3
Item 26 – fraction area model	27.0 (24)	56.2 / 16.9	39.2 (20)	52.9 / 7.8	23.5 (12)	66.7 / 9.8	28.9 (37)	56.3 / 14.8
Item 32 – fraction of amount	27.0 (24)	59.6 / 13.5	43.1 (22)	41.2 / 15.7	27.5 (14)	64.7 / 7.8	25.8 (33)	53.1 / 21.1
Item 22 – equivalent fractions	37.1 (33)	52.8 / 10.1	49.0 (25)	41.2 / 9.8	25.5 (13)	58.8 / 15.7	53.1 (68)	35.9 / 10.9
Item 2 – percentage of number	40.4 (36)	40.4 /0.0	58.8 (30)	31.4 / 9.8	27.5 (14)	72.5 / 0.0	53.9 (69)	40.6 / 5.5
Item 5 – decimal multiplication	50.6 (45)	48.3/1.1	56.9 (29)	41.2 / 2.0	33.3 (17)	66.7 / 0.0	48.4 (62)	46.1 / 5.5
Number Sense								
Item 18 – Elevator problem	6.7 (6)	78.7 (14.6)	7.8 (4)	76.5 / 15.7	3.9 (2)	74.5 (21.6)	8.6 (11)	85.9 / 5.5
Item 11 – 4 times the number	43.8 (39)	42.7 / 13.5	52.9 (27)	41.2 / 5.9	29.4 (15)	58.8 (11.8)	47.7 (61)	43.0 / 9.4
Item 3 – multi-digit subtraction	73 (65)	27.0 / 0.0	64.7 (33)	29.4 / 5.9	68.6 (35)	31.4 / (0.0)	57.0 (73)	36.7 / 6.3
Measurement								
Item 31 – Salt average weight	1.1 (1)	73.0 (25.8)	2.0 (1)	58.8 / 39.2	0.0 (0)	66.7 (33.3)	1.6 (2)	59.4 / 39.1
Item 15 – Area garden path	20.2 (18)	74.2 / 5.6	13.7 (7)	80.4 / 5.9	11.8 (6)	80.4 (7.8)	10.2 (13)	81.3 / 8.6
Item 34 – Shaded rectangular area	16.9 (15)	69.7 (13.5)	21.6 (11)	52.9 / 25.5	17.6 (9)	68.6 (13.7)	16.4 (21)	60.9 / 22.7

Overall, PS teachers found operating with, ordering, representing, and generating equivalent, fractions; decimal multiplication; and interpreting complex diagrams (measurement) difficult. These findings raise concerns as these content areas are specific to primary mathematics and therefore would require explicit remediation to enable PPS teachers to competently mediate student learning of the same concepts in primary classrooms.

Main Findings and Implications

Main findings demonstrate that more mathematically capable students are not choosing teacher education at foundation level with the preservice group's performance at least one standard deviation lower than the "other" foundation groups. Also, the mathematical performances of consecutive preservice intakes illustrated moderate improvement with the diploma education trend consistently lower than that of foundation education albeit effect size is small. Further, the PPS teachers' foundation-diploma-education transition remains problematic as illustrated by the statistically non-significant difference and trivial effect size between the two groups despite "satisfactorily" completing foundation-education. The university needs to determine best strategies to monitor and sustain improved mathematics competency levels from year to year, towards mastery competence ideally before exit. A view of areas of difficulties was gleaned from an examination of item analysis data on content specific to primary mathematics. Evidence suggested preservice teachers in both programs experienced difficulties solving word problems and items on ordering, modelling, operating with, and generating equivalent, fractions; decimal multiplication; percentage; and determining area of geometric shapes embedded in complex diagrams. Collectively, these findings have implications for policy making (a) to improve the quality of intakes into teacher education by reviewing minimum entry requirements to attract more capable students, (b) to review the appropriateness of support for mathematically at-risk PS teachers, (c) to make the bridging mathematics course compulsory for PS teachers without Year 12 mathematics background to upgrade their content knowledge, (d) to include content specific to primary mathematics in the bridging mathematics course, (e) to incorporate the specific pedagogical content knowledge (Hill, Rowan, & Ball, 2005) required to teach word problems, fractions and measurement in the content of the two compulsory mathematics methods courses. On the other hand, the education ministry needs to consider systemic strategies to enhance teaching as an attractive career choice for capable school leavers and to provide sufficient support to the university to explicitly remediate the mathematical needs of PPS teachers as a result of lowering entry requirements to ensure a constant quota of incoming PS teachers and outgoing graduate teachers. Once the students are admitted then the university is obligated to address their mathematical needs.

In addition, the identified content areas of difficulties require explicit remediation as teachers' knowledge has a significant impact on students' learning of mathematics (Hill, Rowan, & Bill, 2005). From a social constructivist perspective, if teachers are to mediate mathematical meaning, facilitate and support students' mathematical thinking and reasoning, then teachers need to understand the mathematics children are dealing with and need to be aware of the many opportunities that present themselves for the learning of mathematics. (Perry & Dockett, 2002). Furthermore, "low levels of content knowledge and the resulting lack of confidence about mathematics limit teachers' ability to maximise opportunities for engaging children in the mathematics learning embedded within existing activities as well as their abilities to introduce more focussed intervention activities designed to cater for diverse learners" (Anthony & Walshaw, 2007, p. 45). This study contributes empirical data to inform the university of the need to develop policies to ensure its graduate teachers have the requisite content knowledge to teach primary mathematics competently and confidently so that they are able, in turn, to maximise opportunities for engaging children in learning mathematics and to effectively design intervention activities to cater for the diversity of student abilities in their classrooms. Finally, findings from this empirical study contribute to the scarce literature on Samoan preservice teacher education in particular and general preservice teacher education. Investigating in-depth the transitions from foundation-to-primary-diploma-education and from final-year-to-first-year-teaching in schools are areas worthy of further research in Samoa's educational system.

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Teachers' Motivation to Attend Voluntary Professional Development in K-10 Mathematics

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Teachers choose to attend professional development courses and workshops for a range of reasons. New mathematics syllabuses in NSW with increased expectations for student learning outcomes created opportunities for providers to support the implementation of the new syllabuses. The participants at four six-week professional development courses completed surveys indicating motivation for attendance. This paper considers the reasons 109 teachers chose to attend these events and the type of knowledge they value as reported in evaluations of the courses. Responses were categorised using Shulman's knowledge categories with pedagogical content knowledge, curriculum knowledge and knowledge of learners as most valued.

According to Grundy and Robison (2004), professional development serves three functions (extension, renewal and growth) and is usually initiated through two drivers (systemic and personal). Systemic professional development is typically associated with renewal whereas personal professional development may serve all three functions. Exploring the reasons for a "personal desire and motivation by teachers to sustain and enhance their professional lives" (Grundy & Robison, 2004, p. 147), particularly in mathematics education where there is a shortage of qualified teachers, may help providers to plan appropriate content and knowledge building experiences to enrich and retain more teachers in the profession (Martinez, 2004).

The content of professional development courses varies from addressing mathematical content knowledge, pedagogical content knowledge, knowledge about how children learn mathematics, or a combination of these (White, Mitchelmore, Branca, & Maxon, 2004). If teachers do not receive the support they require at school, they seek support and collegiality through other means such as professional associations, local networks and professional development courses offered by providers or "knowledge vendors" (Kennedy, 2005, p. 212). To explore teachers' motivation to attend professional development, the introduction of new mathematics syllabuses for primary schools (Kindergarten to Year 6) and secondary schools (Years 7 to 10) in New South Wales (NSW) (Board of Studies NSW, 2002a, 2002b) provided a context for the investigation.

At the University of Sydney, three courses were developed to support teachers' knowledge and understanding of the new syllabuses – the *Certificate of Primary Mathematics Education* (the Primary Course), the *Certificate of Middle Years Mathematics Education* (the Middle Course), and the *Certificate of Secondary Mathematics Education* (the Secondary Course). The courses discussed in this paper were offered during the period from 2005 to 2008. For each course, participants from a range of schools attended a one-day conference followed by six consecutive Wednesday evening workshops. Evaluations through reflective surveys were used to gather data about teachers' motivation to attend and knowledge preferences.

While the investigation focused on teachers' knowledge preferences in professional development, it is acknowledged that knowledge is filtered through beliefs, experiences and the social context of teaching (Anderson, White, & Sullivan, 2005). These aspects were recognised in the design of the courses, as were the types of knowledge required of quality teachers. This paper addresses the questions:

1. What motivates teacher attendance at voluntary professional development courses?
2. What type of knowledge are teachers seeking and what knowledge do they value?
3. Is there a difference in the knowledge sought by primary and secondary school teachers of mathematics?

The Knowledge of Mathematics Teaching

Shulman (1987) sought to identify the particular knowledge required of teachers, which distinguished it as a profession and hence enabled the articulation of a set of standards for quality teaching. In the late Eighties, Shulman worked with a group of colleagues to design an assessment for the certification of teachers against "well-grounded judgements and standards" (p. 6). The identification of standards has relevance today given the development of the *AAMT Standards* in 2002 and in NSW, the *Professional Teaching Standards* (NSW Institute of Teachers, 2005). Shulman's work led to the identification of seven categories of knowledge (Table 1).

Table 1

Categories of the Knowledge Base of Teachers (Shulman, 1987, p. 7)

Content knowledge;
General pedagogical knowledge, with special reference to those broad principles and strategies of classroom management and organization that appear to transcend subject matter;
Curriculum knowledge, with particular grasp of the materials and programs that serve as "tools of the trade" for teachers;
Pedagogical content knowledge, that special amalgam of content and pedagogy that is uniquely the province of teachers, their own special form of professional understanding;
Knowledge of learners and their characteristics;
Knowledge of educational contexts, ranging from the workings of the group or classroom, the governance and financing of school districts, to the character of communities and cultures; and
Knowledge of educational ends, purposes, and values, and their philosophical and historical grounds.

Some professional development courses focus on content while others focus on pedagogy. White et al. (2004) compared several courses from each category to identify the advantages and disadvantages of each. The authors noted perceived relevance and teacher enthusiasm as criteria for success although these factors are clearly influenced by contextual and cultural factors such as working conditions, funding support, and accreditation. They conclude that a blended model, which incorporates both content and pedagogy is most desirable although it may not be what teachers are seeking. Their recommendation supports the advice of Loucks-Horsley, Love, Stiles, Mundry, and Hewson (2003) who advocate professional development should focus on knowledge of content, knowledge of students, and knowledge of instruction and assessment. However, is this the knowledge teachers are seeking when they choose to attend particular professional development courses, particularly when offered outside of school time?

If professional development is aimed at extension, renewal and growth, what sources provide the best opportunities for this to occur? Kennedy's (2005) detailed observations and interviews with 45 upper primary school teachers revealed improvements to practice are usually developed privately by teachers at the teacher's own initiative. She reported that almost all teachers' references to improvements were motivated by new ideas from three main sources – informal (including experience, colleagues, own materials), institutional (including curriculum, textbooks), or "knowledge-vending" (p. 212) (including professional development and university courses). To ascertain the importance of each source, she "cross-tabulated the sources of teachers' new ideas with the areas of concern that they addressed" (p. 214) (Table 2). Kennedy (2005) noted teachers typically did not seek out professional development to solve specific problems with several enrolling in professional development for "casual" reasons (p. 218) including recommendation of a colleague, proximity to the school, convenient timing of the program, or for accreditation.

Table 2

Percent of References to Sources of Ideas by Area of Concern (Kennedy, 2005, p. 214)

	Source of Ideas		
Area of concern	**Informal**	**Institutional**	**Knowledge vendors**
Defining learning outcomes	3	38	10
Fostering student learning	22	32	63
Maintaining lesson momentum	25	15	3
Fostering student willingness to participate	28	9	7
Establishing the classroom as a community	16	3	10
Attending to personal needs	6	3	7
Total	100	100	100

Contrary to Kennedy's findings, an earlier investigation into secondary mathematics teachers attending a professional development course by Anderson and Moore (2006) revealed more than half of the participants reported a desire to gain new teaching ideas, improve skills, and learn more about the curriculum. To further explore these aspects, this investigation seeks to identify the types of knowledge both primary and secondary teachers seek when they choose to participate in professional development activities.

Designing and Evaluating Professional Development

To address the introduction of new syllabuses in NSW from Kindergarten to Year 10, the three certificate courses were designed for different audiences – primary school teachers (K to 6), middle school teachers (5 to 8) and secondary school teachers (7 to 10). The design of courses was informed by Clarke's (1997) ten principles to guide planning of professional development as well as a large-scale national survey into the effects of different characteristics of professional development on teachers' learning (Garet, Porter, Desimone, Birman, & Yoon, 2001). In particular, each course aimed to develop teachers' knowledge and understanding of mathematics content as well as curriculum requirements, incorporated ongoing discussions with colleagues and opportunities to try teaching ideas between meetings. Each course included a conference and weekly workshops over a six-week period and encouraged more than one teacher from each school to attend.

Presenters embedded a combination of curriculum knowledge, pedagogical content knowledge, knowledge of learners, as well as content knowledge into workshops. The remaining categories of knowledge identified by Shulman (1987) – general pedagogical knowledge, knowledge of educational contexts, and knowledge of educational ends, purposes and values – were not considered explicitly in planning. However, aspects of these did arise in discussions between teacher participants and between teachers and presenters.

While it is a challenge to meet all teachers' needs given the diversity in experience and leadership responsibilities, providers need to determine whether their planning and organization are aligned with current issues for teachers in schools as well as their personal goals and expectations. As the courses aimed to develop teachers' knowledge and skills, it was appropriate to seek participants' views about their level of satisfaction in relation to identified learning needs. Guskey's (1999, p. 78) "five critical levels of professional development evaluation" were considered in the design of reflective surveys. These include:

1. participants' reactions,
2. participants' learning,
3. organisational support and change,
4. participants use of new knowledge and skill, and
5. student learning outcomes.

Evaluation of the certificate courses was primarily qualitative, conducted through self-report surveys. While questions related to all five of the critical levels listed above, there was a particular focus on Levels 1, 2, and 4.

Data collection for all four courses was extensive with the use of eight separate surveys seeking detail about teachers' learning, and confidence before and after each workshop. Teachers were also required to report the following week on any actions taken from the previous meeting. As the focus for this paper is to explore why participants chose to attend the courses, what knowledge they were seeking, and whether the course met their learning needs, the data included here relates to the first and last surveys only. While each of the certificate courses has been offered on at least two occasions since 2004, the courses of focus for this paper are the primary and secondary courses offered in 2005, and the middle years courses offered in 2007 and 2008.

Exploring Teachers' Motivation to Voluntarily Attend Professional Development

As part of the reflective survey administered at the end of the first day of the course, teachers were asked to provide background information (name was optional, years of teaching experience, role in school, and source of funding support), reasons for attending the course, as well as questions about each of the keynote presentations and workshops. The final summative survey sought information about whether the course had met teachers' needs as well as overall impressions of the organisation and support materials.

When considering teachers' motivations for attending the certificate courses another focus is to ascertain whether primary school teachers were seeking different knowledge to their secondary colleagues. It is possible primary teachers wanted to enrich their content knowledge because of the increased expectations in the new syllabus. Secondary teachers might be more interested in pedagogical content knowledge in relation to using technology to support teaching and learning or to further explore curriculum knowledge to implement Working Mathematically.

This section presents background information about the participants in each course as well as the data from particular questions on the first and last reflective surveys. The questions of interest include:

- Why did you decide to attend the certificate course? (first survey and last survey)
- Has the course met with your expectations? (last survey)

Participants

Background information revealed a range of teaching experience with the majority of participants in the primary and secondary courses having spent ten or fewer years in the classroom. In contrast, the middle years certificate had a greater proportion of teachers with 16 or more years experience and with almost half (26 out of 60) of the participants in leadership positions (Table 3). Overall, 59 of the 109 (54%) participants were classroom teachers. However, 42 (39%) reported being in the leadership positions of consultant, principal, assistant principal, mathematics coordinator or assistant mathematics coordinator. Such roles bring added responsibilities and potentially different needs. The middle years certificate attracted only 9 secondary teachers compared to 51 from the primary sector. In all courses and with only a few exceptions, there were at least two teachers from the same school with several schools sending more than two.

Table 3

Identified Roles of Participants in each of the Three Courses

Roles	Primary (N=22)	Middle Years (N=60) Primary	Middle Years (N=60) Scondary	Secondary (N=27)	Total
Consultant (non school based)	2	3	-	2	7
Principal	-	3	-	-	3
Assistant Principal	4	9	-	2	15
Mathematics Coordinator	1	6	1	7	15
Assistant Maths Coordinator	-	-	1	1	2
Classroom teacher	13	27	7	12	59
Other	2	3	-	3	8
Total	22	51	9	27	109

Additional background information included gender and source of funding to attend the course. Reflecting the greater proportion of males teaching mathematics in secondary schools, the ratio of males to females attending each course was primary (3:19), middle years (8:52) and secondary (10:17). The source of funding for attendance indicates that for 80% of participants, attendance was funded by a system or by a school. This might suggest there was some coercion into participation. However, most participants indicated their attendance was voluntary except for some encouragement to attend with a colleague from the same school.

Motivations to Attend the Certificate Courses

The question on the first and last surveys asking why teachers decided to attend the course aimed to identify motivations for attendance as well as the types of knowledge teachers were seeking. Initially the responses to this question on both surveys were read to identify categories. There were six main reasons for teacher attendance: quality of the program; personal development; a love of mathematics; accreditation or recognition; casual reasons; or to gain knowledge and skills for teaching (Table 4). As some participants had several reasons for attending, their comments were placed into more than one category.

Motivations for participation ranged from personal growth and recognition to a desire to learn new ideas for implementation of the syllabus. The 'personal development' category included comments like "I want to do something for myself" and "for my own needs" reflecting a strong motivation to sustain and enhance teachers' professional roles (Grundy & Robison, 2004). Accreditation was a goal for some participants in each course and took three forms: recognition as a teacher of mathematics; acknowledgement of participation in a university endorsed course; and accreditation to satisfy the NSW Institute of Teachers requirements for new-scheme teachers. For example:

My 9th consecutive year on Kindergarten – I need to see where I'm heading (as I don't have a BEd) I need to reaffirm for myself I'm on the right track as younger teachers with a BEd and MEd consider you're past it and they are the only ones connected – some are very confronting (Primary)

I have been teaching in a maths department for four years. I wanted something to help me [to] get reclassified to include maths teacher (Secondary).

Table 4

Number of Comments (first and last survey responses combined) for Each Attendance Reason for Each Course

Reason for attendance	Primary (N=22)	Middle Years (N=60)	Secondary (N=27)
Quality of the program	3	18	5
Personal development	5	25	13
Love of mathematics	4	8	5
Accreditation or recognition	5	10	10
Casual reasons, e.g., go with a friend	2	2	5
Gaining knowledge and skills for teaching	42	100	34
Total	61	163	72

The majority of reasons given for attendance related to knowledge, indicating teachers were seeking to develop knowledge, skills and understanding of teaching and learning mathematics. Unlike the teachers in Kennedy's (2005) study, the majority of participants in all courses reported a desire to extend or renew their knowledge, with few indicating they were attending for 'casual' reasons. To identify the type of knowledge teachers were seeking, the comments categorised into 'gaining knowledge and skills for teaching' were further classified using Shulman's (1987) knowledge types. The knowledge types represented by teachers' comments included (Table 5):

- content knowledge,
- curriculum knowledge,
- pedagogical content knowledge,
- knowledge of learners, and
- knowledge of educational contexts.

Table 5

Number of Comments for Each of the Knowledge Types for Each Course and Percentage of Comments for Each Knowledge Type

Gaining knowledge and skills for teaching	Primary (N=22)	Middle Years (N=60)	Secondary (N=27)	Overall Percentage
Content knowledge	7	9	1	10
Curriculum knowledge	11	16	13	23
Pedagogical content knowledge	12	28	17	32
Knowledge of learners (transition issues)	3	32	0	20
Knowledge of educational contexts (references to leadership and sharing)	9	15	3	15
Total	42	100	34	100

Teachers' comments about the knowledge they were seeking from the courses reveal similarities and differences between primary and secondary participants. Both groups of teachers reported seeking all five types of knowledge, particularly pedagogical content knowledge. All but one of the comments from teachers wanting to develop mathematical content knowledge, were from primary school teachers. This is not surprising, as most secondary school mathematics teachers have studied more mathematics in their teacher education programs. The greater number of comments about 'knowledge of learners' from middle school teachers involved comments about transition issues as well as the engagement and motivation of middle school students. Examples of teachers' comments for each knowledge type are presented in Table 6.

Table 6

Examples of Teachers' Comments for Each Knowledge Type

Knowledge Type	Examples of teachers' comment
Content knowledge	To extend/reinforce my prior learning in mathematics. (primary) I've just switched from primary to secondary maths teaching and I would like a refresher on maths concepts and principles (middle years secondary teacher).
Curriculum knowledge	Desire to learn more about the syllabus (secondary) Chance to develop better quality resources (secondary).
Pedagogical content knowledge	Love of maths and a need for more practical ideas and recent research and understandings in maths ed. I want to take these into the classroom and into my teaching (primary)
Knowledge of learners (transition issues)	To teach mathematics to my class of children at its full potential. To give my children the maximum opportunities to learn mathematics (primary) I am teaching Stage 3 this year. I have noticed the decline in engagement and motivation of Stage 3 students over a couple of years (middle years primary teacher)
Knowledge of educational contexts (references to leadership and sharing)	We would like to use the course to help us with whole school professional development. (primary).

Teachers' comments frequently revealed their passion and desire to improve their knowledge and understanding to support students' learning of mathematics. The following comment by a participant in the primary course reveals a desire to support children's learning in a way that she was not supported at school.

Passion for helping children understand as I never understood ANY maths as a kid (primary teacher).

This sentiment has been reported in earlier studies (e.g., Anderson et al., 2005) and possibly applies to more primary than secondary school teachers. However the comment reinforces that "passion" can drive teachers to attend professional development for deeply held personal reasons.

On the last survey participants were asked whether the course had met their expectations. For secondary teachers, 26 of the 27 teachers answered "yes" with many commenting that the course exceeded expectations. While almost all participants were satisfied, requests for additional support related to using technology and teaching mathematics to students with special needs. For middle years, 55 of the 60 participants answered "yes" with many additional positive comments. Participants wanted more input on implementing technology sessions and addressing transition issues from primary to secondary school. All primary participants indicated the course had met or exceeded their expectations with one stating,

> Yes. I have extended my learning in a lot of areas and feel more comfortable with a lot of the concepts that have been covered.

Capturing Teachers' Motivation to Attend Voluntary Professional Development

We are all familiar with teachers attending professional development sessions to obtain worksheets and 'good ideas' for use in class the next day. They will discuss useful professional development sessions as those which were 'practical'; some of the teachers attending the courses described here were no exception since several participants made comments about a particular session being 'more practical' than another. While a good teaching idea might be useful, focusing on interesting tasks in short workshop sessions may not substantially build teachers' knowledge for teaching. These courses aimed to go further by providing opportunities for teachers to engage in a sustained way with important issues in relation to the implementation of a new curriculum. Course evaluations provided evidence that teachers were seeking more than practical knowledge or even curriculum knowledge. Most participants wanted to learn more about content knowledge, pedagogical content knowledge, knowledge of learners and knowledge related to leading teachers in their schools. While the data is taken from self-report surveys, follow-up interviews with participants will enable a more detailed investigation of teachers' motivations.

As noted in Kennedy's (2005) investigation of primary teachers areas of concerns in classrooms, teachers want advice and support about fostering student learning. Providers or knowledge vendors should focus on the ways teachers' knowledge about student learning can be better supported. Further support needs to be provided which combines theoretical and practical knowledge for teaching. Capturing the enthusiasm and passion teachers bring when they voluntarily attend professional development outside of school hours is an opportunity for all who provide professional development for teachers including those who work in post-graduate teacher education courses.

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Using National Numeracy Testing to Benefit Indigenous Students: Case Studies of Teachers Taking Back Control of Outcomes

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This paper focuses on the disempowering nature of Years 3, 5, and 7 *Aspects of Numeracy* testing undertaken by the Queensland Studies Authority, in relation to teachers of Indigenous students. It describes a Deadly Maths project with four far North Queensland regional schools to take back control of outcomes by empowering teachers to undertake their own analyses of the tests and use them for immediate intervention programs. It describes students and teachers' perceptions of the tests and teachers' reactions to the project, indicating difficulties in overcoming teacher resistance and reporting success for teachers who did take back some control.

As in many countries, Australia requires its states and territories to develop and administer a numeracy test for students in Years 3, 5, and 7. The tests are state-specific in content but all tests across Australia are expected to include items from the National Numeracy Benchmarks. The Queensland tests are called *Aspects of Numeracy*. This paper reports on a project to support teachers in four far North Queensland regional schools with Indigenous students use the 2007 *Aspects of Numeracy* results to improve their students' mathematics outcomes.

Background

In Queensland, the *Aspects of Numeracy* tests are administered in August each year. Following strict government guidelines, the Queensland Studies Authority (QSA), a government agency, collects and grades the test papers and provides a report to each school at the end of the school year (late November-early December) and a publicly-available report when it has ministerial approval (usually at least 12 months later). The report provides information on every student for every test item and compares overall students' outcomes with the general state average and a state average for "like" (comparable in terms of size and student background) schools. Schools are then required to report these outcomes to their communities.

Because the reports of student results are not made available to teachers until the end of the year (and the teachers are unlikely to have the same students in the next year), there is little chance, if any, for the reports to provide feedback to teachers that could affect instruction. Furthermore, the reports are in a format that was inaccessible to many teachers. Although teachers are not afforded the opportunity to provide input to the form of the test items, their performance and the performance of their students are judged by the test results in a government-sponsored initiative called "making teachers accountable."

In recent years, the Queensland *Aspects of Numeracy* testing program has tried to accommodate real-life experiences of a variety of students (including Indigenous). However, the test items and strict testing procedures of the past years have not take into consideration the personal attributes of the students, such as their social and racial background, remoteness, the quality of the school staff, facilities and plant, and community support available to them. The only position that appears to be important is comparison with the state average. As a result of this focus on outcomes compared to other schools, school results have generally corresponded to the socioeconomic status of students. Thus, lower socioeconomic schools, particularly those with significant Indigenous populations, have been disempowered, because language, background, and culture made it difficult for their students to understand many of the test items. In particular, Indigenous students' average test performance has lagged two years behind that of non-Indigenous students (QSA, 2004, 2005).

Our, the Deadly Maths group's, solution to the problem of disempowerment, particularly in Indigenous schools, was to work collaboratively with teachers to take back some control of the testing process. The solution recommended that teachers photocopy the test papers, mark their students' answers within a week of the test, and then use a spreadsheet to analyse the results. Based on student results, teachers can then make well-informed decisions about the ways in which they teach numeracy to their students and implement appropriate modifications to the curriculum to more effectively address numeracy outcomes. This paper

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describes a collaboration undertaken with four schools, called A, B, C and D, in far North Queensland with Indigenous populations and reports on teachers' and students' perceptions of the *Aspects of Numeracy* tests and teachers' reactions to the Deadly Maths group's solution of taking back control of the tests by marking and analysing photocopies and using the data to improve teaching and students' mathematics outcomes.

Testing and Indigenous Students

Indigenous students' performance on the Aspects of Numeracy tests was largely attributed to a number of factors other than their actual ability and mathematics knowledge. One of these factors was language. English is a second language for most Indigenous students, particularly those in rural areas. Most Indigenous students use a kreol called Aboriginal English outside of the classroom. The numeracy tests covered number, space, and measurement, but, except for a section on mental computation, were based on reading, interpreting, and solving word problems. In fact, many test items were based on a colour magazine with written information, tables, graphs, pictures, and diagrams. Because the language used in the test items and magazine was Standard English, the Indigenous students had difficulty understanding what to do. As a result, in one Indigenous boarding school in another Deadly Maths project, not one student was able to correctly interpret word problems in the 2006 tests.

The second issue affecting the performance of Indigenous students on the Aspects of Numeracy tests was culture and context. The everyday lives of regional and remote Indigenous students are very different to non-Indigenous urban students. Because the real-life applications of test items were designed for experiences typical of non-Indigenous, urban, middle-class students, Indigenous students often could not make a connection to the test items with their own experiences. A principal from another Deadly Maths project stated that the best way for her remote Indigenous students to do better in the tests would be for them to spend six months living in Brisbane. She described a particularly extreme example in which students had been asked a problem about the "rover" pass, a special ticket that allows a day travel on busses, trains, and boats in Brisbane. Her students had interpreted a "rover" pass to have something to do with a dog, as this was the only thing they knew that might have been named "rover". Another teacher in the same project said that the magazine itself was the problem, because her Indigenous students had so little experience with magazines that they spent much of the test time looking at the magazine and not actually answering the questions.

The teaching support received by many Indigenous students was the third issue that influenced their performance on the numeracy tests. Most teachers in Indigenous schools are non-Indigenous, young, newly graduated, and inexperienced in their teaching practice and knowledge and understanding of Indigenous culture. Therefore, many tend to construct Indigenous culture in accordance with western ideals and preconceptions and, instead of integrating learning, they compartmentalise learning using techniques that are unfamiliar to Indigenous students (Christie, 1994, 1995; Grant, 1997; Rothbaum, Weisz, Pott, Miyake, & Morelli, 2000). In particular, these teachers do not attempt to contextualise mathematics into Indigenous culture (Frigo & Simpson, 2000) which has a negative impact on the confidence of Indigenous students in their ability to do mathematics (Howard, 2001; Matthews, Watego, Cooper, & Baturo, 2005). Similarly, because teachers have low expectations of Indigenous students (Raeburn, 1993), Indigenous students' participation and success are marginalised (Sarra, 2003), resulting in limited educational impact on Indigenous student learning (Department of Education, Training and Youth Affairs, 2001). As a result, teachers of Indigenous classes are inclined to focus on algorithm practice with worksheets that do not prepare students for the word problems in the numeracy tests. Many non-Indigenous teachers believe that, because classrooms are places where they are in charge (McFadden, Munns, & Simpson, 1999), they are reluctant to allow Indigenous participation in decision-making (Beresford, 2001). This dominant attitude impacts on teacher-student relationships and results in negative beliefs and feelings (Groome, 1995). As concluded in Warren, Cooper, and Baturo (2007), education in Indigenous schools is socially unjust, particularly in remote communities.

The consequence of these issues is that *Aspects of Numeracy* testing is a very difficult time for Indigenous schools; students strongly disliked the testing regime, actively agitated against the administration conditions, and became very depressed by their inability to understand what to do. For most of our projects, we had avoided attending schools during testing, because it was a period of great disruption. Most teachers in Indigenous schools expect the results that the students achieve on the tests and believe that they are not responsible for the level of the results (Sarra, 2003). They see the test process is not a positive experience but rather one of disempowerment. Therefore, we have advocated that teachers retake control of the testing and use it for their own purposes and their students' futures. This paper describes our first formal attempts to work with schools to achieve this goal.

Design

The methodology was qualitative and longitudinal (the schools were part of the project for one year), with the researchers collaborating with teachers in action-research case studies (Kemmis & McTaggart, 2001) in which they use the *Aspects of Numeracy* test results to improve the numeracy learning of their classes, particularly their Indigenous students. The participants were the administrators, Years 3, 5, and 7 teachers (13 in all), and their students from four schools in a regional centre in the northern part of Queensland. The four schools were independent, run by religious organisations. Only School A, a boarding school covering Years P to 12, had a totally Indigenous student population. Schools B, C, and D were Years P to 7 day schools that taught both Indigenous (less than 30%) and non-Indigenous students. School B was a catholic primary school and taught students predominantly from its local community. Schools C and D were independent, represented particular religious persuasions and taught students from across the regional city. Schools B and C was middle sized and D was a small school.

The procedure involved five stages. In the first, the Deadly Maths researchers and the Years 3, 5, and 7 teachers had a professional learning day (PL1) on: (i) *Aspects of Numeracy* testing (its form, use, and implications); (ii) interpreting QSA reports; (iii) using spreadsheets to analyse student tests responses; and (iv) translating student performance data into remedial and preventative instruction. In the second stage, the researchers assisted the teachers to: (i) analyse their 2006 Years 3, 5, and 7 test data (using ideas from PL1); (ii) develop a program to use these results to look at their 2007 class; (iii) photocopy and mark their students' 2007 test responses and enter the data on a spreadsheet; and, (iv) analyse this data and determine possible areas for further work with their class. In the third stage, the researchers and teachers had a second personal learning day (PL2) on: (i) implications of teachers' individual 2007 data analyses for numeracy learning; and, (ii) mathematics instructional approaches for Indigenous students. In the fourth stage, the researchers assisted the teachers to develop and implement a teaching program to improve numeracy weaknesses identified in their 2007 test responses. In the final stage, the teachers shared the results of their trials with the other teachers and the researchers.

The data gathering processes used were: (i) observations of the seminars, the sharing and classroom trials; (ii) interviews with administrators and teachers; (iii) interviews with a sample of each teachers' students; (iv) surveys of teachers; and (v) the collection of artefacts (for example, teachers' plans, students' work).

Findings

The video and audio recordings of the observations and interviews were transcribed, organised, and combined with the field notes and descriptions of the artefacts to give a rich description of the project. This paper considers data from observations across the project and interviews in Stage 4 and at the end of the project. Findings cover students' perceptions of testing, teachers' perceptions of testing; and teachers' responses to the Deadly Maths project.

Students' Perceptions of Testing

The students' interviews in the four schools gave rise to the following responses with regard to their feelings about mathematics, the tests, and their success in the tests. Only some of the data is displayed in Table 1. Note that, in Item 7, many students recorded more than one response when explaining strategies employed to solve problems.

Table 1

Yrs 3, 5, & 7 Students' % Responses (N = 38 - Yr 3, N = 38 - Yr 5, N = 64 - Yr 7)

Item	[Form of answer for which % is given]	Yr3	Yr5	Yr7	All
How do you feel about maths generally?	[Students who said "happy"]	61	55	48	54
How did you feel before the test?	[Students who said "happy"]	37	34	23	30
How did you feel when finished the test?	[Students who said "happy"]	84	71	84	81
How did you think you went?	[Students who said "got most right"]	21	29	33	29
	[Students who said "got some right"]	32	37	38	36
You got answers right because?	[Students who said "good at maths"]	50	38	27	28
	[Students who said "tried hard"]	42	55	53	56
Got answers wrong because?	[Students who said "not good at maths"]	25	16	16	19
	[Students who said "did not try hard enough"]	5	30	22	18
How did you work out tricky questions?	[Said "kept working"]	47	50	70	61
	[Said "worked it out wrong"]	8	24	17	16
	[Said "just guessed"]	29	8	17	18
	[Said "gave up" or "didn't do it"]	16	13	16	15

In general, a small majority of students overall (54%) felt happy about mathematics (more younger than older). Before the test, students across all ages (30%) were either unhappy or unsure of their feelings about the tests prior to the test day. However, after the test, most students (81%) indicated they felt happy, but it is uncertain whether their feelings were related to their performance or just that they were relieved to have finished the test. Most students felt that they did not do very well with only 29% indicating that they felt they "got most right" and 36% believing that they "got some right" (older more confident than younger although younger had a more favourable attitude). Whilst overall, only 28% of students believed that they are "good at maths" (more younger than older) some 56% indicated that they had "tried hard" (more older than younger). The older students considered extrinsic factors, such as easy questions or lucky guessing, to have contributed to their correct answers but, unfortunately, they were more likely to blame their failure on themselves or not trying hard enough. There was a propensity for students to believe that they had tried their best in sitting the test. With regard to more complex (or "tricky") questions, the majority of students (61%) identified that they "kept working" until they were confident of their answer, whilst only a small percentage (15%) "gave up" or "didn't do" them. This trend shows that students were inherently motivated to achieve success despite the external factors that were negating their abilities to do well.

As is described in the next section, only 8 of the 13 teachers completed all stages and, particularly, did a complete analysis of students' responses to the tests. However, the teachers that did complete analysis of their students' test responses reported a discrepancy between students' beliefs about their success and their actual success, particularly with respect to problems. One teacher was sufficiently concerned about what she saw as students interpreting problems to be simpler than they were to build her remedial program around this. Overall, in all schools, students' performance was not as high as the students had expected, particularly in School A. However, we need to consider whether the expectations of the students originates from a different individual benchmark, such as an improvement in the students' ability, understanding, and confidence at this point in time compared to at the beginning of the year.

Teachers' Perceptions of Testing

The teachers at School A, the Years P-12 Indigenous boarding school, felt that the tests were unfair for their students, stating that the language and problem contexts used in the tests were unfamiliar to many students (not being in Aboriginal English and not relating to rural/regional Indigenous settings). They stated that the tests were therefore onerous and stressful for the students as the students received poor results despite lots of effort. They were very committed to the education of the students, however, they felt that the tests were unfair to them, arguing that low test results could be interpreted as reflecting poor teaching, rather than a consequence of external factors. They believed that their students started from such a low base that the students' results would still be below average even if they, in their teaching, achieved significant improvements in their students' mathematics outcomes.

The teachers at Schools B, C and D, where Indigenous students made up less than 30% of the population, were not as strong in their dislike of the Years 3, 5, and 7 tests as School A, although some teachers of School B did feel that the testing time was onerous on students and time consuming and that, in their opinion, the tests were a waste of time. Similar to School A, teachers at Schools B, C and D were concerned that the test results may reflect poorly on their individual teaching ability. However, there was no mention of their schools' programs as a contributing to students' results.

Teachers' Perceptions of Deadly Maths Project

Initially all teachers in the four schools seemed enthusiastic about being able to gather and use test data from their students for a relevant purpose. They felt that the time spent administering these test served no real purpose in the everyday running of their classroom and, therefore had little relevance to their individual classrooms. No teachers had previously photocopied, marked and analysed their students' test responses and very few had even used the QSA's reports to determine students' understanding and inform teaching. They had restricted themselves to organising for all their Years 3, 5, and 7 students to complete the tests and then waited for results at the end of the year.

In practice, the teachers' performance did not match their original enthusiasm. One of the major reasons for this was that all participating teachers had very limited knowledge of working with data. None of the teachers had any data entry skills (not even knowing what data was useful and how to find class percentages etc). There were varying levels of skills with using excel (ranging from very limited to moderate – no one was confident using excel for data analysis or to produce graphs). After data had been collated and graphed, teachers could see the trends in the students' understanding. However, none appear able, unsupported, the use these trends to develop remedial programs. Only 8 of the 13 teachers (all 3 from Schools A and C and one each from Schools B and D) completed all five stages of the project.

The schools undertook the data entry and analysis differently. All teachers in School A had their data entered analysed by one teacher who was released for this purpose. Thus, although all the teachers had results, they had not become empowered by the task, and had little data analysis skills at the end of the project. Worse than this, the resulting analyses often had incorrect calculations (e.g., percentages of over 4,000) which were not evident to the teachers because they had not worked with the data. However, the teachers, with support from Deadly Maths researchers, did complete analyses and develop and trial remedial programs, and all presented at the sharing conference. The teachers in School B became reluctant to undertake the task. They became apprehensive about how much time the process of photocopying, marking and analysing would take, clearly giving the impression that they thought it was not really worth the personal time it would take to do it. They employed a casual teacher to mark and enter data but analyses were incomplete. Only one teacher followed through, completed the analysis, and developed and trialled a remedial program. The other teachers seemed to become preoccupied with data entry and analysis becoming system based and undertaken by experts external to their school. The teachers of School C seemed enthusiastic and had good intentions but had little knowledge of how to use test responses to improve teaching. With support (there were many mistakes to correct), they carried out all tasks and were able to develop their own analyses. They all came to see the benefit at the end of the process. Some teachers found that the data simply affirmed what they already understood to be the weaker areas; however, they felt that it was good to have this confirmation. All the teachers from School C followed through the process for the 2007 tests and all presented at the sharing conference. The teachers at School D were initially eager to analyse the results for them selves to find areas that they could improve on. However, their good intentions and enthusiasm for the project did not translate into analyses and remedial programs. The severe lack of administrative staff at the school made it difficult for the school to carry out the data analysis tasks (teaching principal was the only one really involved with the project) and only one teacher completed all stages (but did not present at the sharing conference because of timetable problems).

Teacher Perception of the Project

In surveys, all teachers marked 4 or 5 out of 5 for usefulness, indicating that they felt that the process was useful in some way. However, although 8 of the 13 participating teachers attempted to complete to Stage 5, in practice, 5 teachers were not willing to spend the out of school time to complete their analyses, some because it was taking too long, others because they felt it should not be their job, and one because it became too difficult. Many of the teachers seemed to be in a mind set of just wanting classroom activities or resources rather than wanting to understand the theories behind the ideas. This made the delivery of some aspects of the project difficult as they seemed to 'tune out' or not really engage with the aspects of the project that they thought they would not directly use in the classroom. Feedback showed teachers giving highest priority to practical hands on activities that they could immediately use in their classrooms.

The teachers who did the work appeared to reap the benefit. All the teachers who completed Stage 5 and attended the sharing conference felt that the process had been worthwhile and beneficial to their teaching and had generated an improvement in the students' outcomes in the selected intervention area. Two teachers did restate the position that the project involved a lot of time just to reaffirm what they already knew about the students' knowledge. They seemed to miss the idea that the test analysis not only indicates areas of weakness and strength, but provides detail for the intensive follow up interventions that enhance students' mathematics outcomes. Some teachers did raise three other considerations. First, they were concerned with the relevance of some items in the tests to what they considered important in their students' mathematical knowledge, and felt their students' responses to these items were unimportant. Second, they repeated their difficulties with test-item language and context not being Indigenous and rural or remote. Third, they felt, at times that the tests contained insufficient questions on specific concepts to determine if students understood the concept, or were those particular questions difficult or not true indicators of the students actual knowledge in that area.

Conclusions

Aspects of Numeracy testing is a difficult time for students and teachers. The students feel stressed and very disappointed when their results do not meet their expectations, a result that was higher in schools with larger Indigenous populations. The teachers feel that they are being judged for results that they believe are largely out of their control, although this may not be as true as they feel. This leads to feelings of disempowerment.

All the 8 teachers who completed all stages (all 3 from Schools A and C and 1 each from Schools B and D) stated that they had gained a lot from the process and so had their students. The 7 who were able to present at the sharing conference showed effective remedial activities and good student progress. However, the support given to these 8 teachers and the resistance showed by the 5 teachers who did not complete showed the difficulty that would be had implementing a program in which teachers mark and analyse their students' government numeracy tests and use the results to undertake immediate remedial programs. The teachers in this project simply did not have the knowledge and skills to do this. On top of this, some teachers felt that doing this analysis was not their job and were reluctant to do any out-of-class and after-school work to achieve it. For 5 of the teachers, these became huge barriers to overcome.

Some teachers gave the impression that they already knew their students' weaknesses and felt that our process was too long and time-consuming and that the data entry and analysis was tedious. These opinions also stemmed from the teachers' lack of skills and familiarity with basic uses in technology, including spreadsheet software, in which simple tasks required a really long time to complete. In particular, teachers tended to demonstrate insufficient skills in each of the following three areas: (i) data entry (having little idea what numbers were important or what numbers to compare and, therefore, not knowing how to lay out the data); (ii) data analysis (not knowing which scores should be combined and averaged to provide an overview of responses that would help interpretation and not having the skills to use a spreadsheet, e.g., to produce graphs easily); and, (iii) interpretation (having little idea what the graphs and tables of data meant in terms of what students know and not know); and (iv) translation (very little idea of how knowledge of what students know and not know can be translated into effective programs for remediation and/or intervention. In particular, teachers were unsure of what strategies to employ to improve students' understanding in the areas of weakness identified in the data analysis.

Notwithstanding, all the teachers who completed the entire process (from data entry to analysis and implementation of an intervention program) achieved improved student outcomes. Those teachers who returned the follow-up survey from the sharing conference indicated that they would follow the process again in the next year. However, the impression was that perhaps they would not comply with the same level of detail due to time constraints. Unfortunately, because the time and effort involved with the process overwhelmed teachers, they did not have an opportunity to feel more empowered about the testing process. Instead, they demonstrated a "tick-box mentality" whereby they felt as though they had accomplished something with the test results if they had merely looked them over, regardless of what the outcomes actually were.

Members of Deadly Maths have unsuccessfully approached QSA in the past with a request that the *Aspects of Numeracy* tests be redesigned so that they are developed in relation to a diagnostic profile and source of remedial activities that can be accessed by teachers when the QSA's analysis comes out or when teachers produce their own analyses. With so much money being spent on testing, it is an utter waste that the tests do not come with accessible diagnostic and remedial material. This, of course, does not even take into account the issue of the tests themselves and their role in social reproduction of Australia – this has to wait for another paper.

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Recollections of Mathematics Education: Approaching Graduation and 5 Years Later

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As part of an evaluation of the mathematics education strand of a bachelors degree program a sample of a cohort were interviewed on completion of their mathematics education studies, and another sample from the same cohort were interviewed 5 years after graduation. Interviewees were asked about their perceptions of the value of the course and its impacts on their beliefs about teaching mathematics. Comparison of the responses revealed changes in priorities and evidence of the shifting status of memories. There are implications both for mathematics educators and for understandings of the development and structure of teachers' knowledge and belief systems.

Longitudinal studies in education typically involve preservice teachers in their final year, moving into the first or second years of their teaching careers. Levin (2003), in introducing her exceptional 15-year study, lists many examples of studies of this kind and the situation in relation in mathematics education is similar.

Short term views of course effectiveness are also typical, with evaluations of various aspects of mathematics education programs usually conducted immediately at the ends of units or courses (e.g., Beswick, 2006; Hart, 2002). The preservice teachers who provide data for such evaluations have necessarily had limited opportunities to test the ideas and experiences with which they have engaged at university. Furthermore, it is well-established that the main opportunities that preservice teachers have to apply their university learning, namely practica, have the effect of negating, at least to some extent, the changes in their beliefs about how mathematics is learned and is best taught that their studies appear to have achieved (Hart 2002; Beswick, 2006). The problem, identified by Ball (1990), of beginning teachers reverting to teaching practices that are familiar from their own experiences as students is well known. It has been linked with the firm beliefs about teaching that preservice teachers bring to their university studies (Bobis & Cusworth, 1995) and the difficulty of changing beliefs (Lerman, 1997).

This study was designed to take a longer term view of the impact of the mathematics education units in a bachelor of education (B. Ed.) (early childhood and primary) program by considering the views of teachers who had graduated 5 years earlier. The evidence cited above suggested that teaching experience may have had a significant impact, possibly eliminating the initial effects of the units. On the other hand, a longer period in the field may have afforded opportunities for the teachers to reflect more carefully on their university mathematics education units, free of the pressure of being assessed that is inherent in the practicum, and having had time to come to terms with the demands of classroom life. In either case, it was recognised as imperative that teacher education practices be based on evidence of their effectiveness. As the Committee for the Review of Teaching and Teacher Education (CRTTE, 2003) stated, "Teacher education courses need to demonstrate overall quality and effectiveness in preparing highly competent teachers. To meet diverse student requirements, they need to be flexible and responsive." (p. 119). At the same time there are ongoing calls for the adoption of standards for teaching excellence (e.g., Australian Association of Mathematics Teachers, 2006) and moves towards national accreditation of teacher education programs (Teaching Australia, 2006).

With these factors in mind, the research questions that formed the study's focus were:

1. How do graduates of a B. Ed. program perceive the mathematics education strand of their course 5 years after graduating?
2. How do these perceptions compare with those of members of the same cohort prior to graduating?

Theoretical Framework

The theoretical underpinnings of this study are based in understandings of beliefs as incorporating knowledge (Beswick, 2007), and belief systems as described by Green (1971). In this section these ideas are introduced, and elaborated to account for explanations of memory including the phenomenon of forgetting.

Beswick (2007) defined beliefs as anything that an individual regards as true, and this is the definition adopted in this study. Since constructivists maintain that our only access to reality is by way of our senses, there is no basis for establishing the absolute truth of any proposition. Nevertheless, there are understandings that are more or less common in any given society at any given time and that tend to be referred to as knowledge within that context. They are distinguished from beliefs only in being judged to be supported by better or more evidence and to have greater explanatory power in relation to perceived events in the world (Guba & Lincoln, 1989).

It is well-established that beliefs constitute systems rather than existing as isolated entities. Of particular relevance to this study is Green's (1971) notion of the centrality of beliefs. The more central a belief the more intensely it is held as a result of its greater connection to other beliefs, and hence the more difficult it is to change. However, belief systems are also dynamic and beliefs are contextual. The particular beliefs that are most centrally held vary according to the context and this fact can explain apparent contradictions between beliefs professed in one context with those inferred from observations in another (Beswick, 2003). Remembering that beliefs and knowledge are equivalent from a constructivist viewpoint, knowledge that is less relevant in a particular context is less central. That is, it is less well connected with other knowledge. Knowledge that remains peripheral in this way would become increasingly less central by the further loss of connections and at some point could be described as forgotten. Such a view is consistent with the well known fact that the ability accurately to remember, declines with time but that such loss of memory is reduced by review (Basden, Reysen, & Basden, 2002). In terms of belief/knowledge systems, review amounts to bringing knowledge to a place of centrality thereby strengthening its connections within the system.

Beswick (2004) argued that an individual's beliefs about themselves are likely to be among his/her most central. They are also likely to maintain their centrality across a broad range of contexts and hence to be powerful drivers of behaviour in many circumstances. Beswick (2004) used the particular centrality of beliefs about self to explain the intransigence of the beliefs and practice of one mathematics teacher, and Wilson and Demetriou (2007) pointed to the special importance of teacher's identity at the beginnings of their careers. Connections between memory and emotion are likely to be related to the centrality of beliefs about self. For example, Mather and Johnson (2003) found that when subjects reviewed their affective responses to an observed event their ability accurately to recall its details at a later time was impaired, but there was an increased tendency to articulate false memories that were in line with their existing beliefs. Focussing on one's own emotional responses when reviewing an event creates a context in which one's most centrally held beliefs (i.e., those about oneself) are likely to be connected with memories of the event, thereby incorporating these memories inextricably with existing central beliefs. Sfard and Prusak's (2005) equating of identity with stories about the individual concerned highlights the largely social nature of identity and helps to explain the fact that memory is strongly influenced by the recollections of others and that false memories introduced through social interaction are sustained (Basden et al., 2002). In Sfard and Prusak's terms, beliefs about self amount to that part of identity which is the stories that one tells oneself about oneself.

The Study

The study was conducted as part of an evaluation of the mathematics education strand of the B. Ed. at the University of Tasmania. At the time that the participants were enrolled, the strand comprised three half-units of mathematics education offered in years 2-4 of the program. The first two involved weekly 1-hour lectures, and 1-hour tutorials designed to exemplify the lecture content, throughout a 13-week semester. The final component was taught as weekly 2-hour tutorials for a semester. In all cases tutorial numbers were approximately 25 and tutorial activities involved considerable group work on relevant mathematics, and use of manipulatives. In addition to these compulsory units, students could elect to study up to three mathematics education elective modules each comprising a series of six 2-hour tutorials. The three were entitled: *Investigations in Space and Number*, *Problem-solving in Maths*, and *Maths for Middle School*. Almost all students in the cohort elected to do at least one of these and many chose two. This was the first cohort for whom the mathematics education components were the same for all students, regardless of their specialisation (early childhood (K-2), or Primary (3-6)). Each component was designed to integrate mathematical content and pedagogy and, in tutorials, the lecturers endeavoured to model teaching that was consistent with a constructivist view of learning.

Participants. The 15 students interviewed in 2001 reflected the gender proportions of the cohort in that two were males. In 2006/07 one of the eight teachers interviewed was working in a child care centre, and the remainder were employed in government schools. Two were male. Four (three women and one man) had been members of the 2001 sample.

Interviews. The audio-taped interviews in both 2001 and 2006/07 were of 30-60 minutes duration. The questions relevant to this study asked participants to: nominate two things that they liked about the mathematics education strand of their course and two things that they did not like; and describe ways in which they felt the course had impacted their beliefs about the nature of mathematics, mathematics teaching, and mathematics learning. In 2006/07 participants were also asked about their current employment, their specialisation (early childhood or primary), the number and placement in the course of mathematics components, and to describe if and how they had used ideas from the course.

Procedure. In 2001 every ninth student on an alphabetical list of the 141 students enrolled in the final mathematics education component of the program was invited to be interviewed. The next student on the list was invited in cases where the initially invited student declined. In 2006/07 as many of the 2001 sample as could be located were invited to participate and then, using the same list as in 2001, attempts were made to locate successive students following each 2001 participant. The University's student administration mailed invitations to those who could not be located on the publicly accessible government database. Difficulties resulting from out-dated addresses at student administration, name changes, and the fact that many graduates do not find employment in Tasmanian government schools limited the size of this sample.

Interviews in 2001 were conducted immediately following the final mathematics education component of the B. Ed. program, and again in the period from the end of 2006 to early 2007. Several of the 2006/07 interviews were conducted by telephone.

Results and Discussion

Table 1 provides data about the 2006/07 participants' employment, specialisation, and recollections of the number and placement of mathematics components in their course. Only two, Susan and Stuart, correctly recalled the structure of the mathematics strand, and five held positions that did not correspond to their specialisation. The latter fact vindicated the decision that had already been made to ensure that all students studied mathematics education covering the range K-6.

Table 1

Personal data and recollections of the 2006/07 sample

Pseudonym	Number of mathematics education components studied	Where in the course were the maths ed. components?	Current position	B. Ed. Specialisation
*Susan	3 + Maths for middle school	Years 2-4	Teacher Learning Support Leader, K-10, for cluster of schools	Primary
*#Eve	All compulsory units + early childhood maths in year 4	Years 1-4	Teacher, grade 3/4	Early childhood
*#Ewan	Maths for middle school + something else	Year 3	Teacher, grade 9-12 mathematics	Primary
*Clare	8, 1 per semester	Years 1-4	Child carer, K & pre-K	Early childhood
Emily	Core units + maths for middle school	Years 1-4	Teacher, grade 5/6	Primary
#Laura	4	Years 1-4	Teacher, grade 9/10, not maths	Early childhood
#Mandy	Can't remember	Can't remember	Teacher, grade 10, not maths	Primary
#Stuart	3 + a couple of extras	Years 2-4	Teacher, grades 8-10 including maths	Primary

* Also a 2001 participant # current position does not match specialisation

Responses to the question about likes and dislikes in relation to the mathematics strand were coded and categorised as shown in Table 2. The table includes examples of responses in each category and shows the numbers and percentages of responses in each in both 2001 and 2006/07. On both occasions participants most commonly referred to liking the teaching staff, course content, or the nature of activities, however over time it appears that positive recollections of staff became relatively more common while references to liking aspects of content declined. Indeed, content was the most commonly used category of dislike in the follow up interviews. In 2001 dislikes most commonly related to assessment but this was a less prominent concern in the later interviews. The time allocated to mathematics education was not mentioned in 2006/07.

Table 2

Likes and dislikes about the mathematics education strand

		Likes		Dislikes		Likes & dislikes	
Category	**Example(s): like, dislike**	**No. 2001 (%)**	**No. 06/07 (%)**	**No. 2001 (%)**	**No. 06/07 (%)**	**No. 2001 (%)**	**No. 06/07 (%)**
Staff	approachable, this year's tutor	6 (13)	6 (29)	1 (7)	0 (0)	7 (11)	6 (21)
Content	activities you could use in the classroom, 1st year not primary focussed	22 (46)	5 (24)	3 (20)	4 (50)	25 (40)	9 (31)
Nature of activities / delivery mode	Working on tasks as groups, would have liked lectures in 4th year	14 (29)	7 (33)	1 (7)	2 (25)	16 (25)	9 (31)
Assessment	Lesson plan assignments, that we had to do assignments	5 (10)	0 (0)	6 (40)	2 (25)	10 (16)	2 (7)
Time allocation	Should be in 1st year too, 2-hour tutes a bit tiring	0 (0)	0 (0)	4 (27)	0 (0)	4 (6)	0 (0)
Other	I just like maths	1 (2)	3 (14)	0 (0)	0 (0)	1 (2)	3 (10)
Totals		48	21	15	8	63	29

The nature of positive comments about staff were similar on both occasions and included references to the lecturers "relating to us like people", using humour, being enthusiastic and passionate about mathematics, and not being patronising or intimidating. Many of these were emotionally laden in that they related to the participants' own feelings as they engaged in the course. Several comments were indicative of course contributing to participant's identities in relation to mathematics. For example,

> At school I was labelled and pigeonholed as brilliant at English and all that and just vegie maths … so when I got to Uni and had to do maths I was really, really nervous and (Maths Ed. lecturer name) made us draw something in the first lesson about what we feel about maths and I drew someone with their mouth open and scared. So it's changed my whole perspective and it has also made me feel fantastic about teaching maths. (2001)

It is not surprising that such emotionally focussed, and central (in that they relate to self) recollections were retained.

Criticisms of course content in 2006/07 included "not learning all about the content (of the Tasmanian curriculum)" (Eve), that the course "didn't always cover how to introduce something for the very first time" (Emily), provided, "nothing to help with teaching high school" (Laura), and that "more 7/8 and less early childhood would have helped" (Mandy). As shown in Table 1, Laura and Mandy were both teaching in secondary schools.

Responses to being asked about the impact of the course on their beliefs are categorised in Table 3. All of 60 statements made in 2001 described changes in beliefs attributed to the course and all were in line with the aims of the course. However, in 2006/07, four of the 36 statements described the course in terms of building on or confirming existing beliefs. Three of these statements were made by Susan (see Table 1). Comparison of her perceptions of the impact of the course on her beliefs in 2001 and 2006/07 suggests that, with time, the novelty of new ideas had faded to the extent that they were perceived as having been always there. In 2001 Susan said,

> I used to … think maths was purely memorising, like you'd just memorise all these formulae … Whereas now I think of it more as problem solving and more patterning.

In 2006/07 she said,

> Through doing college maths (year 11/12) I'd started to develop that understanding of maths as being patterns and it really built on that I'd say, just confirmed that maths was enjoyable and also helped me further explore the pattern based maths.

Susan's view of mathematics as patterning had become so enmeshed with her other beliefs about mathematics and mathematics teaching and learning that when asked to reflect on the impact of the course she could not disentangle ideas originating there from others.

Table 3

Perceived impacts on beliefs

Category	Example(s)	No. 2001 (%)	No. 06/07 (%)
Importance of maths	I can see the importance of it more	1 (2)	1 (3)
Multiple methods	There are different ways to get to the same end point	2 (3)	2 (5)
Enjoyment	It doesn't have to be boring; Maths teaching can be enjoyable	6 (10)	3 (8)
Student diversity	Students are very different; I teach children not grades	4 (7)	7 (18)
Applications, relevance, hand-on tasks	I think statistics and percentages are useful but you don't do algebra everyday; children learn via real life experiences	11 (18)	6 (17)
Conceptual understanding, problem solving	I've looked more deeply into the thinking aspect of maths; maths teaching should be about developing conceptual understanding	9 (15)	7 (19)
Memorisation, repetition, text books, intelligence	I used to think maths was about right and wrong answers; used to think you just use text books and chalk	10 (17)	3 (8)
Confidence, competence	More confident now; I feel fantastic about teaching maths	7 (12)	1 (3)
Comparison with own experience	It doesn't have to be taught the way it was taught to me	3 (5)	1 (3)
Teacher	Teacher enthusiasm is important	1 (2)	0 (0)
Other	I had negative beliefs because of experiences at high school; my previous beliefs were completely wrong	6 (10)	5 (15)
Totals		60	36

In 2006/07 there were also two critical responses, both related to student diversity which was one category for which the number of responses increased between the two sets of interviews. The two statements were, "... hasn't helped me understand how children think about maths" (Laura), and "there wasn't much about how to deal with that (differences between students)" (Stuart). In its 2007 survey of beginning teachers, the Australian Education Union (AEU) found that 69.1% of respondents believed that preservice teacher education had not adequately prepared them to teach diverse groups of students but this issue did not feature among their most commonly held concerns. Instead, workload, pay, behaviour management, and class sizes were of highest priority (AEU, 2008). This is in line with the preoccupation of preservice and beginning teachers with behaviour management evident in the research literature (Bobis, 2007). The concern for professional learning expressed by 31.7 % of respondents (AEU, 2008) may or may not have included a perceived need for professional learning related to catering for student diversity, and caution is needed in comparing these data with the responses of just eight teachers interviewed in the current study in 2006/07. However, it seems that, for these teachers, after 5 years of experience, behaviour management may have subsided as an issue to be replaced with a heightened concern about how to teach individual students. If this is the case then it may be that after the beginning phase teachers are able to revisit ideas encountered at university.

The 2006/07 interviews provided some evidence that this might be the case. Five of the eight interviewees reported having used and found helpful, ideas that they learned at university. In contrast to this, Eve said she had tried some ideas in her first year but they'd not "worked". Eve and Laura were the only teachers who had discarded all of their university notes and materials, and Mandy had not had the opportunity to teach mathematics but still had "some things". It was evident that some had simply used various activities and tasks but some also provided encouraging indications of having taken deeper lessons with them. Examples included the following:

> When I was a beginning teacher, whenever I was introducing a new topic … I'd go back to the notes and remind myself of what the key concepts were that I should be focussing on developing … use some ideas from the tute notes that I thought were relevant to the groups I was working with and build on those. (Susan)

> All you want to do when you are at university is go out and teach and you really are not so enthused about the theoretical aspects of it, but then as you become a teacher you realise that you need all that theoretical stuff actually, you know, it's got to be in the back of your mind when you're actually teaching and planning and assessing kids. (Emily)

> I actually get the kids to use a lot of reflective journal writing these days because I find that that actually tells me more about what they understand and what they don't understand, about their mathematical thinking … (Emily)

> I can see the value in a lot of the activities I'm doing that maybe I couldn't have unpacked before. (Clare)

> A lot of water has passed under the bridge since then. I can remember (Maths Ed. lecturer name) saying that the process was extremely important and I found out since that she was 100% correct. (Ewan)

Conclusion

Overall the mathematics education strand of the B. Ed. course was perceived positively by this cohort, both immediately at its completion and 5 years later, and it appears to have had a lasting impact on the thinking of at least some of them. It was possible to reconcile relevant findings about memory with Green's (1971) description of beliefs systems, accompanied by a broad definition of beliefs, and this was useful in explaining the kinds of things that were more likely to be remembered. Essentially, knowledge that becomes central in the sense of having many connections with other beliefs, and particularly to those about self, is most likely to be retained. This includes knowledge that has an emotional meaning to the individual concerned and that constitutes part of his/her identity. There was also evidence that such knowledge becomes so entangled with existing knowledge that its source is lost. Although this is not a concern in terms of improving mathematics teaching, it is probably not helpful in developing positive views of preservice teacher education in the wider community. Furthermore, although not evident from this study, Basden et al.'s (2002) finding concerning the social transmission of distorted recollections suggests that participation in a community, such as a school, where inaccurate memories of teacher education (which were noted in this study) are likely to be shared, could lead to such beliefs becoming widely held.

Only a small sample of the cohort was interviewed in 2006/07 and the selection process militated against the inclusion of graduates working outside of Tasmania, in Catholic or independent schools, or not working at all. Although many encouraging comments were made it was not possible to observe teaching and thereby gain further insight into what the various comments meant. Nevertheless, this study has provided a rare long term evaluation of a primary mathematics teacher education program and offers some encouragement that the preoccupations of beginning teachers may give way to a focus on more substantive issues of practice for at least some teachers. Further research of this nature is needed, including studies in which data collection includes classroom observations and is continuous over many years.

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Using Paper-Folding in the Primary Years to Promote Student Engagement in Mathematical Learning

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Traditionally, craft activities have been incorporated into early childhood mathematical learning experiences as an effective means of fostering curiosity and interest, and introducing abstract mathematical concepts through the use of concrete materials. However, in the primary schooling years creative and active mathematical learning experiences often give way to approaches such as computational drills and rote learning which are less than inspiring for students and often do not result in meaningful understanding. This paper shows that craft activities, in particular paper-folding, can also be valuable in the primary years as a means of promoting affective, behavioural and cognitive engagement in the mathematical learning.

In the primary classroom, the inclusion of paper-folding activities in mathematics can offer an appealing and creative means of addressing some of the goals for teaching and learning mathematics in the 21st century as identified by English (2002), namely: engaging students effectively in mathematical modelling, visualising, algebraic thinking and problem-solving. Paper-folding activities can provide a hands-on, active experience that contributes to the development of mathematical ideas, thinking and concepts; skills in communicating mathematically; and group interaction skills. The particular focus of this study is to investigate how paper-folding in primary mathematics can also be used to promote affective, behavioural and cognitive engagement in mathematical learning.

Theoretical Background

Essential to arguing the use of craft activities, most specifically paper-folding, as a valid approach to be incorporated into effective mathematical teaching and learning strategies, is the view that a learner can engage with mathematical concepts and construct their own knowledge through building and creating objects. The knowledge construction that takes place when learners build objects is emphasised in the constructionist learning theory. The theory of constructionism emerged in the 1980s from work undertaken by Seymour Papert. It builds on the "constructivist" theories of Jean Piaget that knowledge is not simply transmitted, but actively constructed in the mind of the learner. As an extension of this, constructionism suggests that learners are particularly inclined to generate new ideas when they are actively engaged in making some type of external artefact, which they can reflect upon and share with others.

> Constructio*n*ism - the N word as opposed to the V word - shares constructivism's connotation of learning as "building knowledge structures" irrespective of the circumstances of learning. It then adds the idea that this happens especially felicitously in a context where the learner is consciously engaged in constructing a public entity. (Papert, 1991, p.1)

Constructionism involves two intertwined types of construction: the construction of knowledge in the context of constructing personally meaningful artefacts, which Papert (1980) referred to as "objects-to-think-with" (p. 11). A second important aspect of a constructionist learning environment is audience, emphasising that knowledge construction is also embodied in the creation of an artefact which will be viewed and valued by others: "what's important is that they are actively engaged in constructing something that is meaningful to themselves [and] to others around them" (Resnick, 1996, p. 281).

Review of Literature

Student Engagement

A prevalent view of engagement to be found in the literature is that it comprises three interrelated components: behavioural, affective, and cognitive. Affective engagement refers to a student's emotional responses when participating in a learning activity and may be demonstrated through their enthusiasm, optimism, curiosity or interest (Chapman, 2003; Fredricks, Blumenfeld, & Paris, 2004). Student behavioural engagement has been found to be evident in their involvement in learning tasks, together with effort and persistence (Russell, Ainley, & Frydenberg, 2005). The third component, cognitive engagement, has been defined by Helme and Clarke (2001) as "the deliberate task-specific thinking that a student undertakes while participating in a classroom activity" (p. 136).

The influence of task factors is an aspect of student engagement that is of particular relevance in this study. Research undertaken by Marks (2000) identifies that *authentic* tasks contribute strongly to student engagement. Additionally, Helme and Clarke (2001) argue that engagement is more likely when students work on *novel* tasks and a strong association between *challenging* tasks and student engagement has also been identified (Marks, 2000). Finally, Blumenfeld and Meece (1988) emphasise the significance of *hands-on* tasks in promoting engagement in primary students.

Paper-Folding in the Primary Mathematics Classroom

The literature reveals many benefits that arise from including paper-folding activities in primary mathematics classroom. The principal benefit is that it contributes to the development of mathematical ideas and thinking, and the understanding of mathematical concepts (Cornelius & Tubis, 2006). Paper-folding lends itself ideally to spatial visualisation and geometric reasoning (Cipoletti & Wilson, 2004) but is also valuable in enhancing problem-solving skills (Youngs & Lomeli, 2000). Paper-folding also provides a range of opportunities for students to develop their language skills and proficiency in communicating mathematically (Cipoletti & Wilson, 2004). Sze (2005) identifies the value of paper-folding in promoting the correct use of geometric terminology, whilst Jones (1995) suggests that paper-folding can also be used to encourage writing in the mathematics classroom. Some further benefits that arise from the use of paper folding in the primary mathematics classroom are that it encourages group interaction and cooperation (Levenson, 1995) and promotes the development of fine motor skills and manual dexterity (Tubis & Mills, 2006). Of note is that investigations regarding the value of paper-folding in promoting *student engagement* in mathematical learning are seemingly absent in the previous research.

Research Focus and Significance

This study describes a constructionist approach to teaching and learning whereby paper-folding is used to create and explore mathematical models and investigates whether using this approach in primary mathematics can promote affective, behavioural and cognitive engagement in the mathematical learning.

The significance of this study is, firstly, that it extends the range of benefits that can arise from including paper-folding activities in the primary mathematics classroom in that student engagement in the mathematical learning inherent in such activities has not been addressed in previous studies. Additionally, paper-folding has been incorporated into mathematical learning activities in this study in original and inventive ways not readily evident in previous studies. In some cases (Coad, 2006; Turner, Junk, & Empson, 2007), whilst the paper-folding activities incorporated mathematical explorations, they do not involve the creation of a model. Whilst other examples (Cipoletti & Wilson, 2004; Cornelius & Tubis, 2006) did involve the creation of a model to be used for various mathematical investigations, once the explorations had been completed the paper model afforded no on-going mathematical purpose. This study takes the use of paper-folding one step further in that the students create various models that are used for mathematical explorations during the course of the learning activities, including ultimately the pieces of the Soma Cube puzzle, which then continues to offer a mathematical challenge as the students explore solutions to the puzzle. Finally, the significance of this study also lies in its contribution to research about constructionist approaches to teaching and learning. Kafai and Resnick (1996) contend that constructionist researchers are continually seeking new educational activities and tools to expand their endeavours. Previous constructionist research has focused overwhelmingly on the digital learning environment (Papert, 1980; Stager, 2001). This study contributes a fresh perspective on new constructionist educational activities through a non-digital construction medium: paper.

Research Method and Procedures

A classroom teaching experiment approach to research was adopted for this study. This research method involves a teaching agent (either the researcher or the collaborating teacher) conducting a sequence of teaching episodes over a period of time (Cobb, 2000; Kelly & Lesh, 2000). What transpires during each episode is recorded and this data is used in preparing subsequent teaching episodes, as well as in conducting a retrospective analysis of the teaching experiment (Steffe & Thompson, 2000). In this study, the research site was a Year 5 classroom. This site was purposefully selected because the classroom teacher and the researcher had a well-established communicative and collaborative relationship. The class consisted of 26 students, 14 boys and 12 girls, aged between 9 and 11 years. The mathematical ability of the students in the class varied significantly. Some students would be considered to be of high intellectual potential ("gifted") whilst others had diagnosed learning disabilities. During the classroom teaching experiment the researcher acted as the teacher, to order to draw upon personal expertise in the paper-folding techniques that were required. The teaching sequence, which was devised and reviewed in collaboration with the classroom teacher, incorporated three modules and comprised one or two teaching episodes each week over eight school weeks.

In the first module, the students learned how to construct paper cubes using unit origami techniques, then small groups of students paired with small groups from another Year 5 class to teach them how to construct the paper cube. The mathematical focus in this module was to review of the features of geometric figures such as faces, vertices and edges, together with line and angle properties. This occurred incidentally whilst the construction techniques for the cube assembly were being demonstrated. Additionally, the appropriate use of mathematical language was emphasised, particularly in designing the set of written instructions for the peer teaching episode. The second module involved the students extending their paper cube to create paper-folded rectangular prisms in order develop further their paper-folding skills. The mathematical learning in this module took the form of an investigation that focussed on patterns and algebraic reasoning. In this investigation the students exploring the "growing" pattern associated with the number of squares to be found on the faces of the rectangular prisms they had created by joining one, two or three cubes, that is the sequence $\{6,10,14,\ldots\}$. The students were asked to predict, extend and generalise the pattern by considering the total number of squares on the faces of larger rectangular prisms, without constructing such prisms from paper. The mathematical learning in the third module focussed on spatial visualisation and geometric reasoning. The module commenced with the students constructing one piece of the Soma Cube puzzle, a three cube L-shaped model. This model was used to examine the property of irregular figures: the possibility of joining two points on the surface of the figure with a line that lies outside of the figure. Using small wooden blocks, the students investigated the six different irregular figures comprising four cubes, which are the remaining pieces of the Soma Cube puzzle. This module culminated in the students working in groups to paper-fold the seven pieces of the Soma Cube puzzle. Rather than distributing written and diagrammatic instructions for the construction, each group was just provided with a colour photograph of each of the seven pieces. The successful completion of the Soma Cube pieces required that the students utilise their skills in spatial visualisation to construct the three-dimensional object associated with a two dimensional photograph. To culminate the teaching sequence, the whole class collaborated to plan and organise a Paper Folding Expo for parents and students from other classes, where they exhibited the models they had created and demonstrated the skills and understandings they had acquired.

Data Sources and Analysis

Two data sources were utilised to provide evidence of the students' affective, behavioural and cognitive engagement in the mathematical learning inherent in the activities. Firstly, qualitative student self-report data in the form of written reflections were collected at the conclusion of each module in the teaching sequence. In the written reflections, the students were provided with guiding questions designed to elicit their affective, cognitive and behavioural responses to the learning activities. The indicators of student engagement used to analyse the written reflections have been drawn from overlapping elements found in the work of others. Several previous studies have identified indicators of affective engagement in learning activities, such as: enthusiasm, optimism, curiosity and interest (Klem & Connell, 2004), together with enjoyment, happiness and excitement (Marks, 2000). Concentration and persistence have been characterised as indicators of behavioural engagement (Skinner & Belmont, 1993), and student claims to have learned something is one indicator of cognitive engagement utilised in the work of Helme and Clarke (2001). The students' written reflections were initially coded using these previously identified indicators for affective, behavioural or cognitive engagement. Further analysis of the written reflections highlighted additional indicators of student engagement, seemingly absent in previous research. Helme and Clarke (2001), however, recommend that sole reliance on student self-reporting ought to be avoided when seeking evidence of student engagement. Some researchers have also used student work samples to determine engagement in learning tasks (Chapman, 2003). Student work samples, completed in the second and third modules, contribute particularly to the evidence of cognitive engagement in this study. These work samples were analysed to illuminate the students' willingness to explain procedures and reasoning which has been identified as another indicator of cognitive engagement (Helme & Clarke, 2001).

Results and Discussion

Affective Engagement

The most apparent evidence of affective engagement associated with the paper-folding activities was the sense of *enjoyment* that the students expressed: "It is fun and you don't feel as if you are learning maths at all"; and "I didn't even notice it was maths I was having so much fun". For one student paper-folding also enhanced an *interest* in mathematics: "I enjoy it because it is something I wouldn't usually learn because it teaches me many maths skills I didn't learn in school maths". Pride and contentment as indicators of affective engagement are seemingly absent in the findings of previous studies. Yet in this study a number of students expressed their *contentment* upon completing their first cube: "…it was a good moment"; and *pride* in their achievement: "I felt really good and proud of myself".

In reflecting the key principles of the constructionist approach to learning, the creation of artefacts and the opportunities to share them with an audience elicited the most significant expressions of affective engagement. Upon successfully completing their first paper cube many students clearly indicated their *happiness* with the artefact they had created: "I felt happy because it was hard"; and "I felt glad and happy because I thought I was never going to make it". Whilst constructing the first paper cube was a significant accomplishment for many students, completing the Soma Cube pieces and solving the Soma puzzle was the ultimate goal and the *happiness* and *excitement* expressed in the student reflections once this had been achieved was palpable: "I felt really excited and really happy that we accomplished that goal"; "I WAS SO EXCITED!!!!!"; and "Soooo happy! I was laughing so much I cried! The best feeling!". The sense of excitement and achievement in the classroom as one-by-one student groups solved the Soma Cube puzzle was almost euphoric.

The data also exposes the significance of audience in promoting affective engagement. The student responses to the peer teaching experience range from expressions of *contentment*: "I felt good to share my knowledge with others"; *happiness*: "I felt happy because I had taught other people things that could make them happy"; *pride*: "I got a bit frustrated, but in the end I felt proud and glad"; and simply *enjoyment*: "It was fun teaching them". The Paper Folding Expo elicited an equally rich range of indicators of affective engagement in the student reflections. Expressions of *pride* in their achievements were common: "I felt proud that other people looked at what our class had achieved"; and "I felt proud and special because only we knew". Others expressed their *enjoyment* of the occasion: "I really had fun showing my mum and dad things we've made in our paper-folding class"; their *happiness*: "I felt happy that I could show my skills to my family"; and their *contentment*: "I felt good because everyone was congratulating us on our good job".

Behavioural Engagement

The data revealed that, by their own recognition, the paper-folding activities provided the students with a challenge: "It's fun but hard to teach and learn". However, *persistence* was evident as a key indicator of behavioural engagement with the challenges in the learning activities: "It can be difficult the first time but after a while it's easy"; "I felt glad and happy because I thought I was never going to make it"; and "You don't always get it the first time you try". *Concentration* was also a significant factor in the students' willingness to persist in the face of the challenges that arose, and photographs taken during the learning activities provided clear visual evidence of student concentration levels.

Cognitive Engagement

Indicators of cognitive engagement, namely a willingness to explain procedures and reasoning and most evidently student claims they have learned something, provide evidence that the students connected with the mathematics inherent in the paper-folding activities. Many students identified mathematical concepts they had learned: "I learnt about irregular shapes, normal shapes, angles and 3D shapes"; and "I learnt number patterns". The responses also indicated that the students had learned something about the nature of mathematics: "That all maths has patterns"; or "Maths can be stuff other than numbers and sums". Some students identified that they had learned new mathematical skills: “I learned how to make a cube and explain how to do it”; and “I learned how to draw shapes on isometric paper”, whilst others recognised more generic skills they had learned through the paper-folding activities, such as: “How to write instructions”. One student clearly articulated his cognitive engagement by simply declaring “I put my mind to it”. Analysis of the students’ written work samples provided evidence regarding their willingness to explain procedures and reasoning. Most students were prepared to explain the procedures they used in the mathematical investigations, for example how they determined all of the irregular figures that comprised four cubes. Some described a trial and error approach: “Get some blocks and shuffle them around to see what irregular shape I can get”. Others were more systematic: “I would make a regular shape and then move 1 block, then another block and then another block to make irregular shapes. Then I recorded it”.

The written work samples, though, were less conclusive with respect to the students’ willingness to explain their reasoning. For example, the students were asked to predict the total number of squares on the faces of a rectangular prism that comprised four cubes. In extending the growing pattern {6,10,14,…} almost all of the students correctly offered eighteen squares as their response. However, only a few were able to explain their reasoning: “Each time you add on a cube the square faces go up by 4 and the last one was 14 and 14 + 4 = 18”; or “There will be 4 square faces on each side which is 16 and then 2 bottom and top makes 18”. One the other hand, whilst many other students were able to correctly extend the numerical pattern they offered only vague explanations of their reasoning: “I followed the pattern and added 4”; or “The number goes up by 4”. Possible reasons for the unwillingness of the students to clearly explain their reasoning in writing can be found in the work of others. In her study, Warren (2005) encountered similar difficulties in encouraging primary students to rationalise and generalise patterns in writing, instead the students in that study found it easier to verbalise generalisations. Others have also pointed out that encouraging students to write in mathematics can be difficult, and that success in doing so appears to be determined by the students’ ability to express their thinking in the written word (Anderson & Little, 2004).

Sustained Engagement

This study does not make the claim that paper-folding is a guaranteed means of ensuring that mathematics will be more engaging and meaningful for all primary students. Whilst the data in this study have revealed the value of paper-folding in promoting affective, behavioural and cognitive engagement in mathematical learning, not all of the students will wish to approach mathematical learning through this means on future occasions, and nor should they. However, some students in this study continued to explore and extend the classroom learning activities in out-of-class contexts, exposing an additional mode of engagement: sustained engagement. In describing sustained engagement, Eisenberg and Nishioka (1997) use “musical instruments” as an analogy. They explain that whilst most students have the opportunity to try a musical instrument during their schooling, the instrument will become a passion for some. Paper-folding could be similarly considered as a mathematical “instrument” capable of fostering a long-term passion for creative mathematical practices.

Over the course of this study the sustained engagement of some students with the paper-folding activities became increasingly apparent. It was first evident when a group of students asked that I hold optional lunchtime sessions so they could learn to construct more complex models. Included in this group was Tom, a student with a diagnosed learning disability and a form of autism. The collaborating teacher was amazed that Tom not only attended an optional lunchtime session but successfully completed a reasonably complex model with little assistance. Tom had never been known to remain in the classroom for a minute longer than he had to! Several weeks later Tom proudly brought to school a “super-sized” cube that he had designed and decorated at home. Other members of the lunch time group also continued exploring and extending their skills at home with students excitedly bringing into class increasingly complex models, some exquisitely small, and others large and decorative. Matt was a particular student whose sustained engagement with the paper-folding classroom activities transformed into a growing passion for this “mathematical instrument”. Matt’s skills and fascination blossomed over the teaching sequence and in recognising his passion Matt’s parents purchased several of the resource books I had been using in order for him to continue on his paper-folding journey. At the conclusion of the study, Matt’s parents wrote: “It’s been great seeing him come alive with the paper folding”.

Conclusion

The purpose of this study was to investigate the value of paper-folding in the primary mathematics as a means of promoting affective, behavioural and cognitive engagement in mathematical learning. Drawing on overlapping elements in the work of others, a range of indicators were utilised to discern evidence of student engagement in the data sources: written student reflections and student work samples. Some indicators of affective engagement, as identified in previous research, were similarly evident in this study, specifically: enjoyment, interest, happiness and excitement. Further, additional indicators of affective engagement, seemingly absent in the findings of previous studies have also been revealed, namely: contentment and pride. The data have also exposed persistence and concentration as the key indicators of behavioural engagement with the learning activities and the students' willingness to describe their mathematical learning and explain the procedures that they used in the mathematical investigations provide strong evidence of their cognitive engagement with the mathematics inherent in the paper-folding activities. Finally, the study has revealed an additional mode of engagement in some students, sustained engagement, evident when they continued to explore and extend classroom learning activities in out-of-class contexts.

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The Case of Mathematical Proof in Lower Secondary School: Knowledge and Competencies of Pre-service Teachers

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A case study of the preparedness of pre-service mathematics teachers to teach proof at lower secondary is reported. Data collected by questionnaire and problem-centred interviews were subjected to in-depth qualitative content analysis. Knowledge and competencies demonstrated by two future teachers are examined. Both displayed high affinity with proving and a formalist image of mathematics but their inferred abilities to convey this affinity to students at this point in their careers differed. Implications for teacher education and university mathematics are outlined.

A renewed interest in proof and proving in school can be observed worldwide. This is reflected by the planning of an ICMI study focusing on proof. In the call for papers of this study Hanna and de Villiers (2008) state that this "renewed curricular emphasis on proof" (p. 1) in school mathematics is occurring at all grade levels in many parts of the world. However, others point out that "many school and university students and even teachers of mathematics have only superficial ideas on the nature of proof" (Jahnke, 2007, p. 80). Often secondary students prefer procedures of validation in mathematics that are empirically based even when they know they are expected to produce a deductive argument (Healy & Hoyles, 2007; Heinze & Kwak, 2002). The fact that this preference occurs for even high performing students leads Jahnke (2007) to conclude that "the usual teaching of mathematics is not successful in explaining the *epistemological meaning* of proof" (p. 80). Indeed, when the word "proof" is used in schools this might not mean a formal mathematical proof. As in mathematical research these days, proving in school "spans a broad range of formal and informal arguments" (Reiss & Renkl, 2002, p. 29).

Thus, as Hanna and de Villiers (2008) point out, "mathematics educators face a significant task in getting students to understand the roles of reasoning and proving in mathematics" (p. 1). This applies as much to tertiary mathematics educators as it does to teachers in schools. One of the questions raised by Hanna and de Villiers in the ICMI study discussion document is: "How can we design opportunities for student teachers to acquire the knowledge (skills, understandings and dispositions) necessary to provide effective instruction about proof and proving?" To be able to do this it is first of all necessary to be able to determine the competencies and the nature of the knowledge (mathematical or pedagogical content) pre-service teachers possess with regards to proof. These could have implications for both teacher education and tertiary mathematics courses that pre-service teachers take. Some studies involving pre-service teachers have been carried out (e.g., Selden & Selden, 2003; Simon & Blume, 1996; Stylianides, Stylianides, & Philippou, 2007). There is, however, a dearth of research literature in this area especially in the case of pre-service mathematics teachers for lower secondary school.

Selden and Selden (2003), for example, investigated the ability of 8 undergraduate students (including 4 secondary pre-service mathematics teachers) in a "transition to proofs" course (p. 18) to judge the correctness of four student "proofs" of a mathematical statement. The researchers concluded that the undergraduate students tended to focus on the surface features of arguments and their ability to determine whether an argument proved a theorem was limited. They point out that eventually most mathematics students learn to do this but cannot say when this happens. "Validation of proofs is part of the implicit curriculum, but it is a largely invisible mental process. Few university teachers try to teach it explicitly" (p. 28). This is somewhat of a concern for the preparation of secondary mathematics teachers who will be expected at some point in time to be able "to judge the correctness of their own students' proofs or novel solutions to problems" (p. 29). A further complication is that the reducing nature of mathematics requirements for entry into teacher preparation courses might mean that many of these future mathematics teachers have ceased their mathematics studies before they have learnt, or even experienced, how to carry out validation in a mathematical context.

In the recent report from the international study, *Mathematics Teaching in the 21st Century* (MT21), where the goal was to examine how lower secondary mathematics school teachers were prepared to teach in six countries (Schmidt et al., 2007), there was reference to one of the objectives of mathematics being "student understanding of mathematics as a formal discipline" (p. 17) but no explicit mention of proof. There were several items focussing on proof in this study but these were not mentioned in the report. As a supplement to MT21, a collaborative study between researchers at universities in Germany, Hong Kong, and Australia has begun and one of the initial areas of interest is pre-service teachers' preparedness to teach argumentation and proof in lower secondary school.

In the various Australian state curricula, the emphasis on proof differs at the lower secondary level. In the Victorian Essential Learning Standards (VELS) (VCAA, 2005), for example, proof forms part of the working mathematically dimension, as well as being mentioned in the various content dimensions. In working mathematically, level 6 (usually completed by students in Years 9 and 10), students are expected to

> formulate and test conjectures, generalisations and arguments in natural language and symbolic form. … They follow formal mathematical arguments for the truth of propositions (for example, 'the sum of three consecutive natural numbers is divisible by 3'). (p. 37)

Within the structure dimension at level 6, students are expected to "form and test mathematical conjectures; for example, 'What relationship holds between the lengths of the three sides of a triangle?'" (p. 36) whereas Pythagoras' Theorem is mentioned in the space dimension learning focus (VCAA, 2005). The following case study examines the preparedness of pre-service teachers to teach such content.

The Case Study

Participants

In the case study to be described here the sample from one Australian site (a university located in Victoria) is the focus. Mathematics entry requirements for the secondary teaching course at this university exceed those required for teacher registration in Victoria. Eleven (out of a possible 14) pre-service secondary mathematics students, at the end of their year-long postgraduate diploma course, volunteered to participate in the study. Only 9 of these students completed the section of the data collection which dealt with argumentation and proof. From these 9, 6 were chosen for in-depth interviews. Interviewees were chosen on the basis of their mathematical competencies and pedagogical content knowledge being consistent with their espoused beliefs about mathematics and the teaching of proof or there being inconsistency between these. The responses and experiences of 2 of the interviewed students are the focus of this paper.

Instruments

The study used a questionnaire (approximately 90 minutes to complete) for all participants followed by a problem-centred interview (approximately 60 minutes) for respondents selected for in-depth study. Both instruments evaluate the professional knowledge of future mathematics teachers and were designed and evaluated by the third author for his ongoing doctoral project (cf. Kaiser, Schwarz, & Tiedemann, 2007). The items in the questionnaire are open requiring extended written answers. These items bridge the domains of mathematical knowledge, pedagogical content knowledge, general pedagogy, and mathematical beliefs. Items test several components of the professional knowledge domains as well as beliefs, linking different knowledge facets in an action-based design.

During the interviews, the pre-service teachers were asked about why they chose to be a teacher, their beliefs about mathematics and knowledge requirements in general for teaching mathematics in secondary school, and the adequacy of their university courses to address these needs for them. In addition, further questions probing aspects of the teaching and knowledge of the area of focus are included based around problem tasks presented to the interviewee. "The problem-centred interview (PCI) is a theory-generating method that tries to neutralise the alleged contradiction between being directed by theory or being open-minded so that the interplay of inductive and deductive thinking contributes to increasing the user's knowledge" (Witzel, 2000, p. 1).

The mathematical topics which are the focus of both instruments are the teaching of mathematical modelling and argumentation and proof in Years 8-10. This paper will focus only on argumentation and proof. The questionnaire contained one item consisting of several sub-items related to a proof of the proposition that doubling the length of the side of a square also doubled the length of its diagonal and another multi-part item about the proof of the sum of three consecutive natural numbers being divisible by 3. The interview included questions related to the proof of the Triangle Inequality Theorem. As the data collection is still in progress, the actual items have not been released at this point.

Analysis

The questionnaires were evaluated using qualitative content analysis methods (Mayring, 2000) which is a much more in-depth process than superficial quantitative content analysis (Silverman, 1993). Extensive coding manuals were developed for the questionnaire for Australia, Germany, and Hong Kong based on first trials. Joint coding with coders from all three countries was used to develop the coding manual, avoiding the reflection of cultural biases. The coding of data for the Australian sites was done independently by a team of German coders and one of the Australian authors. There were thus two codings in which any inconsistencies were then discussed and resolved. Variables of interest were the affinity variable of affinity to proving in mathematics lessons and the cognitive variables of subject-related adequacy of a formal proof, adequacy in construction of proofs, didactical reflection on proving, and diagnostic competencies with respect to proof.

The responses of the future teachers in the questionnaires and transcribed interviews were used to develop patterns of misconceptions of the future teachers, their strengths and weaknesses in terms of their ability to explain concepts in both deductive and inductive ways, to explore innovative teaching methods as well as to generalise and make use of knowledge to solve related problems. The relationship, if any, to beliefs about mathematics and mathematics teaching was also examined.

The Cases

Based on their responses to the questionnaire, the majority of the students at this site had a high affinity with proving in mathematics lessons and a formalist image of mathematics when it came to proof, “seeing mathematics as an abstract system that consists of axioms and relations” (Schmidt et al., 2007, p. 14). Two of the interviewed students, Ling and Gabby (pseudonyms), were selected as exemplars. They had formalist images of mathematics but showed differences in consistency between demonstrated mathematical competencies and their affinity with proving in mathematics teaching at lower secondary.

Ling has a Bachelor of Science (Honours) from an Asian university, where she majored in Chemistry but her degree included substantial mathematics studies in both first and second year. After graduating she worked for seven years as an industrial chemist in Asia before coming to Australia five years ago. She is now returning to the workforce but as a teacher because teaching is a “good job for a young family”. Ling’s questionnaire responses showed she had the highest possible level of affinity with use of proof in mathematics teaching. She believed that proof in mathematics lessons in secondary school was “important for training students to think clearly and logically and be able to argue the case but it could be too difficult for the general population. However, if it was taught early, perhaps in elementary school, it could become easier in secondary school.” However, the length of time between her formal mathematical studies and her going to teach, and her previous experiences with proof as a mathematics student at university meant she was restricted in her abilities to convey this effectively in her teaching at the point in time when she was about to begin her transition into the teaching workforce.

Gabby has a four-year undergraduate degree in pure mathematics from an eastern European university. The final year not only included mathematics subjects but also included mathematics teaching method, educational psychology and professional teaching practice. After graduation she worked for three years as a secondary school teacher before emigrating to Australia. However, before she could teach in Victoria, she was required to complete a graduate diploma in education, the course she had just completed at the time of data collection. She did not believe she had developed her teaching knowledge during her overseas undergraduate degree at all as “it is not like here so, umm, it’s not as intense as here in Dip Ed .… It was not as useful as this year here.” She also stated her first two years of teaching in a private school only taught her: “What I don’t want to do”. Her third year, this time in a state school, “was absolutely everything” and “nothing can surprise me anymore". Gabby's responses on the questionnaire showed she had a high level of affinity with proof in mathematics teaching. However, in the problem-centred interview (PCI), despite saying "formal proof, that is the proof I like and prefer", she revealed that

> when I was in school I didn't like it and then when I started teaching about it I was insisting on that. I hated it because I was learning all the proof and all congruencies and similarities off by heart because it was so hard to come up with formal proofs. … But proving using rules or axioms or some proved prior axiom, we can use it like working out from this and this. Yeah that's formal proof.

Gabby's most recent mathematical studies, her teaching experiences in Europe, and her previous experiences as a student with proof both in secondary school and university meant she had more resources immediately available to convey this notion effectively in her future teaching but some of her responses were enigmatic. She believed lower secondary students should be exposed to proofs done by teachers but not construct these themselves or even take much from the experience as the following PCI excerpt shows:

> I definitely believe they should be exposed to formal proofs all the time, so they get a feeling of it and being familiar with it. They will say, "Oh it is too hard" and "Why are we learning this?" and "What is this?" And probably, very likely, they are going to copy it only to copy it from the board but I think as long as something stays in their mind anyway. So I think that it is very important.

Competencies in proof construction. Ling exhibited very high competencies in formal proving when using Pythagoras' theorem for proving the proposition about doubling the sides of a square and its generalisation to rectangles but as she revealed in her interview the only theorem she could recall at all was Pythagoras' Theorem. On all other occasions when required to construct proofs whether geometric or algebraic her attempts were categorised as demonstrating low competency. She was unable to show algebraically the divisibility proposition for the sum of three consecutive natural numbers could not be generalised for k numbers. Her response showed she believed it was able to be generalised. When working with the Triangle Inequality Theorem she clearly believed, demonstrating with reference to a triangle, that it was possible to construct a triangle where the sum of the lengths of two sides was equal to the length of the third side. She then used an example of such a non-constructible triangle (with side lengths of 1, 9, and 10 units) to "disprove" the Half Perimeter Proposition (the half perimeter of a constructible triangle is always longer than each side of the triangle). Gabby, on the other hand, consistently demonstrated she had high competency in this area in all contexts. For instance, she was able to use her formal representation of the statement of the Triangle Inequality Theorem using algebraic notation to prove that the Half Perimeter Proposition for a constructible triangle was indeed true.

Mathematical content knowledge about different kinds of proof. Whilst Ling has a very high affinity for the use of proof in mathematics teaching, her own mathematical experiences have in some cases presented a negative view of proof and have not given her the basis to always successfully translate this affinity into her teaching practice as the following excerpt from the interview shows.

> Don't know much proof, didn't learn much about proof during my studies, and I remember I did proof only a very short experience in the first year of … maths in university, undergraduate study, and couldn't understand much of it. The lecturer just says, "Given this statement, this is how you prove it", and then [we] go to tutorial where we're given a set of questions. The tutor just walks around, and he would call any one of us to show the proof. If any can't show, they will just embarrass [laugh] and everyone will go to the classroom with a lot of fear [laugh]. "Have I found the answer or not?" And, um, yes, when anyone mentions proof, it's a fear [laugh] we don't know how to do it. It is too hard. We were shown how to do a proof to a certain statement, but how to get there, we don't understand at all, we just look at it. … No proof at all in high school…so the only experience was in first year of uni, and after that I dropped maths!

Gabby showed a high affinity with using proof in mathematics teaching and her previous experiences in secondary school and university where "we put lots of emphasis on proof" supported this. However, her pre-service diploma course experiences and exposure to pre-formal proofs in the questionnaire and interview problem tasks caused her to reconsider what she considered as "proof". When discussing the use of a rubber band on a board with three non-collinear nails to demonstrate the Triangle Inequality Theorem, she said,

> If I let them do it by themselves, it is not formal proof ... but it is like a certain degree of proving. At the junior level, it is more likely that's the proof for them which is not a proof of course, but this is empirical evidence. And what I have seen here, I have never seen before … It's like when they are proving things with that dynamic computer software, stretching or whatever ... I don't know if that can be considered as proved. I don't consider it as proof but if we do like, I think we can consider it as proof. Presently not, but I should leave a space for not so absolute 'only this is a proof', it might be accepted as a proof as well, the same thing with nails, and with that computer software.

However, she was fully aware from her teaching experiences that an over-reliance on empirical methods was misleading and allowed students to indulge in their preference for validation procedures that were empirical over formal methods (Jahnke, 2007).

> As long as something is measurable, they have no problem, or obviously visible or whatever. So the advantage is that you can actually show how it works. … it's misleading in the sense that students start thinking this is an actual proof, that it's not like evidence how this theorem works or an application. … So if you do that, then "Let's do the formal proof now", well they will start saying, "We already proved it! That's the proof!" So it's very potentially misleading that they got the impression that this is the proof and they carry out that impression like later on, that that's the proof.

Declarative knowledge about the different structures of proving. As a teacher it can be expected that in addition to being able to construct proofs, teachers will need to draw on their mathematical knowledge about the different structures of proving such as special cases/experimental "proofs", pre-formal proofs, and formal proofs (Blum & Kirsch, 1991) and pedagogical content knowledge when planning teaching experiences and when judging the adequacy or correctness of their, and their students', proofs in various mathematical content domains. The term "pre-formal proofs" means "substantial argumentation on a non-formal basis" (p. 184) where "the conclusions must be capable of being generalized directly from the concrete case" (p. 187). An example could be a geometric diagram used as a visual proof, such as the early Chinese proofs of the Gou Gu problem (cf. Pythagorean theorem) (Joseph, 1991, p. 180). Little attention had been given to pre-formal proving in the pre-service course at this university and the term "pre-formal proof" was met by the pre-service teachers for the first time in the questionnaire. A clear example was given but not all students realised it was able to be generalised and thus different from a special case.

Ling's questionnaire responses appeared to show she understood formal proof and possibly special cases, but she did not use these terms explicitly. In the PCI she thought pre-formal proof "would be using trial and error". She then continued: "informal proof is considering specific examples to show your statement is true or not true". With respect to formal proof, she stated that the prover "really [has] to go through these writing steps: If this, then that. Uhm, if this, therefore." Thus, Ling's responses showed only low levels of declarative knowledge about proving structures. Gabby showed evidence of high declarative knowledge about the structures of proving clearly distinguishing between empirical evidence, formal proof, and verbal argumentation that fell short of proof. However, she was unable to see the difference between pre-formal proof and the mere use of special cases. The pre-formal proof presented for doubling the diagonal of a square when its side is doubled, for example, was not sufficient proof for her "because it refers to a single case, in other words, this is proof for the case that is pictured in the associated diagram above. [It] does not cover all cases (squares) in general."

Feedback and diagnostic competencies. For the proof of the sum of three consecutive natural numbers being divisible by 3, a statement describing the proposition was presented along with several Year 9 students' "proofs" of this. The pre-service teachers were asked how they would respond as a teacher to each student in such a way as to both assess the "proof" presented and to provide motivation for further exploration or correction. For a student who had presented special cases to prove the given statement, Ling was able to evaluate the student's ideas: "You have proven for three sets of numbers. Very good." In addition, she suggested a path for the student to consider in moving more towards the notion of a generalisable proof: "Have you thought about other natural numbers that exist? Do you think this will always hold true? Why?" For a second student who provided an adequate algebraic proof, Ling assessed the response and provided meaningful feedback, "Excellent! You have used a generalisation and this is exactly what we do for proofs. It has to hold in general, for all situations." In this case she made no suggestion to help the student progress further. However, for an unfamiliar adequate diagrammatic proof she was unable to evaluate it or provide feedback as she could not understand it. It appears she was not expecting a visual proof to this proposition and hence could not make sense of one when presented. Clearly, when Ling was able to make sense of the student's work, she was able to provide both an evaluative response and suggestions for future progress. Gabby on the other hand was able to judge the responses correctly and provide meaningful feedback to these three students, not at all being perturbed by the unexpected format of the diagrammatic proof: "Very interesting solution. Correct, as well. Really, we can go on and on like that and there would be no remainder. What do you think, how can you prove correctness of the statement for any three consecutive numbers?" However, neither Ling nor Gabby provided meaningful feedback for a fourth student who had indicated that "you can't say" whether or not the statement was true and then attempted to argue why this was so, clearly not showing an understanding of the general nature of proof.

Didactical reflections about proving. To evaluate pre-service teachers' ability to reflect in a meaningful way about proof, they were asked firstly to judge if a pre-formal proof could be sufficient as the only kind of proof in mathematics teaching and explain their position. Ling firmly stated it could not be as "maths is like science, a hypothesis has to go through rigorous tests before it can be accepted as a proof. It has to be systematic." She was clearly interested in conveying an image of mathematics as formal and rigorous. Gabby's answer was coloured by her equating pre-formal proofs with empirical evidence which she clearly thought was not sufficient. Neither Ling not Gabby reflected on the variation in cognitive abilities of students or the complexity of theorems likely to be presented at this level of schooling. As further opportunities to reflect in a didactical manner about proving, the pre-service students were asked in the questionnaire and in the PCI the advantages and disadvantages of a formal and informal proof. Ling was unable to provide any meaningful examples in the questionnaire, whereas in the PCI she mentioned that a pre-formal proof "gives you the confidence that you are right". Gabby, on the other hand, showed high competencies in this area in both the questionnaire and the PCI, noting that pre-formal proofs were "easier to master" and could "show how it [e.g., a proposition] works" but could be "very misleading" if adopted as "actual proof" whereas "deductive (formal) proof is [much] more difficult for students to master, but once they understood it; it is conceptual understanding that lasts and can be retrieved and applied in future easily."

Discussion and Conclusion

Having a high affinity with proving in mathematics teaching at the lower secondary level and possessing an apparently adequate mathematical background well beyond that required for teacher registration are clearly not sufficient preparation for teaching the rather small amount of proof explicitly identified in the VELS. Whilst short pre-service courses must prioritise some topics at the expense of others, clearly time needs to be devoted to ensuring that future teachers are aware of their needs to revisit critical topics such as proof in order to activate dormant knowledge. Although both future teachers in this study demonstrated at least average competencies in the area of diagnosis of misconceptions about the nature of proof and providing meaningful feedback to enable students to develop their understanding of proof from their current status, the authors concur with Selden and Selden's (2003) suggestion that pre-service courses include opportunities for students to reflect on, evaluate, and provide feedback on a variety of student generated "proofs" to propositions at the lower secondary level. We also concur with Blum and Kirsch's (1991) "plea for doing mathematics on a preformal level" providing all students with opportunities to engage deeply with "preformal proofs that are as obvious and natural as possible especially for the mathematically less experienced learner" (p. 186). The present study also suggests that students even with strong mathematical backgrounds from tertiary studies are not necessarily experiencing proof in such a manner that they can convey a complete image of proving at the lower secondary level. Although one of the future teachers realised that students at this level had a propensity to accept empirical evidence and be dismissive of a necessity for "further" proof, her expectations did not include the necessity for students to be able to construct such proofs for themselves or "follow formal mathematical arguments" (VCAA, 2005, p. 37). Both future teachers in this study have a significantly difficult task ahead of them if they are going to be able to convey to their future students that "proof undoubtedly lies at the heart of mathematics" (Hanna & de Villiers, 2008, p. 2) and a school mathematics without proof is no mathematics at all.

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Employing Mathematical Modelling to Respond to Indigenous Students' Needs for Contextualised Mathematics Experiences

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This discussion paper highlights some of the issues related to mathematics underachievement experienced by Indigenous students. Mathematical modelling can be implemented as a rich way to provide contextualised mathematics learning experiences able to be couched in significant contexts that Indigenous students can relate to and enjoy. This paper proposes that mathematical modelling will enhance Indigenous students' motivation to engage in Western concepts of mathematics, hence improving cognition and achievements in this Key Learning Area.

A Chronic National Problem

Underperformance in mathematics directly influences Indigenous students' access to successful social, fiscal and educational futures, for example, it obstructs their ability to meet minimal Vocational Education Training (VET) mathematics requirements for apprenticeships and traineeships. The Queensland Studies Authority's (QSA, 2006) Queensland State wide numeracy tests are administered to all schools in Queensland and have indicated annually, since their initial implementation in 1998, that Indigenous students in Years 5 and 7 lag behind non-Indigenous students by up to two years in their understandings of numeracy.

Previous literature and research has mainly focused on Indigenous students' learning environments and preferred learning styles, literacy and general mathematics learning. Further research is required that specifically targets three things: Indigenous students' perceived value of mathematics, their conceptualisations of the usefulness of their school mathematics beyond the classroom, and the ways in which their underachievement can be improved. Insufficient research has been performed to reveal new or alternative mathematics teaching strategies that can contextualise mathematics for Indigenous students. Contextualising the mathematics curriculum would be an effective means to significantly improve both Indigenous and non-Indigenous students' understandings of mathematics. In particular the utilitarian aspects of mathematics that will augment their future learning and achievement in this Key Learning Area (KLA), thus promoting greater employment prospects, could be made clear for students through the implementation of real world mathematics applications in mathematical modelling.

Examining This National Concern: A Historical Perspective

Historically, colonists judged the intellectual capacity, civility and worthiness of Australian Indigenous cultures and societies by the degree to which they could be compared to European modes of thinking, norms and customs (National Health and Medical Research Council, 2003). Perceived as competent in concrete thought yet incapable of making abstractions and generalisations or performing analytic thought, these cultures were perceived by proponents of Scientific Racism to be of lesser intelligence and not capable of formal logic or reasoning (Ascher & Ascher, 1986; Eckermann, Dowd, Chong, Nixon, Gray, & Johnson, 2006; Nakata, 2002). Ascher and Ascher (1986) state that "in any context these descriptions are heavily judgmental; in the context of mathematics they are condemning" (p. 128). Hence, in terms of owning mathematics, history perceived Indigenous cultures as being incapable of developing or using mathematics and indeed were viewed as having little or no real use for mathematics in their inferior societies (Nakata, 2002). In the past, Indigenous students have been taught no more than basic numeracy skills; Bin-Salik (1990) and Morgan and Slade (1998) propose that the purpose of this was to disallow intellectual integration into Australia society.

Ascher and Ascher (1986) clearly state that there are no studies or studies that have been repeated that support the myth of the childlike primitive. Walkerdine (1997) claims that it is usual for mainstream teachers to reflect the dominant culture's views via the curriculum and some, albeit unwittingly, promote the idea that Indigenous students are childlike and simplistic in their thinking. It is important to note here that cultural differences can be thought to reside more in the circumstances to which particular cognitive processes are applied, rather than in the existence of a mental process in one cultural group and its absence in another (Ascher & Ascher, 1986). Negative attitudes, values and misconceptions formed about Aboriginal and Torres Strait Islander people are shaped around the concept of scientific thought. These stereotypes persist in contemporary society and continue to marginalise Indigenous people (Eckermann et al., 2006).

What is Happening for Indigenous Education in Contemporary Society?

In recent times, Australia has witnessed the creation of new Acts and policies to improve educational outcomes for Indigenous students, committing resources to improve Indigenous students' numeracy. The literature provides a plethora of reasons as to why Indigenous students do not succeed at school compared to their non-Indigenous peers. However, what the literature does not explain is why, when armed with new and informed syllabus documents, research knowledge and government allocated funding, Indigenous children continue to experience difficulties in becoming numerate citizens. An example is the Australian Commonwealth Government's recent National Indigenous English Literacy and Numeracy Strategy (2000-2004), endorsed by all Commonwealth, State and Territory Ministers, that states all Indigenous students should achieve English literacy and numeracy at the same level as other Australian youth (DEETYA, 2005). To achieve this, it is necessary for students to have appropriate educational resources in their home environments as this has been positively correlated with student performance (De Bortoli & Creswell, 2004). The availability of a dictionary, a space conducive to study that has a desk, necessary textbooks and calculators are necessary to ensure that Indigenous students do not continue to fall behind their non-Indigenous peers in numeracy (De Bortoli & Creswell, 2004). Other factors, such as parents' education and employment histories, travel time to school, availability and usage of computers in out-of-school settings, homework skills and support all impact significantly on students' underperformance in mathematics (De Bortoli & Creswell, 2004).

Nationally, the gap between Indigenous and non-Indigenous students in mathematics, and particularly in numeracy, is alarmingly high and persistent (Schwab & Sutherland, 2001). In their attempt to cure the problems in mathematics education, recent syllabus documents support the contemporary learning theories that emphasise the importance of acknowledging the students' social and cultural contexts and teacher and student relationships and challenge educators to implement a mathematics curriculum that is meaningful, relevant and culturally appropriate for Indigenous students (Coleman-Dimon, 2000; Frigo & Simpson, 2000). Yet the problem escalates and Indigenous students' low mathematics performance is of major concern for Australia, particularly as it relates to low retention rates in secondary school and an inability for Indigenous students to meet the minimal Vocational Education Training (VET) mathematics requirements for apprenticeships and traineeships and entry to university studies.

Many Indigenous students accept as personal failure their inability to achieve acceptable academic outcomes in literacy and mathematics in urban, rural and remote schools, thus limiting their access to the benefits that education provides (Jude, 1998). Low performance and motivation to engage in Western mathematics' practices appears to begin in the primary school years (Tripcony, 2002). A multilevel assessment program conducted by Tripcony (2002) identified that Indigenous students in rural and remote schools perform 3 to 7 years behind urban Indigenous students of the same age in numeracy tests. Tripcony (2002) states that the urban Indigenous student is roughly 2 times more likely than their non-Indigenous peer to be identified in the Queensland Studies Authority's (QSA) Year 2 Net, a ratio that increases to 3 or 4 times for rural and remote Indigenous students.

The QSA (2004) statistics from their statewide tests clearly indicate that Indigenous students continue to fall behind in numeracy in Years 3, 5, and 7. In their findings published in their Overview of Queensland Statewide Student Performance in Aspects of Literacy and Numeracy, the QSA (2004) report that "across all years from 1999 to 2003 the average performances of Indigenous students were generally equivalent to those of non-Indigenous students two year levels lower" (p. iii). This report also found that Indigenous students as a group performed lower than non-Indigenous students where "the mean scale scores for all strands of Numeracy for Indigenous students were appreciably lower than the mean scale scores for the other groups", adding too that these students lag behind the cohort as a whole (QSA, 2004, p. 17). Comparisons were made between 2003 and 2004 Year 3 students' average results and indicated that the 2004 cohort performed slightly lower in all strands of numeracy (QSA, 2004). The report admits that students who perform below the minimum standard deemed acceptable may have difficulty progressing at school. Frigo and Simpson (2000) found similar difficulties for Indigenous students in the New South Wales Basic Skills Test for Years 3 and

5 students, claiming that underachievement of Indigenous students in that State continues, on average, to be significantly lower than for their non-Indigenous peers.

Wright, Martland, and Stafford (2000) agree, noting that their studies revealed that "children who are low-attaining in their early years tend to remain so throughout their schooling, and the knowledge gap between low-attaining children and average or able children tends to increase over the course of their years at school" (p. 2). A recent long term study, conducted by Reynolds, Temple, Robertson, and Mann (2001), indicated that intervention in general cognition in the early years of education for low-income students and students having at risk backgrounds continued to have a positive impact on their educational and social achievements up to the age of 20 years. Intervention programs resulting from the Year 2 Diagnostic Net in Queensland have worked someway towards bridging this gap.

Indigenous culture is still viewed by many as being too primitive to contribute to contemporary society (Howard, 1995; Mathews, 2003). As in the past, Indigenous cultures today are still considered by some as having no valuable mathematics and science knowledge and that any perspectives Indigenous students had on mathematics or science would inevitably be influenced by contact with Western perspectives (Hooley, 2000). The Queensland Mathematics Syllabus (QSA, 2004) states that the concise language of this KLA, both verbal and symbolic, enables communication of shared mathematical understandings within and among communities. It states too that an understanding of mathematical knowledge, procedures and strategies will empower the individual so that they may become effective participants in the interdependent world (QSA, 2004). Mathematics implemented in the Western education system effectively locks out Indigenous students who are unable to speak this concise language of Western mathematics and engage daily in Western education practices that marginalise them.

The Diagnosis

Matthews, Watego, Cooper, and Baturo (2005) claim that one of the significant problems existing in contemporary mathematics education is the lack of relevance. Indigenous students find that their mathematics curriculum undervalues their culture, teaching methods, and worldview. Mathematics as a KLA is viewed as providing students with a futures perspective that involves developing knowledge, practices and dispositions that will enable all students to identify possible, probable and preferred individual and shared futures, leading to insights and understandings about the roles of individuals and groups in visualising and preparing for those preferred futures (QSA, 2004). The QSA (2004) claims that students who hold a futures perspective are empowered as they have an outlook that enables them to take responsibility for their actions and decisions, thus allowing them to participate confidently in the progression of social innovation, recovery and renewal. Indigenous communities often have opposing worldviews to the futures perspectives from the dominant Western culture, which causes disequilibrium for Indigenous students negotiating their educational futures.

Howard and Perry (2007) express it nicely when they pronounce that "all education, including mathematics education, needs to be a place of belonging for Aboriginal students" (p. 402). Unable to perceive its usefulness, Indigenous students require curriculum developers and educators to find a means to instruct Western concepts of in new ways that enable them to make connections between mathematics and various real-life phenomena (Munakata, 2005). Mathematical modelling can be offered to introduce contextualised mathematics experiences for Indigenous students, additional to the necessary, more prescriptive, teacher-led teaching episodes. Today's cognitive learning theories suggest that learning is not linear, rather it develops in various directions at once and at a uneven pace (Lingefjärd, 2002). So, rather than teaching approaches that focus on how to make the learner fit the system, the preferred focus with mathematical modelling would be on how the system can better fit the learners' needs (Frigo & Simpson, 2000).

Working Toward a Cure

The question posed countless times but yet to be satisfactorily answered is how to address the problem of continued underachievement in mathematics for Indigenous students. From their New South Wales study with remote rural Aboriginal students, parents and school staff, Howard and Perry's (2005) results indicated that many Indigenous students believed they attained mathematical knowledge by the teacher depositing it into their heads and also through writing it down. Furthermore, the Aboriginal students believed they acquired mathematical knowledge by redoing incorrect work or through work practised in textbooks; the students were not aware of their own mathematical competencies (Howard & Perry, 2005). Howard and Perry (2005) claimed that although the Indigenous students enjoyed learning new things in maths they found the learning of mathematics concepts difficult, holding negative views about themselves as learners of maths.

It is known that seated bookwork is not conducive to the learning needs of Indigenous students. In fact Hamilton (2007) notes that that there is little evidence to suggest that textbook problem solving will lead to improved mathematical performance beyond the classroom. Mathematics programs that accentuate Aboriginal students' life experiences and contexts bring relevance to their learning, thus providing purpose and in turn increased levels of motivation and engagement. Mathematical modelling and problem solving can inject curiosity into what is sometimes considered by students to be a boring subject: when the two are properly combined, they can improve students' attitudes towards mathematics (Falsetti & Rodríguez, 2005).

Mathematical modelling involves realistic and complex situations where the participant takes on the role of problem solver to work mathematically in ways that move beyond the traditional school experience (Lesh & Zawojewski, 2007). The modelling tasks generate products that often include complex artefacts or conceptual tools required for some purpose to achieve some goal (English, In press, 2009; Lesh & Zawojewski, 2007). Mathematical modelling is dissimilar to customary mathematics instruction as it provides experiences for students to educe their own mathematics (English, In press, 2009). In modelling activities the mathematics is embedded and must be established by the students, thus enabling them to develop mathematisation skills that can become generative resources in life beyond the classroom (English, In press, 2009). Mathematical modelling is much more than merely an inquiry based approach to the teaching and learning of mathematics concepts; mathematical modelling can sustain the integration of key concepts from other KLAs so that mathematics becomes one of the key foci for successful resolution of the task. Modelling provides excellent opportunities for interdisciplinary experiences that require students to not only make full sense of the situation, but to personally mathematise the task in a way that makes meaning for them (English, In press, 2009). Modelling is a cyclic process that includes the interpretation of problem information, the selection of pertinent quantities, the identification of necessary operations that may lead to new quantities, and the creation of significant representations (English, in press, 2009; Lesh & Doerr, 2003). English (In press, 2009) states that "because the children's final products embody the factors, relationships, and operations that they considered important, powerful insights can be gained into the children's mathematical and scientific thinking as they work the problem sequence". It is crucial that Indigenous students be provided with various opportunities to demonstrate what they know and can do (Frigo & Simpson, 2000).

Although mathematical modelling problems require more time for implementation than some other problem activities, the benefits for the students can be outstanding. Mathematical modelling can capture students' imagination and may therefore promote an increased life long interest in mathematics for Indigenous students. Students are often fascinated by things that do not work or fit the correct pattern (Humble, 2005) and educators can provide opportunities for students to think critically through real-life situations where modelling can facilitate the defining and creating of solutions and where often more than one answer can be generated (Lege, 2005). Mathematical modelling allows students to seek solutions, explore patterns and formulate conjectures through culturally appropriate contexts; these activities provide the requisite skills and adequate levels of participation in supported environments necessary for school success (Howard & Perry, 2007). English (In press, 2009) makes clear that this learning approach encourages students to work through complex circumstances collaboratively as a member of a team, developing skills for future-orientated learning. What's more, modelling facilitates the metacognitive skills of planning, monitoring and evaluating all of which are measurable (Tanner & Jones, 2002). Boaler (2001) claims that these mathematical modelling experiences do more than enhance individual understanding; in addition these experiences provide occasions for student involvement in practices that are represented and necessary in everyday living.

An Example of Mathematical Modelling Currently Being Researched

The author of this paper is currently conducting research in an urban Brisbane school with Years 4 to 7 Indigenous and non-Indigenous students. Students work in groups on modelling tasks relating to Australia's cyclones and on the topic of chocolate. The students work with visual and written texts, making connections between the mathematics embedded in the activity and their real world experiences. Students investigate the topic from a historical perspective and in this way it is uncomplicated to integrate the KLA of SOSE with those of Mathematics, Literacy, and Science. In some instances, data are presented on graphs for students to analyse, however much of the mathematics is embedded and students often stated that they didn't realise that they were doing maths.

Negotiation is a key factor featured in modelling tasks. It would have been advantageous to construct modelling tasks for this study that catered for student interest, however, those selected for implementation were done so with student prior knowledge and interest in mind. Students who enjoy taking leadership responsibilities did, and in some groups created roles for themselves and for other group members. Of interest is the ways in which some Indigenous students view the modelling tasks as having a job to do, and compartmentalised the task so that certain members of the groups could accomplish specific parts. Having done that, none of the participants were observed teasing apart the task content so as to specifically identify the maths components of the modelling tasks; considering that most Indigenous students in the study judged themselves as mathematically incompetent when interviewed prior to the implementation of the modelling activity, it was interesting that they did not attempt to avoid the mathematics components of the task when it was recognised. The 'having a job to do' is an interesting concept here, as in the researcher's teaching and research experience Indigenous students often take more pleasure in learning tasks that involve being physically active and are perceived by them to have a genuine utility.

After researching their problem, students responded to the task's assessment by preparing their findings for discussion with their peers. Students are offered a variety of modes to deliver their findings and indeed some students have requested to formulate their own. It is this level of student interest that indicates that mathematical modelling can be perceived by students to be a productive and worthy enterprise.

Recommendations

Educators have long been alert to the fact that our education system is failing Indigenous students. What is required urgently is an integrated and contextualised mathematics curriculum that will engage Indigenous students to actively investigate, analyse, and reflect on real world problems they are interested in. It is argued here that without the development and national implementation of such a curriculum, the low retention rates and low levels of mathematics success for Aboriginal students at secondary school will persist, thus continuing to promote the arrestment of their successful life chances.

This paper recommends that curriculum planners and educators embrace mathematical modelling for the academic and social benefits it can deliver for students. Mathematical modelling could be the step forward in academic change, particularly for Indigenous students. It would be advantageous if curriculum developers were to produce modules, similar to those provided by the Queensland Studies Authority to support the current Queensland syllabus documents, as this would support novice users of mathematical modelling as a pedagogical tool. Development of such modules could foster educator mathematical modelling teaching competence and confidence prior to the teacher constructing specific mathematical modelling tasks for their students.

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Reconceptualising Agency Through Teachers Talking About a Sociocultural Approach to Teaching Mathematics in the Classroom

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This paper explores teachers' "agency" as they talk about using a Sociocultural approach to teaching and learning (Collective Argumentation) to mediate activity in the mathematics classroom. The paper examines a re-conceptualisation of teacher agency as evidence in a report by one middle school teacher of a classroom mathematics activity. Employing discourse analysis to examine aspects of teacher activity in the report, the paper relates the development of teacher agency to the appropriation of pedagogical practices and to teacher talk about those practices.

Classrooms operate within institutions and suffer constraints common to institutions, constraints that specify the roles, status and degree of autonomy that teachers and students are accorded. However, due to the diverse ways of living that teachers and students bring to teaching and learning, classrooms are also embedded in contexts in which learning is mediated by the social and cultural identities of participants (Bruner, 1996). A question that confronts teachers of mathematics, therefore, is how best to establish in their institutional spaces (primary, middle-school, senior-secondary) classroom communities where they may participate more fully with their students in the discourse of the mathematics curriculum, that is, in the ways of thinking, saying, writing, and doing deemed essential by a society's culture for a worthwhile way of life. It is a purpose of this paper to explore the affordances that one particular Sociocultural approach to teaching and learning (Collective Argumentation) provides teachers when engaging in what Pickering refers to as the "dance of agency".

In accounting for historical advances in mathematics and science, Pickering (1995) labelled the tension between the agency of the person knowing and doing the mathematics and the agency of the discipline that accredits and conventionalises ways of knowing and doing mathematics as the "dance of agency". According to Pickering (1995) when teachers follow the established patterns of the discipline, they privilege disciplinary agency and when they take initiative through engaging students in open-ended tasks and cross–discipline conversation they privilege human agency. It is in negotiating the forwards, backwards and sideways movements from the human to the discipline, from the "everyday" to the "scientific", that the "dance of agency" can be conceptualised as taking place.

Mediating Agency in the Classroom

From a sociocultural point of view, simply theorising "agency", that is, the capacity to plan, implement and evaluate the attainment of a goal, as originating in the individual or in a collective (see Hernandez & Iyengar, 2001) is insufficient for understanding the mediating role of "agency" in the learning of mathematics. From a sociocultural perspective, agency needs to be understood as being synonymous with a person's way of being, seeing and responding in the world and as being embedded in contexts of activity and interpretive practices (Edwards, 2000). This understanding is commensurate with research findings that suggest that situating teaching and learning within particular activity contexts influences mathematical development. For example, in articulating the role of discourse in the learning of mathematics Cobb and Hodge (2002) note that student participation in the discourse practices of their classrooms (e.g., engaging in an Initiation-Response-Evaluation format of classroom talk or engaging in reflective discourse) influences the identity development of students when learning mathematics. Van Dijk, van Oers, and Terwel (2003) provide evidence that teacher demonstrated and student co-constructed mathematical models lead to different ways of problem solving in students. What these studies highlight is the tension in mathematics teaching and learning that exists between conventional mathematics and novel ways of knowing and doing mathematics – the "dance of agency".

The notion of "mediated agency" recognises the "dance of agency" as it may play out in the classroom by focusing attention on the "irreducible tension" manifested between agent/s (e.g., teachers, students) on the one hand and the mediational means (e.g., ways of knowing and doing) that they employ or have access to on the other (Wertsch & Rupert, 1993, p. 230). Through interpreting the relationship between agent/s and mediational means in terms of "mediated agency", Wertsch and Rupert (1993) promote a view of human agency which positions mental functioning within systems of collective action that are culturally and historically situated.

From this point of view, issues which affect the organisation of mental functioning on the intermental plane (such as authority, membership, and norms which privilege certain ways of thinking and acting) are seen as essential aspects of functioning on the intramental plane.

According to Smith (1996) approaches to teaching mathematics that extend beyond transmission approaches to teaching and learning and that recognise the tension between teachers' and students' ways of knowing and doing often fail to help teachers reconceptualise their sense of agency and thus fail to convince teachers to change their classroom practice. What is needed if teachers are to move their pedagogy beyond transmission are, according to Smith (1996), opportunities for reconceptualising agency in the teaching of mathematics – opportunities that assist teachers to:

- design tasks that support the development of student thinking/understanding;
- predict student reasoning and the language they may use to express it;
- create safe contexts where students can express and justify their own ideas; and
- value students' activity whilst introducing conventional mathematics.

This paper explores teachers' "agency" as they talk about using a Sociocultural approach to teaching and learning mathematics (Collective Argumentation) to mediate teacher and student activity in their classrooms. Specifically, it examines a reconceptualisation of teacher agency as evidenced through a report by one teacher (Sam) of a classroom mathematics activity to a group of peers.

Method

Sam's report on his classroom activity took place during a professional development session that was part of a larger study into teachers' appropriation of the practices of Collective Argumentation into their everyday teaching of mathematics and/or science. The larger study, conducted over a three-year time frame, involves university educators working with 20 elementary and middle school teachers of mathematics and/or science from 6 schools located in South-East Queensland to bring about and reflect upon change in the way they teach mathematics and/or science.

Collective Argumentation (Brown & Renshaw, 2000) is an approach to teaching and learning that is based on five interactive principles necessary for coordinating competing knowledge claims. First, the "generalisability" principle requires that students attempt to communicate their ideas, so that fellow students can participate in sifting relevant from irrelevant ideas. Second, the "objectivity" principle requires that relevant ideas can be rejected only if they can be denied by reference to past experiences or logical reasoning. If ideas cannot be denied then they must remain part of the discussion. Third, the "consistency" principle requires that ideas which are contradictory to each other or that belong to mutually exclusive points of view must be resolved through discussion. Fourth, a principle of "consensus" requires that all members of the group understand the agreed approach to solving the problem. If a member of the group does not understand, there is an obligation on that student to seek clarification, and a reciprocal obligation on the other group members to assist. Finally, the "recontextualisation" principle involves students re-presenting the group response to the other members of the class for discussion and validation. Communicating to class members outside the group, challenges students to rephrase their ideas, to defend their thinking, and to reassess the validity of their thinking.

Research Design

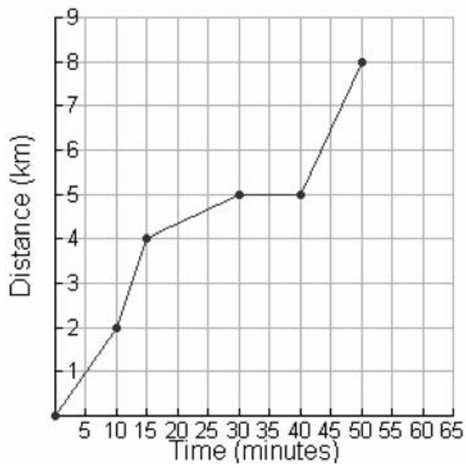
The study employs a sociocultural methodology, based on a "design-experiment" (see Schoenfeld, 2006). The "design-experiment" involves prolonged systematic inquiry into change through engagement in collaborative cycles of analysis, design, implementation, assessment and reflection. The professional development session referred to in this paper was one mechanism used to assist teachers to reflect upon and assess the nature of the activity of their students, their activity as teachers of mathematics and/or science, and the co-constructed activity of their classrooms.

Research context. During a professional development session, involving seven teachers and two mathematics educators, teachers were invited to report on the teaching and learning of mathematics and/or science in their classroom. Each report was video-taped, transcribed and subjected to a form of discourse analysis. Discourse analysis has been used by researchers to, among other things, situate teachers' instructional practices in institutional settings (Cobb, McClain, de Silva Lamberg, & Dean, 2003), and to study the development of students' critical awareness in the mathematics classroom (Wagner, 2007). Informed written consent was provided by teachers for their reports to be used for research purposes.

Research participant. The teacher who is the focus of this paper, Sam, had been using the practices of Collective Argumentation to inform his teaching of mathematics for one school year. Sam taught at a P-12 school located in a middle-class suburb of a major city. Sam started his career, now in its 20th year, by framing his teaching of mathematics within pedagogical practices that reflected a transmission approach to teaching and learning. As such, Sam's agency as a teacher of mathematics, that is, his knowing what to say, when to say it, and how to assess and report student performance, was supported by working in a traditional classroom using tools such as textbooks. However, after a decade of wondering why students performed inadequately when it came to the application of mathematics to novel situations, Sam set out on a journey of professional development which led him to view student learning within a framework that anchored teacher agency to pedagogical strategies that afforded him a focus on student understanding. The following sharing of a classroom lesson with peers provides insights in to the nature of these affordances. The class referred to in Sam's report is a Year 6 class of high-achievers who were being accelerated in their study of mathematics.

The task being reported. The task that was the focus of Sam's report is represented in Figure 1. In the analysis that follows, italics have been used to identify Sam's actual words.

Collective Argumentation Task



a. You are to write a plausible story for this graph. You need to provide a full explanation.

b. You are to develop a suitable title for this graph.

c. Suppose there is need to return back to the starting position by 65 minutes, how can this be shown on the graph? What speed would need to be travelled to allow this to occur? Do you think it is likely that it will be possible to achieve this goal?

Figure 1. Task sheet as presented to a Year 6 class.

Analysis of Sam's Report of a Classroom Lesson

Choosing problems that privilege understanding. *In describing the lesson to peers, Sam commenced by situating the activity of the class within a problem solving context that allowed students to* "give me some information about how they are going in developing … understanding". *Adapting a textbook activity so that it* "allowed them to use collective argumentation" *and so that the students could go* "away and have a bit of a play"*, Sam's purpose was about eliciting* "a variety of responses" *from students. Explaining that this* "play" *was structured by* "focus questions" *(see Figure 1), Sam admitted that he had inadvertently limited student thinking,* "I made a mistake"*, by verbally introducing the problem to the class within the context of* "a runner" *and by referring to* "how fast the runner would need to run". *This contextualising of the problem was interpreted as a mistake by Sam because it was seen as directing student thinking toward a response that* "was very straight forward" *a response that simply required students to* "draw a line from that point *[see Figure 1 coordinates (50,08)]* down to that point *[see Figure 1 coordinates (65,0)]*".

This introduction by Sam to the teaching of a mathematics lesson provides an important insight into how Sam interprets knowing and doing mathematics in this classroom. Teaching mathematics is seen by Sam as being a creative endeavour where the teacher and students are fully engaged in coordinating their interpretations of a task and in establishing co-operative patterns of interaction whilst at the same time having their creativity structured by convention, for example, the use of "focus questions". This view of mathematics evidences a sociocultural approach to teaching and learning mathematics where a balance may be achieved between students' individual ways of thinking and the collective endeavour of the class to learn mathematics. It is in achieving this balance in the classroom that mathematics conventions may be connected with students' inventions (Lampert, 1990). In the process, students may encounter multiple and varied ways in which to participate in mathematics, ways facilitative of the development of understanding and the "higher mental" processes – ways in which Sam's admission that *"I made a mistake"* is interpreted in a developmental rather than self-effacing sense. Sam then goes on to provide his peers with an enthusiastic account of how one group of students responded to the task (see Figure 2).

A Student Group Response

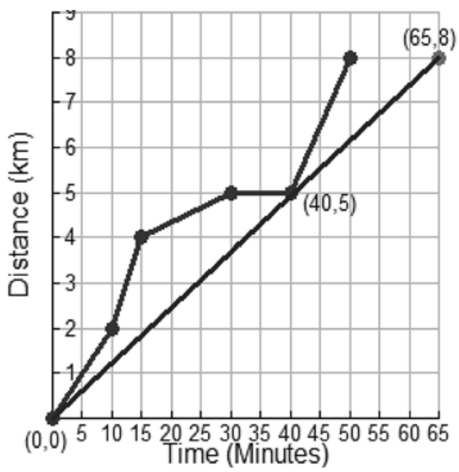


Figure 2. A novel response to a "straightforward" task.

Moving beyond the expected to consider the novel. Sam narrated a "plausible story" provided by one group of students that he considered to be a novel response to the task. The story concerned a bicyclist going for a morning ride travelling at 0.2 kilometres per minute for the first 10 minutes, 0.4 kilometres per minute for the next 5 minutes, 0.07 kilometres per minute for the next 15 minutes, resting for 10 minutes and then travelling 0.3 kilometres per minute for the last 10 minutes. So this group of students had translated the problem in terms of the concept of *"speed",* a translation that Sam considered to be *"pretty cool".* However, what was *"really cool"* was the return journey where the students *"found the average speed. So they said I want to get home in sixty-five minutes, so this is where home is* [see Figure 2 (0,0)]*, so what they found was they drew a point from here* [see Figure 2 (65,8)] *to here* [see Figure 2 (0,0)]*, because this* [points to the coordinates (65,8)] *is how far out they are, so they are eight kilometres away and they want to get back home in sixty-five minutes… So that's the speed he's got to travel to get back home, and they* (the group) *found the equation to that particular line." Sam then went on to say that he had "never actually thought about it* (the task) *like that" and that "for them* (the group) *to interpret it* (the task) *that way, it's really cool".*

This account by Sam of a student group response to the task highlights an important element of the nature of the teacher's agency in this classroom. Sam's knowing what to say, when to say it and how to evaluate a response is anchored to "thinking mathematically", that is, to applying formal mathematical knowledge flexibly and meaningfully to appropriate situations. This is reflected in Sam's utterance that he *"never actually thought about it* (the task) *like that"* and that *"for them* (the group) *to interpret it* (the task) *that way, it's really cool"*. As such, Sam's agency within this mathematics lesson may be said to reside not in his authority as the "teacher" nor in his use of a textbook task, but in his capacity as an expert who is willing to participate in the struggle to understand the novel and to link it to the knowledge of the discipline – to engage in a "dance of agency" with his students. As such, it may be said that Sam's report of a class activity to a group of peers is just as much about how he values thinking about and understanding mathematics as it is about finding equations to particular lines – an essential element of mathematics teaching and learning (Schoenfeld, 1988). This valuing is reflected by Sam's peers as they engage him in discussion about the activity of his class. For ease of presentation only those turns of the 5 minute discussion that are representative of teachers' contributions are recorded in Table 1. Please note that all turns are in chronological order.

Table 1

Teachers thinking about student thinking

Turn	Speaker	Dialogue
01	Julie	The fact that they (the group) sort of interpreted it (the task), the fact that they wanted to get home in sixty-five minutes and so that (7.35km per hr) is the average speed that they needed to travel to get through, I thought that was pretty clever.
02	Jay	It's a realistic way as you said.
03	Sam	I mean they probably could have intersected it (the X-axis) further down, but then you've got the same sort of problem of the speed scenario. Yeah, so they have done their calculations and they have come up with a speed of, the slope of that line is what, 7.35km / hr to reach home. So in terms of the speeds they were travelling all the way through, that is a reasonable type of speed.
04	Julie	What made them (the group) go backwards (work from right to left), I mean every juncture of the journey, they've (the group) always moved from left to right, why are they going that way (right to left)?
05	Sam	Well I think they were just trying to represent the fact that they were going to travel eight kilometres in sixty-five minutes, you know that's what they were trying to do, trying to represent the fact that they were travelling eight kilometres in sixty-five minutes. And of course that's right they couldn't of dropped it down here [to the point (65,0)] because then they wouldn't of travelled it in sixty-five minutes they would of only travelled it in fifteen minutes, so they actually had to go back to that point (0,0) to actually satisfy the condition of travelling the eight kilometres in sixty-five minutes.
06	Jay	Maybe it's my not understanding, but wouldn't that (the question) be, 'Complete the whole thing (journey) in sixty-five minutes?' But they (the group) have done to take sixty-five minutes to get back home, just different interpretation.
07	Sam	That's right they've (the group), it was to return home, well they've sort of read it as (getting home) by sixty-five minutes.
08	Sam	So in terms of what I expected and what I got I thought that (the group response) was pretty cool.
09	Julie	Yeah because scientifically and mathematically we always do it (read graphs) left to right, so um, but it (the group response) is right. I think what they have done is they have reinterpreted to idea of time. Total time as opposed to …
10	Sam	It took me a long time to understand what their interpretation was and why they did it, but when I finally did (understand) it (the group response) was fine.

It is clear from the above excerpts of teacher discussion that this group of teachers privileges thinking about and understanding the mathematics generated by students over replacing students' ways of knowing and doing with disciplinary conventions. This privileging is represented in statements that, although accepting the response as being *"pretty clever"* (turn 01), provide some sense of the "dance of agency" that these teachers are participating in on a regular basis as they struggle to understand a novel student response within an institutional curriculum that privileges the conventions of the discipline. Statements that peppered the whole discussion such as *"what made them* (the group) *go backwards* (work from right to left)*..."* (turn 04), *"maybe it's my not understanding, but wouldn't that* (the question) *be, 'Complete the whole thing* (journey) *in sixty-five minutes?'"* (turn 06), *"…because scientifically and mathematically we always do it* (read graphs*) left to right…"* (turn 09), and *"it took me a long time to understand what their interpretation was…"* (turn 10) provide evidence that these teachers have moved beyond simply managing the tensions between students' goals and the goals of the school classroom. The verbal interactions between Julie, Jay and Sam evidence a desire by these teachers to use students' representations of solutions to tasks as "cultural tools", that is, as thinking devices that may explain and generate understanding. This is evidenced in Jay's statement at turn 02 that the student solution *"is a realistic way"* of addressing the task, by Julie's statement that *"…we always do it* (read graphs) *left to right, so um, but it* (the group response) *is right"* (turn 09), and Sam's statement at turn 08 that *"…in terms of what I expected and what I got I thought that* (the group response) *was pretty cool"*. In so doing, these teachers are struggling to interpret the novel within the conventional, at the same time gaining insights into their own practices as mathematicians and/or scientists.

Conclusion

This paper explores the nature of "agency" as evidenced in a teacher report of a student solution to a task provided to teachers who employ a Sociocultural approach to mediate the teaching and learning of mathematics. As Sam talked to and with the teachers during the presentation a number of characteristics of Sam's "agency" in the mathematics classroom became evident. Firstly, Sam's way of being, seeing and responding in the mathematics classroom is embedded in contexts of activity, contexts that will allow students to go *"away and have a bit of a play"* and that will elicit *"a variety of responses"* from students. Secondly, Sam's capacity to plan, implement and evaluate the attainment of a goal in the mathematics classroom is embedded in interpretive practices that give Sam *"some information about how they* (the students) *are going in developing … understanding".* These characteristics of agency are reflected in statements made by Sam's peers as they partner Sam in a "dance" of linking a novel student response to their own understandings and to ways of knowing and doing privileged by the discipline. In so doing, Sam, Julie, and Jay are provided with opportunities to gain further insights into the efficacy of what they are doing in their classrooms and to further conceptualise a sense of teacher agency that validates their efforts to provide opportunities for students to bring their understandings to bear on tasks and to see the effects that these understandings have on making sense of mathematics.

The development of a sense of agency that utilises the tension between teachers' and students' ways of knowing and doing, a sense of agency as evidenced by Sam and his colleagues, has been shown over time to be supported by the principles of Collective Argumentation. These principles assist teachers to utilise tasks that will permit students to generalise and objectify their thinking, to employ practices that promote consistency in student reasoning and consensus in the ways thinking may be represented, and to create contexts where students can safely discuss their ideas and accept guidance from others. Principles that not only support a Sociocultural approach to teaching and learning in the mathematics classroom, but also assist teachers to reconceptualise their sense of agency in the classroom by talking about and reflecting upon their own practice.

Not all 20 teachers involved in the larger study display conceptualisations of "teacher agency" that are in accord with that displayed by Sam and his colleagues. Teachers' conceptualisations of agency are influenced by many factors such as level of competence with the knowledge of the discipline and the amount of support provided by institutional authorities in implementing different approaches to teaching and learning. However, initial findings of the larger study suggest that talking about classroom practice with colleagues also influences the development of teachers' reconceptualisations of agency in the mathematics classroom.

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Middle School Students' Interest in Statistical Literacy

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This paper presents the results of a research review into the development of middle school students' interest in statistical literacy. In particular it reviews the concept of interest and its motivational influence on learning. Findings reveal that very little research has examined the influence of positive affect such as interest on learning in the middle-school statistics context. Further, these findings suggest that interest development will be the result of a complex interplay of classroom influences and individual factors such as: students' knowledge of statistics, their enjoyment of statistics and their perceptions of competency in relation to the learning of statistics.

Through the learning of those concepts situated in the Chance and Data component of the Australian state (P-10) mathematics syllabi, students should become statistically literate; that is, they should be able to interpret and critically evaluate messages that contain statistical elements (Gal, 2003). A statistically literate person, for example, should be able to recognize bias as a possible source of error in media reports of survey data. Models have been conceptualized that describe the development of statistical literacy in learners (Gal, 2002; Watson, 2006). Whereas these models have acknowledged the importance of affect in this development, little research has actually focussed on affect in this context. The purpose of this paper, therefore, is to review the literature in relation to the development of affect in the statistical literacy context.

Students in a middle school context (years 6 to 9) are typically in early adolescence, which appears to be a critical stage in their affective development. In the mathematics education context, for example, evidence points to a decline in levels of affect as a student progresses through school (Fredricks & Eccles, 2002) with such levels reaching a minimum in year 10 (Watt, 2004). The correlation between student attitudes towards mathematics and their achievement in mathematics, however, appears to be strongest for students in years 8 to 10 (Ma & Kishor, 1997; Ma & Xu, 2004). The influence of affect on learning appears to be more pronounced for this group of students. Such findings are supported by reported physiological changes to the brain that occur during adolescence (Wigfield, Byrnes, & Eccles, 2006), changes that result in the greater likelihood of affective activity. In the statistics context, later high levels of reported statistics anxiety (Onwuegbuzie & Wilson, 2003) and shortages of skilled statisticians (Trewin, 2005) could be the result of students developing a negative affect towards statistical literacy during adolescence.

Interest is an affect that is fundamental in the development of a person's concept of self (Deci, 1992). Moreover, recent research suggests that interest is necessary for psychological growth, with absence of interest in adolescents being linked with psychological disorders such as depression (Hunter & Csikszentmihalyi, 2003). Interest is known to have behavioural consequences and in learning contexts has been shown to be positively associated with achievement (Schiefele, Krapp, & Winteler, 1992). Given the importance of interest development in adolescence and its association with learning, a study of the development of affect in students should include, if not entirely focus on, the development of their interest.

Interest has been regarded as having both trait and state characteristics (Schiefele, 1991). At the trait level *individual interest* is an internalized affective element described as a "person's relatively enduring predisposition to reengage particular content over time" (Hidi & Renninger, 2006, p. 113). Interest at the state level is more transitory; it is typified by high levels of arousal and accompanied by positive emotions. This state can be induced by aspects of the environment and in such instances is termed *situational interest* or can be induced from the individual's predisposition to engage and in such instances is termed *actualized interest*.

Under certain circumstances, it is believed that situational interest will develop into individual interest. Silvia (2001), for example, proposed that such development is essentially the result of the individual resolving the cognitive conflict that occurs when he or she interacts with the object of interest. More specifically, he argued that during the person-object interaction, incoming stimuli are collated with current personal information on the basis of a number of *collative variables* that are associated with the learner's response to the stimuli. These collative variables include; novelty, uncertainty, and complexity. During this interaction, the learner will fail to engage in any significant way with stimuli that are considered routine (low levels of novelty).

Similarly the learner will fail to engage when the stimuli are too unknown or frightening (high levels of novelty). For optimal levels of these variables a state of curiosity will be evoked that is characterised by high levels of arousal and positive emotions, including interest. In this state the learner will be motivated to resolve the conflict created by the particular collating variable. If this conflict cannot be resolved quickly, the learner will be motivated to persist with the object, even returning to it at later times. Such persistence with the object may uncover further stimuli that in turn create a conflict in need of resolution. In such a way it is hypothesized that both knowledge and interest in the object will develop, with people losing interest in simple objects and pursuing those with more complex associated knowledge. This close relationship between knowledge and interest is supported by the Model of Domain Learning (Alexander, 2003), which posits that as individuals move from acclimation to proficiency in a particular knowledge domain, their individual interest in the domain and knowledge of the domain will both increase.

Although the discussion above has identified knowledge as a factor that contributes to the development of students' interest, it has not specifically examined interest development in the current context. Accordingly, this paper seeks to answer the following question: what are the factors documented in the literature that influence middle school students' interest in statistics?

Methodology

The literature review was conducted in two phases, commencing with a search on the specific question and then generalising the search to encompass related contexts. Searches in both phases commenced with databases of academic journals and abstracts including: *A+Education*, *Emerald*, *ERIC*, *Expanded Academic*, *JSTOR Education*, *Proquest*, *PsycINFO*, *SAGE*, *SpringerLink* and *Wiley Interscience*. In addition *Google Scholar* was found to be a particularly useful search engine. Secondary searches of others' bibliographies and searches using citation indexes were also conducted in each phase.

The initial search specifically addressed the research question using keywords: interest, statistics (or statistical), in the article title. In addition to the databases discussed above a search was conducted on specific statistics education journal archives including: *Statistics Education Research Journal*, *Journal of Statistics Education* and *Teaching Statistics*. Further, an archive of statistics education dissertations retained by the *International Association for Statistical Education* was also searched. Only one study located in the search specifically examined the concept of interest as it relates to the learning of statistics in a school context. The search was then expanded to include mathematics contexts and tertiary statistics contexts.

The final search resulted in 36 hits with publication dates that ranged from 1976 to 2007. Thirteen of these could not be readily accessed and in most cases were published prior to 1995. Of the remaining 23 articles, six were discarded as the term interest was used generically to describe a feeling of well-being that was neither defined nor measured. A further three articles were also discarded as they had included interest items in larger attitudinal scales, but had not reported specific interest outcomes. The 14 studies used in this review are described in Table 1.

Results

A review of the themes common to these articles suggests that factors influencing interest development can be broadly classified into environmental and individual factors (see Table 2). Teacher practices, including the types of learning experiences that students encounter, can be classified as environmental factors. Several studies provided evidence to support the notion that teacher practices can enhance student interest (Heinze, Reiss, & Rudolph, 2005; Mitchell, 1993, 1997; Sciutto, 1995). As an example, Mitchell (1993) noted that learning activities that involved puzzles, computers and group work would catch students' interest. Similarly, teaching strategies that promoted student involvement and which students found meaningful were found to hold students' interest. Mitchell was able to provide some evidence to suggest that the individual interest of students in environments high in situational interest would increase in both a mathematics (Mitchell & Gilson, 1997) and statistics (Mitchell, 1997) secondary school context. It is arguable whether changes in interest reported after a period of only 14 weeks, the period used in these studies, reflect changes in individual interest. Nevertheless, teacher practices undoubtedly influence the situational interest in the classroom, which it is argued will ultimately develop into individual interest (see earlier discussion).

The social environment also plays an important role in developing interest. In a mathematics education context, Bikner-Ahsbahs (2004) argued that a type of interest, termed *situated collective interest*, could emerge in a group situation where one by one students become involved in an activity and come to value the activity. Through observations of children she was able to provide some evidence to support this theory. In relation to the social environment, Fox (1982) found that the views of significant others, including parents and teachers, influenced student ratings of *career interest* (the type of career they would be interested in pursuing), but indirectly through their ratings of confidence and the utility of mathematics.

Table 1

Results of Literature Search

Article	Description	Context
(Bikner-Ahsbahs, 2004)	Observational to support theory development.	Sixth grade mathematics class (Germany).
(Fox, 1982)	Empirical study (N = 125).	Junior-secondary (year 7) mathematics (US)
(Heinze et al., 2005)	Empirical study (N = 500).	Junior-secondary (years 7 and 8) mathematics (Germany)
(Koller, Baumert, & Schnabel, 2001)	Empirical study (N = 602).	Secondary (years 7 to 12) mathematics (Germany).
(Lawless & Kulikowich, 2006)	Empirical study (N = 267).	Tertiary statistics (US).
(Lopez, Brown, Lent, & Gore, 1997)	Empirical study (N = 296).	Secondary (age 15 -16) mathematics (US).
(Marsh, Trautwein, Ludtke, Koller, & Baumert, 2005)	Empirical study (N = 7913).	Secondary (years 7 to 12) mathematics (Germany).
(Mitchell, 1993)	Validation study (N = 350).	High-school mathematics (US).
(Mitchell, 1997)	Empirical study (N = 51).	High-school statistics (US).
(Mitchell & Gilson, 1997)	Empirical study (N = 598).	School and tertiary mathematics (US).
(Renninger, Ewen, & Lasher, 2002)	Observational to support theory development.	Primary school mathematics (age 11).
(Sciutto, 1995)	Evaluative study (N = 17).	Tertiary statistics (US).
(Stevens & Olivarez, 2005)	Validation study (N = 724).	School (year 4 to 10) mathematics (US).
(Trautwein, Ludtke, Koller, Marsh, & Baumert, 2006)	Empirical study (N = 14341).	Secondary (grade 9) students (Germany).

At an individual level, several studies demonstrated an association between student achievement levels and their level of interest (see Table 2). The direction of this relationship has also been explored. Koller et al. (2001) found that interest in early adolescence predicted later interest but not achievement. They concluded that age was a factor in interest development and argued that younger adolescents were more sensitive to achievement feedback than older adolescents who presumably have more stable interests. Marsh et al. (2005) on the other hand demonstrated that a reciprocal effects model existed: Levels of interest were predicted by earlier achievement but also predicted later levels of achievement. The strength of the association between achievement and interest is known to be influenced by the structure of the knowledge domain in question. Lawless and Kulikowich (2006), for example, reported a stronger association for statistics than for psychology and argued that the former was a more structured knowledge domain.

Several studies also demonstrated a link between students' conceptions of their competency and their level of interest. Lopez et al. (1997) provided evidence to suggest that students' self-efficacy beliefs predicted their interest in mathematics. The strength of this relationship, however, was dependent on the branch of mathematics being studied: A stronger correlation existed for geometry than for algebra. Marsh et al. (2005) and Trautwein et al. (2006) demonstrated the link between students' academic self-concept and interest in mathematics, with Trautwein et al. asserting that self-concept was a strong predictor of interest, which almost entirely mediated the influence of achievement and tracking (the assigned level of class). Moreover, Trautwein et al. argued that this relationship was influenced by the frame of reference used by students to judge their competency: High achievement students who were in a group of even higher achieving students reported low levels of interest in mathematics while low achieving students in a group of even lower achieving students reported high levels of interest.

Table 2

Content Themes

Common themes	1. Environmental factors that influence interest include: Classroom influences (Bikner-Ahsbahs, 2004; Heinze et al., 2005; Trautwein et al., 2006); the views of significant others (Fox, 1982); learning experiences (Mitchell, 1993, 1997; Mitchell & Gilson, 1997; Sciutto, 1995); and the nature of the domain of knowledge studied (Lawless & Kulikowich, 2006). 2. Individual factors that influence interest include: Students' competency based beliefs and their prior knowledge (Fox, 1982; Koller et al., 2001; Lawless & Kulikowich, 2006; Lopez et al., 1997; Marsh et al., 2005; Trautwein et al., 2006); and also their age (Koller et al., 2001).
Divergent theme	Operationalisation of the interest construct: Assessing both enjoyment and importance (Koller et al., 2001) or assessing levels of interest (Lawless & Kulikowich, 2006).

Differences were evident regarding the operationalisation of the interest construct. The German studies (Koller et al., 2001; Marsh et al., 2005; Trautwein et al., 2006) regarded interest as having both a value and an emotion component, with the former including the importance of the task and the later the enjoyment of the task. Interest, however, is regarded as the underlying affect of intrinsically motivated behaviour (Deci & Ryan, 1985): A student interested in statistics will engage in statistical activities for their inherent value. The concept of importance may assess the usefulness or utility of the task, an extrinsic motivator. Students, who report mathematics as important, may do so because they perceive it to be necessary for future job prospects. Such importance may not reflect interest, although evidence suggests that it may predict interest (Fox, 1982). Other studies operationalised interest through asking students to indicate their level of interest in a given task (Lawless & Kulikowich, 2006; Lopez et al., 1997; Sciutto, 1995). Of concern, is whether students' assessment of interest is similar to their assessment of enjoyment, with some authors suggesting the two are quite distinct emotions (Izard, 1984; Reeve, 1989; Silvia, 2001).

In this section a content analysis has identified factors that influence the development of a students' interest, primarily in a mathematics context. This has revealed that interest development can be attributed broadly to both individual and environmental factors. Further, the empirical studies cited, revealed a high level of complexity between such factors. The content analysis has also revealed differences in the way that studies have operationalised the interest construct. Such differences may have implications for the generalisability of the findings.

Discussion

Self-determination Theory (Deci & Ryan, 1985) provides a unifying framework for interest based studies such as those described in this paper. Deci (1992) argued that a person would experience interest when he/she encountered novel activities in a context that allowed for the satisfaction of their basic psychological needs; that is, competence, autonomy and social-relatedness. A student's need for autonomy (being able to choose what he/she does) and social-relatedness can be met if aspects of the classroom environment, of which the teacher is the primary architect, are conducive. A student's need for competence in statistical literacy, however, will be met if he/she possesses the necessary individual factors; that is, a sufficient knowledge of statistical literacy and positive competency-based beliefs regarding his/her ability to acquire statistical literacy. The studies cited in the last section either examined the influence of individual factors or the influence of environmental factors, but not both. There is a need for research aimed at providing a linkage between these two types of factors.

The relationship between students' competency based beliefs, their achievement and interest, which was explored by several of the studies, could be further clarified if the emotions of enjoyment and interest were disentangled. Reeve (1989) provided some evidence to demonstrate that interest is derived from collative sources (see earlier discussion) and enjoyment from the feelings of satisfaction that accompany task competency. He argued that both emotions were necessary for intrinsically motivated learning. Students enjoy success and are likely to reengage with tasks with which they perceive likely success. With no interest, however, they are likely to tire of the task. It is argued, therefore, that positive competency based beliefs in a learning situation will directly influence achievement and that this relationship will be mediated by feelings of enjoyment. Interest, on the other hand, will directly influence achievement.

Implications

The literature review reported in this paper identified a significant gap in the literature as it applies to interest in statistics and indeed statistical literacy. Related research in the mathematics education context suggested that interest in mathematics is predictive of achievement for senior secondary school students but not for middle school students (Koller et al., 2001). Further, there appeared to be a difference in the strength of this relationship according to the knowledge domain in question (Lawless & Kulikowich, 2006). Such findings suggest that further research into the development of middle school students' interest in statistical literacy is needed. As discussed earlier, such research needs to include both environmental and individual factors in order to obtain a broader view of how students' interest develops.

The empirical studies cited in this paper, broadly operationalised interest in two ways. The first was to assess interest in mathematics through measures of importance and enjoyment, and the second was to assess interest through task related interest ratings. As discussed in the last section, enjoyment and interest are distinct emotions. Given that they can be assessed separately, further research in this area may clarify the complex relationship that appears to exist between students' competency based beliefs, their achievement and their interest.

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One Secondary Teacher's Use of Problem-Solving Teaching Approaches

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This paper reports part of a larger study and examines one teacher's use of problem-solving teaching approaches in Years 7 and 9. Thirteen problem-solving lessons were observed over an 18 month period during which the teacher devised and used 7 different problem-solving tasks. Three tasks are described in detail and analysed in terms of task structure and implementation, and how the teacher managed whole-class discussions. Analysis highlights changes in the design and use of the tasks. They became more open-ended and the teacher improved the quality of whole-class discussions to promote student learning and reflection.

Anderson and White (2004) distinguish between *problem solving*, "the process of students exploring non-routine questions, using a range of strategies to solve unfamiliar tasks, as well as developing the processes of analysing, reasoning, generalising and abstracting" (p. 127), and *problem-solving teaching approaches*, "investigations, open-ended questions, and modelling tasks, as well as providing opportunities for students to pose questions and explore new ideas" (p. 127). Problem-solving teaching is therefore an approach in which "teachers see themselves as guides, listeners, and observers rather than authorities and answer givers" (Norton, McRobbie, & Cooper, 2002, p. 39).

Problem solving is an important part of what it means to do mathematics and students require frequent opportunities to solve complex problems and reflect on their thinking (National Council of Teachers of Mathematics, 2000). In New South Wales, the revised syllabus for Years 7 to 10 (Board of Studies NSW, 2002) highlights the importance of problem solving through its *Working Mathematically* strand which includes student processes of questioning, applying strategies, communicating, reasoning and reflecting as critical aspects of mathematics learning (Clarke, Goos, & Morony, 2007). Despite the prominence of problem solving in recent syllabuses and the professed support of teachers for it, research suggests that many teachers do not use problem-solving activities (Anderson, Sullivan, & White, 2004). The research reported here describes how one secondary teacher began to incorporate problem-solving teaching approaches into his lessons and documents the changing nature of the tasks he devised and how he used them.

Analysing Lessons and Tasks

A number of research studies have examined how to support student learning in reform classrooms. These studies broadly refer to three key elements: explicit teacher actions to communicate expectations and encourage student engagement, mathematical tasks and tools, and "building a learning community" (Sullivan, Mousley, & Zevenbergen, 2004) through the ways that teachers design and structure activities and interact with students to support their learning. Some illustrative studies are now described in more detail.

Artzt and Armour-Thomas (2002) developed a model for examining teachers' instructional practices and distinguished between *lesson phases* and *lesson dimensions*. Lesson phases refer to the ways in which teachers initiate, develop and conclude instructional stages of lessons. Phases include the introduction (establishing students' readiness to learn), investigation (helping students learn new concepts and construct new meanings), and summary (assisting students integrate what they have learned and extend it). Phases provide "a temporal sequence of teaching-learning experiences" (p. 11) occurring over one or more lessons. Lesson dimensions are aspects of instructional practice that foster student learning with understanding and include learning tasks (designed to capture students' attention and sequenced so that students can progress and connect new knowledge to what they already know), learning environments (the social and intellectual climate of the classroom designed to support student engagement) and discourse (including teacher-student and student-student communications, and especially the teacher's use of questioning and wait time to encourage student thinking).

Stein, Grover, and Henningsen (1996) examined the characteristics of classroom tasks used during reform-oriented instruction to enhance students' mathematical thinking. The researchers define a *mathematical task* as "a classroom activity, the purpose of which is to focus students' attention on a particular mathematical idea"

(p. 460). Tasks include students' productive output during a lesson, how they produce it, and the resources which they use. Stein, Grover, and Henningsen differentiate between the teacher's expectations of how the activity should be conducted, including the distribution of materials and the teacher's instructions (the *task set up*), and how students actually complete the task (the *task implementation*). They also define *task features* (aspects of the task which promote high-order thinking and permit the use of multiple representations or alternate solutions) and *cognitive demands* (thinking processes entailed in solving the task). They analysed 144 tasks ranging from 10 to 51 minutes duration and comprising an average 52% of total lesson time. Tasks found to support students' mathematical thinking and reasoning included those with high cognitive demands requiring students to explain or justify their reasoning. Teachers can assist students' mathematical development by designing task set up and features to maintain students' focus on mathematical processes and scaffold student learning, but not in ways that detract from the cognitive demands of the task.

Carpenter and Lehrer (1999) proposed three classroom instructional dimensions which may assist in developing students' mathematical thinking and reasoning: *tasks*, *tools* and *normative practices*. While noting the importance of appropriate task and tool selection, the researchers describe the crucial role of normative practices in fostering understanding. In particular, classroom practices should facilitate students' ability to relate their current understanding to new concepts and support the structuring of new knowledge by allowing students to apply and compare alternate solution strategies, make links between different representations of problems, and build on basic problem-solving skills. Teachers can provide extended opportunities for students to articulate and explain their thinking, both during problem-solving activities and in the whole-class discussions which follow.

The critical role of the teacher in the reflective phase of an inquiry-based lesson was considered by Leiken and Rota (2006). They examined the frequency and duration of teacher behaviours such as listening and watching to begin, continue, or summarise a class discussion. They noted that teachers could support students' mathematical thinking during the discussion phase of lessons by asking open-ended questions and waiting for students to explain their answers fully before attempting to summarise them or move on. Whole-class discussions were found to be more fruitful when the teacher asked questions that related directly to student conjectures, stimulated replies from students, and built on students' explanations to develop new mathematical ideas.

Method

The research reported here is part of a larger study focussing on how secondary mathematics teachers in NSW responded to a new syllabus, particularly the extent to which they implemented its Working Mathematically strand (see Cavanagh, 2006). Data were sourced from questionnaires, classroom observations, student work samples and teacher interviews. One participant from the larger study expressed an interest in developing problem-solving teaching approaches and was invited to take part in a second round of interviews and classroom observations which are the subject of this paper.

Mr Richards

Mr Richards (not his real name) was the head mathematics teacher at an independent, co-educational school in a mid to high socio-economic suburb of Sydney. In 10 years teaching, he had taught all secondary grades, though he stated his preference for middle and lower ability classes in the junior secondary years since he felt there was less pressure from external examinations, allowing him greater freedom to experiment in his teaching. Mr Richards said his teaching was "not very conventional" because he tried to involve students actively in their learning, though at the start of the study he was not always sure about how to do this.

Research Design and Data Collection

An ethnographic case study design (Yin, 1994) was used to investigate how Mr Richards developed and used problem-solving tasks with two classes over an 18-month period. Data collection included lesson observations, teacher interviews, student work samples and researcher field notes. The observed lessons were digitally videotaped and the recordings made available during a stimulated recall interview with Mr Richards which lasted approximately 30 minutes and took place as soon as practicable after the lesson. Each interview was structured in three parts: First, Mr Richards discussed how he had developed the lesson materials and the intended purpose of the task; second, he viewed short segments of the lesson and the researcher encouraged him to describe what he was doing and thinking during the lesson; third, he reflected on the entire lesson, discussing its effectiveness and describing how he might teach it differently if he were to repeat it. The interviews were audio-taped and transcribed.

In total, 13 problem-solving lessons were observed. Each lesson was 50 minutes long and based on a problem-solving task which Mr Richards had devised. Typically, each task required two or three lessons for students to complete in small groups and report their findings, and the lessons followed basically the same structure. After settling the class, Mr Richards introduced the problem and ensured students understood what they were required to do. Students completed the task for the remainder of the lesson while Mr Richards circulated among them to monitor their progress and deal with any difficulties they experienced. In the following lesson(s), students completed their investigations and sometimes Mr Richards chose individuals to present their results or he led a discussion on the mathematical ideas which had emerged from the students' work. This paper reports in detail on three of the tasks, selected because they show how Mr Richards' problem-solving teaching approaches developed over the course of the study.

Problem-Solving Tasks

Task 1. This task, shown in Figure 1, was the first to be observed and comprised three lessons. It was for Mr Richards' Year 9 class during a two-week unit on statistics. In the first lesson, students measured and recorded the length of eight of their body parts (e.g., height, foot length, arm span). That evening, Mr Richards combined all of the students' data in a spreadsheet so that they could work in pairs to analyse it in the next lesson using graphics calculators to display scatter plots and writing individual reports of their findings. In the third lesson, students concluded their reports and Mr Richards asked selected individuals to present their work to the class.

Body Parts Investigation

Choose any two body-part measurements and do the following:

- Enter the data into the List menu of your graphics calculator
- Perform a list division

Consider the results of the list division:

- Does it look like there is a correlation?
- Find the mean of the list division
- Draw a scatter plot on your graphics calculator
- Find the correlation coefficient

Write a paragraph stating what you have discovered about the two measurements.

Repeat the activity for any other two body-part measurements.

Figure 1. Task 1.

Task 2. Mr Richards prepared this task, shown in Figure 2, for Year 9 to revise a unit they had just completed on volumes of solids. It was the second task observed and covered two consecutive lessons. In the first, students worked in pairs to draw designs of water tanks. In the next, they wrote individual summaries of what they had done and Mr Richards used the design created by one pair of students to demonstrate an algebraic solution to find the dimensions of a cylinder when its volume and radius were known.

Waggilendra's New Tank
On average, each person in the small country town of Waggilendra uses 30 litres of water per day. There are 2 500 people in the town. Waggilendra needs a water tank capable of holding 30 day's supply of water. You are a water tank consultant. You are to submit 2 designs for 2 possible water tanks. You are to: • Find how many litres of water the tank should hold. Design 1 possible tank which will be sufficient for the town. The diagram does not have to be to scale. You must include the life-sized measurements. • Design a second tank which uses a different type of shape. Include all life-sized measurements.

Figure 2. Task 2.

Task 3. This task, for Year 7, was observed near the end of the study. Students completed the task, in Figure 3, in two consecutive lessons. They began working in pairs to create open-top boxes by cutting out equally-sized squares from each corner of a piece of grid paper and folding up the sides. Mr Richards asked the students to calculate the volume of each box. In the following lesson, he led the class in analysing their results and used a data projector to display a spreadsheet containing values for the length of the cut-out squares (x) and the volume of the box (V). He also showed students a graph of V as a function of x, which he produced using a graphics calculator. The objective of the analysis was for students to find the dimensions which would maximise the volume of the box.

The Open-Top Box
It is possible to make an open top box by cutting squares from the corners of a rectangular card and then folding up the sides. If the card is 12 cm by 6 cm, what might be the volumes of some boxes you can make?

Figure 3. Task 3.

Data Analysis

Glaser and Strauss (1967) propose a constant comparison analysis of key concepts which gradually surface from the data analysis. Data for each task including lesson observations and post-lesson interviews were analysed prior to subsequent school visits. The lesson videotapes were viewed and sections which the researcher thought to be representative of Mr Richard's approach in the lesson were transcribed and read. A similar method was applied to the post-lesson interview transcripts so that general impressions of Mr Richard's statements could be made before a more detailed coding for recurring themes was undertaken. Emergent themes from the researcher's earlier field notes and observations were then used to interpret the subsequent lesson observations and interviews. Data from subsequent tasks were similarly analysed collectively before a final examination of all of the data from the different sources was then undertaken to cross-reference and confirm the main ideas which had been identified. The purpose of all of the analysis was to identify and track changes in the nature of the tasks which Mr Richards devised and how he used them in the classroom.

Analysis and Discussion

Tasks

There were significant changes in Mr Richard's design of the tasks, particularly in their set up and features (Stein, Grover, & Henningsen, 1996). Task 1 is typical of Mr Richard's early attempts to maintain tight control by carefully structuring student activities so they completed a number of steps in a specified order. Accompanying worksheets and Mr Richard's detailed instructions at the start of each lesson precisely described what students had to do. He also chose task features which provided a narrow focus for students' investigations. For instance, students were not encouraged to make any connections between the summary statistics and the shape of their scatter plots. Mr Richards repeatedly stated that his primary objective in the early tasks was for students to learn to persevere with the activity and produce detailed written summaries which explained their solutions clearly.

Tasks 2 and 3 were more open-ended and, instead of repeatedly exhorting students to follow the steps and set out their working carefully, Mr Richards now encouraged the class to “look for patterns in your answers”. Although Task 2 maintains a narrow focus, Mr Richards took advantage of the potential for multiple representations that were a feature of Task 3 and discussed them to a much greater extent than he did previously. Linking the numerical values for the side of the cut-out squares, x, and the volume of the box, V, to the graph produced on the calculator and then working with students to develop an algebraic formula for V in terms of x allowed Mr Richards to make connections between the different features of the problem and encourage students to relate them back to the original task of creating a box and maximising its volume (e.g., by interpreting the coordinate points on the graph with respect to the numerical values in the spreadsheet and the size of the cut-out squares). Although Tasks 1 and 2 could potentially have been used to demonstrate alternate representations, this did not occur. However, the coordination of multiple representations in Task 3 placed a significantly higher cognitive demand on students and ensured a more productive lesson.

Lesson Phases

The description of lesson phases (Artzt & Armour-Thomas, 2002) is used to report changes in the organisation of Mr Richards’ problem-solving lessons and Table 1 shows the lesson time allocated to each lesson phase in the tasks. While the introduction phase for each task decreased slightly over the three observed tasks, the most salient feature of the lesson phases is the extra time allocated to whole-class discussion over the three tasks.

Table 1

Proportion of Time in Each Lesson Phase

Task	Introduction phase	Investigation phase	Discussion phase
1	20%	74%	6%
2	19%	68%	13%
3	15%	59%	26%

There were qualitative differences, too, in the teacher-student interactions in the discussion phase across the three tasks. In Task 1, seven students presented their findings to the class but each spoke for less than a minute and simply recounted their summary statistics and described the shape of their scatter plot. Occasionally Mr Richards asked a direct question such as “What did you get for the mean?” but otherwise he was silent. He did not further probe the students’ thinking, even when they provided fairly simplistic accounts of what they had done. In Task 2, Mr Richards spent about nine minutes leading the class through the design made by one pair of students in order to demonstrate an algebraic solution to the problem—a strategy which few had employed in their investigations. During that time, Mr Richards asked 22 questions of the class, but only two of these were questions which required more than a straightforward response such as a numerical result or calculation. His questioning adhered closely to the traditional IRE recitation pattern (Mehan, 1979) of teacher **i**nitiation followed by student **r**esponse and teacher **e**valuation as the following excerpt shows:

Mr R: I want to show you another way because most of you just picked a number and then used trial and error, which is perfectly fine, but I want to show you another way that you could have done this. [He uses the student example of a cylinder of radius 5mm]. What is the formula for the volume of a cylinder?

S1: πr^2 times height.

Mr R: So πr^2 is the circle and h is the height. Now, how much does the volume have to be?

S 2: 2 250 [m^3]

Mr R: Good. So 2 250 equals π times, and what’s π?

S 3: 3.14

Mr R: Okay, their r was 5 millimetres so that's $(0.005)^2$, times by h. Now if you were able to manipulate the formula you wouldn't have had to guess and check. So let's have a look. [He writes $2\ 250 = \pi (0.005)^2 h$ on the board] What can we do to this so we can just plug numbers in and find out what h is?

Students: [no response]

Mr R: This is algebra. What do we do to both sides?

S 4: Divide it.

Mr R: By …?

S 4: π.

Mr R: So if we divide by π we get 2 250 over π equals what?

S 5: (0.005)2h

Mr R: [He writes 2 250/π = (0.005)2h on the board] Alright, the last step is to do what?

Students: [no response]

Mr R: What do we need to do to both sides to get h on its own?

S 4: Divide.

Mr R: Divide by what?

S 4: $(0.005)^2$

The conversation continued briefly until the class found the height of the cylinder but Mr Richards made no attempt to compare the solution strategies, nor did he ask students to apply the algebraic method to one of their own designs. The classroom discourse at the conclusion of Task 3 was different in a number of significant ways. Not only were multiple representations compared, but the style of Mr Richards' questioning changed: he asked fewer questions overall, but there were more open-ended questions that called for opinion and conjecture from students; he allowed wait-time for students to respond; and he sanctioned incorrect responses without immediately discarding them. For example, the following exchange took place when the students were discussing the spreadsheet showing values of x, the length of the cut-out squares, and V, the volume of the box:

Mr R: Okay, let's look for a pattern. Someone describe what's happening with those numbers there [values of V and x]:

S 1: Except for 0.3 they are all going up and then down.

S 2: They all increase up to 1.3 and then they drop.

Mr R: Yes …

S 3: The 6.8 plus the 12.9 is almost 18.5

S 4: Just with the decreasing theory, what about 2.1, 2.2 and 2.3? Oh, sorry I thought it was 28.

S 5: If you plus 18.5 with 6.8 it's almost 23.3

Mr R: Yes, ok let's just realise what's going on here. The x-values are what?

Students: [No response].

Mr R: You need to know what is actually happening here. Hands up: what is x? You've just done a whole lot of calculations and if you don't know what x is you're missing the point of the exercise. What are these x numbers? Have a think. Have a think about what you've just done …

Conclusion and Implications

This article describes how one secondary mathematics teacher sought to change his classroom practice by incorporating problem-solving learning activities into his lessons. There were a number of factors which assisted Mr Richards in this process. His personal philosophy of teaching mathematics was conducive to allowing time for student investigations since he was not inclined to rush through the syllabus in order to cover the entire content. He also attended two professional development sessions which he described as very influential in alerting him to reform teaching approaches. His accounts of the workshops echo some of Guskey's (1986) guiding principles for professional development in that Mr Richards regarded the presenters as credible practitioners who offered realistic and achievable ideas, and the feedback on the effects of the changes he was making on student learning was positive; student engagement increased and their reports of what they had done became more sophisticated. Mr Richards also adopted a measured approach in reforming his practice. He quickly realised that he could not change everything at once and was judicious in choosing one or two specific areas on which to focus. In doing so, he gradually learned some of the new skills he needed to shift from a transmitter of knowledge to a facilitator of learning and he assisted his students in progressively becoming more active participants in the classroom. Mr Richards also acknowledged that the work he undertook with the researcher as part of this study had been helpful in focussing his attention and providing a mechanism for reflecting on his practice. There could be value in replicating and documenting a similar model of "researcher as sounding-board" in future studies that cover the professional growth of individual teachers.

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Does Student Success Motivate Teachers to Sustain Reform-Oriented Pedagogy?

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This paper investigates one outcome of teacher professional development, student success, which appears to be an effective mechanism to motivate and sustain teacher change for teachers implementing reform-oriented teaching approaches. The reform-oriented pedagogy is inquiry-based and aims to encourage student mathematical thinking in order to increase student achievement in numeracy. It is this outcome of increased student achievement and student enthusiasm for the new teaching approaches that motivated teachers in this reported study to sustain changes to their teaching.

Introduction

As a result of reform-oriented mathematics being introduced into pre-service and in-service professional development programs throughout the last decade researchers are currently exploring the extent to which teachers have incorporated aspects of these reforms into their classroom practice (Anderson & Bobis, 2005; Cady, Meier, & Lubinski, 2006). This paper reports on one aspect of a similar study that explored the extent to which eight primary school teachers incorporated the mathematical reforms introduced in the New Zealand Numeracy Development Project [NDP] in-service professional development. The study concluded that while, overall, the teachers perceived the NDP program reforms as worthwhile and important, their self-reports indicated the intense nature of the personal content knowledge to be learned and incorporated into classroom practice was challenging. For most of the teachers the reform-oriented, inquiry-based teaching approaches took an extended time of ongoing support to internalise and consolidate (Cheeseman, 2006). However, all but one of the teachers reported their student responses to the new teaching approaches as positive and a key motivator for them to continue with the NDP reforms. This paper focuses on this aspect of teacher incorporation of mathematical reforms when motivated by positive student responses to the reform-orientated teaching approaches.

Background

Reform-oriented mathematics has evolved in western countries because of international concerns about student underachievement in mathematics. The resulting critique led many educators to argue that achievement standards could not improve unless mathematics was taught differently so as to engage more effectively all groups of learners (Duit & Confrey, 1996; Fennema & Franke, 1992). Skemp (1986) claimed that most teachers taught in an instrumental manner, whereby the mathematical procedure was given to the students, rather than in a relational manner, where the students were actively involved in seeking a solution path. Duit and Confrey (1996) concluded that rote learned "procedural knowledge becomes an effective mathematical tool only within the framework of conceptual understanding" (p. 83).

The critique of procedural mathematics pedagogy by researchers led to a growing acceptance within the reform movement of an alternative view of learning mathematics based on Constructivist theories**.** Cobb et al. (1991) saw the constructivist view of mathematics learning as inquiry-based, "a process in which students reorganise their conceptual activity to resolve situations that they find problematic", rather than "a process of internalising carefully packaged knowledge" (p. 5). Such mathematical reforms required a major shift in pedagogy from teaching approaches that focused on a procedural approach to a teacher-facilitation approach focusing on student thinking and reasoning (Stein & Strutchens, 2001).

New Zealand teachers were assisted to accommodate these reform-oriented teaching approaches by participating in the government-initiated NDP in-service professional development program. To be eligible to participate in the NDP program teachers were required to enrol as part of a whole school or syndicate based initiative. The nature of the professional development was long-term and school-based in order to create a community of learners encompassing the management team, teaching staff and the students. The reform approaches included an inquiry-based strategy-teaching model, whereby teachers facilitated student learning through problem-solving.This was to involve higher-order thinking, the use of apparatus, and the encouragment of student explanation and justification of strategies (Higgins, 2004). Importantly, discussions of alternative mathematical strategies served as a model for new learning to occur for the students, and often for the teacher as well (Steffe & D'Ambrosio, 1996). Teachers found by asking students to express their mathematical thinking they, too, encountered problem-solving strategies that were new to them, thereby creating an opportunity for reciprocal learning for both students and teacher.

The NDP strategy-teaching approaches followed the constructivist view that places students central to the teaching and learning process (Hughes, 2002). The teachers in this wider study found the complex nature of the reforms (e.g., coming to terms with understanding multiple strategies and moving away from procedural-based algorithms) led to huge shifts in their content knowledge and teaching approaches. They struggled to accommodate the range of these reform-teaching approaches without ongoing support (Cheeseman, 2006). Similar teacher reports were identified in the nation-wide government evaluations of the NDP (Young-Loveridge, 2004). Other studies investigating teacher change in relation to curriculum and professional development initiatives reported teachers' difficulties accomodating the reforms (Anderson & Bobis, 2005; Cady, Meier, & Ljubinsky, 2006; Stigler & Heibert, 1997). These studies found that teachers accommodated some aspects of the reform pedagogy but not others. This report suggests that teachers in this study were more inclinded to accommodate pedagogical practice that created positive responses from their students resulting in increased student achievement in or enthusiasm towards mathematics.

Methodology

The study used an interpretive approach to investigate the perceptions of a small sample of teachers who participated in the in-service professional development of NDP. The mode of enquiry was in the form of forty-five to sixty minute face-to-face interviews. The in-depth nature of the interviews provided an effective method to collect quality data from the participants, eight teachers, who shared self-reports (Davidson & Tolich, 2003) regarding the extent to which they had incorporated the mathematical reforms introduced in the NDP program. Semi-structured, open-ended interview questions were formulated as a guide for the researcher to follow. The main intention of the researcher was actively to listen so that the interview was shaped by the participant's voice. Davidson and Tolich (2003) describe the participants' layers of understandings as a rich texture of experience and see the use of good quality questions as a key "to get people talking along the thematic lines of the research" (p. 148). They suggested prompting to be used only "to bring out the main points of what they have to say" (p. 148). Probing was used spontaneously on numerous occasions during the interviews to seek clarification, or to encourage the participants to elaborate their ideas. The quotations from the participants included, verbatim, in this paper were derived from two general interview questions: "What aspects of the project helped you most? How and why?"; and, "How has your developing understanding from participating in the project changed your teaching practice?" Some prompts were used with these questions where the researcher thought it necessary to probe further.

The teachers were selected from four urban primary schools, two teachers from each school. The schools were chosen because they were centrally located and their teachers had been involved in the program for a minimum of two years. The researcher was responsible for the selection of two of the participants. The principals of the respective schools selected the remaining six teachers. The teachers selected covered the range of age groups taught, from five-year olds in the junior school to ten-year olds at the senior level of the school. Seven of the teachers were in their second year of using the new reform teaching approaches with the exception of one teacher who was still participating in the formal NDP program. Although the small sample of teachers involved in this interpretive study made generalisation problematic, the in-depth responses from the interviews provided valuable information about the centrality of the students' responses to teachers' learning and valuing of the reform teaching approaches.

Findings

Seven out of eight teachers, without prompting referred to student success and student enthusiasm as key motivators to sustaining the changes they had made to their classroom practice as a result of participating in NDP. The new inquiry-based teaching approaches involved working with students in small groups to allow for interaction between the teacher and students, and interaction among students. The teachers reported that listening to students explain and share their mathematical thinking was a major shift in pedagogy and equally, a shift in learning styles for their students. The teachers noted the teaching approach involved both teaching and formative assessment providing information on their students' achievement and teaching needs. Their excitement about their student achievement is reflected in their comments.

Robyn: On reflection I can see how valuable the program has been; seeing the progress the kids have made and the stages they have moved on to. So it [personal time given to project] was worth it! … I've really enjoyed teaching the program and I can see how [well] my kids have responded to it. I really like the strategies based way and I can see how it will provide a really good grounding for those more advanced concepts further down the track.

Sandra: I think you actually do see kids' lights going on and they actually make those connections, and I think they might have done that in the past but I wouldn't probably have noticed it. But now you know what you're looking for to make the jump and that's quite exciting.

Claire: The NDP program provides a chronological building of skills and strategies which gives the children confidence because 99% of them actually experience success before they are asked to take on a more difficult concept.

Teachers' enthusiasm was further supported when they saw that student achievement in their class was evident in school and class assessment statistics.

Amanda: Our [school wide] statistics show that the children have improved in numeracy over the last couple of years [since undertaking the professional development].

Ruth: Most of them [children in class] have jumped tremendously. One that's most dramatic … she didn't have very good basic maths knowledge at all at the beginning of the year and I tested her on Form A [countable numbers]. She had no idea about proportions and ratios, and now she does and is on Test B [part-whole strategies] – for somebody like her that's a huge jump.

The students' positive responses to and their engagement in the new teaching approaches was another positive aspect identified by two of the teachers.

Ruth: Some of them [children] while we were teaching fractions and decimals were very reluctant at first but once they realised that there was quite a bit of fun involved [referring to less paper and pencil work and increased use of apparatus], you could see they were actually enjoying it. Quite often I would have them sitting down during morning tea still carrying on and I would say, "Don't you want to go now? Can we pack up please?"

Sandra: I guess what we're doing is we're asking kids to explain their thinking which we've never done before. We just asked for a result but now they have to justify what they've done, and say why and it's accepted. They know that people are listening, and the kids are sparking off [each other] and it's totally different, isn't it?

The teachers also spoke about how the students' enthusiasm towards the teaching approaches had helped them sustain the content and pedagogical changes they had made.

Claire: It's changed my attitude – I love maths now. I really like it – the kids say, "Oh can we have some more time?" And they don't want to get off the mat. It's [NDP] very effective. The children absolutely love it. The first thing every morning they say, "Can we have maths now?"

Amanda: The children get excited at the number properties [abstract] stage – everyone bursting to say how they got it [strategy used]. They are getting immediate feedback and they feel so proud! … I'd never go back to just teaching algorithms without teaching children [a range of] mental strategies … it's really doing them a disservice. I just see it as so good, encouraging the children to think. I am sold on it [NDP teaching approaches].

Janet: The children loved the numeracy games! They like the number, the challenge, the practice activities, and because the children explain their thinking it's [NDP teaching approaches provide] very visible achievement!

Sarah: The children love it [maths] because they share their thinking and they, particularly the boys, love the materials and games … I love it because I think that you are much more aware of what you want the children to know and where to take them next – you are not actually giving them the information, you are trying to get it out of them.

The eighth teacher was less enthusiastic about the professional development program and felt her students were not showing such positive signs of achievement or excitement about the new teaching approaches.

Helen: I think I keep kids in holding patterns a bit long because this is my first year. I think when you are learning yourself; the kids have to go on a holding pattern while you read the brochure [booklet] and keep working at what you are going to do.

Helen was the only teacher in the group of participants who was in her first year of the program and she felt her teaching was taking a backwards step before she was able to move the students forward. Her comments could be due to a number of reasons: she was less trustful of the program; she needed to know more theory behind the inquiry-based teaching approaches and or she had older students (10 year olds) who were already highly successful learners of mathematics. Her reservations were further illustrated by her comments about the program and reference to her students' progress.

Helen: I'm not sure about that theory and I would like to know what the theory is. When you start something you really don't know where you are going so you don't know what it is that you are learning [the big picture]. E.g. why we are doing this style? What the benefit is? That it is not just a fad that will come in and disappear and the kids will be a year worse off.

Helen's concern that her students may be "worse off" evolved from her professional development switch to inquiry-based teaching from the procedural-based teaching of algorithms that was expected of the students when they progressed to the intermediate school (11-12 year olds) the following year.

Discussion

The notion of student enthusiasm and increased achievement resulting from the teachers' inquiry-based teaching approaches was a common thread in seven of the eight teachers' self-reports. The firsthand experience of incorporating aspects of the professional development (e.g., eliciting student strategies and the increased use of numeracy games) in their classroom with their students allowed the teachers to see the immediate success the program had in engaging their students' learning. Student success and engagement in the inquiry-based teaching approaches appeared to be an important factor to maintain the teachers' shift in pedagogy and beliefs as illustrated in Amanda's comment, "I'd never go back to just teaching algorithms … it's doing them [the children] a disservice". Research on teacher change reports teachers' beliefs and attitudes affect the quality of engagement in professional development and suggest "the most significant changes in teaching attitudes come after they begin using a new practice successfully and see changes in student learning" (Guskey, 1985, p. 1). This outcome was reflected in the teachers' reported enthusiasm using the new teaching approaches and motivated them to continue with the changes they had made as a result of participating in NDP. Guskey (2002) considers the positive impact of professional development on student learning as reflected in teachers' new learning directly benefiting students in any way as an effective evaluation of quality teacher professional course work.

The school-based, in-class situation of the in-service professional development whereby teachers worked directly with their own students allowed for the possibility of reciprocal learning or "teacher reflexivity". Steffe and D'Ambrosio (1996) described "teacher reflexivity" as how student mathematical development contributes to teacher development. Active listening to student strategies can add to the range of teacher's strategies. Reflexivity also extends to teachers using their students' mathematical thinking to inform their planning and teaching (Simon, 1995; Steffe & D'Ambrosio, 1996). The teachers' discussion about student achievement, although not referred to directly, implied they were indeed using their student responses to inform their planning. Some teachers made mention elsewhere in the study of increasing their own mental agility after listening to a range of student strategies. These were important shifts in teaching approaches made by the teachers after listening to their students' mathematical thinking and observing the students' enthusiasm when sharing their strategies. The teachers' willingness to try out the new strategy-teaching approaches with their students is supported in the professional development literature that regards teachers who are risk takers and demonstrate improving their own mathematical knowledge are more likely to motivate students and to foster their learning (Barth, 2001; Fullan & Hargreaves, 1996; Steffe & D'Ambrosio, 1996). The teacher comments demonstrate they were able to motivate their students mathematically and that their student responses in turn motivated them to sustain teacher change.

Indeed, increased student achievement in and enthusiasm for numeracy was a significant motivation for seven of the eight teachers to continue to use the new inquiry-based teaching approaches promoted by the NDP professional development program. The eighth teacher used the approaches while participating in the program but was less convinced of the value of them. Her less favourable attitude to the new teaching-approaches appeared to be influenced by her students' bored disposition to the strategies introduced and it seemed less doubtful that she would sustain the new learning over time. Cobb et al. (1991) describe the change in teacher beliefs, knowledge and practice as a result of participation in professional development as reflexive in nature. They consider that "it is essential that teachers have reason and motivation to want to reorganise their pedagogical practice, which can only occur if teachers come to view aspects of their current practice as problematic" (p. 8). Helen believed moving away from her current practice (the procedural-based teaching of algorithms) was not beneficial for her students in terms of mathematical expectations at intermediate school and therefore did not see her current practice as problematic. Concern about her students' achievement appeared to influence her beliefs about and acceptance of the strategy-teaching approaches.

The literature on teacher change reports new knowledge and pedagogical experiences are filtered through the teachers' belief system and then interpreted in the teachers' own way (Fennema & Franke, 1992; Stigler & Heibert, 1997). For seven of the teachers, student enthusiasm for the new teaching approaches and achievement as a consequence of these practices led to their feeling successful regarding the changes to classroom practice. This mirrors the professional development literature on teacher change (Barth, 2001; Fullan & Hargreaves, 1996; Higgins, 2004). For one teacher the student responses to the new teaching approaches were not influential in motivating her to change her beliefs dramatically or commit to incorporating the reforms over the long term.

Although this study had a limited sample of participants, seven of these eight teachers were motivated to continue the initiatives presented in the NDP reforms. This motivation was generated by the students' enthusiastic responses to the teaching approaches and their visible increase in achievement, thus providing an effective mechanism to sustain teacher change. The richness of the teachers' voices in recognising the importance of student achievement and enthusiasm as central to successful professional development makes worthwhile academic commentary.

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Year Five Students Solving Mental and Written Problems: What Are They Thinking?

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This study investigated strategies used by Year five students to solve multi-digit whole number problems mentally and in written formats. The students participated in semi-structured interviews and the think-aloud strategy was used to determine the students' metacognition. The results indicated that students experience more difficulty answering oral questions than written questions. In particular subtraction and division questions proved more troublesome for students. The range and type of strategies used was indicative of the level of cognitive understanding of the students.

The increased interest in mental computation shown by researchers since 1990 is due in part to the perceived connections between strategies used by students and their conceptual understanding. Straker (1999) stated that "the ability to calculate mentally lies at the heart of numeracy" (p. 43). Knowledge and use of mental strategies may assist the development of conceptual understanding of multi-digit number calculations (Fuson et al., 1997). Unlike most written procedures, mental computation requires an individual to use conceptual understanding to select appropriate strategies.

According to Northcote and McIntosh (1999), adults predominantly use mental calculations in everyday life. Despite this knowledge, mental computation remains sidelined in many Australian classrooms (Morgan, 2000). Primary teachers often have little appreciation for the importance of mental computation in the curriculum. Traditional written algorithms continue to be emphasised at the expense of mental computation in the majority of classrooms.

This report describes strategies used by Year Five students to solve problems presented in oral and written formats and is part of a larger study that includes practicing primary teachers and preservice teachers. Although there has been an increased focus on mental mathematics over the past 15 years, many teachers remain unfamiliar with the range of strategies that students use to solve oral problems. Knowledge of strategies used to solve problems in different formats may provide teachers with more information about students' conceptual understanding. The strategies described in the following section are considered to be somewhat hierarchical and therefore indicative of the level of number sense development (Foxman & Beishuizen, 2002).

Literature Review

Mental Computation

Multi-digit addition and subtraction strategies. According to Verschaffel, Greer, and De Corte (2007) knowledge of multi-digit mental computation is not as comprehensive as that of single-digit calculations. Investigation of this area can provide valuable insight into students' conceptual understanding of multi-digit number calculation. Heirdsfeld and Lamb (2005) used the work of noted mathematics researchers including Beishuizen, Reys, and Thompson to describe classes of mental strategies used for addition and subtraction. An excerpt from Heirdsfeld and Lamb's table follows (coding is included for reference in the table of results):

- Counting on or back – 28+35 : 28, 29, 30, (c);
- Separation

1. Right to left - 28+35: 8+5+13, 20+30+50, 63 (sr);
2. Left to right - 28+35: 20+30=50, 8+5+13, 63 (sl);
3. Cumulative sum or difference - 28+35:20+30=50, 50+8=58, 58+5=63 (sc);

- Aggregation

4. Right to left - 28+35: 28+5+33, 33+30=63 (ar);
5. Left to Right - 28+35: 28+30=58, 58+5=63 (al);

- Wholistic

6. Comprehension - 28+35: 30+35=65, 65-2=63 (wc);
7. Levelling - 28+35: 30+33+63 (wl);
    - Mental image of pen and paper algorithm – Child reports using the method taught in class, placing numbers under each other, as on paper, and carrying out the operation, right to left (m).

Multi-digit multiplication strategies. Verschaffel et al. (2007) draw on the work of Baek (1998) to describe four multiplication strategies as follows:

Direct modelling strategy – child uses drawing or manipulatives (dm);

- Complete number strategies – child uses techniques such as doubling or repeated addition (cn);
- Partitioning number strategy – child splits the multiplicand or multiplier into simpler numbers and combines answers at the end (p);
- Compensating strategy – child adjusts the multiplicand and or the multiplier to make the calculation easier (cs).

Multi-digit division strategies. Likewise, Verschaffel et al. (2007) describe Ambrose, Baek, and Carpenters (2003) work on division strategies as follows:

Working with one group at a time – child repeatedly subtracts the smaller number from the larger number (og);

- Split dividend – child carries out division of units, tens and hundreds separately (sd);
- Building up strategies – child combines strategies in a complex manner (b).

Mental Computation versus Written Calculation

Verschaffel et al. (2007) explain that "standard algorithms have evolved over centuries for efficient, accurate calculation and for the most part are far removed from their conceptual underpinnings" (p. 574). Mental calculation strategies, however, are typically closely related to the conceptual nature of the calculation. While most researchers advocate for students to be exposed to both mental and written calculation strategies, an inordinate amount of classroom time focuses on the practice of traditional written algorithms (Verschaffel et al., 2007).

An algorithmic approach to mathematical computations promotes passive responses from students (Thompson, 1999). Mental computation in contrast requires more active participation. Students, when using mental computation, are more likely to thoughtfully engage with the problem and apply conceptual knowledge to assist in the selection of appropriate strategies. According to Callingham (2005) further research is needed into the relationship between algorithms and conceptual understanding in the area of mental computation.

Metacognition

Whilst an in-depth discussion of metacognition is beyond the scope of this paper, some comments are pertinent. According to Panaoura and Philippou (2007), "metacognition is a multidimensional construct with two main dimensions: knowledge about cognition and regulation of cognition" (p. 149). Assessing students' competence in appropriately selecting and using strategies to solve mathematics problems is an important component of metacognition.

Wilson and Clarke (2004) describe metacognition as having three parts: (a) awareness of thinking, (b) evaluation of that thinking, and (c) regulation of that thinking. In order to enhance students' metacognition in mathematics, teachers must have knowledge of the process of strategy choice and application. This understanding will potentially result in more effective selection of problems and instruction by teachers.

Purpose

The study was designed to provide teachers with information about the mental and written strategies that students use to solve multi-digit addition, subtraction, multiplication and division problems. The research questions answered in this paper are:

What strategies do Year Five students use when solving mental and written multi-digit whole number problems?

- What metacogonitve awareness of the strategies used do the students exhibit?

Methods

Study

This study took place in an Australian primary school where the researcher had previously conducted professional development and co-taught activities with some classroom teachers. Following a verbal description of the study by the researcher and teacher (for the students), an information letter was sent to parents/guardians and students volunteered to participate in the research. Students were not selected on the basis of ability.

Eleven Year Five students (four boys and seven girls) participated in semi-structured interviews that were conducted in a small private room by the researcher. Fontana and Frey (2005) describe interviews as "one of the most common and powerful ways in which we try to understand our fellow humans" (pp. 697-698). A semi-structured format was used to provide a basis for comparison between the participants while allowing for some flexibility to cater for individual differences. To this end, all students were asked identical base questions but subsequent interaction varied slightly according to the students' responses. Wilson and Clarke (2004) describe a multi-method interview (MMI) as an appropriate method for assessing metacognition. MMI involves an interview, self-reporting, a think-aloud strategy, observation, and video recording. The method was adapted for this study, as the emphasis needed to be on the strategies used to solve the problems rather than on metacognition.

Students were individually asked a series of four oral questions and four written questions (see appendix). The written questions were presented using a horizontal format (for example, 600 – 35) and students were able to rewrite the questions in any way they wanted to. Each interview was videotaped and transcribed for purposes of analysis.

During the interview, students were asked to solve one question at a time and subsequently asked to describe how they solved the problem. In particular they were asked to describe what they were thinking when they solved the problem. The "Think-Aloud Strategy" provides an insight into cognitive processes and is suitable for problems that require more than an automatic response and therefore necessitate some thinking (Van Someren, Barnard, & Sandberg, 1994). It is suggested that the Think-Aloud Strategy be used for a small set of problems, as the technique can be time consuming.

Analysis

Transcriptions of students' answers were analysed and coded using the framework of strategies defined in the literature review. The interviews were scrutinised for the whole group and for individuals. Patterns of strategies used were examined according to accuracy of answers and the metacognition of the students.

A comparison of students' success with oral and written questions provided another perspective for the analysis. Table 1 details the number of correct answers for all problems and indicates the strategies used for the oral questions. In order to maintain anonymity, pseudonyms have been used for student names.

Results

Students were able to provide descriptions for a vast majority of questions, although there were five instances when students could not describe the strategies used. In two of these cases students were unable to answer the question, probably due to lack of conceptual understanding. On the three other occasions, students successfully answered questions but could not describe how they had solved the problems.

Some patterns are discernable from the type of questions answered correctly. The students were far more capable of using mental strategies to solve addition questions than subtraction questions. Likewise oral multiplication questions were answered more successfully than division questions. It is interesting to note that these features were not evident for the written questions.

While an examination of the strategies used by students does not reveal any clear pattern, there are many interesting observations to be made. Seven of the nine students who described a strategy for the oral subtraction question chose to use a separation strategy. In contrast, only four students selected the separation strategy for the oral addition question. The students appeared to experience more difficulty with the subtraction problem and resorted to the lower level separation strategies. All but one of the students correctly answered the addition question and a wider range of strategies was used by the group. In fact four students selected the higher level aggregation or wholistic strategies.

The partitioning strategy was favoured by the majority of students for the oral multiplication question. Of the three students who gave an incorrect answer for the multiplication question, one used the lower level complete number strategy and one attempted to mentally use the written algorithm. The oral division question proved to be the most troublesome for students. Reasons for incorrect answers for the mental division question can be categorised as follows: (a) inaccurate multiplication tables knowledge; (b) unsuccessful attempt to mentally use the written algorithm; and (c) lack of conceptual understanding of division.

Table 1

Summary of Students' Results

Name	**Oral -**	**Oral +**	**Oral x**	**Oral /**	**Written -**	**Written +**	**Written x**	**Written /**	**Total**
Alan	N (sr)	Y (wl)	N (cn)	N (n)	N	Y	N	N	2
Lisa	N (c)	Y (ar)	Y (p)	N (dm)	N	Y	Y	N	4
Serina	Y (sr)	Y (al)	N (p)	N (dm)	N	Y	N	Y	4
Teresa	N (sl)	N (sl)	N (m)	Y (b)	Y	Y	Y	Y	5
David	Y (sl)	Y (sl)	Y (p)	N (n)	Y	Y	Y	N	6
Robert	N (sl)	Y (sl)	Y (p)	N (t)	Y	Y	Y	Y	6
Alice	N (n)	Y (n)	Y (p)	Y (t)	Y	Y	N	Y	6
Georgia	Y (n)	Y (sl)	Y (p)	N (t)	Y	Y	Y	Y	7
Leanne	Y (m)	Y (m)	Y (p)	Y (t)	Y	Y	Y	Y	8
Helen	Y (sl)	Y (m)	Y (m)	Y (t)	Y	Y	Y	Y	8
Edward	Y (sr)	Y (wl)	Y (p)	Y (t)	Y	Y	Y	Y	8
Total	6	10	8	5	8	11	8	8	64

Y = correct answer N = incorrect answer (code) = uses the strategy code defined in the literature review (n) = no explanation given, (t) =multiplication tables, codes such as (m) were used across a range of questions

Students answered more of the written questions correctly than the oral questions. Only 11 of the 22 oral subtraction and division questions were answered correctly. In contrast, 19 of the 22 written subtraction and division questions were answered correctly. Students appeared equally capable of answering oral and written addition and multiplication questions.

Discussion

Success in using written algorithms is not necessarily indicative of students' conceptual knowledge. This is illustrated by the different success rates for oral and written subtraction and division questions. From descriptions given of strategies used for the oral questions, some conceptual misunderstandings were evident. These same students however, correctly answered written forms of the questions and described in detail how to use an algorithm.

Lisa's Response to Questions One and Five

Researcher: A chocolate frog costs about 45c. How much change would you get from $5?

Lisa: $4.95… I just took, I went forty-five, then I went all the way to a hundred and then I just counted and took $1 off.

Researcher: Can you work out the answer to this question? (shows written form of 600-35)

Lisa: (Lisa rewrote the problem in a traditional written vertical ormat) 565 … With the zero you can't do it so you have to borrow a number, except the next number is zero, so I went to the six and then I went to the five. Then I brought the one across and then I cross that out and that became nine and then I took one and added it on to zero. Then I took nine and take three which is six and five take zero, which is five.

There appeared to be some connections between students' success rates with questions and the ability to clearly describe the strategies used to answer oral questions. Panaoura and Philippou (2007) concluded that more capable students exhibit better metacognitive skills**.** The following example illustrates the metacognition of a student who correctly answered all eight questions.

Edward's Response to Question Two

Researcher: There is a school trip for two classes and they need to know how many students will be on the bus. Mr T's class has 19 students and Mrs D's class has 27 students. How many students altogether?

Edward: 47 … I took three from nineteen to get thirty, so it is easier to work out. It's actually forty-six.

This response also supports the theory that more capable students tend to select higher level strategies (Foxman & Beishuizen, 2002). In this example, Edward used a wholistic levelling strategy whereby he subtracted three from nineteen and added three to twenty seven (19-3 = 16, 27 + 3 = 30, 16 + 30 = 46). Conversely, students producing few correct answers tended to select lower-level strategies to answer questions. Alan only successfully answered two questions and his response to question three exemplifies this notion.

Alan's Response to Question Three

Researcher: A local team scored 14 goals. How many points did they score?

Alan: 100 … I did it a kind of long way. I kept adding six until I got to fourteen.

It is interesting to note that some of the more capable students chose to use mental strategies for selected written questions. For example, Georgia (who gave seven correct answers) used a mental strategy to answer question five.

Georgia's Response to Question Five

Researcher: Can you work out the answer to this question? (shows written form of 600-35)

Georgia: 565 … I know that hundred take thirty is seventy and take five is sixty-five and then I just add five hundred.

An unexpected observation was the fact that some capable students solved the oral questions by visualising a written sum in their heads. This is illustrated by Leanne's description of the strategy she used to answer questions one and two.

Leanne's Response to Questions Two

Researcher: There is a school trip for two classes and they need to know how many students will be on the bus. Mr T's class has 19 students and Mrs D's class has 27 students. How many students altogether?

Leanne: I think that would be forty-six students.

Researcher: You worked that out in your head very quickly, how did you do that?

Leanne: Well in my head I pictured a sum and I added the two together and I got an answer.

Some struggling students used written strategies to solve oral problems, because they did not appear to have a repertoire of mental strategies. In contrast to Foxman and Beishuizen's (2002) findings, the type of strategies used do not always equate to a student's mathematical ability. There are numerous factors, including background experiences and teachers that impact the choice of strategy to solve oral questions.

Conclusion

This was a small-scale study and it is important therefore the results are not generalised. It is true, however, that the results do support the findings of previous research described in the paper. Students' conceptual understanding largely corresponds to the strategy choice described to solve multi-digit whole number problems mentally. Furthermore, cognitive ability is evident in the students' metacognitive fluency.

The lack of skills and understanding of multi-digit subtraction and division displayed by some Year five students is concerning. Poor conceptual knowledge has been masked by their ability to perform written calculations. This deficiency has implications for the students' future success with mathematics.

The next step in this study will engage practicing primary teachers with the data. Teachers will view the DVD of the students being interviewed and discuss the strategies used and the implications this has for teaching. The role of mental computation in developing conceptual understanding will be explored by the teachers. Following this, the teachers will develop pedagogy to effectively incorporate mental computation into the classroom.

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Appendix

Verbal Questions

1. A chocolate frog costs about 45c. How much change would you get from $5?
2. There is a school trip for two classes and they need to know how many students will be on the bus.
   Mr T's class has 19 students and Mrs D's class has 27 students.
   How many students altogether?
3. How many points in a football goal? (answer given if the student did not know this)
   A local team scored 14 goals. How many points did they score?
4. You have 56 lollies to share with 6 of your friends. How many will you each receive?

Written Questions

5. 600 – 35 =
6. 6. 79 + 33 =
7. 17 x 6 =
8. 84 / 7 =

Following each question the students were prompted to describe how they solved the problem. Some of the prompts used were:

- How did you work that out?
- Can you tell me what you thought in your head when you worked out the answer?
- How did you know what to do?
- How could you check your answer?

Mathematics for Engineering Education: What Students Say

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In the context of curriculum renewal, 111 engineering students were surveyed about their perceptions of the relevance of topics in their core mathematics and science subjects to their later engineering studies. The mathematics topics rated most highly were trigonometric functions and linear algebra. Differences by student major and by stage through the degree are described. The implications of low ratings and ways to address these are discussed.

This research was undertaken as part of a curriculum renewal project, funded as a Learning and Teaching Performance Fund initiative, in which academics from the Faculties of Science and of Engineering at the University of Technology, Sydney are collaborating to renew the mathematics and science curricula in the Engineering degree course. The curriculum renewal project aims to further enhance the relevance of mathematics and science to engineering students, so that students have the enabling mathematical and scientific knowledge and skills to engage confidently and effectively in their engineering studies and are prepared for life-long professional learning.

The project follows a design cycle of needs analysis, curriculum planning, design and development, implementation, and review and evaluation. This paper reports on part of the needs analysis, in particular the results of a survey of selected students in the degree.

Background

The place of mathematics and science in engineering education is currently under review. In Australia, the UK and the USA this is linked to a concern that at a time of high demand for engineering graduates there is a declining level of interest in choosing engineering courses, and a declining proportion of students studying mathematics and science to high levels in secondary schools (King, 2007). There is debate about the kind of mathematics needed by future engineers, with a shift towards the use of software packages and data handling "on the job" yet to be reflected in mathematics courses that remain based on algebra and calculus. On the other hand, algebra and calculus are used *within* engineering subjects, and a facility with mathematical notation and standard techniques is essential for following the development of the subject material in many engineering subjects. This raises the question of whether students are aware of the connections between the mathematics they study early in their degree and the later engineering subjects.

The literature review conducted for the project *Mathematics Education for 21st Century Engineering Students* (Henderson & Keen, 2008), reports on a range of subject designs and teaching methods that demonstrate adaptations to the needs of 21st Century engineering students. These include (1) using computer based methods such as web-based delivery, computer algebra systems and interactive software, (2) using flexible delivery, and support through tutoring and drop-in centres that are provided to address the issue of variability in students' mathematical preparation, (3) taking a multidisciplinary approach in various ways, such as team teaching of subjects designed by mathematicians and engineering academics working together, and (4) using problem based learning strategies. In recent years at the university where we work, all these strategies have been incorporated to some extent within our offerings of mathematics and science subjects within the engineering degree. As part of our curriculum renewal project, we needed to find out from stakeholders (including students and academic staff) the opinions about the success or otherwise of the teaching and learning in these subjects, especially in terms of revealing the connections between mathematics and later engineering subjects.

Survey of Students

Timing of the Survey

At UTS, all engineering students undertake two semesters of internship, working in their professional fields. These internships are scheduled for the first six months of the second and fourth years of the degree, as shown in Figure 1. All students study two mathematics subjects, usually taken as Core subjects in first year; and Physical Modelling, which is usually taken in first year, first semester as a Core subject.

TYPICAL PATTERN OF PROGRESS IN THE BE, DipEngPrac

YEAR 1		2		3		4		5	
SEMESTER 1	2	1	2	1	2	1	2	1	2
Core	Core	Internship 1	Core	Core	Core	Internship 2	Core	Core	Capstone Project
Core	Core		Field of Practice	Field of Practice	Field of Practice		Field of Practice	Field of Practice	Field of Practice
Field of Practice	Field of Practice		Field of Practice	Field of Practice	Field of Practice		Field of Practice	Elective	Elective
Field of Practice	Field of Practice		Field of Practice	Field of Practice	Field of Practice		Elective	Elective	Elective
	Engineering Practice Preview 1		Engineering Practice Review 1		Engineering Practice Preview 2		Engineering Practice Review 2		

Figure 1. Diagram showing the arrangement of subjects within the Engineering Degree.

It was decided to survey students in the subjects Engineering Practice Review 1 (EPR1) and Engineering Practice Review 2 (EPR2) in Spring semester 2007. These students had just completed either their first or second internship. It was felt that these students would have an opinion about the usefulness of their mathematics and science studies that would be shaped by their recent internship experience; and that those in EPR2 would also have completed more than half of their engineering subjects and have more to say about the way the mathematics and science subject supported their engineering subjects.

Design of the Survey

The survey was anonymous. Some biographical questions were asked, including the student's major. The body of the survey consisted of tables listing the content of core mathematics and science subjects, with columns to tick to indicate a response: either very relevant, somewhat relevant, or not relevant. The survey concluded with summary questions, asking for responses of "Agree", "Disagree" or "Neutral", to the statement:

> I found the current structure and order of topics within Mathematics, Statistics and Science subjects was satisfactory.

In addition, students were asked to rate in a similar way four statements about the connections between mathematics and science subjects and their later engineering subjects:

> I feel there is a clear connection between first year mathematics (statistics, physics, chemistry) and later engineering subjects.

Below these questions were the open-ended questions: "If you have ticked the 'disagree' column for any of these questions, please explain why" and "Any other comments?" with room to write responses.

Analysis and Results

We discuss some of the results and implications for the core first year mathematics subjects, Mathematical Modelling 1 (MM1) and Mathematical Modelling 2 (MM2). Each topic within the subjects was given a relevance rating by the students and the following scores were attached for the data analysis:

1. Not relevant;
2. Somewhat relevant;
3. Very relevant.

Average rating scores were used to get an overall idea of which topics seemed of higher relevance to the students. They were also used to gain insight into the difference of opinions between the students from different majors and stages through the degree. Tables 1 and 2 indicate the student ratings of topics in each of the core mathematics subjects. In each table, bold type indicates the topics with the three highest ratings, italics indicates the topics with the three lowest ratings. In Table 3 the average ratings are shown by the students' major, and in Table 4, by their stage in the course.

Table 1

Student Ratings of Topics in Mathematical Modelling 1

Topics in Mathematical Modelling 1	**Mean**	**N**
Introduction to physical and mathematical modelling	2.33	110
Functions, measurement, and interpreting physical results	2.37	110
Differentiability	2.30	110
Differential equations arising from physical problems	2.23	109
dy/dx = f(x)	2.25	109
dy/dx = k y	2.21	109
dy/dx = a y + c	2.21	109
Idea of solution by series	2.05	110
Growth and decay problems	2.01	109
Oscillatory motion including SHM, damping, forced oscillations, resonance	2.10	109
Trigonometric functions	2.38	110
Inverse functions and inverse trig functions	2.05	110
Complex numbers: polar, Cartesian, exponential form, DeMoivre's theorem	2.08	110
Integration from first principles (Riemann sums)	*1.93*	106
The logarithm function	2.22	109
Methods of integration: by substitution, by parts, by partial fractions	2.25	110
Using a table of integrals	2.11	109
Introduction to non-linear oscillations	*1.96*	111
Computer algebra system: *Mathematica*	*1.94*	111

Table 2

Student Ratings of Topics in Mathematical Modelling 2

Topics in Mathematical Modelling 2	Mean	N
Linear algebra:	2.32	81
solutions to sets of equations resulting from particular problems	2.35	108
the need to develop a variety of ways of solving sets of equations	2.36	107
matrices and determinants	2.14	108
eigenvectors and eigenvalues	1.94	108
Vectors: a standard treatment building on that given in Physical Modelling	2.22	108
Partial derivatives, waves and temperature distributions as illustrative examples	2.07	108
Using partial derivatives for rates of change	2.13	103
Optimisation (critical points and their nature for functions of two variables)	2.02	105
The method of least squares	1.85	105
Multiple Integrals: theory and practice	2.00	106
Multiple integrals: Applications	2.01	106
Computer Algebra System: *Mathematica*	1.88	107
Probability with a focus on the determination of the reliability of a system of components in various engineering contexts	1.96	104
Descriptive statistics – graphical	2.12	106
Variance, measures of location and measures of spread	2.08	104
Skewness and kurtosis	*1.80*	104
Probability distributions	2.02	106
Conditional probability and bi-variate probability	*1.85*	105
Inference of means and proportions in populations	1.88	104
Use of statistics software *Minitab*	*1.73*	108

Table 3

Averages of Student Ratings of Topics by Student Major

	Mathematical Modelling 1		Mathematical Modelling 2	
Engineering Majors	Mean	N	Mean	N
Civil	2.04	35	1.97	35
Mechanical	2.22	31	2.12	28
ICT & Electical	2.20	45	2.03	45
Total	2.15	111	2.03	108

Table 4

Averages of Student Ratings of Topics by Students' Stage in Degree

	Mathematical Modelling 1		**Mathematical Modelling 2**	
Stage in degree	Mean	N	Mean	N
EPR 1 (usually 4th semester)	2.05	62	1.89	60
EPR 2 (usually 8th semester)	2.28	49	2.22	48
Total	2.15	111	2.03	108

As seen in the above tables, there appears to be a slightly higher overall average rating for Mathematical Modelling 1 (MM1) topics compared with Mathematical Modelling 2 (MM2) topics. In both of these subjects, the Civil engineering majors tended to give the lowest ratings, and the Mechanical engineering majors tended to give the highest ratings. When comparing the EPR 1 group to the EPR 2 group, the EPR 2 group rated topics more highly for both of the Mathematics subjects than did the EPR 1 group.

Statistically Significant Results

The Civil engineering majors tended to rate mathematics topics lower on average than students in other majors. They rated significantly lower the topic of complex numbers, with average rating of 1.74 compared with 2.26 and 2.23 for the Mechanical engineering and ICT/Electrical engineering majors. The results are illustrated in Figure 2.

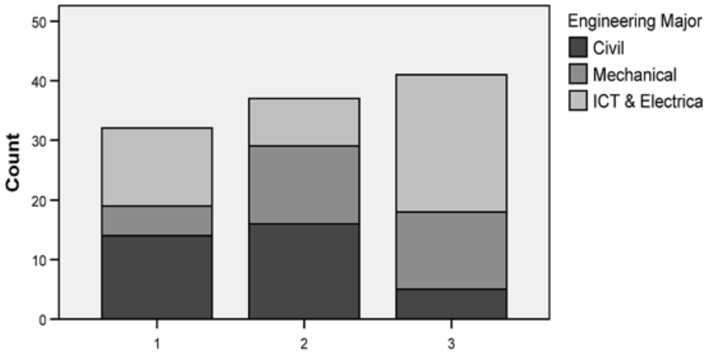

Figure 2. Chart showing different ratings of the topic Complex Numbers by students in different majors.

Similarly, the ICT & Electrical majors gave significantly higher ratings for solving certain types of differential equations with a mean of 2.38 compared to 2.13 and 2.06 for the Mechanical and Civil engineering majors. This is illustrated in Figure 3.

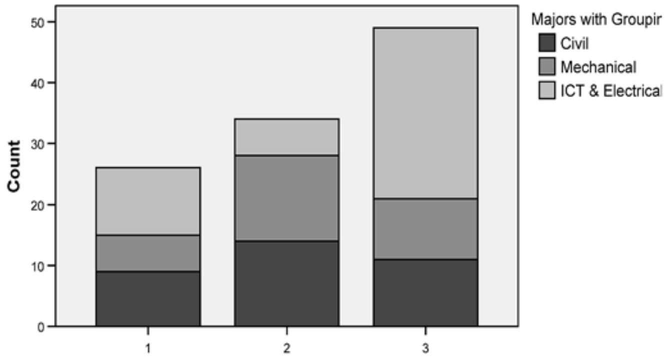

Figure 3. Chart showing different ratings of the topic "Solve dy/dx=ay+c" by students in different majors.

There was not much difference between EPR 1 and EPR 2 ratings of the MM1 topics, however there were significantly different ratings in many of the MM2 topics. The topic "solving sets of linear equations" was rated above average by the EPR 2 group, with a mean rating score of 2.54 compared to 2.20 by EPR 1 students; and the topic "eigenvalues and eigenvectors" was rated below average by the EPR 1 group at 1.78 compared to 2.15 by the EPR 2 group.

The most remarkable finding however, was in the statistics topics within MM2. The EPR 2 students rated every statistics related topic significantly more highly than did the EPR1 students. This is a clear indication that the students early in their degrees have not yet encountered the importance of the use of statistics in engineering.

Results for the Summary Questions and Student Comments

Table 5 shows the responses to the summary questions. It is clear that many students do not appreciate the connections between their experiences in mathematics subjects and in later engineering subjects.

Table 5

Responses to the Final Summary Questions about Mathematics Subjects.

Please rate the following	Agree	Neutral	Disagree
I found the current structure and order of topics within Mathematics, Statistics and Science subjects was satisfactory	62 *60.2%*	38 *36.9%*	3 *2.9%*
I feel there is a clear connection between first year mathematics and later engineering subjects	54 *52.4%*	40 *38.8%*	9 *8.7%*
I feel there is a clear connection between first year statistics and later engineering subjects	30 *29.1%*	51 *49.5%*	55 *21.4%*

To gain some idea of the reasons for these responses, it is useful to consider the range of students' comments. Starting with positive comments, we find:

> "These subjects give the student the analytical skills to tackle any job." ICT, EPR 1

It is clear that the students in different majors see different mathematical needs:

> "Because there is certain subject need math such as CA [Circuit Analysis] & other subject and it should be covered very well - exponential & Laplace & different way of differential equation." ITC, EPR 2

> "Math Mod 1 & Math Mod 2 doesn't seem at all relevant to later civil & enviro eng. Subjects. This also goes with physical modelling. You don't really need to know the stuff in math mod 1 & 2 or physical mod to do civil or enviro eng." Civil/Environmental, EPR 2

Many students were critical of the statistics they were required to study in MM2, however some came to appreciate applications of statistics in other areas:

> "While overwhelmed with stats in MM2, it has played little role after that aside from looking up graphs" Civil, EPR 2

> "I don't recall having really used statistics anywhere else throughout degree. I still think the knowledge of them is important." Mechanical, EPR 2

> "The concepts of MM1 & 2 should be swapped, as the advanced trig I found was not as useful as statistics…. I found that the statistics course has been very important in many of my subjects (both engineering & business (econometrics)). It would be useful to introduce this initially to first year students, as it would assist their later subjects with a greater understanding of it." ICT, EPR 2

For academics about to redesign subjects, these comments are extremely important:

> "I think the Physics and Math Mod 1 subjects could be taught in a more relevant way to engineering. Some of the content is very useful in later subjects however a clearer connection to later subjects should be shown in these early subjects." Civil, EPR 2

> "All Maths is done in first year, so by 4th year some concepts have been forgotten. Maths sometimes meaningless without engineering application." Mechanical, EPR 2

Discussion and Implications

What Does a Low Relevance Rating Mean? What Can be Done About it?

The content of the core mathematical subjects consists of standard topics for engineering degrees. When a topic is rated as "not relevant" by students one needs to ask whether (1) the topic has actually become outdated, (2) the relevance has not been made evident by the style of teaching and type of examples discussed, or (3) perhaps students have not experienced enough of their engineering subjects to see the relevance for themselves. Our approach is to gather more information from engineering academics to assist with answering this question, especially if we suspect the answer is likely to be (1). A topic may indeed be outdated in terms of expectations of pen and paper problem solving in cases where the standard in professional practice is to use a software package. This needs to be investigated topic by topic, as moving too quickly to software tools without understanding basic concepts brings its own dangers. Eric Love (1995) writes:

> Expert users are able to think of such aspects as 'tools' because they project their previous experiences of paper-and-pencil mathematics on to the situation in the computer software and use these tools as surrogates for their previous manual techniques. Learners, of course, do not have this previous experience and thus have the double handicap of knowing neither under what circumstances they might use the tool, nor how it works. (p. 114)

One does see a change towards valuing conceptual understanding over accurate by-hand calculations, however conceptual understanding may require a certain amount of by-hand calculation in its formation.

It is clear from the results presented in this paper that (3) is often a factor to be considered. In this survey, later stage students had experienced more uses for statistics than early stage students. The differences in ratings of some topics by students in different majors could also be considered here. This is where guest appearances by lecturers of later stage engineering subjects may help to motivate and interest students.

As far as (2) is concerned, we can already report on three changes that have interested the students in mathematics classes since this survey was conducted. First, we are trialling the relegation of routine exercises to be done on "hand-in" sheets, which are due at the *beginning* of tutorials. This makes time available in tutorials for discussion of practical engineering applications of the mathematics. Engineering academics are being invited to provide the practical engineering applications. Additional peer-supported study groups are available for students who need assistance with the basic routine exercises, and we also have a drop-in tutor system available through the Mathematics Study Centre. Second, where topics permit, engineering academics are making appearances in mathematics lectures to "team teach". For example, a very popular lecture involves a structural engineer talking about deflection in beams. At the vital point when integration is required to calculate the second moment of area for the cross section of a beam, the mathematics lecturer steps in to explain how the integral is initially set up as a Riemann sum. Third, the balance of topics between revision of high school material, and the first two semesters of university mathematics is being revisited. We hope this will allow more time for the second semester mathematics topics to include the development of engineering applications.

Conclusion

The student survey reported here is one of several methods of data collection to support curriculum change in our situation. The voices of students need to be heard, and their opinions need to be weighed along with those of other stakeholders. Interviews with engineering subject co-ordinators have provided more information, and the actual process of conducting those interviews has started a process of collaboration that we intend to continue. As noted by Broadbridge (2007, p. 18), "Students have a higher satisfaction rating where there is regular consultation between Maths and Engineering faculties on curriculum planning and where space and time are provided for drop-in assistance."

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Advancing Research Into Affective Factors in Mathematics Learning: Clarifying Key Factors, Terminology and Measurement

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The literature on affective factors in the learning of mathematics is difficult to interpret because of differences and inconsistencies in terminology and measurement. To advance research in this field of affect, I compare and clarify terminology, and reconcile scales for measurement by examining the factors and research instruments targeted by four research teams. The findings reveal two distinct broad primary areas of interest, namely self-concepts about mathematics, and intrinsic motivations for learning mathematics. The instruments used to measure a range of underlying factors within these two areas are analysed and reconciled, terminology is clarified, and further recommendations are made.

It is both timely and imperative to explore the potential of self-beliefs and attitudes to inform educational planning and practice. It is known that self-concepts and related attitudes are important influences in the learning choices students make, and play a role in learning behaviour and performance. For example, in a study of the factors that influenced more than 500 first-year Australian students, Cretchley, Fuller, and McDonald (2000) found that self-beliefs about mathematics ability were a major influence behind their choice to study mathematics at university. Given the need to attract students to mathematics, the legacies of low mathematics self-esteem and low interest in mathematics are serious.

Despite the need, there have been few attempts to clarify the role of mathematics self-concepts and attitudes in learning. A number of researchers have explored students' attitudes via case studies and journal entries, but few have accepted the challenge of quantifying affective factors, exploring relationships with learning approaches and progress, and monitoring changes. Reviewing literature in the field of affect in mathematics education, Leder and Grootenboer (2005) found '*few studies in which the difficult task was attempted of exploring the relationship between affect and a range of other important factors including cognition, learning and achievement.'*. This neglect is easy to understand, given the difficulties researchers face in this new field: theories not yet well-developed, terminology used differently and ambiguously, and varying research instruments, some untested, make the literature difficult to interpret, and leave researchers open to criticism.

Research findings also vary. Correlations between affective factors and performance vary widely. Leder and Grootenboer described *'tantalizing'* findings and *'provocative glimpses'* of the interaction between affect, teaching, and learning. and causal directions found in relationships between affective and cognitive learning factors are inconclusive, and little has changed since Marsh (2002) argued that early research supported a model of '*reciprocal effects'*, with prior academic self-concept having a positive effect on subsequent achievement beyond that which can be explained in terms of prior academic achievement; and vice versa, subsequent academic self-concept is affected by priori achievement beyond what can be explained in terms of prior academic self-concept.

Nevertheless, some reports find significant relationships between self-concepts and achievement. Cretchley and Galbraith (2003) found Pearson correlation coefficients of up to 0.60 between mathematics confidence and performance on a range of different types of assessment tasks. It is clear that there is much still to be learned about self-concepts and other affective factors related to learning. To facilitate understanding of students' self-beliefs and learning attitudes, to investigate the ways in which these shape learning behaviour, and to ensure that the research literature is accessible to educators, there is an urgent need to advance the field. This research report aims to:

- identify affective factors that are important for research into mathematics learning;
- examine instruments for their measurement;
- clarify the terminology.

Self-Concept Terminology and Measurement

Researchers and educators frequently refer to students' self-beliefs about their ability to do or learn mathematics as *mathematics confidence*. Some use the terms confidence and self-efficacy synonymously . Clearly we need to distinguish.

Probably because *confidence* generally is widely understood to refer to self-beliefs and judgments about one's capabilities, definitions of mathematics confidence are hard to find. *Self-efficacy* is a term coined relatively recently by the architect of social cognitive theory, Albert Bandura (1977, 2005) for *'people's judgments of their capabilities to organize and execute courses of action required to attain designated types of performances*'.

Bandura's theories propose that there are hierarchies of increasingly context specific self-efficacies, and that individuals have multiple self-efficacies. But he warns strongly that measures of self-efficacy must be closely task-specific. This warning surely provides the strongest means of distinguishing between confidence and self-efficacy, for an examination of the literature reveals that measures of mathematics confidence generally tap broader self-concepts. However, the term confidence is often used within specific contexts, thus making the need for the term self-efficacy highly questionable. We can speak rather of task or context-specific confidence.

How and why do education researchers measure task-specific or context-specific mathematics self-efficacy or confidence? Journal entries, and written or verbal responses to open questions are frequently used to gather qualitative data. In order to quantify and/or monitor mathematics attitudes and confidences however, most researchers use Likert-style questionnaires. For example, asking students to indicate a level of confidence for each task performed provides a valuable indicator/warning of areas in which a student is over or under confident. Aggregated over a topic, indicators of that kind provide a measure of self-efficacy for that set of tasks. Such task-specific measures may not be broad indicators of mathematics self-confidence and feelings, however. Most commonly, confidences are assessed by means of Likert-style self-response scales that invite learners' responses to a set of statements/items. The items are designed to tap underlying factors/constructs that have been chosen for investigation by the researchers, and may vary from context to context.

The shortcomings of self-report instruments are widely known and reported. Importantly, it is widely acknowledged that the wording of items is open to different interpretations by respondents and researchers. Equally importantly, researchers know that respondents may consciously or unconsciously answer with bias or not be frank. Nevertheless, self-report scales remain the most common means of measuring self-concepts because they are phenomenological by nature, and because such instruments enable timely and easy data capture. Clearly, such data must be interpreted with awareness of the shortcomings, and reporting appropriately.

The online ETS and Buros research instrument databases contain some that are not Likert-style. All are of the types noted in Keith and Bracken's (1996) review of self-concept instruments: namely *checklists, Q-sorts and free response questions.* Free response instruments invite respondents to complete partial statements or answer open questions. These do not provide quantitative measures, but offer a rich breadth of views. Data of this kind are often used to develop items for self-report scales. Checklists are useful for gaining a broad qualitative description of the respondent who indicates which items he believes apply to him from a list of descriptors. Q-sorts may be used to establish the level of agreement with each descriptor - the respondent sorts the descriptors into piles that are most and least like him. By far the majority of instruments are Likert-style, however, and hence this study explores and reconciles such instruments via the work done by four sets of research teams.

Analysis of Four Current Sets of Research Instruments

The instruments chosen for this study constitute a substantial body of work in this field by researchers from the cognate disciplines of psychology, mathematics, and education, namely, Galbraith and Haines (2000), Pierce, Stacey, and Barkatsis (2006), Tapia and Marsh (2004), Fogarty, Cretchley, Harman, Ellerton, and Konki (2001). All are self-report scales that invite Likert-style responses, all have been subjected to substantial trial and analysis, and all are currently in use for research into secondary or tertiary mathematics learning.

These instruments advance the early work of Fennema and Sherman (1976) who developed the field significantly. Their nine scales comprised 108 self-report items and measured what they called *confidence in learning mathematics, mathematic anxiety; motivation for challenge in mathematics (effectance motivation), mathematics usefulness, attitude towards success in mathematics, mathematics as a male domain, mother/father support* (2 scales), and *teacher support*. Because *Anxiety* data correlate strongly with *Confidence,* the *Anxiety* scale was dropped. Over three decades, researchers have further refined those scales, but what factors are now targeted as important in mathematics learning?

- Galbraith and Haines developed instruments for use with undergraduates. Three of these are called *mathematics confidence*, *mathematics motivation,* and *mathematics engagement* scales. Computer attitude scales were developed at the same time.
- Fogarty, Cretchley, Harman, Ellerton, and Konki analysed a *mathematics attitudes* instrument designed for use with undergraduates, calling it *mathematics confidence* after the dominant factor. Technology attitude scales were also developed.
- Tapia and Marsh developed what they termed *self-confidence, value of mathematics, enjoyment of mathematics,* and *motivation* scales.
- Pierce, Stacey, and Barkatsas developed *mathematics confidence, behavioural engagement* and *affective engagement* scales for use at school level.

All four teams clearly target what they term *mathematics confidence, motivation, and engagement*. But their instruments differ substantially in wording and length, and they tap mathematics self-concepts, feelings, and beliefs about learning preferences and behaviour under different labels. To reveal the nature and range of the underlying factors tapped by the scales, I classified the items in these instruments into groups that tap similar factors. This strategy reveals the emphases placed by the different researchers, and the nature of their mathematics confidence, motivation, and engagement scales. The classification is subjective to some degree, but the items group readily into ten factors: *innate talent and other mathematics self-concepts (including learning confidence or self-efficacy), anxiety, interest, enjoyment, intellectual stimulation, reward for effort, diligence, valuing mathematics, willingness to do mathematics, and approaches to learning mathematics.*

To facilitate comparison of the statements, Table 1 offers my grouping of the scale items into these factors. Items from the teams' Mathematics Confidence scales are labeled MC, those from Mathematics Motivation scales are labeled MM, and those from Mathematics Engagement labeled ME. Affective Engagement items are labeled AE, and Behavioural Engagement items BE. A few items could be classified in more than one group. Three Mathematics Engagement (ME) statements group well under *diligence* along with the Behavioural Engagement items. Other ME items tap a range of approaches to learning and doing mathematics. Items from the Value scale are labeled VM.

Table 1

Grouping of Scale Items Into Factors and Labels

Factor	Item	Label
Talent, confidences and self-efficacies	*I do not have a mathematical mind.* *I am not naturally good at mathematics.* *I have a mathematical mind.* *I have a lot of confidence when it comes to mathematics.* *I am confident with mathematics.* *I find mathematics confusing.* *When I have difficulties with maths, I know I can handle them.* *I know I can handle difficulties in mathematics.* *I can get good results in mathematics.* *It takes me longer to understand maths than the average person.* *I have never felt myself able to learn mathematics.* *I have less trouble learning mathematics than other subjects.* *Having to learn difficult topics in mathematics does not worry me.* *No matter how much I study, maths is always difficult for me.* *I am able to solve mathematics problems without too much difficulty.*	MC MC MC MC MC MC MC MC MC MC MC MC MC MC MC
Anxiety	*I find mathematics frightening.* *The prospect of having to learn new maths makes me nervous.* *I am more worried about mathematics than any other subject.* *Studying mathematics makes me feel nervous.* *I am always under a terrible strain in a math class.*	MC MC MC MC MC
Interest	*I find many mathematics problems interesting and challenging.* *I can become completely absorbed doing maths problems.* *I am interested to learn new things in mathematics.*	MC MM AE
Enjoyment	*I don't understand how some people seem to enjoy spending so much time on maths problems.* *I enjoy trying to solve new mathematics problems.* *I have never been very excited about mathematics.* *I don't understand how some people can get so enthusiastic about doing maths.* *Mathematics is a subject I enjoy doing.* *Learning mathematics is enjoyable.*	MC MC MC MM MM AE
Intellectual stimulation	*I get a sense of satisfaction when I solve mathematics problems.* *I like to stick at a maths problem until I get it out.* *Having to spend a lot of time on a maths problem frustrates me.* *The challenge of understanding maths does not appeal to me.* *If something about mathematics puzzles me, I would rather be given the answer than have to work it out myself.* *If something about maths puzzles me, I find myself thinking about it afterwards.*	AE MM MM MM MM MM
Reward for effort	*Maths is a subject in which I get value for effort.* *In mathematics you get rewards for your effort.*	MC AE
Diligence	*If I make mistakes, I work until I have corrected them.* *If I can't do a problem, I keep trying different ideas.* *I concentrate hard in mathematics.* *I try to answer questions the teacher asks.* *When learning new mathematics material I make notes to help me understand and remember.* *I don't usually make time to check my own working to find and correct errors.* *I find it helpful to test understanding by attempting exercises & problems.*	BE BE BE BE ME ME ME

Valuing mathematics	*Mathematics is important in everyday life.* *Mathematics is one of the most important subjects for people to study.* *High school math courses would be very helpful no matter what I decide to study.*	VM VM VM
Willingness to do mathematics	*I would like to avoid using mathematics in college.* *I am willing to take more than the required amount of mathematics.* *I plan to take as much mathematics as I can during my education.*	MM MM MM
Approaches to learning mathematics	*I prefer to work on my own than in a group.* *I like to revise topics all at once rather than space out my study.* *I prefer to work with symbols (algebra) than with pictures (diagrams& graphs).* *When studying mathematics I try to link new ideas to knowledge I already have.* *I find working through examples less effective than memorizing given material.*	ME ME ME ME ME

Findings

The four sets of instruments differ substantially in number of items, style of wording, and scope. Two sets balance positive and negative statements to counter bias and careless response, and they tap a few factors with broad scales. For example, Fogarty et al's single 11-item scale assesses self-concepts, confidence, and motivation. The other two teams tap fewer factors, each with just 3 or 4 statements, a strategy favoured for children's scales.

Mathematics Confidence and Self-Concepts

The four *mathematics confidence* scales (so-called) are substantially different, varying from narrow to broad. But all four position broad mathematics self-concepts as central.

- All four scales tap a range of self-beliefs about *ability to do and learn mathematics*.
- Three include self-beliefs about *talent* or innate ability.
- Three include feelings of *anxiety.*
- Two tap mathematics *learning self-concepts*.
- The broadest scale includes *interest, enjoyment* and *excitement,* which others tap under what they call *motivation* or *affective engagement*.

Other Motivations to Do Mathematics

The *mathematics motivation* scales (so-called) are also substantially different.

- Galbraith et al's *mathematics motivation* scale taps the following factors: interest in mathematics, enjoyment of mathematics, and intellectual stimulation (including elements that Pierce et al term *affective engagement)*.
- Pierce et al's affective engagement scale taps enjoyment, interest, intellectual stimulation, which Galbraith tap under motivation. It also taps reward for effort, which Galbraith includes under mathematics confidence. The behavioral engagement scale taps diligence, which may also be a motivational factor.
- Fogarty *et al*'s broad *mathematics confidenc*e instrument is better termed *mathematics attitudes* because alongside self-concepts, it taps motivation factors.
- Tapia & Marsh *motivation* scale taps something different: intentions to avoid or choose mathematics.

It is not surprising that the underlying factors are grouped and termed differently by the researchers: Cretchley et al., (2000) and others have shown that self-concepts are also motivators for doing and learning mathematics. Figure 1 shows the areas of overlap in the factors tapped by the four sets of scales analysed here, labelled T, P, G, and T for brevity.

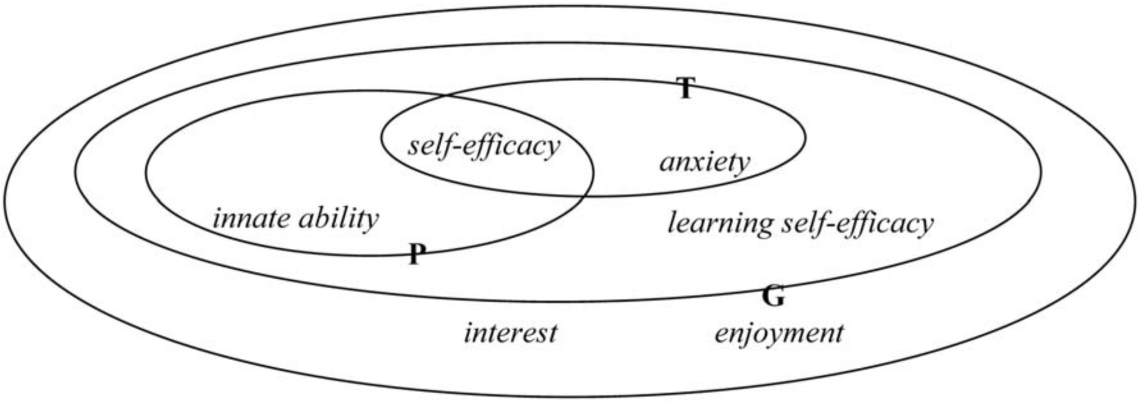

Figure 1: Venn diagram of self-beliefs tapped by four different *mathematics confidence* instruments labeled P, T, G, F.

In summary, these current researchers into affect have placed emphasis on mathematics self-concepts and other motivations for doing mathematics.

- Self-concept factors targeted for research are *talent, confidences*, *self-efficacies* and *anxieties*.
- Other motivation factors targeted for research are *interest, enjoyment, intellectual stimulation*, *reward for effort, valuing mathematics,* and *diligence*.
- Other factors targeted for research are willingness to study mathematics, and approaches to learning mathematics.

Summary and Discussion

While valuable work on affect in mathematics learning has been done with qualitative data, few studies have taken on the difficult task of quantifying and monitoring key affective factors, and assessing their role in mathematics learning. Difficulties with terminology and measurement make research difficult and hinder interpretation of the literature. The field needs accessible terminology and research instruments. To advance this work, this study investigated the factors currently targeted for research into mathematics learning by four experienced research teams, and their research instruments. Analysis revealed two primary areas of research interest: *mathematics self-concepts* and *intrinsic motivations to do mathematics*. Secondary areas targeted by some of these researchers were *willingness to study mathematics* and *approaches to learning*. The underlying factors investigated in these two primary areas are as follows:

- self-concept factors: mathematics talent, confidence, self-efficacy, anxiety.
- other motivational factors: interest, enjoyment, intellectual stimulation, reward for effort, valuing mathematics, diligence.

Terminology

The investigation revealed that scale labels like *confidence, motivation, engagement* are used differently and often too briefly for clarity. In particular, different levels of specificity in the ways in which confidence and self-efficacy have been measured may explain the variations reported in the literature to date (Carmichael & Taylor, 2005). The terms *self-concept, confidence,* and *self-efficacy* need clarifying, and much in the literature (Bandura, 2005; Marsh & Hattie, 1996) supports using these terms as below, general to specific.

- *Self-concepts* refer to the full range of self-beliefs about abilities and potentials to do and learn mathematics, from broad and innate to very specific;
- *Self-confidence* (usually termed just *confidence*) refers to self-beliefs about abilities to do and learn mathematics in some context, not necessarily generally. Hence a learner may be confident within one area of mathematics, but perhaps not another.
- *Self-efficacy* refers to self-beliefs about the abilities to perform specific tasks, in line with Bandura’s position that its measurement be closely task-specific. Hence, a student may have high level of self-efficacy for factorizing a quadratic polynomial, but a low level for a cubic.

Proposals for Research and Measurement

The items used by these four research teams provide a selection from which scales can be built for research into *mathematics self-concepts* and a range of *intrinsic motivations* to do mathematics. Ideally, however, self-report measures of this kind should be supplemented with data of other types, some of which are listed in this report. More specific recommendations are as follows:

- Scales measuring *willingness to study further mathematics* need to be developed. Tapia and Marsh's *motivation* scale comprises just three items tapping this construct.
- Research and theory position beliefs about innate talent as largely entrenched, confidences as less so, and self-efficacies as highly contextual. Hence, for monitoring self-concepts over a learning intervention, a broad *confidence* instrument is recommended. For identifying which tasks are associated with a lack of confidence or *self-efficacy*, a question should be asked for each task.
- Monitoring levels of mathematics self-concept and/or intrinsic motivation is clearly different to identifying the dominant factors for an individual or sample, which may vary with context. A broad scale tapping the factors identified here, *interest, enjoyment, intellectual stimulation*, *reward for effort, valuing mathematics,* and perhaps *diligence*, will address a valued range.

Research into mathematics attitudes and their impact on learning needs to extend way beyond naïve tests of their correlation with performance. Research has already shown the latter are variable and contextual. Deeper understanding of the nature and depth of learners' attitudes to mathematics informs course development and classroom practices.

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Explorations of Early Childhood – New Entrant Transition in Mathematics

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There is current interest in how the mathematics content, understanding, and practices of the new entrant classroom connects with the learning the child experienced within early childhood settings. Positive transitions directly impact on children. They are important to the child, to the parent, to the teachers, and to the centre/school. This paper reports on research investigating the existing transition practices between early childhood settings and primary schools with regard to mathematics learning and teaching. We report on interim findings from four early childhood settings.

Background

Positive transitions are important to the child, to the parent, to the teachers, and to the centre/school (Dockett & Perry, 2001). Children start school with a wealth of mathematical knowledge and experiences (Young-Loveridge, 1989). Recognition of this rich resource by the new entrant teacher may facilitate the smooth transition of the child into school (Perry & Dockett, 2004). We are interested in current transition practices in mathematics. By transition practices we refer to the experiences as well as the information, artefacts, and documentary evidence transferred between the early childhood centre and the school regarding the mathematics understanding of particular children.

Our work builds on a sound research base to engage with provision for student learning. Research has shown that transition can pose difficulties for new entrant children (Eyers & Young-Loveridge, 2005; Perry & Dockett, 2004; Peters, 1998) and has a long-term impact on school achievement (Timperley, McNaughton, Howie, & Robinson, 2003). Kagan and Neuman (1998) suggest there are high costs in not ensuring continuity between sectors; these costs relate to lower success rate at school, difficulties in making friends, and vulnerability to adjustment problems. A successful transition is seamless and ensures the continuity of children's physical, social, and philosophical experiences.

Barriers to smooth transitions vary depending on the individual contexts, on relationships that have developed among ECC (early childhood centres), schools, and parent/caregivers. Neuman (2002) suggests impediments to smooth transitions may develop through different visions and cultures, structural divisions, and communication. Furthermore, it is suggested that the differences between the requirements of these educational settings may invite problems related to adjustment (Kienig, 2002). These requirements are a consequence of different social and academic goals between the school and those of the pre-school setting (Bronström, 2002).

How teaching and learning occurs in an early childhood setting is of paramount interest in the research. According to *Te Whāriki* (Ministry of Education, 1996), the early childhood curriculum, teaching in an early childhood setting involves "reciprocal and responsive interaction with others" building on the "child's current needs, strengths, and interests by allowing children choices and by encouraging them to take responsibility for their learning" (p. 20). This socio-cultural viewpoint suggests learning stretches across people, places, and things and that the classroom is viewed as a community (Cowie & Carr, 2004). In these shared learning experiences learners and experts (peer or adult) co-construct "as they engage in meaning making" (Cullen, 2004, p. 70). Co-construction "emphasises children and teachers together studying meanings in favour of acquiring facts" (Jordan, 2004, p. 33). Thus in the co-construction process the child's own expertise is acknowledged as being as valid as the teachers – together the child and teacher move learning in the child's topic of investigation.

There are, however, changes in the roles, activities and interpersonal relationships between the teacher, parent, and the child as the child transitions to school (Bronfenbrenner, 1979). Successful transition to the school setting has been defined as an ecological transition between two "microsystems" (Bronfenbrenner, 1979). Broström (2002) suggests that young children feel "suitable" in school (p. 52) – in terms of enjoying a feeling of wellbeing and belonging when they successfully negotiate the daily. Concerns have been raised regarding the "very different expectations" of teachers between the early childhood settings and the school (Timperley et al., p. 32). Tensions arise as a result of change from a learning environment based on socio-cultural and co-constructionist ideas of learning (Bronfenbrenner, 1979), to more structured activities and formal instruction (Pratt, 1985).

In our study we used as our theoretical framework Bronfenbrenner's analogy of the child's learning environment as "a set of Russian dolls". More specifically we used *Te Whāriki's* "Levels of Learning" framework (Ministry of Education, 1996. p. 19) derived from Bronfenbrenner's ideas of the learner and their engagement within the immediate environment, situated at the first level of learning. The second level extends to the relationships between the immediate learning environments. In the context of early childhood this relates to the home and family, the early childhood setting, and the people within these contexts. Level three encompasses the influence of the adult's environment on their capacity to care and educate. Wider social beliefs about the value of early childcare and education form the final level. *Te Whāriki* is mainly concerned with these first two levels whilst acknowledging the influences of the other two.

Te Whāriki views the child as a competent learner and communicator and includes "dispositions" to learning as an important outcome for early childhood. "Dispositions are a very different kind of learning from skills and knowledge. They can be thought of as habits of the mind, tendencies to respond to situations in certain ways" (Katz, 1988, p. 30). One of the foundations of these dispositions is Bronfenbrenner's idea of "educational competence", for example, persisting in tasks, thinking, and working together. The nature of assessment in early childhood settings has developed to reflect the child's dispositions towards learning (Carr, 2001). This credit approach to assessment is supported in *Te Whāriki:* "[a]ssessment of children should encompass all dimensions of children's learning and development and should see the child as a whole" (p. 30). Assessment focuses on the child as a learner in specific contexts rather than on achievement objectives and skills. Narratives of incidences of a child's/children's learning is often in the form of a "learning story" (Carr, 2001) and focuses on dispositions such as curiosity, trust, perseverance, confidence and responsibility. Carr further suggests that learning dispositions are one of the key things that children take to school and on into adult life.

It is our contention that what happens at the transition directly influences a child's ability in and dispositions towards mathematics.

Methodology

The research is a one year study which investigates the existing transition practices, in a small town in New Zealand, between early childhood settings and primary schools with regard to mathematics learning and teaching. Our specific research question is: What centre and new entrant practices facilitate positive transitions in mathematics between early childhood settings and primary schools? A case study approach was taken to answer this question. This allowed us focus on "specific instances or situations and to identify, the various interactive processes at work" (Bell, 2002, p. 11). Our principal data gathering approaches include:

- evidence of teacher planning,;
- policies relevant to teaching programmes and transition;
- copies of newsletters;
- information from schools available to parents of children at the ECC;
- copies of assessment;
- photographs of mathematics in action;
- teacher interviews;
- parent questionnaires; and
- observations at the ECC.

To achieve this, in 2007, we focused our attention on four ECC – two kindergartens and two early childhood education and care centres. Each kindergarten employed three teachers and 45 children aged from three years 9 months to five years (five years being school entry age). Children attended the kindergarten five days a week from 9.00am to 12.30pm. The two early education and care centres were privately owned. In the "over two" section of these centres there were three teachers/care givers and 12-15 children in full day care. We report on interim findings from four ECC with a focus on parental perceptions, the structural provisions for mathematics, the assessments that are made with regard to children's mathematical understanding, and how this information is conveyed to the school. Results of this small study are relevant to these project sites and may not be able to be generalised.

Discussion

Structural Provisions

The approach to learning in ECC is holistic in nature based on Bronfenbrenner's idea of the child engaging with the learning environment. Children are immersed in rich learning across a range of subject curriculum areas with a strong focus on the child's interest often embedded in play. Teachers from both centres and kindergartens reflect this approach in their philosophy to mathematics learning:

> children need to have autonomy of their learning and to be able to make some choices for themselves.

Teachers seize a "teachable moment" to progress children's understanding and interest. If a child shows a particular interest the teacher may build on and nurture that interest. This illustrates *Te Whāriki*'s first level of learning, as adapted from Bronfenbenner's ideas, where a "responsive and reciprocal relationship" is developed with the teacher.

> It happens throughout the whole engaged curriculum. It doesn't stand as a solitary stand alone exercise unless it is extending a child's interest. So it is based around a child's interest and we can seize an opportunity and teachable moment and extend it.

> But often it just kind of happens in that moment in the water trough or the sandpit or you use what is there at the time.

The provision of opportunities for rich mathematical learning and language development arose in areas such as the sand pit, block corner, family corner, water play, with farm animals, toy cars, carpentry, and computer games.

> … the first thing that came into my mind was the mathematical language that is used all around the centre in lots of different areas ... you know the measuring, longer, shorter, longer … with the carpentry, water, sandpit. All of those words classifying and sorting words.

Within the learning environment of the ECC setting children worked alone, in solitary play or in parallel play, or played together. The teacher observed, interacted, challenged, scaffolded, co-constructed, or was not present.

> Working with other children never just by themselves either. There would be a group of children invariably come along. You might start with one but you would end up inviting other children to participate and all the turn taking and the sharing.

More formal activities also provided for mathematics learning such as shape matching, bead threading, puzzles, and jigsaws.

> We have a lot of parents that will come in and work with their children at the puzzles, at the play dough table, at one of our tables that we might have a game set up.

At times a teacher would remain at the activity encouraging and extending the learning through conversations and challenges. Displays of children's work, routines such as roll taking utilising category data and information sharing charts for parents on learning and curriculum also provided a maths focus. Mat time or *whanau* [group] time often provided an opportunity for a mathematics focus as a result of child observations or a teacher initiated focus.

> We definitely bring it [mathematics] out in planned times like that.

> I was working with a little boy who was going 1 2 3 and I thought there is a whole heap of stuff there…So that is why I thought of bringing in [at mat time] the actual 1 2 3…It's not giving him the knowledge it's like developing an awareness.

The two centres provided a planned "focus on four year olds" programme. These programmes had either a strong subject focus or allowed children to engage with extra resources less suitable for younger children. Both centres had a strong "preparation for school" approach in these sessions.

Parental Perceptions

Positive relationships between settings are important because they develop continuity between home, ECC, and the new entrant classroom (Bronfenbrenner, 1979). We wanted to unpack the parent/caregiver's view of the transition partnership between "teacher-parent-child". The questionnaire provided information on the parent's perceptions of transition and various aspects of children learning mathematics. Parents believed that in ECC mathematics happens most often as children play with puzzles and games, during mat time, at construction, in water play, and on the computer. According to parents mathematics happens less often in the writing area, with play dough, and in the family corner. Parents think the teachers are working with children on things mathematical most often through conversations, during mat time, and with inside equipment.

Parents are able to articulate quite clearly their perception of their child's mathematics knowledge. This includes the child's capacity to:

- Use and understand positional language (90%);
- Recognise shapes (90%);
- Compare lengths and heights (85%);
- Say the number names in order (varying competencies from 6 to 30);
- Accurately count a group of objects (varying competencies from 5 to 30);
- Perform simple addition (70%);
- Compare volumes (63%);
- Perform simple subtraction (49%);
- Tell you the days of the week, using a calendar (40%);
- Understand fractions (10%); and
- Tell the time to the hour and half hour clock (3%).

Parents held similar views about mathematics learning in a new entrant classroom. They see mathematics as being simple and incorporated into everyday situations, fun, challenging, and engaging children's interests.

> Very basic – using objects for counting and subtracting – then to introduce written number and know their values.

> Very basic. I would expect the transition between formal learning and learning by play to be fairly slow at first to help the child settle into school.

Assessment

We found narrative assessments were the most common form of documentation in the ECC. These tended to document, in written and photographic form, the dispositions exhibited by the child rather than focus on a specific subject. The child is actively engaged in their learning environment and not simply achieving a skill. The foreground of these narratives described the whole experience to ensure that the complexity of the learning was preserved (Carr, 2001). Within the background there was evidence of specific mathematics concepts being developed/practised/achieved. For the audiences reading these – child, teachers, parents and *whanau*/family – there is far more to be found here than a simple mathematics skill. This focus on a broader audience links well to *Te Whāriki*'s second level of learning.

> You enjoyed your time in the water filling bottles using jugs and small containers. You had really good concentration and showed awesome control when pouring the water into the bottles. You lined the new cylinders up from smallest to largest and filled these too. You were not only developing your fine motor control but discovering all about volume. (Learning Story)

Narratives are generally stored in an individual child's portfolio and are available at any time to teachers, children, and family. Within the portfolios teachers attempt to include a variety of voices as this contributes to the development of a rich picture of the child. This highlights the relationship between the two learning environments of home and centre.

> We have just really been promoting the stories from home and the family voices and to try and get them to contribute to children's interests.

Teachers cannot always fully judge the meaning behind a child's action but may get a fuller picture through conversations with the child and/or parent (Carr, 2001).

> She just thrives on painting activities and creates wonderful pieces of artwork. It has been observed that J is very interested in painting circles. … Mum explained to us that in the weekend J was learning about the different shapes. This could link to why she has really enjoyed creating circles. (Learning Story)

Carr (2004) highlights that formative assessment should include "where to next" and possibly some puzzlement.

> H may have a strength with the number system. We will offer more resources to stimulate his interest. (Learning Story)

Te Whāriki suggests a very clear purpose of assessment is to "[f]eedback to children on their learning and development [and] should enhance their sense of themselves as capable people and competent learners" (Ministry of Education, 1996, p. 30).

Anecdotal assessments were made by individual teachers and these were shared at planning meetings. From these assessments, resources and activities were planned and offered to meet the interests of the children; however, children were not required to carry out these activities. At times these assessments formed the basis of the plan for the *whanau* or mat time.

So I suppose the planning for us can we see an interest and then we bring in the resources. That would be in our session evaluation we would look at that and how we would extend it.

Information Sharing

There were no specific policies between any of the ECC or schools determining what should be shared and the format it should take. Teachers in charge of the junior school from both schools visit the kindergartens a few times a year with varying purposes. One kindergarten seemed to view these visits simply as a roll gathering exercise by the schools, while the other had a closer relationship with the school and more comprehensive information was shared and programmes discussed. These relationships indicate the "professionalism and collegial development" of the third level of learning (Ministry of Education, 1996, p. 19).

> The DP, [deputy principal] she will come down once a term but that's more for who's likely, for rolls and stuff. But with the A school we have just got once or twice a term they either come here or we go there, have a catch up about different children.

Early childhood teachers considered that the portfolios contained sufficient information for the new entrant teacher to use as a starting point in getting to know the child.

> I put this in the child's profile book with a link about the learning involved and I thought wouldn't this be great if I could hand it on to the teachers, so they had a knowledge of where they were at. But I don't know maybe they have their own assessment.

They were unsure that the new entrant teacher would use a child's portfolio. It was left to parents to decide if they would take it to school. When asked if they specifically suggested to parents they could take the portfolios into the schools they didn't.

> I haven't actually come to think of it.

One centre provided a written report of the child when they left the centre. These reports reflected the strands of *Te Whāriki* and were left to parents to decide if they would take it to the school.

> He is able to count confidently and is able to match numerals to objects. (Excerpt from written report identifying the Communication strand – Expressing a point of view)

Parents' expectation of what information the new entrant teacher would seek about the child's maths understanding was gathered from the questionnaire. Parents overwhelmingly expected discussion with parent/caregiver (89%), discussions with child (85%), a written report from early childhood teacher (80%), and to a slightly lesser extent the child's portfolio (70%) as prime information sources. Parents considered that verbal reports from early childhood teacher (36%) were less likely.

Conclusions

We noted the diverse and rich mathematical experiences available to the children within each of the ECC. Assessments were very holistic in nature focussing on dispositions to learning. Subject curriculum areas were not emphasised but were evident in the background when reading the assessment narratives. Parents were able to articulate clearly how their children learn maths in ECC and their expectations of how connections should be made to this learning in the new entrant classroom. We anticipate changes in the roles, activities, and interpersonal relationships between the teacher, parent, and the child as the child transitions to school (Bronfenbrenner, 1979).

The richness of mathematical learning experiences that children bring with them to school has been well researched (Aubrey, 1993; Perry & Dockett, 2004; Young-Loveridge, 1989). Perry and Dockett (2005) analysed the many mathematical experiences children have in prior-to-school settings demonstrating "immense knowledge … including mathematics" (p. 36) and the mathematical power of young children's skills in mathematising, making connections, and argumentation. The role of the new entrant teacher is to recognise this mathematical power and to nurture it by providing learning experiences that make connections to their existing mathematical understanding (Perry & Dockett).

The second phase of the research may confirm the recommendation in The New Zealand Curriculum (Ministry of Education, 2007) that "this new stage [the transition from ECC to school] in children's learning builds upon and makes connections with early childhood learning and experiences" (p. 41). We will further investigate transition practices to determine the extent that "schools can design their curriculum so that students find the transitions positive and have a clear sense of continuity and direction" (Ministry of Education, 2007, p. 41).

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Eliciting Growth in Teachers' Proportional Reasoning: Measuring the Impact of a Professional Development Program

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Proportional reasoning is required to operate in many mathematical domains in the middle years' curriculum. It is also a major connecting theme across both mathematics and science. Working together, middle years teachers have the potential to promote students' proportional reasoning through integrated learning experiences. However, building awareness of the connections between these two curriculum domains is an important first step. This paper reports on one aspect of a large project exploring the connections between mathematics and science curriculum. In this paper, first steps to eliciting teachers' pedagogical content knowledge in relation to proportional reasoning are discussed.

Background

Proportional reasoning is the key to understanding and operating in many domains in the mathematics curriculum in the middle years of schooling (typically Years 4-9). Fractions, percentages, ratios, decimals, scale, algebra, probability require proportional reasoning. Proportional reasoning means being able to understand the multiplicative relationship inherent in situations of comparison (Behr, Harel, Post, & Lesh, 1992). The study of ratio is the foundation upon which situations of comparison can be formalised, as a ratio, in its barest form describes a situation in comparative terms. For example, if a container of juice is made up of 2 cups of concentrated juice and 5 cups of water, then a container triple the size of the original container will require triple the amounts of concentrate and water (that is, 6 cups of concentrated juice and 15 cups of water) to ensure the same taste is attained. Proportional thinking and reasoning is knowing the multiplicative relationship between the base ratio and the proportional situation to which it is applied. Further, proportional reasoning is also dependent upon sound foundations of associated topics, particularly multiplication and division (Vergnaud, 1983), fractions (English & Halford, 1995) and fractional concepts of order and equivalence (Behr, et al., 1992). Although understanding of ratio and proportion is intertwined with many mathematical topics, the essence of proportional reasoning is the understanding of the multiplicative structure of proportional situations (Behr, et al., 1992).

Not only is proportional reasoning fundamental to many mathematics topics that students study in the middle years of schooling, it also underpins many middle school science topics (e.g., density, speed, acceleration, force, molarity, machines). The topics of ratio and proportion are typically studied in mathematics classes, and ratio and proportion in fact, have been described as the cornerstone of middle years' mathematics curriculum (Lesh, Post, & Behr, 1988). However, research has consistently highlighted students' difficulties with proportion and proportion-related tasks and applications (e.g., Behr, Harel, Post, & Lesh, 1992; Ben-Chaim, Fey, Fitzgerald, Benedetto, & Miller, 1998; Lo & Watanabe, 1997), which means that many students will struggle with topics within both the middle years mathematics and science curriculum due to their lack of understanding of ratio and proportion.

Proportional reasoning, as a major connecting theme across mathematics and science, suggests the potential of integrated teaching, a key principle of middle years' reform philosophy (e.g., Pendergast & Bahr, 2005). The potential is apparent for mathematics and science teachers to mutually support each other in developing rich learning experiences to promote students' proportional reasoning skills.

The Context

A current research project being undertaken by the authors brings together middle years' mathematics and science teachers around this important topic, with the intention to provide an opportunity for teachers to explore the proportional reasoning linkages between topics in both mathematics and science, and to create, implement and evaluate innovative and engaging learning experiences to assist students promote and connect essential mathematics and science knowledge. One of the main aims of this project is to promote teachers' mathematics and science content knowledge around new curriculum in these disciplines, as well as their knowledge for teaching (pedagogical content knowledge). A targeted program of professional learning is being incorporated within the teacher meetings, in which new ideas and ways of teaching to connect essential learnings in mathematics and science are shared with teachers.

In this project, the impact upon teachers' understanding of proportional reasoning is of major interest to the researchers. A current trend in evaluating the impact of professional development programs for teachers has been the creation of instruments designed to measure teachers' knowledge for teaching (e.g., Baker & Chick, 2006; Hill & Ball, 2004; Watson, Beswick, Caney, & Skalicky, 2005/6), based specifically around pedagogical content knowledge (Shulman, 1986). Such research studies have provided suggestions for the development of pen and paper surveys and questionnaires to gather data on teacher professional learning. One of the key research questions in this project, and the focus of this paper, is as follows:

How might teachers' proportional reasoning and knowledge of related key ideas be profiled, and how does this knowledge relate to their classroom practice?

The Study

In this project, a Background Teacher Survey (BTS) was designed along similar lines to Hill and Ball's (2004) instrument for measuring teacher knowledge for teaching mathematics. As emphasised by Hill and Ball (2004), knowledge for teaching cannot be equated to one's level of content knowledge, but encompasses knowledge of students, knowledge of how students learn, and knowledge of strategies for improving learning. The survey also drew upon Shulman's (1986) descriptions of four elements of knowledge for teaching: subject matter knowledge, pedagogical-content knowledge, curricular knowledge and knowledge of students as learners. The survey contained classroom scenarios and asked teachers to comment on given student responses.

The time required for teachers to adequately review and reflect upon given students' responses was a major factor in considering the number of scenarios presented. For each scenario, between four and five student responses were developed. If teachers spent approximately 5 minutes analysing and responding to each of the given student responses, each scenario would require approximately 20-25 minutes minimum for analysis. Also, the content of the scenarios was a further factor in development of the scenarios. Because this project is about the connections between mathematics and science curriculum in the middle years of schooling, the authors were keen to develop scenarios that equally favoured each curriculum area. Taking the number of student responses and content of the scenarios into consideration, the resulting survey included two classroom scenarios, one about making mixtures (recipe) and one about density (floating balloon baskets). Even though both topics equally fall within the science curriculum, the first scenario was regarded as more common-place in a mathematics classroom and the second as more linked to a science classroom. The given number of student responses was five and four respectively for the two tasks.

The issue of asking teachers to undertake a 'test' of their knowledge for teaching was a further major factor in the development of the survey. This issue was identified by Watson, Beswick, Caney, and Skalicky (2005/06) who reported on the reluctance of teachers to complete a pen and paper profile as a component of a mathematics teacher professional development project, and has also been highlighted by Hill, Sleep, Lewis, and Ball (2007) as a major limitation of pen and paper tests for teachers. Early tests of teacher knowledge, as summarised by Hill et al. (2007), were unashamed tests of mathematics content, with more recent tests designed to assess content as well as pedagogy, often in the form of a specific mathematics topic presented with a series of approaches to teaching the topic, or a student's erroneous response to a mathematics item with a series of ideas for teaching to assist the student overcome his/her misconception (e.g., Hill, Schilling, & Ball, 2004). Such multi-choice items can result in respondents locating the 'best' response through a process of elimination which provides little insight into their pedagogical content knowledge. An alternative approach was taken by Watson, Beswick, Caney, and Skalicky (2005/06) with the presentation of a classroom task (e.g., 90% of 40) with teachers required to (a) list as many appropriate and inappropriate ways students might solve the task, and (b) describe how the task might be used in the classroom.

Taking these issues into consideration, we wanted to move away from multiple choice items that forced teachers to look for the 'best' thinking from a list of given responses, and we also wanted teachers spend time thinking specifically about students' proportional reasoning. Hence, our survey provided teachers with specific responses from students to direct their analysis. We were hoping that provision of specific student responses to given classroom scenarios rather than a multi-choice format would also lessen the potential of the survey being directly interpreted by teachers as a 'test' of teacher knowledge. In the instructions for completing the survey, we informed teachers that we hoped the classroom scenarios would raise issues around the teaching of proportional reasoning that are useful for the project, and that their comments and ideas would help us in our planning for the rest of this project.

Despite these considerations, during construction of the survey there was a considerable feeling of being impertinent in asking teachers to display their knowledge for teaching, particularly as teachers in Australia do not have a tradition of being formally assessed on their mathematics knowledge and understanding of teaching. The resulting survey is entitled innocuously as the Background Teacher Survey and specifics of the survey are presented in the next section.

The Background Teacher Survey

The survey BTS is a pen and paper survey containing two classroom scenarios in which students are engaged in tasks requiring proportional reasoning. A problem is posed to the students in each scenario and a series of student responses to the problem are presented. To complete the survey, teachers are required to comment on the given responses, providing reasons for why they think the students answered the way they did. The first scenario is entitled *Sticky Mess* and describes a cooking activity where students must alter a recipe to make a bigger mixture to the one given in the original recipe. The second scenario is entitled *Ballooning* and describes students making hot-air balloons using balloons and baskets of various sizes and recording those that fly and those that do not, with class results presented as a data set and the mass and volume of the teacher's untrialled balloon given. After teachers have commented on the given students' responses to each scenario, they are asked to complete the following three questions in relation to the Ballooning scenario:

a) Place the students' responses in order of increasing quality.

b) Explain the criteria you used for your ranking.

c) Do you see this as a mathematics or a science lesson (or both?). What types of concepts potentially could be developed through such a lesson? Please elaborate.

Participants

Fourteen teachers from six different schools completed the BTS both at the start of the year and at the end of the year. All teachers were teachers of students in the middle years of schooling, but came from a broad range of schools and teaching situations. Some teachers taught secondary mathematics classes only, some taught secondary science classes, some taught both secondary mathematics and science classes, and some were primary teachers teaching both mathematics and science. The schools include single-sex schools, low socio-economic status schools, high socio-economic status schools, Catholic schools, Independent schools, and State schools. The teaching experience of participating teachers ranges from less than one year's teaching experience to over 25 years teaching experience.

Survey Administration

On the first day of the meeting with teachers, the BTS was distributed. The teachers were asked to individually work through the presented scenarios, making notes and writing comments as requested without sharing their ideas with anyone else. Teachers were assured that their comments on the surveys would only be read by project personnel, and their surveys would be de-identified by replacing their name with a code. It was suggested that teachers should aim to spend about 35-40 minutes on the survey.

Surprisingly, all teachers approached the survey in a positive manner with no verbal objections raised or further questions asked. The teachers completed the survey in silence and appeared to take a serious approach to completing the survey individually and comprehensively. All teachers managed to complete the survey in the allotted time.

At the end of the first year of the project, and after the teachers had attended 5 full professional development days and 4 afternoon workshops throughout the year, the teachers completed the BTS a second time. The teachers approached the survey in the same manner as noted in the first administration. Several teachers who did not complete the first survey but who were present for the second survey completed the survey for the first time, but their responses are not included in this data set.

Results

The first item, *Sticky Mess,* provided teachers with five student responses, only one of which was correct. The *Sticky Mess* scenario is that a recipe requires 4 cups of sugar and 10 cups of flour, but a larger amount is made with 6 cups of sugar and students must determine how many cups of flour are required for the new mix. Duane suggests that the answer is 12 cups, noting that an increase of 2 cups of sugar will require an extra 2 cups of flour. Eva suggests that the answer is 15 because you need 2½ times as much flour as sugar. Ivy suggests that the answer is 18 because you double 4 and add 2. Tyrone says that you need 12 because you need 6 more cups of flour than sugar, and Teresa sets up a proportional equation, (incorrectly) solves for x and decides that the answer is 2.4.

The fourteen teachers in the study diligently responded to each of the students' answers and identified that Eva was the only student who correctly gave the right answer and that all other students were focusing on other things. Several teachers congratulated the students on actually providing a response ("the student has a least provided a reason for their answer"; "the student's response is encouraging because he is making connections and seeing patterns in the quantities of ingredients"), and some teachers made other comments about things that could be done in the classroom ("I would suggest a cooking lesson"; "I'd like to let the student make the mix using both recipes, then compare them") and other comments ("obviously doesn't help cook in the kitchen or help mix concrete"). Teacher responses on the second survey provided a much more focused analysis of students' responses and directly responded to students' understanding of ratio. For the students who clearly used an additive strategy (Duane and Tyrone), the teachers couched their discussion using the word 'addition'. In the second survey, all teachers repeatedly referred to 'additive thinking' and 'multiplicative thinking' in relation to students' responses.

The second scenario *Ballooning* provided four student responses showing varying levels of data analysis. The first student, Josh, looked only at data from one balloon that was similar in mass and volume to the teacher's balloon. Josh stated that he thought the teacher's balloon would not fly because the located balloon didn't fly. The second student, Jess, reorganised the data table showing lightest to heaviest mass and, ignoring one balloon that did not fit the pattern, reasoned that the teacher's balloon would not fly because, in general, light balloons fly but heavy ones do not. The third student, Jamie, drew a graph to show the two values (mass and volume) of each balloon. Drawing a line between the balloons that could fly and those that did not, reasoned that the teacher's balloon would be located above the line and hence would probably fly. The fourth student, Jim, included an extra column in the given data table to show the ratio of the mass to the volume of each balloon and reordered the table to show ratio in ascending order. Jim reasoned that balloons with ratio greater than 1.2 did not fly and that because the ratio of the mass to volume of the teacher's balloon was 1.2, it should fly because another balloon in the data set also had a ratio of 1.2 and it could fly.

The teachers' responses to the Ballooning task were not as focused as the Sticky Mess, either in the first or the second survey. Ballooning is a task that is a specific exploration of density and, even though there were 6 teachers of secondary school science in the group, none of the teachers used this word in discussing the students' responses in the first survey. The use of the word density was peppered throughout the second survey, but not by all teachers. Most teachers mentioned that two of the students' responses showed a means of exploring the relationship between volume and mass and that both these factors affect flight. All teachers identified the flaw in the first student's (Josh) response (finding a balloon with dimensions similar to the teacher's) and the limitations of the second student's (Jess) response (focusing on mass only) where anomalous data was dismissed, but commented on the value of reorganising the data table in increasing mass for ease of analysis. The third students' (Jamie) response (plotting each balloon on a mass/volume graph) was described as an "innovative approach", as showing "excellent reasoning and manipulation of data", and “a useful starting point”, but spontaneous graphing was questioned: “I couldn’t see a student opting to plot a graph voluntarily”. Several teachers commented on the appropriateness of the line of best fit drawn by Jamie, suggesting that a different line could give a different conclusion: “if the line of best fit was drawn slightly raised it would give a different answer”; “is this a good (valid) line of best fit? Maybe the line would be more sloped and sit above teacher point”; “there are actually a number of ways she could have drawn this line”. The fourth student’s (Jim) strategy was generally applauded by all teachers in both the first and second survey. All teachers commented that Jim specifically considered the relationship between the two variables to support his hypothesis of whether the teacher’s balloon would fly or not: “his idea is logical”; “this is the right way to go about it”; “conclusions drawn from supporting data”. Two teachers specifically questioned whether such a student would take such an approach to data analysis: “Student must have had experiences with density and ratios to think to do what he did here”, and “Where did you get this kid? This is higher level thinking as he can see a relationship between the two variables”.

Following analysis of each student’s response to the Ballooning task, the teachers were asked to rank order the students’ responses in terms of increasing quality. On both the first and second survey, all teachers except one, ranked Josh lowest (who focused on only one other balloon like the teacher’s balloon), Jess second (who reordered the table in increasing mass and concluded that mass determines ability to fly), Jamie third (who plotted a graph), and Jim highest (who calculated ratio of mass : volume). In explaining their ranking, all teachers emphasised the logical nature of Jim’s approach and his search for a relationship upon which to base his conclusion. Several teachers mentioned the graph as a potential valuable strategy, more frequently in the second survey than the first, but also mentioned errors in the line of best fit and how alternative conclusions could have been drawn. The dissident voice rated Jamie as the student with “no idea” and Jess as “almost there”, thus suggesting that the graph was a very flawed approach, but that focusing on mass only was a useful approach.

The teachers were asked to state whether they felt the Ballooning task was a science or a mathematics activity, and to list concepts that could be potentially developed through the Ballooning task. In both surveys, teachers rated the task as both a mathematics and science task, and provided a comprehensive list of concepts that could be developed through Ballooning. However, in the first survey, the word ‘density’ was only listed by 3 teachers, and data handling was mentioned repeatedly (fair testing, controlling variables, graphing). In the second survey, the word ‘density’ was mentioned by 8 teachers and the list of concepts that Ballooning could develop was more focused, with further density activities mentioned that could complement and consolidate the ballooning task.

Discussion

The data generated from the BTS raised several issues that overshadowed its capacity to evaluate teacher pedagogical content knowledge of proportional reasoning. Comparing the results of the first survey to the second survey showed only marginal differences in teachers’ proportional reasoning, although there was a noticeable difference in the preciseness of teachers’ language when they discussed each of the scenarios. It is evident that the project has provided teachers with the language to discuss students’ proportional reasoning, and the differences between additive thinking and multiplicative thinking were clearly enunciated by teachers. This finding is similar to that of Watson et al., (2005/06) who reported that their pre and post profile showed a greater use of specific language that had been used throughout the professional development program by project teachers.

The actual design of the items in the BTS was an issue in terms of data yield. As outlined previously, the items were specifically presented to focus teachers’ attention on students’ proportional reasoning skills and to favour equally science and mathematics. However, the teachers’ written discussion of each student’s response resulted in teachers repeating the strategy of the child and evaluating it in terms of its appropriateness. No teachers specifically mentioned teaching approaches to assist students develop their proportional reasoning, except in a very general sense (e.g., “need to develop their understanding of ratio”; “given them cooking classes”). The survey did not ask teachers to comment about possible teaching approaches or reasons for why the students answered the way they did, and this may have been a reason for the limited description of the students’ responses. Also, the number of responses for each scenario that teachers were required to respond to may have lead to rather stilted discussion. This suggests that the directions for teacher completion of the survey require attention. In the design of the survey, the authors were keen to move beyond multi-choice responses, although Hill & Ball (2004) stated that such items can be used to measure teacher's pedagogical content knowledge. Other researchers have combined pen and paper surveys with interviews (e.g., Chick, Pham, & Baker, 2006) to elicit greater depth of responses, or used other data sources (e.g., Watson, et al., 2005/06).

The issue of 'testing' teachers' professional knowledge was a major consideration in the design of the BTS. The format of the survey did not appear to make any teachers uncomfortable and all teachers completed the survey in a professional manner. Other researchers have highlighted the issue of 'testing' teachers and how teacher reactions to 'testing' often reflect the degree to which teachers volunteer to take the 'test' (complete the survey). For example, Hill and Ball (2004) discussed the low return rate of teacher surveys, but how incentives such as certification and teaching contracts encourage completion and return. Watson et al. (2005/06) discussed the choice that teachers have in terms of survey completion in relation to the funding body of the research project – a project funded by the employing body ensures that all teachers complete required tasks. Why did teachers in our project readily complete the first and second survey? The answer to this question may be that the survey was not perceived as a test, or that the items were designed in a manner that aligned with their professional knowledge. A further reason may be because they felt comfortable with the researchers, although this is unlikely on the first day of a professional development program. This issue of the actual design of survey items is not a trivial one, as it can impact teachers' approach to the survey and therefore quality of response. However, in the BTS, it is clear that more specific questions need to be posed to further elicit teachers' pedagogical content knowledge in terms of proportional reasoning.

Concluding Comments

This paper has reported on one aspect of a current research project exploring the connections between topics in mathematics and science through proportional reasoning. The focus of this paper was on teacher knowledge of proportional reasoning and measuring growth of teacher knowledge as a result of participating in a targeted professional development program. Analysis of data from the teacher survey suggests that further work is required on the survey items to create a useful instrument, although the necessity of combining interview and other data, including classroom observations, to determine growth of teacher professional knowledge is clearly apparent. This is a first step towards eliciting teachers' proportional reasoning across both mathematics and science.

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Links Between Children's Understanding of Multiplication and Solution Strategies For Division

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This paper reports on strategies 26 Year 3 students used to solve a range of division word problems in a one-to-one interview, following the participation of half of the group in a teaching experiment. The focus here is on the strategies used by case study students to solve *equivalent groups* and *times as many* division tasks. Results suggest that young children are capable of solving complex division problems given experience with a range of semantic structures for multiplication and division.

Studies on children's solutions to multiplication and division problems indicate that children as young as kindergarten age can solve a variety of problems by combining direct modelling with counting and grouping skills, and with strategies based on addition and subtraction (Anghileri, 1989; Bryant, 1997; Carpenter, Ansell, Franke, Fennema, & Weisbeck, 1993; Kouba, 1989; Mulligan, 1992). While this may be true, it is a widely held belief that multiplication and division are conceptually complex both in terms of the range of semantic structures (Anghileri, 1989; Kouba, 1989) and conceptual understanding (Steffe, 1988). To illustrate this point, consider solving 4x3. In using additive thinking, a child adds four collections of 3 (perceived as three ones) together. This involves only one level of abstraction, as the child makes inclusion relations on only one level. Multiplicative thinking on the other hand involves making two kinds of relations:"(a) the many-to-one correspondence between the three units of one and the one unit of three; and (b) the composition of inclusion relations on more than one level. Making three units of one into one unit of three is an abstraction at a higher level than thinking only of units of one" (Clark & Kamii 1996, pp. 42-43). Therefore to understand multiplication and division a child needs to "coordinate a number of equal sized groups and recognise the overall pattern of composites (view a collection or group of individual items as one thing), such as 'three sixes'"(Sullivan, Clarke, Cheeseman, & Mulligan 2001, p. 234). Sullivan et al. (2001) and Killion and Steffe (2002) suggested that the acquisition of an equal-grouping (composite) structure is at the core of multiplicative thinking.

To understand division requires more than knowledge of sharing out a collection equally; it requires an awareness of the relationship between the divisor and the quotient (Bryant, 1997). Knowledge of amounts each receives varies according to the number of recipients is not a consideration by many young children when sharing. Bryant maintained that a young child may be able to share using one to one correspondence but is unlikely to have an understanding of this relationship.

Fischbein, Deri, Nello, and Merino (1985) proposed two aspects of division: partitive and quotitive. In partition division (commonly referred to as the sharing aspect) the number of subsets is known and the size of the subset is unknown, whereas in quotition division (otherwise known as measurement division), the size of the subset is known and the number of subsets is unknown. Partition division has traditionally been taught before quotition (or measurement) division because the sharing aspect was considered to relate much more to a child's everyday life (Bryant, 1997; Haylock & Cockburn, 1997). However, Brown (1992) found that children in grade 2 performed better on quotition problems and tended to solve partitive problems using grouping strategies, rather than sharing strategies. Other research indicated that the sharing aspect of division is limited and of less relevance in the long term than quotition division (Correa, Nunes, & Bryant, 1998; Haylock & Cockburn, 1997).

Over the past decade, a number of studies have focused on children's solution strategies to multiplication and division (Anghileri, 1989; Brown, 1992; Kouba, 1989; Mulligan, 1992; Mulligan & Mitchelmore, 1997; Oliver, Murray, & Human, 1991; Steffe, 1988). These studies have provided evidence that children's solution strategies begin generally with direct modelling and unitary counting, progress to skip counting, double counting, repeated addition or subtraction, then to the use of known multiplication or division facts, commutativity and derived facts. As they progress, they are moving from relying on direct modelling to solve problems to partial modelling through to developing multiplicative thinking, at which point they are operating on problems abstractly. Prior to this point they are unable to form composites or integrate the composite structure with their counting strategies. Mulligan and Mitchelmore (1997) found that children build up a sequence of increasingly efficient intuitive models derived from previous ones, and rather than switch from one model to the next, they develop an increasing range of models to draw upon when solving a problem. Kouba (1989) found children used two intuitive strategies when solving quotition problems: either repeated subtraction or repeatedly building (double counting and counting in multiples). For partitive division, children drew on three intuitive strategies: sharing by dealing out by ones until the dividend was exhausted; sharing by repeatedly taking away; and sharing by repeatedly building up. Kouba (1989) suggested the need for further research on whether individual children have more than one model for multiplication or division, and whether they consistently employ similar models for partitive and quotitive division.

Mulligan and Mitchelmore (1997) used four categories listed by Greer (1992) namely: equivalent groups, multiplicative comparison, rectangular arrays and Cartesian product, in their study, Anghileri (1989) on the other hand examined children's responses and strategies to these six categories: equal groups, allocation/ rate, array, number line, comparison (times as many) and Cartesian product. In some studies these categories of multiplication situations are referred to as semantic structures (Kouba 1989; Mulligan & Mitchelmore, 1997; Schmidt & Weiser, 1995). Kouba identified two semantic factors specifically related to one-step multiplication and division word problems, which may influence children's solution strategies. The first relates to the differences in the interpretation of the quantities. For instance the interpretation of 3 in each of the following: equal groups (e.g., 3 cherries per plate, given 4 plates); comparison problems (e.g., 3 times as many); Cartesian product (e.g., possible combinations with 3 shirts and 2 ties), may prompt quite different solution strategies. The other semantic factor relates to the quantity that serves as the unknown in a problem. In partition division (number of subsets in each set is unknown) whereas in measurement division (number of sets is unknown). For example, 12 divided by 3 interpreted as a partition problem translates to a situation such as, 12 lollies shared between 3 people how many each? Interpreted, as a quotition problem would be: 12 lollies in bags of 3, how many bags?

The degree to which children's solution strategies may be influenced by the semantic structure of a problem or the quantities used, is inconclusive from the research (Anghileri, 2001; Clarke & Kamii, 1996; Mulligan, 1992; Mulligan & Mitchelmore, 1997). This paper reports on one aspect of a larger study that investigated the effects of providing students with a broad range of multiplication and division word problems based on different semantic structures, on their developing understanding of multiplication and division. The particular aspect being reported on in this paper focuses on *equivalent groups* and *times as many* word problems for both partitive and quotitive situations. The research questions that guided this aspect of the study are (a) Do children think flexibly about division, in particular 'times as many' using multiplicative thinking? (b) To what extend do children use their knowledge of multiplication and intuitive strategies in solving division problems?

Methodology

The study was conducted from March to November 2007, and involved students aged eight and nine years from two grade 3 classes of two primary schools in a middle class suburb of Melbourne. One grade (EG) was part of a teaching experiment (TE); the other was used as a control group (CG). The TE occurred in two 12-day blocks, the first in May with the focus on multiplication, and the second in October when the focus was on division. The selection of the time frames was governed by the schools' schedules and the availability of the teachers. During these periods, the researcher and classroom teacher worked collaboratively.

The researcher planned the learning experiences each day in response to insights gained from the children's performance and strategies used. The teacher and researcher met for 30 minutes prior to each lesson and debriefed at the end of each lesson. The teaching approach involved a problem or question being posed and the students discussing possible strategies or methods for solving it. Word problems and open-ended tasks were the main context with some use of games. While the students were working the teacher and researcher roved and questioned them about the strategies they were using and their thinking. Often students were challenged to think of other ways they could solve the tasks and how they might check if the solution was correct.

Participants. While all 27 children in the EG were part of the TE, only 13 were selected as case studies from each grade, using a maximum variation sampling strategy (Patton, 2002). This enabled the researcher to gain a cross-section of each class according to their mathematical achievement. Prior to the TE, both grades were interviewed using the counting, addition and subtraction, multiplication and division domains of the Early Numeracy Research Project Interview (ENRP, Clarke, Cheeseman, Gervasoni, Gronn, Horne, McDonough, Montgomery, Roche, Sullivan, Clarke, & Rowley, 2002). These data were coded using a research-based framework of growth points, to identify the growth points reached by the students, and then students were ranked. The group was then divided into three bands and four children from the top and bottom band were chosen and five from the middle band. The classroom teacher in the TE had more than 20 years' teaching experience; the CG teacher was in her second year of teaching.

Instruments. The main sources of data collection were interviews. Two one-to-one, task-based interviews (Goldin, 1997) were used to probe and gain insights into students' understanding of multiplicative structures and strategies used in solving multiplication and division problems. There were three levels of questions for each of the following semantic structures: equivalent groups, allocation/rate, arrays, times as many, identified by Anghileri (1989) and Greer (1992). For each question, there were three levels of difficulty (rated by the researcher as easy, medium or challenge from pilot testing).

Multiplication word problems can be written as division problems (Greer, 1992; Mulligan & Mitchelmore 1997). The division interview consisted of 10 division word problems devised using the multiplication questions from the earlier interview. Each category included both a partitive (sharing) and quotitive (measurement) question to identify whether there was a relationship between the strategies students chose and the division type. Table 1 lists the questions chosen for discussion in this paper, categorising each as times as many or equivalent groups, noting the aspect of division (partition or quotation) and the rated level of difficulty. These were selected to show the contrasting strategies used for two quite different structures and because times as many is considered more difficult than the more commonly used equivalent groups (Haylock & Cockburn, 1997). In most instances, the contexts were the same for each problem type and the numbers chosen varied according to the level of difficulty.

Table 1

Division Word Problems Used in the Study

Semantic structure	Aspect of division	Level of difficulty	Problem
Equivalent groups	Partition	Easy	I have 12 cherries to share equally onto 3 plates. How many cherries will I put on each plate?
		Medium	I have 18 cherries to share equally onto 3 plates. How many cherries will I put on each plate?
		Challenge	I have 48 cherries to share equally onto 3 plates. How many cherries will I put on each plate?
	Quotition	Easy	There are 12 children in the class. Three children sit at each table. How many tables are there?
		Medium	There are 24 children in the class. Four children sit at each table. How many tables are there?
		Challenge	72 children compete in a sports carnival. Four children are in each event. How many events are there?
Times as many	Partition	Easy	Sam read 20 books during the read-athon, which was 4 times as many as Jack. How many books did Jack read?
		Medium	Sam read 36 books during the read-athon, which was 4 times as many as Jack. How many books did Jack read?
		Challenge	Sam read 72 books during the read-athon, which was 4 times as many as Jack. How many books did Jack read?
	Quotition	Easy	The Phoenix scored 18 goals in a netball match. The Kestrels scored 6 goals. How many times as many goals did the Phoenix score?
		Medium	The Phoenix scored 28 goals in a netball match. The Kestrels scored 7 goals. How many times as many goals did the Phoenix score?
		Challenge	The Phoenix scored 48 goals in a netball match. The Kestrels scored 16 goals. How many times many goals did the Phoenix score?

Interview Approach

The case study students in both schools were interviewed using these instruments three weeks after each twelve-day classroom intervention. The multiplication interview was administered to both cohorts in July and the division interview only in November. Each interview was audiotaped and took approximately 30 to 40 minutes, depending on the complexity of the student's explanation of the strategies used. Responses were recorded and any written responses retained. Students had the option of choosing the level of difficulty. Each question was presented orally, and paper, pencils and tiles were available for students to use at any time. If a student chose a difficult task and found it too challenging, there was an option for the student to choose an easier task.

Method of Analysis

Initially, the author coded the students' responses as correct, incorrect, or non-attempt as well as the level of abstractness of solution strategies, drawing upon the categories of earlier studies (Anghileri, 2001; Kouba, 1989; Mulligan, 1998; Mulligan & Mitchelmore, 1997). Those chosen, listed and defined in Table 2 according to the level of abstraction, include direct and partial modelling, building up, repeated subtraction, doubling and halving, multiplicative calculation and wholistic thinking. For the purpose of this paper, the term abstractness refers to a student's ability to solve a problem mentally without the use of any physical objects (including fingers), drawings or tally marks. Where a student solved two tasks for the one question (easy and hard), the code for the more sophisticated strategy was recorded only. If a student used a strategy that reflected lack of understanding of the task, this was coded as an unclear strategy.

Table 2

Solution Strategies for Whole Number Division Problems

Strategy	Definition
Unclear	Strategy reflects lack of understanding of task, or is unrelated to task.
Direct modelling	Uses sharing or one to many grouping with materials, fingers or drawings and calculates total by skip or additive counting.
Partial modelling	Partially models situation with concrete materials, or drawings using sharing or one to many grouping. Consistently uses skip or double counting to find the total.
Building up	Skip counts using the divisor up to the dividend. May use fingers to keep track of number counted. Records a number sentence in symbolic form.
Repeated Subtraction	Repeatedly taking away a specific number from the dividend until reaches zero, or skip counts back in multiples of the divisor from the dividend. Partial modelling in some instances. Records a number sentence in symbolic form.
Doubling and Halving	Derives solution using doubling or halving and estimation, attending to the divisor and dividend. Recognises multiplication and division as inverse operations. Records a number sentence in symbolic form.
Multiplicative Calculation	Automatically recalls known multiplication or division facts, or derives easily known multiplication and division facts, recognises multiplication and division as inverse operations. Records a number sentence in symbolic form.
Wholistic Thinking	Treats the numbers as wholes—partitions numbers using distributive property, chunking, and or use of estimation.

Giving the students a choice and the need to identify the particular strategies used for both partitive and quotitive division tasks added to the level of complexity in presenting the data, as reflected in the tables and figures in the following section.

Results and Discussion

Table 3 shows the comparison of results for both grades on equivalent groups and times as many partitive and quotitive whole number division problems. The responses for all 13 EG case study students were correct for both equal group tasks, and only one student in the CG response made an error. The number of correct responses varied on the times as many tasks. As indicated in Table 3, many more students in the CG gave incorrect responses for both the partitive and quotitive tasks, compared to the EG students. Only two students from the CG gave correct responses for the challenging tasks in the TMQ item during the interview, compared to nine students in the EG. Indeed, 12 EG students were able to give correct responses to items in the TMQ category compared with 3 CG students. This may be attributed to the fact that the students in the CG had little experience with such tasks prior to the interview, as their classroom learning sequence on division focused on equal group partition and quotition using direct modelling with materials leading to symbolic recording.

Table 3

Comparison of Correct Responses for Whole Number Division Problems for Both Grades

Grade	Task	EGP	EGQ	TMP	TMQ
Experimental	Easy				
(n=13)	Medium	4	3	3	3
	Challenging	9	10	8	9
Control	Easy			1	
(n=13)	Medium	20	6	1	1
	Challenging	3	6	3	2

Note. EGP Equal group partition, TMP Times as many partition, EGQ Equal group quotation TMQ Times as many quotition

Table 4 shows the frequency of strategies by each grade on the different whole number division tasks. For space reasons, the strategies are not shown for each level of difficulty.

Table 4

Distribution of Strategies Used for Times as Many and Equivalent Groups Tasks for Both Grades (Experimental (EXP) and Control (CON))

Grade	Task type	Unclear strategy	Direct modelling	Partial modelling	Building up	Repeated subtraction	Doubleing or halivng	Miltiplicative Calculation	Wholistic thinking
EXP *(n=13)*	EGP			2	2		1	6	2
	EGQ			1	2		3	6	1
	TMP				2	2	4	3	2
	TMQ				3	2	3	3	2
CON *(n=13)*	EGP		3	5	1			3	1
	EGQ		3	1	6			3	
	TMP	1	1	2	1	5	1		2
	TMQ	2	5	1	3			1	1

Multiplicative calculation was the preferred strategy of the EG (six students used this for the equivalent groups tasks), whereas the CG students used a wider range of strategies. In focusing on the times as many tasks, EG students predominantly used multiplicative strategies (wholistic thinking-four, multiplicative calculation-six, or doubling and halving-seven) and gave correct responses, whereas students in the CG predominantly used some form of modelling or a strategy that reflected little or no understanding of the task (building up-three, partial modelling-three, direct modelling-six, unclear strategy-three) with only eight correct responses (as indicated in Table 3) over the two times as many tasks. Over the four task types, multiplicative calculation was used most frequently (15), doubling and halving (eleven), building up (11) and wholistic thinking (8), in the EG. By contrast, direct modelling was used most frequently (12) in the CG, building up (11), partial modelling (8), multiplicative calculation (7).

Students in the CG rarely drew on their knowledge of multiplication when solving division tasks, and when they did, it was in the form of skip counting. They tended to use sharing in the partition tasks and repeated subtraction or count back, in the times as many tasks (as indicated by Kouba 1989). Students in the EG, on the other hand, tended to draw on their knowledge of multiplication facts as a starting point, rather than repeated subtraction or count back. They were able to record a division number sentence and explain the relationship between the numbers in both a multiplication and division number sentence. Many students in the EG during the interview commented that division is easier than multiplication because "you know how many you have to start with and how many groups or how many to divide it between".

The following excerpts from the students exemplify some of these strategies. To solve the equivalent groups quotitive problem, Bianca used known multiplication facts and doubling and Jack used doubling.

> Bianca: I started with 12x4 then doubled it to get 24x4 and that's 96, but that's too much. So I took away 20 from 96 and that gave me 76, and that's 19x4 but I still need to take away another 4 so it would be 18 events or 72 ÷ 4 =18.
>
> Jack: 4 children in each event and 72 children is 18 events, 'cause 4 plus 4 is 8 (that's 2), 8 plus 8 is 16 (that's 4), 16 plus 16 is 32 (that's 8), 32 and 32 is 64 (that's 16), 64 and 2 fours more is 72, so it is 18 fours. The number sentence would be 72 ÷ 4 = 18.

To solve the times as many partitive problem, Samantha used wholistic thinking by splitting the product and using the distributive property. Jack on the other hand used halving and his place value and fraction knowledge.

Samantha: I need to think of 4 times something equals 72. 72 take away 40 (which is 4 times 10) is 32, and 32 is 4x8. 4x20 is 80, but that's too much. I know 6x12 equals 72, but 12 would be 6 times not 4 so it can't be that. I need to take 8 off 80 to get 72 so it would be 18, because 4x10 is 40 and 4x8 is 32 and together that's 72. So Sam read 18 books, 72 ÷ 4 = 18.

Jack: Half of 7 is 3 and a half, so half of 70 would be 35 and half of 72 would be 36 because half of 2 is 1. Half of 3 is one and a half, so half of 30 is 15 and half of 6 is 3 so it is 18 books. 72 ÷ 4 = 18 because you halve 72 two times, which is the same as saying 18, four times.

The numbers used in the equal groups partitive and times as many quotitive tasks were the same, as were the numbers in the equal groups quotitive and times as many partitive, but none of the children recognised this. These students solved times as many tasks more efficiently (and in less time) than the equal groups tasks, which was surprising. In each instance, they were clearly drawing on their problem solving skills and knowledge of number as they were thinking about the problem. Jack consistently used doubling and halving; the others used multiplication.

Conclusion

The results suggest that young children are capable of solving complex division problems when provided with a problem solving learning environment that encourages them to draw on their intuitive thinking strategies and knowledge of multiplication. Given an opportunity to experience a range of semantic structures for multiplication provides a solid basis for children's developing understanding of division. This finding resonates with the work of Mulligan and Mitchelmore, (1997) but this study adds to the body of knowledge an intensive look at the *times as many* structure of multiplication as applied to both partitive and quotitive aspects of division.

Children need a variety of experiences with different semantic structures and contexts to understand fully the operations of multiplication and division. These experiences need to include both partitive and quotitive aspects of division in the early years, using contexts that relate to children's everyday lives. This implies allowing students to draw on their own intuitive strategies to solve both partitive and quotitive division word problems, prior to formal teaching. Placing emphasis on the relationship between multiplication and division and the language associated with both operations before any use of symbols or formal recording needs to be a priority. The study also provides an argument to support delaying the introduction of any formal algorithm for division until children have a sound conceptual understanding of division, and are confident in solving division tasks beyond the multiplication fact range mentally. A possible question for further research: Do children retain the richness of mental strategies when taught the formal written algorithms?

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Intervention Instruction in Structuring Numbers 1 to 20: The Case of Nate

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Nate was one of 200 participants in a research project aimed at developing pedagogical tools for use with low-attaining 3rd- and 4th-graders. This involved an intervention program of approximately thirty 25-minute lessons over 10 weeks. Focusing on the topic of structuring numbers to 20, the paper describes Nate's pre- and post-assessments, including major gains on tests of computational fluency. Relevant instructional procedures are described in detail and it is concluded that the procedures are viable for use in intervention.

The Numeracy Intervention Research Project (NIRP) aims to develop pedagogical tools for intervention in the number learning of low-attaining third- and fourth-graders (8- to 10-year-olds). Within the NIRP we have developed an experimental learning framework for whole number knowledge comprising five key domains: number words and numerals, structuring numbers 1 to 20, conceptual place value, addition and subtraction 1 to 100, and early multiplication and division (Wright, Ellemor-Collins, & Lewis, 2007). For each of these domains we are developing a set of instructional procedures. The case study reported here is part of the research program developing a detailed set of instructional procedures for the key domain Structuring Numbers 1 to 20. The main purpose of the case study is to document the instructional procedures that appeared to contribute to the learning of a student who made significant progress in knowledge of Structuring Numbers 1 to 20.

Background

Addition and Subtraction in the Range 1 to 20

In early number learning, children use strategies involving counting by ones, for example solving 8+7 by counting on seven from 8, using fingers to keep track. Children achieve facile additive thinking when they develop a rich knowledge of number combinations, and solve additive tasks with non-counting strategies, for example 8+7 as: 8+8 is 16, less 1 is 15 (Fuson, 1992; Steffe & Cobb, 1988; Wright, 1994). Facile strategies in the range 1 to 20 are foundational for further arithmetic, and for developing number sense (Bobis, 1996; Heirdsfield, 2001; McIntosh, Reys, & Reys, 1992; Treffers, 1991). Thus, the development from counting to facile non-counting strategies in the range 1 to 20 is a critical goal in early numeracy (Wright, 1994; Young-Loveridge, 2002).

Some children do not achieve this facility. Instead, they persist with strategies involving counting by ones for addition and subtraction in the range 1 to 20, and in turn use counting strategies in the higher decades. Persistent counting is characteristic of children who are low-attaining in number learning (Denvir & Brown, 1986; Gervasoni, Hadden, & Turkenburg, 2007; Gray & Tall, 1994; Treffers, 1991; Wright et al., 2007). Persistent counting can disable students' progress with numeracy. An extended review of this literature is available elsewhere (Ellemor-Collins & Wright, 2008).

Numeracy is a principle goal of mathematics education and there are calls for intervention in the learning of low-attaining students to bring success with numeracy (Bryant, Bryant, & Hammill, 2000). In developing numeracy intervention, there is a pressing need to design instructional procedures that are likely to progress students from counting strategies to non-counting strategies. Designing such procedures is a central goal of the present study.

Instructional Design of Structuring Numbers 1 to 20

Structuring Numbers 1 to 20 is an instructional topic designed to build rich knowledge of number combinations and develop facile non-counting calculation strategies in the range 1 to 20, developed from the work of Treffers (1991); Gravemeijer, Cobb and colleagues (Gravemeijer, Cobb, Bowers, & Whitenack, 2000); and Wright and colleagues (Wright et al., 2007). Drawing on the *emergent modelling* heuristic (Gravemeijer et al., 2000), instructional design involves devising instructional procedures which foster students' *progressive mathematisation* from informal, context-bound knowledge to more formal, generalised knowledge. We seek to devise instructional *settings* in which students can first, develop context-bound knowledge of combinations and non-counting strategies—such as identifying five red and three green dots as eight dots—and then, reflect on their activity, and generalise toward more formal reasoning about numbers—such as partitioning eight into five and three to solve the written task 8-3 without counting. Building knowledge of combining and partitioning numbers supports the development of facile calculation (Bobis, 1996; Fischer, 1990; Young-Loveridge, 2002). For addition and subtraction in the range 1 to 20, informal non-counting strategies commonly develop around doubles combinations, combinations with 5 and 10, and tens-complements (9+1, 8+2, 7+3, 6+4, 5+5) (Gravemeijer et al., 2000). Useful settings for these combinations are the *ten frame* for the range 1-10 (Bobis, 1996; Treffers, 1991; Young-Loveridge, 2002), and the *arithmetic rack* for the range 1-20 (Gravemeijer et al., 2000; Treffers, 1991) (see *Instructional Settings* below).

Method

The NIRP adopted a methodology based on design research (Cobb, 2003), with three one-year design cycles. In each year, teachers and researchers implemented and further refined an experimental intervention program with students identified as low-attaining in their schools. The program year involved (a) in term 2, a range of pre-assessments of the students; (b) in term 3, a ten week instructional cycle; and (c) in term 4, post-assessments. In total, the project has involved professional development of 25 teachers, interview assessments of 300 low-attaining students, and intervention with 200 of those students.

Assessments

The primary assessment instrument was an individual task-based interview, approximately 40 minutes in length, videotaped for later analysis. Tasks addressed the five key domains of the learning framework. The interview assessment informed instruction, and enabled an assessment of progress sensitive to the intended instruction. A written screening test was used to help identify students low-attaining in their cohort. It included numeral sequence tasks, horizontal addition and subtraction tasks, and word problems. Among other shorter assessments, significant for the present case study were the *One Minute Tests of Basic Number Facts* (Westwood, 2003), for addition and subtraction. The addition test uses a page listing 33 additions of two single digit numbers written in horizontal format; the student is given one minute to write answers to as many as possible. The subtraction test is structured similarly. Each of these assessment instruments was administered as pre-assessments in term 2 and as post-assessments in term 4.

Instructional Cycle

In each school, two students were taught individually, six in trios. The instructional cycle consisted of approximately thirty 25-minute lessons across 10 weeks. Each lesson typically addressed three or four domains of the learning framework. Individual lessons were videotaped, providing an extensive empirical base. The analysis of the learning and instruction is informed by a teaching experiment methodology (Steffe & Thompson, 2000).

Instructional Settings

Ten frames 1-10. A 2x5 frame with a standard configuration of dots for a number in the range 1 to 10, either pair-wise (such as four dots on each row) or five-wise (five and three).

Tens-complements cards. Ten frames with ten dots of two colours in the combinations: 9 and 1, 8 and 2, 7 and 3, 6 and 4, 5 and 5, configured pair-wise and five-wise.

Ten frame addition cards. The 25 frames having 1-5 red dots on one row and 1-5 green dots on the other.

Arithmetic rack. Two rods, each with five red and five blue beads. Like on a counting frame, beads can be moved to one end of the rods to present certain configurations, such as 5-and-1 on upper and 5-and-1 on lower; or 10 on upper and 2 on lower.

Expression card. Two addends in the range 0 to 10, in horizontal format (such as 2+7). The set of expression cards includes all 121 such expressions.

Case Study Overview

Nate was eight years old and in the 3rd grade. His intervention teacher was Louise. Nate was selected as a case study of Structuring Numbers 1 to 20 for three reasons. (a) He participated in the third year of the study, when the instructional procedures in Structuring Numbers 1 to 20 were most developed. (b) He began the intervention with some knowledge of non-counting additive strategies in the range 1-10, so he had some scope for progress in the range 1-20 within the ten weeks of instruction. (c) He made major progress in non-counting strategies and timed computational fluency over the course of the intervention. The purpose of the case study is to document Nate's progress, and to describe the intervention instruction that seems to have been significant. The study informs our knowledge of the potential usefulness of the instructional settings and procedures.

The Case of Nate

Below, we compare Nate's knowledge of addition and subtraction in the range 1-20 in his pre- and post-assessments. We then describe relevant episodes of the instruction.

Nate's Assessments in the Range 1-20

Pre-assessments. Nate's pre-assessments were in April and May. In the interview, he could say how many more to make ten for 5, 9, and 7 fluently, and for 2 and 4 with four seconds of thinking each. He found partitions of 7 using his fingers, but then found partitions of 6, 12, and 19 without fingers or counting. Asked the doubles 8 plus 8, 9 plus 9, and 7 plus 7, he thought for five seconds for each answer, but did not apparently use counting. For 9 plus 9 he answered "19". He added ten to each of 7, 5, and 1 successfully, without counting by ones. Five tasks were presented in written horizontal format. He solved 6+5 using a near-doubles strategy. 9+6, 8+7, and 11+8 were solved by counting-on by ones, using fingers to keep track of his counts. 17-15 was solved counting back 15 counts from 17. The written screening test included 12 addition and subtraction tasks in the range 1-31, in horizontal format. Nate incorrectly answered four subtractions: 15-9 (9), 14-11 (2), 9-4 (3), and 23-17 (7). In summary, in the range 1-20, Nate knew some useful combinations such as tens-complements, doubles, and tens-combinations, but still tended to use counting strategies for unknown calculations, and was error-prone to some extent.

Post-assessments. Nate's post-assessments were in October. In his interview, Nate was more fluent on the tens-complements, partitions, doubles, and tens-combinations tasks, and did not use counting on these. He made one error, initially stating 9 and 11 as two numbers to make 19 and then saying "18 and 1". Asked for another partition of 19 he answered "12 and 6". He answered the five tasks presented in written format quickly and successfully, using non-counting strategies. On the written screening test, he answered all 12 addition and subtraction tasks correctly. In summary, in the range 1-20, Nate showed increased fluency with combinations and partitions, a shift from counting to non-counting strategies, and increased facility and success with calculation.

Progress on One Minute Tests of Basic Facts. These tests were not intended as proximal measures of the intended learning, rather as potential benchmarks of computational fluency. Useful for comparison, Westwood's study (2003) provides mean scores and standard deviations for ages 6 to 11 from a large cohort (n=2297) of students in Adelaide in 1995. Nate's one minute tests make a striking comparison from pre- to post-assessment. In April, Nate scored 12 of 33 for addition, and 12 of 33 for subtraction. In October, he scored 30 and 31, respectively. Comparing Nate's progress with the results for Westwood's 1995 cohort (see table 1), Nate has progressed in addition from half a standard deviation below the mean to two standard deviations above, and in subtraction from a mean score to three standard deviations above the mean. In both cases he has almost attained the maximum score, and made a marked atypical leap in computational fluency.

Table 1

Nate's Scores Compared with Westwood's 1995 Mean Scores and Standard Deviations

		Addition			Subtraction		
Month	**Age**	**Score**	**1995 mean**	**Std. dev.**	**Score**	**1995 mean**	**Std. dev**
April	8.5	12	15.54	6.43	12	12.28	5.59
Oct	9.0	30	16.79	6.10	31	13.11	5.79

Instruction–Structuring Numbers 1 to 20

The intervention included 24 individual lessons over 10 weeks, from July to September. Instruction in Structuring Numbers 1 to 20 was mostly restricted to the 19 lessons of the first six weeks. Below we describe episodes from weeks 1, 2, 5, and 6, to show some key instructional activities that seemed to be productive, and to indicate the progression of the teaching and learning across these weeks of intervention.

Weeks 1 and 2: Ten frames. Louise used a set of ten frame cards to present a pattern identification activity. She would show each card, and Nate's task was to name the number. Later she changed to flashing each card. In each of lessons 1-9, Louise worked twice through the pair-wise 1-10 set, or the five-wise 1-10 set, or both. Nate was generally successful and facile on these tasks. In lesson 5, Louise introduced the tens-complements ten frame cards. Nate's task for each card was to say how many of each colour, for example "6 and 4; 2 and 8." Louise showed the set twice, then flashed the set twice. During the first turn showing the five-wise tens-complements set, Nate made one error, identifying the 8&2 card as "8 and 1". He made the same error when the set was flashed. He commented that the five-wise set was hardest.

Lesson 6, week 2: How many more to make ten? In lesson 6, after flashing the five-wise 1-10 set, Louise extended the task, asking Nate to say for each card both the number of dots and how many more to make ten. Nate rapidly answered the first three cards with "Five and five! Nine and one! Two and… two, eight." He then commented that the task was like the tens-complements cards. After this activity, Louise showed and flashed the pair-wise tens-complements set, and flashed the five-wise tens-complements set twice. Nate answered without errors, and with increasing fluency.

Lesson 6, week 2: Ten frame addition cards. Later in lesson 6, Louise introduced the ten frame addition cards. Nate's task was to say how many dots of each colour, how many altogether, and how many more to make ten. Thus, the activity involved identifying dot patterns, combinations less than ten, and complements to ten—a rich activity in structuring numbers 1 to 10. Louise posed 11 tasks, and Nate was successful with each task. Recognising the two numbers was easy for him. Some sums and tens-complements were apparently more difficult for him. In solving the sum of the first card (3&4), Nate answered "Three and four make nine…no (looks away from the card), three…three and four make…(looking up)…s-…seven". On the remaining tasks, Nate sometimes looked away from the card, and sometimes at the card, and may have used some counting by ones.

Lesson 13, week 5: Doubles and 10-plus. Louise used the arithmetic rack to present doubles and 10-plus tasks. Three sets of tasks followed a sequence: (a) not screening the rack, (b) screening and momentarily unscreening the rack, and (c) keeping the rack screened and posing the tasks verbally. Next, Louise posed doubles and 10-plus tasks verbally, without the rack. Nate was generally successful and fluent on these tasks.

Lesson 16, week 6: Structuring 1 to 10 consolidated. Lesson 16 included five sets of tasks involving structuring 1 to 10. Louise flashed each of the pair-wise and five-wise tens-complements cards. Nate was fluent with these despite that Louise had not used them since week 3. Next, Louise used the ten frame addition cards, as she had in lesson 6, but now only flashed them. Nate's task was to say the two numbers and the sum. By contrast with his effort in lesson 6, he was generally fluent on these tasks.

Following the addition card activity, Louise posed tens-complements tasks verbally: She would say a number in the range 1-9, and Nate's task was to say how many more to make ten. Nate answered fluently. Next, Louise introduced a set of expression cards with sums in the range 6 to 10. As she placed each card on the desk, Nate's task was to say the sum as quickly as possible. Nate was fluent and successful with all 31 of these tasks, and did not use counting by ones. The written setting (expression cards) is typically more challenging for students than structured settings; Nate's facility with number combinations to 10 in this activity was significant. Following the expression card activity, Louise presented tasks on jumping to the nearest decuple, which involved applying knowledge of tens-complements with numbers beyond 20. Nate was successful both in a screened ten frame setting, and with purely verbal tasks.

Lesson 19, week 6: 9-plus and 8-plus. Following an initial segment involving doubles and 10-plus tasks presented verbally, Louise presented a set of 9-plus tasks, using the arithmetic rack. First, with 9 on the rack, Louise asked "9 and 3", and moved 1&2 on the rack. Nate seemed to be confused. After four such tasks, they discussed why Louise was making the addend in a 1&X form, and she presented three more tasks. Next, Louise screened the rack. She called out the additions—"Nine and four more"— and after Nate answered, she lifted the screen for him to check. Nate was successful with these tasks and became more engaged, looking up from the screen and thinking hard. He was not using counting by ones, and it seems likely he was adding through ten and perhaps visualising the rack in doing so. Finally, Louise posed eight 9-plus tasks verbally. Nate was successful with these, generally answering within one second. Following these 9-plus activities, Louise presented similar tasks involving 8-plus, first on the arithmetic rack, and then verbally. Nate was successful on these tasks, apparently with less certitude than he showed with the 9-plus tasks. On the verbal task, 8 and 5 more, Nate nodded his head three times before answering "13", then stated that he had counted by ones. This suggests that his use of non-counting strategies was not yet fully routine. Rather, using a non-counting strategy was, to some extent, engendered by the setting, that is the arithmetic rack.

Discussion

Over the course of the intervention, Nate made significant progress with addition and subtraction in the range 1-20, changing from using counting to using non-counting strategies, and attaining high computational fluency. The instructional procedures in structuring numbers 1 to 20 appeared to be significant for his learning. His progress is evident within single lessons, for example in lesson 19 establishing a non-counting strategy for 9-plus tasks. Progress is also evident in activities repeated over a series of lessons, for example in fluency with the tens-complements cards from lesson 5 through to lesson 16.

As part of the design research, the case study informs the design of the instructional procedures for Structuring Numbers 1 to 20. We describe four features of the instruction which seemed to facilitate Nate's learning. First, Louise generally focused each lesson segment on a particular aspect of structuring numbers. Aspects in the range 1-10 were: five-wise and pair-wise patterns for numbers 1-10, tens-complements, and combinations less than ten. Aspects in the range 11-20 were: doubles and 10-plus patterns for 11-20, 9-plus and 8-plus combinations, near-doubles combinations. These aspects are roughly identifiable as subsets of the combinations in the range 1-20, some are also identified with particular sets of ten frame cards or expression cards. In organising the instruction by aspects, the range of any one activity did not become unmanageable for Nate, and Louise could target the cutting edge of his knowledge of that aspect.

A second feature of instruction is the use of the ten frames and arithmetic rack. Over the first five weeks working in the ten frame setting, Nate became facile with combining and partitioning numbers in the range 1-10. In the 9-plus episode in lesson 19, the setting of the arithmetic rack seemed critical in supporting Nate's attention to reasoning without counting by ones. The case study affirms the usefulness of these settings.

Table 2

Settings Used in Each Week, for Each Aspect of Structuring Numbers 1 to 20

Week	Patterns 1-10	Tens-complements	Sums ≤ 10	Doubles, 10-plus	Sums > 10
1	Show TF →Flash TF				
2	→Flash TF	Show TF →Flash TF	Show TF		
3	→Flash TF →Flash AR	Show TF →Flash TF		View AR →Flash AR	View AR →Flash AR
4			→→→Exp cards		
5		→→Screen TF →→→Verbal		→Flash AR →→Screen AR →→→Verbal	
6		→Flash TF →→Screen TF →→→Verbal	→Flash TF →→→Exp cards	→→→Verbal	View AR →→Screen AR →→→Verbal

Note: TF=A ten frame set. AR=arithmetic rack. Exp cards=a set of expression cards. → =setting distanced.

A third feature of instruction is the progressive distancing of the settings, through the use of flashing and screening. Progressive distancing is evident at micro- and macro-levels of the instruction. Within a lesson segment with ten frames, Louise would move from showing to flashing the cards. Extended lesson segments, such as the 9-plus episode in lesson 19, consisted of a progression: Showing the setting, then flashing, then screening, and finally removing the setting and posing tasks verbally. This progressive distancing enabled Nate to apply his knowledge of patterning and partitioning to reasoning without counting by ones, on verbal tasks. On the macro-level, over the weeks, Louise moved from showing the structured settings towards verbal and written settings. Table 2 shows the progression of settings Louise used for tasks on five different aspects of Structuring Numbers 1 to 20. Progressive distancing of the setting is evident down each column.

A fourth feature of instruction is that the aspects were not treated as discrete or disconnected. Aspects were connected within lessons, for example, in the lesson 6 episode "How many more to make ten?", an activity identifying patterns 1-10 became an activity of reasoning about tens-complements, and in lesson 19, Louise reviewed 10-plus tasks before a set of 9-plus tasks. Further, the instruction in these aspects was not in lock-step fashion. As seen in Table 2, the introduction of each aspect is staggered across the weeks, and attention to the aspects overlaps, with each aspect at different stages of progress.

This case study provides a detailed overview of instruction that resulted in significant learning in the domain of structuring numbers to 20, and major progress on two timed tests of computational fluency. We conclude that the instructional procedures described here are viable as a basis for intensive intervention and we will continue to refine them.

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Interdisciplinary Problem Solving: A Focus on Engineering Experiences

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We are facing a serious skills shortage in mathematics, science, and engineering—our efforts to remain globally competitive will be severely hampered if this shortage continues. Numerous recent calls for improving students' learning in these disciplines and for raising our nation's levels of innovation and creativity have been made. In response, this discussion paper argues for a future-oriented interdisciplinary approach to mathematical problem solving, one that draws upon engineering. Consideration is given to engineering as a problem-solving domain, the interdisciplinary knowledge and processes that are fostered, and the role of mathematical modelling in solving engineering-based problems. An example of such a problem for the primary/middle school is analysed.

Worldwide demand for new mathematical solutions to complex problems is unprecedented and has led to an appreciation of the power of cross-disciplinary research within the mathematical sciences and with other disciplines. (National Strategic Review of Mathematical Sciences Research in Australia, December, 2006, www.review.ms.unimelb.edu.au).

Numerous recent reports have highlighted the need to review our teaching of mathematics and science if we are to remain globally competitive. The key findings of the above review alert us to the challenges we face in providing a strong base in the mathematical sciences. Without such a base, "our options for solving complex problems, adding intellectual value to new technologies, spearheading innovation and continuing to compete globally will be severely hampered" (www.review.ms.unimelb.edu.au). In a similar vein, the Business Council of Australia (2007) has argued that, while significant and far-reaching changes have been made in the way the wider community operates and communicates, "many aspects of our school system have not changed since the 1960s" (BCA, 26 Aug., 2007, www.bca.com.au). Of particular concern is the skills shortage facing the Australian workforce. Not surprising, there is a serious skills shortage in the sciences and mathematics, with a predicted estimated shortfall of 19,000 scientists and engineers by 2012 (Department of Education, Science, and Training, 2006).

The above concerns are echoed in the recent call for submissions to *Australia's Innovation Challenge* (Cutler, 2008, www.innovation.gov.au). Cutler has identified seven basic questions for consideration, including the following:

- Can we imagine a better world? Are we asking the right questions?
- Could we do everyday things better?
- How do we educate and equip our people to be creative and innovative, life-long?

The first question addresses the "generation of novel and fresh ideas," which involves "pushing boundaries, and questioning the status quo." The second question focuses attention on creative problem solving "everywhere by everyone," while the last question highlights the need for innovation, creativity, and design to be engendered throughout life. These questions provide grist for exploring ways in which we might advance the teaching and learning of mathematics to address the skills shortages we face.

In this discussion paper I offer one approach to advancing the mathematics curriculum, namely, through interdisciplinary problem solving. In particular, I consider how engineering experiences can enrich the primary and middle school mathematics curriculum by promoting greater student interest in, and appreciation of, mathematics and science in solving real-world problems. Such experiences also help students appreciate how engineering shapes so many facets of our world and how society influences, and is influenced by, engineering. An early introduction to engineering as a career is also a significant component here. I begin by briefly reviewing recent calls for new perspectives on mathematical problem solving.

Calls for New Perspectives on Problem Solving

Research on mathematical problem solving has received a good deal of attention in past decades. Among the notable developments have been Polya's (1945) seminal work on how to solve problems, studies on expert problem solvers (e.g., Anderson, Boyle, & Reiser, 1985), research on teaching problem-solving strategies and heuristics and fostering metacognitive processes (e.g., Charles & Silver, 1988; Lester, Garafalo, & Kroll, 1989), and, more recently, studies on mathematical modelling (e.g., Lesh, in press; English, 2008a). Despite these decades of research, it seems that students' problem-solving abilities still require substantial improvement (Kuehner & Mauch, 2006; Lesh & Zawojewski, 2007; Lester & Kehle, 2003). This current state of affairs has not been helped by the noticeable decline in the amount of problem-solving research that has been conducted in the past decade. As Lester and Kehle (2003) noted, such a decline is not surprising given the increased complexity of problem solving and the multiple categories of interdependent factors that contribute to problem-solving performance.

On the other hand, the decline in this research could also be attributed in part to the mathematics education community's complacency with problem solving, assuming that we know all there is about problem solving and need only refer to established curriculum documents to justify any such research (e.g., *Standards* documents of the National Council of Teachers of Mathematics). Much-needed recent calls for new perspectives regarding the nature of problem solving and its role in the mathematics curriculum have appeared in the literature (English, 2007; Lesh & Zawojewski, 2007; Lester & Kehle, 2003). One such perspective is the interdisciplinary collaboration of researchers in their search for a more comprehensive understanding of human cognition and problem solving (English, 2008b; Lester & Kehle, 2003).

Future-Oriented Interdisciplinary Problem Solving

Future-oriented problem-solving experiences in mathematics and science increasingly require interdisciplinary contexts and approaches (English, 2008b; Lesh, in press). Concerns have been expressed by numerous researchers and employer groups that schools are not giving adequate attention to the understandings and abilities that are needed for success beyond school. For example, potential employees most in demand in mathematics/science related fields are those that can (a) interpret and work effectively with complex systems, (b) function efficiently and communicate meaningfully within diverse teams of specialists, (c) plan, monitor, and assess progress within complex, multi-stage projects, and (d) adapt quickly to continually developing technologies (Lesh, in press). Research indicates that such employees draw effectively on interdisciplinary knowledge in solving problems and communicating their findings. Furthermore, although they draw upon their school learning, these employees do so in a flexible and creative manner, often creating or reconstituting mathematical knowledge to suit the problem situation, unlike the way in which they experienced mathematics in their school days (Gainsburg, 2006; Hamilton, 2007; Lesh, in press; Zawojewski & McCarthy, 2007).

These findings present interesting challenges for mathematics and science educators. For example, how might we help students better understand and appreciate how their mathematics and science learning in school relates to the solving of problems outside of the classroom? How can we broaden students' problem-solving experiences to promote creative and flexible use of mathematical ideas in interdisciplinary contexts? How can we help address the skills shortage in mathematics/science related fields? A promising approach to addressing these questions is through the discipline of engineering.

Engineering as a Problem-Solving Domain

Why should mathematics curricula consider engineering as a problem-solving domain? Australia, along with many other nations, has experienced a significant decline in the number of graduating engineers, an overall poor preparedness for engineering studies in tertiary institutions, and a lack of diverse representation in the field (Dawes & Rasmussen, 2007; Downing, 2006; Lambert, Diefes-Dux, Beck, Duncan, Oware, & Nemeth, 2007). Internationally, the number of Australian engineering graduates per million lags behind most of the other OECD countries (Taylor, 2006). The availability of engineers, mathematicians, and scientists has been identified as "one of the notable competitive disadvantages" for Australia with respect to its level of innovation; Australia remains well behind other nations in this sphere (The World Economic Forum Global Competitiveness Report; http://www.weforum.org).

The need to capture students' interest in the engineering domain before they embark on tertiary education has been highlighted in many recent documents. For example, the Queensland Government's discussion paper, *Towards a 10-year Plan for Science, Technology, Engineering and Mathematics (STEM) Education and Skills in Queensland* (Bligh & Welford, October, 2007) and the *Australian School Science Education National Action Plan 2008-2012* (Goodrum & Rennie, 2007) are illustrative of the increasing concerns being expressed over Australia's need to rebuild engineering and the mathematical sciences.

The proportion of year 12 students studying suitable enabling subjects in mathematics and science has continued to decline at the same time that shortages in engineering domains have emerged (Dawes & Rasmussen, 2007). Furthermore, the representation of women in engineering is still low, despite some efforts at the tertiary level to attract more female students (e.g., Dhanaskar & Medhekar, 2004). More than ever before, we need to increase the profile and relevance of mathematics and science education in solving problems of the real world, and we need to begin this in the primary and middle schools (The Business Council of Australia, 2007). Indeed, the middle school has been identified as a crucial period for either encouraging or discouraging students' participation and interest in mathematics and science (Tafoya, Nguyen, Skokan, & Moskal, 2005). Engineering provides an exceptional context in which to showcase the relevance of students' learning in mathematics and science to dealing with authentic problems meaningful to them in their everyday lives.

The domain of engineering builds on students' curiosity about the natural world, how it functions, and how we interact with the environment, as well as on students' intrinsic interest in designing, building, and dismantling objects in learning how they work (Petroski, 2003). By incorporating engineering problems within the primary and middle school mathematics curriculum, we can: (a) engage students in creative and innovative real-world problem solving involving engineering principles and design processes that build on existing mathematics and science learning; (b) show how students' learning in mathematics and science applies to the solution of real-world problems; (c) improve preparedness of senior subjects; (d) help students appreciate how society influences and is influenced by engineering; and (e) promote group work where students learn to communicate and work collaboratively in solving complex problems (English, Diefes-Dux, Mousoulides, & Duncan, submitted; Zarske, Kotys-Schwartz, Sullivan, & Yowell, 2005).

In summary, given the increasing importance of engineering and its allied fields in shaping our lives, it is imperative that we foster in students an interest and drive to participate in engineering from a young age, increase their awareness of engineering as a career path, and better inform them of the links between engineering and the enabling subjects, mathematics and science.

Interdisciplinary Knowledge and Processes Fostered by Engineering-Based Problems

Engineering-based problem-solving experiences for the primary and middle school need to build on and complement existing core mathematics and science curricula content. Such problems should be designed so that multiple solutions of varying sophistication are possible and students with a range of personal experiences and knowledge can participate (English et al., submitted). *Interdisciplinary knowledge and processes* fostered in solving these problems include the following (adapted from Cunningham & Hester, 2007):

Interdisciplinary knowledge of: (a) what engineering is, what engineers do, and the different fields in which engineers work; (b) core engineering ideas and principles and how these draw upon mathematics and science; (c) the nature of engineering problems and their multiple solutions and approaches; (d) engineering design processes in solving these problems; (d) the role of mathematical models in solving engineering problems; (e) how society influences and is influenced by engineering; and (f) ethical issues in undertaking engineering projects.

Interdisciplinary processes involving: (a) applying engineering design processes; (b) applying mathematics and science learning in engineering; (c) employing creative, innovative, careful, and critical thinking in solving problems; (d) envisioning one's own abilities as an engineer; (e) trouble shooting and learning from failure; and (f) understanding the central role of materials and their properties in engineering solutions.

Mathematical Modelling in Solving Engineering-Based Problems

At the heart of engineering is an understanding of engineering design processes (Cunningham & Hester, 2007) and the creation, application, and adaptation of mathematical/scientific models that that can be used to interpret, explain, and predict the behaviour of complex systems (English et al., submitted; Zawojewski, Hjalmarson, Bowman, & Lesh, in press). A basic engineering design process involves the following cyclic components (Cunningham & Hester, 2007). ASK—What is the problem? What have others done? What are the constraints? IMAGINE—What are some possible solutions? Brainstorm ideas. Choose the best one. PLAN—What diagram can we draw to help us here? Make a list of materials needed. CREATE—Follow your plan and create it. Test it out. IMPROVE—Discuss what works, what doesn't, and what could work better. Modify your design to make it better, Test it out.

The cyclic processes of modelling and design (see Figure 1) are very similar: a problem situation is interpreted; initial ideas (initial models, designs) for solving the problem are called on; a fruitful idea is selected and expressed in a testable form; the idea is tested and resultant information is analysed and used to revise (or reject) the idea; the revised (or a new) idea is expressed in testable form; etc. The cyclic process is repeated until the idea (model or design) meets the constraints specified by the problem (Zawojewski et al., in press). Engineering-based problems thus fit very nicely within existing mathematics curricula, in particular, those that incorporate the important strand of models and modelling.

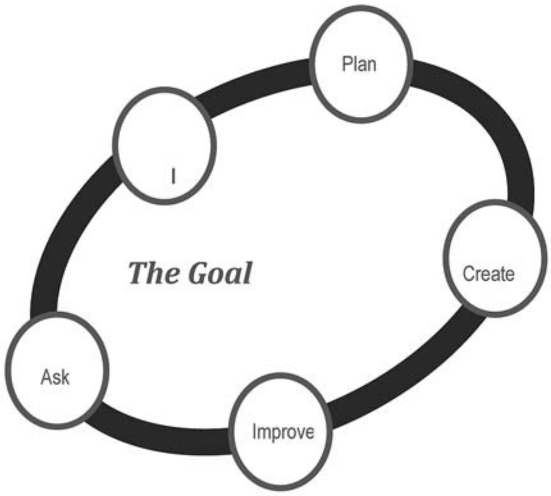

Figure 1. The cyclic processes of modelling and design

Engineering-Based Problem Resources: An Example Exploring Food Packaging

Engineering education in the primary and middle school is a fledging, yet rapidly developing, field of research in the United States, with numerous resources available to teachers and researchers. One of the foremost institutions that are introducing engineering into the primary/middle school mathematics and science curricula is the National Centre for Technological Literacy at the Museum of Science in Boston (Cunningham & Hester, 2007). The Centre's *Engineering is Elementary* program is also being implemented as part of the INSPIRE program at Purdue University (Institute for P-12 Engineering Research and Learning; Diefes-Dux & Duncan, 2007). The Women in Engineering ProActive Network (WEPAN) (www.wepan.org/) has likewise developed sets of rich resources (*Making the Connection*) for introducing hands-on engineering activities to students in year levels 3-12. The goals and activities of these engineering education programs are well suited for integration within primary/middle school mathematics and science curricula and provide fertile ground for interdisciplinary research.

An example of one engineering-based problem from the WEPAN site is "Snack Attack: Food Packaging," the main components of which appear in the appendix (more comprehensive information on the processes of implementation of the problem activities appear on the WEPAN website). This particular problem is targeted at year levels 5-6, but there are several problems addressing this theme for students at other year levels (e.g., for year levels 9 and 10, students redesign and justify the packaging currently used in some consumer products). This hands-on problem activity explores the design process and materials used to package food—students assume the role of an engineer by designing and testing a package for a snack. In doing so, students (a) experience engineering in terms of decisions related to advantages and disadvantages of process and product; (b) identify relevant design features in developing a model to solve a given complex problem; (c) identify materials used to accomplish an engineering problem based on specific properties; (d) rate packaging according to how it performs under test conditions; and (d) consider ways to minimise costs while at the same time produce effective packaging.

An important component of this problem activity is students' sharing of their packaging design model with their peers, who in turn evaluate the models. Significant issues for students to consider here include: (a) the feasibility of the models created and their efficiency (e.g., which ones protect best against heat, water or contaminants?); (b) the amount of packaging used; (c) the cost factors involved; and (d) whether the materials are environmentally friendly.

Follow-up discussions can address the role of mathematical and scientific factors in the decisions companies make on the kinds of packaging materials to use. For example, the size of the packaging, together with the costs and weight of the required materials, bear heavily on decisions made regarding the protection of both the food and the environment. Larger, heavier packaging increases shipping costs. In addition, the more material used, the less environmentally friendly and cost effective the packaging will be.

Charting Research Directions

The theme of this 31st annual MERGA conference is "Navigating Currents and Charting Directions." This paper has highlighted some of the currents we are presently navigating in our efforts to increase participation in key mathematical and scientific domains. As the MERGA website reminds us, "Although we are constantly pushed to account for the quality and impact of our research, we need to assert some control over our work by making our own research futures" (www.addon.edu.au/merga31/welcome.html). Here, I have presented a case for exploring future-oriented interdisciplinary problem solving, one that incorporates engineering-based problems within the primary and middle school mathematics (and science) curriculum. Clearly, substantial research is required to further explore and document the issues that I have raised.

There are several broad avenues of research that need to be investigated here. These include studies that address: (a) the changing nature and role of problem solving in our society; (b) the ways in which problem solving is being addressed in our schools today, including the types of problems presented and the instructional approaches adopted; (c) the ways in which students' problem-solving experiences in mathematics and science can draw upon other disciplines; (d) the developments in primary/middle school students' learning in solving engineering-based problems; (e) the ways in which the nature of engineering and engineering practice can best be made visible to young learners; (f) the types of engineering contexts that are meaningful, engaging, and inspiring for these learners; and (g) the teacher professional development opportunities and supports that are needed to facilitate interdisciplinary problem solving within the curriculum.

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Appendix

Snack Attack: Food Packaging (adapted from WEPAN; www.wepan.org/)

Background

The aim of the problem activity is for students to understand the basic engineering involved in designing food packaging. Packaging engineers have to ensure that food arrives in the best possible condition while using materials that are cost effective as well as environmentally friendly.

Introductory Problem

Snack food has to survive a rigorous journey from its place of manufacture to the point when it is consumed. Discuss how food gets from the manufacturing plant to your home. Brainstorm the different situations food can encounter during transport and the different types of packaging used to protect it. (In addressing this problem, students evaluate the packaging of 3-4 different snack foods such as crackers, chips, and chocolates and document their findings with regard to which materials protect the food from various conditions [e.g., water, breakage, heat]).

Main Problem

A new sweets company wants to package individually wrapped, ready-to-eat snack packs consisting of two squares of cream crackers, a piece of chocolate, and a marshmallow. These packs are often taken on camping trips and are thus subjected to a variety of environmental conditions. After some testing, the condition that the engineers are having most difficulties with is making their packaging WEATHER PROOF. Design a package to protect these snack packs from heat and water. Use the experiment below to help you.

Problem Experiment

Create a package that will keep your snack packs COOL and DRY. The package will need to keep your chocolate and marshmallow from melting during the heat test (45 seconds under a hair dryer on high). Your package will also need to keep your crackers dry when 1 cup of water is poured over it.

RATE YOUR PACKAGE on how well it performs in these two tests, as follows:

Heat Test for Chocolate: Not melted (solid)—10 points; Partly melted—5 points; Completely melted—0 points. *Heat Test for Marshmallow*: No browning—10 points; Partly or completely brown—0 points

Water Test for Cream Crackers: Dry—10 points; Damp—5 points; Wet—0 points

Heat Test for Cream Crackers: Dry—10 points; Damp—5 points; Wet—0 points

The COST OF THE PACKAGING is another concern. Engineers want to design effective packages at the lowest possible cost. You have a budget of $2.00, which means you cannot spend more than this amount on your package but you can spend less. The cost sheet to be completed is below (see Table 1).

Table 1

Supplies to Purchase

Item	Quantity	Amount	Total Cost
Cardboard pieces		40 cents each	
Aluminium foil 15cm square		35 cents each	
Waxed paper 15cm square		20 cents each	
Plastic wrap 15cm long piece		40 cents each	
Small foam plate cut in half		60 cents each	
Toothpicks		5 cents each	
		TOTAL (not more than $2.00)	

Your package will receive a COST RATING from your teacher or the “store manager.” The package that costs the least, which is what you want, will get the most points while the package that costs the most will get the lowest number of points. You will then have a total score for your packaging (the results from your heat and water tests plus the results from your packaging costs).

Validation of an Assessment Instrument Developed for Eliciting Student Prior Learning in Graphing and Data Analysis

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This paper reports on the validation of an assessment instrument used to elicit student prior learning in relation to reasoning about data in an ICT environment. A paper-based assessment instrument developed using a theoretical framework about statistical thinking and reasoning in relation to data analysis, graphing, and graph-sense making was completed by year 4, 5, and 6 students. Student responses were analysed using the framework to determine if the assessment instrument elicited responses for all aspects of the framework. The results identified that one aspect of the framework had not been addressed; informing further development of the theoretical framework and changes to the assessment instrument.

In recent times, it has become apparent that evidence-based research has the potential to influence education policy and has a role to play in transforming educational practices (Slavin, 2002). This has resulted in the call for research to be scientific and rigorous in terms of scientific inquiry (Kalantzis, 2006). The National Research Council (2002) noted that there are a variety of scientific research designs utilised in education research and caution that the design of research does not necessarily make it scientific, noting:

> To be scientific, the design must allow direct, empirical investigation of the research question, account for the context in which the study is carried out, align with a conceptual framework, reflect careful and thorough reasoning, and disclose results to encourage debate in the scientific community. (p.6)

It is, therefore, pertinent at this time to select research methodologies that account for these ideas of scientific inquiry. It is also important to ensure the methodology uses theoretical frameworks to inform the research approach, provides the evidence required to answer the research questions, and determines if the data collection instruments employed provide the evidence required to ensure the results are valid and credible (Shavelson, Phillips, Towne, & Feuer, 2003) .

A review of the literature revealed three distinctive research methodologies that are considered evidence-based research. The first is based on traditional scientific experimental design. The second is based on qualitative research and is termed correlational and descriptive research and the third, design-based research, appears to be a combination of the first two. Of particular interest are the design-based research methods as they can provide a lens for understanding how theoretical claims about teaching and learning can be transformed into effective learning in educational settings (The Design-based Research Collective [DRC], 2003) and are characterised as being "iterative, process focused, interventionist, collaborative, multileveled, utility orientated, and theory driven" (Shavelson et al., 2003, p. 26). They are based on design-analysis-redesign cycles that move toward an understanding of learning and activity or artefact improvement. Importantly, they are theory driven as they test and advance theories through interrogation and repetition of design-analysis-redesign cycles and are particularly suited to the exploration of significant education problems within technology-based learning environments (Seeto & Herrington, 2006).

These principles of design-based research have been incorporated into a project investigating ways in which technology contributes to the development of statistical thinking and reasoning for students when using a data analysis program, *TinkerPlots Dynamic Data Exploration* (Konold & Miller, 2005). The purpose of the research was to explore aspects of the learning environment that contributed to students' development of statistical thinking and reasoning in an ICT environment, from an holistic perspective. Both qualitative and quantitative data were collected. Student interviews, researcher observations and reflections, student work samples, student survey, and video of teaching and learning episodes were used to collect evidence. Analysis of the data involved cluster analysis, and application of descriptive statistics (Miles & Huberman, 1994).

As part of the project, an assessment instrument was developed to evaluate student prior learning (Appendix A, Fitzallen, 2006). It was developed using a theoretical framework, *Model of graphing in an ICT environment* (Fitzallen, 2006), which was developed from theoretical models of statistical thinking and reasoning that were directly related to data analysis, in particular, graphing. It was also influenced heavily by the learning environment afforded by *TinkerPlots*.

This paper reports on the validation process undertaken to (a) put into action the design-based research methodology and (b) determine if the assessment instrument provided the empirical evidence expected.

Method

Participants. The participants for this part of the larger research project were drawn from a grade 4/5 class (n = 21) at a district high school (grades K – 10) and two grade 5/6 classes (n = 50) at a primary school (K-6). The schools were selected as they were in involved in an Australian Research Council Linkage Project, *Providing the Mathematical Foundation for an Innovative Australia within Reform-based Learning Environments* (MARBLE), which was a professional learning program. All students in the three classes completed the paper-based assessment instrument, which took approximately 45 minutes. Of the 71 students, one student from the grade 4/5 class withdrew participation in the research project. The assessment instrument collected from that student was returned to the classroom teacher and subsequently returned to the student.

Data Analysis. As the purpose of this paper is to validate the assessment instrument, quantitative data only will be analysed and reported in this paper. It should be noted that the student responses on the assessment instrument provided rich, descriptive information about their understanding of graphs and graph making. They provided valuable information about the prior learning of the students, which was used to plan the implementation of a learning intervention, designed to explore students' development of statistical thinking and reasoning in a technology environment. Qualitative analysis of the data was undertaken, informed changes to the assessment instrument, and will be reported elsewhere.

Results

It was anticipated that questions on the assessment instrument would address multiple categories of the theoretical framework used to construct the assessment instrument (Fitzallen, 2006). The shaded areas in Figure 1 indicate the connections between the theoretical framework and the assessment instrument items. The unshaded areas indicate when no data collection is anticipated for that key element. The shading is to assist in identifying easily the disparity between the anticipated data collection points and the actual data collection points. As the assessment instrument was a paper-based survey, the key element *Understanding how to use the features of technology* was not addressed in any of the questions. The assessment instrument included items that were similar to the data representations used in *TinkerPlots*.

The student responses were collated to determine a count of how many students provided information on the assessment instrument according to each of the key elements of the categories of the theoretical framework. At this stage, no qualitative analysis was conducted. The results are presented in Figure 1.

Category	Key elements	Item no.						
		1	2	3	4	5	6	7
Generic Knowledge	Speaking the language of graphs	29	17	7	32		63	51
	Recognising the components of data and graphs.	43	25	20	85	87	70	81
	Understanding how to use the features of technology.							
Being creative with data	Reducing data to graphical representations or statistical summaries.		93			4		
	Constructing different forms of graphs.		17					
	Translating verbal statements into graphs.							
Understanding data	Making sense of data and graphs.	5			60	10		51
	Understanding the relationship among tables, graphs, and data.	5					45	18
	Identifying the messages from the data.	1			7			20
	Answering questions about the data.							47
	Recognising appropriate use of different forms of graphs.	10	2	25	75		70	
	Describing data from graphs.	52	43	20				81
Thinking about data	Asking questions about the data.					87		
	Recognising the limitations of the data				15			
	Interpreting data and making causal inferences based on the data				20	2		15
	Looking for possible causes of variation					1		12
	Looking for relationships among variables in the data.					1		20
	No or inappropriate response	16	10	55	15	13	23	19

Figure 1. Percentages of responses provided according to each key element of the theoretical framework.

Discussion

The data presented in Figure 1 show clearly that the assessment instrument provided a great deal of information about students' prior knowledge of graphing. Although the data collected for this report are only a count of student responses for each key element, it was obvious from Figure 1 where there were gaps in the data. Of particular note is that there were no data collected for *Translating verbal statements into graphs* from any of the items on the assessment instrument. Examination of the assessment instrument in Appendix A revealed that there are no items requiring students to perform that task. It is recommended that an assessment item be developed to address this omission.

Another gap in the data was evident for item 4, for the category of *Being creative with data*. The item required students to construct a graph; therefore, it was inappropriate for that category to be highlighted for the item. The item required students to construct a graph based on the data in a table. Seventy percent of the students were able to draw a graph from the data, six students constructing covariation graphs. Most students produced bar graphs, many of which did not reflect the data appropriately.

Item 3 had the highest percentage of *no or inappropriate responses*, with 55 percent of the students stating that they had not used graphing software to create graphs. This item also required students to describe the type of graphs constructed, with only 20 percent of the students able to describe graphs constructed. Items 5 and 7 required students to read information from graphs and make inferences based on the graphs. The students were able to draw information from the graphs but had difficulty making inferences and determining the trend in the data.

Using the theoretical framework to analyse the data from the assessment instrument highlighted two aspects of the framework that required modification. The student responses to item 2 and 4 required students to reduce data to graphical representations; however, none of the items provided the opportunity to reduce data to statistical summaries. Based on this it is recommended that the key element *Reducing data to graphical representations or statistical summaries* be separated into two separate key elements. It also became apparent that the student responses related to *Describing data from graphs* only elicited low level responses and was more appropriately placed in the *Being creative with data* category. Figure 2 reflects these modifications to the original *Model of graphing in an ICT environment* (Fitzallen, 2006).

<table>
<tr><th colspan="2">Categories</th><th>Key Elements</th></tr>
<tr><td>Generic Knowlege</td><td rowspan="2">Being creative with data</td><td rowspan="2">Reducing data to graphical representations.
Summarising data.
Constructing different forms of graphs.
Translating verbal statements into graphs.
Describing data from graphs.</td></tr>
<tr><td rowspan="3">Speaking the language of graphs and data.
Recognising the components of data and graphs.
Understanding how to use the features of software and technology environments.</td></tr>
<tr><td>Understanding data</td><td>Making sense of data and graphs.
Understanding the relationship among tables, graphs, and data.
Identifying the messages from the data.
Answering questions about the data.
Recognising appropriate use of different forms of graphs.</td></tr>
<tr><td>Thinking about data</td><td>Asking questions about the data.
Recognising the limitations of the data.
Interpreting data and making causal inferences based on the data.
Looking for possible causes of variation.
Looking for relationships among variables in the data.</td></tr>
</table>

Figure 2. Adapted model of graphing in an ICT environment.

Conclusion

The changes proposed to the theoretical framework and the assessment instrument suggested in this paper were determined as part of the design-analysis-redesign cycle of the research project. Further analysis of the qualitative data from the student responses on the assessment instrument will be used to reiterate the design-analysis-redesign process. This may inform further development of the theoretical framework and suggest other changes to the assessment instrument, potentially strengthening the validation of the assessment instrument. In addition, the potential of applying the Rasch model (Bond & Fox, 2007) to the quantitative data to increase internal validity of the research project will be explored.

The results from the research reported in this paper provided empirical evidence that was used to plan a teaching and learning intervention to develop students' understanding of covariation using *TinkerPlots*. It was determined that the students had limited experience using graphing software to construct graphs; were experienced at reading individual data points; were less experienced at determining the trend in graphs or the messages in data; were inexperienced at describing graphs and the purpose of graphs; and previous experiences of constructing graphs were predominantly focused on bar graphs, with little experience using a variety of graph types.

The assessment instrument was designed so that the key elements were addressed on multiple occasions across the range of items in the instrument. On the whole, this was the case. It is, however, important to evaluate the responses to determine the range of level of responses to get an accurate picture of the statistical thinking and reasoning of the students in the study.

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Appendix A: *Assessment instrument items* (Reproduced from Fitzallen, 2006, pp. 207-208)

Item no. **Item description**

1 What is a graph? What are graphs used for? Where have you seen graphs used?

2 Draw an example of a graph. Any type will do. Put as much detail on the graph as possible. What does the graph show?

3 Have you used the computer to draw graphs before? What programs did you use? Describe what sort of graphs you drew. What were the graphs used to show?

4

A science class was studying temperature. They used a thermometer to measure the room temperature every 5 minutes for 30 minutes.

First they turned a heater on for 15 minutes.
Next they turned the heater off for 10 minutes.
Lastly they opened the window for 5 minutes.

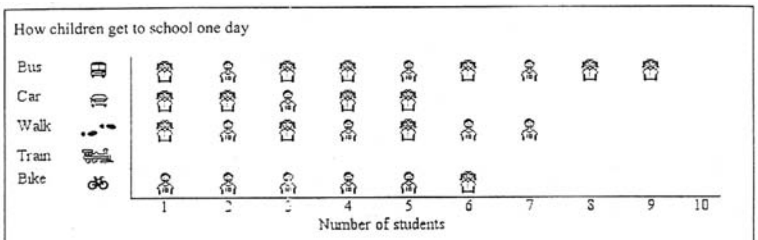
They wrote down these numbers.

Time (Minutes)	5	10	15	20	25	30
Temperature (°C)	15	20	25	25	25	15

Draw a graph to show how the temperature changed over time.

Moritz (2003b, p. 234).

5



a) How many children walk to school?
b) How many more children come by bus than by car?
c) Would the graph look the same everyday? Why or why not?
d) A new student came to school by car. Is the new student a boy or a girl? How do you know?
e) What does the row with the Train tell about how the children get to school?
f) Tom is not at school today. How do you think he will get to school tomorrow? Why?

Watson & Kelly (2003, p. 722).

6 The information about individual students is on separate cards, like the ones below. What questions about the group of students could be answered by using the information on the cards?

Student	1	**Student**	2	**Student**	3
Gender	Male	**Gender**	Male	**Gender**	Female
Height	1.3m	**Height**	1.3m	**Height**	1.2m
Eye Colour	Blue	**Eye Colour**	Brown	**Eye Colour**	Brown
Hair Colour	Blonde	**Hair Colour**	Brown	**Hair Colour**	Brown

Adapted from Chick & Watson (2001).

7

Some students were doing a project on noise.

They visited 6 different classrooms. They measured the level of noise in the class with a sound meter. They counted the number of people in the class.

They used the numbers to draw this graph.

Q1. Pretend you are talking to someone who cannot see the graph. Write a sentence to tell them what the graph shows. "The graph shows...

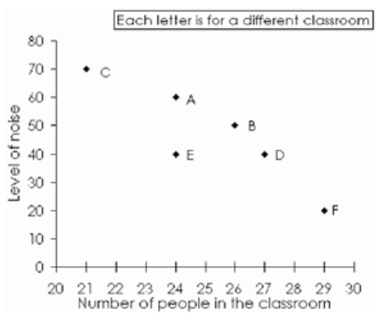
Q2. How many people are in Class D?

Q3. If the students went to another class with 23 people, how much noise do you think they would measure?

Please explain your answer.

Q4. Jill said, "The graph shows that classrooms with more people make less noise". Do you think the graph is a good reason to say this?

❑ YES or ❑ NO Please explain your answer.



Moritz (2003, p. 525).

Using Valsiner

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Students develop within rich, complex cultural community settings involving teachers and the environmental surrounds. To investigate the multiple perspectives in such a teaching and learning setting a suitable framework incorporating sociocultural practices is needed. The developmental works of Valsiner and associated double stimulation methods of Vygotsky, are proposed here which, it will be argued, assists in the identification and analysis of developmental issues. The application of the theoretical framework presented in this paper is exemplified in the development of numeracy in a 1st year nursing

Introduction

In recent years, Valsiner's zone theory has been used in a number of contexts in mathematics education (Galbraith & Goos, 2003; Goos, 2005; Warren, Cooper & Lamb, 2006; Blanton, Westbrook & Carter, 2005). However the use to which the theory has been put has been relatively narrow, often in relation to teacher development or teacher practice (Brown, 2005). In this paper it will be argued that Valsiner offers much more, and that his human development theory could be applied in wider contexts (Galligan, 2005) including the development of mathematical understanding in children and adults. In this paper specific reference will be made to the development of adult mathematical skill development in university and how Valsiner's theory can be applied in this context.

The Theory

Valsiner first wrote about his theory of human development, incorporating zone theory in 1987 and then expanded his theory in a second edition of his book (Valsiner, 1997). Since then he has extended the notions that underpin this theory, particularly in the methodological considerations (Valsiner, 2000; 2006); theory of dialogic self (Valsiner, 2005) and personal cultures (Valsiner, 2007). Valsiner himself said in an interview in 1998 that he "didn't want to make these fragile concepts [i.e. zone theories] into an orthodoxy for myself" (Mey & Mruck, 1998, p. 17), and in a personal communication (Valsiner, 2008, personal communication, 21 March) he seemed hesitant to use the ideas further as in reality these terms seemed deceptively superfluous. However, these original notions do permeate his later work. This paper will first explain the human development and zone theory from his 1997 work; summarise some of his later work, then argue that these notions, far from being 'fragile' are important concepts for investigating human development, particularly in educational settings.

Valsiner (1997) theoretical system conceptualises three aspects of human development:

- Organisation of person/environment relations in context of everyday actions (interpersonal);
- Relation between actions and reflection on actions in the process (intrapersonal);
- Experiences transfer to the general life-course development.

Valsiner's theory is based on a concept of development as a "change in an organisational system in time which is maintained (rather than lost) once the condition of its emergence disappear" (Valsiner, 1997, p.3). Valsiner argues that:

A person involved in mastering a skill is no longer lacking that skill, nor is the skill present in its fully-fledged form. The skill is coming into existence. The phenomenon here is quasi structured. Rudiments of the skill can be detected in the flow of conduct, yet nobody can say for sure that the skill as such already exists. (2000, p.105)

He focuses on microgenetic studies (i.e. the immediate processes of emerging new phenomena) as he is interested in the whole set of possibilities that may or may not actualize (not just the actual emergent possibilities). In the 1997 edition of his book he developed this in set notation as:

$P = \{a, b, c, d, e \,\&\, f \,\&\, g, h \,\&\, b, ?\}$ - (Valsiner, 1997, p.177)

Here P is a set of possibilities. The person may actually go from *b* to *h&b* and then to *a* which is finally actualised. The failed set of possibilities, the tried ones as well as the successful ones are all informative, not just the successful ones (as would be the case in a competence-performance study). So now at the time (time = t +1) that has been actualised we have another set of possibilities:

$P(t+1)/(\text{given } a(t)) = \{b, c, d, e \,\&\, f \,\&\, g, h \,\&\, b, i, j, \&?\}$ - (Valsiner, 1997, p.178)

Upon this model, which at the moment is just a set of possibilities at different times, Valsiner constructed a canalization process of development through the concept of zones.

Valsiner's theoretical framework includes three Zones: the Zone of Proximal Development (ZPD) from Vygotsky (1978), the Zone of Free Movement (ZFM), originating from Lewin (1933; 1939, in Valsiner, 1997), and the Zone of Promoted Action (ZPA).

Valsiner's three zones constitute an interdependent system between the constraints put on the environment of the learner and the actions being promoted for the learner. Both the constraints and the promotions are usually imposed by others so that:

> the developing child is conceptualised in the context of his relationships with the culturally and physically structured environment, where the child's actions upon that environment are guided by assistance from other human beings - parents, siblings, peers, teachers etc. The particular physical structure of the environment of a human child is set up by the activities of other human beings, and modified by them over time. (Valsiner, 1997, p. 76)

It is tempting to conceptualise Valsiner's theory by visualising the three zones as intersecting (as in Venn diagrams). This is what Goos, Warren et al. and Blanton et al. have done. Goos (2005), for example has used the ZPA/ZFM/ZPD as if these three zones exist at the same point in time. However Valsiner specifically set up the ZPD as an emerging next set of possible actions from the ZFM/ZPA interaction. These ZPD cannot be predicted so could be inside, outside or intersecting with the current ZFM/ZPA interface. Despite this difference between Valsiner's approach and the one taken by Goos, the model developed by Goos is helpful in school environments to help explain, for example, why development in one school has not occurred while in another, with different ZPA/ZFM interactions, development can be identified.

However, I believe Venn diagrams simplify the theory. Valsiner didn't use a Venn diagram model for his theory. In particular, in 1997 he used set notation to show the relationship between the three zones. In his later work, he also tentatively uses topology to help model parts of his theory (Valsiner, 2005; Diriwachter & Valsiner, 2006). This next section will detail the three zones and use the set notation to model the theory.

Zone of Free Movement

While acknowledging students' freedom of action and thought, the Zone of Free Movement represents a cognitive structure of the relationship between the person and the environment, seen in terms of constraints that limit the freedom of these actions and thoughts. This environment is socially constructed by others (teachers, administrators, the curriculum writers) and the cultural meaning system they bring to the environment, but the ZFM's themselves can either be set up by these 'others', the students themselves or through joint action, but are ultimately internalized. Thus ZFM structures access to areas and objects such as technology, time, curriculum, and class rules as well as the teachers' and students' expertise, experience, beliefs and values. By the time a student enters university, the adult student has in place a set of constraints that cannot be discarded, but have experienced other constraints that, given the right circumstances, are replaceable. For example a nursing student may have been disengaged with maths at school but in the context of drug calculations, sees arithmetic application and may become excited about learning the division algorithm.

Valsiner suggested zones such as:

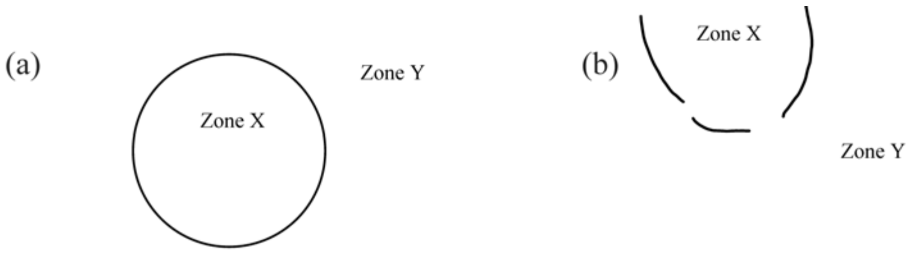

Figure 1. Two explications of the zone concept (Valsiner 1997, p. 187).

Scenario (a) is made up of two zones X and Y but this situation is rare. In reality there are often fuzzy, impermeable, or undefined boundaries (like scenario b). Hence the move from Venn diagrams to set notation to clarify the elements. At a given time (t_1) a ZFM can be identified by:

$ZFM(t_1) = \{a^*, b^*, c, d, e, f, ?, ??, g^*, h^*\}$ - (1: Valsiner 1997, p.190)

Here *c, d, e, f* are interior areas and *a*, b*, g** and *h** are areas in contact with the boundary. The ? and ?? are areas that are unknown to the students or the teachers. They may exist but they haven't been detected. Valsiner refers to this as a "reserve area of possible experiences" (p.192) if the ZFM remains static for a period of time. This notation thus allows for fuzziness not available in a Venn diagram.

In an extension:

$ZFM(t_2) = \{a^*, b^*, c, d, e, f, m*, ??, g^*, h, j^*\}$ - (2: Valsiner 1997, p. 191)

Notice the *h* is no longer on the boundary and a new element *j** is now seen on the boundary. The ? is now replaced by an *m** as it has actually been detected. (e.g., having a greater understanding of division (perhaps in context) and then seeing long division maybe useful or just being interested to learn).

Zone of Promoted Action

While the ZFM suggests which teaching or student actions are possible, the Zone of Promoted Action (ZPA) represents the efforts of a teacher, or others to promote particular skills or approaches. For example, a nursing department promotes students to go to numeracy classes. However the ZPA is not binding; thus students may not wish to actively participate in this course. The ZPA should also be in a student's ZPD (see next section), so having very poor mathematics skills in a class which assumes basic mathematics skills, may result in students' inability to participate or learn. On the other hand those students who believe (and may have) the skills already, may not participate. This is what Valsiner calls an "illusionary construction" (1997, p.193). This concept was developed further by Blanton et al. (2005) as the Illusionary Zone (IZ).

In terms of set theory we first look at the ZPA in terms of the ZFM. Taking the example (2) ZFM/ZPA at time (t_x):

$\mathrm{ZFM/ZPA}(t_x) = \{a^*, \underline{b^*}, c, d, \underline{e}, f, ?, ??, g^*, h, \underline{j^*}\}$ - (3 Valsiner 1997, p. 192)

To complete the microgenetic examination, the model is furthered by, for example, promoted action outside the zone (say *k*), and an action/reflection stage. Thus the model, perhaps now incorporating *k,* can become:

$\mathrm{ZFM/ZPA}(t_{x+1}) = \{a^*, B\underline{b}^*, c, d, B\underline{e}, f, ?, B??, Bg^*, h^*, Bj*, k\}$

Here at Bb* the action is promoted and there is sign mediated reflection but this reflection is not promoted. Thus with each action there are both promotions of actions (or not) and reflections on actions (or not), depicted in Figure 2.

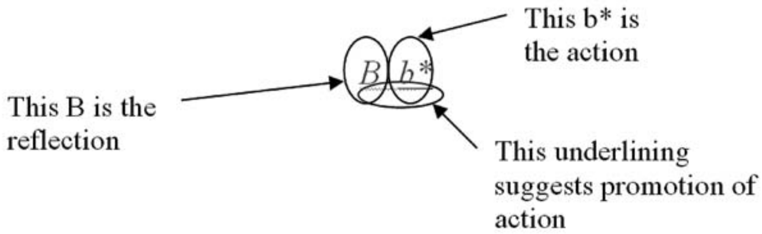

Figure 2. Model of an action, reflection and promotion of action.

In later articles, Valsiner expands on the notion of semiotically mediated reflection (Valsiner, 2005)

Zone of Proximal Development

The third Zone concept Valsiner uses is the Zone of Proximal Development (ZPD), borrowed from Vygotsky. For Vygotsky it was the difference between a learner's "actual development level as determined by independent problem-solving" and the level of "potential development as determined through problem-solving under adult guidance or in collaboration with more capable peers" (1978, p. 86). So if teachers or curricula promote tasks (e.g. numerate practices) that are too far away from the student's present capabilities (both too high and too low), then the tasks are unlikely to be assimilated. However Valsiner's concept is more narrowly defined as the "set of possible next states of the developing system relationship within the environment" given the current state of the ZFM/ZPA interface and thus the ZPD becomes subservient to the other two concepts (Valsiner, 1997, p. 200).

When these three zones merge, what emerges is, not so much a Venn diagram that is promoted by Goos (2005), Blanton et al. (2005) and Warren et al. (2006) suggesting interdependent circles, but a "family of possible novel forms of change" where the ZPD is dependent on the ZFM/ZPA complex. It can be depicted as:

ZFM/ZPA(t) ZPD(t+1)

$$\{a^*\} \rightarrow \quad \{a, a^*, Aa, Aa^*, A, ?\}$$

$$\{B\underline{b^*}\} \rightarrow \quad \{b, b^*, Bb^*, Bb, B, ?\}$$

(Valsiner 1997, p.203).

Here there is a set of ZPD's that represents a further mapping of a time extended kind. It consists of the options for each ZFM/ZPA. In the ZFM/ZPA of $\{a\}$, There could be no change {a}; a change in boundary status $\{a^*\}$; semiotic mediation on these $\{Aa$ or $Aa^*\}$; or further open ended options $\{?\}$. The actual development emerges from the negotiation process within this set of possibilities. With this set of microgenetic contexts, over time a more stable pattern with a more ontogenetic flavour may emerge.

In this model Valsiner examined semiotic mediation as being more complex than simply reflection on action. It also becomes a set of possible new senses. While the initial ZFM/ZPA complex describes the primary field of action, there is also a secondary "personal sense"; the connection between the primary and the secondary, and also a metafield emerging from the interacting fields. This concept of mediational fields appears more strongly in Valsiner's later work on semiotic mediation and dialogic self in 2001 and 2005 (Valsiner, 2001; 2005; Lawrence & Valsiner, 2005). In 2001 he describes a developing semiotic regulatory system:

> A person feeling something (but it is not yet clear to oneself what that something is)…then at some instant, the person realised "I am angry" (i.e. creating a sign…to reflect upon the feeling process). (p. 93)

In 2005 he brings in the concept of scaffolding into the theory of dialogic self (i.e. "construction of meaningfulness of the self in relation with others" (Valsiner, 2005, p. 199). Here Valsiner uses scaffolding as a semiotic mechanism that allow people to regulate, construct and reconstruct their "I-positions".

The investigation of these human developing systems within a framework of the zone theory and dialogic self theory may have importance for research into the ways adults learn mathematics. If these theories are placed within the context of adults developing mathematical concepts that may have been troublesome in the past, then new ways of approaching adult learning may be possible. Dialogic self is focussed an adult development (Valsiner, 2001) and zone theory links the environment (including past feelings, and context, self-efficacy etc) and promotions within that environment (for example universities needing mature aged students to learn academic skills; or adult self-promotions of learning). Finding out how adults re/learn these concepts is important in many areas of both university and everyday life. This learning occurs first at the microgenetic level. How is the microgenetic captured? What methodology fits this theory?

On Methodology

In mathematics education, where a researcher aims to change or improve students' understanding of concepts, action research or design experiment is often used (Kelly & Lesch, 2000; Collins, Joseph, & Bielaczye, 2004). However there have been a number of criticisms of this approach (Dede, 2004; Engsetröm, 2007). Engsetröm rejects design experiment, particularly as it emphasises completeness and ignores the "contested terrains that are full of resistance, reinterpretation, and surprise from the actors in the design experiment" (p. 368). Instead Engsetröm advocates the use of Vygotsky's double stimulation as a basis for interventions as it aims to elicit "new, expansive forms of agency in subjects … making subjects the masters of their own lives" (p. 363). Valsiner also believes the method of double stimulation in a developmental quasi-natural experimental setting is useful when researching human development. In double stimulation experiments, subjects are provided with a richly structured environment which can be restructured in a goal oriented way. Figure 3 shows the general structure of the method concentrating on the unfolding of the intermediate forms, both the ones that eventually turn into final forms and those that don't. Microgenesis at this level may be limited to developmental transformations in very short periods of time, even measured in microseconds (Diriwachter & Valsiner, 2006).

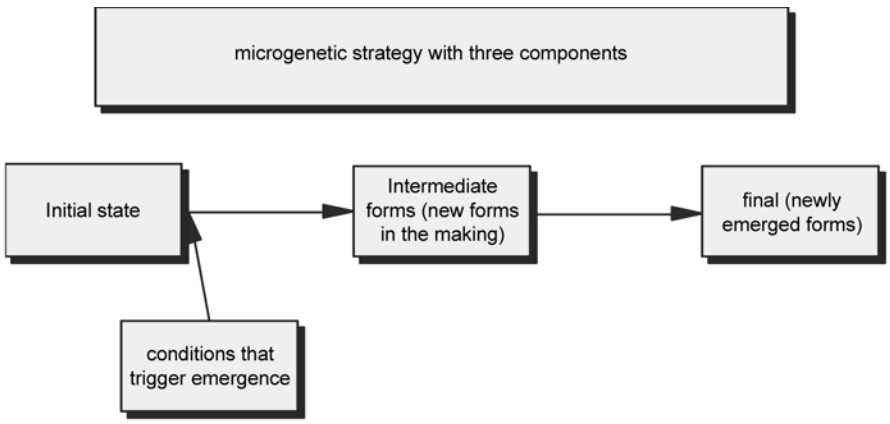

Figure 3. General structure of the method: Development studied as it unfolds (Valsiner, 2000, p 78).

Valsiner depicted Vygotsky's approach diagrammatically in Figure 4. In his original work, Valsiner exemplified this double stimulation with the development of young children learning. However this method can be used in many contexts. Engeström, for example, uses the method in the context of post office workers in the redesign of delivery work of mail carriers (Engeström, 2007). Here it will be exemplified with adult nursing students learning mathematics. The researcher asks the nurse to solve a drug problem (Researcher STIMULUS-OBJECT). The student aims to answer the problem (Actual STIMULUS-OBJECT). When the student selects the task the researcher suggested, the task is reconstructed, even if the STIMULUS-OBJECT seems to be the same. In the room there may be books, pens, formulae, calculator, perhaps even models (environment). She/he may see the formula (stimulus X). She decides to use the formula to solve the problem (STIMULUS-MEANS 1). The nursing student thinks about how to solve the problem, as well other thoughts may be involved, fear anxiety, self-efficacy etc (STIMULUS-MEANS 2). It is this "moment of human interpretation" (Valsiner, 2000, p.79) i.e. that moment of present-to-the-future development, that Valsiner highlights from Vygotsky's work as important for investigation. In later articles on the *Social Mind* (2000) and *Dialogic Self*

(Valsiner, 2005), Valsiner appears to focus on the growing importance of this reflective social nature of the phenomena (i.e. the STIMULUS-MEANS 2). This reflection is particularly important in the study of adults. While the traditional notion of ZPD links teachers or others to scaffolding student learning, Valsiner develops the idea of self-scaffolding where

> [t]he developing person constantly acts above his or her actual-already mastered-developmental competencies and through such constant probing into the domain of incompetencies-expands the competence. (Valsiner, 2005, p.203)

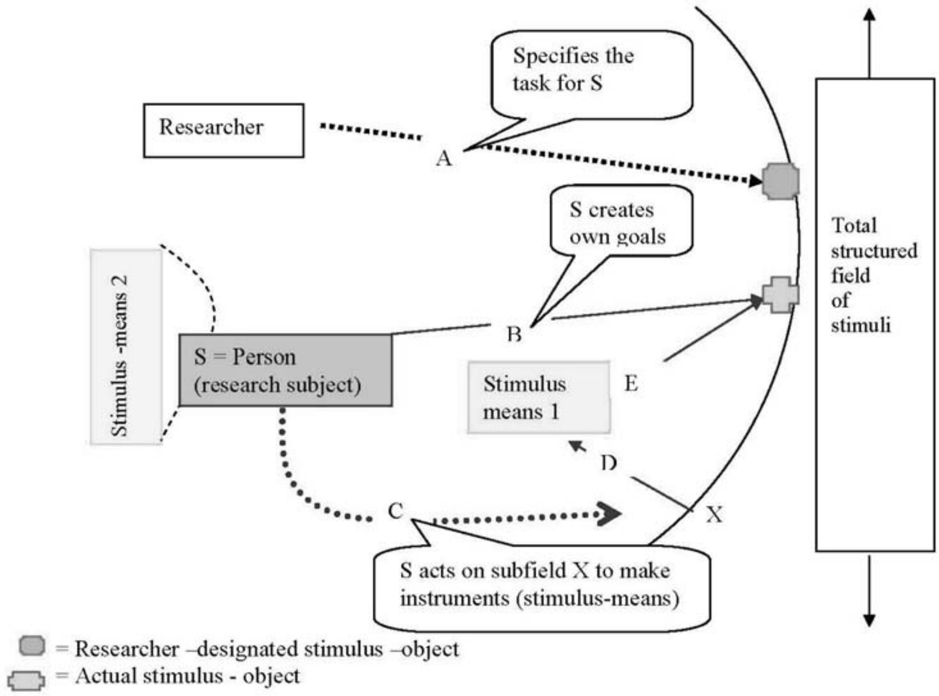

Figure 4. A schematic depiction of Vygotsky's methods of double stimulation (from Valsiner 2000, p. 79).

In researching adults learning mathematics, a series of double stimulations involving a richly structured environment, and means for self-scaffolding and may provide a "constant flow of microgenetic episodes" (Valsiner, 2005, p.204) which may lead to new understandings of how ontogeny emerges for adults re/learning mathematics.

Discussion

Throughout Valsiner's work, his passion for a "second psychology" (Cole & Valsiner, 2005, p. 293) can be felt. He advocates to move away from statistically oriented sciences that treats humans as acultural beings, that measures subjects "as they are" rather than "as they become" (Diriwachter & Valsiner, 2006, para 8). Rather, he proposes methodologies that centre on careful observation in microsettings, where development may be observed. His approach to the total environment through ZFM; to scaffolding through the ZPA and later self scaffolding in his work on dialogic self; and his take on the role of the ZPD as a set of possible future actions that result from the ZFM/ZPA interaction (both successful and not), all fit, I believe in the methodology of double stimulation. The challenge is to move the theoretical approach into an experimental setting to see how well it can explain human development.

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CAS Enabled Devices as Provocative Agents in the Process of Mathematical Modelling

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This paper considers the potential of Computer Algebra Systems (CAS) to enhance the processes associated with mathematical modelling and application tasks. In doing so, the role of technology in the cyclical development of mathematical models is considered in relation to current literature in this area. The analysis of data drawn from a one year study of three different secondary school classrooms indicates that CAS enabled technologies have a role to play as *provocateurs* of productive student-student-teacher interaction in both small group and whole class settings.

Background

This paper investigates the nature of the potential partnership between two powerful thinking tools – mathematical modelling and computer algebra systems (CAS) – in teaching and learning mathematics in the senior secondary school. Mathematical modelling – formulating a mathematical representation of a real world situation, using mathematics to derive results, and interpreting the results in terms of the given situation – is a significant element of the senior mathematics syllabuses in Queensland, Australia and appears, as applications of mathematics, in the curriculum documents of most other Australian states. As CAS enabled technologies are developing increasing acceptance in mainstream mathematics instruction there is need to explore and understand the synergies that might be developed between these technologies and current curriculum objectives. This paper reports on initial findings of a project designed to explore the potential of CAS enabled technologies to enhance mathematical modelling activities.

CAS and School Mathematics

While there is significant research centred around solving contextualized problems through the use of the multiple representational facilities offered by mathematically enabled technologies (e.g., Doerr & Zangor, 2000; Huntley, Rasmussen, Villarubi, Santong & Fey, 2000; Yerushalmy, 2000) and substantive argument to support the use of CAS to enhance the process of mathematical modelling (Kissane, 1999, 2001; Thomas, 2001), typically, the potential of modelling and CAS have been considered separately in mathematics education research. Further confounding this issue, there is still "very little is known about the issues which arise when teachers use CAS in their classrooms" (Thomas, Monaghan, & Pierce, 2004).

At present applications of CAS in Australian secondary school classrooms are largely limited to developing students' understanding of abstract mathematical concepts and this appears to be reflected in research designs in this area. For example, a recent review of Australian research into calculators and computer algebra systems between 2000 and 2003 (Forster, Flynn, Frid, & Sparrow, 2004) did not include a single study which focused on the role of CAS enabled technologies in mathematical modelling or the solution of contextualized problems.

There is, however, an emerging literature directed towards mathematical modelling within the context of CAS active learning environments. Thomas (2001), for example, claims CAS capable digital devices are uniquely able to support the process of mathematical modelling by enabling exploration, representation and analysis of authentic data in ways that cannot be achieved with pencil and paper, or with standard graphics calculator technology. Thus, CAS has the potential to provide access to more sophisticated life-related problems. However, because research has usually lagged behind implementation of CAS active curricula (Zbiek, 2003), there remains much for researchers and teachers to do before we understand how to best use CAS to enhance students' capabilities in the area of mathematical modelling.

The Role of Technology in the Process of Mathematical Modelling

Based on a three year longitudinal case study of a class of students studying mathematics in technologically rich environment Galbraith, Renshaw, Goos, and Geiger (2003) provide a description of the role of technology in the process of working with applications of mathematics and mathematical modelling.

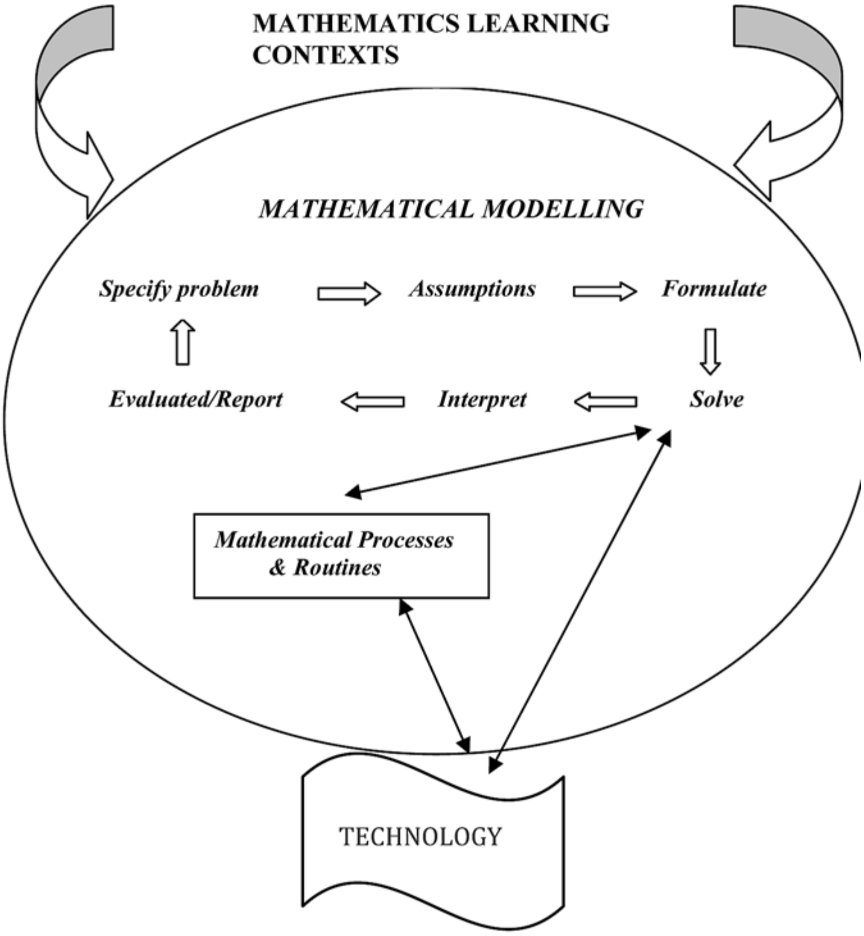

Figure 1. Some technological and mathematical interrelationships from Galbraith, Renshaw, Goos, & Geiger (2003, p. 114)

In this description, illustrated in Figure 1 above, mathematical routines and processes, students and technology are engaged in partnership during the *Solve* phase of a problem, which follows from the abstraction of a problem from its contextualised state into a mathematical model. This view identifies the conceptualization of a mathematical model as an exclusively human activity while the act of finding a solution to the abstracted model can be enhanced via the incorporation of technology. Thus technology is seen as a tool used to interact with mathematical ideas only after a mathematical model is developed rather than as a tool for exploration and development of a model or its validation as a reliable representation of a life related situation.

Context of the Study

The data described below is sourced from a 12 month study of the use of CAS enabled technologies in authentic senior secondary classroom settings. Three cohorts of students (1 Year 12 group and 2 Year 11 groups) from different schools were observed on three different occasions each for periods of time ranging from 45 minutes through to 90 minutes. All classes were studying Mathematics B, a subject that includes substantial elements of calculus and statistics. The schools included one government school and two non-government colleges. Data collection instruments included observational field notes, video and audio recording of small groups of students working on specified tasks, and video and audio recording of episodes of whole class activity. In addition, individual student and teacher interviews were conducted after each class session in order to ascertain their perceptions of the benefits offered by CAS enabled technologies to learning mathematics in general and specifically to working on mathematical modelling tasks.

On the majority of occasions, students and teachers worked on tasks that incorporated some element of mathematical modelling. While the three teachers' experience with the use of the Nspire handhelds, and CAS in general, varied from novice to expert, all were experienced users of other technologies (e.g., graphing calculators, spreadsheets, statistical packages) as tools to teach mathematics. Students' experience in the use of technology to learn mathematics also varied across the three classes with one group having very limited exposure and the other two groups with extensive previous use. None of the students had used the Nspire handhelds before the beginning of the year in which the study was situated.

Each class was equipped with a set of Texas Instruments Nspire handheld devices (at least one for each student and the teacher) and one licence for software that mirrored the facilities of the Nspire handheld device. These technologies possessed all of the features of a typical graphing calculator, such as function and graph plotting modules, but also included a CAS capability that is highly integrated with other calculator facilities. Other features included a fully functional spreadsheet (again with CAS integrated capability) and a well developed feedback mechanism for reporting on input errors. The data analysed in the two vignettes following are based on observational field notes of whole class activity and audio and video recordings of students and teachers working together in both small group and whole class settings.

Two Vignettes Where CAS Promotes Contention

The observation related to technology use and mathematical modelling reported above stands in contrast to classroom observations in a current study which aims to investigate the role of CAS based technologies in enhancing the process of modelling in upper secondary mathematics programs. The two vignettes reported below come from classrooms in two different schools – one a government school and the other a private college. In both cases the teachers challenged students to make use of CAS based technology, Texas Instrument's Nspire handheld devices, as an aid to working with problems that required skills related to some aspect of mathematical modelling.

Vignette 1

The vignette described below took place in a Year 12 (final year of secondary school) mathematics classroom where students were investigating the nature of population decay towards extinction. These students were using TI-Nspire CAS handheld technology which had been introduced to them only a few weeks earlier. During one lesson observation the teacher set the students the following question:

When will a population of 50 000 bacteria become extinct if the decay rate is 4% per day?

One pair of students developed an initial exponential model for the population y at any time x and equated this to zero - - in the belief that the solution to this equation would give them the number of periods, and hence the time, it would take for the population to become extinct. When they entered this equation into their handhelds, however, the device unexpectedly responded with a *false* message, as illustrated in Figure 2 below.

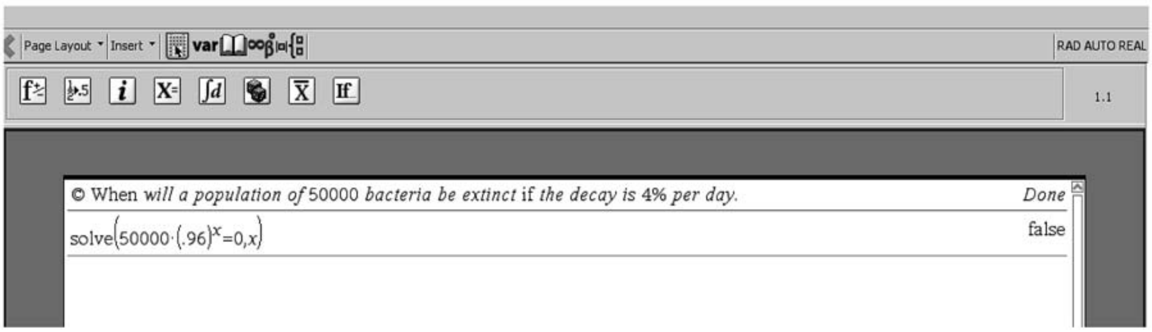

Figure 2. Nspire display for the problem $y = 50000 \times (0.96)^x$

The students were initially concerned that this response had been generated because they had made a mistake with the syntax of their command. They re-entered the instruction several times and tried a number of variations to the structure of the command but did not consider that there was anything at fault with the parameters they had entered. When the students asked their teacher for assistance, he looked at the display and stated that there was nothing wrong with the technical side of what they had done but they should think harder about their assumptions.

After further consideration, and no progress, the teacher directed the problem to the whole class. One student indicated that the difficulty being experienced was because "you can't have an exponential equal to zero". This resulted in a whole class discussion of the original pair of students' assumption that extinction meant a population of zero. The discussion identified the difficulty as equating an exponential model to zero and then considered the possible alternatives. Eventually the class adapted the original assumption to accommodate the limitations of the abstracted model by accepting the position that extinction was "any number less than one". Students then made this adjustment to their entries on the handheld and a satisfactory result was returned.

Vignette 2

In the second classroom students were asked to work on the following task.

The CSIRO has been monitoring the rate at which Carbon Dioxide is produced in a section of the Darling River. Over a 20 day period they recoded the rate of CO2 production in the river. The averages of these measurements appear in the table below.

The CO2 concentration [CO2] of the water is of concern because an excessive difference between the [CO2] at night and the [CO2] used during the day through photosynthesis can result in algal blooms which then results in oxygen deprivation and death of the resulting animal population and sunlight deprivation leading to death of the plant life and the subsequent death of that section of the river.

From experience it is known that a difference of greater than 5% between the [CO2] of a water sample at night and the [CO2] during the day can signal an algal bloom is imminent.

Table 1

Rate of CO_2 Production versus Time

Time in Hours	Rate of CO_2 Production
0	0
1	−0.042
2	−0.044
3	−0.041
4	−0.039
5	−0.038
6	−0.035
7	−0.03
8	−0.026
9	−0.023
10	−0.02
11	−0.008
12	0
13	0.054
14	0.045
15	0.04
16	0.035
17	0.03
18	0.027
19	0.023
20	0.02
21	0.015
22	0.012
23	0.005
24	0

Is there cause for concern by the CSIRO researchers?
Identify any assumptions and the limitations of your mathematical model.

Students were expected to build a mathematical model, initially by inspecting a scatterplot of the data which in turn helped them determine the general form of function that would best fit the data. This general form was then to be adapted to fit the specific data presented in the question and the resulting equation used to answer the questions at the end of the task. Students had studied strategies for determining if a particular function type was most suited to a data set. Most recently, students were introduced to a technique where ln versus ln plots of data sets was used to determine if a power function was an appropriate basis on which to build a mathematical model. This appears to have influenced the actions of two students as the transcript below indicates.

Researcher: So you are up to now where you are building the model are you?

Student 1: Well we worked out a plan of what we are going to do, we are just putting it on paper.

Researcher: So do you want to tell me what the plan is?

Student 1: The plan is to do the LOG LOG plot of both the data see if they are modeled by a power function. We have previously seen that the …

Researcher: So that is something you have learnt to do over time? When ever you see data look like that you check to see if it's a power function of Log/Log and if it is what was the graph look like of a Log/Log?

These students experienced problems with this approach, however, as the technique involves, in this case, finding the natural logarithm of 0.

Student 1: 0.44 zero… (entering information into calculator). Don't tell me I have done something wrong. Dammit. Mummbles … Start at zero is it possible to do a power aggression? I don't think so!

This comment was in response to the display which resulted when the students attempted to find the natural logarithm of both *Time* and *CO2 output* data using the spreadsheet facility of their handheld device (Figure 3 below). Students were surprised by the outputs they received for both sets of calculations, that is, the #UNDEF against the 0 entry in the *Time* column and the lack of any entries in the *CO2* column. In addition, an error message was produced indicating results where not available in the current settings of the handheld device.

After a little more thought students realised where the problem lay.

Student 1: L N time is going to be equal to the L N of actually time. Tim … oh is that undefined cause it's zero?

Student 2: Yep.

Student 1: Right now if I go back to my graph … Enter

Student 2: If you try zero fit, it will just go crazy.

Students eventually identified the problem with their approach and realised their initial assumption, that is, the best model for whole data set as power function, was at fault. Eventually, they realise it was best to model the data with two separate functions.

Student 1: So we have fitted a linear model for the top data and then we fitted a power function to the bottom data given we take the absolute value of those the question asks, the difference greater than 5% we need to look at the actual CO2 produced, now what we have got is the rate, to go back to the actual CO2 absorbed we need to integrate the model or both models and then use the percentage difference formula – predicted minus actual divided by actual or in this case night minus day divide by day x 100 to look at whether for any x or any t there is any percentage difference greater than .05.

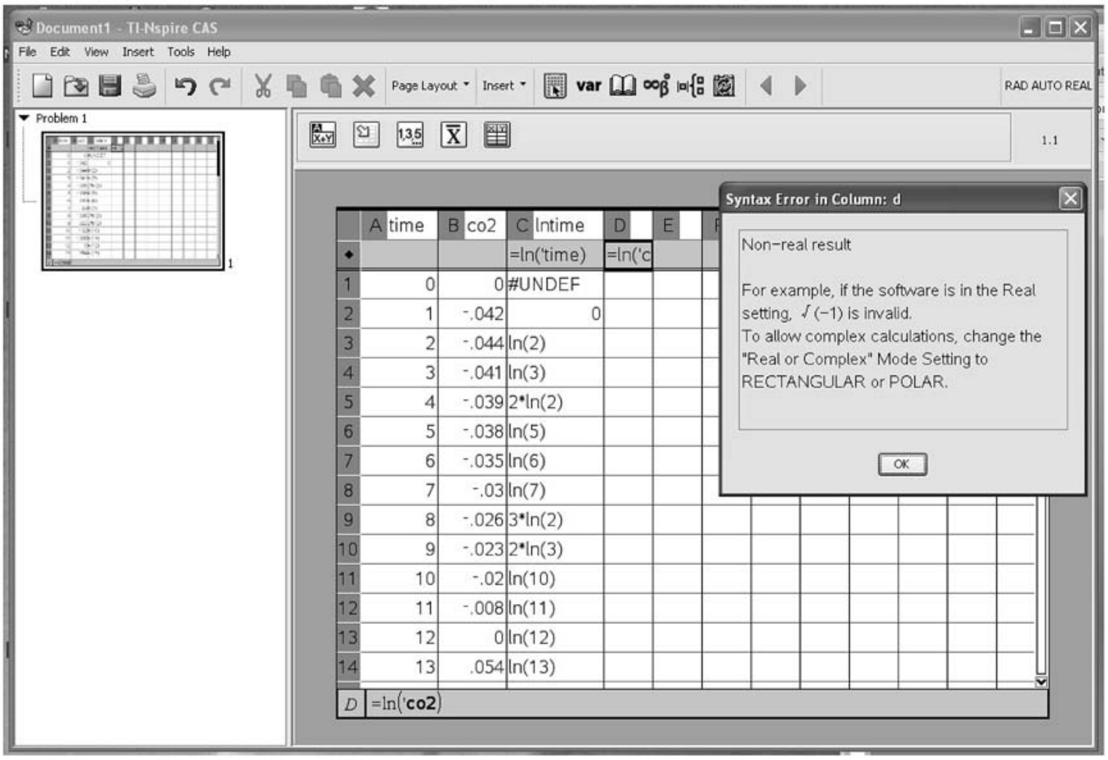

Figure 3. Nspire display of spreadsheet for natural logarithm of time and CO2 output data.

Follow-up Whole Class Discussion

In each of the vignettes described above, teachers found other students in their classes were experiencing the same or similar blockages to their progress and used this feedback as the centrepoint for a whole class discussion in which the problematic issue was explored and then resolved. It is important to note, also, that during both small group and whole class discussions students themselves contributed to the improvement of knowledge and understanding. Technology has therefore played a role in catalysing student participation in their own learning through small group and more public interactions with the teacher and their peers.

Discussion and Conclusion

In contrast to the role attributed to technology in mathematical modelling by Galbraith, Renshaw, Goos, and Geiger (2003), in the episodes described above, the electronic output forced students to revaluate fundamental assumptions they had made within the context of the described problems. This means that technology related activity takes place during the *assumptions* phase rather than only at the *solve* juncture outlined in Figure 1. Consequently, this assigns a role to technology in the conceptualisation of the model rather than simply as a tool which is used to solve a mathematical problem after it has been abstracted – a position more consistent with a view of technology as a *partner* or *extension-of-self* in technology enhanced mathematical activity (Goos, Galbraith, Renshaw, & Geiger, 2003).

The unexpected output on the handheld devices in both vignettes acted to *provoke* students to rethink their original assumptions and to make adjustments to their initial approaches solving these problems. Students were forced to reshape their early thinking to satisfy the demands of both the context and the limitations of their abstracted model. Thus technology, in this case, has fulfilled a more interactive role than simply that of a powerful computational tool.

These *provocations* also represent opportunities for teachers to gain an awareness of students' misconceptions and then to provide appropriate scaffolding in order to move the students forward in their understanding of the issue that was proving problematic. As reported above, both teachers used the consternation generated by the error messages recorded on students' handhelds to structure a forum in which student-student-teacher interaction played an important role in resolving the issue of concern.

The role of CAS based technologies as a *provocateur*, as reported in this paper, of productive student-student-teacher interaction in both small group and whole class settings is therefore an area worthy of further research in relation to mathematical modelling and the field of technologically rich learning environments.

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Researcher-Teacher Relationships in Mathematics Education

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University-based mathematics educators typically rely on gaining access to teachers and students in schools or teacher education settings to conduct their research. In these circumstances it is more common for teachers to be co-opted into the research agenda than for genuine researcher-teacher collaboration to be realised. This paper proposes a framework for examining research spaces created by such relationships, illustrates its use by comparing three of my own research projects, and discusses implications for researchers working with teachers on mathematics education projects.

Most university-based mathematics educators would claim that the aim of their research is to improve the quality of mathematics teaching and learning; yet education research is often criticised for its lack of impact on, and relevance to, classroom practice. This so-called "research-practice gap" has sometimes been explained by reference to the different processes used by researchers and teachers to improve educational practice, and the different forms of knowledge that result. For example, Wiliam (2003) compares the analytic rationality of formal research that seeks to develop generalisations about educational phenomena with the practical inquiry of teachers who need to address immediate day-to-day problems. Thus the object of research, unlike in teaching, is not to solve problems but to create knowledge that helps us to understand a problem (Labaree, 2003). This tension between the aims of formal research and the needs of teachers is also evident in the often unequal relationships between researchers and teachers who participate together in classroom based studies. Breen (2003) argues that true collaboration can only be realised if there is sharing of control and decision-making between the participants. However, this is an uncommon occurrence as teachers are usually co-opted into the research agendas of university academics because the latter have greater access to power and resources. The result is a "bidirectional disconnect" (Heid, Middleton, Lawson, Gutstein, Fey, King, Strutchens, & Tunis, 2006, p. 79) where the complementarity of researchers' and teachers' knowledge remains unexplored.

The purpose of this paper is to examine some of the issues raised above and discuss implications for mathematics educators. To do this, I will outline a framework for analysing researcher-teacher relationships and then illustrate its use by comparing ways in which I worked with teachers in three of my own research projects. The analysis highlights characteristics of successful research collaborations and leads to questions about the role of university-based researchers working with teachers on mathematics education projects.

Theoretical Framework

The issue of researcher-teacher relationships has been of interest to mathematics educators for some time. For example, a *Teachers as Researchers* Working Group first met at a PME conference in 1988. This was followed by various PME Discussion Groups (e.g., Novotná, Lebethe, Rosen, & Zack, 2005), a Research Forum (Novotná, Zack, Rosen, Lebethe, Brown, & Breen, 2006) and a Working Session (Novotná & Goos, 2007). At the latter Working Session, a framework was developed for analysing ways in which university academics and teachers might conduct research together (Table 1). The three elements of this framework arose from the questions and issues identified by Working Session participants in discussing their own experiences in conducting research with teachers.

Table 1

Framework for Analysing Researcher-Teacher Relationships

Beginning the partnership	Participants	Purposes of the research
How?	Roles & expectations	Topic (who chooses?)
• Seeking a teacher	Language	Research questions (whose?)
• Teacher seeks you	Trust/relationships	Benefits (for whom?)
• Enforced participation	Communities & asymmetric needs	

The main question considered in *beginning the partnership* was how teachers enter into this process and who initiates the research. At times, for example, a university-based researcher seeks out one or more teachers to participate in a project that has already been planned. Occasionally a partnership might be initiated by a teacher who seeks out a university-based researcher. Some Working Session participants noted that teachers could be encouraged or required by a school administration or government education department to enter into a university-based research project. In all these instances we need to consider "why" as well as "how" such research partnerships are initiated.

Several questions relate to the *participants* themselves. The extent to which roles are shared between teacher and researcher came under scrutiny; for example, there may be benefits and disadvantages in either maintaining strong role boundaries or sharing/swapping roles (teacher-as-researcher or researcher-as-teacher). However roles are determined, it was agreed by Working Session participants that expectations need to be made clear from the start as a foundation for building trust and mutual respect. Malone (2000) has argued that since teachers and researchers create and act upon two different types of knowledge (practical inquiry in particular contexts versus theoretical inquiry aiming at generalisation), they belong to two distinct communities, or sub-cultures, with intersecting interests but asymmetric needs. That these two communities also use and value different forms of language presents a challenge for communicating the findings of research to non-researcher audiences, such as teachers and policy makers.

Purposes of the research may depend upon how the partnership is initiated, since this often influences the choice of topic, negotiation of research questions, and realisation of any benefits for theory, practice, or policy development.

In the next part of the paper I use the framework sketched out above to compare researcher-teacher relationships in three contrasting research projects I have conducted over the last 14 years. The first project highlights the development and gradual transformation of a long term collaborative relationship between myself as a university-based researcher and a school teacher as we carried out classroom research together. The second project was a longitudinal study of the transition from pre-service to beginning teaching, and the third project was commissioned by the government to support implementation of a new mathematics curriculum by working with teachers to expand their pedagogical and assessment repertoires.

A Collaborative Research Relationship: Project #1

Since 1994 I have carried out research with a teacher (Vince) who shares my interest in secondary school students' mathematical thinking (see Goos & Geiger, 2006; Geiger & Goos, 2006 for extended discussions of this collaboration). I conducted most of my PhD research in Vince's classroom, and we have since collaborated in other projects. *Initiation of the partnership* came about when we were introduced by our former pre-service teacher education lecturer, who had become my PhD supervisor. At the time, Vince had recently completed a Masters degree and was motivated to participate in my research by his desire to resume regular professional conversations with someone like his former university supervisor. Thus there was some equity in the partnership from the start in terms of its initiation and the underlying motivations of the participants.

As *participants*, although we agreed to keep our roles separate – myself as non-interventionist researcher and Vince as teacher – the nature and distinctiveness of these roles changed over time as we developed mutual trust. I was a novice researcher as well as a novice teacher, and thus I was conscious of the kind of respectful relationship that needed to be established with this very experienced teacher if the research was to be productive. Vince later explained how he valued my presence as "someone who can see with non-judgmental and different eyes who views the world of the classroom through an analytical lens that seeks to understand rather than to prescribe action" (Geiger & Goos, 2006, p. 256). However, my efforts to understand did eventually lead Vince towards specific actions so that over time I became more of a participant than a passive observer. For example, our post-lesson discussions about classroom events and my conversations with students often led Vince to modify his teaching plans for the next lesson. He explained:

> The interesting thing for me as a teacher was to think about what made it happen in that way, can we replicate this? … Could we manipulate what was happening to bring about particular types of learning and interaction between students? (Goos & Geiger, 2006, p. 38)

Vince and I explicitly negotiated issues related to power and what each of us wanted to achieve out of the collaboration as we began to write and present papers together at research conferences. Vince believed that "teachers' voices … have to be heard if research is going to make a difference to teaching and learning in schools" (Goos & Geiger, 2006, p. 38), and he saw jointly authored publications as acknowledging his equal contribution to creation of the new knowledge reported therein. Likewise, I gained credibility with practising teachers through joint presentations at professional development conferences where Vince was well known because of his leadership and advocacy roles in teacher professional associations. This was how we introduced each other into the distinct sub-cultures of mathematics education to which we separately belonged – the community of educational researchers and the community of teachers – and how we learned to communicate with different audiences using the language of research and the language of practice. Thus our needs, although different, were mutually recognised and valued.

Initially the *purposes of the research* were determined by my own interests in that I proposed the topics and research questions. This situation has evolved into a more equal arrangement since Vince enrolled in a PhD, under my supervision, and later began to formulate his own research plans. He has now left his job as a school teacher and moved into a new position as a university academic, but our collaboration continues on a range of other projects.

Research with Pre-Service and Beginning Teachers: Project #2

My second example is typical of research conducted by mathematics teacher educators with their pre-service students. The aim of project #2 was to investigate and compare the pedagogical practices and beliefs of pre-service and recently graduated teachers in integrating digital technologies into the teaching of secondary school mathematics. This was a longitudinal study over three years (2002-2004) in which I followed three successive cohorts of my own pre-service students into their early years of teaching (see Goos, 2005; Goos & Bennison, 2008, for details).

In a sense this *partnership was initiated* as soon as the students enrolled in my course and therefore became prospective research participants. The research design had two components: (a) a survey-based cohort study of practicum experiences in technology integration experienced by the group as a whole; and (b) individual case studies of selected pre-service teachers that allowed snapshots of experience to be captured at developmental stages during practicum sessions and the first year after graduation. I invited all my students to participate in the surveys, and then selected a small number for individual case study (lesson observations and interviews) based on research criteria that involved sampling a range of different school settings, including government and independent schools in capital city and regional locations, with differential access to technology resources.

There was no sharing of teacher and researcher roles amongst *participants*, but role boundaries became blurred in another way in that I filled the dual roles of teacher educator and researcher. Adler, Ball, Krainer, Lin, and Novotna (2005) pointed out that this personal investment in teaching makes it difficult for us to take a critical stance towards the research we do with prospective and practising teachers, even though it may assist in building the trust that is needed to establish productive research relationships. While recruitment of students was carried out in accordance with the university's ethical guidelines, their participation may have been motivated by their relationship with me as the teacher-educator-researcher, and in such circumstances it is difficult to negotiate the power relationship that exists between teacher educators and their students. Although I helped the pre-service teachers to publish their technology integration work in professional journals (e.g., Quinn & Berry, 2006), I did not seek out opportunities for them to share their research experiences with a wider audience. Instead, I filtered their experiences through my own research perspective when I communicated findings from this project to the research and professional communities (Goos, 2005, 2006).

The *purposes of the research* arose from my experiences of teaching previous cohorts and my observations of the potential for technologically knowledgeable pre-service and beginning teachers to act as change agents in schools (Marcovitz, 1997; Weinburgh, Smith, & Smith, 1997). Thus the teacher-participants unknowingly influenced the topic and research questions without having any direct input into their formation. Several of these beginning teachers later approached me to volunteer for other research projects, which may indicate they gained some benefit from participation.

Research-Based Professional Development: Project #3

The third contrasting example was a professional development project carried out in 2005-2006 that supported a group of secondary mathematics teachers in planning and implementing mathematical investigations, consistent with the intent of a new syllabus (see Goos, Dole, & Makar, 2007, for details). The main challenge for teachers implementing the syllabus lay not only in using a new structure for curriculum planning, but also in designing learning experiences and assessment tasks that take an investigative approach to working mathematically. Four pairs of teachers in four schools located in the same geographical region participated in the study. We made three visits to the region over a five month period to work with the whole group of teachers, each time for two consecutive days.

This *partnership was initiated* by the State education department, which provided funding to several universities for professional development projects to support initial syllabus implementation. The department nominated the education district in which the project was to be conducted, and asked school principals in the district to recruit teachers. As a result, we did not meet the teachers until our first visit. Although we played no part in initiating the researcher-teacher relationships in this project, we nevertheless discovered that all the teachers had volunteered to participate because they were looking for ideas about taking a more investigative approach to their classroom practice, and some were already experimenting with investigative approaches to mathematics teaching.

We maintained clear role distinctions as *participants* who were either teachers or researchers. However, as with Project #2, the researchers filled dual roles, this time as professional developers who were expected to bring about change in teaching practice. A key element to our research design was to align the project goals with the needs of the teachers. In particular, we recognised the importance of providing teachers with authentic, practice-based learning opportunities that included examples of mathematical investigations, opportunities to experience these investigations as learners themselves, and opportunities to share their ideas and experiences with colleagues, including the challenges encountered and their insights into the process. We discussed these principles at the start of our first visit, together with expectations regarding the iterative research approach that required teachers to plan, implement, and evaluate at least two units of work. It was difficult to build trust between researchers and teachers due to the very short duration of this project and the limited nature of our interactions with the teachers. However, by the time of our third research visit, we had established sufficient familiarity and trust to go to the teachers' schools, observe and videotape lessons, and use the videotapes as a stimulus for a post-lesson discussion.

Similarly to Project #2, as researchers we have communicated our findings to the research and professional communities (Goos, 2007; Goos, Dole, & Makar, 2007). Additionally, at the end of the project the State education department planned a forum where participating researchers and teachers could co-present their experiences to an audience of senior teachers and education district officials. Several of the teachers who participated in the project have since taken leadership roles within their district in providing professional development to their peers. The project has thus provided the teachers with opportunities to share their practical inquiry with colleagues in other schools.

The broad *purposes of the research* were determined by the State education department, but the pedagogical focus of the project was at the discretion of the researchers and teachers. The teacher participants identified their own personal goals (topics) and these shaped their planning, implementation, and refinement of units of work. When discussing overall impact some teachers claimed they now had greater understanding of mathematical investigations. A few commented that it was unlikely significant change would have occurred without the impetus provided by this project, because the opportunity to participate validated the changes in teaching practice that they wanted to achieve. Several of the teachers also mentioned that working with university researchers had enhanced their status as professionals in the eyes of their colleagues. While all the teachers seemed to benefit in some way from their participation, at times we – the researchers – felt challenged by teacher beliefs and school cultures that were not supportive of investigative teaching approaches.

Conclusion: Implications for Mathematics Educators

Table 2 summarises features of the researcher-teacher relationships in the three projects and highlights *mutuality* of motivations, roles, purposes, and links between communities as a key characteristic of genuinely collaborative researcher-teacher relationships (project #1).

Table 2

Comparison of Researcher-Teacher Relationships in Three Projects

Feature of researcher-teacher relationship	Collaborative research relationship (project #1)	Pre-service & beginning teachers (project #2)	Professional development (project #3)
Beginning the partnership • How? • Motivation?	Initiation: mutual colleague. Motivation: mutual interest in research topic and process.	Initiation: researcher seeks teachers (own students). Motivation: students' relationship with researcher-teacher-educator?	Initiation: State education department. Motivation: desire to improve teaching and learning?
Participants • Roles • Expectations • Language • Trust/relationships • Communities • Asymmetric needs	Separate roles but some blurring; explicit expectations. Trust established over extended time. Complementary knowledge explored; researcher & teacher communicate findings to both communities.	Researcher as teacher educator; implicit expectations. Trust relies on teacher-student relationship. Separate researcher & teacher knowledge; researcher communicates findings to both communities.	Researcher as professional developer; explicit expectations. Difficult to build trust over short time span. Separate researcher & teacher knowledge; some shared communication of findings to both communities.
Purposes of the research • Topic • Research questions • Benefits	Topic & research questions defined by researcher but evolve over time. Mutual academic & professional benefits.	Topic & research questions defined by researcher. Academic benefits for researcher, implicit professional benefits for teachers.	Topic defined by government, research questions by researchers & teachers. Academic benefits for researcher; some professional benefits for teachers

This brief exploratory analysis suggests a way of mapping out the research space occupied by university-based mathematics educators who carry out research with pre-service or practising teachers. It also raises questions about the role of university-based researchers, especially when they are also teacher educators or professional developers in relation to the teachers participating in the research project.

Implications for Researchers as Teacher Educators

1. How can pre-service teacher educators negotiate ethical issues (unequal power) in researching with their own students?
2. How can pre-service teacher educators develop a critical stance (distance and scepticism) towards the research they conduct with their students?

In reflecting on the emerging field of mathematics teacher education Krainer noted that teacher educators have the dual roles of "intervening and investigating … of improving and understanding" (Adler et al., 2005, p. 371). In the same article Adler suggested that in order to fulfil our dual roles as teacher educators we need to develop effective theoretical languages to distance ourselves from what we are looking at. Perhaps scepticism and distance might also be achieved by subjecting our own practice to the scrutiny of teacher education colleagues in other institutions and countries via collaborative projects.

Implications for Researchers as Professional Developers

3. Who has the right to "transform" teachers and teaching practice?
4. How can researchers working with teachers balance transformation with critique in ethical and intellectually honest ways?

Each of these questions reflects the challenges of conducting professional development projects where there is an expectation that teaching practice will be transformed for the better. Labaree (2003) pointed out that researchers and teachers speak different languages and work within different paradigms (analytical vs practical), and these cultural differences need to be negotiated with care in order for researchers and teachers to build mutual understanding of, and respect for, each other's knowledge.

Implications for Linking Research and Practice

5. In communicating findings from research with teachers, who should speak for whom and to whom?
6. What conditions are needed for researchers and teachers to explore each other's roles and understand how their respective communities develop generalised versus particularised knowledge of teaching and learning?

The final two questions invite readers to consider how researchers and teachers – representing two different communities with intersecting interests and asymmetric needs – might work together to develop and communicate "principled practice" (Heid et al., 2006, p. 78). The framework for analysing researcher-teacher relationships presented in this paper may provide a starting point for this important endeavour.

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Towards a Sociocultural Framework for Understanding the Work of Mathematics Teacher-Educator-Researchers

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Within the mathematics education research community there is growing interest in theories that view teachers' learning as a form of participation in social and cultural practices. This paper explores what we can learn from research that takes a sociocultural perspective on learning to teach, and how such research might provide a framework for understanding the work of mathematics teacher-educator-researchers. These questions are addressed with particular reference to Valsiner's zone theory, drawing on studies that take contrasting approaches to its application.

Recent reviews of research in mathematics teacher education have noted increasing attention to the social, cultural and institutional dimensions of teachers' learning as well as attempts to integrate social and individual levels of analysis (da Ponte & Chapman, 2006; Lerman, 2001; Llinares & Krainer, 2006). Lerman (1996) defined sociocultural approaches to mathematics teaching and learning as involving "frameworks which build on the notion that the individual's cognition originates in social interactions (Harré & Gillett, 1994) and therefore the role of culture, motives, values, and social and discursive practices are central, not secondary" (p. 4). In the light of these theoretical developments, this paper considers two questions:

1. What can we learn from research that takes a sociocultural perspective on learning to teach mathematics?
2. How might socioculturally oriented research provide a framework for theorising the role of mathematics teacher educators who conduct research with prospective and practising teachers?

Following a brief review of some of the main themes in socioculturally oriented research in mathematics teacher education, these questions are addressed with particular reference to Valsiner's (1997) zone theory as a potentially useful framework.

The Sociocultural Landscape in Mathematics Teacher Education

Sociocultural perspectives on learning and development grew from the work of Vygotsky in the early 20th century. Vygotsky's theoretical approach refers to the social origins of higher mental functions, and the mediation of these functions by tools and signs, such as language, writing, systems for counting and calculating, algebraic symbol systems, diagrams, and so on. Vygotsky also introduced the concept of the Zone of Proximal Development (ZPD) to explain how social phenomena are transformed into psychological phenomena. Since the 1970s, education researchers have applied Vygotsky's ideas to the study of social interactions in classroom and institutional contexts. Contemporary sociocultural theory proposes that learning involves increasing participation in socially organised practices, and the notion of a situated learning in a community of practice composed of experts and novices (Lave & Wenger, 1991; Wenger, 1998) has been fruitfully applied to education settings.

Recent socioculturally oriented research on teachers' learning has drawn on both *discourse* and *practice* perspectives. The discourse perspective focuses on the dynamics of mathematical communication in classrooms, where interest centres on the role of semiotic mediation. Representative of this approach is research by Blanton, Berenson, and Norwood (2001a, 2001b), who used Vygotsky's concept of language as a mediating tool in an individual's development to investigate the discourse of prospective teachers and their university supervisors. The pedagogy of supervision that emerged from this research was claimed to open up a Zone of Proximal Development where the nexus between theory and practice could be explored.

The practice perspective in sociocultural research links activity structures with learning and identity. Here, the notion of learning in a community of practice has been invoked in research on teacher learning via professional collaboration (e.g., Graven, 2004) and in studies of the effects of participating in different communities on the development of beginning teachers. For example, Bohl and Van Zoest (2003) analysed discontinuities between a beginning teacher's facility in talking about reform based mathematics teaching during her teacher education program and her difficulty in translating her knowledge and beliefs into classroom practice.

Similarly, situative approaches have helped researchers understand how context makes a difference to the development of mathematics teachers and their professional identities (e.g., Peressini, Borko, Romagnano, Knuth, & Willis, 2004).

Krainer has noted that teacher educators have the dual roles of "intervening and investigating … of improving and understanding" (Adler, Ball, Krainer, Lin, & Novotna, 2005, p. 371). Studies of the type referred to above demonstrate that sociocultural research can enhance our *understanding* of how teachers learn from their experiences in different contexts, such as the university pre-service course, the practicum, and the school of employment. Sociocultural perspectives have perhaps been used less effectively to inform research on *improving* teachers' opportunities to learn, and this has left the role of the teacher educator largely untheorised. A more elaborated sociocultural theory of teaching is therefore needed to complement sociocultural language and concepts used to describe learning in a community of practice or in the ZPD. In this regard, some researchers have turned to Valsiner's (1997) zone theory – which re-interprets and extends Vygotsky's concept of the Zone of Proximal Development to incorporate the social setting and the goals and actions of participants – to develop stronger sociocultural frameworks for teacher education research. Use of zone theory in this research is illustrated in the next section.

Using Valsiner's Zone Theory to Understand and Improve Mathematics Teachers' Learning

In his theory of child development, Valsiner's (1997) sees the ZPD as a set of possibilities for development that are in the process of becoming actualised as individuals negotiate their relationship with the learning environment and the people in it. He then proposes two additional zones, the Zone of Free Movement (ZFM) and the Zone of Promoted Action (ZPA). The ZFM structures an individual's access to different areas of the environment, the availability of different objects within an accessible area, and the ways the individual is permitted or enabled to act with accessible objects in accessible areas. The ZPA comprises activities, objects, or areas in the environment in respect of which the person's actions are promoted. Valsiner explains that the ZFM and ZPA are dynamic and inter-related, and are constantly being re-organised by the adult in learning interactions with the child. Mathematics teacher educators have taken two contrasting approaches to applying this theory in their research, one of which defines the zones from the perspective of the teacher-as-teacher and the other from the perspective of the teacher-as-learner.

Approach #1: Focus on Teacher-as-Teacher

Applying Valsiner's ideas to classrooms, the teacher's instructional choices about what to promote and what to allow establish a ZFM/ZPA complex that characterises the learning opportunities experienced by students. This is the approach taken by Blanton, Westbrook, and Carter (2005), who used Valsiner's theory to interpret novice teachers' ZPDs. They compared the ZFM/ZPA complexes organised by three mathematics and science teachers in their respective classrooms as a means of revealing these teachers' understanding of student-centred inquiry and hence establishing the potential for development within their own ZPDs. Two of the teachers created the appearance of promoting discussion and reasoning when their teaching actions did not actually allow students to experience these, and the researchers explained this apparent contradiction by theorising the existence of an illusionary Zone of Promoted Action (IZ). For one of these teachers, the IZ appeared to signal a transitory state as she eventually changed her practice to both promote and allow student interaction. Thus existence of an IZ may indicate that inquiry based teaching practices are within the teacher's ZPD but have not yet been enacted in the way intended or perceived by that individual. Nevertheless, Blanton et al. note there is no guarantee that this transition will occur and they acknowledge the need for further research on external factors, such as the role of teacher educators, that contribute to teachers' development.

Approach #2: Focus on Teacher-as-Learner

My own approach to the use of zone theory (see Galbraith & Goos, 2003; Goos, 2005a, 2005b, 2005c) differs from that of Blanton and colleagues in that all zones are defined from the perspective of the teacher as learner. When I consider how teachers learn, I view the teacher's ZPD as a set of possibilities for their development that are influenced by their knowledge and beliefs, including their disciplinary knowledge, pedagogical content knowledge, and beliefs about their discipline and how it is best taught and learned. The ZFM can then be interpreted as constraints within the teacher's professional context such as students (behaviour, socio-economic background, motivation, perceived abilities), access to resources and teaching materials, curriculum and assessment requirements, and organisational structures and cultures. While the ZFM suggests which teaching actions are *allowed*, the ZPA represents teaching approaches that might be specifically *promoted* by pre-service teacher education, formal professional development activities, or informal interaction with colleagues in the school setting. For learning to occur, the ZPA must engage with the individual's possibilities for development (ZPD) and must promote actions that the individual believes to be feasible within a given ZFM. It is significant that prospective teachers develop under the influence of two ZPAs, one provided by the university program and the other by the supervising teacher(s) in the practicum school, which do not necessarily coincide.

The vignettes in Figures 1 and 2 illustrate how I applied this theory to mathematics teacher learning in two studies involving prospective and practising teachers, both of which involved integration of digital technologies into secondary mathematics teaching (for details of these studies see Galbraith & Goos, 2003, and Goos, 2005b).

Pre-service teaching. Adam completed practice teaching in a school with substantial technology resources (graphics calculators, data logging equipment, software, internet). Some of these changes had been made in response to new mathematics syllabuses that mandated the use of computers or graphics calculators in teaching and assessment programs. *(ZFM afforded technology integration)*. Adam had previously worked as a software designer and was confident in using computers and the internet. Although he had not used a graphics calculator before starting the teacher education course, he quickly became familiar with its capabilities and with the support of his Supervising Teacher began to incorporate this and other technologies into his mathematics lessons. *(ZPA organised by Supervising Teacher was consistent with ZPA I offered in my university course and also with Adam's ZPD defining his potential for development.)*

Beginning teaching. After graduation Adam was employed by the same school where he had completed his practicum. *(Same ZFM, ostensibly affording technology integration)* He now discovered many of the other mathematics teachers were unenthusiastic about using technology and favoured teaching approaches he claimed were based on their faulty belief that learning is linear and teacher-directed rather than richly connected and student-led. Conflicting pedagogical beliefs were a source of friction in the staffroom, and this was often played out in arguments where other teachers accused Adam of not teaching in the "right" way. Compared with his earlier experience as a prospective teacher, Adam now found himself in a more complex situation that required him to defend his instructional decisions while negotiating a harmonious relationship with several colleagues who did not share his beliefs about learning. *(Conflicts between technology-rich ZFM, school ZPA that promoted technology-poor teaching, and Adam's ZPD.)* He responded by paying attention only to those elements of the Mathematics Department's teaching culture *(school ZPA)* that were consistent with his own beliefs and goals *(his ZPD)* and also with what he had learned in the university teacher education course *(university ZPA)*. This was how he was able to reconcile his pedagogical beliefs *(a part of his ZPD)* with his teaching environment *(ZFM/ZPA complex)*.

Figure 1. *From pre-service to beginning teaching: The case of Adam. (Goos, 2005b).*

Early professional development workshops. Lisa was an experienced teacher but a relative novice in the use of technology *(mixed ZPD: strong pedagogical content knowledge but not in relation to technology integration)*. As Head of her school's Mathematics Department she had considerable autonomy in obtaining desired resources and in managing curriculum and assessment programs *(ZFM afforded technology integration)*. She described the early professional development workshops she attended to learn how to use graphics calculators as "off-putting", because the emphasis was on procedural aspects of operating the calculators and the mathematics presented was too difficult for participants to engage meaningfully with the technology. *(early ZPA inconsistent with ZPD)* After several more workshops she felt confident enough to use graphics calculators in her teaching, but only as a replacement for pen and paper.

Research-based professional development program. Lisa's participation in a research-based professional development program was a turning point for her as it emphasised pedagogy rather than "pushing buttons" *(research-based ZPA consistent with her ZPD, i.e., her need to understand pedagogical rather than procedural aspects of using technology)*. Until this time she only saw graphics calculators as a tool to draw graphs and analyse statistics. Now she "started to see different ways of using it that I hadn't thought of before", such as in data collection and analysis, mathematical modelling, and collaborative group work. She began to see how technology could be used to build mathematical understanding rather than just to improve speed and accuracy of calculations.

Figure 2. *A professional development intervention: The case of Lisa. (Galbraith & Goos, 2003).*

What Can We Learn from Teacher Education Research Using Valsiner's Zone Theory?

The elaboration of Valsiner's zone theory outlined above is helpful for analysing relationships between teachers' pedagogical knowledge and beliefs and the teaching repertoire offered by courses for prospective teachers, practicum and initial professional experiences, and professional development programs in order to understand how they learn in multiple contexts. One such configuration is represented in Figure 3; others can be imagined if we allow the overlap between zones to change. This representation implies that learning takes place at the intersection of the three zones.

Figure 3. Representation of relationships between ZPD, ZFM, ZPA.

Analysis of Adam's case and the experiences of other prospective and beginning teachers who participated in this study gave me a better understanding of the scope and limitations of my role as a mathematics teacher educator as I pondered the question of how I could help novice teachers implement the technology enhanced approaches I promoted in my teacher education course. For many years I dealt with this question by addressing separately some of the key factors known to influence technology integration. For example, I had my students carry out an annual technology audit of their practicum schools so that on their return to the university they could report on and debate the significance of *access to resources and technical support* and the effect of *curriculum and assessment requirements* on technology usage. In these post-practicum sessions I also structured small group discussion tasks in which students compared their own *pedagogical beliefs* about the role of technology in mathematics education with the technology-related practices demonstrated (or not) by their *supervising teachers*. These coursework activities have not changed in their classroom enactment. What has changed is the way I now integrate these and other elements of my course into a single zone-theoretical framework that suggests to me how and where I might intervene in the development of prospective and beginning teachers' identities as users of technology.

The case of Lisa illuminated for me some of the issues facing experienced teachers who are unfamiliar with new technologies such as graphics calculators. While her ZFM presented few constraints, she had to search for professional development (ZPA) that would extend and challenge, rather than accommodate, her existing ideas about teaching with technology (her ZPD). As a result of this work with Lisa and other teachers I began to use Valsiner's zone theory to design professional development interventions that give careful attention to discovering the participating teachers' epistemological and pedagogical beliefs, understanding their institutional contexts, and identifying how all these factors interact to potentially influence their learning and development (see Goos, Dole, & Makar, in press). Only by analysing this initial zone configuration do I feel able to engage with their possibilities for development (ZPD) and promote actions they believe to be feasible in their school environments (ZFM).

From a broader perspective, zone theory could be used by teacher educators to improve teachers' opportunities to learn at three stages of development:

- Pre-service education: helping prospective teachers to analyse their practicum experiences (ZFM), the pedagogical models these offer (school ZPA), and how these experiences align with or contradict the knowledge gained in the university-based program (university ZPA);
- Transition to the early years of teaching: creating induction and mentoring programs that promote a sense of individual agency within the boundaries of the school environment (ZPD within ZFM);
- Professional development: designing professional learning programs for more experienced teachers (ZPA to stretch ZPD).

How Can Valsiner's Zone Theory Help Us Understand the Role of Mathematics Teacher-Educator-Researchers?

This paper has shown how Valsiner's zone theory brings teaching, learning, and context into the same discussion, and how the theory can be applied in two connected layers – to the teacher-as-teacher orchestrating classroom ZFM/ZPAs for students (Blanton, Westbrook, & Carter, 2005) as well as the teacher-as-learner negotiating the ZFM/ZPAs offered by the professional environment (Goos, 2005a). At the latter layer the teacher-educator-as-teacher comes on the scene, providing the ZPA. What if we imagine a third layer, with teacher-educator-as-learner? How does our professional context constrain our actions in culturally expected ways (ZFM), and what are our opportunities to learn (ZPA)? Could we describe a set of possibilities for our own development in the near future (ZPD)? In other words, how might zone theory help us analyse our own roles as mathematics teacher educators conducting research with prospective and practising teachers?

Let me sketch out what such an analysis might look like by applying zone theory to my own practice in the dual roles of researcher and teacher educator. As a researcher, my Zone of Free Movement is constrained by academic structures and cultures within and beyond my university. These include:

- guidelines for career development, identifying activities that are formally recognised and rewarded;
- mechanisms for managing academic workloads that seek to balance teaching and research;
- government programs for assessing the quality and impact of university research;
- competitive research grant schemes;
- the process of peer review of articles submitted for publication in scholarly journals.

Closely inter-related with these elements of my professional context is the Zone of Promoted Action represented by my initial research training (doctoral studies, early experiences as a research assistant), participation in research conferences and other activities of educational research associations, and formal or informal mentoring by more experienced colleagues. This ZFM/ZPA complex helps shape possibilities for my development as a researcher (ZPD) by defining what is allowed and what is promoted. The learning opportunities that arise in this way are well charted and form part of the enculturation of novice researchers into academic life.

As a mathematics teacher educator, I must negotiate a different zone configuration. Here, my practice is constrained by a Zone of Free Movement comprising the following elements:

- student characteristics, such as their mathematical knowledge and their beliefs about mathematics teaching and learning;
- curriculum and assessment requirements that are increasingly governed by external teacher registration authorities as well as university course accreditation processes;
- limited access to technology resources in the university;
- reduction of the hours allocated to teaching methods courses in the pre-service teacher education program;
- difficulties in finding suitable practicum placements for prospective teachers;
- perceptions amongst colleagues that teacher education is low status work.

My ZPA as a teacher educator is less clearly defined in that it is difficult to identify people or activities that explicitly promote my development in this role, and thus difficult to describe the ZFM/ZPA complex that shapes my teacher education practice. Llinares and Krainer (2006) point out that the growth of mathematics teacher educators as learners is a new field of study, and research in this area has so far drawn on notions of reflective practice rather than sociocultural theories that take into account the settings in which practice develops. From a sociocultural perspective, I could say that my own research in teacher education acts as a ZPA that informs my practice as a mathematics teacher educator. My research using zone theory has also influenced how I work with prospective teachers – my own teacher education students – to help them analyse tensions between the learning experiences offered by the university course and the practicum. While this approach helps give coherence to my dual roles as researcher and teacher educator, further elaboration of Valsiner's zone theory is necessary to create a conceptual framework that better explains how mathematics teacher educators learn from research into teacher education.

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Identity as a Lens to Understand Learning Mathematics: Developing a Model

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In this discussion article we consider mathematics learning as a process of developing a mathematical identity. This process is constituted by relationships between three key components – the teacher, the students and the discipline of mathematics. It is posited that the teacher's role is to facilitate the development of students' mathematical identities by relationally bridging student and the subject. Fundamentally, this requires that mathematics teachers have well-developed personal mathematical identities.

Research over many years has highlighted a number of persistent issues related to participation and engagement in mathematics (e.g., Australian Academy of Science, 2006; Grootenboer & Zevenbergen, 2007). Despite being well aware of these problems for some time, there appears to have been little progress towards improving the situation and still many students hold unhelpful and unhealthy views of mathematics, and participation in mathematics classes at the higher levels continues to diminish. Recently, some researchers and writers (e.g., Boaler, 2002) have turned to an *identity* conceptual framework to try and better understand learning in mathematics and possible approaches for tackling these tenacious issues.

In this discussion paper we consider a concept of identity as a framework for theorising learning in mathematics. An initial model is presented that simply captures some of the salient features of the learning milieu of the mathematics classroom, and it posits that the teacher's role is to facilitate the development of students' mathematical identity by relationally bridging student and subject. The implications of this view of learning mathematics are then discussed with particular reference to the need for teachers at all levels to have a personally well-developed mathematical identity.

Learning in Mathematics

Many researchers and theorists have considered how mathematics is learned, each foregrounding different aspects of the learning process and viewing it from a range of theoretical frameworks. More recently, the notion of *identity* has been employed to try and bring about greater understanding of learning in mathematics, and perhaps provide some insights into how the perennial issues facing mathematics education may be addressed.

Identity is a term that has been employed by writers and researchers from a range of theoretical perspectives including the psychological (e.g., Erikson, 1968), the socio-cultural (e.g., Boaler, 2002), and the post-structural (e.g., Walshaw, 2004). There are clear epistemological differences between these perspectives, but Grootenboer, Lowrie and Smith (2006) suggested that the plurality of theoretical lenses may indeed provide a richer and more comprehensive understanding of the issues of identity in mathematics education. While we do not want to ignore, simplify or diminish the theoretical debate about the concept of *identity* (see Sfard & Prusak, 2005), in this article we want to focus on the development of student's identity in mathematics, and the relationship between students' mathematical identities and the discipline of mathematics. For this purpose we view identity as "how individuals know and name themselves …, and how an individual is recognized and looked upon by others" (Grootenboer, et al., 2006, p. 612). Identity is a unifying and connective concept that brings together elements such as life histories, affective qualities and cognitive dimensions.

In their mathematics classes, students learn more than just mathematical concepts and skills, and they are involved at more than just a cognitive level (Boaler, William, & Zevenbergen, 2007). Furthermore, Putman and Borko (2000) stated that, "how a person learns a particular set of knowledge and skills, and the situation in which a person learns, become a fundamental part of what is learned" (p. 4). With this in mind, identity is a useful concept to explore and understand mathematical learning because it includes the broader context of the learning environment, and all the dimensions of learners' selves that they bring to the classroom. Furthermore, the goal of mathematics education is to develop students' mathematical identities.

A focus on students' mathematical identities does not diminish the need for the development of mathematical skills and knowledge, but rather it encompasses them alongside other important dimensions such as attitudes, beliefs, emotions and dispositions. Moreover, because the various dimensions are complexly inter-related, a focus on the identity means they can be considered simultaneously.

The development of students' mathematical identities has become increasingly important as technological advancement requires greater mathematical competence and confidence (Kilpatrick, Swafford, & Findell, 2001; Zevenbergen, 2005). Problems of disengagement and non-participation in mathematics have tenaciously persisted for many years, as have concerns about mathematical achievement and poor attitudes and dispositions, and it is hoped that an investigation of mathematical learning as identity development may offer some new insights into how these issues can be ameliorated.

The Learning Milieu of the Mathematics Classroom

As highlighted above, the mathematics classroom is made up of many interacting and complex dimensions. Three significant aspects are the student, the classroom community, and the discipline of mathematics (see Figure 1). The teacher is the key dimension of the classroom community, hence it is highlighted in Figure 1 below. These are not exclusive dimensions or clearly defined, because, for example, the student is part of the classroom community, and each aspect is complexly networked to a diverse range of other individuals and communities beyond the immediate classroom context.

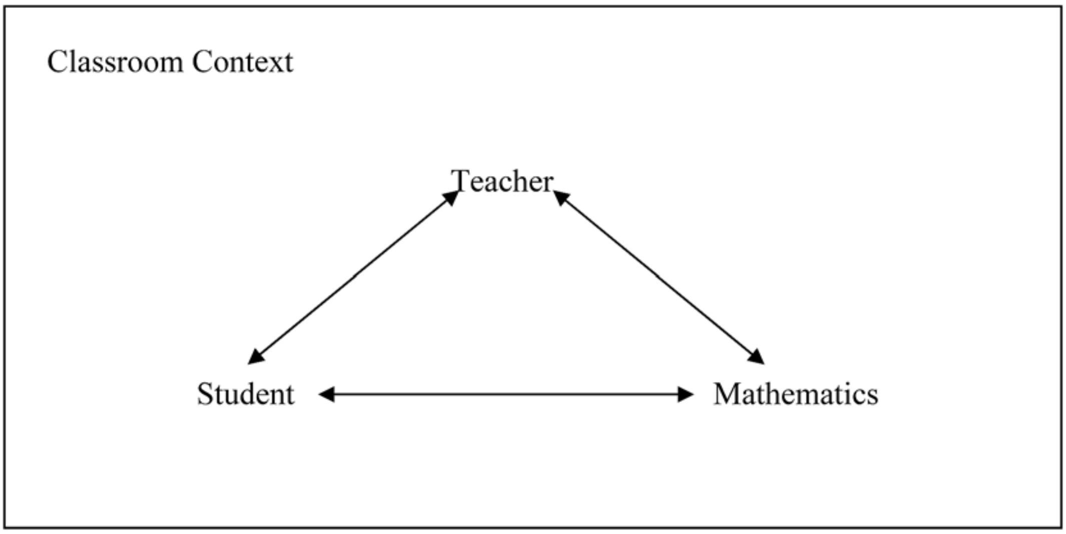

Figure 1: The learning milieu of the mathematics classroom

The Student

The student's identity has many facets (or some would say that they have multiple identities) formed through their life history and engagement with their peers, family, etc. Their identity incorporates their knowledge, abilities, skills, beliefs, dispositions, attitudes and emotions, and these broad conceptions of 'who the student is' are brought to the mathematics classroom. The student comes to the mathematics classroom as a member of some sort of family community and they will be immersed in, and a part of a generational culture, meaning that their perspectives can be quite different from that of their teachers. Zevenbergen and Zevenbergen (2007) discussed the characteristics of *millennials* and the implications of this new generation of students coming to school, particularly in the context of different cultural identities meeting in the mathematics classroom. While there are a range of affordances and issues inherent in this coming together of generational identities, the key point here is students enter the mathematics classroom with dimensions to their identity that are an integral part of their mathematics learning.

As noted above, students' (and teachers') identities incorporate a range of dimensions, including knowledge, abilities, skills, beliefs, dispositions, attitudes and emotions related to mathematics and mathematics learning – their mathematical identity. These will have been significantly influenced by their previous experiences of mathematics education, and, they will be integral to their future learning on mathematics.

The Discipline of Mathematics

The discipline of mathematics is an important dimension because it is the distinguishing characteristic of the learning context. We have been concerned that the *mathematics* of mathematics education is being diminished in recent years, and pedagogy is largely generic with the mathematics only providing the particular flavour. While we agree that there is value in considering pedagogy in general, we think that this is insufficient for the development of mathematics education. Our firm conviction is that mathematical pedagogy is fundamentally different from other subject pedagogies, because the nature of practices of mathematics are fundamentally different from other disciplines. Indeed, we think it is important that greater attention is paid to the *mathematics* of mathematics education if progress is to be made with the perennial issues outlined in the introduction. The focus on the development of students' *mathematical identity* provides an appropriate focus on the discipline of mathematics and the learner.

Previously (Grootenboer & Zevenbergen, 2007) we have discussed the importance of considering the nature of mathematics and the practices of mathematicians when theorising about learning mathematics. After studying classrooms in Britain, Boaler (2003) found that the mathematical epistemology that underpinned the teacher's pedagogy significantly influenced the mathematical identity developed by the students. While the nature of mathematics is a contested notion (Davis & Hersh, 1998), it appears that the characteristic of *school mathematics* is often quite different from the nature of mathematics undertaken by mathematicians. This was noted by Burton (1999, 2001, 2002) when she found that the mathematical identities of mathematicians were characterised by emotional, intuitive, and affective dimensions, and their practices were usually collaborative and about finding connections.

The Classroom Community

The third important aspect is the classroom community, and this has the teacher as a significant feature (hence the significant place of the teacher in Figure 1), but it also includes the other students and the physical environment. The classroom community is complex and is significantly constituted and influenced by factors and connections beyond the immediate 'classroom walls'. Nevertheless, the mathematics classroom is to a certain extent a loosely bounded community where the intended focus is the development of students' mathematical identities.

Wenger's (1998) social theory of learning clearly related the development of a sense of identity and *communities of practice*. Within this perspective, identity is a constant process of becoming, where an individual continually negotiates who they are through their practices as they become and belong to a community (Smith, 2006). In the mathematics classroom, the community of practice includes all the students in the class, but is usually dominated by the teacher. Other defining factors in the community can be the curriculum, text books, and urban myths and stories about mathematics and mathematics teaching and learning.

The three dimensions (student, mathematics, and community) outlined above are each important, but if the goal of mathematics education is to develop a strong mathematical identity, then the critical focus is the relationship between the student and the discipline of mathematics. The facilitating context for the development of this relationship is the classroom community, and specifically the teacher, but the classroom community is temporal, and it will be the mathematical identity (the connection between student identity and mathematics) that will remain.

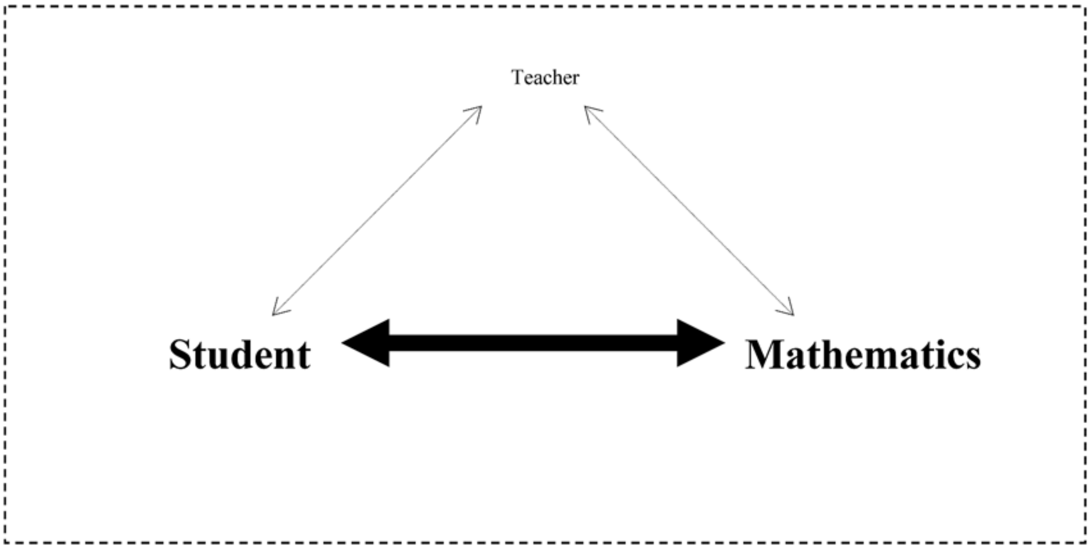

Figure 2: The relationship between student and mathematics is what remains

The Teacher: Bridging Student and Mathematics

At the beginning of a new teaching period when the students and teacher are united within a mathematics classroom setting, even the youngest children begin with a mathematical identity (or identities). Of course, this means that mathematics teachers in their role of developing mathematical identities, are not beginning with a blank slate, but in many respects it is a fresh start.

If a teacher is to be effective in developing students' mathematical identities, then the teacher must themselves have a well-developed mathematical identity. This would include significant mathematical knowledge and skills, but also a positive attitude towards the subject, a sense of joy and satisfaction in undertaking mathematical practices. Furthermore, they must see mathematics as an integral part of their broader identity, and something that has helped define their sense of self and vocation. Mathematics must be more than just an inert body of information and skills that they try to pass onto to students. In describing effective teachers, Palmer (1993) suggested that they have and display a friendship-type relationship between themselves and their subject. Through this, "students are affirmed by the fact that this teacher wants them to know and be known by this valued friend" (p. 104).

The classroom context also requires that the teacher has a relationship with the students. In this sense, the relationship is more than just a social connection and includes pedagogical approaches. There has been much written about the characteristics and nature of quality pedagogical relationships between teacher and student, but here we want to highlight the importance of the student – teacher connection in building students' mathematical identity. Palmer (1993) suggested that effective teachers are able to connect with both student and subject, and in the process they facilitate the students' relationships with the subject – their mathematical identity.

The teacher, who knows the subject well, must introduce it to the students in a way one would introduce a friend. The students must know why the teacher values the subject, how the subject has transformed the teacher's life. By the same token, the teacher must value the students as potential friends, be vulnerable to the ways students may transform the teacher's relationship with the subject as well as be transformed. If I am invited into a valued friendship between two people, I will not enter unless I feel that I am valued as well. (p. 104)

The teacher's role in facilitating the development of students' mathematical identity is one of bridging student and subject. The goal of this bridging is to allow and invite students to develop a strong, enabling and warm relationship with mathematics – something the teacher themselves already enjoys.

Discussion and Implications

There are a number of issues that emerge form viewing mathematics learning as a process of identity development:

- the mathematics integral to mathematics classes must indeed be mathematical, and, the pedagogical approaches employed in mathematics classes need to be consistent with the nature of mathematics;
- the teacher must have a strong personal mathematical identity; and
- the teachers' role is temporal, and at the end of the teaching period it is the students' mathematical identities that will endure.

These issues and their implications will now be briefly outlined and discussed in turn.

Mathematics Education that is Mathematical

While this point would appear to be rather obvious, we are concerned that the form of mathematics presented in school mathematics classrooms is not consistent with the practices of mathematicians or the nature of the discipline. There is a concern that participation rates in mathematics are diminishing, and it could be that students are not rejecting mathematics per se, but rather the form of mathematics they experience at school. While we accept that there is much conjecture about the nature of mathematics, even amongst mathematicians, we believe that there needs to be philosophical discussion and debate about the epistemological foundations of mathematics education practice. This debate needs to extend beyond the walls of the university and engage with practitioners and teachers who ultimately enact and embody the forms of mathematics that pervade the classroom.

An inspection of most curriculum or syllabus documents will reveal some form of statement about the nature of mathematics and mathematical activity, often explicitly in an introductory section. These will often present mathematics as a unified, coherent and engaging discipline. However, what then ensues is a list of outcomes and objectives that, perhaps inadvertently, atomise mathematical knowledge into a series of small discreet pieces. Mathematics teaching that attends to the small details of the outcomes and ignores the 'big picture' of the introduction, leads to a form of mathematics that is disjointed and inconsistent with the nature of the discipline. In this way students can know 'bits of mathematics', but not know 'mathematics' in the same way that one can have pieces of a jigsaw without ever seeing the big picture of the puzzle.

The recent promotion in Australasia of numeracy, as the application of mathematics to real-life situations, has arisen to partly address this problem. The utilitarian epistemological foundation of numeracy attends to the misconception of many students that mathematics is useless and irrelevant. If indeed, students can see mathematics as integral to their life-world beyond the classroom, then they are more likely to engage with the material and develop a mathematical identity grounded in utilitarianism. However, mathematics is more than just numeracy and school experiences based only on the utilitarian value of mathematics will be denuded of much of the richness of the discipline. While there is not scope here to discuss this fully, we believe that it is crucial at this time to consider the impact of the numeracy-based approach to mathematics education and the developing mathematical identities of students.

Teachers' Mathematical Identities

As was highlighted previously, the mathematics teacher's role is as a relational conduit between the student and mathematics. To facilitate the development of students' mathematical identities the teacher shares their relationship with mathematics through their pedagogical relationship with the students. Of course, this is difficult if the teacher does not have a healthy and fond relationship with mathematics themselves – their mathematical identity. One wouldn't consider a music teacher who has no musical aptitude or appreciation, or a physical education teacher who is personally unhealthy and adverse to physical activity, and similarly a mathematics teacher who dislikes mathematical tasks and does not engage in mathematical activity should be an anathema. However, this is problematic at both the secondary and the primary level.

At the secondary school level, the shortage of well-qualified mathematics teachers has been known for some time and clear solution paths still appear elusive. Often mathematics classes are taught by non-specialist teachers, particularly at junior levels, and so their commitment to mathematics is at best divided, and frequently limited. Indeed, the problem is perhaps compounded by the decreasing number of students studying mathematics at higher levels, thus creating a vicious cycle as there are less mathematics graduates who can become mathematics teachers. Furthermore, it appears that some mathematics teachers who have robust mathematical identities are perceived as being less committed to their pedagogical relationships with students (Picker & Berry, 2000, Grootenboer, 2001), thus they are still limited in their capacity to bridge student and subject.

Primary school teachers have the difficult task of teaching a range of disciplines. The idea of the teacher as a relational conduit between student and discipline sees the primary school teacher as needing to have a well-developed identity in a range of subject areas, including mathematics. While primary teachers do have strong discipline-based identities in a range of areas, it appears that often mathematics is not one of the favoured ones (Schuck, 1997). Again, this is not a problem that will be easily resolved, and quick solutions that are based on limited conceptions of mathematics need to be avoided. Perhaps, if their was a greater focus on the development of personal mathematical identities in preservice teacher education and professional development programs in mathematics, then teachers could appreciate and understand mathematics, and relate to it in a more personal manner.

The Temporal Role of the Teacher

The goal of learning in the mathematics classroom is the development of students' mathematical identities – their relationship with the discipline of mathematics. At the completion of the teaching period, the teacher is in a sense, removed from the triangle of relationships as shown in Figure 2, although they remain in the memories of the students and in the influence they have over the mathematical identities the students have developed. Because the teachers' role is ultimately finite and temporary, they need to teach their students so that they can become obsolete, and this implies that their pedagogical task will change over the course of the teaching period. What remains are the students' mathematical identities, and if they then go on to study further mathematics, this will be the foundation from which their new learning experiences will ensue. Regardless, it is important that students develop a robust and enabling mathematical identity.

Concluding Comments

In this paper we have tried to present some ideas and thoughts about learning and teaching mathematics using an identity framework. These ideas are not seen as comprehensive, but it is hoped that they may add another perspective to our understanding of students' mathematical learning. The relational model of mathematical identity development presented in this paper highlighted some issues of particular concern, particularly related to the nature of mathematics and the mathematical identities of teachers. We suggest that it is vital that classes that are named *mathematics* are indeed based on experiences that are consistent with the nature of mathematics. Also, we think that it is essential that teachers of mathematics (at all levels) have well-developed personal mathematical identities.

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Capturing Students' Thinking about Strategies used to Solve Mental Computations by Giving Students Access to a Pedagogical Framework

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Capturing evidence of the strategies that students use when completing mental computation is difficult to do in everyday classroom practice. Teachers do not have access to or time to analyse verbal protocols as researchers do. This study shows that meaningful written recordings of strategy choice and student thinking while completing computations are possible when a scaffolded framework for teaching the strategies is used and the students are given access to this pedagogical framework in a way they can understand and apply.

Goals and content of school mathematics are changing and the change is not restricted to Australia. The focus is not only on the delivery of content knowledge, skills and procedures but also on assisting students to develop deeper understanding of mathematics concepts and the processes; for example, thinking, applying, communicating and reflecting. The National Statement on Mathematics for Australian Schools (Australian Education Council, 1991) which is still the most recent "policy" document about mathematics education in Australia, stated that "learning mathematics involves both its products (body of knowledge) and processes (ways of knowing)" (p. 26). The new Queensland Essential Learnings framework describes "the knowledge, understanding and ways of working that students need for ongoing learning, social and personal competence and participation in a democratic society" (Queensland Studies Authority, 2008). This aligns with Skemp's (1976) description of relational understanding (How does this make sense with in relation to what I understand?) as opposed to instrumental understanding (What do I have to do?).

Recording Student Thinking

To make judgements about student learning of mathematics, teachers need to consider both aspects of mathematical learning – knowledge and processes. It is easier for teachers to focus on the mathematical knowledge and skills aspect of learning that are visible and can be evidenced through observation, focussed analysis, checklists etc. Evidencing thinking and ways of working is more difficult. In research situations the use of individual student interviews with video or audio recording can assist in capturing evidence of student thinking and processes. Techniques like asking students to "think-aloud" to explain strategies used or what students understand about a concept can be utilised to structure the data collected. Teachers do not have easy access to these techniques in their everyday practice, nor do they have time to analyse them closely so they tend to rely on written evidence in the form of checklists of observed behaviours, student work samples, written tests etc. Also these records can be collected in the form of a student folio and reflected on for reporting to parents and to evidence progression of learning. Evidence of thinking, strategy choice and conceptual understanding can be recorded as anecdotal records based on observation or discussions with the student. Often these assessments are seen as subjective and not "hard evidence". A method for collecting written forms of students thinking and strategy choices that does not require transcribing recorded speech would be a useful tool for teachers' assessment of mathematical learning in their classrooms.

Researchers have considered student written recordings as a way of capturing student thinking. Rose (1989) described writing as a valid way of "thinking aloud on paper". Pugalee (2004) compared verbal and written descriptions of students involved in problem solving activities and described situations where students were asked "to record any working for the problem on the paper provided….". One group was specifically asked to "write everything which comes to mind during the solving of the problem" while the other group were told "please think out loud by telling everything that comes to mind while you are solving the problem" (p. 33). He found that the strategies used by students did not vary greatly between those who provided written or verbal descriptions of the problem solving processes. The students who wrote about their processes produced correct solutions at a statistically higher rate than those using the think-aloud method.

Vygotsky (1987) described writing as involving deliberate analytical action on the part of the writer requiring the writer to maximally compact inner speech so that it is fully understandable. He also viewed writing as important in forming associations between current and new knowledge, helping the writer organise ideas in order to make connections between prior and new concepts. Research has shown that writing provided a level of reflection that promoted students' attention to their thinking about mathematical processes (Carr & Biddlecomb, 1998; Pugalee, 2001). Pugalee (2004) stated that this awareness and self regulation appeared to play an important part in students' selection of appropriate information and strategies.

Effective Mental Computation

The focus of computation instruction has shifted from developing students' proficiency with the traditional written algorithms to a focus on strategy use and development of number sense. Research into identification of effective mental computators has taken different approaches to the identification of successful students. Many early studies equated success with speed and accuracy only. A common research method was timed tests (Reys, Reys, & Hope, 1993). These studies were unable to identify strategies used by the students as students were asked to write down answers only after calculating mentally. Their thinking and processes were not able to be captured by the researchers. It has been noted that accuracy by itself is not sufficient as a model for successful mental computation (Heirdsfield, 2001; Thompson, 1999). In some studies, for example, Hope and Sherrill (1987), this type of testing was used in conjunction with further interviews to identify student strategies that gave information on more than one component of mental computation.

Heirdsfield (2001) concluded that mental computation is calculating using strategies with understanding, and thus, proficiency in mental computation was not confined to accuracy, but also included flexibility of strategy choice. Thus successful mental computators need a variety of strategies with which they are comfortable and understand the application of, as well as flexibility to choose from known strategies according to the problem context.

To enable teachers to judge how effective students are in regard to mental computation their thinking about strategy choice and application needs to be captured as well as their answers. Panaoura and Philippou (2005) noted that asking young students about their cognitive processes involves some particular problems. Young students have limited experience with the world and limited vocabulary on which to draw, and as such their experience with certain maths concepts is limited to what they are able to articulate. Panaoura and Philippou supposed that children's answers may reflect not what they know or believe, but rather what they can or cannot tell to the interviewer.

There have been studies where young students verbalised strategies used for computation but few studies involving young students recording their thinking in writing. McIntosh (2002) developed informal written recording processes for mental computation, which showed that it was possible for primary school students to record their strategies.

Asking students to record responses to computation questions in writing does not automatically elicit their thinking strategies. Asking students to show their thinking on paper could elicit the traditional written algorithms rather than the targeted thinking and strategy choice used for mental computation. Younger students are still developing their writing skills and knowledge of mathematical symbols and as such their recording methods may not capture the metacognitive processes they are using. Scaffolding the process of strategy instruction, discussion and recording thinking could assist these students to be able to record their thinking and choice of strategy. This would enable teachers to "see" students' thinking and make assessment judgements not only about the content they know and the types of problems they can solve, but how they are doing this and what they do or do not understand about computation concepts and number sense.

A Framework for Mental Computation Strategies

Hartnett (2007) proposed a categorisation framework of mental computation strategies. The intention of the strategy categorisation was to create a small number of general categories with intuitive labels that would make sense to teachers and also to the students. A list of sub-categories made clearer the variations that could be a focus in each category. In all, five major categories and 21 sub-categories were identified (see Table 1). With the labels for the categories kept in simple language it was intended that these would be used in the classroom as the focus of lessons and to facilitate the discussion of strategies used by students. By presenting a coherent way of thinking about the possible mental computation strategies the teacher and the students would have a common language for discussions about strategies. This framework provides structure learning for activities, student thinking and strategy choice and also a structure for the recording of student thinking during computation activities and assessments.

Method

A focus class was chosen from a suburban Catholic primary school in Brisbane, Australia. Teachers at this school had a shown an interest in the development of mental computation strategies. The subjects were one class of 27 Year 3 students (8-9 year olds). A Year 3 class was chosen as the focus class due to this being the year when computation, particularly on two digit numbers, is a major topic of learning. The students had not had any exposure to the traditional written algorithms that are commonly introduced in this year level and often before.

The researcher planned for and taught one strategy development lesson each week based on Hartnett's (2007) strategy categories (see Table 1), and the classroom teacher followed this with further lessons during the week. The strategies were explained and referred to consistently by their category and sub-category names; for example, a series of lessons focussed on teaching the "breaking up two numbers using place value" strategy. The researcher and the class teacher modelled the recording of thinking during lessons. They also provided and modelled the use of structures for supporting computation strategies, for example, number boards, empty number lines, arrows indicating progression of thinking etc.

Table 1

Categorisation of Mental Computation Strategies (Hartnett, 2007)

Strategy category	Strategy sub-categories
Count On and Back	▪ Count on to add
	▪ Count back to subtract
	▪ Count on to subtract
	▪ Count on to multiply
Adjust and Compensate (Change and Fix)	▪ Adjust one number and compensate
	▪ Adjust two numbers and compensate
	▪ Adjust two numbers
Break Up Numbers	▪ Break up two numbers using place value
	▪ Break up two numbers using compatible nos.
	▪ Break up one number using place value
	▪ Break up one number using compatible nos.
Double and /or Halve	▪ Use a double or near double to add or subtract
	▪ Double to multiply by 2
	▪ Double, double to multiply by 4
	▪ Double, double, double to multiply by 8
	▪ Half to divide by 2
	▪ Half, half to divide by 4
	▪ Half, half, half to divide by 8
	▪ Double and halve
Use Place Value	▪ Think in multiples of ten
	▪ Focus on relevant places

Much of the research on strategy use in computation has focussed on students working with examples involving small numbers (to 20). In this study a deliberate focus was made on using larger numbers that were appropriate as a computational instruction focus for this year level. The focus operations were addition and subtraction on two and three digit numbers as outlined in the current Mathematics syllabus (Queensland Studies Authority, 2004). A variety of number combinations, including those requiring the bridging of ten (e.g., 19+12 and 100–36), were included as they suited a strategy focus for computation. They also provided a change in focus for teaching as traditionally in Year 3 examples requiring regrouping were left until after students could complete examples without regrouping when being taught the traditional written algorithms.

The students completed a pre- and post-test of a range of computation questions chosen to allow for a range of strategies to be used across both single and two-digit examples at the beginning and the end of a school year. They also completed mid-year assessments that included some items from the pre-test and some others. For this paper a comparison of the pre- and post-tests looking for evidence of recording of strategy choice, including use of the category labels, was the focus. The students were asked in both the pre- and post-tests to record their thinking so that someone reading their response would understand how they had worked out their answer.

To determine how the students had described the strategies they used their descriptions in the pre- and post-tests were reviewed for a change in the number of students who were able to clearly describe the strategy that they had used from the pre- to the post-tests. Special note was taken of students who used the actual strategy category names from the framework.

The class teacher was interviewed informally throughout the study and the researcher kept field notes of these discussions. She was also interviewed formally at the end of the study.

Results and Discussion

The students in the class chose to alter the “Adjust and Compensate” category to call it “Change and Fix” which they thought was a better description of the strategy. This showed that they felt comfortable with the labels and had a deeper understanding of their usage than just remembering the name.

In the pre-test many students only recorded an answer (58.2%). This was likely to be due to inexperience with this way of working and/or lack of the language to describe them. There was no way to deduce which strategy or strategies a student had utilised. In the pre-test 30% of the students made no response to pre-test questions and 12.5% of the students attempted to record their thinking. Of the students who attempted to record their thinking only 2% of these managed to make the explanation of their strategies clear. In the post-test 87% of the students made an attempt to record their strategies with 63% of these doing so in a way that was clearly understood. Although the number of students whose strategy recording was unclear also rose, along with the increase in attempted recordings this shows that a large proportion of the students were growing in confidence with their ability to record their thinking and strategies used.

Figure 1 summarises the change in the percentages for each type of response in the pre- and post-tests for the focus class.

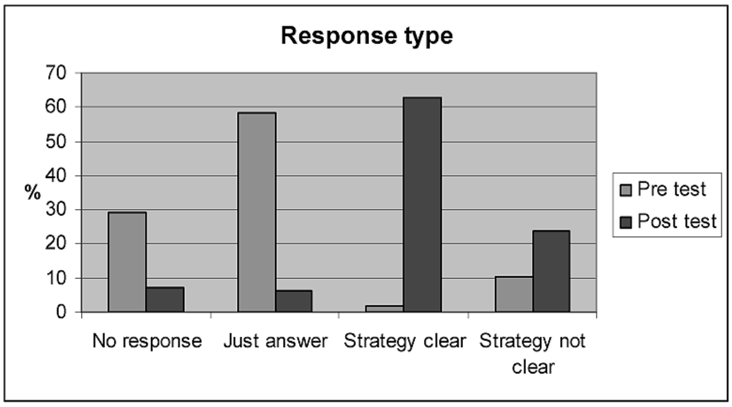

Figure1. Percentage of students who used each type of response in the pre- and post-tests.

The number of students who just provided the answer reduced from 58% in the pre-test to 6% in the post-test. The questions that elicited just responses in the post-test tended to be ones referred to as retrieval strategies (Siegler, 1987) and the students recorded “I just knew it” or similar beside their answer. Siegler (1988) noted that more knowledgeable students tend to use retrieval more often and to answer more quickly and accurately. This was the case in this study where the students who exhibited such responses tended to be the more capable students, identified by consistently high success with the questions on all tests and who had been identified as above average by the class teacher. For this study students who wrote such responses were categorised as providing just answers.

The descriptions that the students used to explain their strategies in the post-test were analysed according to whether they used the actual name of the strategy from the framework (see Figure 2); whether they used a method of showing the strategy used that had been demonstrated during the lessons, for example, empty number line (see Figure 3); or whether the strategy used was evident through a personal explanation that allowed the teacher to determine the strategy (see Figure 4).

Figure 2. Student use of categorisation framework labels.

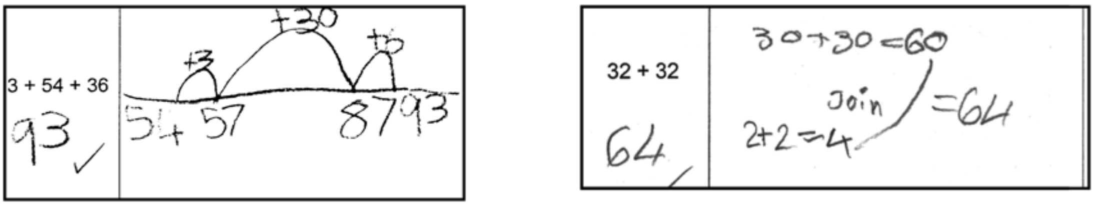

Figure 3. Use of recording matching as demonstrated or discussed during lessons.

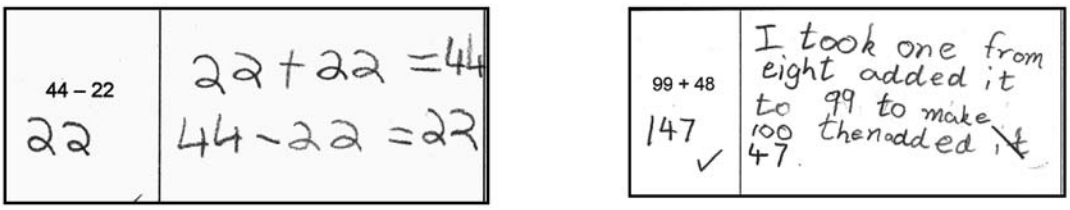

Figure 4. Student's own description of the strategy used.

Table 2 shows the percentage of students across all the questions in the post test who showed each of these descriptions. The remaining percentage was students who gave no response or just an answer.

Table 2

Variations in Strategy Descriptions by Focus Class Students

	Use of strategy categorisation labels	Strategy as demonstrated in lessons (without label)	Strategy obvious - student used own descriptions
Post-test	12.35%	63.43%	10.49%

Conclusions

The aim of this study was to trial the use of ways to capture the strategies used by students when completing mental computation questions. The results show that it is possible for young students to record their thinking and the strategies used so others can understand what they did. The use of the categorisation framework (Hartnett 2007) provided a structure for the teacher to organise learning activities for the students providing comprehensive coverage of a range of strategies appropriate to the numbers that were the focus of computation for this year level. The framework labels provided a structure for classroom dialogue about strategies used and the students were able to show this in their recording of their thinking. Over 75% of post-test strategy descriptions used the framework labels or strategy applications that had been demonstrated in lessons. The use of the strategy framework labels was never a requirement for the students but the number of the students who used them without prompting shows an understanding of the framework and that they must have felt this would help explain what they had done.

The teacher was pleased to have "hard" evidence of the students' understandings that she was able to keep and refer to for planning further lessons, discussing with the students, making assessment judgements, and reporting to parents. She was able to see development of strategies and concepts across the year and was able to identify misunderstandings in the methods some of the students were using. An example of this remedial use of the student descriptions was when some students had recorded just an answer of 10 for 32–18 in the mid-year test. The strategy recording provided by one student (See Figure 5), allowed the teacher to identify the misconception. The students had been wrongly applying the "Breaking up two numbers using place value" strategy in the subtraction. After discussion with these students and consequently the whole class, the error was clarified and deeper understanding of the potential difficulties with the application of this strategy particularly for subtraction was discussed and alternative strategies were proposed.

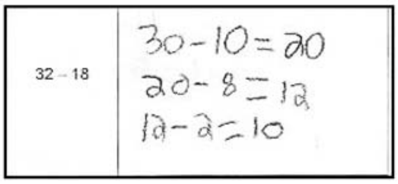

Figure 5. Evidence of a misconception about a strategy in a student's recording.

In this study, the students were given access to knowledge that is usually provided only for teachers – namely the strategy categories. They coped very well with this and the teacher commented that she thought they seemed "proud of themselves" to be able to use what they considered teacher talk with her and the researcher.

Implications for Further Research

The study was only for one year and as such there were limitations in the internalisation of the strategies by the students. The students were really only beginning to gain familiarisation with computation with two-digit and larger numbers as well as with the framework. It would be interesting to follow students who have worked with the strategy categories for all of primary school noting differences in strategy use and thinking as evidenced through their recordings across all operations and with other numbers, for example, decimals.

The school involved has contracted to work with the researcher to develop a whole school approach to teaching computation strategies based on this work. With further scaffolding of strategies in future years it is hoped that more of the students would be likely to develop proficiency at justifying their thinking and communicating their methods in writing thus providing assessment data of this nature for the teacher.

The use of the strategy category framework to further examine communication about thinking and to guide student recording of their thinking and strategies could be examined in relation to student to student communication.

The recording of thinking in this study focussed on capturing computation strategies. There is scope for the capturing of other thinking strategies like those used with problem solving strategies especially if a framework for these strategies was part of the instruction.

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A Review of Recent Research in Early Mathematics Learning and Technology

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The proliferation of technological tools in Australian mathematics classrooms has not been well supported by evidence-based research, particularly in early mathematics learning. This paper reports two stages of document analysis; a review of recent meta-analyses in early mathematics education and technology, and a quantitative analysis of research published in selected mathematics education research journals over the last five years. The initial review highlights potential affordances of technology for mathematics learning predominantly with older students. The quantitative analysis supports this finding but highlights the limited quantity and scope of publications focused on mathematics learning and technology with young children.

Technology has been lauded as a potential tool to improve mathematics learning; and technological tools, in all forms, are becoming more prevalent in many Australian classrooms. Lynch's definition of technology is utilised here to incorporate all "electronic computing media" (2006, p. 30). The theoretical framework of Hoyles and Noss (2003) centres on the impact of digital technologies that have the potential to alter and enhance students' cognitive infrastructure. There is a growing research field that investigates the advantages of technology use to enhance mathematics learning (Gutiérrez & Boero, 2006). While studies on the use of technology in mathematics learning have explored the pedagogical ramifications and outcomes for older students, few studies have examined technology use in early mathematics education. Perry and Dockett (2007) describe a recent surge in early childhood mathematics education research. However, other reports assert that there is an absence of studies focused on the role of technology (Groves, Mousley, & Forgasz, 2006; Mulligan & Vergnaud, 2006; Perry & Dockett, 2004).

This paper originated from a review of recent, pertinent, international journal articles that examined the impact of technology in early mathematics learning and repeated claims lamenting a paucity of research in this area (Clements & Sarama, 2003; Yelland, 2000, 2005). Research in early mathematics learning and technology is scant and so judgements about potential affordances in mathematics instruction are, to a large extent, purely speculative. This paper seeks to quantify and systematically account for the proportion and scope of articles dedicated to early mathematics learning and technology, profiled in international mathematics education journals, as identified by Australian researchers. This review aims to inform the design and scope of further research in this area, and to situate Australasian research within an international context.

Previous Reviews

An analysis of several previous reviews of research in early childhood mathematics education, educational technology in early learning and/or early mathematics learning with technology highlight three recurring themes: (i) calculator use and effectiveness, (ii) computer use and effectiveness and (iii) research in early childhood mathematics learning.

1. Calculator Use and Effectiveness

There is a significant corpus of research devoted to exploring the implementation and effectiveness of sustained calculator use on early mathematics learning (Groves et al., 2006; Perry & Dockett, 2004). A surge of Australian research has documented the potential for calculators to allow profound changes in teaching and learning mathematics. However, more recent research has suggested that studies of actual classroom implementation are limited (Groves et al., 2006). Scrutiny of previous meta-analyses suggests that the initial euphoria surrounding calculator use has not been sustained, as the most recent study cited in Groves et al. (2006) was conducted in 2000.

2. Computer Use and Effectiveness

A common theme emerging from meta-analyses examining technology use and mathematics is the crucial role of the teacher and accompanying pedagogy which support effective technology integration (Groves et al., 2006; Laborde, Kynigos, Hollebrands, & Strasser, 2006; Yelland, 2005). Regardless of the technology used, appropriate teacher intervention has been consistently identified as an essential element for successful mathematics learning. It has been suggested that technology, per se, does not improve student learning. It is the curriculum in which it is embedded, and the accompanying pedagogy, which may determine the ultimate effectiveness of technology implementation in mathematics classrooms.

Screen-based tools such as Logo, Microworlds and dynamic geometry environments have been identified as applications in which there has been significant research in mathematics education (Ferrara, Pratt, & Robutti, 2006). These tools afford representational expression and shape mathematics learning, but there are few studies that describe the representational processes of young learners. Plowman and Stephen (2005) echo this shortcoming in existing research, sighting that much of this work is confined to screen-based technology. Most studies have examined the role of screen-based tools with students in elementary and secondary school but there are few studies with younger students.

Previous reviews of research, examining the use of technology in mathematics learning, have been limited in scope to particular mathematics domains. It appears that geometry, algebra and calculus have been well researched as domains exploiting the potential affordances of technology. These studies suggest technology may have a positive impact on student learning outcomes in specific domains (Laborde et al., 2006; Yelland, 2005). However, these meta-analyses and reviews have not articulated whether technological tools may also enhance learning outcomes in other mathematical areas such as rational number and measurement.

Computer games have also been identified as potential tools for enhancing mathematics instruction (Perry & Dockett, 2004). The permeation of educational software and web-based resources designed specifically for mathematics instruction is striking, yet there is a notable absence of studies which evaluate the impact of this software on early mathematical learning. Research is required to review the range of software applications used in classrooms and to assess their effectiveness in capturing and conveying mathematical content (Groves et al., 2006). New technologies, particularly games-based environments, may alter learning trajectories for young learners, but research has not validated this claim (Perry & Dockett, 2004).

3. Research in Early Childhood Mathematics Learning

The reviews of research, specifically related to early mathematics learning, are dominated by studies exploring the domain of numeracy. There are a plethora of studies, in Australian and New Zealand, documenting the implementation of systematic numeracy initiatives with young learners, such as Count Me In Too and First Steps (Perry & Dockett, 2004).

These reviews continue to indicate that the use and impact of technology in early mathematics learning does not appear to have been a widely researched area (Fox, 2007; Groves et al., 2006; Mulligan & Vergnaud, 2006; Perry & Dockett, 2004, 2007). This is despite an overall surge in early childhood and neuro-scientific research, with recent policy and curricula reflecting the view that young children are capable learners (Clements & Sarama, 2007a; Fox, 2007). In summary, previous meta-analyses and reviews conclude that further research is needed to explore the role of technology in young children's development of mathematical concepts.

Method

The first stage of this document analysis involved an evaluation of previous reviews and meta-analyses of mathematics education research published within the last five years. Common themes were identified and are summarized in the preceding section. The second stage comprised an analysis of research published in five international mathematics education research journals over the same time frame (January 2003-November 2007). The five journals selected were for analysis were identified by Australian mathematics researchers as significant The five significant mathematics education research journals selected for analysis were: 1) Educational Studies in Mathematics; An International Journal; 2) Journal for Research in Mathematics Education; 3) For the Learning of Mathematics; An International Journal of Mathematics Education;4) Mathematics Education Research Journal (Journal of the Mathematics Education Research Group of Australasia Inc.); and 5) The Journal of Mathematical Behavior.

The authors chose to investigate only those articles published during the last five years from each of these journals, both to ensure currency of research and for pragmatic reasons. The five-year span also enabled any emerging trends and current developments in technology use to manifest.

Three phases of analysis allowed articles to be systematically examined and categorized (editorial comments, book reviews and letters to the editor were not included). The first phase of analysis focused on each journal index separately and then sorted articles by title, abstract and keywords. In this phase, each article was categorized according to age group and whether technology was a focus. This enabled the researchers to compile a succinct list for further analysis in the second phase. The second phase of the analysis identified those articles that focused on early childhood education (in Australia, this is from birth to eight years). These were examined more closely for identification and coding of key mathematical concepts and processes. The third phase analysed those articles incorporating technology as a learning tool and identified the specific technologies employed in each study.

Discussion

The following section provides an overview of four key findings arising from the second stage of this analysis.

1. Age Group Categorization

In total, 512 journal articles were reviewed. Of these articles only seven (1.4%) investigated mathematical learning processes of children prior to school age. Another 34 (6.6%) articles included children in their first three years of school. However, of this group of articles 29 (5.7%) reported studies of children included in a larger cohort with older students as subjects of research. In relation to the entire body of articles those investigating mathematics learning of young children is disproportionately low (see Table 1) and warrants further investigation.

These findings confirm assertions made in previous meta-analyses (Fox, 2007; Groves et al., 2006; Mulligan & Vergnaud, 2006; Perry & Dockett, 2004, 2007) that research on young children's early mathematics learning is under represented. Other research (Baroody & Lai, 2005; Clements & Sarama, 2007) highlights increased emphasis on the fundamental importance of early mathematics learning; yet this is not reflected in the selected mathematics education research journals within the given time frame. It is possible that the publication of such research is disseminated in early childhood, technology education and/or generic education journals but this work was not prominent in either the current analysis or previous meta-analyses.

Table 1

Age group of participants represented in journal articles (n=512)

Age group of participants	Percentage of total (n=512)
Children prior to school (aged 0 - 5)	1.4%
Children in the first three years of school (aged 5-8)	6.6%
Children in the first three years of school (aged 5-8) as part of a larger cohort	5.7%
Children in the first three years of school (aged 5-8) as sole cohort	0.97%

2. Mathematical Content Domains

A review of the studies conducted with young children identified numeracy as the most commonly reported mathematical domain, supporting the finding of previous meta-analyses (Mulligan & Vergnaud, 2006; Perry & Dockett, 2004, 2007). These studies focused on cardinality, counting, number concepts, addition, subtraction, multiplicative thinking and rational number. Another frequently reported area of investigation related to teaching, curriculum and/or assessment issues. A third domain frequently discussed was problem solving. The focus on curriculum and problem solving is in contrast to the findings of previous reviews (Yelland, 2005). Other themes evident, albeit limited in frequency, include generalized mathematical processes, mathematical language and discourse analysis, and spatial and geometry concepts. Of concern was the lack of research investigating young children's use of technological tools in mathematics teaching and learning. Whilst these findings may reflect the emphases of the particular journals and/or the agendas driving international policy and curriculum reform, it is clear that a wider range of studies is needed.

3. Technology in Mathematics Education Research

Of the 512 articles, approximately 10% (n=51), identified educational technology as a focus of investigation. These data represent varied groups and only a few (n=4) relate solely to research with young children. These four articles described the potential affordances of screen-based technologies such as Building Blocks (Clements & Sarama, 2007b), the Ameritech Classroom (Davis & Hyun, 2005), graphical application software (Åberg-Bengtsson, 2006) and a technology games-based environment (Lowrie, 2005). An analysis of the types of technology indicate a dominance of screen-based software, a finding which is consistent with the work of Plowman and Stephen (2005). In these journals thirty-nine of the 51 articles with a technology focus utilized screen-based technologies. These studies predominantly dealt with older students. This perpetuates the trends evident in previous reviews where older students were the focus of research (Ferrara, Pratt, & Robutti, 2006; Laborde et al., 2006). These focus areas included but were not limited to, Geometer's Sketchpad and other dynamic geometry programs, graphing software, spreadsheets, computer-algebra systems (CAS) and web-based resources. Eleven of the articles specifically investigated the use of calculators with a dominance of research on graphic calculator use. This may be attributed to the rising popularity and affordability of these tools.

4. Trends Over the Five-year Period

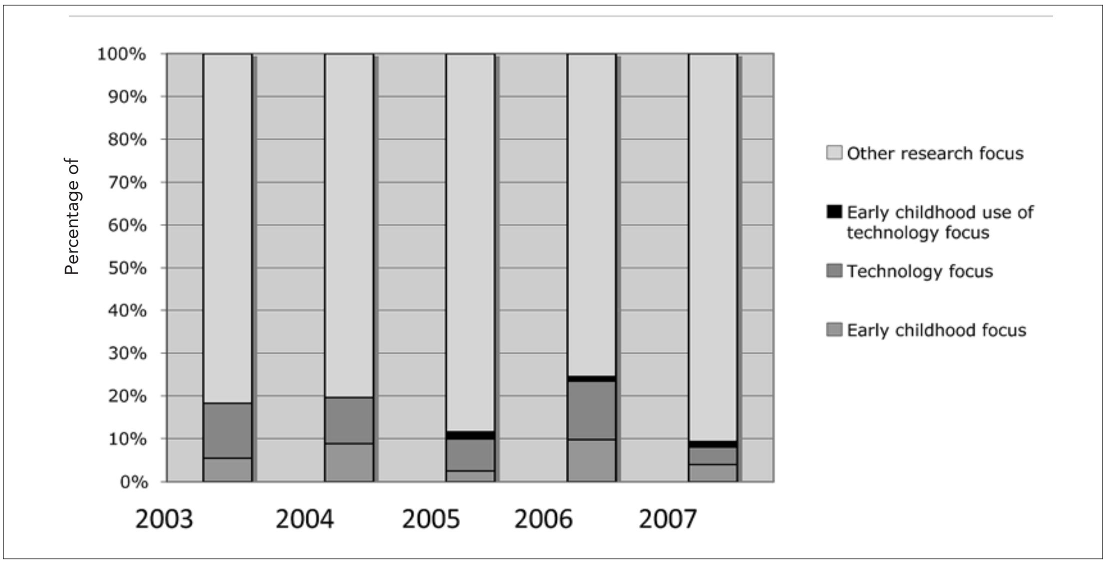

Figure 1: Percentage of articles published in selected mathematics education research journals with an early childhood and technology focus.

Whilst, as suggested by Perry and Dockett (2007), there is some growth in research that focuses on young children's mathematical development and technology, these results suggest that it would be premature to propose any emerging trends over the period investigated in these journals. Figure 1 (above) shows the proportion of articles pertinent to early mathematics learning with technology. Again, these articles utilized screen-based technologies, corroborating findings from previous reviews (Ferrara et al., 2006). It is promising to note that there has been some research published in this area over the last three years. In Australia, there is currently a strong government agenda resulting in the rapid implementation of technological tools in classrooms. However, this agenda is not supported by a sufficient research base.

Conclusions, Limitations and Implications for Further Research

This review is limited by both the number of journals selected for review and the subjective nature of selection. The identification of these significant journals by Australian mathematics education researchers is to some extent subjective. Further, the exclusive selection of articles published in English limits the breadth of investigation. The time frame for analysis was the recent five-year period and any pertinent articles or emerging trends prior to this were omitted from the second stage of analysis.

It is clear from these data that there is a shortage of research pertaining to young children's mathematics in the selected journals, which is even more pronounced when technology is the modality for learning. However, despite these findings it is not possible from these data to accurately speculate on the reasons behind this paucity. A review of the broader research corpus in early childhood does provide some insight and suggests that there may be several factors that may account for this lack of published research. It is plausible that research investigating the use and effectiveness of technology in early mathematics learning has been conducted, but published elsewhere. There are multiple avenues for research of this nature to be published: results may have been disseminated in early childhood journals or technology journals. A proposed, expanded review of journals would confirm or reject this postulation. Broadening the present analysis to include early childhood journals and technology education journals will provide opportunity for a fuller review and potentially insight into the reasons behind the limited research in this area.

It is also possible that the absence of articles pertaining to early mathematics learning and technology may be a direct result of a lack of research being conducted in this area. There has been an historical reluctance in the early childhood field to embrace digital technologies and this has translated in the disappointing uptake of technology in prior-to-school settings and the early years of formal schooling (Cordes & Miller, 2000; Dwyer, 2007). If the available technologies are not being harnessed in classroom settings, then it is unlikely that research would be conducted on their use. This is surprising for two reasons: first, technological tools are permeating schools and to a lesser extent, prior-to-school facilities and second, the early years have been identified as a crucial stage of learning. There is a growing body of empirical research which clearly delineates the importance of the early years in terms of cognitive development (Aubrey, Dahl, & Godfret, 2006; Ginsburg, 2002) and studies suggesting that young learners are capable and competent mathematicians (Baroody, 2004; Perry & Dockett, 2002). Perhaps, one of the difficulties faced by researchers is the limited opportunity to engage in investigations with young children who are using technological tools in naturalistic settings. Young children may also be reluctant to articulate their thinking when they are focused on activity with technological tools.

In summation, an analysis of the selected mathematics education research journals provides quantitative evidence to support claims that there is a paucity of research examining early mathematics learning with technology. At an international level this is reflected in previous meta-analyses and echoed by publications of the PME group (Mulligan & Vergnaud, 2006). These findings have implications for both research development and dissemination in early mathematics education. The analysis calls for new research agendas and supports current work conducted at the Centre for Research in Mathematics and Science Education (CRiMSE) at Macquarie University. Here, a suite of new studies on young children's early mathematical development and the use of technology, such as programmable toys, dynamic interactive software and interactive whiteboards, are in progress.

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The Development of Students' Use of Justification Strategies

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This paper examines young students' development and use of justification strategies when engaged in numeric patterning activities. Drawing on findings from a three-month teaching experiment aimed to improve students' early algebraic understanding we show how student use of justification strategies can be extended through the use of specifically designed tasks and pedagogical actions. In particular, we examine how students' ability to participate in mathematical argumentation can be supported through use of justification which triangulates numeric, verbal and visual strategies.

Introduction

Difficulties in developing student understanding of algebraic concepts are well-documented in New Zealand and internationally (Irwin & Britt, 2005; Knuth, Stephens, McNeil, & Alibabi, 2006). A key difficulty identified within recent research studies relates to the need for students to justify generalisations using arguments which support their development of mathematical proof (Carpenter, Levi, Berman, & Pligge, 2005). Many researchers (e.g., Blanton & Kaput, 2005; Carraher, Schliemann, Brizuela, & Earnest, 2006) argue that student participation in practices of conjecturing, generalising, and justifying mathematical reasoning are fundamental to the development of foundations of algebraic reasoning. The recent New Zealand curriculum document advocates that students learn to "justify and verify, and to seek patterns and generalisations" (Ministry of Education (MoE), 2007, p. 26). However, developing such practices in mathematics classrooms is challenging for many teachers and students, particularly because they may not have previously experienced classrooms in which justification of mathematical reasoning takes a central role. The research reported in this paper examines how a classroom environment and the teachers' pedagogical actions facilitate primary school students to use a range of justifications to develop generalisations of functional patterns.

Teachers take a significant role in scaffolding the specific questions and prompts which move students from explaining their solution strategies to justifying, defending, and generalising their solution strategies (Hunter, 2007). Justification is a critical element of the generalisation process. However, determining the legitimacy of a general statement is a demanding task for young students (Lannin, 2005). Most often, studies have found that elementary students initially view specific examples or trying a number of cases as valid justification (Carpenter et al., 2005; Lannin). Research has also shown, however, that in classrooms where students have opportunities to participate in mathematical argumentation and justification, the quality of students' reasoning, explanations, and justification can be enhanced (Manoucheri & St John, 2006; McCrone, 2005; Wood, Williams, & McNeal, 2006). Through establishing classroom social and socio-mathematical norms which require students to justify through triangulation of verbal, numerical and graphical strategies, students can be supported to learn ways to provide further, more sophisticated forms of justification (Kazemi, 1998).

Within this process, research has revealed the importance of students learning to make connections across representations. In particular, the use of visual and numeric patterns has been found to support students to identify, communicate, and justify functional rules. For example, Carpenter and his colleagues illustrated in a 6th grade classroom that students could be scaffolded to provide concrete justification through the use of materials. Other studies (e.g., Beatty & Moss, 2006; Healy & Hoyles, 1999) highlight the importance of teachers drawing student attention to the visual representations of functional patterns. This, coupled with a teacher press for students to communicate and justify their generalisations using the geometric context, resulted in a more robust student understanding of functions and ability to find, express, and justify functional rules.

In our study, drawing on the emergent perspective of Cobb (1995), students' mathematical learning is recognised as both an individual constructive process and as social negotiation of meaning; neither is given more significance than the other. Combining both Piagetian and Vygotskian notions of cognitive development the person, cultural, and social factors are all viewed as important features within the students' learning environment. From this theoretical perspective we attempt "to offer a developing picture of what it looks like for a teacher's practice to cultivate students' algebraic reasoning skills in robust ways" (Blanton & Kaput, 2005, p. 440) through specific focus on the development of justification strategies.

Method

The findings reported in this paper are one section of a larger study (Hunter, 2007) which involved a 3-month classroom teaching experiment (Cobb, 2000). The research was conducted at a New Zealand urban primary school. The 25 participant students were between 9 and 11 years old. The student group came from a predominantly middle socio-economic home environment and included a range of ethnic backgrounds.

All students participated in pre- and post-interviews. Working as collaborative partners, the researcher and teacher used data gathered from the pre-interview to develop an initial sequence of learning activities informed by a hypothetical learning trajectory focused on developing early algebraic understanding. The learning trajectory was used to establish learning activities involving tasks and participatory practices and associated learning climate that required students to make conjectures, justifications, and generalisations. The initial tasks and activities focused students on exploration of the properties of number and associated computations. The students were then provided with problems designed to develop algebraic reasoning through the use of linear functional problems and patterning activities including tasks with a geometric context.

Throughout the teaching experiment data were generated and collected through participant observations, video records and classroom artefacts. Subsequent on-going data analysis and collaborative examination of classroom practices by the researcher and teacher led to modification of the instructional sequence. Findings of the one classroom case study were based on retrospective data analysis that used a grounded approach identifying categories, codes, patterns, and themes.

Results and Discussion

Before we focus on the nature of the teaching experiment with regard to developing students' understanding of functional relationships, we first present a summary of the pre- and post-test results of participants' understanding of functional relationships. Table 1 overviews the number of students able to use a functional relationship[3] at the first interview conducted at the start of the classroom study.

Table 1

Percentage of Students (n=25) Correctly Using the Functional Relationship

	Correct response	Incorrect response	No response
Part A	40%	52%	8%
Part B	28%	52%	20%

Comparable data from the final interview (see Table 2) indicate that there was an increase in the number of students able to correctly use the functional relationship to solve the problem—thus demonstrating impact in terms of performance measures.

Table 2

Percentage of Students (n=25) Correctly Using the Functional Relationship

	Correct response	Incorrect response	No response
Part A	88%	12%	0%
Part B	84%	12%	4%

The focus of this paper is to explore one aspect of how such learning was occasioned. Tasks were designed

3 1) To make copies of a CD, a store charges a set-up fee and an additional amount per CD. The store charges $2 as a set-up fee and an additional $3 for each copy.

A) What is the cost to make 10 copies of a CD?

B) What is the cost to make 21 copies of a CD?

and selected to promote students' algebraic reasoning through the explicit use of numerical, verbal, and visual schema and to support their development of justification. Of equal importance to the design of the tasks were the planned pedagogical interventions. In the next section, we consider the pedagogical practices—involving teacher scaffolding, modelling, and a press for all students to interact and justify their reasoning through argumentation—evidenced in the study.

Developing Justification Beyond Numeric Examples

Initially, many students maintained a view that using lots of different numbers justified their reasoning. For example, when challenged by the teacher Hayden explained:

> Hayden: We tested it with this number, it works, this number, it works and we are just finishing this one and so far it is working.

To shift students beyond the use of multiple examples the teacher required that they integrate visual and numeric schema and justify their generalisations using the geometric problem representation. In a large group discussion Hamish explained his group's generalisation with reference to the problem context:

> Hamish: Thirty-two people sit at the table … you get the ten and times it by three and the two people who are sitting on those ends, one of them stays there and the other one gets moved to the end of the new table.

The teacher pressed him to justify his contextual explanation using the geometric pattern. In doing this, she demonstrated how to justify the link between the functional rule and the geometric representation through modelling (see Figure 1).

> Teacher: Hamish can you show … the times three part of your model there and the plus two part? … This is the first table here so I'm going to use blue for the times three and the plus two here [uses pink counters on each end] … so two tables, two groups of three do you see that? [Points to blue counters.] Plus two [points to pink counters].

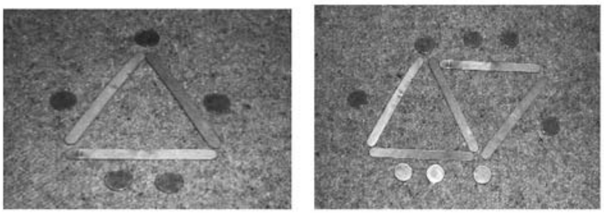

Figure 1. Using a geometric representation to justify a functional rule.

Positioning Students to Justify Algebraic Reasoning

The teacher modelled how participants were required to justify their reasoning. She explicitly positioned the students to take a stance to explain and justify their reasoning and provided space for participants to agree or disagree mathematically. For example, in a large group discussion Mike stated an incorrect generalisation for a problem[4]:

> Mike: You would times it by five then you would minus one because of the six … it would be one over so you would have to minus this to make it fair.

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4 Jasmine and Cameron are playing "Happy houses". They have to build a house and add onto it. The first one looks like this….. /\
|_|
The second building project looks like this… /\/\
|_|_|
How many sticks would you need to build four houses? How many sticks would you need to build eight houses? Can you find a pattern and a rule?

The teacher revoiced but did not evaluate the conjecture:

Teacher: [revoices] Mike said the number of houses times five minus one because of the six you get at the start. Does everyone agree? Could someone show us why or why not you agree?

The teacher, by asking the students to take a position to agree or disagree and provide convincing evidence of their stance, effectively increased the press for justification. Subsequently, a student argued her position through drawing a house that acted as a referent for her generalisation:

Ruby: [draws a house] That is one house and if you added another one, that is always going to be a six when you times it by five you would actually add one because you have timesed that by five and it's still a six so you would add it on.

This provided an avenue for the teacher to step in as a participant and provide a counter argument that modelled how to justify a position using equipment:

Teacher: [builds representation of two houses] I could show you another way why it doesn't work. Now I have to times by five and two times five is ten, now if I take it away I am going to have an incomplete house. I have to add one so that is my two times five, to make it complete I need to add one.

Through teacher modelling, a sustained press to explain and justify reasoning, and explicit space provided to rethink conjectures, the students recognised that they were required to not only explain and justify their reasoning but also to be prepared to validate it. This was exemplified when a group of students recognised an emerging pattern and one member challenged:

Susan: We have got to make proof that it actually works because if you think it is eleven but you don't know that it is eleven.

Increasingly, as the teaching sequence progressed, students demonstrated awareness of the need for explanatory justification to extend beyond the provision of multiple examples. This was further illustrated in a lesson when a student argued that justification required more than multiple examples:

Ruby: It's not really saying anything … do you think it would be more convincing if we used equipment [for our explanation]?

The argument presented by Ruby pressed other group members to shift towards connecting the visual pattern and functional generalisation.

Triangulating Numeric, Verbal, and Visual Strategies to Justify Functional Generalisations

The previous exemplars illustrated that concrete materials, as well as inscriptions of mathematical reasoning, were important thinking and communicating tools. The teacher requirement that the students triangulate their numerical, verbal, and graphical strategies appeared to become an accepted practice—a practice that facilitated more sophisticated and shared understanding of the justification process. For example, she required that they link their use of a t-chart with visual representations when justifying their explanations:

Teacher: You need to be able to show and prove that by drawing a table and using equipment.

On several occasions the teacher was also observed emphasising the importance of constructing links between the numeric pattern in the t-chart and the geometric models. During student provided explanation she frequently drew attention to how the explainer had linked the numeric and geometric patterns as illustrated in the following response:

Teacher: This is really important this part, what he is going to show you will help you make the links between that and the table of data.

The combination of the use of the designed linear functional problems involving geometric patterns and the teacher press prompted students to use equipment to support their justifying processes. Furthermore, requiring students to link numeric and geometric patterns encouraged them to construct reasoned explanations which incorporated multiple representations. This is illustrated in a large group discussion when Ruby justifies her explicit generalisation of the house problem using a geometric pattern:

Ruby: [builds model] The first one is six but then when you add another house it is only five because you don’t need another wall … if you wanted to see how many for eight you could just go eight times five and then plus the one, you are plusing the one because you have to still understand that that is six [points to first house].

Again in a later lesson, Josie and Matthew repeatedly refer to the geometric context of the problem to convince their other group members of their explicit generalisation and to justify their reasoning. When other group members remain unconvinced, they use multiple forms to justify their reasoning. After examining the model (see Figure 2), Josie begins by pointing with her hands to the already drawn model:

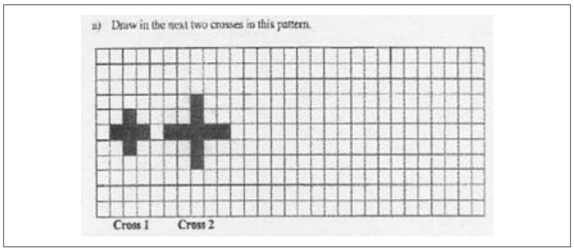

Figure 2. Cross problem.

Josie: This is cross one. There is one on each side plus one in the middle. This is cross two, so two here and two here and one in the middle so that makes five. So you double it and then add one to get the number across.

Matthew builds on it using a visualised representation to explain how to find the number of squares across for cross twenty:

Matthew: [Indicates a vertical line with his hands] It would be twenty and twenty plus one so that would be forty-one.

However, the other group members Steve and Rani remain unconvinced and continue to press for justification:

Steve: So why does it equal forty-one?

Josie and Matthew again refer to the already drawn model to provide additional justification for their explicit generalisation:

Matthew: [points to each side] Because like cross one has one there and one there.

Josie: Plus one in the middle.

Matthew: Two has two there and two there and one in the middle and three has three there and three there plus one.

The questioning stance taken by other group members prompts Matthew to redraw two models using different colours (see Figure 3):

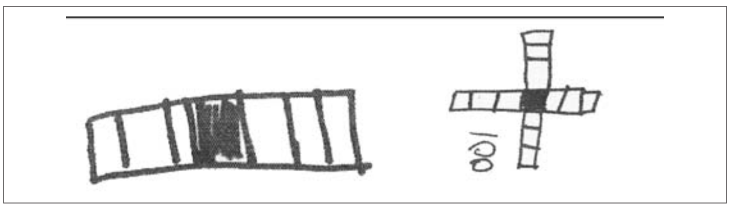

Figure 3. Matthew’s justification for his generalisation.

Matthew: One, two, three and the one in the middle is black. So this is for cross three. Right, have you noticed a pattern yet?

Steve: No I haven't.

Matthew: Cross three has got three, cross two has got two on the outside.

Josie: Each part that sticks out has the amount, the number of the cross then there is one in the middle to join it up … it is the number of the cross doubled and then you add one in the middle to get that amount there [covers the model so only the horizontal row is showing].

The participants pause to examine the explanation; however Steve and Rani recognise their right to question until convinced. They reframe their questions as they analyse the reasoning:

Steve: So when you double it, what are you actually trying to get to by doubling it? ...

Josie: [covers the vertical row so only the horizontal row is visible] The number of squares in that line there.

Rani looks closely at the inscription and then she revoices the generalisation using the model of the cross to clarify her understanding.

Rani: [points to one arm of the cross] so you double that and add one.

Josie: You double the number of the cross and add one.

Rani: [points to the arms of the cross] So, you double these?

Josie: [points to right horizontal arm] That little bit here [points to upwards vertical arm] this little bit here is also three squares wide and this is three squares wide [points to left horizontal arm] and that is three squares wide [points to downwards vertical arm] so to get the bit across here in the middle [points to the horizontal row] you do times two plus one.

At Josie's statement all the students nod in agreement. Through the extended analytical discussion the students illustrated that they recognised the need to determine the legitimacy of the generalised reasoning beyond immediate justification.

Conclusion and Implications

This study sought to explore how students were supported to use justification to develop rich understandings of early algebraic reasoning. In particular, focus was placed on how students can be scaffolded to use justification strategies which triangulate numeric, verbal, and visual strategies. Similar to the findings of other researchers (e.g., Carpenter et al., 2005; Lannin, 2005), many of the students initially viewed trying a number of cases as valid justification. Specially designed tasks coupled with the purposeful teacher actions led to students developing understanding of the need to justify beyond examples and determine the legitimacy of the justification.

As other researchers have previously described (Manoucheri & St John, 2006; McCrone, 2005; Wood et al., 2006), opportunities to engage in argumentation and justification enhanced both the quality of students' reasoning, and their ability to determine the validity of a justification. The small sample of emblematic learning activities which were used in the teaching experiment demonstrate how the activities involving functional problems with a geometric base, in combination with specific pedagogical actions, can successfully extend student communication of justification strategies. The task design involving geometric patterns and use of multiple representations provided many opportunities for students to explore, justify, and validate their reasoning using equipment. A further pedagogical press from the teacher including questioning and positioning of students and the requirement that they take a stance led to them justifying their reasoning through an increased use of concrete materials, inscriptions, and geometric representations.

Implications of this study suggest that a triangulated approach of teacher press for justification, task design, and use of materials scaffold students to develop multiple forms of justification. However, further research is required to validate the findings of this study due to the small sample of participants involved.

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Using Task-Based Interviews to Assess Mathematical Thinking of Primary School Students

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This paper describes part of a larger study of an intervention program involving strategies designed to develop the mathematical thinking of upper primary students. It considers the effectiveness of task-based interviews in identifying the extent to which different modes or levels of thinking were used by eight students in a multiple case study following the implementation of an intervention program. The modes of thinking are based on mathematical, contextual and strategic knowledge.

Introduction

There is no universally accepted definition for numeracy though the myriad attempts to describe it have a number of common features. It has been equated with the idea of 'statistical literacy' (Watson, 1995, 2004), 'quantitative literacy' (Steen, 2001), 'mathematical literacy' (Steen, 2001), and 'realistic mathematics education' (van den Heuvel-Panhuizen, 2001). As well, it has been described in the *Numeracy Framework* (Morony, Hogan & Thornton, 2004; Willis, 1998) that outlined three modes of thinking, or components of numerate behaviour, and these were:

- Mathematical knowledge – the skills, techniques and concepts necessary to solve quantitative problems encountered in a real context.
- Contextual knowledge – an awareness and knowledge of how the context impacted on the mathematics being used.
- Strategic knowledge – the confidence, disposition and skills to find out what needs to be known in order to act numerately (Morony et al., 2004).

All of these descriptions of numeracy have in common the notion that students are able to understand mathematical ideas in various contexts and to apply those ideas to learn more about the context in which they are embedded.

Theoretical Framework

The study on which this paper is based (Hurst, 2007) interpreted the *Numeracy Framework's* description to develop indicators for each of the three modes of thinking that characterise numerate behaviour. Each mode of thinking is considered to be evident when a student exhibited some of the following respective sets of criteria.

Mathematical knowledge

- Recalls or identifies specific items of mathematical information.
- Recognises and reiterates examples of mathematical information.
- Uses statistical information to suggest or perform a mathematical operation.
- Poses questions that require the use of a mathematical operation.

Contextual knowledge

- Interprets specific data contained in the context and/or poses questions that require the interpretation of specific mathematical data.
- Describes, in one's own words, the main mathematically related ideas contained in the contextual information.
- Interpolates or extrapolates from aspects of the data.
- Poses questions that suggest connections between different aspects of the data and uses data to explain such connections.

Strategic knowledge

- Predicts or suggests how the data could be used to develop a new idea.
- Develops a scheme or method for representing the data, other than that already presented.
- Evaluates aspects of the data for consistency and validity and/or identifies anomalies in the data or misleading information.
- Evaluates aspects of the data to clarify related issues and make decisions.
- Poses or responds to questions which require substantial evaluation of aspects of the data.

The original study was based on the notion that there exists a cyclical relationship between the three modes of thinking. That is, a student may approach a situation with embedded mathematical ideas by initially using mathematical knowledge in an attempt to primarily understand the nature of the context. Alternatively, the situation could be approached with contextual thinking where a student might have experience with the particular context and use some of the embedded mathematics to connect his/her understanding with related aspects of the context. Finally, an experienced user of mathematics might exercise strategic thinking and immediately identify an apparent anomaly in some embedded data, or a misuse of that data, and then use mathematical knowledge to prove how that was so. Figure 1 reflects this cyclical relationship.

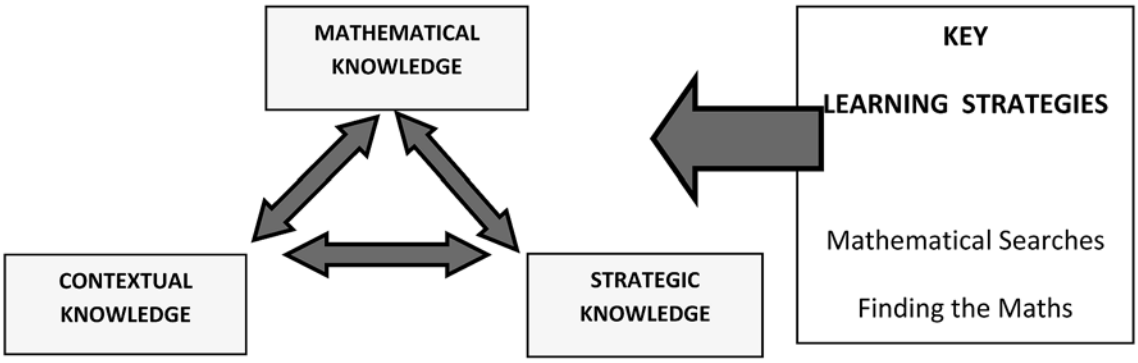

Figure 1: Cyclical relationship between modes of thinking.

Design and Methodology

The original qualitative study investigated the effect of using key learning and teaching strategies as shown in Figure 1 and sought to answer the following research question:

To what extent does the Mathematical Search enhance student capacity to recognise mathematical ideas embedded in a written context, and to display contextual and strategic thinking about mathematical ideas embedded in written contexts?

One of the data gathering tools used was the task-based interview. Task-based interviews were chosen as the main data gathering strategy for the original project because it was felt that the potentially 'data rich' environment they afforded would provide the best context for assessing and probing for the presence of the three modes of thinking both before and following the intervention phase of the project. As well, task-based interviews reflected the same type of contexts containing embedded mathematical ideas as did the key strategies used during the intervention phase. These strategies were:

- Mathematical Search – Students were given pieces of text and were asked to identify and describe the embedded mathematics, to say how the embedded mathematics helped them to understand aspects of the text, and to develop questions that could be asked using the embedded mathematical information.
- Finding the Maths – Students were asked to collect samples of published material (newspaper articles, brochures etc.) containing mathematical ideas and perform the same type of analysis as with the Mathematical Search.

During these interviews conducted at the beginning and end of the data gathering phase, students were shown a range of artefacts containing embedded mathematical concepts. These artefacts included maps, pictures of signs, shopping dockets, newspaper advertisements, and newspaper articles containing statistical information (see artefact samples in Figures 2 and 3). A program of questions and prompts was established to allow for as wide a range of student responses as possible and to enable the researcher to probe the thinking of students. Some of the questions asked were as follows:

- If we are shopping, coming to school, or watching TV, we see numbers around us. Do you take notice of them or wonder what they are telling you about?
- If someone gave you a page out of a book or newspaper, and asked you to describe the mathematics that you saw, what sort of things would you look for?
- Show the student the sample of the shopping docket/receipt and ask 'What mathematical ideas can you see and what do they tell you?'
- Show the student the map samples. What mathematical ideas can you see in the maps? How could you use those ideas to help you work out or learn something?
- Show the student the furniture advertisement. Ask 'What mathematical ideas can you see in that?' Ask 'How could you use that to help you work out something?' or 'What could you work out from that?' or 'How might that be useful to you?'

Figure 2. Artefact sample – computer advertisement.

Figure 3. Artefact sample – netball advertisement.

When shown the above artefacts, students were asked questions such as 'Is there a question you could ask about this?' or 'Does anything in there make you think of a question to ask?'

Eight female Year Six students from six different primary classes, aged eleven or twelve years, were selected as subjects for a multiple case study. Task-Based Interviews were conducted with these eight students at the beginning and end of the project to determine changes in their modes of thinking over the six month period of the intervention phase of the project. In addition, pre-project and post-project benchmark tasks based on interpretation of graphic and tabular information, and pre-project and post-project interviews with teachers of the case study students were conducted so that data triangulation could be achieved. As well, benchmark task samples were collected from the remaining 112 students in the six classes, hereafter termed the General Sample, to see whether or not results from them differed from those of the multiple case study.

Interviews were transcribed and student responses were categorised as mathematical, contextual or strategic, according to the criteria listed in the 'Theoretical Framework' section of this paper. Instances of each mode of thinking were compiled in table form to enable comparison of student thinking before and after the intervention phase of the project (see Table 1, following later).

Results and Discussion

The following excerpts from interview transcripts provide examples of how the student thinking was categorised. The excerpts are illustrative examples only and are not exhaustive representations of the responses of any of the students. Rather, they serve to indicate typical responses given and how those responses were interpreted by the researcher using the criteria previously listed in the theoretical framework section of this paper. The particular criterion used in each transcript sample is shown in parentheses in the second line of each sub-heading.

1. Pre-project interview with student 'Kerryn' showing *mathematical thinking*.
   [i.e., Uses statistical information to suggest or perform a mathematical operation.]

Interviewer: [referring to the furniture advertisement] OK, good. I'm going to show you one more thing; this is an advertisement about furniture. Have a look at that [showed Kerryn the furniture advertisement]. What mathematical things can you see there?

Kerryn: You can see the prices; you can see how much you've saved, and you can get the prices and compare the differences between them. It tells you when the day was, and what year it is [pause] when it's open from Monday to Friday, Saturday, Sunday [pause] then it tells you how tall the actual furniture is.

Interviewer: How could you use something like that?

Kerryn: To do length times width to see how big it is [pause] 100% leather means that it's all leather. There's like, five pieces so there's one whole and five parts [pause] and 1.65 metres long and 3700 metre squared for the store.

Interviewer: How would you use something like that information, that the table is that long?

Kerryn: In your house, you'd have to measure 1.65 metres to see if it would fit. Also, it has a map here to tell you where to get it; that's also mathematical.

2. Pre-project interview with student 'Tania' showing *contextual thinking*.
   [i.e., Interprets specific data contained in the context]

Interviewer: [referring to statistical charts taken from a newspaper]. I took these charts out of the newspaper. If you saw those charts, would there be any of them that you would look at and say "I wonder what that's telling me about"?

Tania: I'd look at … probably… this one (pointed to the fuel prices)

Interviewer: Why would you look at that one?

Tana: To look at the prices for petrol and to tell Mum where to go.

3. Pre-project interview with student 'Louise' showing *contextual thinking.*
   [i.e., Poses questions that require the interpretation of specific data]

Interviewer:	[referring to the furniture advertisement] And what's that bit telling us?
Julia:	It's telling you the trade hours that they're open for.
Interviewer:	How would you use that to help?
Julia:	I would probably look at the trade hours and look at the time now and think "How long do I have in the shop?"
Interviewer:	OK, if you wanted to buy some furniture there, which one of the shops would you go to?
Julia:	I would probably go to that one.
Interviewer:	Why that one?
Julia:	Because it doesn't look as busy.
Interviewer:	How do you know it's not as busy?
Julia:	Because it doesn't have as many streets around it.

4. Post-project interview with student 'Mary' showing *strategic thinking*.
   [i.e., Poses questions that require substantial evaluation of aspects of the data]

Interviewer:	[referring to the netball advertisement – see Figure 3] Finally this is a piece about a netball team, the Perth Porcupines. Have a read of that (paused while student read the sample). Is there anything that makes you think about asking something?
Christy:	Umm, say if I wanted to go join it, there's no website or address or phone number to ring. Where would I go, and because they've been playing netball for fifty years, would they be a really good team, because they'd been together and it said they had great success. Can adults play it and would they still be free? Is there a training date?

5. Post-project interview with student 'Sara' showing *strategic thinking*.
   [i.e., Evaluates aspects of data to clarify related issues and make decisions]

Interviewer:	[referring to the computer advertisement – see Figure 3]. Read this advertisement about a shop selling you computers (paused while student read sample). Is there something that you want to ask about.
Sara:	Umm, sometimes when they do the 60 months, they might think why they are doing it when sometimes it's more expensive to do it that way. But, if you didn't have the money at the time, it would be better.
Interviewer:	Is there anything else there?
Sara:	Would you still get the huge monitor [with the 40 Gb]? In a way, I'd get that one [the 40 Gb] possibly because, if it wasn't for school and just for home, we don't really use 80Gb.
Interviewer:	So you ask yourself if you needed all those gigabytes eh?
Sara:	Yeah, that's what I do when I'm buying discs… when I'm buying discs for it, whether I need a big disc or a small disc.

The five transcript samples provided indicate that task-based interviews can be used to assess the mathematical thinking of students, particularly when artefacts such as advertisements and maps are used. In each of the transcript samples, only one particular mode or level of thinking has been identified though it could be considered that degrees of mathematical and contextual knowledge are shown when a student displays strategic knowledge such as in Sample 5. As well, not all of the indicators for identifying the various modes or levels of thinking, as listed in the theoretical framework, are evident in the transcript samples, as those indicators are representative of the types of criteria used to categorise students' thinking, rather than being a finite list.

The interviews were also used to assess the success of the intervention program conducted during the project at large and to see whether the strategies used in the intervention assisted in the development of the students' thinking. On the basis of information gathered from the pre-project interviews, all eight students in the multiple case study displayed both mathematical and contextual knowledge and during the post-project interviews all students displayed all three modes of thinking – mathematical, contextual and strategic.

This suggests that the intervention program using the *Mathematical Search* strategy was successful in helping to develop student thinking, and hence, the research question has been addressed. To reiterate, this question asked: *To what extent does the Mathematical Search enhance student capacity to recognise mathematical ideas embedded in a written context, and to display contextual and strategic thinking about mathematical ideas embedded in written contexts?*

The *Mathematical Search* strategy had not been encountered by students previously as was verified by responses during post-project interviews conducted with both teachers and students. Hence, it is reasonable to claim that it had at least some effect in enhancing student capacity to display different modes of thinking. Further evidence to support this claim is contained in Table 1 which illustrates the gains in thinking shown by the eight case study students over the course of the project. The abbreviations 'M', 'C' and 'S' correspond to 'mathematical', 'contextual' and 'strategic' thinking respectively.

Table 1

Gains in Modes of Thinking for Students from Pre-project to Post-project interviews

Student	Pre-Project	Post-Project
Mary	M/C	M/C/S
Sara	M/C	M/C/S
Jenny	M/C	M/C/S
Tania	M/C	M/C/S
Kerryn	M	M/C/S
Louise	M/C	M/C/S
Lexie	M/C	M/C/S
Sonia	M/C	M/C/S

Other aspects of students' thinking and attitudes about mathematics were also explored during the task-based interviews. For example, students were asked about whether or not they identified mathematical ideas in everyday situations and what sorts of things they looked for. The following two responses from students Tania and Sonia during pre-project interviews are typical in that the emphasis is on number aspects.

Interviewer: If somebody gave you a page out of a newspaper, or a book, that wasn't a maths book, and asked you to describe the mathematics you could see, what sorts of things would you look for?

Tania: Numbers and, you know, place value, like 300 000 type things.

Sonia: Umm,…I'd look maybe for some numbers, if there were any on there, and maybe some maths related words…. Words and symbols.

However, the following excerpt from a post-project interview with student 'Kerryn' indicates some different thinking.

Interviewer: So where would you be likely to find mathematical ideas?

Kerryn: Umm.. everywhere, yeah basically everywhere … The other day I was looking at a magazine called "Just Kidding" and I saw something that just grabbed my eye and I thought about maths so much and I related it back to the tasks that we were doing.

Conclusions and Implications

Task-based interviews can be useful tools for helping teachers assess the mathematical thinking of their students, particularly when mathematical concepts are embedded in everyday or 'real life' contexts. Given the time constraints of individual contact with students, it may be difficult for teachers to use task-based interviews on a large scale but the apparent benefits of their use suggest that they should at least be used with students identified as being 'at risk' or with students identified as being 'gifted and talented'.

In order to overcome the issue of time constraints, it should be possible to train students in interview techniques so that pairs of students could participate in informal interview situations. Interviews could also be conducted as small group activities with a teacher interviewing say three children at a time. This technique would provide opportunities for students to learn from one another and to see how other students interpret mathematical ideas embedded in a range of contexts.

The task-based interview format could also be modified to become a teaching tool. Having identified a student as being possibly 'at risk', a teacher might use the process with purposeful questioning to lead the student to recognise the embedded mathematical ideas and to begin to develop contextual and strategic thinking in applying and questioning the use of those mathematical ideas. Data from the project at large on which this paper is based suggest that the eight students in the multiple case study showed greater gains in their thinking than did their classmates who formed part of the General Sample. The only tasks in which the case study students participated that were not experienced by the General Sample students were the pre-project and post-project task-based interviews. This might indicate that the interviews were partly instrumental in the case study students showing greater development of their mathematical thinking, particularly with regards to contextual and strategic thinking.

The use of task-based interviewing would be particularly relevant where teachers use integrated programming. Specific mathematical concepts could be purposively embedded in artefacts and text samples based on content and themes related to associated learning areas such as Society and Environment, Science and Health. Hence, students would be more likely to appreciate the application of mathematical ideas to other areas, an essential element of numerate behaviour.

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Who a Student Sits Near to in Maths: Tension between Social and Mathematical Identities

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This paper reports on an investigation into the seating arrangements of a mathematics classroom, and the effect of these arrangements on students' affect and learning. A seating arrangement is successful depending on whom a student is sitting near. Students need to be surrounded by others whose behaviour does not disrupt or distract them, and who they like and feel comfortable with. The study suggests that adolescent students do not have the power or control to stop other people's behaviour affecting them, nor do they have the power to sit where they want to ensure their academic identities are being fulfilled. By instituting seating plans, teachers can ensure students' academic and social needs are met therefore improving student learning through positive discussion and help-seeking.

Introduction

At all levels of schooling, mathematics classrooms are social places where students pursue both social and academic goals. This social environment is likely however to be particularly important to young adolescents, 13-15 years. Not only do they have increasingly strong social needs (Ryan, 2001), but they are also self-conscious and sensitive as "more so than at other ages, young adolescents doubt their abilities to succeed … question the value of doing their schoolwork, and decrease their effort towards academics" (Ryan & Patrick, 2001, p. 439).

The relationships between students impact on the adolescent social environment. This paper describes one contextual dimension of these relationships, the effect of seating arrangement on students' mathematical identities and affective responses. It begins by outlining previous research surrounding interactions between students in close proximity to each other, presents the theoretical framework used in the investigation, and then describes the effect of seating arrangements in the context of a mathematics classroom.

Seating Arrangements

Sullivan, Tobias, and McDonough (2006) surveyed and interviewed 50 students, aged 13, investigating students' perceptions of success in mathematics. One, unanticipated finding was that a significant minority of students suggested "students deliberately do not try in order to comply with a particular classroom culture or avoid the perception of trying due to threats of sanctions by peers" (p. 96). In later research (Sullivan & McDonough, 2007) students further suggested how sitting in close proximity with another may affect them. "Some people who are sitting next to smart people felt like being smart and doing it, but sometimes there's a dumb group and they don't want to look like a nerd in front of everyone" (Sullivan & McDonough, 2007, p. 703).

Students sit near each other in the mathematics classroom according to the desk arrangement, teacher direction, and social norms. It is probable that many adolescent students, if allowed, choose to sit with friends because of the high value placed on friendship in adolescence (Crosnoe, Cavanagh, & Elder Jr., 2003). Whom the student sits next to exposes them to a number of influencing beliefs and behaviours allowing a context to emerge with regards to the "norms, values, and standards that concern academic motivation and achievement" (Ryan, 2001). Crosnoe et al (2003) highlight that research into adolescence is often focussed on the negative influence of friends, whereas research into other stages of life view friendships as a social resource. They, therefore, introduce the concept of adolescent friendship as *social capital* which they define as "the resources accessed through relationship ties" (p. 333). When two individuals form a relationship, they are gaining access to resources such as their values, social support, knowledge and skills relating to schooling and academic subjects, and emotional support for the meeting of challenges.

Affect and Identity

Not only is learning fundamentally social, it is an *emotional practice* and studying the affective domain, which includes constructs such as beliefs, attitudes, emotions and feelings, values, confidence, anxiety, and motivation, is important. It is difficult to assume a direct link between positive affect and mathematical achievement (Leder & Forgasz, 2006), or even to know the direction of influence between them (Zan, Brown, Evans, & Hannula, 2006). Nonetheless, affect has been linked to declining participation in mathematics (Norton & Irvin, 2007), and affective processes are now understood to be an integral part of problem solving and learning (Op 'T Eynde, De Corte, & Verschaffel, 2006). The domain has unique methodological issues and limitations (Leder & Forgasz, 2006, p. 404), and developments in educational, psychological and social psychological research have encouraged a variety of new theoretical perspectives (Hannula, Evans, Philippou, & Zan, 2004). Students' socio-cultural backgrounds are a focus, and there is interest in the notion of identity. Sfard and Prusak (2005) equate identities to reifying, endorsable, and significant stories about a person. People have a number of stories told about them by a variety of people, including themselves. Each person has multiple identities, split into two sets of *actual identities* (stories about the actual state of affairs) and *designated identities* (a state of affairs expected to be the case). Sfard and Prusak suggest that when there is a perceived and persistent gap between these sets, there is likely to be a sense of unhappiness in that person. In this research, Sfard and Prusak's notion of identity has been used and the existence of a gap between a student's actual and designated identity signalled by their affective responses (positive or negative).

Methodology

To capture reifying, endorsable, and significant stories about the emotions and feelings of students is difficult, especially for adolescents in the complex environment of a mathematics class. The research methodology must have length, breadth, and depth. It needs to be longitudinal because students become less resilient to negative emotions and feelings about mathematics as they move through school (McLeod, 1992), and therefore their dynamic identities need to be captured over time. The research needs breadth through the use of a broad range of rich data collection methods, with a range of data sources and identity narrators. Depth is gained through the richness of the analysis that inductive data collection allows. This research has been informed by a grounded theory approach, which is the derivation of theory from data "systematically gathered and analysed through the research process" (Strauss & Corbin, 1998, p. 12). Decisions made about each stage of the data collection process are grounded in the data itself and the emerging categories and themes (Strauss & Corbin, 1998). Supporting the grounded theory approach, the analysis software NVivo helped to manage and analyse the large quantity of qualitative data.

Three strands of data were collected. These were students' mathematical identities that relate to seating arrangements, related instances of affective responses, and descriptions of the mathematical context. Data were collected over two years and included classroom observations, teacher interviews and feedback, student interviews and written responses (questionnaires, autobiographies, metaphors, drawings, personal journey graphs, journals, evaluations), parent written responses, and school documents. Students were also asked to comment on seating plans specifically and, more generally, on how who they sat next to affected their feelings and learning. For every observed lesson, a plan of the classroom was drawn and each student's seating noted.

Sfard and Prusak's structure was used to differentiate between multiple identities of an individual; for example, a story about a student told to the researcher by a teacher in 2007 would be identified by ($_{\text{Teacher}}\,\text{Student}_{\text{ResearcherInterview2007}}$), situating each story in both time and space. In this paper, the recipient has been assumed to be the researcher. The identities were analysed for indicators of affect using as a guide the work of Evans, Morgan, and Tsatasroni (2006). Indicators sought included verbal expressions of feelings, body language, physiological reactions, gesturing, or resistance to authority figures. A seating arrangement in this research was deemed successful for a student when their mathematical identities and related affective responses were positive.

Describing the Context

The research was conducted in an urban coeducational school in New Zealand. The 31 students involved in the study were New Zealanders of European descent, and, in 2006, were together in a Year 10 class (aged 13-14 years) for their core subjects Mathematics, English, Science and Social Studies. In 2007, the students were split into seven different mathematics classes.

During the first 10 weeks of 2006, the research students were seated in pairs in an alphabetic seating plan in their mathematics class. Seating plans are not unusual across subjects or schools in New Zealand, especially in the first term of the year. They enable the teacher to learn the students' names and separate social groups, thus contributing to a perceived improvement in classroom discipline. For the rest of the year, the students remained in pairs and chose where to sit, except during a research intervention when they were seated in groups. In 2007, the students were in self-choice seating in pairs or groups, depending on desk arrangement.

Results and Discussion

In this research, students entered their mathematics classroom and either sat according to a prescribed seating plan or sat where they wanted. During the lesson, students interacted at some level with all the students in the classroom, but they were mainly aware of and interacted with those students immediately sitting next to them *and* those in close promixity. In this research, when the desks were arranged in pairs for example, roughly one third of the students were against a wall and many of them leant against it, changing their orientation 90° and giving them more frequent interactions with students in front and behind them. This research suggests that who a student sat *near* was related to how they felt about the mathematics they were doing, the amount of mathematics they did, and on their mathematical discussions. Regardless of whether students were in self-choice seating or in a seating plan, seating arrangements were successful when two conditions operated; *other students' behaviour did not negatively affect the student* and the student *liked and felt comfortable with the others they were sitting with.* The following examples provide evidence for these conditions, and then other aspects of seating arrangments are discussed.

Other Students' Behaviour

For the students who had strong learning goals and a good work ethic, it was important that the people sitting near were not disruptive. "If I was next to people who didn't concentrate on their work and were loud … it would be hard to concentrate" ($_{\text{Corrina}}$Corrina$_{\text{Interview2007}}$). More commonly, for other students, this same behaviour was distracting, rather than disruptive. "Who I sit next to totally affects me. I don't do anything. I find it hard to focus. I get distracted really easily"($_{\text{Moira}}$Moira$_{\text{Interview2006}}$). "If I'm sitting next to someone who works hard, I'll work hard. If I'm sitting next to people that don't, I just won't" ($_{\text{Susan}}$Susan$_{\text{Interview2006}}$). Students largely felt powerless to control others' behaviour and could only work if a person's behaviour *allowed* it.

Liking and Feeling Comfortable with Others

The second condition is that seating arrangments appear more positive when the student is near someone they like and feel comfortable with. While the amount of mathematics done sometimes increased if the student sat near someone they did not like, mathematical discussion often did not occur, and, in general, if the students were not near to someone they liked, they felt less positive and became bored because of the lack of social talk. "If I'm sitting next to someone who I don't really like then I'll just get on and do the work because I don't really want to talk to them" ($_{\text{Katrina}}$Katrina$_{\text{Interview2007}}$).

Importantly, how comfortable a student felt with another student seemed to make a difference to whether or not a student asked for help. "If I sit next to friends … it makes me feel more comfortable because … we help each other. If I sit beside someone I don't like or don't know … I don't feel comfortable asking them" ($_{\text{Bridget}}$Bridget$_{\text{Interview2006}}$). Students at a similar level mathematically were noticeably more comfortable using words like "discuss" and "figure it out" rather than "help" and "ask". Students at a lower level mathematically to their neighbours were often uncomfortable.

> If I sat next to Colin I would feel stupid ... he's really smart ($_{\text{Cheryl}}$Colin$_{\text{Interview2006}}$).

> I know [Angela] can hear [Jason and I discussing] ... and I can hear [her] thinking you're supposed to be bright ... I avoid asking her for help because I don't want her to know that I need it … sitting next to a brainbox makes me feel intimidated and stupid ($_{\text{Robyn}}$Angela$_{\text{Interview2006}}$).

Other students would not necessarily feel negative pressure, but would be aware of the different level of perceived mathematical ability. "You sort of look over every now and then to see what they're up to" ($_{\text{Connor}}$Connor$_{\text{Interview2006}}$). Sitting near to a stronger mathematician can be positive. "I always ask Katrina because she knows how to do maths. I just ask her … and ... she helps me" (DebbieKatrinaInterview2006).

The stronger mathematicians sometimes enjoyed and saw the benefits in helping others. "Explaining … is beneficial as it helps clarify things" (KatrinaKatrinaEndofTrig2006). The experience of helping others can be negative for regular helpers and the following stories no doubt contribute to the negative affect of the people being helped. Note the "responses" to Robyn and Cheryl's earlier concerns.

> If I'm trying to do my own work and Robyn's like Angela how do you do this and I'm like you'll have to ask the teacher ... I don't have a lot of patience as a person. Sometimes it is a little frustrating when I can see the answer and no one else can (AngelaAngelaInterview2006).

> Cheryl is quite a challenge for me to explain something to 'cause sometimes it's really funny that she doesn't know it and I'm laughing on the inside. I take a deep breath and explain it to her (ColinCherylInterview2006).

> … other people in the class always ask me what to do because I … get it. I understand. I say I don't get it either ... because then I'd just be helping them the whole time. I don't get paid to help them. The teacher does (RuthRuthInterview2007).

> They just keep bugging me (PeterPeterInterview2007).

Seating Plans and Self-Choice Seating

Most of the students stated they did not like seating plans. Indeed, only two of the students stated an unreserved approval of seating plans. They were perhaps well aware of the *socially correct* answer. In an individual interview, one student who had stated in the written response that she did not like seating plans said " ... if I sit next to someone I don't really like I concentrate more and I'll do my work. I shouldn't be saying that [trails off in a small voice]" (CorinnaCorinnaInterview2006). Even without a seating plan, an adolescent rarely has a choice about where to sit. "We're expected to sit together because we're such good friends" (RobynAngelaInterview2006). In interviews, however, the students often confessed that sitting with friends was often difficult because of social disruption, their own distraction, or negative affect due to being at different ability levels. They acknowledged that they may need to sit somewhere else but seemed generally powerless to do so, often admitting they needed the teachers help to prevent others from affecting their learning.

> If I wanted to do my work really well, I wouldn't sit with [my mates](JasonJasonInterview2006).

> Where I sit affects me lots. In the last few weeks, [since a discipline meeting about behaviour] I sort of walk into the room and see who's where and if there's a place (ConnorConnorInterview2006).

> [If I got to choose where I sat all the time] I wouldn't learn anything. That's where seating plans help (BenBenInterview2006).

Mathematics vs Other Subjects

In both mathematics and English, the students were observed to be affected by who they sat near to in terms of other students' behaviour disrupting and distracting them; to a lesser extent in English, perhaps because a greater proportion of the time was spent in whole-class discussion. The students reported a greater level of discomfort in mathematics due to seating arrangments however and this affected student talk, both in terms of discussion and help, particularly if a students' preferred learning style was to talk about the mathematics.

> In maths, it's a subject where talking helps you ... talking to the people beside you helps ... so if you're silent, you don't learn as much (JillJillInterview2006).

Compared with other subjects, students seemed to feel more discomfort in maths when they were at a lower level academically than other students. They discussed how they could write a sentence in English and it was not so obviously wrong, while in maths there seemed to be more visual clues to failure, such as calculator use, working, or the incorrect answer.

Working in Cooperative Groups

For several weeks, the students worked in cooperative learning groups of similar ability levels, with some support from someone they liked and felt comfortable with. Attempts were made to separate the disrupters from the distracted and the dependent students from those with less patience when it came to helping people. A large majority of the students found the explicit expectation and license of support and discussion helpful and this increased their comfort when asking for help from each other. "I *worked with* people that weren't super intelligent and could grasp it instantly. That was good for me because sometimes I feel inferior or not smart enough" (SaskiaSaskiaInterview2006). The atmosphere of mutual support enabled one student, who had been absent, to catch up from his peers on missed work and more importantly, missed understanding. "It made me feel better as I could ask for help. I would have not caught up the same if I hadn't been in a group" (ConnorConnorEndofTrig2006). Journals, designed to collect students' affective thoughts on a day-to-day basis, were used by many students as another avenue for help from the teacher.

When the group work did not go well, the reasons were similar to those encountered with other seating arrangements. One or two students still felt that they could not get on with their own work because of students asking for help. "I don't really like group work, because then we have to go at the pace of the slowest person" (PeterPeterEndofTrig2006). Others were distracted or disrupted by their group members on occasion. Many of the students however stated that what they enjoyed most about the unit was working in groups. It made them feel good about the mathematics they were doing, it was fun, it gave them variety, and they got to know people they previously had not talked to before, despite having been in a class together for nearly two years. This was perhaps because both their social and academic needs were being met.

Accepting and Harnessing the Social

In general, an important focus for this group of students was being social. In Year 10, the students were already settled into social groups, but these changed and mixed as students became more confident with the opposite gender. By Year 11, this had developed and intensified "… there's more like the puberty thing. People more like to have sex and have boyfriends and girlfriends and that's where the focus of school is. You're meant to be focussed on school" (SaskiaClassInterview2006). Both the parents and teachers are certainly aware of it, seeing the social element as negative.

> Moira likes being social and that's more important to her than doing her work. I don't think she's here to learn really ... it doesn't cross her mind at all … talking is more important than doing maths (Teacher1MoiraInterview2006).

> Jason's mother told me at the parent interview I needed to stop him talking because he *shouldn't be* talking and that I should be harder on him and I shouldn't sit him with Ben ... that they'll talk and talk and talk" (Teacher1Jason'sMumInterview2006).

> Works well in class *despite* social nature (Teacher6BenFeedback2007).

This causes tension for many of the students because of conflict between their designated social and academic identities (what they should or want to be achieving both academically and socially) and the gap between their designated and actual academic identities (what they are achieving academically). Despite this tension, a social element is necessary. Talking is something that they "have to do" and they still "need social time".

Conclusions and Recommendations

In summary, regardless of whether the students were in seating plans or in self-choice seating, how they felt about and learnt mathematics was related to who they sat near to in the mathematics classroom. The importance of positive social interactions in a mathematics class is well established. What is significant in this research is that, because of their stage in life, these students were, in general, not able to control the behaviour of others, and were not able to choose where to sit to fulfill their designated mathematical identities because of their strong social needs. This suggests that mathematics teachers may need to account for this by accepting the social needs of adolescents, instead viewing aspects of them as social capital to be harnessed. The relationships between students, in particular, are a resource, and accessing this resource can lead to better mathematical learning through mathematical discussion, emotional support, and the sharing of knowledge and skills.

To harness this resource, and to account for the students' powerlessness to control their own strong social identities, the teacher can ensure the conditions for positive seating arrangements are met every day through sensitive seating plans. Teachers might do this by getting to know the students better in terms of both their social and mathematical identities and the students could be seated in pairs with the pairs in front or behind ready to turn to form a larger cooperative learning group. The teacher also may need to give explicit messages regarding students' interactions with their classmates such as: students are valuable resources to work with to increase learning; and a students' behaviour affects others. Sullivan and McDonough (2007) usefully suggest that teachers seek responses from individual students about negative influences of peers, and use these responses as a basis for class discussion, story-writing, or role play.

By ensuring the seating arrangements are positive both academically and socially, the students are more willing and able to work towards fulfilling their designated mathematical identities.

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Social Constructivism in the Classroom: From A Community of Learners to A Community of Teachers.

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This paper reports on a Head of a Mathematics Department (the author) in a Brisbane suburban High School and her attempts, over a two year period, to implement a new teaching program involving innovative curriculum resources and a new pedagogy into the first year of high school. The goal of the Head of Department was to support staff to create learning environments that followed a social constructivist philosophy. What was found was that there was a parallel process that occurred for the teachers. The teachers were learners and they needed a community of learners for support within which to reflect, learn and grow.

Introduction and Background

How students learn to think, reason, and problem solve mathematically informs teaching practice. Booker (1996) contends that it is a "commonly held belief among mathematics educators" (p. 381) that students learn mathematics by "the individual construction of ideas, processes and understandings" (p. 381). Students bring to the classroom knowledge and thinking that has proven successful for them in the past. These knowledges and thought patterns guide their replacement by new thinking and in this way, students can be seen to be "active meaning makers" (Booker, 1996, p. 382).

> Mathematical thinking is based not on the symbols that are used, but on the meaning on which they are based and come to represent. (Booker et al., 2004, p. 24)

Thus, a teacher's role is not to transmit "the" mathematical knowledge to the students but to assist in the reconstruction of particular ways of thinking. Selection of relevant models, materials, patterns will assist learners in constructing the "mathematical ways of knowing compatible with those of the wider society" (Cobb, Yackel, & Wood, 1992, p. 27) but are not enough as "meanings do not reside in words, actions and objects independently" (Booker, 2004, p.16). Students gain meaning through discussion and debate – speaking and hearing mathematics – with each other and with teachers. In this way "mathematics can be viewed as a social practice or a community project" (Cobb, Yackel, & Wood, 1992, p. 27) and, so, classroom environments need to be created that cultivate active and engaged discussion (Booker, 2004).

One mode of instruction to consider is the instructional game and it is the pedagogy used in the curriculum reform of this study. Gough (2000) defines a mathematics game as "something you play to learn or practise mathematics." Ainley (1990) refines this by distinguishing the effective game as being born out of and inherently relying on mathematical ideas and understanding in order to run. The research is overwhelmingly in favour of using instructional games in the mathematics classroom. Booker (2000, 2004) argues that instructional games provide an explicit vehicle for discussions about mathematical concepts in that, for play to proceed, a mathematical dialogue has to have occurred. This may happen in the traditional classroom described by the TIMSS (1999) video study (in Hiebert et al., 2003), with students working on sets of problems whilst seated near other students, but it is an accidental by-product, not a planned occurrence. It is a planned occurrence when the teacher uses instructional games in their teaching.

Randel, Morris, Wetzel, and Whitehill (1992) reviewed the literature and they present a powerful case for changing the mode of instruction in the mathematics class. They found that the use of games in teaching mathematics is superior to traditional instruction in terms of achievement, retention, and motivation for learning. For students using instructional games in the classroom, the literature shows an increase in achievement, improved retention of information, heightened motivation to learn, greater participation, increased opportunities to learn, more complete engagement in the task, greater interest, improved confidence, and more risk taking/problem solving (Booker, 2000; Booker, 2004; Booker et al., 2004; Gough, 2000; Onslow, 1990; Ortiz, 2003; Pearn & Merrifield 2004; Randel et al., 1992; Rowe, 2001; Schmidt, 1995). The benefits of positive cooperative social interactions are also well documented (Booker, 2000; Ortiz, 2003; Rowe, 2001). The positives for teachers focus on the benefits to students. Booker (2000) also mentions increased opportunities to communicate with students one on one while they play and varied assessment opportunities via observation of student interactions during the game.

Teachers need professional support to integrate such pedagogy into their teaching and it is this aspect of a curriculum reform that is reported on in this paper. A comprehensive investigation into curriculum reform efforts has been conducted by Wilson, Peterson, Ball, and Cohen (1996) and it provides direction for educators who are endeavouring to bring about change to the way mathematics is taught. After various attempts to reform the teaching of mathematics in classrooms, policy makers learned that teachers needed considerable time to reflect and share their experiences with the new materials and professional development became meaningful conversations about the reform journey from teachers' perspectives. It was this process that lead to change in these teachers' classrooms. They were able to learn from each other and rethink their underlying assumptions and belief systems of mathematics teaching and learning. The teachers were put in the position of learner and given respect around what it takes to embrace change.

In stark contrast, Wilson et al. (1996) describe the journey of a teacher who does not see the need to change her teaching practice even after three years of participating in reform activities. The style of the courses she attended did not afford the teacher with any opportunity for conversation about what she was thinking and her experiences or the opportunity to hear from her colleagues. She summarised that she hadn't learned anything from the professional development activities and that the assessment reforms shown to her were too much work. Wilson et al. (1996) eloquently describe the situation the teacher is in:

> In the trenches of teaching's dailiness, Mrs. B. constantly faces obstacles to deep change. To learn what she needs to in order to translate these reforms into realities, she needs ongoing support, careful attention, and lots of opportunities to think – and rethink – her assumptions and practices. Yet such opportunities have not been offered her. Her experiences and ideas have been neither invited nor challenged. And the reform effort is the worse off for it in her classroom. (p. 473)

Ball (1993) also contends that teachers' work structure and the "accepted norms of social discourse – in what educators talk with one another about and in what ways" (p. 396) – make opportunities for meaningful discussion around conceptual understandings of mathematics and pedagogical dilemmas to be extremely limited. A suggestion for improvement made by Ball is the creation of a professional community within which teachers discuss and reflect on teacher practice. She cites an example of videotaping her own lesson and the wealth of information and ideas that came to her about her own practice as she examined it. Finding an answer is not the emphasis as such but the provision of the opportunity for choice and reflection on future directions. Lampert (1994) agrees with Ball when she concludes that if teachers had more opportunities to meet and work as a group on their teaching practice then they would gain insight and understanding into what is needed to teach effectively for understanding.

A four year study of six cohorts of preservice teachers by Frykholm (2005) provides evidence that interacting with innovative curricular material can have an extensive effect on preservice teachers' content knowledge of mathematics and their ideas about what good teaching and learning looks like. The preservice teachers reported that the materials that were used in class challenged the way they thought about mathematics and, hence, created a realisation that there was more to mathematics than learned procedures. Teaching became more about concepts than algorithms for these preservice teachers and they reported a determination to teach mathematics themselves in a conceptual way. Frykholm argues that it was through the interaction with the innovative activities from the perspective of learner that was the "catalyst for growth" (p. 31) for these teachers. Thus he emphasises it is not enough to just discuss the pedagogy of the reform curriculum:

> Through the explorations of the materials, the students learned a great deal about how curriculum might be structured, how mathematical concepts build on one another, how students' ideas and intuitions can guide classroom discussions, and the important connection between assessment and instruction. (pp. 32-33)

In conclusion, Frykholm states that contrary to research findings that preservice teachers knowledge and beliefs about mathematics are "fairly rigid and resistant to change" the preservice teachers in his study significantly improved their content knowledge and meaningfully changed their beliefs around the teaching and learning of mathematics.

Other studies emphasise the enhancing effect on teachers' mathematical and pedagogical knowledge of creating a learning environment for teachers. Teachers working together in collaborative groups have been shown to make a difference to their own understandings of mathematics (Jeanpierre & Lewis, 2007; McDiarmid, 1990; Soto-Johnson, Liams, Hoffmeister, Boschmans, & Oberg, 2007) and to those of their students (Siemon, Breed, & Virgona, 2006).

A detailed description of the "typical" classroom environment that many mathematics teachers work and teach in reveals that it is complex and impacted on by many internal and external factors (Ball, 1999). Tomlinson (2000) concurs with Ball when she lists the seemingly unrelenting pressures on teachers of oversized classes, challenging schedules, paperwork, and consistent persistent demands from administration and the community. Lampert (1994) eloquently sums up the situation with "teaching is not about solving problems, it is about managing complexity ... (it) necessarily is about trying to manage a lot of conflicting goals" (p. 30).

In addition to an already complex and pressured situation a program of reform is to be implemented which required teachers to come to terms with several new ideas – new physical materials of the instructional games, new content, and a focus on teaching conceptually. This major step into the unknown creates a lot of anxiety in teachers and it may feel chaotic (Frykholm, 2005). Rowe (2001) reviews several studies incorporating instructional games and highlights the increased workload in preparation and management of resources, the lack of student writing of mathematics, and the increased noise levels in the classroom. Her own study using games in her teaching leads her to conclude that the sessions when the games were played were "hard work" (p. 14).

Method

The researcher views research as a learning experience not a confirmatory process and seeks to uncover the social nuances that occur in a school during a significant change. Lampert (2001) points out that research that examines a case of teaching in detail will "contribute to a conversation about the nature of the work schoolteachers do" (p. 7) because insight is gained into the many facets of daily teaching life. She contends that such research is important because "there are problems of teaching practice that are common across differences in schools, subjects and age groups" (p. 6) and, so, an analysis of one teaching situation can inform another.

The aim of the study reported on in this paper was to investigate a department of teachers in a high school setting as they implemented a new pedagogy and curriculum into their teaching practice. The details of what it took to implement the new program from the perspectives of the leader and teachers involved and how that shaped the future directions of the program in the school are the main sources of enquiry. In this way, this study followed the process of participatory action research as defined by Kemmis and McTaggert (2000).

Participants and Data Collection

The participants in this study were the Head of Department Mathematics (the author) and the teachers of Year 8 Mathematics (four in 2005 and six in 2006 with three of these teachers involved in both years).

The researcher collected primary data in various forms – firstly, researcher field notes of observations of teachers and students; secondly, teacher observations of themselves and the students; thirdly, journal writing by the researcher and the teachers; fourthly, minutes and field notes from weekly teacher meetings and regular teacher professional conversations; fifthly, teacher interviews.

Discussion

It is important to emphasise the position the teachers were in. They decided to put themselves in the position of learner whilst teaching a new curriculum with a new pedagogy. They didn't just interact with the innovative materials and new pedagogy as the Frykholm (2005) students had as university students, they immersed themselves in it by actively endeavouring to teach with it in their classrooms over a five week period. Each lesson involved teaching with instructional games utilising the teaching methods of Numeration ideas as espoused in Booker et al. (2004).

The teaching team, in 2005, decided to meet each week for at least one and a half hours after school hours. Professional development of the curriculum content and resources was the planned focus of these meetings. A shared understanding of what to teach and the management of resources was seen by the researcher and teachers to be essential to the success of a new teaching program. What resulted from these meetings was that the teachers were undergoing a parallel process to the students in the classroom. That is, because the teachers were learners they created a social constructivist environment within which to learn just as they were trying to achieve in the classroom.

A case study illustrating how the teachers discussed with each other in their weekly meetings and, thus, worked through an issue leading to an assessment reform is described. The discussion at the weekly meetings initially focused on content knowledge and operational matters but it gradually became dominated by teachers attempting to find a better way to cater for all the levels of learning that were evident in their classrooms. As a leader, the researcher had designed the games so they could be used in the following way. Each style of game was available at several levels of learning and, so, the lesson could be organised around a particular big idea such as linking language to symbols and then each group of students given a game to play at their level of readiness. For example, the recognition games (shown in Table 1) came in four levels in terms of the size of the numbers and, also, within 4 and 6 digit numbers there were three levels of concept.

Table 1

The Recognition Games

Game Name	Purpose of the game
Recognise Me – 4	4 digit numbers – place value name recognition
Who am I? – 4	4 digit numbers – place value name recognition with saying the number
Recognise Match – 4	4 digit numbers – place value name recognition with comparing numbers
Recognise Me – 6	6 digit numbers – place value name recognition
Who am I? – 6	6 digit numbers – place value name recognition with saying the number
Recognise Match – 6	6 digit numbers – place value name recognition with comparing numbers
Who am I? – 9	9 digit numbers – place value name recognition with reading 9 digit numbers from the symbols
Who am I? – 12	12 digit numbers – reading 12 digit numbers written in words and then writing the symbols

In theory these seemed achievable but in practice teachers were not doing it and the researcher, as classroom teacher, was not able to achieve this in her classroom. Observation of classrooms showed that the teachers were tending to use the games towards the end of the lesson as a replacement for the traditional textbook exercises. A typical lesson involved teaching the concepts via direct instruction from the front of the classroom at the whiteboard followed by whole class practice from the whiteboard. Then the students would work on the games. This highlights that providing teachers with the resources to achieve a desired teaching outcome is not enough to make it happen. Teachers had to work through the issues together and come to their own conclusions as to how to proceed.

The effect on the students was marked as those who could do the work from the whiteboard became bored with the games very quickly as they had already mastered the skills and were ready for new challenges. The students who had not grasped the new concepts did not interact well with the games either as they were not ready for the ones they had been given. Teachers reported increased behaviour management issues. The field notes of a weekly meeting highlight this:

> One teacher reported her class as saying: “Do we have to do this? Why are you punishing us?” She said that the rest of the lesson was spent copying “times tables” from the board and that there was silence – yippee.

Further evidence to support this can be found in the researcher’s diary entries:

> I was out of my comfort zone. We are all out of our comfort zone and this could be one of the reasons. The cohort’s behaviour across the school has erupted of the last week or two and this makes the classroom a challenging place at the best of times. Hence, it is a gut reaction to revert to what was always done. It’s my teaching from the board that’s not working. The games are good.

By the fifth week of the program the weekly meeting was characterised by spirited discussion, a sense of frustration and concern over the direction of the program. Various teaching strategies had been tried over the weeks including “enrichment” worksheets and problem solving yet the need to cater better to the students’ learning needs was a recurrent subject of debate.

Staff still wanted to continue to try to work better with the instructional games and provide a learning environment so that all students had the opportunity to extend their understandings (minutes). This is not unexpected according to Lampert (1994) and Ball (1993) who espouse the need to give teachers time to meaningfully reflect on their practice. The fact that the staff was so greatly concerned over trying to teach so as to meet multiple levels of learning needs simultaneously was a major paradigm shift from the traditional Year 8 lessons described by the TIMSS (1999) video study (in Hiebert et al., 2003).

The way forward was seen to be a formalisation of the differentiation process into the end of unit assessment (field notes). Students were required to submit a portfolio of work (refer Irvin & Booker, 2005). There were three difficulty levels of portfolio and each student was issued a personal list according to the progress of their learning as ascertained by screening tests (Booker, 2005) and class teacher observations. Incorporated into each portfolio were games to be played by students whilst a teacher witnessed them, two written assignments, and a written report. One assignment involved making an own designed universal board game. Its inclusion emphasised to the researcher the value the teachers had placed on the instructional games and highlighted to the students the importance of the game in mathematics learning.

Feedback at the next weekly meeting was much more positive and indicative of the change created by the introduction of the portfolio task. Once the students began working on their individual portfolios the teachers reported that the behaviour issues inside the classrooms significantly decreased. The staff noted (in interview) a sense that they were teaching the students and that the students liked having choice and the work set at their level of understanding. Observations of teachers' lessons and the researcher's own lessons indicated that there was much less teacher talk from the front of the class.

The curriculum program was taught again in 2006 and provides a comparison. Weekly meetings quickly discontinued in 2006 due to several contributing factors. Firstly, three of the six teachers had been on the 2005 team and they reported (in interview) that they felt very confident with what and how to teach as well as believing they could mentor the other three teachers new to the program on an as needed basis in the staffroom. Secondly, substantial lesson resources had been created the previous year to support teachers in their individual preparation. Thirdly, after the second meeting there was considerable dissension among teachers about the continuation of weekly meetings after school hours so as to cause considerable disharmony in the staffroom.

Even though formally arranged meetings were not occurring it is important to note that the researcher observed frequent discussion among the teachers of 2006. This was entirely left to the initiative of the teachers and provides a stark contrast to the weekly meetings of 2005. Two teachers' experience around collegial support provides insight.

Firstly, one teacher, new to the program, designed a menu of the instructional games for student use. This was along similar lines to the Portfolio Task with one important difference – the same menu was given to every student and every student had to demonstrate competent playing of every game on the list. In this way games appear to have been seen as akin to a set of exercises in a text – all to be done by all at the same time. Consequently, even though a number of students had demonstrated deep understanding of numeration ideas for up to six digit numbers (screening test, Booker, 2005), all had to play games such as "Who am I? – 6" which required players to state place names and read out loud six digit numbers. Observation of students indicated that the lessons were not engaging and that they complained loudly about how boring the games were. Feedback from the teacher interview was that the generic menus had not been such a good idea yet he was enthusiastic about using instructional games in the future.

Secondly, another teacher new to the program in 2006 noted, at interview, that she had never understood why the weekly meetings had been cancelled and, as a beginning teacher, had felt a bit unsupported and in need of some collegial help with pedagogical and mathematical matters. This teacher's desk was in a different staffroom which meant that she had had to rely on corridor conversations in the classroom block if she had happened to encounter one of the other teachers of Year 8 or made a special visit to the mathematics staffroom which she found difficult to find time to do. This does highlight the day to day busyness of teaching affecting the teachers that the literature describes (Ball, 1999; Lampert, 1994; Ma, 1999; Tomlinson, 2000; Wilson et al., 1996). Neither teacher had been a part of the discussions and experiences in 2005 nor been regularly included in the informal staffroom professional conversations in 2006 (researcher observations). These two teachers can be seen to have operated in isolation of the teaching team.

Conclusions

Attempts to implement a curriculum and pedagogical reform into a suburban high school by a Head of Department can be seen to have been successful. The Program was taught in 2007 and 2008. The pedagogy of instructional games has been incorporated into the way mathematics is taught at the school – they are central to units in Year 8 Numeration, Year 10 financial literacy and number, Year 12 prevocational mathematics and Year 9 Algebra. Teachers have made games involving multiplication facts, operations with negative integers, chance and data, and algebra. Another teacher has designed a "make a game" assignment in Science and students have made games for assessment in other subjects of their own initiative.

The Head of Department (the author) found that the learning process for teachers seemed to be aligned with social constructivist theories of learning. That is, teachers needed an environment where they could express their thinking and listen to others' understandings to compare with their own and coincidentally this was the environment they were trying to create for their students in their classrooms. The formalised weekly meetings of 2005 were more effective than the informal professional conversations of 2006 as staff had to be in the right place at the right time to be included. Some staff were left expressing a need for more support. As a result, the weekly meetings were re-established in 2007 and continue in 2008.

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Primary Teachers' Beliefs About the Use of Mathematics Textbooks

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This paper describes a small-scale investigation of the beliefs teachers hold about student textbooks and the factors that influence their decision to use them in mathematics lessons. Findings suggest that teachers' decisions to use textbooks in mathematics are influenced by external factors, the teachers' perceived educational value of textbooks and, the teachers' personal confidence and competence to teach mathematics. However, there is some evidence to suggest that contrary to previous studies, the 34 Queensland teachers surveyed appear to make less frequent use of textbooks and are more discerning about the manner in which they use textbooks in their classrooms.

In contemporary primary mathematics classrooms the existence of commercially produced resources such as concrete materials, tuition software and student textbooks is widespread. Such resources have been designed to complement and augment mathematics teaching and learning. Research in recent times has shown Australian teachers are dependant on a wide variety of these commercially produced curriculum resources, particularly the student textbook, for delivering the curriculum (Shield, 1991; Watt, 2002). However, teachers' heavy dependence on the mathematics textbook as a means of improving the educational outcomes of students is problematic, as the value of textbooks to student learning is presently ill-defined (Zevenbergen, Dole, & Wright, 2004). Taking into account the lack of conclusive evidence supporting the efficacy of textbooks, the reasons for Australian teachers' commitment to implementing student textbooks in mathematics classrooms is unclear. This ambiguity is further pronounced when one considers the widely held belief amongst educators that teaching is enhanced when underpinned and informed by research and theory (Knobel & Lankshear, 1999). If not research, then on what are teachers basing their decision to use textbooks in primary mathematics classrooms?

Throughout history, mathematics textbooks have been synonymous with mathematics education (Fauvel, 1991; Gray, 1990; Love & Pimm, 1996). In many instances, historical mathematics textbooks are often all that remains as a record of mathematics programs of the past (Love et al., 1996). One of the earliest documented mathematics textbooks written with a scholarly approach to mathematics is Robert Record's "The Ground of Arts", published in 1543 (Fauvel, 1991). Written entirely in dialogue between Master and Scholar, the purpose of "The Ground of Arts" was to instruct readers (scholars) in the basics of elementary arithmetic: numeration, addition, subtraction, multiplication, and division. Record's mathematics textbook was widely successful and was reissued a number of times both during his lifetime and after his death. This book's immense success became a catalyst for the authoring and publishing of numerous versions of mathematics textbooks, a trend which continues to the present day.

The technology of printing allowed the ideas of educators and academics to be shared and made readily available to the general population. Educators welcomed this reformation and adapted their procedures to make best use of the means by which education was now being disseminated (McClintok, 1999). Such success has led to mathematics textbooks dominating both the perceptions and practices of school mathematics.

Textbooks have remained significantly unchanged, and as with schooling of the past, contemporary schooling is characterised by a heavy dependence on textbooks (Chambliss & Calfee, 1998; Woodward, Elliott, & Nagel, 1988). The teaching of mathematics relies on textbooks more than any other subject area (Johansson, 2006). Used in up to 90% of mathematics lessons, the mathematics textbook is often the teacher's source of content, sequencing and instructional activities and ideas for lessons (Johansson, 2006; Reys, Reys, & Chaves-Lopez, 2004; Woodward et al., 1988).

Results regarding the quantity of use of student textbooks in Australian schools are consistent with the findings of published research from international studies, which find textbooks are basically used as the defacto curriculum (McNaught, 2005; Watt, 2002). Findings from a recent nationwide study conducted on the role of curriculum materials in Australian Schools indicated that textbooks, used 86.6% of the time during scheduled lessons, play an important role in the development of the curriculum (Watt, 2002). It is a reality of modern times that despite the lack of Australian evidence endorsing their positive value, student textbooks are a mainstay in primary mathematics classrooms, and in many instances are used daily by the classroom teacher. Despite this reality, very little research carried out in the Australian context, particularly in recent times has been dedicated to examining the efficacy of textbook use in primary mathematics classrooms, nor the reasons teachers rely on them so extensively.

Teachers' Beliefs About Mathematics Education and Textbook Use

The literature suggests that a teacher's beliefs about mathematics education are a primary factor influencing their decisions to use student textbooks. Specifically, the beliefs a teacher holds about mathematics education potentially influences both how frequently textbooks are used, in addition to the manner in which they are used (Manouchehri et al., 2000; Stipek et al., 2001). Teachers who hold beliefs about mathematics education consistent with a traditional approach to teaching and learning are more likely to use student textbooks, and they tend to follow the pedagogy and sequences embedded within the student textbook and the accompanying teacher's guide (Stipek et al., 2001). On the other hand, teachers whose beliefs correlate with a constructivist approach to mathematics are more likely to either teach without, or modify the activities within student textbooks (Kagan, 1992; Stipek et al., 2001).

However, this relationship between the beliefs about mathematics education a teacher holds and their use of student textbooks in mathematics is not consistent, and there are inconsistencies between teachers' professed beliefs and their classroom practices (Handal, 2003; Speer, 2005). The foundations for such inconsistencies is likely due to the complexity of the belief construct itself and the fact that a teacher's overarching beliefs about mathematics education and their subsequent classroom practices are mediated by a number of internal and external factors. The decisions teachers make about the implementation of student textbooks in mathematics are influenced by their beliefs regarding their own level of confidence and competence to teach mathematics, perceived pressure from external forces such as administrators, parents, and fellow teachers, as well as their judgements about the educational quality and value of student textbooks (Handal, 2003; Kagan, 1992; McNaught, 2005; Pehkonen, 2004; Stipek et al., 2001).

The research study reported in this paper investigated teachers' commitment to using student textbooks in primary mathematics classrooms in two Queensland primary schools. The aim of the study was to determine what the contributing factors are that influence the teachers' decisions to use student textbooks in primary mathematics classrooms. The study also investigated the frequency and pattern of use of student textbooks by these teachers.

Method

From a comprehensive review of the literature pertaining to teacher beliefs about the use of textbooks in mathematics education a teacher survey was designed. The survey contained three main parts: Part (a) comprised 10 items asking about the teachers' demographics, Part (b) comprised 14 items that used a 5-point Likert scale (Strongly disagree, Disagree, Neither agree nor disagree, Agree, Strongly agree) to investigate teachers' beliefs about mathematics education and student textbooks, and Part (c) comprised 6 items investigating the frequency and patterns of student textbook use in mathematics. Part B of the survey instrument was specifically designed to measure the following three teacher belief dimensions gleaned from the literature:

- Beliefs about the external influences that impact on teachers' use of student textbooks in primary mathematics classrooms.
- Beliefs about the teachers' perceived educational value of student textbooks in primary mathematics.
- Beliefs about the teachers' personal confidence and competence to teach primary mathematics.

The subjects involved in this study were 34 teachers from 2 urban primary schools at the Gold Coast, Australia. The teachers' demographic data are displayed in Table 1.

Table 1

Demographic information detailing teacher numbers by school, gender, years of teaching experience, and year level taught

Demographic Descriptor	Number of teachers	%
School 1	12	35
School 2	22	65
Female	32	94
Male	2	6
0-5years experience	10	29
6-10 years experience	3	9
11-20 years experience	8	24
20+ years experience	13	38
Preschool	2	6
Year 1	6	18
Year 2	10	29
Year 3	5	14.5
Year 4	2	6
Year 5	3	9
Year 6	5	14.5
Year 7	1	3
Total	**34**	**100**

As indicated in Table 1, teachers across all primary year levels, from preschool to Year 7 were represented. The early childhood sector was particularly well represented, with 67.5% of the teachers indicating their current year level being Preschool to Year 3. In terms of experience, 62% of teachers in the study had 11 or more years of experience, thus the majority of teachers could be described as very experienced. The majority of teachers were female (94%) and this reflects the overall demographic distribution in the two schools.

The survey was administered to the teachers at each school as a group during a scheduled staff meeting in order to minimise disruption and inconvenience for teachers. Research indicates that face-to-face administration of surveys is quicker, more accurate and obtains more returns than a mailed self-completion survey (Burns, 2000). Also, questions to clarify the items can be addressed immediately to ensure that all respondents understand the items. The data collected from the surveys were then analysed using the Statistical Package for the Social Sciences (SPSSI5).

Results and Discussion

This small pilot study aimed to explore the relationship between teachers' beliefs about mathematics education and their use of student textbooks, and was guided by the following research question:

What is the relationship between the frequency and patterns of use of student textbooks in primary mathematics classrooms and teachers' beliefs about their use?

Frequency and/or Patterns of Textbook Use As Reported By Teachers

The data collected from the 34 teachers involved in this study provides an interesting insight into both the frequency and the patterns of student textbook use in two Queensland primary schools. Although it was expected that textbooks would be used frequently based on the findings of international research which suggests textbooks are used in up to 90% of lessons, and Australian research which indicates textbooks are used 86.6% of the time (Reys et al., 2004; Watt, 2002), there is an indication from the data that teachers in this study are less dependant on textbooks than the literature suggests. In response to item 15 (My students use a mathematics textbook as part of the planned mathematics lessons most days), over 70% of teachers surveyed in this study indicated that they 'Sometimes', 'Occasionally' or 'Never' use the textbook this frequently. The remaining 30% of teachers who responded with 'Usually' or 'Always' to this item appeared to use textbooks as frequently as the previous studies have indicated – basically in nearly every mathematics lesson. Of the five teachers who did respond in this manner, only 1 teacher reported that he/she 'Always' used student textbooks in mathematics lessons. Thus, it might be inferred that Queensland primary teachers, as represented by the teachers surveyed, are not as reliant on textbooks as is reported in the literature.

Evidence of this lesser dependence on student textbooks in mathematics was also found in the teachers' responses to item 19 (I prefer to teach mathematics lessons with a mathematics textbook). For this item, over 80% of the teachers indicated that they do not 'Usually' or 'Always' prefer to teach mathematics with a textbook.

Further, as with the findings related to the frequency of textbook use, there is an indication that how textbooks were used in the two schools was inconsistent with the findings of previous research. Based on the results of this study it appears that the teachers showed flexibility and changeability in their use of textbooks. While the literature suggests that a teachers' sequence of instruction often matches the sequence laid out in textbooks (Johansson, 2005), over 55% of the teachers surveyed reported 'Never' following the textbook's sequence of lessons, with no teachers indicating they 'Always' follow the textbook sequence. Furthermore, while the literature also implies that the topics presented in textbooks are the topics most likely presented by teachers (Johansson, 2005), the teachers surveyed not only reorganise the sequence of the textbook, but 70% proactively select what they feel is appropriate from the textbook to present in their mathematics lessons.

Lastly, items 17 (All of my students use a mathematics textbook regardless of their ability), and 18 (My students all work on the same activity at the same time from their textbook) pertain to the patterns of textbook use in relation to the different abilities of students. Of the thirty-four teachers, nineteen teachers reported that the students in their class work on the same activity from the textbooks at most 'Sometimes', with only 3 teachers reporting their students 'Always' work on the same activity. Combined with the data from item 17 to which over one third of the teachers reported that they 'Never' have students working from textbooks regardless of ability, it seems that the teachers in these two schools cater to individual abilities and student needs when using textbooks in mathematics lessons.

Relationship Between Frequency and/or Patterns of Textbook Use and Teacher Demographics

The literature suggested that teacher demographics such as gender, year level taught, and years of experience have been found to influence the classroom practice of individual teachers (Nisbet et al., 2000). To investigate the relationship between teacher demographics and textbook use, a series of Chi-square (χ^2) tests were conducted. The results of the Chi-square (χ^2) tests for teacher gender and years of experience obtained non-significant results, and as such in this study it is implied that teacher gender and years of experience do not significantly influence teachers' decisions to use student textbooks in mathematics.

However, for item 18 (My students all work on the same activity at the same time from their textbook), the results indicated that the school at which teachers were employed significantly related to how the textbooks were used. Of the teachers from school 1, 11 out of 12 reported that their students 'Never' or only 'Occasionally' work on the same activity at the same time. Conversely, of the teachers at school 2, 15 out of 21 reported that students are 'Usually' or 'Always' working on the same activity at the same time. Thus, even though it appears that the teachers in this study cater for individual student abilities and needs more than has been reported in previous studies, this result was statistically related to the variable 'school'. These two schools indicated different patterns of use with respect to whether students all work on the same activity at the same time or not. However, as there were only two schools in this study, significantly more data would be required before making generalisations about the impact of the 'school' on teacher use of mathematics textbooks.

Significant results from the Pearson Chi-square tests were also obtained for year level taught in relation to item 15 (My students use a mathematics textbook as part of the planned mathematics lessons most days), item 16 (My students use the mathematics textbook for supplementary educational activities e.g., homework, time-filler, enrichment activities), and item 19 (I prefer to teach mathematics lessons with a mathematics textbook). For each of these items there appears to be a relationship between year level taught and the patterns of textbook use, with teachers in higher year levels using textbooks during scheduled mathematics lessons and for supplementary educational experiences more frequently than teachers in lower year levels. Additionally, the teachers in higher year levels preferred to use textbooks in the teaching of mathematics more so than the teachers in lower year levels. Once again however, the small data set does not allow definitive conclusions to be made.

However, the inconsistencies between previous national and international studies which suggest that textbooks are used frequently and often without modifications, and the data from this study which suggests teachers use textbooks more moderately and display initiative in modifying both the sequence and the content of the textbooks, are possibly related to the lack of Australian research at the primary school level. The difference between this study's findings and those of previous studies would be worthy of further investigation.

Relationship Between the Teachers' Perceived Educational Value of Student Textbooks and the Frequency of Their Use in Mathematics

The results pertaining to the educational value of student textbooks in mathematics were indicative of previous research which suggests teachers have strong positive beliefs about the educational value of student textbooks. When surveyed, these teachers typically reported positive beliefs about the educational value of student textbooks in mathematics.

The results indicate that there is a particularly strong belief held by teachers in this study in regards to the value of the textbook as a source of opportunities for students to practice mathematical skills. The pedagogy of contemporary mathematics textbooks is described in the literature as being focused on skill acquisition through a heavy emphasis on procedural practice (Ball et al., 1988; Chavez-Lopez, 2003). Results from this study seem to indicate that teachers place positive value on this aspect of textbook pedagogy, with half of the teachers surveyed reporting that they believe mathematics education should include the use of a textbook to allow students to practice mathematical skills. Interestingly, 38% of the teachers were undecided as to the value of student textbooks in providing students with opportunities to practice mathematical skills. In fact, despite the overriding positive beliefs in regards to the textbook's educational value, there was a significant portion of the teacher sample who reported they were undecided for each of the items relating to the educational value of student textbooks. An obvious question is therefore: 'Why do significant numbers of teachers (30% in this study) 'Usually' or 'Always' use textbooks in mathematics lessons if they are undecided about the value of the textbook to student learning?' This question alone could provide the foundation for future research investigations. Overall however, the teachers surveyed believe the student textbook is a valuable teaching and learning aid in the mathematics classroom, particularly as a resource to provide students with opportunities to practice various mathematical skills.

Relationship Between External Influences Perceived by Teachers and Their Use of Student Textbooks in Mathematics

A Kruskal-Wallis Non-Parametric test provided evidence of a positive relationship between teachers' beliefs about external influences and the frequency they use student textbooks in mathematics lessons; a result which supports previous studies. Policies in place within schools and the perceived pressure from administrators and parents to use student textbooks are external factors recognised in previous research as potentially influencing teachers decisions to use textbooks (Handal, 2003; McNaught, 2005; Perso, 2005). Furthermore, there is the suggestion in the literature that the increasingly heavy workload demands placed on teachers encourage them to rely on the textbook's pre-planned lessons. Data from this small study supports this position, with only 23% of teachers disagreeing with survey item 8 (I use mathematics textbooks with my students because I believe it makes teaching mathematics easier). Also, a third of the teachers reported that they use a textbook in mathematics because it is expected by parents (item 12).

Relationship Between Teachers' Personal Confidence and Competence to Teach Mathematics and Their Use of Student Textbooks in Mathematics

Overall, the teachers surveyed hold very positive beliefs about their levels of confidence and competence to teach mathematics. Over 50% of the teachers indicated they either 'Agreed' or 'Strongly Agreed' with item 1 (I enjoy teaching mathematics), item 2 (I teach mathematics to my class confidently), item 3 (I would feel confident to teach mathematics to any primary year level) and item 4 (I believe I am a competent teacher of mathematics). In fact, for items 1, 2, and 4, 70% of teachers were in agreement with these statements.

Previous studies suggest that teachers with high self-efficacy regarding mathematics education are more likely to modify or teach without textbooks (Stipek et al., 2001). This result is supported by this study in which the majority of teachers have positive beliefs about their levels of confidence and competence to teach mathematics and at the same time make only a moderate use of student textbooks during mathematics lessons, with 20% indicating they 'Never' use student textbooks. Furthermore, as suggested earlier these teachers indicated that they modify the textbooks by reorganizing the sequence and selecting only what they feel is appropriate from the book for use in their mathematics lessons.

A prevalent suggestion throughout the literature is that low self confidence and competence in teaching mathematics leads teachers to develop a heavy dependence on textbooks as a means of overcoming their perceived shortcomings (Ball et al., 1988; Chambliss et al., 1998). As such, the strong positive beliefs held by these teachers about their confidence and competence to teach mathematics may provide some insight into the inconsistencies between previous research suggesting frequent use of textbooks in mathematics, and the results of this study which finds the majority of teachers from these two schools use student textbooks at most only 'Sometimes'.

Furthermore, the literature indicates that teachers with low levels of confidence in their subject matter knowledge in mathematics often hold the belief that textbook authors possess more mathematical expertise (Ball et al., 1988; Chambliss et al., 1998). It is inferred that teachers who hold this belief become textbook dependant as they believe the authors of the textbook are better able to develop a mathematics program. However, 38.2% of teachers surveyed, disagreed that the authors of mathematics textbooks know more about teaching mathematics than they do, compared with 14.7% who agreed with the statement (item 7). The teachers in this study were relatively confident about their ability to teach mathematics to their particular year level and the results indicated that teachers do not rely on the authority of the textbook when deciding the sequence of mathematics lessons, or which topics are appropriate for students in their class.

In conclusion, the results of this study suggest that student textbooks are used less frequently and in different ways than has been previously reported. However, teachers' beliefs about student textbook use in mathematics align reasonably well with the literature. The teachers involved in this study appear to be making use of student textbooks in their mathematics lessons, as on the whole they possess beliefs that student textbooks are of sound educational value. Also, the strong positive beliefs held by these teachers in regards to their personal confidence and competence to teach mathematics suggests that these teachers are often proactive and thoughtful about their use of textbooks. The teachers report that they reorganise the sequence of the textbook in addition to selecting only what they feel is appropriate from the textbook for use with their class.

However, results of the study also demonstrated that for each of the survey items relating to the factors deemed influential on teachers' decisions to use student textbooks in mathematics: the perceived educational value of student textbooks, external influences, and teachers' perceived levels of confidence and competence to teach mathematics, a large number of teachers reported that they were undecided as to the reasons they use textbooks. This result is of interest, particularly in regards to the large numbers of teachers reporting they are undecided about the educational value of textbooks. Coupled with results which indicated the majority of teachers did not use textbooks because of school policies, or because parents expect textbooks to be used, the result is even more perplexing. It could be questioned from this result why the teachers at these two schools continue to use student textbooks, even to a moderate degree, if they are undecided about the educational value of the textbook and there is not a strong influence on them from external sources to use textbooks. This question could underpin future research investigations.

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Abstraction in Context, Combining Constructions, Justification and Enlightenment

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The nested epistemic actions model of abstraction in context has been used to analyse a solitary learner's process of justification. In previous work, we have shown that this process gave rise to the phenomenon of parallel interacting constructing actions. In this paper, we analyse the interaction pattern of combining constructions, and show that combining constructions indicate an enlightenment of the learner. This adds an analytic dimension to the nested epistemic actions model of abstraction in context.

Introduction and Background

Abstraction has been a central issue in mathematics and science education for many years. The classic work by Piaget, Davydov, Skemp and others has in recent years been succeeded by research fora, symposia and discussion groups at various conferences, as well as several special issues of research journals, most recently the *Mathematics Education Research Journal* (Mitchelmore & White, 2007).

One of the approaches to research on abstraction presented on these occasions is *abstraction in context,* or AiC (Hershkowitz, Schwarz, & Dreyfus, 2001). This approach considers abstraction as a process of emergence of knowledge constructs that are new to the learner. In order to describe such processes at a fine-grained level, abstraction in context makes use of a model, the RBC model, which is based on three epistemic actions to be described below. The RBC model has been used for this purpose by different research teams with students of different ages learning about different mathematical topics (including square roots, algebra, probability, rate of change, function transformations, and dynamical systems), in a variety of social and learning contexts (see e.g., Hershkowitz, Hadas, Dreyfus & Schwarz, 2007, and references therein).

In particular, when analysing a solitary learner's construction of a justification for bifurcations in dynamical systems, Dreyfus and Kidron (2006) found an overarching constructing action, within which four secondary constructing actions were nested. These secondary constructing actions were not linearly ordered but went on in parallel and interacted. Interactions included branching of a constructing action from an ongoing one, combining or recombining of constructing actions, and interruption and resumption of constructing actions. The aim of the present paper is to exhibit a facet of the analytic power of the RBC model for abstraction in context, by building on the research by Dreyfus and Kidron, and showing that combining constructing actions indicate crucial steps in the justification process, which lead to an enlightenment of the learner.

Abstraction in Context

Freudenthal has brought forward some of the most important insights to mathematics education in general, and to mathematical abstraction in particular, and this has led his collaborators to the idea of "vertical mathematization" (Treffers & Goffree, 1985). Vertical mathematization points to a process of constructing by learners that typically consists of the reorganization of previous mathematical constructs within mathematics and by mathematical means. This process interweaves previous constructs and leads to a new construct.

AiC adopts this view and defines abstraction as a process of vertically reorganizing previous mathematical constructs within mathematics and by mathematical means so as to lead to a construct that is new to the learner. The genesis of an abstraction passes through a three stage process, which includes the arising of the need for a new construct, the emergence of the new construct, and its consolidation. The need may arise from the design of a learning activity, from the student's interest in the topic or problem under consideration, or from a combination of both; without such a need, however, no process of abstraction will be initiated.

We note that this view of abstraction follows van Oers (2001) in negating the role of decontextualisation in abstraction, and embraces Davydov's dialectic approach (1990) in that it proceeds from an initial unrefined first form to a final coherent construct in a dialectic two way relationship between the concrete and the abstract (see Hershkowitz et al., 2001; Ozmantar & Monaghan, 2007).

Furthermore, we found that activity theory (Leont'ev, 1981) proposes an adequate framework to consider processes that are fundamentally cognitive while taking into account the mathematical, historical, social and learning contexts in which these processes occur. In this, we follow Giest (2005), who considers activity theory as a theoretical basis, which has an underlying constructivist philosophy but allows avoiding a number of problems presented by constructivism.

According to activity theory, outcomes of previous activities naturally turn to artefacts in further ones, a feature which is crucial to trace the genesis and the development of abstraction throughout a succession of activities. The kinds of actions that are relevant to abstraction are *epistemic actions* – actions that pertain to the knowing of the participants and that are observable by participants and researchers. Pontecorvo and Girardet (1993) have used this term to describe how children developed their knowledge on a historical issue during a discussion. The observability is crucial since other participants (teacher or peers) may challenge, share or construct on what is made public.

The RBC model

For the above reasons, Hershkowitz et al. (2001) have chosen to use epistemic actions in order to model the central second stage of the process of abstraction. The three epistemic actions they have found relevant and useful for their purposes are recognizing (R), building with (B) and constructing (C). *Recognizing* takes place when the learner recognizes that a specific previous knowledge construct is relevant to the problem he or she is dealing with. *Building with* is an action comprising the combination of recognized constructs, in order to achieve a localized goal, such as the actualization of a strategy or a justification or the solution of a problem. The model suggests constructing as the central epistemic action of mathematical abstraction. *Constructing* consists of assembling and integrating previous constructs by vertical mathematization to produce a new construct. It refers to the first time the new construct is expressed by the learner either through verbalization or through action. In the case of action, the learner may but need not be fully aware of the new construct. Constructing does not refer to the construct becoming freely and flexibly available to the learner. Becoming freely and flexibly available pertains to the third stage of the genesis of an abstraction, consolidation. Examples for constructing actions will be given below, in the subsection entitled "Combining Constructions".

The RBC model constitutes a methodological tool used for realizing the ideas of abstraction in context. In this sense, it has a somewhat technical nature that serves to identify learner actions at the micro-level. On the other hand, the model also has a definite theoretical significance; Hershkowitz (2007) has discussed the theoretical aspects of the model, its tool aspects, and the relationships between them.

Combining Constructions and Enlightenment

In this section, we use the RBC model, and in particular the notion of constructing action, in order to describe and analyse the knowledge construction process of one mathematician, to be called L, learning about bifurcations in a logistic dynamical system. We have presented the description of this process elsewhere (Dreyfus & Kidron, 2006). While we were acutely aware that the core of the process is a justification, we did not pay attention to the question what justification means for L, nor did we analyse the relationship of this meaning of justification to the constructing actions and the interactions between them. This is the focus of the present paper.

Methodological Considerations

Gathering data about learning processes is methodologically non-trivial. Gathering data about the learning process of a solitary learner presents even greater challenges because there is usually no need for the learner to report about her learning. In the present study, L's epistemic actions were inferred from the detailed notes she took, her Mathematica files, and her computer printouts. Like many mathematicians, L wrote, graphed, drew and sketched a lot, some by hand and some by computer. As is her habit, she carefully dated and kept these notes as well as all computer files and printouts. These documents later served as a window into her thinking for the researchers.

A priori, L's collection of her notes, files and printouts had nothing to do with a plan for the present or any other research. In fact, the idea of using them as raw data for a research study has only been conceived several months after they had been collected. The researchers then constructed a report of the learning process, following an elaborate procedure of several cycles of description by the first author and challenges by the second author. The accuracy of the report may be verified by observing its close correspondence with the raw data, some of which have been published (Dreyfus & Kidron, 2006).

Once the report was agreed upon, we adopted the RBC methodology for identifying epistemic actions (Hershkowitz et al., 2001). The report of the learning experience was divided into episodes, numbered 1-16. Each episode forms a cognitively coherent unit. For the purpose of analysis, each episode was further divided into subunits, called events, and denoted by Latin letters a, b, c, and so on. Events are the equivalent to utterances for the case of a solitary learner; they form the minimal units that can be categorized as epistemic actions according to the RBC model.

One of the tasks of the researchers when using the RBC model is to decide which constructing actions to focus on. For this purpose, each of us independently proposed constructs, which we saw as emerging during the learning process, as candidates for constructing (C) actions. Agreement between us was fairly high, once we agreed on grain size. We identified one overarching C action and four regular C actions whose relevance was obvious to both of us (for more detailed methodological considerations, see Dreyfus and Kidron, 2006). Next, we independently marked each episode according to which of the four regular C actions were active in the episode. There was full agreement between us concerning active C actions. The analysis of the resulting web of interweaving C actions forms the topic of the next subsection.

Combining Constructions

L was interested the following iterative process: Given the quadratic function $f(x) = x + r x (1x)$, where r is a real parameter, consider the sequence of values $\{x_n\}$ produced from an initial value x_0, $0< x_0 <1$, by successive application of f, that is $x_{n+1} = f(x_n)=f^n(x_0)$, for all $n \geq 0$. L discovered empirically that for certain values of r, the sequence of values $\{x_n\}$ converges to a fix point; for somewhat larger values of r, it approaches a process of period 2, for even larger values of r, a process of period 4, and so on. With some support from books and internet sources, she soon computed that the transition from the fix point regime to the 2-periodic regime occurs at r=2 and that this can be computed on the basis of a quadratic polynomial with parameter r, by showing that r=2 is the smallest value for which the discriminant of this polynomial vanishes, and thus the polynomial has a double root.

In the episodes of interest in the present research, L set out to understand where and why the transition from the 2-periodic to the 4-periodic regime occurs. The corresponding constructing actions and some of their interactions are described in this subsection and illustrated by means of the diagram in Figure 1. The time axis of the figure runs from top to bottom.

L spent a considerable number of hours, spread over about two weeks, investigating this question. While the question is analogous to the one concerning the previous transition from fix point to 2-period, the polynomial $p_r(x)$=0 of interest is now of order 12. Web resources led L to the notion of discriminant for a general polynomial (episode 5), which she used with the help of Mathematica (episode 6) to find the numerical value $r = \sqrt{6}$ for the transition point to the 4period. This value of r neatly corresponded to the empirical evidence she had collected. Encouraged by this numerical success, she began to search for the mathematical reasons behind it (episode 7).

One of L's constructing actions, denoted C_1, is the process of finding the four solutions of the polynomial equation $p_r(x)$=0 in the case of period 4. The solution process is considered algebraically and numerically. The focus is on the solutions for fixed values of the parameter r and on relationships between the solutions for different values of r.

Another constructing action, denoted C_2, is the process of constructing algebraic connections between the transition point from the 2-periodic to the 4-periodic regime, the existence of multiple roots of the equation $p_r(x)$=0, and the zeros of the discriminant of $p_r(x)$.

As can be seen in the diagram in Figure 1, at the beginning of episode 7, construction C_1 branches off from the ongoing construction C_2. This happened when L attempted to algebraically connect between the zeros of the discriminant and the transition point, but saw no way to reach this goal because of the complexity of the equations involved. This led her to take a more familiar approach, using numerical calculations belonging to C_1 that she expected to eventually lead to the same goal.

This branching of C_1 from the ongoing C_2 can be explained by means of a refinement of the classification of building-with (B) epistemic actions. Specifically, a class of B-actions was introduced whose purpose it is to organize the problem space so as to make its further investigation possible. Such actions can lead to the requirement of additional constructions and thus branching.

It has been shown how interruptions, resumptions and combining of constructions can be similarly explained by means of refined and/or modified R- and B-epistemic actions. The reader is referred to Dreyfus and Kidron (2006) for details. Here, we focus on combining constructions, such as the combining of C_1 and C_4 in episode 10.

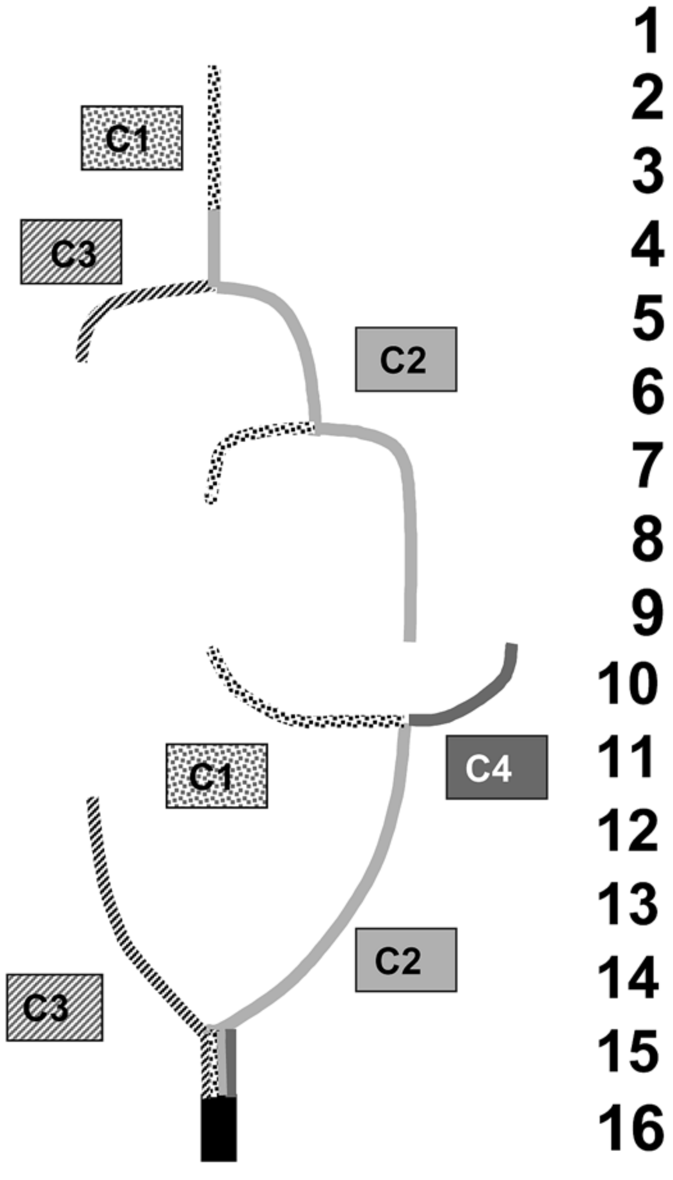

Figure 1. L's interacting parallel constructions.

C_4 denotes the construction of a dynamic view of the bifurcation in which the final state values of the process (the solutions of $p_r(x)=0$) are considered as functions of r. This construction started for L when she had exhausted all her algebraic and numerical resources at the end of episode 9. She considered various graphic representations and focused on the transition of interest:

> 10d Looking at these values in the bifurcation diagram, my attention was focused on the transition from the 2period to the 4period. This focus was different from the one I had had previously when each time series plot gave a partial picture corresponding to a specific value of the parameter *r*.

10f I looked at the fork-like shape and associated its splitting with the fact that the discriminant vanishes. Suddenly, the bifurcation diagram seemed different, endowed with a new meaning. I looked at it and I could not understand how I failed to see it this way before.

The gradual approach between C_1 and C_4 in Figure 1 expresses the connection in L's thinking of the numerical mode and the graphical mode, which during this approach changed from being static to being dynamic. L's integration of the two modes of thinking results in her view of the transition as a dynamic graphic – numerical process: Her conception of the nature of the parameter r changed from being discrete to being continuous. The two constructions C_1 and C_4 have combined.

While this was intuitively satisfying, it only constituted a first stage since it did not provide the justification L was looking for. She had no algebraic handle on the discriminant, which would connect its zeros to the transition point. She returned to the algebraic mode of thinking C_2, which was now strengthened by her graphic-numerical insight, but she made little progress until, in episode 13, she resumed construction C_3, the process of linking between the derivative of a polynomial, in this case the derivative of $p_r(x)$, the zeros of its discriminant, and the stability of fix points and periods.

This was of interest to L because she vaguely remembered that fix points are stable if the derivative of f is smaller than 1 and unstable if the derivative of f is bigger than 1. Thus 1 is the limiting value of stability, and at this value a transition occurs.

L carried out a straightforward but rather technical computation showing that $p_r'(x)=0$ (and thus $p_r(x)$ has a multiple root and its discriminant equals zero) if and only if the derivative of $f^4(x)=f(f(f(f(x))))$ equals 1. Possibly without being fully aware, she extended her vague knowledge that the value of 1 of the derivative of f is the limiting value of stability for fix points to the same value 1 of the derivative of $f^4(x)$ being the limiting value of stability for a 4-period.

13i At this moment, I connected the last equality with my previous vague knowledge that fix points change stability when the (absolute value of the) derivative moves across the border 1 as the parameter *r* varies.

13j At last, I found some connection between the fact that there exists a multiple root (therefore, the discriminant equals 0) and the way fix points change stability.

It turned out that this value of 1 of the derivative contained an unwanted minus sign, and L needed more time and effort to clear this up; this is the reason why in Figure 1 the combining of constructions C_3 and C_2 in episode 13 is only partial.

Justification and Enlightenment

In the episodes of interest in the present research, L set out to understand where and why the transition from the 2-periodic to the 4-periodic regime of the logistic dynamical system occurs. At the beginning of episode 8, and again at the beginning of episode 11, right after the first combining of constructions, she expressed what she was looking for, and it was not a formal proof:

8a My aim was to justify why the transition from 2-period to 4-period occurs for the smallest positive real number for which the discriminant equals zero.

8b I felt the need to explain why the requirement that the discriminant equals zero permitted to find the value of the parameter *r* for which the 4-period begins.

11a I was interested in a mathematical explanation why the transition point from 2-period to 4-period is obtained by setting the discriminant equal to zero.

Her question thus was how the value of the discriminant $D = 0$ was connected to the transition between regimes of the dynamical system. Her aim was to convince herself, not others. She felt the need to explain because she wanted to gain more insight.

L's drive to understand the transition was to some extent satisfied intuitively and visually in episode 10, through the combining of the C_1 and C_4 epistemic actions, and she expressed this as follows:

> 10g Now, it seemed to me intuitively clear that at the bifurcation points there must be double solutions and therefore the discriminant should equal zero.

She similarly expressed added insight at the end of episode 13, and again at the end of episode 16:

> 13k Now I understood why at the bifurcation points the discriminant should equal zero. The different elements fit together nicely, like in a puzzle.

> 16d Now I was sure that my mathematical construction will not collapse any more … I was absolutely confident in my justification why the discriminant equals zero, even if the fact that $f^4{}'(x)=1$ was demonstrated at this stage only numerically (based on intuition and authority).

In this sense, L's use of the word justification was very close to Rota's view of enlightenment in the sense of insight into the connections underlying the statement to be justified:

> Verification alone does not give us a clue to the role of a statement within the theory; it does not explain the relevance of the statement … the logical truth of a statement does not enlighten us as to the sense of the statement. … every teacher of mathematics knows that students will not learn by merely grasping the formal truth of a statement. Students must be given some enlightenment as to the sense of the statement. (Rota, 1997, pp. 131-132)

and

> Mathematical proof does not admit degrees. A sequence of steps in an argument is either a proof, or it is meaningless. Heuristic arguments are a common occurrence in the practice of mathematics. However, heuristic arguments do not belong to formal logic … . Proofs given by physicists admit degrees. In physics, two proofs of the same assertion have different degrees of correctness … . A great many characteristics of mathematical thinking are neglected in the formal notion of proof. (ibid., pp. 134-135)

Thus the combining of the C_1 and C_4 actions is an expression of L reaching a first degree of enlightenment, and a feeling of having to some extent explained and justified the structure of the double solution at the transition by means of a dynamic view of the bifurcation. Similarly, the combining of the algebraic mode of thinking of C_2 with the analytic mode of thinking of C_3 in episode 13 expresses L's second degree of enlightenment, which is reinforced by the fact that her previous, if vague, knowledge about the stability of dynamical systems directly confirmed the connection she had established computationally.

Finally, she reached a third degree of enlightenment in episode 16, when she was able to numerically link the dynamic view of the transition – the C_1/C_4 construct from episode 10 – to the link between stability and derivatives – the C_2/C_3 construct from episode 13 – and thus achieve an integration of all four constructing actions.

Conclusion

We remind the reader that the interacting parallel constructions diagram in Figure 1 was obtained by considering L's process of justification as a process of abstraction in context and analysing it by means of the epistemic actions of the RBC model. Only after this description of the process was complete, did we realize the particular meaning, which L associated with the notion of justification, and discover that each additional degree of enlightenment occurs with a combination of two constructions, and each combination of two constructions indicated an additional degree of enlightenment. This enriches the analytic power of the RBC-model: It allows researchers to use the epistemic actions of the RBC model in order to identify a learner's enlightening justification.

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How Humanism Can Foster Mediocrity in Early Years Mathematics Education: A Poststructuralist Comparison

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In this paper I argue that humanist understandings of learners can underscore mediocrity in mathematics learning in the early years. Although many children come to school ready and eager to learn mathematics, it can happen that their classroom experiences alienate and disenfranchise them. This sometimes occurs when teachers, deferring to humanist understandings of learners as naturally capable and competent and learning as experiential, teach little mathematics but concentrate on fashioning the learning environment to supposedly make it non-threatening, 'enjoyable' and 'relevant'. In contrast I use the poststructuralist notions of positioning and subjectification to suggest that learners can not be positively positioned in the discourse of mathematics education if they are not given the opportunity to construct robust mathematics and generative and idiosyncratic ways of thinking and reasoning in mathematics.

When they come to school young children are quite fascinated by mathematical ideas and have an easy and energetic confidence in working out simple problems (Askew & William, 1995; Hughes, 1986). Many of them count purposefully with few mistakes (Lambert, 2000). Over time, though, an initial fascination fades and these very same students are not backward in asserting that they neither like mathematics, nor the learning of it. As stated in *A National Statement on Mathematics for Australian Schools* (1990, p. 31) "there is considerable evidence that children come to school enthusiastic and eager to learn mathematics" and "leave school with quite negative attitudes". How can it be that an initial desire to use and learn mathematics waxes and wanes and initial interest and confidence turns to aversion and dread?

This is an important question for those teaching and researching in the early years of schooling, where I suspect feelings of alienation and frustration take root. Although policy documents such as those from the Queensland Studies Authority (2005) recommend learners' active engagement in environments of investigation and play, little has changed in classrooms where routinised computation and worksheet or textbook work prevail (Askew & Williams, 1995; Hardy, 2004; Willoughby, 2000). Although teachers, and in this paper I refer to student teachers in a preservice program in regional Queensland, hope to improve learning environments, their own constituted sense of what mathematics is and humanist, psychological notions of learners scuttle their best intentions. This is because proposed changes at the classroom level do not merely involve 'sugar coating' established teaching practice, but qualitatively change teaching-learning relationships to emphasise the active and productive role of students (even very young students) as initiators of learning and creators of knowledge. The new ways of being a learner (and teacher) of mathematics are premised on new power relationships and new conceptions of learners that are not considered in humanist assumptions about learning. At the moment meaningful participation is denied many students and active engagement becomes little more than a ruse, or sham, as the mathematics is cosmetically enhanced, though stripped of its reasoning processes and robustness.

To try to better understand this issue I asked preservice teachers intending to teach in the early years to describe some strategies they could use to enhance learning in mathematics, and to say how each strategy would actually boost their students' learning. I wanted to know first of all which discourses they found seductive and convincing, and then to analyse their comments to contemplate the possible effects on learners in their care. For example, I anticipated that these preservice teachers who had completed a semester long subject engaged in exploring mathematics as a science of pattern and order would stress teaching strategies that ensured engagement in learner-generated reasoning processes, leading to understanding and the construction of robust mathematics. I felt that some at least of the preservice teachers were keen to make a difference; I set out to find out the assumptions that would guide their teaching practices and the possible consequences of these actions for their pupils' learning and identity construction.

A poststructuralist, analysing learning environments, assumes that learners are de-centred and at the mercy of intersecting relationships of power which inhere in instructional (discursive) strategies and are productive of identity. For example, Dahlberg et al., (1999, p. 31) reiterate how Foucault cautioned that discourses speak us into existence, they "shape our understandings of what is possible and desirable". So learners of mathematics take up discursive positionings as their own, and come to know themselves as competent, confident and authoritative (in the sense of having *author*ship of ideas and practices) in performing mathematical tasks and applications or as marginal to the operation of the discourse. A poststructuralist turns to the operation of the discourse (the knowledge produced therein and the relative positionings available to learners) when evaluating outcomes. On the other hand, the researcher (and teachers) wedded to psychological (humanist) readings of learners assume a rational, autonomous individual 'naturally' able to engage in learning tasks. When learning outcomes are not met, attention turns to the individual learner rather than the regulatory and constraining teaching practices. For example, a teacher might ask why Trudy can figure out some mathematics and Tom can not. Has Tom not been listening? Has he neglected to do the homework? Is he just not good at figuring out? Each question is laden with some sort of implied deficit on Tom's part and leaves the teacher nowhere to go; other than to position Tom as 'not good at figuring out' and in need of help. Although we are not likely to be able to dispense with humanist ways of reading the world, a poststructuralist analysis attempts to make visible how the use of language, as in Tom's case, produces what is taken to be real (Weedon, 1987); in this case, that Tom is mathematically deficient in some way.

Mathematics classroom worldwide operate on humanist understandings of learners. Mathematics education is informed by Piaget's child development through stages, Vygotsky's social interaction is a key force in the development of mind, and Lave's (Lave & Wenger, 1991) 'situating' learning in socially supportive contexts; each of these is framed by notions of the rational, autonomous learner of mathematics and the principles of developmentally appropriate practice (DAP). As Yelland and Kilderry (2005) point out, these intersecting notions and teaching principles comprise a meta-narrative informing education in the early years, and it is difficult to understand children and learning outside this discursive frame. However, while the theories above make important epistemological contributions regarding the construction of mathematical ideas, they do not recognise how learners themselves, and what counts as mathematics, are *produced* in teaching-learning interaction (which often privileges adult control and direction and ignores diversity). These theories are silent on the ontological dimension of *how* it is (rather than why) that so many young students are not confident, 'turned off' mathematics and wouldn't do it even if they could (Willoughby, 2000). As suggested by Yelland and Kilderry (2005), if developmental theories such as Piaget's could be removed from positions of *primacy* in the field, new ways of conceptualising and engaging with learning in the early years might emerge.

In the table below humanist and poststructuralist notions of the learner and learning are compared. Humanism takes for granted rational, autonomous learners "competent and capable" as in the Early Years' Curriculum Guidelines (Queensland Studies Authority, 2005). On the other hand, poststructuralism posits a contradictory, multiple, multi-layered self, constituted through engagement in a range of discourses over one's life. In this research I analyse the preservice teachers' discourse to contemplate their seduction by humanist understandings of the learner and the possible later effects on teaching practice.

Table 1

Humanist and Poststructuralist Notions of the Individual

	HUMANISM	POSTSTRUCTURALISM
LEARNER	Rational, coherent, autonomous being. Ability and attitude are personal attributes.	One's identity (subjectivity) is constituted in discourses such as mathematics education through one's own and others' acts of speaking and writing. The learner seeks to be recognisable (by oneself and others) as a legitimate participant in the discourse and discursive practices.
LEARNING	Learning mathematics is about constructing knowledge. Learning choices are based on rational thought. Learners have a choice, and those who do not make the 'correct' choices are somehow at fault.	Intellectual and self knowledge are constituted in the learning process. Learning is rhizomatic, rather than linear, a process of establishing oneself as competent and confident in a particular discursive field. Co-requisites include: Space to make personal sense of discursive 'truths' (mathematical knowledge) and practices; Having access to a subject position in which one has a right to speak and be heard, including the right to initiate new discursive threads (ideas or actions).
AGENCY	All persons have agency; they are autonomous.	Agency is not a personal attribute; it is constituted in discourse (discursive practices). Agency is a discursive position available to some persons some of the time. Agency includes (the dot points above, plus): Having a constituted sense of oneself as able to go beyond the given to forge new/innovative ways of being or acting in a discursive field.

A poststructuralist analysis focuses on discourse and discursive and regulatory practices (Davies & Gannon, 2005). Any setting where discourses are mobilised can be used for research; in this case I have analysed second year preservice teachers' comments about what they consider to be responsible pedagogical practice. The data that I present are examined not as if they described the 'real world' of preservice teacher education, but as indications of the constitutive work that has formed these particular prospective teachers and their taken for granted assumptions. The preservice teachers were responding to the request to:

> Describe briefly some strategies you could use to enhance learning in mathematics in the early years; say how each strategy would be beneficial to your students' learning.

Analysing the Preservice Teachers' Discourse

A first analysis of the preservice teachers' comments found that most of them focused on 'making maths fun/enjoyable' and 'making it relevant/something they (their students) are interested in'. For example, the discourse the novice teachers used demonstrated their strong desire to make the learning of mathematics an enjoyable experience for those they would teach. It was almost as if they intended to teach in an environment where 'having fun' would be unproblematic and easily translate into competent, numerate persons. Some of their statements about the strategies they would use to enhance learning included the notion that 'Best learning comes from children having fun' (a statement of one of the preservice teachers):

> To enhance children's learning in mathematics I would try and make it appealing to them by using a hands on approach, not just writing sums in a book. Students will be more involved and interested in maths.
>
> Use games because this shows that maths doesn't have to be boring! Games can be a very helpful teaching tool because they get the children excited and interested in learning.
>
> Make maths fun and interesting, using fun resources, hands on things.

> Use small groups to give variety and fun; work stations were also mentioned for their ability to make learning enjoyable
>
> Use hands on as the children remember more through participation in experience.
>
> Use concrete resources
>
> Make students feel comfortable when asking questions and encourage questions
>
> Only one student (out of the 37) concentrated on the mathematics and suggested the use of ICTs to encourage and enhance the learning; that teachers encourage and emphasise mathematical relationships, have students explore their own daily activities to see where maths and numeracy are relevant, and when, how and why it is used.

From a poststructuralist perspective what was missing from the preservice teachers' talk was any mention of strategies that would ensure their students' engagement in mathematics, in mathematical reasoning processes, such as those of representation, justification and generalisation, necessary for the construction of robust knowledge, the foundation of meaningful participation in the discourse of mathematics.

A second concern of many of the preservice teachers was that the mathematics should be 'relevant' or 'authentic'. One student stressed that mathematics should be related to real life and used a 'relevance=interest' equation. Again, in their discourse the preservice teachers make it clear that they would do what is necessary to make learning mathematics palatable, positioning their students' interests (rather than the students themselves, and their participation as agentic, generative) as of utmost importance. Some of the ways they wrote about 'relevance' included:

> Rather than using the traditional methods for counting for example, some children may benefit from counting familiar /favourite objects. Eg: If Ben loves trucks, allow him to bring in some trucks to practise counting with. Helps to develop a love for mathematics.
>
> I could incorporate maths into other subjects and activities, in the curriculum. This way the children can see how maths links with everyday concepts and becomes part of everyday life. They may also find it more interesting than just copying from the blackboard.
>
> Use authentic activities as these keep interest.
>
> Students enjoy and relate to outside use (by showing maths in everyday situations the better the students understand)
>
> Relate it to everyday life (relevance=interest)
>
> Real life examples make maths a lot easier to understand especially for younger children.
>
> Make it relevant and something they are interested in (caters to needs of different learning levels)

In these discursive events there is an underlying notion, held by the preservice teachers, that they are sharing their power with the students, using contexts that are 'relevant' and 'authentic' for their students means that they are harnessing the students' (assumed) interest, and interest is assumed to invoke an inherent and unquestioned competence. The assumption is that if the students are interested they will learn; however, the question could be asked just what are they learning and are they really interested in the mathematics, or something else? Some miscellaneous comments, which satisfy a full representation of the kinds of comments the preservice teachers made included:

> Reinforce learning by repetition
>
> Teach math early in day

Even If I Hate Maths, Don't Let the Students Know

The quote I have used to introduce this section of the paper signifies again a preservice teacher's desire to make mathematics something the students regard favourably, and the humanist assumption that s/he can manipulate the learning environment to make it so. The last thing prospective teachers would want to engender is a hatred of mathematics, yet in the expression of constituted desire (to have 'fun' and 'relevance' as key components of learning) this may just be the outcome. One (double sided) reason for this might be that the external cosmetic tampering with 'the environment' for learning mathematics does little for the 'learning' and even less for the mathematics. The teacher operating under humanist assumptions assumes that negative effects of power can be extracted from the learning process, rendering the students free to act according to reason and choice. The 'humanist' teacher can assume much more of her teaching than it delivers, and epithets such as those below from the preservice teachers more likely than not go unfulfilled and unchallenged:

Make maths fun and interesting

Make students feel comfortable

Make maths a lot easier to understand

Make it relevant and something they are interested in

One consequence, to do with the learning process, is that it is as manipulated and regulated by the teacher as it ever was. The teacher sets out to make everything OK and in so doing chooses tasks that will be enjoyable for and relevant to, all students . However, a postmodern world is characterised by difference, heterogrneity and contradiction and any chosen task can not appeal to all learners. Since this appeal is taken for granted though, the teacher does nothing to focus on students' idiosyncratic ways of making sense of mathematical ideas that would render them participants in the learning community. A second problem is that the students learn to be suspicious of mathematics since it needs so much dressing up to make it palatable; students are 'turned off' even before they are granted entry to mathematics' order and pattern. Teacher dominated regulatory practices are maintained and the learner of mathematics is alienated and frustrated by not being able to make sense and participate in personally meaningful ways.

Another consequence of humanism, to do with autonomy, is that because it is assumed to be a commodity available to all, nothing is done to lessen the effects of power relations that delimit students' active engagement. Mathematics today is not viewed as 'a set of correct answers but a method of reasoning, a way of figuring out a certain kind of system and structure in the world' (Department of Education, QLD; 2001, p. 898). Consider the learning process appropriate and fruitful for young learners; it is one where grappling with rich mathematical ideas is paramount, where learners have a real 'presence' and license to ask questions and initiate lines of inquiry, one where they are encouraged to explore new ways of thinking and reasoning as they struggle to establish themselves as numerate subjects. The ontological in learning can not be denied, and new power relationships are needed that recognise the contingencies of productive learning, that recognise the productive quality of all pedagogic encounters. After all, mathematics education is a discursive field in which the discourses of mathematics and education come together in teaching strategies that structure the learning experience; the way in which mathematics education is played out in any context affects the extent to which learners can establish themselves as competent and confident, numerate subjects.

A related issue to a cycle of mediocrity circulating through mathematics education is that any sort of change is likely to be slow in coming. One reason for this is that a discourse privileging 'enjoyment' and 'relevance' is very convincing and could surely not be problematic. But more is left out of this discourse than is said, and it is likely that this sort of talk is likely to reign in early years education for some time. The teacher and parent deferring to humanist perceptions of the child will favour 'enjoyment' and 'relevance' in education, while the poststructuralist might insist that the opportunity to learn some robust mathematics and actively participate in the discourse might be more 'relevant'. Where humanists see 'relevance' in the external environment, poststructuralists consider it to be visceral, internal to the student who senses a certain capability, a desire to learn more about some aspect of mathematics. A second reason that change is unlikely is that teachers' identities stand strong in the humanist tradition. Any problems can be sheeted home to the students, and the teachers only gain in prestige and power from their attempts to make learning so appealing for their students. They often lack mathematical knowledge themselves, and in concentrating their energies on the external environment, on physical resources and students' active participation in games and play, they manage to keep their teacher identities in tact.

Conclusion

In this paper I have argued, from a poststructuralist perspective that recognises the constitutive power of discourse, that in spite of teachers' best intentions contemporary learning environments can be quite banal, with dour consequences for learners, especially in the early and primary years of schooling. I am concerned that preservice teachers' allegiance to 'enjoyment' and 'relevance', as currently constructed and played out in schools does not necessarily enhance the learning of mathematics; indeed I have argued that the opposite may indeed be the case. The humanist inspired assumption that learning is experiential encourages teachers to employ teaching strategies that focus on active engagement and play; where students are supposedly 'free' to take the learning in directions they choose, according to individual effort and drive. Power relations are seen to be negative and denied, their continued invisibility ensuring the maintenance of mediocrity in learning experiences that do little to inspire and mathematically engage young learners.

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Preservice Teachers and Numeracy Education: Can Poststructuralism Contribute?

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Numeracy is the confident and critical application of mathematical ideas in a variety of contexts. Numeracy has an ontological dimension, it is a state of being numerate, that has its genesis and growth in mathematics education and life. Preservice teachers are expected to teach their students for numerate participation in a global world, although they themselves oftentimes lack the necessary mathematical foundations and strategic and critical skills. In this paper I describe how an awareness of the poststructuralist processes of subjectification and positioning informed my teaching in numeracy education, and the possible effects.

In keeping with changing times, where robustness and flexibility of mathematical thought are of the utmost importance in problem solutions, definitions of numeracy go beyond the notion of computational competence to stress effective participation in life and work. As stated by the Queensland School Curriculum Council (QSCC) (2001, p. 2):

> Numeracy involves abilities that include interpreting, applying and communicating mathematical information in commonly encountered situations to enable full, critical and effective participation in a wide range of life roles (Queensland School Curriculum Council, 2001, p. 2).

Beyond the mind and the purely mathematical, numeracy signifies a state of being where one manifests "practices and dispositions that accurately, efficiently, and appropriately meet the demands of typical everyday situations involving number, space, measurement and data" (QSCC, p. 3). The notion of 'confidence' has now also come to the fore, with Education Queensland (2007), for example, stressing that teachers must be well enough equipped to ensure that their students have the confidence to apply mathematics in their everyday lives (Minister's Foreword). Indeed, their *Framework for Action*, *2007-2010 is titled Numeracy: Lifelong Confidence with Mathematics (Education Queensland, 2007).* Given that many preservice teachers, especially those keen to teach in primary and early years' classrooms themselves lack robust mathematical knowledge, confidence and a positive disposition in application of mathematical ideas and strategies, one wonders what magic wand can be waved in teacher education to bring about their enlivened and appropriate use of mathematics in the learning context of a classroom.

While a variety of cognitive, psychological theories can well inform the construction of robust mathematical knowledge, they tend to fall short of adequately explaining that dark chasm that exists between knowledge construction and responsible application, especially in regard to teaching practice. Here I am referring to the tenuous link that often exists between teachers' knowing about and approving more inquiry based teaching practices and the actual facilitation of "cultures of sense-making", where the child is at the centre of the learning process, constructing personally meaningful solutions (QSCC, 2001, p. 4). Oftentimes it is assumed that there is a linear link between knowing and doing, but teachers often revert to traditional ways of working even when they are well versed in, and approve of, more innovative and inquiry-based methods (Foss & Kleinsasser, 1996). Related to this is the phenomenon whereby some teachers and preservice teachers feel uncomfortable with proposed 'new' methods and resist them from the start. For example, as Nicol (2006, p. 31) found in her research with preservice teachers, they collectively agreed that "teaching in ways that respect students' thinking and sense-making was not worth the time, the effort or the consequences". Clearly pedagogic challenges present themselves for teacher educators who hope that their students will teach a rich and robust mathematics in ways that inspire their pupils with confidence and a passionate regard for mathematics and its responsible use in the world. Although this appears to be a tall order indeed, it may be that new philosophies and ontologies can overlay (but not obscure) the cognitive and psychological in teacher education to inject new energy and understanding into teaching for numeracy.

We teach in postmodern times characterised by change and uncertainty. In this paper I use the poststructuralist concepts of subjectification and positioning to unsettle the taken-for-granted meta-narratives of learners (in this case, preservice teachers) as rational and autonomous agents of change. The concept of subjectification reminds us that all preservice teachers have been and still are *subject(ed) to* teaching strategies and practices that make imaginable what they might be as teachers in the future. That is, the way in which novice teachers are able to establish themselves as active participants in the mathematics education discourse in teacher education has implications for their future teaching. It is an empowerment issue, and important that they are able to recognise themselves as having a real presence, and the knowledge and skills to be able to teach in generative ways. Taking heed of the imperative to position preservice teachers positively in learning to teach for numeracy, I attempted to turn around the broader discourse of 'deficit' which currently frames much of what is done in teacher education to embrace the potential that each novice teacher brings.

Methodology

Poststructuralist thought with its emphasis on subjectification and positioning of learners within a discourse such as mathematics education makes visible how relationships of power support or suppress learners' recognition of themselves as capable participants in that discourse. As Foucault (1982, 231-2) cautioned, when one looks at relationships of power "everything is dangerous". All practices and interactions are dangerous because they are constitutive of the learners' developing identities as numerate beings (or teachers of numeracy). In numeracy education where it is hoped that positive dispositions, confidence and critical application of mathematical ideas will be facilitated, it is crucially important that teaching strategies and interactions comply. In school and at university, strategies are needed that make available to all learners robust mathematical knowledge, that respect the learner as active participant and initiator of reasoning and sense-making in mathematics, and that help the learner make connections between mathematical ideas and life. These are the strategies I attempted to employ in my teaching; they have to do with positively positioning the learner as an active agent in sense-making in mathematics (and learning to teach for numeracy) and could be summarised as comprising:

- Robust mathematical and pedagogical knowledge (needed to be powerful in the mathematics education discourse)
- The sense that one has a legitimate 'presence' as active learner able to 'struggle' through/engage in a growth process
- The encouragement to go beyond the given, to initiate and follow new pathways.

Researchers in teacher education will note that these recommendations are very similar to those that have influenced mathematics education at the level of policy for many years. However, in poststructuralist thought subtle differences are seen to exist in the interpretation of learners and learning that frame these strategies. For example, the learner in poststructuralism is in no way thought to be an autonomous, rational being; s/he is buffetted and challenged by the practices, rules and regulations of the institution, within which there is a scrambling to establish oneself as competent. The learner is de-centred, in no way at the centre of learning but at the mercy of pedagogical strategies and practices, including the robustness of the knowledge presented for construction; collectively these support or suppress a realisation of self as numerate. On the other hand, Education Queensland (2007, p. 4) takes a more autonomous view of the learner and says: "All students can succeed in mathematics and develop a positive attitude and confidence in using mathematics." While the intention is good, in a way this reading of the learner as naturally competent if only s/he chooses to put in the effort, is dangerous; this is because the onus is put on the learner to be positive and competent when really the machinations of the classroom can render this unlikely. For example, if in my attempts at 'value adding', say in having preservice teachers investigate and work through self-generated computation methods, I cause them to 'lose face' (perhaps due to their not knowing number facts, or not being able to demonstrate flexibility of thought when using number combinations), any cognitive gain is eradicated by a constituted sense of not being able to cope in a learning context where "Everything is dangerous" (Foucault, 1982, 31-32).

Then, too, there are different views of the productivity of the learning process. Education Queensland (2007, p. 4) takes a long term stance and says: "Students' enjoyment in classroom mathematics influences their confidence with numeracy outside the classroom." This makes it sound as if students 'have' enjoyment, as a commodity, that ensures confident application of mathematical ideas after school. Again, this is dangerous because enjoyment does not ensure anything; a difficult issue in mathematics education is that teachers and students can spend untold hours 'enjoying' tasks that have nothing mathematical about them at all. The poststructuralist position is that as intellectual knowledge is constructed, so too is a mathematical identity (a sense of oneself as active, competent participant, or not) and tampering with the external environment, for enjoyment, is always problematic. Rather, it could be said that students need appropriate levels of challenge and satisfaction in doing and using mathematics within a culture of sense-making (QSCC, 2001,

p. 4). Learning to establish oneself as numerate does ideally have a measure of satisfaction about it, but this comes from within, from the acquisition of robust knowledge, rigorous and flexible thinking processes and a recognition of oneself as capable, competent and agentic in doing and using mathematics appropriately. In teacher education, then, the aim is not to 'have enjoyment' but for the preservice teachers to find an 'at homeness' and satisfaction in ways of doing mathematics that focus on thinking and reasoning, that encourage them to justify and generalize as they struggle out solutions for themselves.

Building Repertoires of Teaching Practice

Education Queensland (2007, p. 4) in *Numeracy: Lifelong Confidence with Mathematics* stipulated that teachers need various knowledges when they are to teach for numeracy. They need knowledge of mathematics as a discipline, of how students learn and transfer mathematical knowledge and skills, and how teaching impacts on student use of numeracy. I attempted to develop the mathematics through these pedagogical emphases, rather than the other way around, where various 'methods' might be used to achieve the construction of mathematical knowledge. I began in the first year numeracy subject by tackling the preservice teachers' knowledge of mathematics as a discipline. My hope was that by coming to appreciate the relationships and orderliness of mathematics these preservice teachers would learn some mathematics, learn to appreciate mathematics as a discipline worthy of study and have a structured approach to their teaching later on. Perhaps they could construct some foundational knowledge so that they could begin to recognise themselves as active agents in teaching numeracy without being put 'on the spot' too often.

I hoped that these students would construct some 'key ideas' that would frame their teaching, and that they would engage in and be able to make explicit details of the reasoning process. Mathematics as a science of pattern and order was highlighted, and the novice teachers were encouraged to 'see' and appreciate that order. The National Council of Teachers of Mathematics (NCTM) (2007, p. 2) proclaimed "It is striking that given the robustness of the link between instructional attention to important relationships and students' level of understanding, typical classrooms in the United States focus on low-level skills and rarely attend explicitly to the important mathematical relationships (Hiebert et al., 2003; National Advisory Committee on Mathematics Education, 1975; Rowan, Harrison, & Hayes, 2004; Weiss et al., 2003)." I considered it important to redress this issue and placed 'up front' the mathematical relationships that students in school would need to construct. I encouraged the preservice teachers to expand on them and suggest tasks to develop this knowledge as well as 'real world' applications. For example, when dealing with 'Number' they were given an A4 sheet of paper that read:

These are some key ideas you will want the students in your class to construct (adapted from Van de Walle, J. 2007):

- Every number is related to every other number in a number <u>relationship</u>. For example, 8 is 2 less that 10; made up of 4 and 4 (or 3 and 5); and is ten times 0.8.
- Number <u>relationships</u> are the foundation of strategies that help children remember number facts. For example, knowing 4+4=8 allows one to quickly work out 4+5= . If one knows 2X5=10, then 4X5 and 8X5 can easily be calculated.
- Each digit in a written numeral has a 'place' value which shows its <u>relationship</u> to '1'. For example, in 23.05 the value of the '2' is 20 ones, while the value of the '5' is only five-hundredths of one.
- Fractional parts are equal-sized shares or <u>portions of one whole</u>. The whole can be one object or a collection of things. More abstractly the unit is counted as 1. On the number line, the distance from 0-1 is the unit.
- Fractions can be represented as common ($^3/_4$) or decimal (0.75) or as a percent (75%).

Throughout the subject the students struggled with developing a numeracy for teaching; they took each dot point and, in groups or on a chatline constructed mathematical expressions and models that represented these ideas. For example, they made a resource that could be used in the classroom to demonstrate the three representations of fractional amounts above, and completed mental computations based on the use of number relationships and computational strategies (making explicit their thinking and reasoning strategies). The mathematical ideas and the process of their learning were unfamiliar to many of them, as they were more used to being shown a procedure in learning mathematics. Nevertheless, the responses on the chatline were generally positive:

> Maths was my least favourite in school (primary and high school) yet for the first time in my life I'm thoroughly enjoying the subject. It's great! As I said to my husband, I feel as though at school I 'did' maths but now I'm 'learning' maths … and it's a massive difference.
>
> This is a basic maths subject yet I'm not only learning how to teach maths concepts – in some cases I'm actually learning the material for the first time. I totally endorse teaching students to understand concepts through exploration of relationships and with practical examples – it gives the material both 'life' and relevance, therefore motivates them to pursue it.
>
> I find it a bit enlightening that the theories of maths can be related to so much of our planet. You can find math in nature, in the universe, in street directories even!!! I had never really thought about that before – as much as I have struggled with my fears of this subject (to the point of being physically sick one night) I am beginning to appreciate the fact that maths is not something only 'clever' people do … .
>
> For my part, I can't believe how much I've enjoyed this subject - & I never thought I'd say that. I absolutely loathed maths at school because for my entire schooling maths was never fully explained, no enthusiasm was injected, we just had to keep up and as long as we passed the exam, that was it.

The knowledge construction that I have mentioned above had to do with investigating and getting to know the relationships and pattern of mathematics and the thinking and reasoning processes that rely on and cement these relationships. This knowledge involved personal (and group facilitated) investigations of mathematical ideas. However, there was another arm to knowledge production in the form of the procedural knowledge that would be needed for teaching; for example, novice teachers need to know how to denote and track generalisations in algebra, how to write measures correctly, how to speak and write the language of mathematics. To this end I produced a web based interactive program that introduced them to the basic language and concepts that would be taught in the primary years of schooling, stressing the language and recording of mathematical ideas and connections to the world. Many of the prospective teachers had not been at school for many years, and even those who had, were not adept in using and correctly recording the language of mathematics. While much of the procedural in teaching mathematics could be dealt with in tutorial groups for the on-campus students, those on-line had to rely fully on the program on the web.

Analysis and Implications

An interesting assumption of poststructuralism is that 'relevance' in learning pertains to any interaction that helps ensure a student's admission to and generative participation in, a learning community. Because identities are seen to be constituted in discursive practices, it is important that these practices collectively provide:

- Robust knowledge (needed to be powerful in any discourse)
- The knowledge that one has a legitimate 'presence' as active learner able to 'struggle' through/ engage in a growth process
- The encouragement to go beyond the given, to initiate new pathways.

These were the aims framing the development and implementation of the numeracy subject. Although the comments above indicate that most student teachers seemed to feel quite positive about their involvement and learning in the subject, there are many issues for the teacher educator to explore further.

The first issue might be that it is very difficult if not impossible to get anything like an 'objective' reading of what was really going on here with the students' learning. Power relations always pertain and the comments students make are nuanced by the knowledge of the lecturer's reading of chat room and tutorial conversations. It is unlikely that students reliant on a pass in the subject would complain too bitterly about the content or delivery; one reason for this might be their previously constituted notion that when you fail or find things difficult in mathematics it is your own fault, in that you are just not 'good' at it.

Although change can come from changing the discourse and discursive practices, there was also the chance that in changing them I would run the risk of totally alienating those students firmly wedded to previously constituted notions of what mathematics is and how it should be taught. It became clear to me during tutorials that many of the students really struggled with the transition from 'doing sums' to the use of mental computation strategies, for example. This notion was further reinforced in the exam where some students resorted to known algorithms even though the instructions were to use a variety of mental strategies and show thinking and reasoning processes as we had done in chat groups and tutorials. This I saw as a problem with two distinct implications. On the one hand these students had not constructed new knowledge for teaching, which included the mathematical as well as ways of learning that focused on thinking and reasoning mathematically. I had wanted them to come to know new ways of 'being' in mathematics, new ways of constructing mathematical knowledge; through active collaboration in groups where they 'made' new knowledge together. Because of their reticence I also became aware that no matter what one does in the name of inducting novice teachers into a numeracy for teaching, there can be no certainty as to who will pick up and run with new ideas and who won't. Any change is always piecemeal and diffuse, depending on past and present discursive engagements. One has to learn to live with uncertainty.

The second dot point above indicates the importance of the affective dimensions of the learning environment; in poststructuralist thought this involves not so much the emotions but positive positioning in the discourse. It is taken to be important, if students are to be well positioned and agentic in educational discourses, that they feel themselves to be fully 'present' with the ability and right to speak and be heard. This is a problem in so many ways for the preservice teachers. There are some who do not come to tutorials because they find the experience too embarrassing and draining because of their lack of mathematical knowledge and their anxiety. Then there are the practicum teachers who will not permit the prospective teachers to take an authoritative and equal position; rather than assume they can do it, they sometimes assume that they can not (and quite often, of course, the preservice teachers can not). On top of all this, they are so used to having to come up with what the teacher wants that they may find any deviation from the norm stressful. And last but not least is the intriguing question of whether the on-line students are as well positioned in the discursive relations as the face-to-face; or are they potentially better served because they do not have to 'endure' lecture and tutorial sessions where they stand to lose face amongst their peers and the lecturer?

Conclusion

Preservice teachers are expected to teach their students for numerate participation in a global world, even though they themselves oftentimes lack the necessary mathematical foundations and strategic and critical skills. In this paper I have used the poststructuralist concepts of subjectification and positioning to contemplate what I considered to be a 'value adding' to numeracy education in preservice programs. That 'value adding' comprised an explicit concentration on mathematical patterns and relationships and connections to the classroom and life, which the preservice teachers developed and investigated in the numeracy subject. As well there was an emphasis on 'new ways of being a learner of mathematics' with the explicit recognition and application of thinking and reasoning processes and the communication of mathematical and pedagogical ideas. However, not everyone agreed with the 'value' of the value adding, though some students appreciated the new emphases:

> I think if we had been taught mathematics this way it would have been a subject I would have enjoyed more and subsequently been better at. When you understand what you are doing you enjoy the work much more = just like this subject.

> Numeracy is a contraction of 'numerical literacy' … it's not just about number-crunching, but about understanding mathematical purpose, patterns and relationships. Many of us have spoken about how much this numeracy subject differs from the maths we did at school. Think about it- what is the difference? The answer is the answer to this exam question. We aren't just being told how to do maths anymore … we're being taught how and why to do maths and to show our reasoning. It's this deeper understanding which brings mathematics to life as it makes it meaningful.

While 'bringing mathematics to life', on many levels, was certainly the aim, the process remained fractured, arduous and contingent. Poststructuralism tells us that we should expect it to be so. The preservice teachers with whom we work have been subjected to discursive strategies that made them feel ill; rather than gain a sense of empowerment through self-initiated reasoning processes they have come to dread any further encounters with mathematics:

> Am a little nervous about this subject. Always remember being at school and dreading the double maths period. I recall spending much of it writing notes and planning the weekend, why oh why didn't I pay attention?
>
> Just reading the subject outline made me break out in a cold sweat.

This does not augur well for their teaching for numeracy in school; regardless of what they say about their studies in teacher education, we should perhaps be circumspect about their ability and willingness to challenge and support their students in the construction and confident application of rigorous mathematical ideas. It is more likely that some at least of these young teachers will avoid mathematics at all costs, or dress it up as 'play' or 'active learning' which can become empty replicas of the real thing, though less threatening (to them, the teachers). Speaking poststructurally, what we have here are young teachers who have not been able to establish themselves as numerate subjects at school and found themselves marginal to the operation of the discourse and the discursive practices. Because they were at the periphery they have come to know themselves (were positioned) as 'poor' at mathematics and much more research needs to be done to establish how or if this can be turned around. Certainly, it seems unlikely that the program of study at university could successfully overwrite already constituted discursive alienation, even though they may construct some new teaching and mathematical ideas. However, one thing poststructuralism contributes is to put the ball firmly in our court; it is up to researchers in teacher education to remain vigilant, to take nothing for granted and to track conscientiously the effects of our teaching on numeracy education in schools.

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High Achievers in Mathematics: What Can We Learn From and About Them?

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Success in mathematics is widely regarded as an important gate keeper for many courses and occupations. But does success in mathematics at school influence educational and career paths? Do talented mathematics students have distinctive working habits, are they attracted to a mathematics intensive field or more likely to turn to other areas? These and related issues are explored through information gained from students recognized at secondary school as high achievers in mathematics.

Review of Previous Research

The development of exceptionally talented individuals, including high achievers in mathematics, has attracted sustained and diverse research attention. The *Study of Mathematically Precocious Youth* [SMPY] founded by Julian Stanley in 1971 has spawned a huge amount of literature, ranging from publications in which the rationale for the program and early findings pertaining to SMPY participants were described (e.g., Stanley, Keating, & Fox, 1974) to more recent documentation of longer term personal growth, educational and vocational adult achievements. As noted by Lubinski, Benbow, Webb, and Bleske-Rechek (2006) many of these latter publications focus on students who "before the age 13, … scored within the top 0.01% for their age on either SAT mathematical reasoning ability (SAT-M ≥ 700) or SAT verbal reasoning ability (SAT-V ≥ 630)" (p. 194). Others to explore the development and working preferences of highly able mathematics students and of mature and successful mathematicians include Burton (2004), Csikszentmihalyi, Rathunde, and Whalen (1993), Gustin (1985) and Wieczerkowski, Cropley, and Prado (2000). The collective findings from these studies have revealed the importance of parental, educational and peer support, the mathematics students' and mathematicians' willingness to work hard, satisfaction from their mathematical achievement, their appreciation of the aesthetics of mathematics – its "succinctness, compactness, or conciseness" (Burton, 2004, pp.187-188), and the way they were "motivated by the order and beauty they see in mathematics" (Gustin, 1985, p. 330). Other findings from these works are interwoven with the reporting of data from the current study.

Aims

The overall aims of this study are to examine how exceptionally high achievers in mathematics perceive mathematics, and to gain insights into their background, motivations, work habits, and occupational choices. This information is particularly timely, given the drift away from demanding mathematics courses and the widespread concerns about the declining popularity of mathematics.

Method

Csikszentmihalyi, Rathunde, and Whalen's (1993) study of talent development and Eccles' (1985) model of academic choice were particularly influential in shaping the focus of this study. These authors highlighted personal qualities and characteristics (e.g., subject specific and broader attitudes and beliefs, expectations, motivations, self-perceptions) and environmental factors (e.g., the cultural milieu, the home, peer and educational environments) as significant predictors of success. Aspects of these factors are tapped through the Web based survey used in this study.

Sample Selection – Preamble

Purposeful sampling was used to select information-rich cases for this project. Focusing on students who have demonstrated exceptional performance in a limited domain, that is, purposeful sampling to obtain optimally intense descriptions of the phenomena of interest, is an approach frequently selected by those working within a framework of gifted education (e.g., Bloom, 1985; Csikszentmihalyi et al., 1993; Lubinski, Webb, Morelock, & Benbow (2001). Participants for the current study comprised past and current high achievers in the Australian Mathematics Competition, which is described more fully below.

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The Australian Mathematics Competition [AMC].

The first AMC, held in 1978, attracted some 60,000 students from 700 schools. Since then the competition has grown substantially and, according to the Australian Mathematics Trust [AMT] (2007), it has become "the largest single event on the Australian Calendar". In recent years some 400,000 students from 40 countries have entered the competition. The AMC is open to students of all standards and each entrant receives some form of acknowledgement of success (AMT, 2007). Each year a small number of students, about 1 for every 10,000 students entered, receives a medal. "These are awarded on the judgement of the committee to students who are outstanding within their region (Australian State or Territory or other country), within their year group and internationally" (AMT, 2007). Thus the level of achievement required for receipt of a medal is as stringent as that for entry into the SMPY program, some of whose participants are described by Lubienski et al. (2006).

Between 1978 and 2006, 690 medals were awarded to secondary school students within Australia. Since some students win a medal in more than one year the actual number of recipients is less. Many of the earlier medallists can be reached as they maintain contact with the AMT through personal communications or the AMT website. For others, older address details are still available. There are few females among the medallists (Leder, Forgasz, & Taylor, 2006) even though roughly equal numbers of females and males enter the AMC, especially in the earlier secondary school years.

Sample Selection – Further Details

For this study, the potential sample was restricted to students who attended schools in Australia and were awarded a medal between 1978 and 2006. To comply with ethics requirements, potential participants were contacted as follows.

A letter, containing a brief outline of the study, support for it from the Executive Director, details about the principle investigator, and a request to complete a survey posted on the AMT Website, was sent by the AMT to medallists for whom contact details were available. For recipients of multiple medals, the letter was sent to the most recent available address. Using this approach, 420 letters were mailed. (This ensured almost full coverage of those awarded a medal over the period of interest – see footnote 2). Fifty-two letters were, however, returned as undeliverable.

At the time of writing this paper (early February 2008), the survey had been accessed by 94 individuals – 84 males, 9 females, 1 gender unknown. To ensure that the data gathered were restricted to AMC medallists[5], surveys were discarded if items relating to winning the AMC medal were not answered.

The Instrument

The survey covered five broad areas: background; school and university; career/vocation (actual or intended); work habits; and some general issues about self. Some items were open ended: for example, "What did winning a medal mean to you?"; "What are your favourite leisure pursuits?" Others were in 5-point Likert format : for example, "Once I undertake a task, I persist"; "I'd rather work alone than with others to complete a task", with possible responses ranging from strongly agree to strongly disagree. More of the survey's content can be inferred from data reported in the results section.

Data Gathering and Synthesis

SurveyMonkey (http://www.surveymonkey.com/) was used to create the online survey and to validate, collect and summarise responses.

5 Although only medallists were specifically directed to the survey site by letter, others who perused the AMT Website also had access to the survey and might have explored it out of curiosity.

Results and Discussion

Response Rate

Surveys were still being completed some two months after the original posting. This could be a reflection of the wide geographic distribution of former medallists, and possible delays in the letter being forwarded to them by their family. For this study, there are different ways of calculating the response rate. Two are given here.

As already indicated, at most 368 of the letters recommending AMC medallists to complete the survey reached their intended destination. The 79 useable responses thus represent a response rate of at least 21%. As is discussed in the next section, the full range of AMC medallists – from the beginning of the competition to the last year surveyed – were represented in the 79 completed surveys. Of the 94 who accessed the survey, 79, that is, 84%, completed a survey useable for the study. Thus most of those for whom the survey was intended found it sufficiently relevant to finish it. These response rates are within acceptable limits (see, e.g., Hamilton, 2003; Kaplowitz, Hadlock, & Levine, 2004)

Sample Details

Of the 79 useable sets of responses, 74 were completed by males; 5 by females. The low number of responses by females prevents any gender comparisons being made. Just over 40% of the respondents were multiple medallists[6]: 25 won two medals, 9 won more than 2 medals, and 45 indicated that they had won one medal. Those who completed the survey included medallists from 1979 to 2005, whose dates of birth ranged from 1960 to 1994. All but five of the respondents indicated that they had completed or were close to completing their first degree. Just under two-thirds of these had completed at least one additional degree. Thus medallists now well into their chosen career as well as students still at school or university were among those who completed the survey.

Background Information

Twenty-one of the medallists (27%) were born out of Australia. This is somewhat higher than the corresponding figure of 22.2% for Australia's population, reported in 2006 Census data (http://www.abs.gov.au/websitedbs/d3310114.nsf/Home/census). Though not directly comparable, it is of interest that 30% of Lubinski et al.'s (2006) SMPY sample had at least one parent who was born outside the USA.

Parents

Occupations were listed for most of the mothers. Fifteen percent were described as home makers. Of the remainder, over one-quarter (27%) were teachers, ten percent were nurses, ten percent were medical practitioners or dentists, six percent were pharmacists, and a further 16% were in a mathematics or science related field, for example, accountant, computer analyst, laboratory pathologist, scientific programmer, and statistician. The other working mothers had diverse occupations including: business manager, editor, librarian, personal assistant, and sales assistant.

Many of the fathers (20%) worked in a mathematics or science related field, for example, computer scientist, biochemist, metallurgist, mathematician, scientist, or statistician. A further 14% were engineers. Other occupations included business/manager (14%), medical practitioner (12%), teacher (8%), and accountant (7%), as well as cook, home trader, minister of religion, and public servant. Overall, it can be inferred, the medallists' parents were tertiary educated.

The mathematicians and able mathematics students investigated respectively by Gustin (1985) and Csikszentmihalyi et al. (1993) similarly had well educated parents. In contrast, 30% of the 70 mathematicians in Burton's (2004) sample "came from a working class family where there was no history of university attendance" (p. 38).

6 If a similar proportion of the 690 medals awarded over the period of interest were to multiple medallists then the 420 letters mailed out represent virtual full coverage of the 1978-2006 medallists. This is consistent with the comprehensive data base maintained by the AMC.

Medallists

Favourite subject at school. Almost two-thirds of those who responded to this item (61%, i.e., 45 out of 74) gave mathematics as their favourite or joint favourite subject. Reasons for the choice of mathematics varied, as can be seen from the representative comments that follow.

Being good at mathematics was a reason given by many: "the ability to obtain 100% marks in tests"; "able to answer the questions and the certainty of the answers"; "I was good at it and enjoyed that"; "I often did well".

Some focussed on the beauty of mathematics: "mathematics is elegant"; has "inherent beauty"; is "methodical but exquisite" were some of the phrases used.

Others liked the logic of mathematics and wrote: "everything fits together; "logical"; "logic, structure, precision!"; "logical thinking, problem solving; mathematics requires logic and understanding much more than hard work and good memory"; "It was the most intellectually rigorous subject".

The perceived certainty of mathematics was also given as a reason: "clear cut answers", "it was unambiguous. You were either right or wrong".

Many enjoyed the problem solving component and the challenges: "solving difficult problems", "finding and fixing problems", "The challenge of problem solving, the satisfaction that came when a new concept was understood", "intellectually stimulating, great satisfaction from solving problems", "I was able to solve things by myself", "the pleasure of figuring something out that was not initially obvious". The thrust of many of the answers given is neatly captured in the following comment: "The interesting non-routine problems that required extensive exploring and creative, logical thinking".

Especially noteworthy is the fact that a number of respondents gave being able to do extension work beyond the regular syllabus as their reason for liking mathematics.

Of those who did not select mathematics as their favourite subject, 11 nominated other science subjects. Six selected English or another language, with the subject's perceived logic and challenge again being the attraction: Latin because "it was the most challenging subject; logical thought processes"; Japanese because of its "blend of science skills and arts skills". Also nominated was music because it provided "the opportunity to study something that is universal. I enjoyed playing the piano", and outdoor education because "it consisted mainly of exciting things like rock climbing, kayaking, and rafting … . It was very social. It was something physical that I was good at, so I felt less geeky."

Careers. As can be seen from the data in Table 1, the majority of the medallists were in, or intended to aim for, mathematics or fields heavily dependent on mathematics.

Table 1

Medallists' Careers

Career: actual or clearly stated as intended	Frequency
Academic/Professor (field not stated)	3
Actuary	8
Artificial intelligence researcher	1
Astrophysicist /Mathematical physicist/scientist (field not stated)	3
Doctor (general practitioner/various specialists)	12
Economist	2
Engineer (various fields)	6
Hedge Fund trader/finance	4
IT/computer science	5
Management	3
Mathematician (academic/researcher)	10
Software engineer/developer	6
Statistician	3
Student (currently & no clear career indication)	8
Other (e.g., music teacher, lawyer, video game developer)	5
Total	**79**

Most of the respondents indicated that mathematics was important in their work. In some cases the reason for this was clearly tied to the demands of their job. For example: "Yes. I am a mathematics professor (!), so I teach AND research it" (Similar answers were given by another nine respondents); "Finance is underpinned by understanding of math and statistics"; "Yes. I am a Statistical Methodologist for [a company] and my role depends on my mathematical comprehension". Other comments included:

> Vital; both in my current work as a statistician and my previous work in biomedical research, I'm essentially an applied mathematician, so it comes into almost every aspect of what I do.

> Yes, being a hedge fund trader required keen understanding of probability, expectation and risk, and most importantly not to be fooled by the randomness of many daily events. People from a mathematical background are much more likely to have an appreciation of these things.

Some responses illustrated how subtly mathematics is integrated into other occupations:

> Yes. Sales forecasting requires trend analysis and pattern detection.

> I am grateful at its practical application in medication dose calculation and working out people's kidney functions. But in a broader sense the statistical analysis of drug trials influence our daily prescribing decisions.

> Yes but less than I thought it would be when I thought I was going to be an engineer. I'm now officially an artist, but I work with computers and it really helps to have a basic knowledge of calculus and a good knowledge of techniques associated with 3 dimensional geometry.

Even those who considered mathematics to have limited applicability to their current job often appreciated the broader skills they had gained from studying mathematics:

> Not directly, but I do think the logical processes and problem solving skills are used all the time in any sort of work, and certainly in teaching and music making. …I think quite mathematically often.

> While doing maths I learnt formal logical skills and ability to learn in a flexible way. These are used regularly at work. However the technical mathematical skills are very rarely used.

Collectively, the medallists' attitudes to mathematics and choice of careers and working preferences reflect the qualities identified by Wieczerkowski et al. (2000) as characteristics of high mathematics achievers: "placing a high value on mathematics", and "seeing oneself as capable of being successful in the mathematics field" (p. 419).

Leisure occupations. The medallists' leisure time pursuits were eclectic. They included sport (e.g., football, golf, hiking, rock climbing, running, soccer, squash, swimming, tennis, volleyball), music (including guitar, piano, singing, violin, and writing music), card games, playing chess, photography, reading, socializing/ spending time with family, and writing. The list of leisure activities closely mirrors those nominated by the students in Csikszentmihalyi et al.'s (1993) study. Stanley et al. (1974) and Lubinski et al. (2001) have also pointed out the diversity of interests of students in the SMPY program, the diverse fields to which they are attracted, and the variety of careers in which they ultimately engage.

Winning a medal. None of the medallists mentioned negative aspects of winning a medal. Many indicated that winning a medal gave them great satisfaction and pride in having their mathematics achievement recognized. Attending the actual award giving ceremony was also prized by many. New or continuing opportunities to attend special courses and do advanced mathematical work with others who liked mathematics and were good at it were seen as particular benefits. Others talked of specific doors being opened, or of longer term benefits.

> A source of pride – we were immensely competitive in a good-natured way at school and there were 3 or 4 students in my year who won AMC medals in various years. We still get together every year to do the Westpac / AMC competition paper over dinner (our 15th year this year).

Working preferences and motivation. Table 2 contains aspects of the medallists' working preferences and motivations.

Table 2

Medallists' Working Preferences and Motivations (in Percentages)

Item	Stornly Agree/ Agree	Neutral	Strongly Disagree/ Disagree
I enjoy working competitively with others	67	14	19
I thrive when I work at something which is challenging and difficult	96	4	-
When working in a group I would rather be in charge	43	43	8
Good relations with my fellow workers are more important than task performance	37	37	26
I worry that my success may cause others to dislike me	25	19	56
It is important for me to perform better than others on a task	48	32	20
I try harder when I am cooperating with others on a task	57	28	15
I sometimes work at less than my best in case others resent my performance	7	10	83
I'd rather work alone if at all possible	40	27	34
Once I undertake a task, I persist	81	14	6
It annoys me when others do better than me	34	28	38
I prefer working in situations that require a high level of skill	93	6	1

Several features stand out. Medallist thrive on doing difficult, challenging, and highly skilled work. Once started, they persist with a task. Their motivation and task commitment is high. Working competitively but also working cooperatively was liked by many. These qualities mirror those of Csikszentmihalyi et al.'s (1993) sample who were described as having "high achievement motivation and endurance" (p. 207).

Being in charge of a group seemed not to be important. Those who preferred to work alone slightly outnumbered those who did not. In general, the medallists wanted to perform well, irrespective of the reactions of their peers. About the same number agreed, as disagreed, that they were annoyed when others did better on a task.

Self descriptions. Many of the medallists (75%) provided a description of themselves. Space constraints allow only a few (representative) comments to be included:

> Very laid back and even-keeled. I like to solve problems, whether they be arguments (or miscommunications) or technical problems about how we can make something in our game look better. I tend to get very absorbed in what I'm doing and spend too much time on it.

> An affable introvert. Keen to please people, scared of confrontation. A good listener but reluctant to open up to others. Able to dig through a lot of dull work provided there's some nugget to excite my imagination. A good teacher, able to communicate concepts simply. Modest. Reasonably self-assured. Tend to fantasise about possible achievements although I lack the ambition to define and pursue my own goals. I could go on ...

> Quietly fun, easily entertained, patient; caring, principled, sensitive to some emotions; proud, perhaps a little arrogant, attention-loving; thinking, logical, analysing; careful, shy, well-meaning.

> Irritable, passive aggressive, a little obsessive compulsive, a bit reserved, mercurial, a little anxious, a little paranoid, worry a little too much, a bit of a hypochondriac, complain a lot, not easily offended or shocked, vulgar at times, easy going, depressive at times, irreverent, not too serious, good sense of humour...

Concluding Comments

Many of the findings of the current study, based on questionnaire responses, are as expected – with respect to family background, the medallists' reasons for liking mathematics, their varied career choices in areas in which they saw mathematics used directly or indirectly, their delight in a challenge, their high level of motivation and persistence, and their wide range of leisure activities. The next stage of the study, that is, interviews with participants involved in the first phase, is an opportunity to probe contextual nuances and individual differences subsumed in the more general findings.

Particularly provocative for those involved in mathematics education is a recurring theme that permeated many of the responses: the most exciting and fulfilling mathematics came from opportunities to do advanced mathematical work with mathematically talented peers outside the regular school curriculum. This finding, too, can be explored in more depth during the interviews.

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The 2007 Common Technology Free Examination for Victorian Certificate of Education (VCE) Mathematical Methods and Mathematical Methods Computer Algebra System (CAS)

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In November 2007 around 14 000 Mathematical Methods students and around 1 500 Mathematical Methods (CAS) students sat a common one hour technology-free examination. The examination covered the same function, algebra, calculus and probability content with corresponding expectations for key knowledge and key skills for both studies. This paper provides some analysis of the examination and items and student performance with respect to both cohorts.

Background

For the period 2006 - 2009, Mathematical Methods Units 1 – 4 and Mathematical Methods (CAS) Units 1 - 4 are parallel and equivalent Victorian Certificate of Education (VCE) mainstream function, algebra, calculus and probability studies, with an approved graphics calculator, and an approved CAS, as their respective assumed enabling technology (see VCAA 2005). Units 3 and 4 are typically studied in Year 12, and have corresponding end-of-year examinations. There is a common one hour short answer question and some extended answer question technology free examination (see VCAA 2007a); and a two hour multiple choice and extended response technology assumed examination, which has substantial common questions as well as some distinctive questions (see VCAA 2007b, VCAA 2007c). From 2009, all students undertaking the mainstream function, algebra, calculus and probability study in Victoria will be enrolled in Mathematical Methods (CAS) Units 1 and 2, in preparation for the final stage of transition for the whole cohort to CAS enabled version of the study at Units 3 and 4 in 2010. Over the past decade or so, various researchers have considered aspects of assessing mathematical capabilities via examinations where students choose to use mental, by hand or technology assisted approaches, or a combination of such approaches, to tackle a range of questions (for example, Kokol-Voljc, 2000; Brown, 2003; Flynn, 2003; and Ball & Stacey; 2007). Particular aspects of student performance on common short answer, multiple choice and extended response questions from examinations for Mathematical Methods and Mathematical Methods (CAS) have been reported on previously by Evans, Jones, Leigh-Lancaster and Norton (2007) and Evans, Leigh-Lancaster and Norton (2003; 2004; 2005).

Other systems and jurisdictions around the world also employ a technology free and technology active examination structure, where the use of CAS is permitted (for example, the College Board *Advanced Placement Calculus Program*) or assumed (for example, Danish *Baccalaureat Mathematics* and CAS based New Zealand National Qualifications Framework Level 1 and 2 *Achievement Standards for Mathematics*) for some component(s) of examination assessment (see CAME, 2007). From 2010, the revised Western Australian mainstream (Mathematics) and advanced (Specialist) calculus based senior secondary courses will be CAS enabled, with a technology free and technology active examination structure (see Curriculum Council WA, 2008).

The areas of study and topics for Mathematical Methods (CAS) encompass those of Mathematical Methods (hereafter referred to as MMCAS and MM respectively) include common specification of key knowledge and key skills in relation to mental or by hand approaches to mathematical routines and procedures (VCAA 2005, p. 67-68, 73-74, 156-157). This paper discusses aspects of the common technology free examination, and student performance across the two cohorts. It should be noted that the group of students taking Mathematical Methods CAS in 2007 is not necessarily a representative sample of all students undertaking the Mathematical

Methods study in 2007. However, the mean scores for both cohorts on the Mathematics component of the 2007 General Achievement Test (GAT) completed in June 2007, that is, prior to the examinations: MM, μ = 24.4, σ = 6.0, and MMCAS, μ = 25.1, σ = 5.9 (from a total available marks of 35); and the Victorian Tertiary Admissions Committee (VCTAC, 2007) scaling report (which makes a post-examination comparison of the performance of all students in a given study with the rest of the student cohort across studies on a truncated normal scale of 0 - 50): MM, μ = 35.73, σ = 7.1 and MMCAS: μ = 35.84, σ = 6.7, indicate that the *overall* level of ability of the two cohorts is very similar.

Comparing Student Mean Performance on the 2007 Paper

In 2007, the common one hour technology free Examination 1 comprised 18 items receiving credit or partial credit (that is a question or part of a question allocated one or more marks), with a total of 40 available marks. A simple comparison of the mean scores is shown in Table 1.

Table 1

Comparison of the Mean Raw Scores for MM and MMCAS on Examination 1

	Group	N	Mean	Std. Deviation	Std. Error Mean
Raw Score	1 non-CAS	13705	16.9248	11.89609	0.10162
	2 CAS	1528	18.6165	11.91652	0.30485

On average, the MMCAS group scored 1.7 points higher than the MM group (Table 1). The independent samples t-test analysis confirmed that this difference is statistically significant (t =-5.272, p = 0.0001). A comparison of the mean scores of the MMCAS and MM groups for each item shows either no difference, or a positive difference in favor of the MMCAS group, as can be seen from Figure 1.

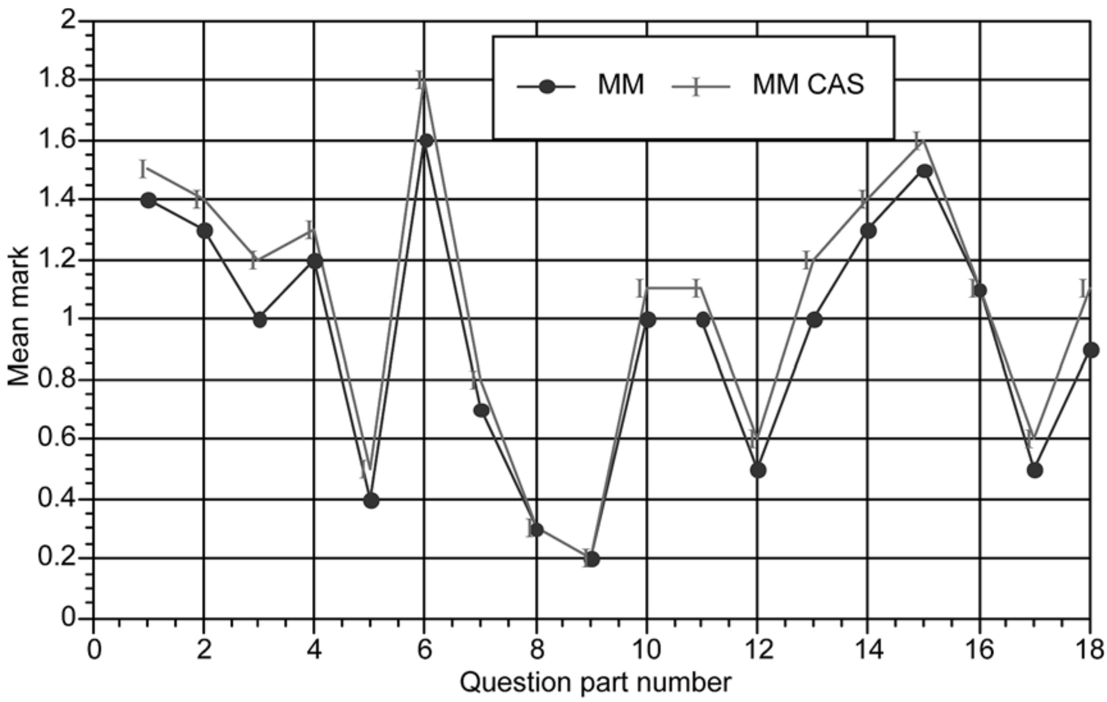

Figure 1. Mean question-part mark by population (MM CAS and MM).

These results contrast with those for Examination 1 in 2006 (see Evans, Jones, Leigh-Lancaster, & Norton, 2007), which showed essentially the same performance on all items, and a corresponding mean overall score of 22 for both groups. The lower overall mean raw score for 2007 is consistent with expert judgment and practitioner views that the Examination 1 paper was more challenging than that of the previous year - assuming that no significant variation in the overall ability of the 2007 student cohort with respect to that of 2006 and earlier years.

Several considerations may be relevant - the 2007 MMCAS group could, on average, be of slightly higher ability than the 2007 MM group. To date, 'high-achieving schools' across the sectors have generally been more inclined to continue with the MM study, where the good performance of their student cohort has been historically well established, intending to make the change to MMCAS at a later stage in the transition process. On the other hand, low-achieving schools' may also have been cautious about taking up MMCAS at an early stage, and hence also make a later transition. It may be the case that as teachers of students in the MMCAS group continue to develop their familiarity and expertise with the technology as a pedagogical tool, there is increased benefit in terms of the scaffolding it affords students in learning mental and by hand computational skills with respect to function, algebra and calculus. Such considerations, and their possible or likely impact, will be able to be more fully explored as further longitudinal data becomes available, and the number of students in the MMCAS cohort continues to increase.

Comparison of Students' Performance (CAS and non-CAS groups) on Each Individual Item

At individual item level, one can test to see if the MMCAS group performed differentially from the MM group. The detection of differential performance between the MMCAS and MM groups on individual items is carried out using item response theory, or IRT (see Lord, 1980). Item response theory was applied to control for differences in students' abilities, where ability can be regarded as a monotonic transformation of the test raw score. That is, there is a one-to-one correspondence between each raw score on the test and the estimated ability on the IRT scale. In general terms, the IRT is concerned with the probability of success when someone attempts multiple-choice or short answers questions. The basic idea behind that theory is that there exist quantities b_i – the difficulty of item *i*, and θ_j - the ability of student *j*, and that p_{ij} , the probability of student *j* responding correctly to item *i* is given by an increasing function of $(\theta_j - b_i)$. The Rasch model (see Rasch, 1960) for the probability of success for a student *j* on an item *i* can be expressed by:

$$p_{ij} = \frac{\exp(\theta_j - b_i)}{1 + \exp(\theta_j - b_i)}$$

In the case where an item has partial credit scoring (that is, the maximum score is greater than 1), the IRT partial credit model (Masters, 1982) can be applied. Using *ConQuest* software (Wu et al., 1998) graphs have been produced for all 18 items. Each of these items was scaled in the item response modelling as the same item for both cohorts. Figure 2 shows a selection of graphs for these items, the blue dotted curve (darker) shows the observed average score of students from the MM group (cas1), while the green dotted curve (lighter) shows the observed score of students from the MMCAS group (cas2), as a function of the "ability level" (labelled as Latent Trait on the horizontal axis). The latent trait level for each student can be regarded as derived from a nonlinear, monotonic, transformation of the total test score, based on the IRT model. The solid line shows the expected score of students as a function of latent trait. There is only one solid line in each graph, as each of these items was modelled as the same item whether it appeared in the MMCAS paper or the MM paper (both papers were identical). For each item the relevant mathematical content has been briefly identified.

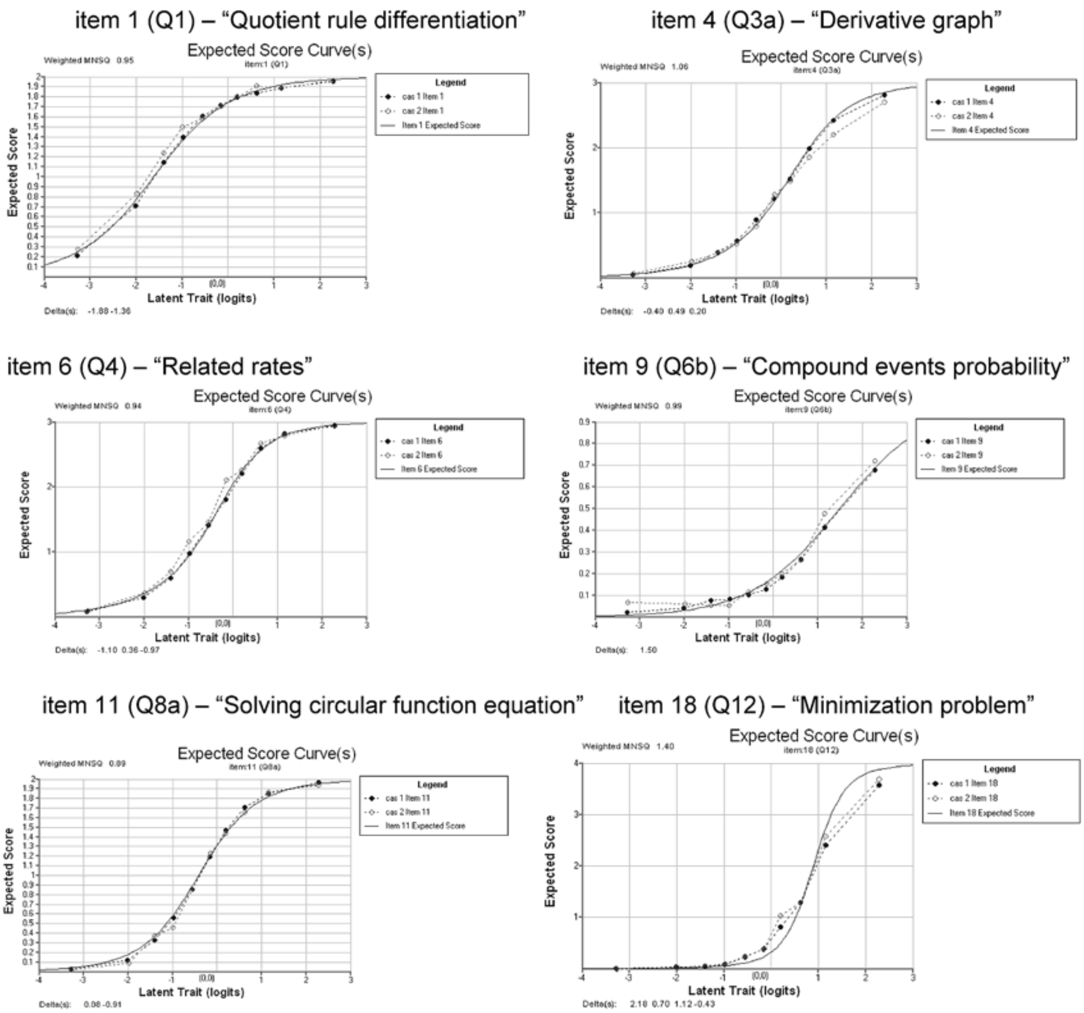

Figure 2. Expected score curves for all test items by the MMCAS and MM groups.

The results reveal that overall, two groups performed similarly across each item, after controlling for the total score on the test, for example, for Items 1, 6 and 11 the theoretical and cohort curves almost overlap each other. Marginal differences are observed between the two groups’ performance on Items 4, 9 and 18. Whereas the higher ability students from the MMCAS group have a slightly lower score than higher ability students from the MM group on Item 4, the reverse is the case for Items 9 and 18 where higher ability CAS students slightly outperformed higher ability MM students. Also, lower ability CAS students performed better on Item 9 than lower ability MM students.

In comparing the observed curves with the theoretical (expected) curves, for Item18, there is a slight but noticeable difference. The observed curves for both are not as steep as the expected curve, indicating that the item does not discriminate between lower ability and higher ability students as well as the model predicts. The other five selected show a close match between the observed scores and the expected scores (both dotted lines almost overlap with the solid line), indicating that the items fitted the IRT model well.

The application of IRT facilitates the comparison of performance of MMCAS and MM groups at individual item level. In general, the two groups of students performed very similarly on almost all items after controlling for the total score on the test. There are some marginal differences for a few items, but these differences are small. It is re-assuring that, overall, there is no strong evidence of differential performance on individual items between MMCAS and MM groups.

Conclusions

The two cohorts display essentially the same *profile* on almost all questions considered using IRT partial credit model of analysis on an item by item basis. While the mean total score on the paper for both cohorts was similar, there was a small but significant difference in favour of the CAS group. On an item by item basis, the difference in mean score MMCAS – MM was non-negative in all cases, and generally positive. This suggests that the common curriculum requirements (content and expectations) for *both* studies (in terms of key knowledge and key skills specified in the study designs) with respect to mental and by hands skills of the type tested on the common Examination 1, provide a robust basis for very developing similar overall levels of student performance for this type of assessment, where students do not have access to technology, irrespective of which cohort they come from.

The view that access to CAS as an enabling technology is likely to *diminish* student mental and by hand capabilities of the kind assessed by the common Examination 1 (where the enabling technology is not available) relative to students who do not have such access, is not supported by the data from 2006 and 2007. Similar observations have been made in other studies over the years (see, for example, Heid, 1984; Park & Travers, 1996, Dunham, 2000), including within school testing at both Unit 1 and 2 and Unit 3 and 4 levels during the VCAA Mathematical Methods (CAS) pilot study 2000 – 2005.

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Focusing Year 8 Students on Self-Regulating their Learning of Mathematics

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This paper tells of a Year 8 intervention in a school where students are reported to typically have low engagement in learning mathematics. Aiming to help students self-regulate their learning of mathematics, a tool was developed for students to set short-term goals and select, and reflect upon, their developing command of related strategies for organisation and persistence. The results from the intervention were mixed, suggesting the tool has most potential to benefit middle and low achieving students.

The under-participation of students in learning in the middle years (students aged 10 to 14) is a phenomenon that some teachers face on a daily basis. Indeed, such under-participation in Australia is persistent and widely reported (e.g., Hill, Holmes-Smith, & Rowe, 1993; Russell, Mackay, & Jane, 2003).

In an earlier paper that addressed the issue of participation of middle years students, it was argued by Sullivan and McDonough (2007) that, ideally, to promote student engagement in learning, two sets of factors must align. The first set of factors includes that the students have the requisite prior knowledge, the curriculum is relevant to them, the classroom tasks interest them, and the pedagogies and assessment regimes match their expectations. The second set of factors relate to student goals for learning, their willingness to persist, and the extent to which they see participation in schooling as creating opportunities. While it is acknowledged that teachers of students in the middle years have the challenge to address both sets of factors, the focus of the present paper is on assisting teachers to address the second, with a particular emphasis on student self-regulation of learning through setting goals and identifying strategies related to persistence and organisation.

In earlier research, the persistence of students was studied through posing them a range of increasingly difficult problems in mathematics (Sullivan & McDonough, 2007; Sullivan, Tobias, & McDonough, 2006) and English (Sullivan, McDonough, & Prain, 2005) with the intention that eventually nearly all students would confront a task which was difficult for them. Students were asked to rate their self confidence and achievement, their persistence, their perception of the value of schooling, and what constitutes successful learning. We found that the students were surprisingly confident in their own ability, they perceived effort as important and themselves as trying hard, and saw these as linked. The students seemed to have goals focused on aiming to please the teacher by getting questions correct and scoring well on tests.

It is interesting that, in contrast to the student perceptions, the teachers of these students reported low levels of persistence and significant difficulties in engaging students in learning. Based on observations in classes, the students overall seemed neither confident in their learning nor did they try hard (Sullivan & McDonough, 2007). This disparity was perhaps due to students being willing to try harder within a one-on-one interview situation, or perhaps due to students having a different understanding from their teachers of what "working hard" in the classroom might entail. Students also may not be aware of, or may have little knowledge of, self-regulatory strategies, thus limiting their ability to use them (Ames, 1992).

It is possible also that Year 8 students are not always consciously paying attention to their type and degree of involvement and the implications each might hold. For example, as reported by Sullivan and McDonough (2007), some students indicated that while they know that mathematics is important to their future careers, they do not think about that every time they sit down in a mathematics class. The present study was designed to address the lack of engagement of students, with the aim of making them more aware of possible strategies to assist them to more actively take part in their learning. It addressed two specific areas of need identified by the teachers: organisation and persistence.

Orientation to Learning

Informed most particularly by the work of Dweck (2000) and Ames (1992), student orientation to learning provided an overarching theoretical perspective for the present study. Dweck and Ames categorised students' approaches to learning in terms of whether they hold either *mastery* goals or *performance* goals, with each form of goal giving "different conceptions of success and different reasons for approaching and engaging in achievement activity" (Ames, 1992, p. 262).

Students with mastery goals associate effort and outcome, tend to have a resilient response to failure, remain focused on mastering skills and knowledge even when challenged, do not see failure as an indictment on themselves, and focus their attention on the intrinsic value of learning. Students with performance goals focus on ability and self-worth. They are interested predominantly in whether they can perform assigned tasks correctly as defined by the endorsement of the teacher, seek success but mainly on tasks with which they are familiar, avoid or give up quickly on challenging tasks, derive their perception of ability from their capacity to attract recognition or do better than others, and feel threats to self worth when effort does not lead to recognition.

Dweck (2000) also distinguished two perspectives on intelligence: a fixed perspective termed *entity* theory that refers to students who believe that their intelligence is genetically predetermined and remains fixed through life; and an *incremental* perspective in which students believe that they can change their intelligence and/or achievement by manipulating factors over which they have some control. Students with incremental perspectives tend to hold mastery goals, while an entity view can result in performance goals.

Self-Regulated Learning

Patrick, Ryan, and Kaplan (2007) stated that "an orientation to mastery goals involves a focus on personal improvement and gaining understanding or skill, with learning seen as an end in itself" (p. 85). Students giving attention to self-improvement for learning through continuous self-reflection are self-regulating learners (Van Grinsven & Tillema, 2006). Where learning is viewed as an activity is which students are proactive, "self-regulation refers to self-generated thoughts, feelings and behaviours that are oriented to attaining goals" (Zimmerman, 2002, p. 65). According to Zimmerman, self-regulating students personally set goals and task-related strategies, are aware of their strengths and limitations, and monitor behaviour and undertake self-reflection. Component skills within this process include students re-structuring their physical or social context so that it is compatible with their goals, self-evaluating their methods, managing their time more efficiently, and adapting future methods. Self-regulation has also been described as a "multi-component, iterative, self-steering process" that implies student choice of goals (Boekaerts & Cascallar, 2006).

As self-regulation includes planning and monitoring (Patrick et al., 2007; Zimmerman, 2002), and orienting oneself before starting a task (Boekaerts & Cascallar, 2006), a tool that helps students set goals for their learning of mathematics and identify potentially helpful strategies, as well as facilitating reflection and further strategy identification, has the potential, at least in some respects, to help students become more self-regulating of their learning.

Research Context and Data Collection

This study involved the development and use, over two iterations, of a tool that allowed students to record goals they developed for one unit of work in mathematics, select strategies related to organisation and persistence to address those goals, reflect on the effectiveness of their efforts, and identify future foci related to organisation and persistence skills. The tool thus addressed Zimmerman's (2002) three phases within the self-regulatory process: forethought, performance, and self-reflection, and promoted self-regulation as an on-going cyclical (Zimmerman, 2002) or iterative (Boekaerts & Cascallar, 2006) process.

The study reported in this paper is part of a larger project related to the engagement and motivation of students in the middle years. Data collection for the present study occurred in one Victorian regional city government secondary school which serves predominantly lower socio-economic families. The regional city is prosperous, overall community infrastructure is good, and there are ample further education and employment possibilities for school leavers. Data for the case study are drawn mainly from work with one Year 8 class, taught by Teacher A, as described below. Two lessons were observed, six students were interviewed individually (three males and three females: two high, two middle, and two low achievers in mathematics), and Teacher A was interviewed. Feedback from some other users of the tool is also reported briefly where relevant.

As part of a larger study of student engagement in learning[7], the Year 8 teachers identified persistence and organisation as two aspects that they believed had potential for further development, could assist students to be more self-regulating in their learning, and could lead to a less teacher-centred learning situation. Their views are in line with those of a range of researchers who identify mastery orientation, persistence, self-management, and planning among helpful characteristics for student learning (e.g., Dweck, 2000; Martin & Marsh, 2006). Inspired by the work of a fellow researcher that was shared with the teachers as part of the larger project (Drane, personal communication), the teachers also expressed interest in a practical tool that gave some structure to the students, that allowed the students to write responses, and that facilitated reflection. As a result, the researcher developed a two-stage version of a recording and reflection tool. While this tool was used by more than one teacher within the school, Teacher A agreed to have the researcher visit Wellbeing lessons with his Homeroom class in which he used the tool asking the students to focus on their learning of mathematics. Teacher A was also able to meet with the researcher to reflect upon the value of the tool. The 17 students involved in this case study joined with another class for mathematics lessons and were team-taught by Teacher A and Teacher B. Teacher C, another Year 8 teacher who team-taught mathematics, but to a different group, had also offered to be part of the study but due to other commitments could only offer his and his students' feedback in the first iteration.

Describing the Development and Features of the Tool

The initial two-part tool was developed for use at the beginning and end of a school week but, following input from Teachers A and C, was changed in the second iteration to a three-part tool and to apply to a topic. Thus Sheet A was designed for use at the beginning of the topic, Sheet B mid way through, and Sheet C at the end of the topic, having the potential to help students reflect in a more focused way on their growing development. The description in this paper will now focus mainly on use of the three-part tool.

To assist students in identifying goals for the topic, three spaces titled "A summary of what I will be doing" were provided on Sheet A for students to fill in as directed by the teacher. In preparing students to think about possible goals, they were also asked to identify on a 1 to 7 scale how much they presently knew about the topic and how confident they felt. This was repeated on Sheets B and C. Scales regarding knowledge and confidence were included to bring students' attention to the progress they potentially would be making.

Students then recorded a goal or goals that they developed for their learning of that topic in mathematics. The decision to include this resulted partly from reports by some Year 8 teachers at the school that although students had at the beginning of the year identified goals for the whole year these were quite general and had not given the students sufficient direction. Thus a need for short term goals had been identified. Such an approach concurred with the view of self-regulation in the literature discussed above. The identification of goals was also included in an attempt to make the focus on organisation and persistence strategies meaningful in that they were directed towards a purpose. It was also hoped that students would associate reasonable effort and a willingness to apply the effort with the task of achieving their goal (Ames, 1992). To facilitate the writing of goals the following prompt was included in the tool: "My Goals: What I most want to accomplish or achieve in this topic (Try to be as specific as possible. For example, to be able to … ; to understand … ; to complete … ; to feel … ; to ……)". This definition of goals was adapted from that used by Bernard (2003) as a result of feedback from Teachers A and C and students following use of the first version of the tool.

Practical strategies related to organisation and persistence were then listed for students to choose from with the heading "How I will help myself to achieve my goal/s" and the instruction "Choose the strategies you most need to work on". These were informed by ideas from the *PEEL Project* (Baird & Mitchell, 1987) a later related project (Corkill & Mitchell, 2002), and the *You Can Do It* program (Bernard, 2003), and most importantly, from feedback from students and Teachers A and C in the first iteration of the tool. These included: Have all my materials ready; Make good notes; Make a plan and spread out the work; Try not to get distracted by others; Listen even when I am tired or bored; Look at the previous question; Ask someone who can explain it simply. An "Other" option was also provided and space was also included for listing places to find help and for self-talk for when one loses focus.

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7 The WHOLE project is the result of a collaboration between three schools, and three universities (Monash, La Trobe and Australian Catholic) and is funded by the Australian Research Council LP 0668937. The views expressed are those of the authors.

On Sheet B students were asked to tick as appropriate for "Have been very helpful" and "Focus for next week" for each of the same strategies as on Sheet A. Students were also asked to write responses "Why or in what way did the strategies help?" and "How I can improve working towards my goal/s".

Sheet C built on the previous sheets by asking students to identify strategies they had tried, those they were now good at using, and those they still needed to develop. Thus the intention was that the three-part tool (Sheets A, B, & C) would help students choose foci, recognise what they had achieved, and continually look to further development. The list of strategies would also bring student attention to ways in which they could play a role in regulating their own learning of mathematics in relation to organisation and persistence to help meet their goals.

Teacher A added a further step to the procedure, asking students in his class to write a reflection on the strategy choice, effectiveness of the strategy use, and whether they had achieved their goal.

Key Findings

When I visited the school to observe use of Sheet C in the second iteration of the tool, the students gave some, but limited, positive responses about its value. However, the students who were interviewed later that day were generally more positive about its use and adaptability although they had not communicated this in class. Findings are illustrated below by reference to a selection of student responses.

Within the whole class group, Teacher A asked his students to reflect upon the value of the tool after the second iteration. When asked why they had used this procedure, responses were that it allowed them to know their goals and to plan out ways to achieve a goal.

The class was also asked whether they had changed the way they worked because of filling out the sheets. One student stated that he tried to listen more, another student referred to doing less talking, and another said she sat away from others and did her work more. For one student for whom the goal was to not be distracted, the strategy of not sitting next to people she talks to "did not work so well" as she "did not really think about it and still sat next to those people". Interestingly, this high achieving student achieved nonetheless, but acknowledged that her talking may have made learning difficult for others. However, another student who had essentially the same goal gave herself a success rating of 7 out of 10.

A related goal, "to not talk as much", was identified by the low achieving female who was interviewed individually. She stated that when she found she was having some success with that goal she set a further goal of sitting away from people who talk a lot. She chose to sit next to a certain person "because she does her work as well and she already knew what my goal was and she knows that I'm a really bad talker and so she tried to help me as well to try and do my goals". This low achiever in mathematics thus demonstrated that she could reflect on what was hindering her learning and take some control over her learning of mathematics. She also had chosen a person who would be supportive of her efforts: "I talked to her and then she'd just sort of zone out from me and keep on doing her work". As a result this low achieving student found she could

> figure out a lot of things in algebra … I actually got to know a bit about the topic and I actually got to understand what to do. … I would usually get the sheet and go, these letters, I don't understand this, and just talk to my friends and wouldn't bother asking the teacher for help or anything but after I set myself the goal and got the sheet I would ask for help or ask my friends for help or something and then I would complete it. [The most helpful thing in learning algebra was] probably sitting there and listening to what the teacher had to say about how did you get the answer and all that sort of ….

She believed she could potentially be a good student like her friend but that when she talks she does not achieve well. She said that while she used the goal in a lot of other subjects, she focused most on algebra. She felt she would also remember this for the next year at school and stated that her next goal might be "to get a certain amount on tests and study a lot more".

Unlike the high achieving student in her class who reported that she did not think about her goal, this low achieving female demonstrated that she could focus on what she wanted to achieve and work at implementing strategies to assist her learning of mathematics. Her conclusion that she could potentially achieve better suggests an incremental perspective on intelligence and her response suggests a "hardy, can-do mentality", associated by Dweck (2000, p. 3) with mastery goals.

The middle achieving male who was interviewed identified his goals as "to be able to learn more about algebra" and "to be confident". In his written reflection he stated that he chose three main strategies: "Try not to get distracted by others; Aim to finish my work; and Listen even when I am tired or bored". On whether his strategies helped, he wrote:

> I listened alot (*sic*) more and I was understanding it well because I was listening. Instead of giving up when it got hard I kept going and I always got my work done. It was hard not to get distracted but when I ignored everyone I did nothing but my work.

He wrote that he had achieved his goal as he could "successfully do an algebraic sum or expression". Although he had not spoken up in the whole class discussion because he is "not really a talkative one", he did give further insights in the interview. He spoke of achieving by moving away from his friends: "I'd sometimes go off and sit by myself so I didn't have anyone to talk to. So I just sat there and did all my work and that helped." He added, "I sort of did have a bit of a plan because sometimes we get more than one sheet and I thought well do the hardest sheet first and if I'm sitting by myself I can get through that fast enough and the rest of them are going to be easy as". He recalled not being successful with algebra in Year 6 but felt that in Year 8 it was very important that he did not get left behind in his learning of algebra "because you knew it was going to be a big thing".

Interestingly, he spoke also of setting goals at home: to practise his musical instrument daily and to exercise regularly. He was inspired to do this after hearing of another student in his class who had set personal goals as a result of the focus on goals in mathematics at school. In addition, he said

> I've written up my own sort of plan like that and I've stuck it up on my corkboard so it's there for me to look at so if I do encounter something bad or hard I can go home, I see that, I think alright I can set a goal. So that's always there to remind me that I can set goals… I wrote what I wanted to do, how I could achieve it, places I could find help, … self-talk – Don't lose focus.

The tool not only had short term impact both in this student's learning of mathematics and for other non-school activities, but also potential to empower him for future learning.

In contrast the high achieving male interviewee commented on the procedure: "My reaction is that I don't like it because I find goals hard to set". This does bring to mind Dweck's (2000) discussion of students with performance goals who seek success mainly on tasks with which they are familiar and avoid or give up quickly on challenging tasks. He felt the procedure could be helpful "if you are struggling in some subject and you want to set a goal to accomplish something you can do it to help you". But he did not see himself as usually struggling in subjects. So it appears that he saw goals as more relevant to others. He felt he could work on "try[ing] not to get distracted by others or distracting others which I do a lot" but did not plan to set such a goal as "I'll just wait for it to happen". This student's reaction is clearly different from those of the low achieving female and middle achieving male interviewees.

From this discussion of just a small number of student responses it becomes evident that there was mixed reaction and commitment to the use of this procedure. It is noteworthy that some positive assessments of its value or potential to stimulate further goal setting that had not been shared in the class discussion became apparent in the interviews. Analysis from all six interview students indicated that themes included not wanting to fall or be left behind in a mathematics topic, wanting to remain focused, thinking about who to sit next to so as to minimise distractions, and associating with peers or a teacher who give good motivational messages. It is clear that some students consciously made changes, and indeed the impression is that the procedure not only brought strategies into focus for the students, but in a sense gave them permission to use such strategies for learning mathematics and even adapt, privately, for school-related and personal goals.

The majority of the students felt that the procedure was worthwhile and could be used with the following Year 8 class but not too often; suggestions included once a term or a couple of times a year.

Teacher A and a number of the students felt that it would be worthwhile to paste a sheet at the front of one's mathematics workbook to be reminded of one's goal/s and strategies. Teacher A thought this would be easier as it would show the students some of the things they could try. It is possible also for additional strategies to be listed on such sheets, perhaps targeted at the particular mathematics topic.

In making suggestions for future use, Teacher A stressed that he thought the tool would best be integrated into one curriculum area, and as one small part of a total approach to giving students more control of their learning: "I think it needs to be smaller and subtle and tied in with something you are doing in class, not something you pull out and do for a long time".

Discussion

Initial impressions from the teachers, prior to use of the tool was that it had potential for use with these Year 8 students who are generally considered less engaged in their learning of mathematics than desirable. Positive comments included "It is specific, not nebulous and gives quick feedback" and "It's simple: what I will do; how I will do it". It was seen to have potential for use at a parent interview, and could assist reflection for students as "at the end of the year on the report [they] have to be able to reflect". There was the comment also that "Too often will kill it" and indeed, that sentiment was repeated after the use of the two iterations of the tool. It was recommended that it is best used within one curriculum area, giving the students a clear focus for their goals and application of strategies.

As an aside, an interesting outcome of the use of the tool occurred within a reflective conversation in Teacher A's Homeroom group where students spoke of their need for briefer explanations in mathematics, particularly from Teacher B (who team-taught mathematics with Teacher A). When this was shared with Teacher B he stated that the students with whom he had used the tool had given the same feedback:

> We talked about it, it was really good. But then [student] put her hand up and said "Gee, Mr [Teacher B] sometimes you tell us too much. I had a handle on everything with zero the other day when you were talking about it in maths and then you mentioned this thing about measuring zero. It just confused me, too much information".

So while not necessarily a direct purpose of the tool, reflection on its use provided the opportunity for students to express views regarding what they perceived as their needs in learning mathematics. It also caused Teacher B to reflect upon the approach he took in wanting students "to make bigger connections" in mathematics.

Self-regulated learning is multi-dimensional (Boekaerts & Cascallar, 2006; Zimmerman, 2002), and organisation and persistence for mathematics learning are just two small aspects. However, the results from this intervention suggest that for at least some of the students in this Year 8 class, this was an appropriate place to start as the tool allowed students to address goals that they saw as relevant and gave practical strategies that they could implement and adapt. Students also experienced success and saw the possibility of setting further goals. Student awareness, participation, and decision making for learning mathematics were enhanced for at least some members of this Year 8 class. The tool also gave an opportunity for teachers to reflect on their teaching of mathematics.

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Feedback About Professional Growth for Teachers of Mathematics: A Developmental Perspective

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This paper provides some insights into teacher professional growth as part of the ten-day Australian Government Summer School for Teachers of Mathematics Programme (the Program). These insights are revealed in three surveys completed during the Program. Preliminary analysis has provided information about participants' expectations related to classroom practice and the professional growth that took place during the program. This growth represents a shift in focus from the acquisition of new knowledge to thinking more deeply about its applications.

Introduction

An important characteristic of a successful professional development program has been articulated as working with, rather than doing to, teachers (Loughran & Gunstone, 1997). This approach is recognised as providing an appropriate mix of influences that contribute to professional growth. Affirming teachers as central agents is also recognised by Clarke & Clarke (2005) in ten key principles that increase the likelihood of long-term and effective professional development. These principles can be used to identify the processes that teachers engage in during professional development programs. For example, addressing "issues of concern and interest, largely (but not exclusively) identified by the teachers themselves and involve a degree of choice for participants" (principle number 1, Clarke & Clarke, 2005). Failure to place teachers at the centre of any plans for reforming practice or to implement innovations, can only lead to disappointment in the achievement of positive outcomes particularly if those plans do not incorporate sustained activity (Gore & Ladwig, 2006; Supovitz & Turner, 2000; Van Driel, Beijaard, & Verloop, 2001).

There is no shortage of guidelines for effective professional development. A typical example in the context of improving assessment practices is that of Black and Wiliam (1998). They have identified a four-point scheme for development, some important features of which include:

- Teachers being supported to work together;
- Teachers incorporating ideas into classroom practice;
- Teachers balancing the requirements of curriculum imperatives and meaningful learning; and
- Teachers gaining feedback from peer/external review of their practice.

Such a scheme provides a working model for professional development that is based on establishing communities of learners who can reflect on their practice. In addition, teachers are provided with clear statements of what it means to be an excellent teacher at various stages of their careers, e.g., The National Professional Teaching Standards for Advanced Teaching and School Leadership (Teaching Australia, 2007), and the AAMT Standards for Excellence in Teaching Mathematics in Australian Schools (AAMT, 2006).

Elsewhere, Loucks-Horsley, Love, Stiles, Mundry, and Hewson (1998) have attempted to simplify the enormously complex process of crafting professional development. In their Design Framework for Professional Development in Science and Mathematics, four key components of Knowledge and Beliefs, Context, Critical Issues, and Strategies interact with a sequence of implementation phases for professional development programs. This framework sets out a developmental sequence that can act as a useful guide, not only for the developers of programs, but also for the way participants interact productively with the 'diet' of materials which make up any program.

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Some essential features of effective professional development would therefore need to include teachers playing an active role, and an environment that is conducive to the sharing of knowledge about teaching and learning in the classroom. In addition, the content and structure of a program would need to facilitate planning for the improvement of teaching and learning outcomes, and critical reflection by teachers about their roles.

Background

The Australian Government Summer School for Teachers Programme (the Program) was announced as part of the Australian Government's 2007-08 Budget Package: Realising Our Potential. Funding of $101.7 million over four years was committed to this initiative. A key feature of the Programme is the recognition of up to 4,000 of Australia's high performing teachers, over four years, by providing them with opportunities to further enhance their knowledge and skills in one of five priority areas: Literacy and Numeracy; English; Mathematics; Science; and Australian History. Indirectly, through follow-up professional development activities and funding through the Australian Government Quality Teaching Programme (AGQTP), many more teachers will benefit from the Summer School experience. The University of New England (UNE) and Flinders University were selected to develop and deliver ten-day Summer School courses for 2008 to approximately 400 teachers in mathematics and science.

Five two-day modules were developed to deliver the Summer School materials with opportunities for teachers to have some choice within the modules to focus on particular aspects that might be more relevant to their needs and interests. The course content showcased current thinking in pedagogical practices, subject knowledge (including pure research and industrial applications), educational leadership, and curriculum issues. In brief the modules were:

- *Frame Works* explored 'big picture' neurocognitive frameworks and how they help teachers understand, practice and assess learning, teaching and problem solving in mathematics.
- *Potential Difference* examined the advantages and challenges associated with student diversity in mathematics classrooms, and how to cater for such diversity.
- *Cutting Edges* explored what it means to be a mathematician in the 21st century by providing participants with opportunities to hear from people who work with and apply mathematics in their daily work.
- *Next Praxis* focused on teaching and learning strategies that integrate information communication technologies within mathematics curricula.
- *World Class* provided an opportunity to share exemplary practices in mathematics education.

During the 10-day Summer School, participants completed three surveys – one at the beginning (Day 1), one mid-way through (Day 5), and one towards the end (Day 9). A fourth survey is scheduled for three months after completion of the Program. The remainder of this paper provides a preliminary analysis of the responses of the participants over the Summer School with the view of identifying the nature of teacher growth.

Participant Feedback

The tender accepted by the Australian Government for the Summer School incorporated two evaluation components. The first of these was a Government initiated evaluation of all Summer School Programs, and the second was a commitment by the Program providers to conduct an internal evaluation. The issues to be addressed by this second evaluation were prescribed by the Department of Education Employment and Workplace Relations (DEEWR) and were drawn from the evaluation criteria of the Australian Government Quality Teaching Programme (AGQTP) Reporting Framework (2007). From these criteria, a schema was constructed setting out the questions to be addressed throughout the Summer School.

The comments obtained from surveys were the most important feedback as they represent, first-hand, the expressions of participants' concerns and interests. This feedback provided mainly qualitative data consisting of responses to questions in each of the surveys. Two questions in the final survey seeking to measure participants' satisfaction with the Summer School required teachers to respond using Likert scales. All free response feedback from the survey forms was transcribed and forwarded to an independent evaluator for analysis to identify emerging themes. The first of the surveys was designed to elicit participants' expectations and motivations in relation to attendance at the Summer School. It provided a convenient reference point for subsequent feedback. Subsequent surveys provided important information concerning the efficacy of the materials making up the Program and the extent to which they supported teachers' expectations, engagement, and further professional learning.

Participant Expectations (Survey 1)

There were two free-response questions in the initial survey of the Summer School that asked participants about their motivations and expectations. An additional question, "Any other comments?" was provided for participants to raise any other issues they saw as relevant. There was significant cross-over in responses with some participants referring to learning about particular aspects of their teaching as a motivation and others as an expectation or area they hoped to address. Consequently, no attempt has been made to report motivation separately from expectations in this preliminary analysis. What emerged from their comments were five broad themes which relate to classroom and professional practice:

- *Theme 1*: To improve teaching and pedagogy.
- *Theme 2*: To update the currency of knowledge in specific areas, e.g., integration of ICT within the curriculum, catering for student diversity.
- *Theme 3*: Opportunities for professional interaction and networking for the purposes of personal learning or sharing learning with colleagues.
- *Theme 4*: The enhancement of student learning and student engagement.
- *Theme 5*: Learning about curriculum and assessment.

The five main themes listed above are in broad agreement with the results that were identified in the recent SiMERR National Survey (Lyons, Cooksey, Panizzon, Parnell, & Pegg, 2006). This survey set out to obtain a deeper understanding of the particular professional development needs of rural teachers, the learning needs of their students and the expectations of their communities.

Participant Feedback Related to Expectations (Surveys 2 and 3)

This section provides an overview of participant responses from the second and third surveys administered during the Summer School. These comments have been selected as indicators of professional growth for each of the five themes that represent participant expectations of the Program. Each of the themes is discussed separately drawing on feedback covering the two separate time periods. The relevant open-ended questions that made up the second survey, and which were repeated as the first of four questions in the third survey, were:

1. Has participation in the Summer School sessions thus far increased the currency of your knowledge of issues in mathematics education? (Please comment)
2. Were the issues presented in the Summer School sessions thus far relevant to the teaching of your students? (Please comment)
3. To what extent have the sessions thus far challenged you to think about new directions and strategies for the teaching of mathematics in your classrooms/school? Please comment on the aspects of the sessions that have been most helpful to you.
4. Any other comment?

Theme 1: To improve teaching and pedagogy. In the second survey, there were many sessions that participants drew attention to as being particularly useful, for example, those related to low-achieving students, indigenous students, engaging students in the middle years of schooling, the SOLO assessment framework (Pegg, 2003), and the *QuickSmart* Program (Pegg & Graham, 2007). Some commented on being taken out of their teaching comfort zones and several voiced their intention to make changes in their practice, or felt just "revitalised". Underpinning the majority of comments in Survey 2 was a focus on being motivated, the acquiring resources and identifying useful strategies that could be applied in the classroom. Some representative examples of comments (from Survey 2) included:

> I am challenged to rethink strategies in dealing with low-achieving students and indigenous students. I have new resources for engagement of students.

> The low achieving/diversity choice has given me ideas I'd like to try in my classroom.

> They have challenged me by stepping out of my comfort zone of teaching to better my students. The aboriginal/middle school sessions gave me many strategies to help with my student's education.

In the third survey on Day 9 of the Program, responses took on a more personal professional relevance with many participants indicating that they felt empowered by sessions. Typical responses referred to what participants intended to do, for example, restructuring lessons, planning significant changes in teaching, re-thinking classroom practice, and restructuring approaches. Some talked about "bad habits" that they had fallen into over the years. They talked about "new ways to re-engage and structure my teaching", "new teaching techniques to bring to the classroom", or being helped to "develop a stronger teaching framework". Some participants began describing plans for implementing changes in their schools indicating that the sessions were "invaluable in supporting my arguments as an agent for change in the mathematics teaching and curriculum development". Another stated "I am looking forward to taking time to meet with my head of department and principal to discuss possible avenues from here". It is clear from these comments that participants' thinking became oriented towards how to improve their own practice – and that of others, as the Summer School progressed. Some representative comments (from Survey 3) included:

> I thought that before I came that I was an experienced and capable teacher. I now realize just how much I have to learn.

> It has made me realize that I need to look through the eyes of the students more often and allocate more time to the reflection of my lessons … I can see that technology in the classroom will open up so many more opportunities for teaching and more importantly learning. I have also got to inspire my staff to think of themselves as learners.

> I'll have to try and convince some of my colleagues that their teaching and learning programmes and ideas may need a little 'adjustment'. I'll have to change my ways a bit too! We are meant to be 'change agents' and I'll have to be fearless in my pursuit to set up the Best Possible situation for the students in my classes!

Theme 2: To update the currency of knowledge in specific areas, e.g., integration of ICT within the curriculum, catering for student diversity. In the second survey, participants commented positively about specific areas of interest, for example, the *QuickSmart* Program for low-achieving students (Pegg & Graham, 2007), the SOLO assessment model (Pegg, 2003), models of brain functioning (Geake, 2004), and the importance of focusing on mental computation and open-ended tasks. Some comments were more general, mentioning that the Program had "sown many seeds", helped improve the "conceptual knowledge", or helped to "rethink the validity of some of the fundamentals on which my practice is based". The underlying benefit for participants at this stage in the programme related to the ideas and strategies that could be applied. Some representative comments (from Survey 2) included:

> The *QuickSmart* program have/will have an impact on my thinking and structure of classroom programs. Some sessions have given me extra ideas for adding more 'creativity' in my classroom programs.

> I now have better contextual knowledge, some clever ideas, more ideas about mathematical careers… The focus on neuroscience has been interesting for me: a new area.

> One particular session has opened my eyes to the importance of focusing on mental computation and open-ended tasks.

In the third survey of the Program, there were still many positive responses about being able to update knowledge in specific areas. Some comments were more general indicating that sessions "challenged our teaching philosophy as well as our school priorities", "allowed me to challenge my current thinking about teaching within a classroom", and provided "a better understanding of what works". Some comments indicated appreciation for the "evidence-base for thinking on current knowledge" and the encouragement to be "a leader". As participants experienced more of the Summer School Program, they were able to 'dig deeper' into their knowledge about teaching to consider underlying frameworks. Some representative comments (From Survey 3) included:

> The feedback session was a real challenge. The need to find out what students know and don't know is essential to providing lessons that have meaning for the students.

> They have been hugely challenging. While some sessions gave small ideas to use, others have turned my thinking 180 degrees. How can I arrange for my school's teachers to watch others teaching? How can I arrange my department to analyse teaching strategies? How can I ensure that feedback to students is regular and real?
>
> I have been challenged every step of the way, but my biggest challenge (and unsolvable, I think) will be finding the time to create and implement some changes. I know I should take small steps, but even that seems to be a large hurdle (oops! Mixing metaphors). Finding the time in an incredibly filled school schedule will mean either I go mad or have a marriage breakdown! The Summer School has filled a need that I have felt for a long time; we teachers need to have our trees shaken every so often: the apple may hit you on the head but it's still good to eat!

Theme 3: Opportunities for professional interaction and networking for the purposes of personal learning or sharing learning with colleagues. Responses at Day 5 of the Program (Survey 2) relating to opportunities for professional interaction and networking invariably referred to workshops, tutorials and informal discussions. As one respondent put it, they "allowed me to share and refine ideas raised in lectures/keynote presentations". Another referred to a "great culture of sharing and professional debate". Some participants were able to obtain resources from other participants, especially interstate participants. Some representative comments (from Survey 2) included:

> The workshops and tutorial discussions have resulted in terrific ideas and a great culture of sharing and professional debate.
>
> The sessions have been good to introduce new ideas but the most helpful part has been the chance to talk about it in workshops and tutorials to get other people's interpretation of the issues.
>
> The workshops, tutorials and informal discussions have been invaluable. I've gained great information from these, together with terrific contacts who are also life long learners. Having the opportunity to hear new ideas and research findings from the presenters and discuss current classroom practices and experiences with other teachers, will certainly impact on my teaching and thus the learning of my students.

Comments at the end of the Summer School (Survey 3) were again very positive and reinforced how participants appreciated "the collegial nature of the conference", being able "to make contact with other professionals Australia wide", of being able "to network with others and discuss with others". Once again, the workshops, tutorials and informal discussions were venues where sharing of ideas took place. As before, with comments from the third survey there is a sense that participants are thinking more deeply about issues. In this context, this represents going beyond the sharing of resources during discussions to making judgements about current teaching styles and practices. Some representative comments (from Survey 3) included:

> While it will take some time to synthesise all that I have been exposed I hope I do not teach in the same manner during this year as if I had not attended the SS. The network and sharing of ideas are invaluable.
>
> The workshops have shown me models of great teaching practices. I have experienced sitting as a student in the classroom and been able to make judgements on what I have taken away what worked for me. The tutorials where I have been able to network with others and discuss what others have learnt.
>
> The material presented, backed up by valid data and research, has provided me with additional knowledge and affirmed my practices in the classroom. Since I want my students to always do better I need to continue to incorporate new ideas into my teaching. This summer school has allowed me to make contacts with other professionals Australia wide to think about and take on other practices and to challenge my existing practices.

Theme 4: The enhancement of student learning and student engagement. In the second survey, participants commented positively on how particular sessions had equipped them with specific strategies or insights that would support student learning. These sessions included those related to assessment and the way they "make one think about student levels of understanding", or those about "engaging and challenging students in the middle years". Many comments appreciated that they would have "more tools to engage … students", that "the sessions were very relevant to the teaching of students", or that they provided "a direct connection with the instruction and implementation of mathematical ideas and concepts". Some voiced their intention to "aim to improve outcomes for all students" in their own practice or through in-servicing other teachers in their schools. Some examples (from Survey 2):

> The engaging students in the middle years of schooling presentation and workshops were not only brilliant and relevant but enabled me to take a plethora of exciting, challenging and fun ideas back to my classroom and my school.

> The initial sessions were useful in how I approach assessment of my students and hence the planning I would use in developing units. The later sessions have given plenty of practical suggestions for problem-solving/ rich tasks that I can use to supplement my current practices.

> I will try to fit in the newfound strategy/ies especially on the students with learning difficulties. I am referring to *QuickSmart* program. I now have a full understanding of the brain anatomy and need to be MORE understanding and receptive to these students.

Comments at the end of the Summer School (Survey 3) once again tended to be more focused on changes that could be brought about as a result of personal reflection and of taking particular strategies and thinking deeply about how to use them to best effect. In many cases participants referred appreciatively to "the idea of seeing the lessons through the children's eyes" and "learning through the eyes of the students". Compared to their responses in the survey half-way through the Summer School, there is a greater sense of a more focused desire to implement changes in their own classrooms in a way that benefits the students. Some representative comments (from Survey 3) included:

> The concept of student learning became a focus. It was good to consider how we as teachers can self-reflect in order to improve.

> The idea of seeing the lessons through the children's eyes had particular impact on me. I am sure I need to give more valuable feedback to my students.

> Probably the most challenging aspect of teaching for me is the significance of feedback on learning, both for the student, and for me as a teacher. I think I need to find some ways to improve in this area hopefully in some "measurable" way. I would also like to find some ways to get the students to reflect more on their own learning process.

Theme 5: Learning about curriculum and assessment. Once again, participants commented positively in the second survey about specific areas of interest, for example, the messages contained in results from PISA and TIMSS, and the SOLO assessment model. The underlying benefit for participants at this stage in the program related to the insights and ideas that could be further developed. Some representative comments (from Survey 2) included:

> … the update of [the SOLO assessment] framework has been excellent in that I can now see much more clearly how it can be useful to me in formative assessment of my students.

> PISA and TIMSS added so much to my knowledge.

> PISA and assessment … I will be able to identify their (students') level of thinking and target scaffolding from there.

In the third survey on Day 9 of the Program, there were still many positive responses about being able to update knowledge about assessment. Some comments were more general indicating that sessions provided "better understandings", and "a real challenge. As highlighted above for other themes, as participants experienced more of the Summer School Program, they were able to 'dig deeper' into their knowledge about teaching and to consider how they might change practice. Some representative comments (from Survey 3) follow:

> … having a better understanding of what works, how effective strategies are, etc, based on research etc gives us ideas and makes us hopefully, constantly evaluating our classroom practices, reasons for using an activity, fine-tuning on assessment or evaluating the way we give feedback information to our students.

> I have learnt that I must concentrate on the teaching practices that actually improve student achievement, not just the ones that just make me feel I have done a good job.

> The assessment parts; trying to move away from quantitative tests will be a challenge for the school where I teach but gave me great insights into why we should.

Conclusion

In working with, rather than doing to, teachers, the Summer School Program provided a productive context in which to obtain commentary about their issues of concern and interest. The feedback from the participants presented above represents a 'snapshot' of their engagement with the Program materials. An important component of that engagement was the commentary about how relevant information might inform improvements in student learning outcomes. The preliminary analysis has identified five areas of professional relevance for participants and for each of these areas there was an observable shift in how they reflected on the sessions. This shift was from a recognition of the many ideas, strategies and insights that presenters delivered to a deeper thinking about the issues in terms of where to next – both personally and professionally, and the processes associated with improving student learning. This shift is summed up in one comment from the end of programme survey:

> I always thought I was a "good" maths teacher. However, looking back there are things that I need to improve or address in order to meet the learning needs of my students more effectively. I now feel that I have a more theoretical base for my beliefs that I can extend into developing worthwhile mathematical (and also other KLA's) activities.

This identified shift to a more reflective professional focus highlights two aspects of the Program. The first is the professionally supportive structure that was provided by the sequence of modules. Whilst evidence-based research was common to all modules, sessions that focused on the incorporation of ideas and strategies into the classroom, and the planning of future activities in schools were a feature of the last days of the Program. The second was the opportunity for discussion that facilitated the transition from "new knowledge" to its "practical application". The associated workshops, tutorials and informal networking times encouraged the sharing of information about approaches to teaching and learning, and about what contributes to enhanced learning outcomes for students.

The professional development provided by the 10-day Summer School Program was intensive and indications are that the collegial interactions were highly productive for providing insights into teaching and learning. The preliminary analysis of participant evaluations echoes some of the findings in the literature concerning components of effective professional development (e.g., Black & Wiliam, 1998), the need for sustained professional development to bring about change (e.g., Supovitz & Turner, 2000), and the importance of providing research-based knowledge to support improvements in student learning outcomes (e.g., Ingvarson, Meiers, & Beavis, 2005).

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Fraction Number Line Tasks and the Additivity Concept of Length Measurement[8]

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The performance of 29 Grade 6 children on eight number line tasks is descriptively analysed with regard to three types of misconceptions. The first is conceptual; over-generalised part-whole unit-forming. The other two are semiotic misreadings of inscriptions, namely counting lines and decimalising. Success on these number line tasks, possible examples of the measure sub-construct of fractions, is descriptively compared to the children's performance on tasks assessing conservation and the additivity concept of length measurement, and on multiplication and division items.

Theoretical Framework

The transferability of knowledge within and across domains in mathematics is seen as important to the development of relational understanding. Understanding of fractions leads to proportional reasoning, and fraction understanding is assumed in the study of algebra, probability and geometry (Kieran, 1993). What are the concepts from other domains that support the development of fraction understanding? Links between the domains of fractions and measurement are made from both areas of the research literature. Rational numbers are necessary to describe leftovers that are the result of a non whole number count when measuring (Kieren, 1995). Researchers in the measurement domain also recognise analogous concepts in some measurement and fraction tasks and call for more research on the intersection of the two domains (Lehrer, Jaslow, & Curtis, 2003). The theory of constructivist learning would also suggest cross-domain interaction in understanding, as well as intra-domain trajectories. The connection between performance on integer number lines and measurement tasks has been reported (Pettito, 1990), but the present paper addresses this connection with number lines in the rational number context.

One type of misconception that can occur within a domain, traceable to conceptual over-generalisation, is an error in accounting for the whole during unit-forming. An example of this is in number line tasks, and may appear as finding *half of* a number line with pre-marked partitions rather than where the *number half* goes (Kieren, 1993). This same instrumental part-whole understanding is evident when the child makes their own partitions on number lines, for example, when drawing a number line from 0 to 6 and labelling 4 as 2/3 (Clarke, Roche, & Mitchell, 2007).

Another family of misconceptions can be attributed to semiotic misreadings of mathematical diagrams or inscriptions. Inscriptions are written representations including both diagrams and symbolic notation, and excluding mental representations (Roth & McGinn, 1998). These misconceptions are a by-product of the use of visual texts to convey information. Mathematical diagrams, both inscriptions and mental visualisations, convey meaning though agreed semiotic conventions on how they are to be decoded (Presmeg, 2006). A behaviour that may link to a semiotic misconception in both the measurement and the fraction domain is counting lines, including the zero-point, and not spaces on scales. In broken ruler tasks, some low ability students revert to counting unit markers rather than linear sub-units (Bragg & Outhred, 2000). Successful students on number line tasks attend to the parts, rather than the vertical lines used to create the equal parts (Bright, Behr, Post, & Wachsmuth, 1988; Pearn & Stephens, 2007). For example, 3/4 is called 4/5 if the mark at zero is included in the count. However, strategy use is inconsistent on rational number number lines and similar measurement scales (Drake, 2007). Other problems with nomenclature when identifying fractions on number lines may also have a semiotic component.

The model used in the present research as a criteria for fraction task selection and analysis, is Kieren's sub-construct model which subsumes part-whole relational knowledge into the measure and quotient sub-construct, leaving four sub-constructs- measure, quotient, operator and ratio (1993). The model includes three underpinning concepts- partitioning, equivalence and unit forming. Fractions, decimals and percentages are the notations that can be used to describe mathematical contexts in these sub-constructs. Research on number

8 The authors wish to thank Professor Doug Clarke and Ms Anne Roche for their advice and assistance in the development of the fraction interview protocol and record sheets.

lines in rational number has concentrated on the distinction between making partitions and reading pre-marked partitions, and proper and improper fractions on number lines labelled 0 to 1 and 0 to greater than 1 (Novillis Larson, 1980; Bright, Behr, Post, & Wachsmuth, 1988; Ni, 2000). There has been an assumption in the literature that as a number line can be an example of the measure sub-construct of rational number, that all number lines are measure sub-construct tasks (Ni, 2000). A task is only an example of the measure sub-construct of fractions if that is the reasoning that the child uses when attempting the task. A child may over-generalise from number lines marked 0 to 1 that the ends of the number line form the whole, rather than using the scale to help with unit forming. This can contribute to the first type of misconception described above.

The model for length measurement used as criteria for task selection in this research is based on Lehrer's eight key concepts for spatial measures (2003). We have categorised these eight key concepts into four concepts that need to be co-ordinated; attribute, additivity, units, and proportion. The *attribute* concept includes distinguishing between multiple attributes of a figure, and recognising attributes in increasingly complex formations such as bent paths and perimeters in length measurement. The concept of *additivity* requires an understanding of the zero-point and conservation, that the whole is the sum of the parts including in bent paths. The concept of *units* includes describing both fractional parts of units when count is not a whole number of units, and the inverse relationship between the size of the unit and the count. The concept of *proportion* is important to spatial measures (length, area, volume and angle) involving the recognition that the size of the count is represented proportionally in visual representations of spatial measures.

Number lines are an example of a mathematical context that may enable reasoning from the measure sub-construct of rational numbers and the additivity concept of measurement, because the zero-point is important to both contexts. The inclusion of the semiotic component to the decoding of inscriptions adds another layer to the analysis of children's performance on number line tasks.

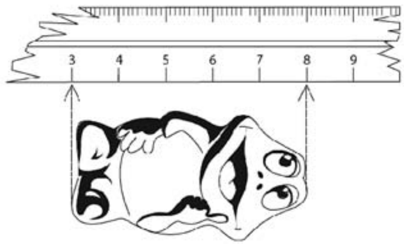
Methodology

A one-to one task-based interview, similar in format to the Early Numeracy Interview (Department of Education and Training, 2001), was developed and covered the domains of a) multiplication and division, b) fractions, c) measurement and d) dynamic visualisation or geometric reasoning. The results reported here are preliminary both in terms of sample size and in the narrowing of the scope of this paper to length measurement and fraction number line tasks. The data collection took place in February and March 2008 and included 29 Grade 6 children from two schools in metropolitan Melbourne.

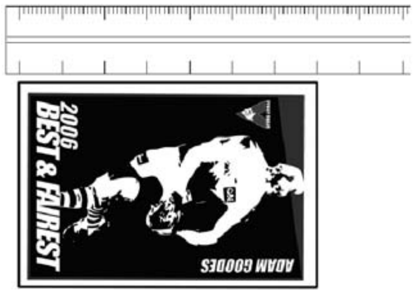
The tasks in the measurement section, analysed in this paper, were chosen to assess relational understanding in the conceptual areas of *additivity*, see *Figure 1*. The interview was piloted in 2006, enabling refinement in all sections and providing evidence that tasks in each of the measurement conceptual sections were hurdle tasks. This enabled the protocol of designating a threshold task for each of the four measurement concepts that every child was asked. Then, if successful they were asked a high benchmark task or if unsuccessful at the threshold task, they were asked the low benchmark task. This divided the cohort, in each of the four conceptual areas, into four sequential groups.

<table>
<tr><td>41. Threshold task

This centimetre ruler is broken. It is measuring a Freddo frog. How long is the Freddo frog? How did you work that out?

Adapted from Bragg and Outhred (2000).</td><td></td></tr>
<tr><td>42. High benchmark task

This ruler measures in centimetres but there are no numbers on it. How long is the footy card? How did you work that out?

Adapted from Bragg and Outhred (2000).</td><td></td></tr>
<tr><td>43. Low benchmark task

These are two pieces of wire that can be bent and straightened. Between the dots is the same length. If the wires were straight, would they be the same length or would one be longer than the other? How did you work that out?

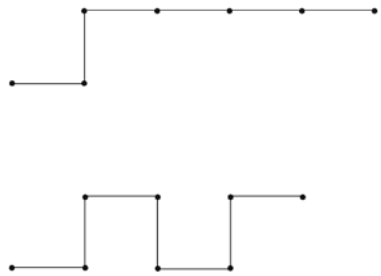
Adapted from Battista (2006).</td><td></td></tr>
</table>

Figure 1. Additivity conceptual tasks in the measurement interview.

Number line tasks reported in this paper, see *Figure 2*, were one type of task used to assess relational understanding in the measure sub-construct of fractions. Other length and area inscriptions were used for this sub-construct in the interview. Table 1 shows the criteria identified in the research literature that directed the choice of number line tasks. The multiplication and division tasks were taken directly from the Early Numeracy Interview (Department of Education and Training, 2001), but unlike the standard protocol for the early years, after omitting the first two questions, this section of the interview was used in its entirety.

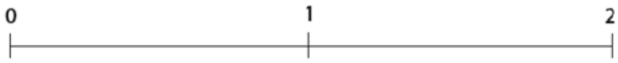
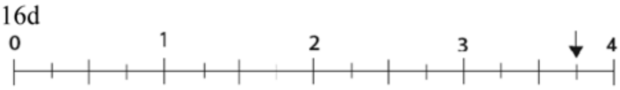
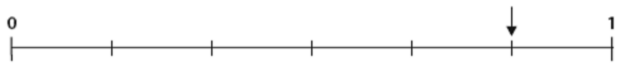
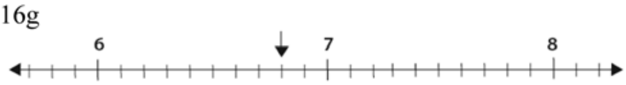
16a. *Give the child a blank piece of paper and pen*. Please draw a number line and mark two thirds on it. *If the child does not mark 0 or 1, ask,* where does zero go? Where does one go? How did you work that out? (Clarke, Roche, & Mitchell, 2007)	
If this is half, *point*, where would one and a half be on this number line? Adapted from Bright, Behr, Post, and Wachsmuth (1988).	16b 
Please mark where one quarter would go on this number line. Adapted from Pearn and Stephens (2007)	16c 
Point to arrow, what number or fraction is that point on the number line? Adapted from Lesh, Landau, and Hamilton (1983).	16d 
Point to arrow, what number or fraction is that point on the number line? Adapted from Novillis Larson (1980).	16e 
Point to arrow, what number or fraction is that point on the number line? Adapted from Pearn and Stephens (2007).	16f 
Point to arrow, what number or fraction is that point on the number line? (Ministry of Education, 2007)	16g 
Point to arrow, what number or fraction is that point on the number line?	16h 

Figure 2. The eight number line tasks used in the data collection interview.

Table 1

The Selection of Number Line Tasks to Represent Research-based Criteria

Task requirements	Number lines 0 to 1	Number lines 0 to >1
Make partitions	16a	Proper fractions: 16c Improper fractions:16b
Read partitions	16e, 16f (non-equal parts)	Improper Fractions: 16d, 16g, 16h

A ranking system was developed because the sample was quite small, in order to compare children's success on different tasks. A weighted score was calculated for the eight number line tasks based solely on the performance of the cohort and not on outside standards for this age group, even though these were available for some of the tasks. The frequency of success (fs) on each number line task was calculated as a percentage, then, if a child was successful on a task, their score was 100 – fs, i.e. they performed better than (100-fs)% of the students. In this way, they accumulated points for correct answers but more difficult questions were weighted more highly. An overall number line weighted score was generated. The same protocol was applied to their performance on the tasks from the multiplication and division section of the interview. These different weighted scores enabled the children to be individually ranked from one to twenty-nine on these separate criteria. The ranking process for the additivity tasks was different as the tasks had been chosen because they were sequential. Children's success on the tasks determined to which of four levelled groups they would belong. Each child within a group was equally ranked to the other children within the same group on the additivity tasks.

Results and Discussion

Permutations

There were 25 different patterns of correct and incorrect responses from the 29 children on the eight number line tasks. As four children were successful on all eight tasks and two were not successful on any, there were 21 different patterns of correct and incorrect responses for the 23 children remaining. The pattern of correct responses did not fall into a sequential pattern, although some questions were clearly more difficult (16a) than others (16b). While the children who were successful on four of the eight number line questions all answered 16b correctly, their incorrect responses did not fit a pattern.

Misconception 1: Instrumental part-whole knowledge (difficulties with unit-forming).

Of the 19 children who were unsuccessful at drawing a number line and marking 2/3, (16a) six constructed a number line using another ratio of 2/3, for example marking 2/3 at 6 on a number line marked from 0 to 9. On a separate question, 16c, 11 of the 15 children who placed 1/4 incorrectly on the number line, marked it at 1/2 and offered a part-whole explanation that indicated that the whole was the whole number line from 0 to 2. This misconception was not uniformly displayed with only three children using this strategy on both number line tasks 16a and 16c. Conversely, three children demonstrated this unit-forming misconception only on 16a and another eight demonstrated it only on 16c. The frequency of this misconception supports the conjecture made in the theoretical framework section above, that not all number line tasks assess the measure sub-construct of fractions.

Misconception 2: Counting lines not spaces.

There were six children who demonstrated the misconception of counting the lines rather than spaces on one or more of the number line tasks, with one child demonstrating it consistently across four of the tasks. One child, knowing the interviewer could not indicate correct or incorrect answers, verbalised this misconception and rhetorically asked whether he should count the lines or the spaces. This misconception provided the opportunity for two children to get the right answer for the wrong reason (marked incorrect in the data analysis). These two children did not imagine the missing quarter mark on task 16f and counted the zero mark and the two lines marked on the number line. They then called the indicated mark 3/4 for the wrong reason. Of course, many children counted the vertical lines successfully, either instrumentally knowing not to count the zero or whole number mark, or relationally using the vertical lines as markers of both the end of one unit and the beginning of another. Similarly to Drake's results (2007), demonstration of misconceptions is not consistent, and only one of these six children also counted the lines instead of the spaces on either the Freddo or the footy card task in the additivity section of the measurement interview. There were numerous other errors due to nomenclature issues but they did not indicate the counting of the mark at zero as part of the count, except in some cases of the misconception described below.

Misconception 3: Decimalising the count.

During the coding of responses, a third misconception emerged which also appeared to be a semiotic misreading of the number line inscriptions. The child read the mark as if the whole was divided into tenths. Generally, they counted left to right, so five sixths (16e) became *point five.* One child self-corrected her response of *point five* to *point nine,* because she reinterpreted the number line by reading right to left, and counted back from the whole. The non-equally partitioned 3/4 (16f.) became *point two* or *point three* depending on whether the counting lines misconception was present or not. Five children demonstrated this misconception but only on one or two questions each. Unsurprisingly, all but one of them gave a correct decimal response to question 16g. This decimalising the count misconception may be an indication of an interference from the measurement domain, as Australian children use metric measuring instruments and scales may prompt a decimal response. One other child had a specialised decimalising nomenclature system and has not been included in the numbers for the description above. His naming system involved identifying the fractional marking and the whole numbers, for example, 5/6 became *zero (the first whole number), five (a correct count of the lines) point one (the second whole number).* On the four tasks that he used this nomenclature, he did not display the counting lines misconception. Equally, other children used decimal notation successfully to describe the points on the number line, most notably in the last two questions.

Transfer between Domains

If the misconceptions that appear similar in measurement tasks and fractions tasks are linked, it is difficult to tell in which direction the transfer might go. It is possible that the transfer might come from either domain, possibly reliant on the classroom activities and mathematical sense making that the child has encountered. It would seem reasonable that there would be a lag been the acquisition of an understanding in one domain, and its transfer to another. If the transfer can happen in either direction, the data may not show a strong correlation.

Success on Number Line Tasks

A comparison between the children's performance on number line tasks and their additivity grouping showed that this correlation may be worth pursuing with further research, see *Figure 3*. A similar comparison between children's overall performance on the eight number line tasks, and their performance on the multiplication and division tasks was performed, also shown in *Figure 3*. It is important to remember that the children's rank order in additivity and in multiplication and division is different. In reading the figure below, the children's ranking stays consistent down a column but not across the two columns.

Success at the threshold task has been achieved by children 1-20 in the additivity concept grouping, and the eight highest weighted scores for number line tasks are by those children. Conversely, several of the top eight successful children on number line tasks fall in the third quartile of the multiplication and division individual ranking. This showed that the idea of the *successful child* in one domain also being generally successful across several domains was not confirmed by the research. This gives further weighting to the importance of the finding above that success on the number line tasks was correlated with success at the threshold task in the additivity concept tasks.

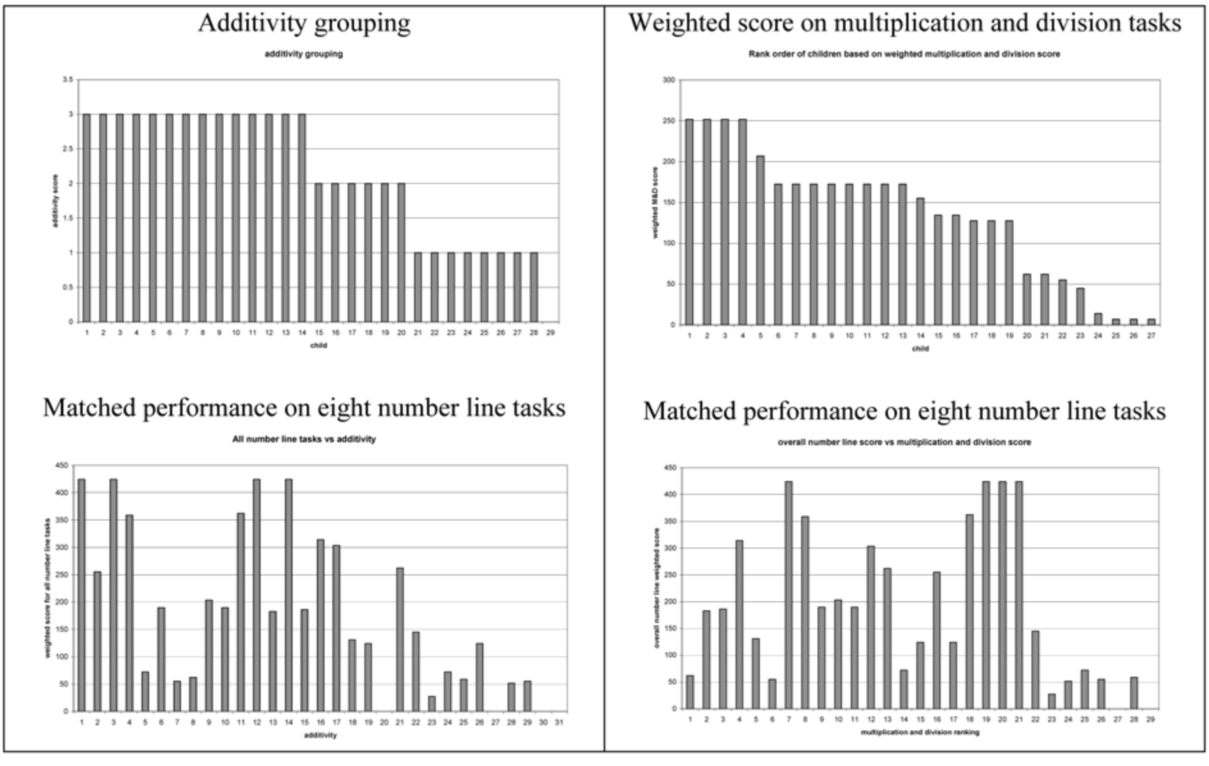


Figure 3. Correlation of number line tasks to additivity and multiplication and division performance.

Conclusions

The variation in patterns of response to the number line tasks highlights the complexity of the task facing teachers in our schools. The range of understandings and strategies amongst our students is broad. Two misconceptions due to semiotic factors were in evidence, counting lines, including the zero-point, instead of spaces, and decimalising the count. A unit-forming misconception related to an instrumental understanding of part-whole knowledge of fractions was also evident in the sample. Success on, as opposed to misconceptions about, number lines correlated more highly with a conceptual understanding of the additivity concept of length measurement than to success at multiplication and division. However, while some children displayed the misconceptions consistently, others only employed them in specific situations, indicating that the counting lines misconception may not be a major contributor to the correlation between additivity grouping and performance on number line tasks. It is unclear whether there is transfer between the two domains of the understanding of the semiotics of lines and spaces. If there is transfer possible in both directions, correlations would be unlikely to be evident. The four conceptual areas of measurement are co-ordinated in an ideal understanding. Number line tasks that use non-equal spacings might draw on the measurement concept of proportion. Number line tasks that use improper fractions might draw on the measurement concept of the unit. Number line tasks in general, because the zero point is crucial to the inscription, may draw on the measurement concept of additivity. Success would seem to be related to a combination of factors rather than the absence of specific misconceptions.

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"Zero is Not a Number": Teachable Moments and their Role in Effective Teaching of Numeracy

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This paper reports on the use of "teachable moments" and the role they play in effective teaching of numeracy. Transcripts from three case study teachers' numeracy lessons were examined to identify teachable moments; qualitative descriptions illustrate the nature of these teachable moments and their potential to enhance students' understanding. It was found that teachers "missed" teachable moments, incorporated them into the discourse or actively ignored them. The findings indicate that teachers need to identify when to act upon teachable moments to avoid the likelihood of students forming misconceptions about important mathematical concepts.

Background

Studies such as the the Effective Teachers of Numeracy Study (Askew, Brown, Rhodes, Johnson, & Wiliam, 1997) have contributed much to our understanding as to what constitutes effective teaching of numeracy. Askew et al. identified effective teachers of numeracy in a range of schools in the UK by looking at mean test scores of students over time. Their identification of effective teachers of numeracy was based on rigorous evidence of increases in pupil attainment, not on presumptions of "good practice" (Askew et al., 1997, p. 8), using data collected from over 2000 pupils and evidence gathered from a sample of 90 teachers. According to Stephens (2000), there appear to be no comparable studies of numeracy undertaken on a similar scale in other countries, but a study designed to identify the effective teaching of numeracy by early childhood teachers was undertaken in Australia by Clarke et al. (2002). Like Askew et al., Clarke et al.'s identification of effective teachers was based on data related to students' mathematical growth. Both studies identified a number of characteristics that were common to effective teachers of numeracy.

The characteristics that were common to both studies were identified by the author, with the term "principles of practice" adopted to describe the commonalities: make connections, challenge all pupils, teach for conceptual understanding, purposeful discussion, focus on mathematics and positive attitudes. As an illustrative example of how the principles were derived from the two studies, "make connections" was derived from "used teaching approaches which connected different areas of mathematics and different ideas in the same area of mathematics …" (Askew et al., 1997, p. 1) and "used teachable moments as they occurred and made connections to previous mathematical experiences" (Clarke et al., 2002, p. 13). The other principles were derived in a similar way.

In order to understand the craft of teaching, exemplary cases against which to measure or model performance are needed (Leinhardt, 1990). The studies conducted by Askew et al. (1997) and Clarke et al. (2002) allowed the author to identify "critical elements" (Leinhardt, 1990, p. 19) that characterised effective teaching of numeracy. Reports from the two studies, however, did not detail specific observable teaching behaviours that illustrated these principles in action. Following initial data collection and a further review of the literature, the author identified six teaching actions which could be directly observed in the classroom and which provided illustrative examples of the principles in action. These actions were: choice of examples, choice of tasks, questioning, use of representations, modelling and teachable moments. In this paper, teachable moments are specifically discussed, using examples from classroom observations, and the following research questions are addressed:

- To what extent did the teachers in this study recognise and act upon teachable moments?
- When and how did teachers use teachable moments in the course of their numeracy lessons?

Theoretical Framework

Teacher knowledge and beliefs

According to Askew et al. (1997), understanding why some teachers were more effective than others required an examination of the relationship between teachers' knowledge, beliefs and classroom practice. Shulman (1987) proposed that a teacher's knowledge base was comprised of seven types of knowledge. Although acknowledging that all knowledge types are important and interact with each other to impact on effective teaching of numeracy, teachers' content knowledge and pedagogical content knowledge (PCK) are particularly applicable to the study discussed in this paper. According to Shulman (1987) the teacher has a special responsibility in relation to content knowledge and should possess depth of understanding in order to communicate what is essential about a subject and be able to provide alternative explanations of the same concepts or principles. Having a well-developed content knowledge, however, does not necessarily result in effective teaching. Teachers also require well developed pedagogical content knowledge (PCK) which entails "the blending of content and pedagogy into an understanding of how particular topics, problems or issues are organised, represented and adapted to the diverse interests and abilities of learners, and presented for instruction" (Shulman, 1987, p. 8).

Together with teacher knowledge, teacher beliefs have also been found to significantly influence classroom practice. Thompson (1992) distinguishes beliefs from knowledge in that they can be held with varying degrees of conviction and although independent of their validity, are valid for the individual who holds them. Askew et al. (1997) found that the teachers in their study differed in their beliefs about what it is to be a numerate pupil, how pupils learn to become numerate, and how best to teach pupils to become numerate. According to Askew et al., the implicit beliefs or theories that teachers have, combined with their knowledge, influenced the way that teachers interpreted classroom events. For example, if a teacher believed that being numerate involves "the ability to perform standard procedures or routines" (p. 31), then pupil errors were more likely to be interpreted as the result of pupil carelessness or lack of attention (transmission belief). If however, a teacher believed that pupils were trying to make sense of information, then errors may be interpreted as arising from misunderstanding, rather than carelessness (connectionist belief).

Constructivism

Students will often construct knowledge in unplanned ways, which may not be consistent with the intention of the teacher. Constructivism is based on the notion that learners "construct their own knowledge" (Van de Walle, 2007, p. 22) and assumes that learning takes place as students process, interpret, and negotiate the meaning of new information (Newmann, Marks, & Gamoran, 1995). This is heavily influenced by the students' prior knowledge and their assimilation of new information depends heavily on whether that information helps them explain or extend their past experience (Newmann et al., 1995). The construction of an idea will therefore vary from individual to individual and even with the same teacher and within the same classroom (Van de Walle, 2007). In the classroom, teachers can assist students with constructing accurate understandings through the use of the six principles of practice and the teaching actions previously identified. Teachable moments in particular, can be used to make connections to previous mathematical experiences (Clarke et al., 2002) and to identify possible student misconceptions which can then be explicitly addressed and worked on (Askew et al., 1997).

Teachable Moments

In this context, a "teachable moment" refers to a teacher's simultaneous act in response to a student's answer, comment or suggestion and is utilized to either address a possible misconception or to enhance conceptual understanding. Clarke et al. (2002) used this term in their study, stating that effective teachers used teachable moments as they occurred and made connections to mathematical ideas from previous lessons or experiences. In order to capitalize on a "teachable moment" the teacher may recognize the connections between different aspects of mathematics (Askew, 2005) such as the connection between fractions, decimals and percentages, the opportunity to provide a context or "real-life" situation and the links with previous learning.

Direct reference to teachable moments in the literature is rare, but Arafeh, Smerdon, and Snow (2001) provided an example of how teaching lessons can be analysed to explore teachable moments. In their study, Arafeh et al. used TIMSS videotape classroom data to identify teachable moments, which they defined as "the set of behaviours within a lesson that indicated that students are ripe for, or receptive to, learning because they express confusion, misunderstanding, uncertainty, struggle, or difficulty with a mathematical problem, concept or procedure" (p. 3). Unfortunately when reporting on their findings, the authors focused on the methodological approaches and the evaluation of the software used to code the video footage, rather than providing illustrative examples of actual teachable moments. The study discussed in this paper addresses this "gap" through documenting classroom examples of teachable moments, and examining their potential to impact on students' construction of understanding.

Methodology

A case study approach (Stake, 1995) was used to document the numeracy practice of three upper primary teachers. The teachers were selected using purposive and opportune sampling (Burns, 2000); years of teaching experience ranged from six to eighteen and their classroom practice in general was highly regarded by the Principals in the different schools in which they taught. They were not selected because they were recommended as being particularly effective teachers of numeracy, but were considered "good practitioners"; this was an important consideration as the author was interested in focusing on factors specifically related to developing students' numeracy, rather than generic factors such as behaviour management. There was an assumption, therefore, that the teachers possessed other effective qualities, such as good rapport with students, organisational skills and effective management techniques. The researcher observed and videotaped a total of 17 numeracy lessons. Parts of the lesson involving teacher led discussions were transcribed within hours of observation and field notes were also used to document aspects of the lessons which were not captured on videotape. Following each lesson, the video footage was viewed by the author and the teacher; discussions related to viewing the footage were audio-taped and transcribed within hours. The transcripts of the lessons were analysed to identify the "principles of practice" in action through the teachers' choice of examples, choice of task, modelling, questioning, use of representations and teachable moments. Each of these actions were then further analysed and their effectiveness evaluated, sometimes through the use of specific criteria. For example, teachers' choice of tasks was evaluated through using the same framework as Arbaugh and Brown (2005) to determine the levels of cognitive demand. Lesson and audio transcripts were manually analysed and coded to identify teachable moments – instances when the teacher or a student made a comment that indicated a misconception or there was an opportunity to enhance conceptual understanding.

Results and Discussion

The teachable moments identified from the transcripts were able to be classified in three ways: the teachable moment was "missed", with no acknowledgement made by the teacher; the teachable moment was incorporated into the lesson; the teachable moment was actively ignored. Examples of each of these responses will now be discussed, using excerpts from the lesson transcripts to demonstrate the nature of these responses.

Teachable Moments Missed

The first excerpt has been extracted from a lesson involving modelling of the guess and check strategy used to solve problems. The whole class of grade 5/6 students were seated on the floor in front of the teacher (Sue) and students took turns to volunteer their "guesses" and record their answers on the whiteboard. Students were read aloud the following problem:

> A family set out on a 5 day trek. Each day they travelled 50 kilometres less than the day they had before. Total distance that they travelled was exactly 1500 kilometres. How far did they travel each day? So they went 1500 kilometres for 5 days – each day they travelled 50 kilometres less than the day before, so how could we work it out?

One of the students, Mandy, initially guessed 900 kilometres and recorded this in a table on the whiteboard. The guess was too high and Randall then volunteered to try his guess:

Randall: 200

Sue: 200, all right

Randall: [starts filling in table, beginning with 200]

Sue: So they're not traveling – whoops – they're not traveling anywhere

on Friday? They're going to stay at home. OK, so is that going to add

up to 1500?

Randall: [indecipherable]

Tr: Is it Randall? What does it add up to? 3, 4, 500?

The discussion with Randall ended there, and Sue then asked another student to try with a different guess:

Sue: Tyler, you're going to have a go? What is your guess?

Tyler: 600

Teachable moment

Sue could have asked Ryan why he selected '200', to make a connection with the first guess being too high

In the above excerpt, Sue could have incorporated the teachable moment, without disrupting the flow of the lesson. As the aim was to develop an understanding of the "guess and check" process, discussion could have occurred around how to decide which number to "guess" next, based on what occurred with previous guesses. Tyler could have been asked to justify why he had selected 600, making the connection that 900 was too high and 200 was too low.

The next excerpt occurred when Sue introduced students to the task of writing their own problems. After some discussion, the following occurred:

Sue: Well, yes – you see if you can make it appropriate to grade 5 and 6 people. OK, it's got to be age appropriate so it can't be what am I – a number between 4 and 5

[some students laugh]

David: There's not a number.

Sue: No

Teachable moment
Both Sue's comments and the student's response indicates a misconception – that there are no numbers between whole numbers

The above exchange provides an example of when a teachable moment can be used to address misconceptions. It appears that both Sue and the student (and possibly the rest of the class) did not consider fractions and decimals as numbers. The example of choosing a number "between 4 and 5", would actually have been appropriate for this age group and would have enabled students to challenge themselves to think of numbers that would "qualify". Sue's failure to capitalise on this teaching moment indicates both a lack of content knowledge and PCK and the likely construction of a (mis)understanding that there are no numbers between whole numbers.

Incorporating Teachable Moments

At other times, teachable moments were recognised by teachers in the course of the lesson and incorporated accordingly. In the following excerpt, John, another case study teacher, was modelling the process for dividing numbers and expressing the remainder as a decimal. The example used was 764 divided by 15 and John modelled the process on the whiteboard, with students contributing answers at different stages of the process (the correct answer was 50.93, with the "3" recurring). The following discussion occurred part way through the modelling, and occurred specifically in relation to expressing the remainder as a decimal by placing a decimal point at the end of 764 and adding zeroes.

John: So we've got 45 – Monty I reckon that's pretty close to it

Monty: You put down 3 next to the 9 and then you put a little dot above the 3 because it's recurring

John: So you talk about the recurring – what does that mean?

Chloe: That there needs to be another zero, and the remainder goes on to it

Teachable moment

The use of the term 'recurring' is used for the first time and John recognizes that it requires clarification

John: Listen carefully – I think I'll take some of the blame for this – Brad

Brad: You put a recurring dot – after the dot – the two numbers are the same

John: So if we had – this is an example – how many threes in ten [writes short division algorithm] a million zeros - how many threes in 10, 3 and 3 left over, how many threes in 10, three and three left over [keeps writing 3 next to zeros and 3 along the top] OK and one of my dreams is to be a hermit and live in a cave and just do that for the rest of my life because that pattern continues – and instead of having to write 5 million and zillion and infinity threes, we can just write 3.33 recurring [writes a dot over the second three] – but if the digits aren't the same

Teachable moment

John provides an illustrative example to demonstrate the meaning of recurring

Trevor: But they can't be the same again because there's going to be a million zeros again and it's never going to be 93 again

John: Um, it may do – but just for our purposes, if it's the same digit and you can see the patterns going to continue, you can use a recurring dot, but if not, just leave it – if you want to be super safe, don't use it at all. Tamara?

Tamara: Would you put a recurring dot if it kept on going 9 3, 9 3?

John: OK – you're testing me – it's something I'll have to find out – but somewhere in the back of my mind from my grade 11 maths, I think you put 2 dots, but I'll have to find out the answer – I'll have to check with the experts – but until I get back to you, don't put anything there – but remember the important thing we're looking at is the process, OK?

Teachable moment

John recognizes that the concept will need more explanation for some students

While this excerpt also illustrates the need for teachers to carefully consider their choice of examples (see Muir, 2007), it also demonstrates how teachable moments can be incorporated into the teaching exchange. John had probably anticipated through previous teaching about decimal remainders that the term "recurring" would arise and that it had the potential to confuse students. While this indicates that John possessed PCK in relation to this, his final comment reveals that he was not as confident with the content knowledge and felt inadequate with answering Tamara's question. After viewing the video footage of the lesson, John commented:

> I think personally it's really poor on my part not being confident enough to use the specific mathematical terms, which I know is really important um cause we're trying to get the mathematical language through that um so that's not an oversight, but probably lack of planning on my part um and knowledge about using the recurring [decimals] and so it's something that I need to brush up or re-brush up on make sure I know it for next time, so they've got those answers there which they're genuinely interested about …

While John recognized the teachable moments inherent in the above excerpt, it seemed that his lack of adequate content knowledge may have impacted on his ability to address the moment in the most appropriate way.

Actively Ignoring Teachable Moments

Viewing of the video footage shortly after the lessons observed provided the teachers with opportunity to clarify their responses to teachable moments. The following excerpt occurred in a lesson conducted by Sue, and involved a discussion about square numbers. Students had volunteered a number of examples, then one of the students mentioned zero.

> Sue: OK, think of another one. David?
> David: Zero
> Sue: Zero?
> David: 0 times 0 is zero
> Sue: Zero? We've never actually had zero as a square number have we? No.
> Scott: Zero isn't even a number
> Sue: No, it's not
> David: It is, zero times zero

The above exchange highlights two different potential misconceptions: that zero is a square number and that zero "isn't even a number".[9] It is logical that the question arose because it depends on one's definition of a square number. If it is taken to mean a number multiplied by itself, then zero is a square number, however, if the requirement is that the number can be displayed as a square, then zero does not qualify as a square number. These students had not been exposed to the visual representation of a square number, hence they were basing their answer on the definition of a number multiplied by itself. Sue later explained her reluctance to incorporate this teachable moment:

> Didn't want to go there then – would go back and do it later – didn't want to distract from what they're doing…I would have left it at that stage as it would have disrupted the whole flow … it depends on whether or not it's going to matter to what we're doing – if it was going to affect what they were doing, then I would – but this wasn't. I'd probably come back to it and finish – on that day – otherwise David and the likes would use it as a ploy to distract you from what you're doing.

Sue justified her decision to ignore the teachable moment but recognized that it was important in that she would address it later on. While it is debatable about "whether or not it's going to matter", Sue's comment revealed that the decision as to whether or not to incorporate a teachable moment depends very much on the teacher's judgement, and is influenced by factors such as context, stage of the lesson and knowledge of, and beliefs about, students.

Conclusions

In any given lesson, it is likely that several opportunities will arise which could be interpreted as teachable moments. Whether or not these moments are addressed depends very much on a combination of the teacher's knowledge, beliefs and professional judgement. While teachers may be reluctant sometimes to interrupt the "flow" of the lesson, sometimes students' comments reveal serious misconceptions which need to be addressed. If teachers possess inadequate content knowledge and/or PCK, then these moments may not be recognised and misconceptions may not be explicitly addressed and worked on (Askew et al., 1997). In this study, teachable moments provided illustrative examples of the principles of practice in action, such as *make connections* and *teach for conceptual understanding*. Implications from the findings indicate that teachers need to firstly recognise teachable moments and then capitalise upon them to enhance student understanding. This paper has contributed to the current research through describing what "teachable moments" look like in the classroom and accounting for some of the differences in the approaches teachers take to responding to teachable moments.

9 Zero is a number as it denotes the absence of units, and/or the initial point or origin (James & James, 1966)

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Students' Attitude Towards Using Materials to Learn Algebra: A Year 7 Case Study

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This paper examines the affective responses of students to an algebra intervention in the last year of primary school. The intervention was based upon extensive use of materials, discussion activities and specifically designed algebra games. Data on student attitudes and beliefs were collected with an eight dimension Likert scale, student interviews, analysis of student sketches and classroom observations. While the overall results indicated student appreciation of the use of materials and educational games, some students perceived their value to be transitory.

Introduction

Students with limited capacity to think and operate algebraically are effectively limited from Intermediate and Advanced Mathematics study in the senior school, in effect competency and confidence in algebra acts as a critical filter for more advanced courses in mathematical thinking and problem solving (e.g., MacGregor, 2004; Stacey & Chick, 2004). There is considerable evidence that confidence and competence interact, and low levels in either or both predict reduced participation in more advanced study (Ethington, 1992; Wigfield & Eccles, 2000). TIMSS data (Thomson & Fleming, 2004) indicates that the transition from later primary into early secondary school is accompanied by a sizable reduction in student confidence about mathematics. Much of this decline in confidence can be attributed to the introduction of symbolic algebra in early secondary school (Kaput, 1987; MacGregor, 2004; Stacey & Chick, 2004). The challenges in teaching algebra have been well documented and include and can in part be attributed to the quality of standard texts which tend to direct the nature of algebra learning. In particular algebra resources found in standard texts frequently do not encourage teachers to enact appropriate pedagogy to foster algebraic thinking (Mousley, 2007; Stacey & MacGregor, 1999). It has been recommended that algebra teaching include (a) the explicit teaching of the nuances and processes of algebra a central feature of algebra learning (e.g., Kirshner & Awtry, 2004; Sleeman, 1986; Stacey & MacGregor, 1997; 1999), especially in transformational activities (e.g., Kieran & Yerushalmy, 2004); (b) that algebra study be embedded into contextual themes (National Council of Teachers of Mathematics, 1998) and (c) texts should use multiple representations of key concepts and incorporate the use of technology (e.g., Kieran & Yerushalmy, 2004; Van de Walle, 2006).

With this in mind, the authors implemented a Year 7 algebra intervention that focused on making the nuances and processes of algebra explicit using multiple representations at times within contextual settings, and examined the effect of the intervention upon students' perceptions about the value of the learning activities. The research questions in this paper were: (1) Did students believe the specific learning activities enhanced their understanding of algebra? (2) Did the intervention enhance students' confidence to undertake high school algebra?

Theoretical Framework

Kieran (2004) suggested that school algebra could be said to be of three types of activity: *generational, transformational* and *global/meta-level*. *Generational* activities involve the forming of expressions and equations that are the objects of algebra. This activity is the representation (and interpretation) of situations, properties, patterns and relations and much of the meaning making of algebra is situated in this activity. For example it may involve students developing a 'rule' that describes an expanding pattern and this can be converted into an algebraic expression. *Transformational* activities focus on symbolic manipulation and include activities such as collecting like terms, factorising, expanding, substituting, solving equations and simplifying expressions. *Global/meta-level* activities use algebra as a tool for problem solving and include modelling, noticing structure, generalising, justifying and proving. Clearly the latter is dependent upon the former two activities.

Method

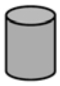
The methodology was essentially a case study (Yin, 2003). The involvement of the researchers as active participants with the teachers in the trials also gave the methodology a participatory collaborative action-research element (Kemmis & McTaggart, 2000).

Participants

Twenty-four Year 7 students participated in this study. The study school was selected on the basis of convenience. The classroom teacher had recently completed a Masters in Education and had a particular interest in the use of material representations in developing students' understanding of number concepts. The classroom teacher and the author wished to extend this teaching approach to the learning of algebra. The school was located in a lower to middle class suburb in outer Brisbane. The students had done a very limited amount of work on patterns in earlier years, so this was their first formal introduction to algebra.

Description of Intervention

Twelve one hour and half hour classes were conducted by the researcher over a 6 week period. In some of the other mathematics lessons during that period the classroom teacher supervised activities that consolidated algebra concepts introduced by the first author (usually through the medium of the algebra games) or on occasions unrelated mathematics work such as the revision of number computations was undertaken. The first researcher had developed an intervention, and taught the twelve critical lessons designed to introduce the following aspects of generative algebra: the concept of variable; connecting materials with tables of ordered pairs, word descriptions, symbolic equations and graphs of functions; understanding and transforming activities including construction algebraic expressions; the substitution of values into algebraic expressions; solving for an unknown in linear equations with the variable on one side only, and; solving linear equations where the unknown was on both sides of the equal sign. Meta-level activities included the construction of linear equations from word problems and solving for an unknown variable. The generative learning activities were drawn primarily from the text *Access to Algebra Book 2* (Lowe, Johnston, Kissane, & Willis, 2001). Transformational activities (algebraic manipulation – Kieran, 2004) associated with algebraic expressions, substitution and solving, used models similar to those described in *A Concrete Approach to Algebra* (Quinlan, Low, Sawyer, White, & Llewellyn, 1987). Materials were used to make the meaning and processes of solving explicit. The example below (Table 1) is taken from the first author's teaching notes. It attempts to link materials, natural language and symbols and emphasises an algebra structure as recommended by Stacey and Chick (2004). In the example below "ones" were represented by blue counters, "negative ones" by red counters and coloured blue and pink cups were used to represent "n" and "-n" respectively. Once familiar with the processes such as those illustrated in Figure 1, students played specifically designed board games (22 different games) to gain competency and solved word problems.

Let n =	Let – n =	Let 1 =	Let – 1 =
			

Say we try to solve 2n + 3 = 3n -1.

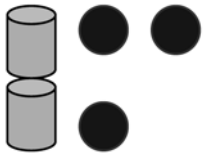
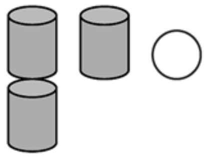
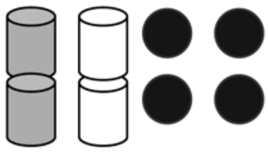
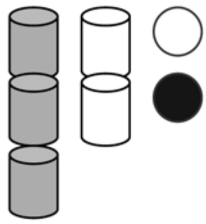
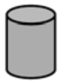
Symbols	Materials and language		
2n + 3 = 3n -1.		=	
It does not really matter what we work on first, the unknowns or the integers. We can add one positive to both sides, this will place all the ones on the LHS, then we could take two n's from both sides. In this instance we can use the zero principle by adding two negative n s to both sides. (I am doing this together simply do save space).			
2n – 2n +3 + 1 = 3n – 2n -1 +1+ 3	The + 2n and the – 2n will add to zero, and the ones will add to 4. 	=	The +3n and the -2n will result in one n, and the – 1 and the + 1 will add to zero. 
Thus  equals  4 = n. This can be checked with substitution.			

Figure 1. Excerpt from teaching notes on solving.

Data Collection

At the end of the intervention the students completed a Likert scale perceptions survey that was developed by the first author. The scale consisted of 32 questions related to eight attributes (with the following five response categories: strongly agree (5), agree (4), uncertain (3), disagree (2), strongly disagree (1). The scoring was reversed for negatively worded items. Each scale attribute consisted of four items, two of which were positively worded and two negatively worded. The first five attributes probed students' evaluation five key learning activities related to critical algebra concepts and processes, the remaining three attributes probed student attitudes towards algebra and mathematics study. The eight attributes and a relevant sample item are shown below. The associated citations indicate the importance of these processes and suggestions on how they might be developed.

1. The role of a balance activity in developing an understanding of the meaning of equals; e.g., *The balance activity helped me understand that you must treat each side of an equation in the same way* (Baroudi, 2006; Horne, 1999; Warren & Cooper, 2005).
2. The role of materials in helping students to develop an understanding of linear functions; e.g., *Using the match sticks to make patterns helped me to understand how to make up algebra rules or equations* (Lowe et al., 2001).
3. The role of materials in helping students to develop an understanding of algebraic expressions; e.g., *Using the cups and counters to make up expressions like 3x + 4 helped me to understand what they meant* (Lowe et al., 2001; Quinlan et al., 1987).

4. The role of materials in helping students to understand the processes of solving linear equations; e.g., *By using the cups and counters I was more able to understand how to find the value of an unknown, like the value of x* (Lowe et al., 2001; Quinlan et al., 1987).
5. The role of algebra games in developing understanding of algebra concepts; e.g., *Playing the algebra games was a good way of learning algebra* (Booker, 2000).
6. The role of the course in developing confidence to do algebra; e.g., *This course has given me confidence to do algebra in secondary school* (Thomson & Fleming, 2004).
7. The role of the course in developing confidence to do high school mathematics; e.g., *My confidence in being able to do secondary mathematics has been improved by doing this algebra course* (Thomson & Fleming, 2004).
8. Student perceptions about the value of learning algebra; e.g., *This course has helped me to appreciate that algebra is important in understanding how variables (amounts of things) relate* (Wigfield, & Eccles, 2000).

Student sketches provided an additional data source on student perceptions. At the end of the study students were asked to draw a picture that summarised their activity and feelings during a normal maths lesson and a picture of an algebra lesson. Sketches were analysed for themes such as those that might emerge from the pictures. The final data set came from student interviews, the eight attributes associated with the survey formed the basis of the interviews conducted with pairs of students at the end of the study.

Results

Survey Results

The means, standard deviations for the survey instrument were calculated. The small numbers in the sample and the limited number of items in each scale limit inferences based on these estimates. That aside, the data provided information about trends and served as a starting point for probing student perceptions via qualitative methods. These descriptive outcomes are presented in Table 1 below. The mean values and standard deviations suggest that on the whole student responses fell somewhere between uncertain and agreement with statements affirming the value of the activities in helping them to understand algebra concepts. For example for the balance activity the mean is almost 4 with a relatively small standard deviation of 0.57 indicating that student responses aggregated about 4 or *agree* to the statements probing students perceptions of the value of this activity. In a small sample such as this a few students can substantially alter the mean and variance. It was found that six students accounted for almost all the 1 and 2 responses indicating a negative view of the value of the materials, almost all the remaining students responding with 4 or 5. The survey data indicate that the class fell into two groups; students who responded on the survey that they found the approaches useful and those who did not. Student interviews shed light on the reasons why these students responded as they did.

Table 1

Means, Standard Deviations, (n = 24)

Scale	Mean	Standard deviation
Balance activity	3.92	0.57
Linear functions	3.66	0.85
Materials and expressions	3.79	1.03
Materials and solving	3.68	1.11
Value of games	3.59	1.11
Algebra confidence	4.17	0.70
Maths confidence	4.30	0.70
Value of algebra	4.17	0.53

Interview Results

Two girls, recent African immigrants, had strongly negative views about the value of materials in algebra learning. Nicola and Marion had similar views (all names are pseudonyms). Nicola summed up the consensus between this pair:

> When you use counters I found it hard, when you write it down or do it in your head I found it easier.

Of the games this student simply said:

> The games did not help me learn much, but I liked the spinners.

Nicola could not articulate why she did not like to use materials, just that they confused her. A second trend emerged among three the boys and two girls, in this instance Chris was typical.

> It was good the first time, but once you understand it is easy, like now I do not need cups and counters and it just slows me down.

This comment shows recognition of the value of materials in establishing understanding, but once the mathematical structure was understood a number of students (e.g., Chris, Ash, Ben, Sally and Sasha) disliked being encumbered with material use. Sally added an additional condition to the use of materials:

> For people who are challenged with algebra you should use cups and counters, but for people who are not, you do not have to.

Each of these students was able to use the materials in problem solving contexts when asked. During the interview process eight students made comments that indicated that the materials were useful for adding meaning to the processes initially, but were not required beyond that initial phase of the learning progression. In four instances this was translated to an overall negative response on the survey items. The following comments are typical of explanations as to how the use of materials helped their learning of algebra:

> Corrine: The matches helped me see the pattern a lot.
>
> Connie: It helped a lot, you can see what you were doing instead of having to keep it in our mind (match stick patterns).
>
> Julia: You can see and feel everything what you are doing.
>
> Joelle: It made it much clearer. If you go step by step it is much clearer, rather than just say "this is how it is done."
>
> Sarah: The cups and counters help you get the picture in your head.

As previously noted, the educational board games were based on material representations such as those illustrated in Figure 1 that link various algebraic representations. Students who responded negatively about games did so for reasons such as those articulated below:

> Ash: The games were not very useful, I would prefer doing sums, the games were too easy, I saw the answer too fast. Otherwise it gets boring.

One conclusion based on these data is that these students sought greater challenges. More generally, the survey data suggests that most students responded positively to the games and the interview data gives explanation:

> Corrie: The games were good; they helped you see the answer. And when I did not know the answer I could ask my friends.

Billy: They were fun to play, and you had to think about it. So it was practicing what you had learnt as well as having fun at the same time.

Themes arising from the survey data indicating approval for the use of games and included "fun", staying on task and an appreciation for the social aspect of learning through playing algebra games.

Sketch Drawing Results

Twenty-three students completed pairs of sketches representing "a normal maths lesson" and "an algebra lesson". Themes emerging from the drawings were categorised by the authors, and also reviewed by peers among the tertiary mathematics education community. Some sketches had a single identifiable theme, others had several. What is clear from the sketches is that most students portrayed a normal mathematics lesson as one dominated by procedural computations and simple representations (15 sketches), none showing the use of materials, a finding consistent with descriptors of the dominant discourse in primary mathematics classrooms (e.g., Vincent & Stacey, 2007). Students reported greater complexity in tasks associated with algebra activities (11 sketches). This complexity was indicated by multiple processes which required a number of different operations to be performed. Representations of materials and algebraic symbols were common in the sketches of the algebra class (14 sketches). Engagement (7 sketches) and understanding (4 sketches) was expressed more frequently in the intervention class as compared to the normal class (engagement 2, understanding 0).

Discussion and Conclusions

The authors are well aware of the inherent limitations in using concrete materials to represent algebraic statements because the inherent particularity of such models can run counter to the generality and abstractness of algebraic statements. However, in this paper we are interested in student perceptions of their learning, including the effectiveness of materials and material based games in helping them to understand the processes and structures of algebra at this introductory level. Student responses to the survey indicated that the class fell into two groups, those that found the materials helpful and a smaller number who did not. The interview data helped to expand upon this finding. With the exception of two girls, those who evaluated the activities based upon materials lowly (1 and 2 on survey items) said in the interview that the activities were useful initially to see the "pattern", and see the process "if you go step by step (with materials) it is much clearer," "see what you are doing…saw the structure…get the picture in your mind" but once they understood the concept or processes they preferred not to use materials since it slowed them down. That is the students recognised that symbolic processes were more efficient than processes tied to materials or material representations. This was especially the case for predominantly generative (Kieran, 2004) activities (such as creating and describing representations of linear functions, e.g., match stick patterns, tables of co-ordinates, graphs and word descriptions of relationships and equations) but also the transformational (Kieran, 2004) activities such as processes that solve for unknowns with the aid of cups and counters. Once students understood the structure they preferred more abstract methods such as mental computations and written algorithms. This implies that while students might gain from material based forms of algebraic activity, for a significant portion of students it is not necessary to labour their use. In the case of the use of mental computations, it is possible that some students were using numerical methods that need not necessarily parallel the formal processes of algebra. The use of more complex numbers (larger numbers and fractions) could have reduced these students' capacity to rely on mental and numerical methods and this will be taken into account in future algebra interventions.

With respect to the first research question, with a few exceptions and some conditions (some students saw initial value in material based activities but did not see the merit in persevering with their use) most students believed the learning activities enhanced their understanding of algebra. Overall student perceptions of their increased competency in algebra were reflected in increased confidence about approaching Year 8 algebra and Year 8 mathematics in general. Given the relationship between self confidence and mathematics achievement (Thomson & Fleming, 2004; Wigfield & Eccles, 2000) this is an encouraging finding.

The merits of learning mathematics in social and collaborative settings has been well documented (e.g., Grootenboer & Zevenbergen, 2007) and the algebra games afforded this learning setting. The comments of students about games in facilitating collaborative social settings support earlier findings (e.g., Booker, 2000; Norton & Irvin, 2007). Insomuch as these activities help to foster student confidence about their capacity to understand algebra, the use of materials and educational algebra games based upon material models warrants further research. The current outcomes indicate that there might well be a case for more widespread use of such approaches before secondary years.

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Teaching Mathematics and Technology through Design Practice

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In many schools throughout the western world teachers are observing increasing student disengagement with the study of mathematics and technology. The Queensland Education Department implemented a curriculum project designed to trial the use of design practice as a tool for integrating the teaching of mathematics and technology. This paper reports on the outcomes of the trail. It argues that differences in regulatory and instructional discourse adopted by the teachers resulted in different and pedagogically important outcomes.

This paper is set against two broad trends in mathematics and technology (M&T) education. The first is a distinct decline in participation in the more advanced learning in M&T, generating serious consequences for the national economy of Australia with a shortage of competent M&T trained graduates holding up critical infrastructure projects (Custer, 2003). Fewer students are taking advanced mathematics in Year 12 (Department of Education, Science and Training, 2003) and opting to study technology-based subjects such as Engineering and IT (Anderson & Gilbride, 2003; Kaiser, 2000). This appears to be caused by a decline in student enjoyment of M&T from Year 4 (Watt, 2005) affecting later participation (Khoon & Ainley, 2005); M&T is perceived by students as being dull, boring, difficult and inconsistent with wanted identity (Barrington, 2006). This connection between perceptions, participation and performance is supported by the findings of Norby (2003), Stepulevage (2001) and Thomson and Fleming (2004).

The second trend is a distinct theoretical shift in what it means to be mathematically and technologically literate from skill development to problem solving (Custer, 2003). With respect to mathematics, the focus on problem solving is excellently justified in the recent submission to the Australian National Numeracy Review by the Mathematics Education Research Group of Australia (Mousley, 2007) where it was stated:

> People need to develop abilities to tackle, interpret and solve problems; to observe, describe, interpret and predict numerical and other sequences; to think abstractly and manipulate ideas; ... to analyse and reason about situations logically; to monitor progress and apply reality checks. (p. 2)

It has been suggested that this shift to problem solving also requires a shift in pedagogy, in particular, by placing an emphasis on connecting mathematical knowledge through investigations in a community of inquiry (Mousley, 2007; National Council for Teachers of Mathematics, 2004).

The shift to problem solving has been adopted by the Years 1-10 M&T syllabi developed by the Queensland Studies Authority (2003, 2004). The Mathematics Syllabus has termed "working mathematically" as an integral component of problem solving, investigation and application, and "connections" across all content strands (QSA, 2004, p. 1). The technology syllabus (QSA, 2003) focuses on design in both content and pedagogy, defining the technology practice strand as embodying the actions of investigation (identifying the problem and gathering information and data), ideation (planning and designing), production (creating and making), and evaluation (testing, judging, and refining).

The overlap of goals and methods in M&T provide clear indicators for the consideration of the learning of these subjects holistically rather than as distinct entities as they have been treated in the past. In order to capitalise on the potential benefits of learning M&T holistically, Education Queensland implemented trials of integrated M&T units of work of 9 weeks duration. Unfortunately, there is little research to guide the processes of curriculum development and implementation with this very specific emphasis in mind. Consequently, the objectives of this paper are to: (i) document and analyse their *regulatory discourse* (Daniels, 2001), a construct associated with the planning and management of the content to be taught, the key outcomes aimed for and the means by which they would be learnt, in particular what models of integration was undertaken; (ii) document and analyse their *instructional discourse* (Daniels, 2001) described as a construct associated with how the outcomes were taught, including learning activities undertaken and the various roles of the students and the teachers and how the students were supported in these activities); and (iii) draw conclusions from the implementation and evaluation processes to guide subsequent planning of integrated learning of M&T.

Method

The methodology was essentially a case study. The involvement of the researchers as active participants with the teachers in the trials also gave the methodology a participatory collaborative action-research element (Kemmis & McTaggart, 2000). However, the participatory role of the researchers was limited to providing a one day seminar on the technology syllabus implementation and some support with initial planning; in particular suggestions as to what types of technology based activities might be rich in opportunities to learn mathematics.

The participants in the trial covered by this paper were a teaching principal, “Alan”, (all names are pseudonyms) of a rural Queensland state school, “Granite Tors”, and 20 students ranging from grades 1 to 7. Only the Years 4 to 7 students participated in the project “Design a Sundial”. The other projects were “Design a Package”, Years 1 to 3; “Design a Bug Catcher”, Years 3 and 4; and “Design Proportional Puppets”, Years 5 to 7. The data were collected over the life of the 9 week trial. Sources of data included observations of the teacher and students’ classroom activity (videotapes, audiotapes, field notes) with particular attention focused on (i) elements of student and teacher activity including language and roles undertaken; (ii) the levels of autonomy and student ownership of the processes involved in the trial; (iii) the time, and the type of support, given to student learning; and (iv) teaching activities to make underpinning concepts explicit. Semi-structured interviews with the teachers at the beginning, mid-point and end of the trial focused upon their perceptions of the teaching potential of integrated teaching of T&M and how they went about implementing the projects. Artefacts including unit plans, lesson plans and student work were collected from the trial and analysed to provide rich descriptions of regulatory and instructional discourse.

The analytical framework for evaluating the interventions focused on the two major constructs, regulatory discourse and instructional discourse, described previously. The various data sources were examined for recurring evidence that allowed the authors to produce rich descriptions about the regulatory and instructional discourse enacted during the life of the technology based projects at the four schools. The results from these case studies were used to highlight pedagogically important differences between Alan’s teaching approaches and those employed by the teachers in the three comparison schools.

Results

The findings related to the nature and types of regulatory and instructional discourses employed at “Granite Tors” and the three comparison schools are summarised in Table 1. It is difficult to generalise across the three comparison schools, not least due to the variation of Year levels and the different project types involved, however, there was strong evidence for each of the descriptions tabled below.

Table 1

Summary of Regulatory and Instructional Discourse

Regulatory discourse for Design a Sundial project (Years 4 to 7) – Granite Tors	***Regulatory Discourse – Other Schools.***
Student derived project based upon a genuine need of the school. Framework of "thinking, discussion, planning, community involvement, making and evaluation, and publication" guided student negotiation of a new "useful project", and student negotiation of design, investigation, ideation, production and evaluation processes. Mathematics focus on space and measurement in design and production of prototypes and a finance focus with the costing and manufacture. Technology focus on ideation, design and dissemination. Project disseminated by students as a digital portfolio linked to school web site. ICT integration central to the project; evaluating various sundial designs and underpinning design theory from web sources analysis and displaying via ICTs. High community involvement, e.g., equipment to clear site and local metal worker to construct according to student designs.	Teachers selected existing projects that might be rich in mathematics and technology learning opportunities and modified them. A travel based SOS project became "Design a Package"; science based insect collection project became "Design a Bug Catcher", and a theatre based production became "Design Proportional Puppets". Mathematics focus on space and measurement in design and production (and also ratio for the Years 5-7 project on puppets). Technology focus on production for the package and puppet projects. The Bug Catcher project also had a strong design element, but the focus was mathematical rather than on the design processes itself. Completed projects shared with class, mostly print and art based display. ICT use auxiliary; limited design sourcing via www, some limited data collection and graphing in Bug Catcher project (Excel). Community involvement through guest speakers talking about the topic (postage and entomology).
Instructional Discourse – Granite Tors	*Instructional discourse – Other Schools*
Extended student reflection and negotiation in design phases. High student ownership. Considerable proportion of time allocated to planning, as well as making and evaluating prototypes, designing a final product, designing and supervising the construction of the base, negotiating with local fabricators to make their sundial design from metal, costing and managing the project, evaluating the effectiveness of the product. Teacher worked "one on one" with students with an area of need. Mostly facilitation role adopted by the teacher. Teacher exhibited high levels of questioning that reflected a focus on developing student ability to be analytical in both mathematical and technological contexts.	Many of the design decision taken by the teachers. Planning proportionally less dominant phase, most time spent on production. Teachers worked "one on one" with students with an area of need. Mix of facilitator and explicit instruction. Teacher exhibited high levels of question and critical analysis in "Bug Catcher"; that focused upon mathematical knowledge; and there was limited critical reflection beyond the material aspects of designs.

Models of Integration

Alan's description of the relationship between the technology based project and mathematics was as follows:

> As mathematics can be a means of viewing the world, to investigate patterns, order, generality and uncertainty, it (the project) is an ideal subject to help students make meaning of the world, but more specifically the "place" they live in...I used the project to draw out their mathematics and two text books in support (*Signpost* and *Jigsaw Maths*) to fill in the light and shade. (Interview data)

Alan had a strong view on the purposefulness of mathematics and that student understanding needed to develop from engaging in "real" activity. The light and shade he refers to are those aspects of measurement and other mathematics that did not "grow out of" the project including much of the foundational number concepts which provide the basis for the other strands. Alan summarised his students' learning in the following words:

> The students in this project see maths as having a purpose and have learnt to apply problem solving strategies. They were immersed in the process of technology practice, learnt lots about suitable materials, managing a project and seeking and transforming information. ICTs were used to access information and to publish their results as digital portfolios. In terms of ICTs, they are streets ahead of most kids their age and they are on their way to becoming autonomous learners.

By comparison the following quotes were typical of the reflections of the remaining three teachers with respect to the relationship between the technology project and the mathematics learning:

> (Bug Catcher) I tended to teach the normal maths number (strand) in the morning…that way we would not miss key concepts… but we would also look at other strands such as measurement. We would develop the concepts in the morning and have an opportunity to apply them in the afternoons (when students were working on the technology projects).

> (Puppets) I was looking at ways of incorporating what I wanted to teach in mathematics. I wanted to teach quite a lot on measurement including length and area, as well as to introduce the concepts of ratio and proportion in a concrete way. The ratio was taught previously using other materials such as blocks, paint and mixing cordial drinks up, to set up (an understanding of) proportion. The students readily transferred the ideas to puppet production including the patterning and making of puppet cloths. I parallel taught other mathematics concepts.

Discussion

All the teachers reported that the students in each respective trial were highly engaged and on task throughout the project, and the integration of M&T worked to harness students' enthusiasm to be creative and constructive for the purposes of learning mathematics. This assertion was supported by evidence from classroom observations.

Mathematics Learning

What was common to each of the four trials was that the teachers had the perceptions that by linking the learning of mathematics to the technology projects, students could see the value in learning mathematics, that is, mathematics has a purpose. All four teachers believed that seeing mathematics as purposeful and useful was a factor in the high levels of student engagement. If we assume the models proposed by numerous authors are correct in that student perceptions are central to their participation and learning of subjects (Ethington, 1992; Khoon & Ainley, 2005; Markku, 2002; Murphy & Gibbs, 1996; Thomson & Fleming, 2004; Wigfield & Eccles, 2000) and that these perceptions are formed early and that early experiences are important (Thomson & Fleming, 2004), the above findings are encouraging. In addition, the regulatory and instructional discourse was reported by all the teachers to have furthered the attainment of mathematical literacy in ways consistent with reform definitions of mathematical literacy (Anderson, 1999; Mousley, 2007; NCTM, 2004).

However, the findings including those in Table 1 indicate that there was a difference between Alan and the other teachers with respect to the manner of developing of understanding. The dominant regulatory and instructional discourse used by the three comparison teachers resulted in the use of an approach that involved parallel instruction, that is, mathematics concepts were taught explicitly in the morning either using textbooks or other resources and this knowledge was applied and built upon in the afternoon session when technology practice was enacted. Alan's approach was somewhat different. The problem based technology project became the focus of mathematical learning and supporting textbooks were used to provide "light and shade", "extension work", "cover missed concepts", and as a resource for one-to-one tutoring of concepts that particular students struggled with. As previously signalled, the integration model used by Alan in teaching mathematics through engagement with the technology project had the added dimensions of seeing mathematics as a tool to "make sense of the world" and in seeing the integrated M&T project as "providing a platform to explore the world of maths". This made Alan's trial and project different in that while integrated M&T projects were seen as opportunities to essentially apply mathematics to authentic contexts in the other three schools, they were seen more as opportunities to generate mathematical understandings at Granite Tors State School.

For both Alan's and the other teachers' models of integration, the effect was increased time and opportunities for students to engage in mathematical activity in addition to time that would normally be spent upon mathematics lessons. All teachers cited examples of the use of mathematics concepts while students planned and constructed their artefacts. Further, the teachers in each trial reported that the harnessing of teaching mathematics with and through technology made mathematics more "fun" and more "understandable." It is likely that the two are intertwined since, as reported above, students consider learning that is "too hard", is also "boring". Put simply, students like to have the tools to understand.

In constructing their artefacts, the students were using mathematics in planning and working with materials that they might not normally engage with in more formal mathematics teaching and learning episodes. Thus, it was evident that the integrated teaching of technology and mathematics gave teachers increased representational opportunities to explore mathematical concepts in very practical situations. An example of this was the use of proportional reasoning in the puppet construction, in addition to representations with blocks, mixing cordial drinks to set ratios and ratios of colours or objects, as well as ratios in purely numerical representational forms. The successful linking of a number of representations is central to the Queensland mathematics syllabus (QSA, 2004) and to reform mathematics learning in general (Booker, Bond, Sparrow, & Swan, 2004; Mousley, 2007; Van de Walle, 2007). In this study, the increase in both time and representational opportunities was accompanied by increased frequency of student engagement with specific mathematical concepts. All teachers were aware that it was important to make the connections between representations explicit. However, in Alan's trial these links were made more obvious to the students as they worked on their projects, in part through their greater participation in the ideation and design actions of technology practice which afforded greater opportunity to engage with spatial concepts, in particular. A simple measure of this is that the students spend more time working with mathematical concepts during the extended design process.

With regard to instructional discourse, the main mathematics implication is the relationship between mathematics and technology teaching. Alan's use of mathematics as a support and extension to the project, not a preparation, meant that he was able to bring the mathematics to bear when the students needed it. His approach was much more a "just-in-time" use of mathematics teaching as against the other teachers' "just-in-case" use. This approach built stronger connections between representations, particularly in relation to artefact design and, thus, potentially deeper mathematics understanding. Observations indicated that making appropriate connections was far from fully refined in the other trials, particularly with regard to technology concepts.

Technology Learning

The difference between Alan's trial and those of the other three teachers is more evident with regard to technology learning. Although all four teachers reported encouraging progress, it was clear that the other three teachers (overall) struggled as to how to go about assisting students to learn technology practice processes and technology concepts, particularly those associated with the design (ideation) and evaluation actions. At Granite Tors State School, Alan was more explicit with linking technology practice as enacted by the students and the actions of technology practice outlined in the technology syllabus (QSA, 2003). He achieved this by having his students use the framework of "thinking, discussing, planning, community involvement (and making), and evaluation and publication". Alan and the class invested a considerable proportion of the allocated time to investigation and ideation (planning). This was a negotiated process and students' input was valued and critiqued by each other with scaffolding provided by the clearly documented instructional discourse that Alan developed and worked with. The student activity was consistent with recommended technology education practices that have moved from a skills focus towards the development of more generic problem solving skills (Custer, 2003; NACCCE, 1998; QSA, 2003). In addition, in Alan's trial, there was a conscious effort to document and critically examine design process through the use of ICTs. The processes of making technology practice explicit were facilitated by student construction of digital portfolios.

The ICT policies and processes of the Granite Tors State School with respect to accessing and processing information stand as a model for the integrated use of ICTs as part of the information strand of the technology syllabus and for integrated M&T projects. Alan's students' proficiency with the use of ICTs was acknowledged when they were invited to the local high school to mentor secondary students in ICT learning. However, despite this finding, the researcher found little evidence of planning by Alan to make explicit key concepts associated with the other strands of the technology syllabus (QSA, 2003) such as systems and materials. For example, while concepts associated with materials were used, the theory behind the nature of materials and their uses was not discussed or assessed. Hence, even with the excellent progress being made by Alan's students, it is likely that key technology concepts remained implicit and that more technology learning opportunities could have been utilised which may have increased the generalisability of the concepts studied.

The differences between Alan and the other three teachers have implications for M&T learning and projects. In terms of regulatory discourse (planning management), the principal implication is the importance of an explicit focus on ideation and evaluation actions. The investigation of existing plans and appreciation of materials and production techniques were largely missed as prominent opportunities to develop creative problem solving and design opportunities in the three comparison schools. The major technology implication is the importance of allowing students to make decisions in design. In the other three trials, the teachers determined the project focus and to a greater extent provided the design plans. The teachers taking responsibility for these processes in these trials limited students' opportunities to engage in authentic design. Alan's opposite policy led to greater in-depth design and technology learning.

Conclusions

In summing up, the following appears to be evident from the study. First, all four teachers believed that the integrated M&T teaching could: (i) potentially facilitate powerful mathematics learning, in part, by offering a multiplier effect through greater time, exposure to additional representations, and increased frequency of engagement with mathematics concepts; and (ii) do so in ways consistent with emerging definitions of what it means to be mathematically literate. Carefully planned regulatory and instructional discourse is necessary to capitalise on these opportunities. Alan's approach in allowing the students control in the selection of the project and resultant design of the artefact appears to offer the greatest potential.

Second, two models of regulatory and subsequently instructional discourse were identified. The three comparison teachers used an integration model which involved teaching the mathematics primarily as they had previously, and used the technology design project as an opportunity to apply in "authentic" contexts what students had previously learnt. In contrast, Alan's model was one of learning mathematics through engagement in the design process and "just-in-time" mathematics teaching. While both approaches are likely to enhance transferable skills and generic problem solving (Anderson, 1999; Mousley, 2007; NCTM, 2004), Alan's model potentially offers stronger connections between mathematical representations (including authentic experiences in the technology project) and deeper understanding of design and technology, in part due to the more extensive engagement in the investigation and ideation actions of technology practice and the use of mathematics in these processes.

Third, it was evident that all the teachers in this study needed assistance in implementing the technology syllabus as they missed opportunities to connect technology syllabus outcomes (QSA, 2003) to a coherent schema based framework. However, this assistance was of greater need for the three comparison teachers to prevent them continuing to empty their technology tasks of creative and innovative opportunities for the students. Their teaching oversights tended to have the effect of developing more craft based technological literacy rather than a more generic problem solving technological literacy. In contrast, the technology practice framework used by Allen (thinking, discussion, planning, community involvement, making and evaluation, and finally publishing) was evidence of the actions of technology practice being made explicit and was more successful. This was particularly so for the student construction of a digital portfolio to record the concepts and processes involved in the project.

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Engaging Mathematics Teachers in Professional Learning by Reflecting on their Pedagogical Practice

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This study engages mathematics teachers in reflecting on their practice by collecting data electronically from their students during the course of a mathematics lesson. This method of data collection eliminates the burden of manual collation and produces immediate feedback for analysis while the lessons are still fresh in the teachers' minds. It is possible to pose a number of standard questions across all lessons as well as questions formulated by the teachers as being of interest.

This paper is an unusual format in that it represents a proposed research project rather than being a report on completed research. The intention is to engage with others with an interest in using innovative data collection instruments to facilitate teacher learning, and to seek feedback that can help to optimise the use and utility of these instruments. At the time of writing the instruments have been prototyped, but at the time of presentation some piloting will have been completed and preliminary results will be available.

Background and Rationale

The following section gives the background and rationale for the project in terms of canvassing possible ways of improving teacher learning, the constraints imposed on these by the secretive nature of teaching, and the role that teacher reflection could play in forming the basis for professional communities.

Processes for Improving Teacher Learning

The constraints that teachers face in introducing changes to their practice are manifold, and the teachers themselves may not be fully conscious of them. Arguably such constraints do not come into sharp focus until a teacher is attempting to make a change to their teaching. It is within this context that the research proposed here will take place, under the auspices of the Australian Research Council funded Task Types and Mathematics Learning (TTML) project. The TTML project works with approximately 50 teachers from 17 schools in Victoria who are interested in improving their mathematics pedagogy. As a result it is well placed to explore these constraints more fully since the participant teachers have already demonstrated a willingness to implement changes to their practice, which will plausibly bring the constraints of their circumstances into sharper relief.

One suggestion for improved teacher learning draws upon the educational practices of other professions such as law and architecture where 'case knowledge' is commonplace. Shulman (1986) proposes that a case literature could be used in teacher education via simulations and teaching laboratories to help inculcate the kind of professional judgements and practices required of good teachers. Shulman (1986) believes such an approach could form the basis of professional teacher examinations – controlled by teachers rather than bureaucrats – and that it could also inform research programs by incorporating both content and process knowledge to amass a body of case literature. Further, given the inherently accessible nature of cases, teachers themselves would be able to make valuable contributions from their own practice, and empowered as research contributors to their own profession.

Reflective Practice and the Secret Lives of Teachers

Such an approach seeks to formalise the kind of stories Connelly and Clandinin (1995) characterise as *secret*. On their analysis teachers are unlikely to be willing to share anything more than *cover stories* because of the hostility they perceive to be present in the educational *landscape* they inhabit. And even if teachers did reveal their secret stories they would likely become frozen in time and place, losing the dynamic, spontaneous, and transformative qualities they might once have possessed. Instead it may be preferable for teachers to reflect on their own practice.

The importance of engaging teachers in reflexive practice is acknowledged by Sullivan and Leder (1992) who propose peer observation as one strategy for improving practice, but also advocating the investigation of whether self-reflective teachers are more or less directive, more or less experienced, and whether reflexive teaching is trait or skill based. It is worth noting that these could prove to be possible limitations for this study in terms of its applicability to the broader teaching profession given the targeted cohort is teachers who appear to be committed to long term professional development.

Feinman-Nemser (2001) claimed that the problems with conventional teacher education and professional development are that teacher training is "weak … compared to teachers' own schooling and on the job experience" and that professional development is usually "sporadic and disconnected" (p.1014). She advocates for an overhaul of teacher learning in order to bring about content rich student-centred teaching which encourages and enables teachers to develop their own curriculum, their own knowledge of practice, and to become practical intellectuals.

Feinman-Nemser (2001) surveyed a number of "promising" reform programmes and catalogues the qualities she sees as what makes them promising approaches to teacher education. Feinman-Nemser (2001), echoing the views of Connelly and Clandinin (1995), acknowledges the private nature of teaching, and the inherent lack of opportunities teachers have to observe colleagues or discuss pedagogy with them, but then goes on to expound the deleterious effect of these aspects of teaching have on inducting graduate teachers into the profession. In effect new teachers' mentors have little or no experience of mentoring, and the culture of teaching they are being inducted into is one of finding one's own way in isolation.

Reflection and Professional Communities

As far as professional development is concerned Feinman-Nemser (2001) advocates new approaches which replace external experts with teachers doing the talking and thinking – with a particular emphasis on conversation that involves detailed descriptions of practice, evidence and alternatives. Teachers would form professional communities to share, encourage, critique and support each other and could form partnerships with universities to draw on their resources. Feinman-Nemser (2001) proposes that teachers would design their own curriculum and leverage their professional community affiliations to refine their efforts and increase both their performance and conceptual understanding of pedagogy, producing problem-based, student centred mathematics lessons.

It is worth noting that the TTML project appears to deliver on many of these suggestions, making it an ideal context within which to explore teacher learning. Gaining entry into the classroom of a teacher known to be interested in enhancing their practice is the first step in this project, but the second is to engage these teachers in reflection on their practice.

This project hopes to achieve this by providing a means of collecting teacher-centric data. This approach will be used as the basis for then exploring how such information is viewed by teachers – whether they find it useful for reflecting on their practice, whether or not such customised feedback encourages them to look at modifying their practice, and whether such a process of data collection and analysis brings to teachers a sense of being empowered/disempowered, stressed/relieved, or interested/disinterested in collecting further data on the impact of their teaching.

Zimmerman (2006) has identified a number of factors explaining why teachers appear to be resistant to changing their pedagogical practice including fear of the unknown, feeling threatened socially/professionally/politically, habitual practices, having experienced failed attempts at change previously, and not perceiving there being any need for change. This study would hope to explore the extent to which immediate student feedback alleviates or exacerbates the influence of such factors.

Method

Overview

The proposal here involves firstly working with teachers to formulate two questions they wish to obtain student feedback on based upon areas the teachers perceive to be strengths and weaknesses. Feedback on these and two other standard questions will be obtained in real time from students intermittently during a mathematics lesson. After the lesson the collected data will be analysed with the teacher, followed by a short semi-structured interview to collect data on the teacher's experience of the process and their views on its utility.

The TTML project has recruited teachers from three clusters of Victorian schools. The clusters are located in Berwick (a burgeoning outer suburb in a growth corridor 45 km South East of Melbourne), Malvern (a well established inner suburb 5 km East of Melbourne), and Geelong (a regional centre 80 km South West of Melbourne). The project was designed to run over the course of three years, incorporating regular professional development meetings for participants.

Participating schools belonged to either the State or Catholic sectors, with considerable levels of support for the project being shown from within the Victorian Government Department of Education and Early Childhood Development and the Melbourne Catholic Education Office. The TTML project targeted the middle years of schooling (Years 5 to 8). Fifteen of the participating schools are primary schools and three are secondary colleges.

Teachers from one TTML cluster of schools will be invited to take part in this study, consisting of approximately 15 teachers, with levels of experience ranging from first year out through to several decades of teaching practice. The pool of potential teachers is predominantly female with approximately 30% of participants being male.

Teacher Feedback Questions

It was hoped that having teachers reflect on their practice and being able to test their assumptions would be of interest and benefit to the participating teachers. Teachers often have a strong sense of what they do well and where they struggle, so this approach would provide them with an opportunity to obtain student feedback directly and quickly.

To these ends teachers were asked to think about and nominate two areas they would like feedback on from their students – one which they felt catered to a strength of their teaching (e.g., I relate well to the students), the other which addressed an area they felt less confident with (e.g., I struggle to explain fractions clearly). These areas of interest would be expressed as two questions that teachers could have their students answer every five minutes throughout a lesson. For example the two areas suggested above might become;

1. How well do you think Ms Teacher understands your learning needs right now?
2. How well do you now understand what Ms Teacher has been explaining?

The number of questions posed is not restricted by any technical consideration per se, but rather by the desire to minimise the disruption to the flow the lesson by maximising the speed with which students can provide their responses. It is also possible to vary the response rate from every five minutes to any other interval, or to have the responses triggered by the teacher directing the class to submit their data, or having students control their own response rates. These options will be discussed with teachers as possible variations after the initial set of data has been collected, including the possibility of having students nominate questions for the class to provide feedback on.

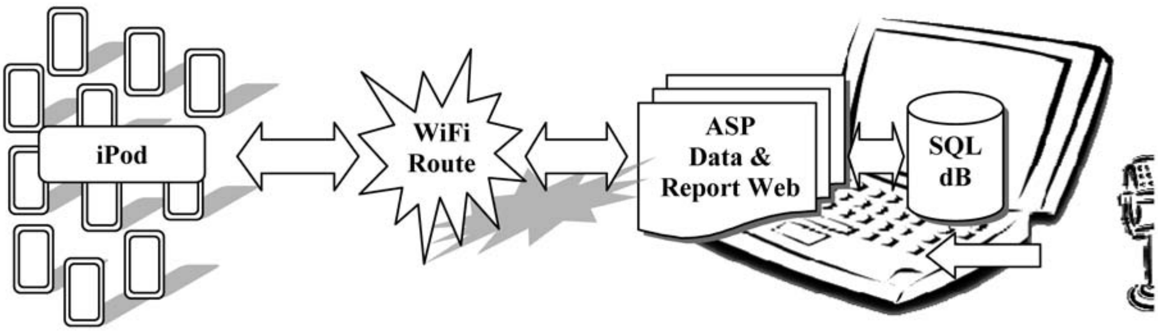

Figure 1. Diagram of the relationships between hardware and software components of the RTFS.

Once the teacher questions have been formulated, they would then be loaded into the Real Time Feedback System (RTFS). This consists of a web page hosted on a laptop computer which can be used to serve the page to a set of 25 iPod Touch devices via an 802.11g Wireless (WiFi) router located in the classroom. Although any portable browsing device would be suitable, the iPod Touch has a unique navigation interface whereby the entire surface is a touch sensitive screen that can be zoomed in or out as desired by either tapping the screen, pinching thumb and forefinger together, or spreading thumb and forefinger apart. Each iPod is configured to browse the locally hosted web page so that each student could be given an iPod to use and to respond by tapping on a visual Likert scale as prompted. The data submitted by students will then be processed by a web application utilising Active Server Pages (ASP) and stored in a relational database also hosted on the laptop. Various triggers and stored procedures within the database enable an administrative web page to produce web based reports that can present the student feedback in graphical formats. Audio of the lesson would also be recorded onto the laptop to provide a timeline for subsequent analysis.

Given the novelty of the iPod devices, it will be important to ensure that students are given an opportunity to familiarise themselves with the navigation system and to have a chance to explore the device generally prior to formal data collection. The iPods ordinarily have a number of other features and functions that have been disabled for the purposes of the project, restricting them to web browsing capabilities only.

A set of other ASP web pages are incorporated into the system as a means of inducting students into the use of the iPod navigation interface. These pages have simple instructions that give immediate feedback when students succeed or fail to tap the correct section of the screen. It will be possible to track student progress on these induction tasks and offer additional assistance as required until all students have mastered the requisite navigation skills. As it may not always be possible to conduct the reflection, induction, data collection, and analysis in a single day it may sometimes be necessary to spread them across two, ideally consecutive, days.

During the feedback sessions students will be provided with the iPods at the start of the mathematics session and given the opportunity to refamiliarise themselves with the browser interface having gone through the navigation induction previously. They will be assured that all of the data they submit will be completely anonymous, and that the iPod will flash every five minutes to remind them to submit another set of answers.

Two additional standard questions are included in the RTFS to collect data on how interesting the students are finding the lesson, and how hard they feel they are trying. It should be noted that the emoticons used as Likert prompts are animated gif images rather than static images.

Others have used technological prompts previously, as reported in Moore, Prebble, Robertson, Waetford, and Anderson (2005) wherein individual students were given tape recorders which played tones every few minutes during a lesson. These audio prompts signalled for students to make entries on an accompanying paper and pencil instrument. The chief difference between such approaches and the RTFS is that the data is also collected, collated, and processed by the same technology which delivers the prompt.

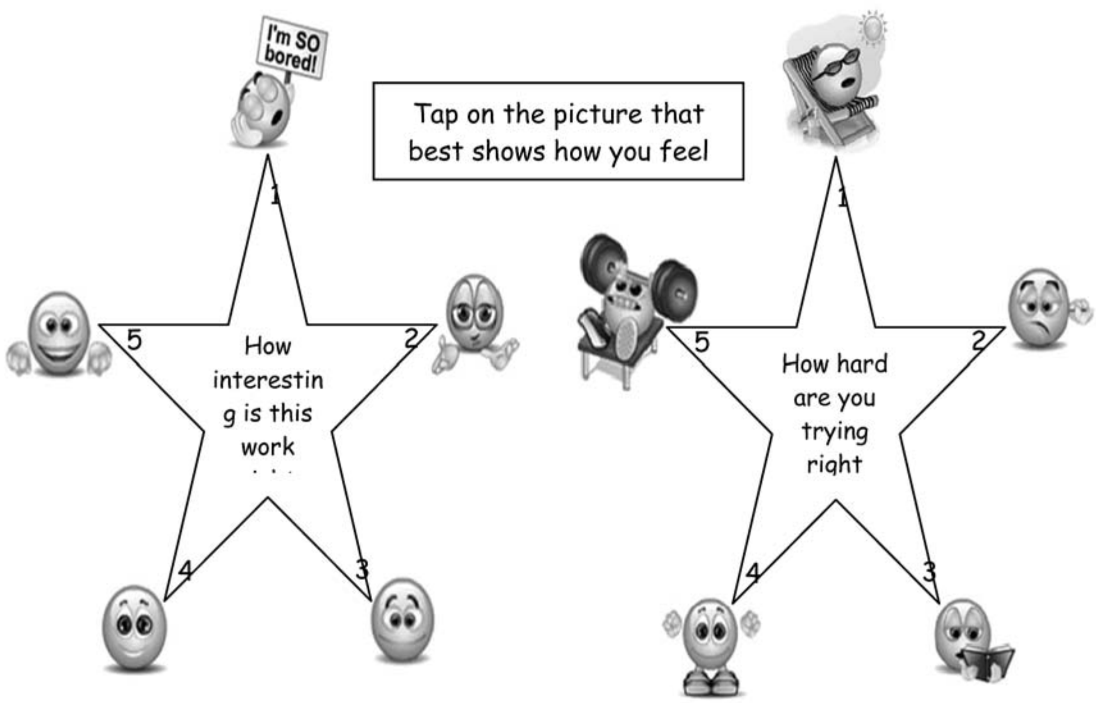

Figure 2. Samples of feedback screens from RTFS.

Analysing and Discussing the RTFS Results

Upon successful collection of student feedback, it will then possible to spend some time going through the data collected with the RTFS. The overall aim of this analysis is to have data on hand that the teachers have collaborated in collecting, the relevance of which will be self evident, and provide a grounding in reality for the ensuing discussions. Initial analysis and discussion will centre on the graphs of student responses to the four questions, mapped against what was happening in the class at the time – as ascertained from the audio recording.

It is possible to view the response graphs on the laptop at any stage of the data collection cycle, so it would also be possible for a teacher to monitor student feedback during the lesson itself and modify their teaching as they saw fit, however the intention initially is to reserve analysis until after the lesson has been completed. It would also be possible to display the results to the entire class by using a data projector if this was thought to be of value, or if the question/feedback stimulus warranted it.

Teachers would be given the opportunity to borrow the equipment for further data collection if they find it of interest and/or use, and one possible measure of the usefulness of the RTFS approach could be the extent to which teachers are interested in taking up this offer to explore other configurations such as using live data monitoring, data projectors, or student generated questions.

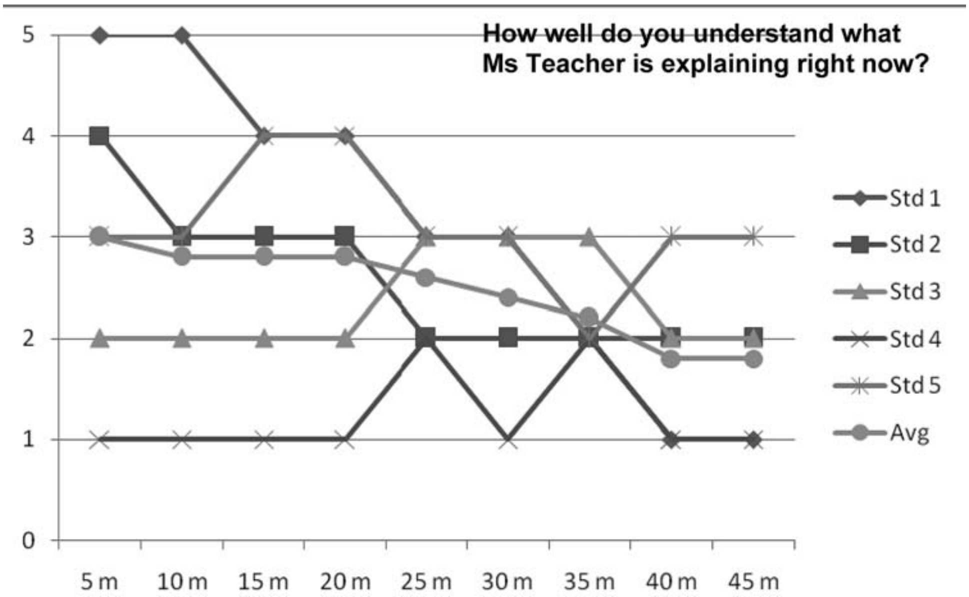

Figure 3. Sample report from RTFS.

The central questions to be explored are:

1. What sense can you make of these results, what do you think we can conclude about your questions?
2. Is there anything here which you find surprising or confusing?
3. How do these responses strike you? Are there any patterns you think are meaningful?
4. Is there anything that you would do differently in light of these results? What obstacles do you think you would have to overcome in making these changes?
5. How useful/helpful do you think this kind of process is in terms of yours or others' professional learning?
6. Could it be made more useful? How? Would live data feedback be of any use to you? Could you imagine using it with a data projector displaying a feedback 'worm' to your class? What sort of situations might that be useful?
7. Would you recommend this to a colleague or would you be interested in borrowing the equipment to run more of your own feedback sessions?

Piloting

At the time of writing only rudimentary testing has been conducted, however a full pilot will have been completed by the time of presentation. It is anticipated that there may be problems with students being overly distracted by the instruments, and as a result the instruments themselves subverting the measurements they are intended to make – a macroscopic parallel to the Heisenberg uncertainty principle.

Although some disruption to classes is inevitable, it is quite likely that students will be less excited by the technology than adults typically are. The rate of technological uptake by students is so high that it is quite possible that many students will have their own iPod Touches or equivalents (Nielsen, 2005). Also by restricting the iPod functionality to web browsing only, and restricting the browsable sites to only those part of the RTFS website, should help to minimise this concern.

Having students contribute questions might be a useful means of harnessing their interest as might be the projection of live feedback results, or displaying a graph of their personal responses and the average response on their iPod after they submit data. It would also be possible to utilise the iPods as dynamic worksheets, using them to provide students with feedback on responses to milestone questions and integrating the RTFS data collection into this process. Interestingly, Moore et al. (2005) found their technique significantly improved the on-task levels and work quality of their subjects, suggesting that the RTFS approach might be adapted to bring about similar improvements in students attending to the mathematics lesson.

In any case, the key purpose of this use of instruments is to provide a vehicle for teacher reflection, so the issue of data accuracy and class disruption is of a second order. Sustained use of the RTFS would rapidly diminish the novelty factor, desensitising students to the recording process, and teachers could have more confidence in the data they collect. The aim of this project is to establish whether using technology in this way holds any promise as an aid to pedagogical reflection.

Conclusion

This nascent nature of this proposal necessarily entails considerable uncertainty about the approach, however the primary goal is to offer teachers a useful tool that will enable them to actively research their own teaching. If successful, this system could prove to be a valuable adjunct to other forms of teacher learning by providing teachers with immediately relevant data to research questions of their own derivation. Arguably any approach that enhances teachers' capacity to reflect on their own practice, based upon empirical data of direct interest to themselves and under their own control, would appear to be worthwhile.

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Primary Teachers' Perceptions of Their Knowledge and Understanding of Measurement

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This study focused on primary teachers' perceptions of their knowledge and understandings of length, area and volume. It also explored their understanding of how children's growth of measurement concepts and processes develops. Data gained from in-depth interviews revealed that teachers' knowledge was often implicit and that they struggled to articulate their knowledge of measurement concepts and children's trajectories of learning.

Over the past decade an increasing amount of research has been concerned with the key concepts and skills that students need to understand measurement–particularly, relating to length and area (e.g., Barret, Jones, Thornton, & Dickson, 2003; Outhred & McPhail, 2000). However, relatively little is known about teachers' understandings of these concepts and how their knowledge impacts on their teaching practices.

The research reported here explores primary teachers' perceptions of their mathematical content knowledge and of how children's understanding of length, area and volume develops. It is also concerned with teachers' perceptions of how this knowledge impacts on their pedagogy.

Background to the Study

Measurement is a central component of primary and secondary school curriculum documents around the world (e.g., National Council of Teachers of Mathematics, 2000). Consequently, there is an extensive body of research relating to the teaching of length, area and volume. The majority of this research centers on the key measurement concepts or principles that need to be understood by students and teachers such as unit iteration, attribute identification and the use of formal and informal units (e.g., Lehrer, 2003). However, a number of studies also explore effective teaching strategies for these content areas (e.g., Bragg & Outhred, 2000; Outhred & McPhail, 2000). Teaching strategies and concepts involving length, area and volume are similar due to the nature of knowledge development. Although some research exists on teacher knowledge of length, area and volume, this area of investigation is still quite limited (Ball, 1990). Current literature suggests that teachers' content knowledge effects student learning and is improved by professional development (e.g., Hill & Ball, 2004). The literature surrounding the development of key concepts in measurement is linked to that of teacher knowledge due to the commonly accepted view that teachers must understand these key concepts in order to be effective teachers of mathematics.

"There is a considerable body of evidence showing many secondary students do not have a thorough knowledge" (Outhred & McPhail, 2000, p. 488) of length, area and volume measurement. This evidence includes results of comparison tests such as TIMSS (Hollingsowrth, Lokan, & McCrae, 2003) and PISA (Lokan, Greenwood, & Creswell, 2001) that show Australian secondary students struggle to understand simple linear measurement (Outhred & McPhail, 2000; Outhred, Mitchelmore, McPhail, & Gould, 2003). In particular, studies have shown that students do not understand the attribute being measured or the units that are used for measurement (Outhred & McPhail, 2000; Outhred et al., 2003). Such evidence raises questions about the effectiveness of the instruction students are receiving within these content areas.

Research in the domain of length, area and volume has highlighted a number of general principles or concepts that underlie the understanding of measurement. These principles are important for both teachers and students. One of the more recent principles that research has emphasized is that of unit iteration (Barrett, Jones, Thornton, & Dickson, 2003; Lehrer, 2003). Unit iteration refers to the knowledge that when measuring, a single repeated unit needs to be used in a way that leaves no gaps and causes no overlap (New South Wales Department of Education and Training [NSWDET], 2003). Recent research has emphasized unit iteration due to the fact that many teachers do not highlight this principle within the classroom (Outhred & McPhail, 2000). Combined with the basic understanding of how to use units when measuring, unit iteration is important in ensuring that children gain more than a just procedural knowledge of length, area, and volume.

In addition to understanding how to use units in measurement, a number of other measurement principles have been identified. These key concepts generally involve:

- Conservation (NSWDET, 2003)
- Attribute identification (Clarke, Cheeseman, McDonough, Clarke, 2003; Lehrer, 2003; Outhred et al., 2003); and
- The use of formal and informal units (Clarke et al., 2003; Lehrer, 2003; Outhred et al., 2003).

Research has also identified a number of skills or measurement processes as being important to the process of measurement. These skills include the ability to:

- Compare measurements (Barrett, et al., 2003; Grant & Kline, 2003)
- Choose appropriate measuring tools (Clarke et al., 2003); and
- Measure from a fixed point (Lehrer, 2003).

In order for students to effectively learn measurement, these concepts and skills, along with unit iteration, usually need to be developed through explicit teaching. Therefore, it is crucial that teachers have a thorough understanding of these principles and skills.

Teacher Knowledge of Children's Growth in Understanding

The key concepts of measurement, outlined above, form the basis for a number of 'learning frameworks' in measurement (e.g., NSWDET, 2003; van den Heuvel-Panhuizen & Buys, 2005). A framework sets out stages of development or conceptual growth points that students normally develop as their understanding of measurement processes and concepts progress to more sophisticated levels. These stages or growth points are a progressive list of mathematical attainments (Clarke et al., 2003). In 1999 the Count Me into Measurement Framework was introduced into NSW primary schools (NSWDET, 2003) and was used to inform the development of the new syllabus *Mathematics K-6* (Board of Studies NSW, 2002). The framework clearly highlights certain levels of thinking that children progress through when learning measurement. These levels of thinking reflect the key concepts that have been identified by research as important to understanding measurement. Research clearly indicates that when teachers have clear understanding of these frameworks or stages of growth in students' development of understanding measurement, it assists them to plan clear and appropriate learning activities (Clarke et al., 2003).

Teacher Knowledge and Classroom Practice

Students' lack of understanding in the content areas of length, area and volume has been intuitively linked to poor or ineffective teaching practices for some time, but growing evidence confirms these links. Research indicates that primary teachers often rely on worksheets, textbooks and inappropriate activities to teach measurement, resulting in an emphasis on procedure rather than process (Bragg & Outhred, 2000; Leinhardt & Smith, 1985; Outhred et al., 2003). Such reliance is considered to stem from a lack of confidence that primary teachers generally have when it comes to the content involved with measurement (Clarke et al., 2003; Sowder, Phillip, Armstrong, & Schappelle, 1998). The use of inappropriate activities may indicate a lack of understanding regarding the key concepts and effective teaching strategies for length, area, and volume. Outhred and McPhail (2000) found that teachers struggled with understanding basic concepts associated with unit iteration and effective teaching strategies. The researchers concluded that there was a reluctance to teach measurement concepts by primary teachers and that this may stem from contextual constraints including teacher knowledge. Such findings build support for the view that student misconceptions may simply reflect teachers' inadequate understandings of measurement concepts (Outhred & McPhail, 2000).

Given the context outlined above, this study sought to explore the following research questions:

1. What are teachers' perceptions of their knowledge and understanding of key mathematical concepts and processes associated with the measurement content areas of length, area and volume?
2. What do teachers know about children's developmental growth in understanding length, area and volume?
3. Do teachers' perceive their content knowledge and knowledge of how children develop understanding of length, areas and volume impacts on their pedagogy? If so, how?

Methodology

This research adopted a qualitative approach, utilising self-report data gained from in-depth teacher interviews. Interviews involved four primary teachers selected from three schools, ranging in teaching experience from one to 26 years. The inclusion of teachers from a range of career stages was based mostly on their willingness to participate in the study, but was also considered a strength of the study since such a range would most likely provide rich data to compare and contrast. However, caution is needed not to generalise findings to all teachers at a particular stage of their career without further study involving a larger sample of teachers. More specific information about each participant is presented in the results section.

The aim of the interviews was to gain information relating to: important biographical and contextual information about each teachers' background to teaching in general and more specifically to mathematics; teachers' perceptions about their understanding of mathematical content associated with length, area, and volume; teachers' perceptions about their knowledge of children's growth in understanding measurement concepts; and the impact this knowledge was perceived to have on their pedagogy.

Interview questions that were concerned with teachers' knowledge were presented in a way that let teachers display their perceptions about their knowledge of key concepts involved with length, area, and volume. Hence, participants were not explicitly asked what they did and did not know, rather they were invited to discuss what they considered to be important concepts, knowledge, and skills necessary for an understanding of length, area, and volume, what they thought were significant stages of development in children's understanding of these measurement concepts and how they perceived their own knowledge of these aspects impacted on their instructional decision-making.

Interviews were audio-recorded and later transcribed to assist analysis. Teachers were interviewed on one occasion for approximately one hour each. The interviews were semi-structured to allow teachers the opportunity to move the interview in any new direction as long as they appeared relevant to the research.

To analyse the data, participant responses were coded to assist in focusing on the essential themes emerging from the interviews that were specifically related to the research questions. Hence, interviews were coded based on the following categories:

1. Teacher knowledge of key concepts and measurement processes relating to length, area, and volume;
2. Teacher knowledge relating to students' growth in understanding about mathematical concepts concerned with length, area and volume; and
3. The types of strategies reported for teaching length, area and volume.

Information falling into each of these categories was then considered in light of their commonalities and differences and is reported in the next section.

Results and Discussion

This section presents and discusses the findings of the study. First, biographical and contextual information for each participant is briefly presented. Pseudonyms have been used for each participant. Following this, the research findings are discussed in terms of the commonalities and differences emerging from the interview data.

The Participants

A summary of the background information for the four teachers and their respective school contexts is presented in Table 1.

Table 1

Summary of Background Information for Teachers

Teacher	Grade currenlty teaching	Years of teaching experience	School context information
Dean	Kinder	1	Public school with approx. 530 students K-6. Forty percent NESB. Situated metropolitan region.
Vicky	Year 1	1	Public school with approx. 530 students K-6. Forty percent NESB. Metropolitan.
Pat	Year 3/4	6	Public school with approx. 200 students K-6. Predominately Anglo-Saxon population. Metropolitan.
Debbie	Year 1/2	26	Public school with approx. 600 students K-6. Students from diverse backgrounds. Metropolitan.

As part of the general background information, participants were asked to rate their general aptitude in mathematics. Both Dean and Vicky rated their mathematical abilities as quite “good”, having done well in the discipline at school. Vicky recalled being accelerated in mathematics at High School and Dean doing mathematics as part of a Physics degree at university prior to undertaking his teacher education. Pat and Debbie, on the other hand, acknowledged that they had struggled in mathematics at school, but both considered that their knowledge had improved due to their teacher education and their experiences teaching it to children.

The rest of the interview asked teachers questions that specifically related to their perceived knowledge of length, area and volume—their knowledge of content, children’s developmental growth and the impact of this knowledge on their teaching practices.

Perceived Knowledge of Measurement Content

Three of the four participants rated their knowledge of content associated with length, area and volume as “good.” While Vicky, rated her knowledge of area as “good”, she rated her knowledge of length and volume as, respectively, “adequate” and “minimal”. She felt that her lack of knowledge of the latter two concepts would improve with more teaching experience. Despite rating his knowledge of mathematics, and measurement concepts in particular, as quite “good”, Dean expressed frustration at “not always understanding” measurement concepts sufficiently himself, thus making teaching it to children more difficult. Thus, it was evident that both Dean and Vicky equated the adequacy of their own content knowledge with that needed to teach it.

When asked what they perceived to be the main concepts and processes involved with developing an understanding of length, area, and volume, all four of the participants provided limited responses. This could be due to the possibility that an interview might not be the most appropriate method to elicit this type of knowledge or that not having had to previously verbalise such knowledge, teachers found it difficult to coherently articulate it. Vicky’s and Dean’s answers possibly reflect their recent university studies, with them naming a number of common concepts and processes prevalent in research literature and syllabus documents such as the importance of comparison and the use of informal units. Dean added that “language” and “measuring from a fixed point” were also important, as was the progression to standard units. Pat also mentioned the need for a standard unit and referred to the “manipulation of units” as important, but was not able to elaborate. Debbie did not mention specific concepts or processes related to measurement. She focused her comments on the need for children to understand that “there is a purpose to it (measurement)”. She confided that she considered her knowledge of “number” concepts to be more extensive. This was attributed to the fact that in her 26 years of teaching she had only ever experienced formal professional development that focused on number.

All four participants considered that they had gained their knowledge of content and student development from their teacher education programs, whether this was at university or at teachers college. Although all participants named their teacher education as important, they also thought that other factors had influenced their knowledge. Dean considered that he had learnt the mathematical processes at school, while university had allowed him to develop the skills to "articulate" these processes. Vicky believed that her knowledge was also developed from the Board of Studies units of work and from other teachers. Pat believed that teaching experience was the most important factor, stating that "at uni you get your general ideas but when you get out into the workforce" you learn a lot more.

It was evident from the interviews that teacher knowledge was both implicit and explicit. While the teachers often struggled to articulate their knowledge of key ideas involving length, area, and volume, it was apparent through their discussion of teaching methods and strategies that many key concepts were being treated within the classroom regardless of the teachers' abilities to explicitly identify key concepts. This may be reflective of the fact that in a classroom situation teachers are not required to articulate their own knowledge of key concepts, rather they are simply required to incorporate these into their teaching practice.

Knowledge of Student Developmental Growth in Measurement

Both Dean and Vicky described their knowledge of student development associated with length, area and volume as "adequate". Both teachers perceived their lack of teaching experience to be a major influence in this rating. Debbie believed that her knowledge of student development lay within the "adequate to good" range for the three sub-strands, stating that she understood more about number than measurement. Pat was the only participant who showed confidence in her knowledge of student development, rating herself as "good." She attributed her confidence to the fact that she had worked with Kindergarten through to Year 5 and was very aware of what measurement concepts children understood at the various grade levels.

When asked about where they thought students encountered the most difficulty in developing an understanding of these concepts and measuring processes; responses reflected the varying experience of teachers. Dean identified that his Kindergarten class had difficulties understanding the concept of starting from a fixed point when dealing with length. Vicky identified that her Year 1 class had struggled with the concept of leaving no gaps when measuring area. The two more experienced teachers, Pat and Debbie, both perceived volume as being more difficult for children to understand than either length or area. They considered that this was due to students' inability to "visualize volume" with Pat claiming that the concept of volume is "hard to imagine". All participants thought that practice and time were the solutions to these difficulties, believing that students would progress to the next stage of understanding when they were "developmentally ready".

Vicky and Dean claimed to be familiar with the NSW DET Learning Framework in Measurement (NSWDET, 2003) having been introduced to it in their initial teacher education programs. Vicky reported not having used it to plan her unit on area although she believed that she would use it for her unit on length in the near future. Dean said that he had "looked at it a bit" but admitted to not being very familiar with it. Dean explained that his mathematics units had been planned in collaboration with other early stage one teachers at the school and they had "started looking" at the Framework in Measurement for future programming, but because the other teachers were not familiar with the Framework, it was perceived as an obstacle to whole-stage planning. Vicky and Dean were only able to vaguely recall some of the Framework's content. Debbie, on the other hand, had no knowledge of any formalised framework for describing children's growth in understanding of measurement, and Pat knew that "frameworks about the way kids learn" existed but could not "specifically think of one".

Teaching Practices

When asked about specific teaching and learning strategies that they employed to teach length, area and volume, all the participants named general teaching strategies such as "hands on" activities using "concrete" materials, rather than provide specific examples of experiences or strategies to address children's misconceptions or to introduce them to more sophisticated concepts of measurement on the basis of developmentally appropriate activities. There was a perception that using a variety of concrete materials allowed students to practice using measurement in real-life settings. This was particularly the thinking of Vicky and Dean who both described class experiences of going "outside into the environment and … giving them (the students) problems they can relate to".

In relation to the third research question about the perceived impact of teacher knowledge on classroom practices, there was a distinct perception that such knowledge had little or no impact. However, the very process of questioning the teachers about their understanding of developmental frameworks seemed to act as a stimulus (or reminder) that such knowledge was available and could inform their teaching.

The Learning Framework in Measurement was introduced into NSW Department of Education and Training primary schools in 1999 and extensively influenced the development of the measurement strands in *Mathematics K-6* (Board of Studies NSW, 2002). There was a distinct lack of explicit knowledge regarding this Framework or any other research-based framework describing children's development of measurement concepts. While the early career teachers in the study were aware of the Framework, their limited teaching experience, combined with the fact that their more experienced colleagues were unfamiliar with such a framework, meant that it was difficult for them to utilise this knowledge in their planning. This raises a number of interesting questions about how and when teachers are introduced to developmental frameworks of this nature. Is there a need for such knowledge to be learnt in a context in which it can be simultaneously applied for this type of knowledge to have more impact on teachers' planning and teaching. In particular, this has implications for the way such information is presented in initial teacher education programs.

Conclusion

The aim of this investigation was to explore teachers' perceptions of their content knowledge and knowledge of how children learn length, area, and volume and how they perceive this knowledge impacts on their teaching. Hence, the data presented does not document actual impact on instruction; such claims would need to be validated through additional data collection methods like observations of teaching.

While based on a small sample of teachers, the interviews revealed a number of issues that point to a need for further exploration of teacher knowledge relating to the measurement strand. The study revealed that the teachers in this study perceived their knowledge of measurement concepts and processes to be generally "good" but that this varied according to the different sub-strands; volume was perceived to cause more difficulties for teachers and learners than length or area. All the teachers struggled to explicitly identify key concepts and processes involving length, area, and volume. This inability may reflect the fact that teachers are not normally required to articulate their own knowledge of key concepts; they are simply required to incorporate these into their teaching practice.

Relying on teachers' self-reported perceptions to identify the impact of knowledge on teaching practices through interviews is less resource-intensive than conducting a large number of case studies involving classroom observations. Nonetheless, such case studies are a necessary component of further study in this area as they will allow exploration of how varying levels of experience and teacher understanding of research-based frameworks of children's mathematical thinking impacts on teachers' practices and ultimately on children's understanding of measurement.

Overall, the data highlights the inability of teachers in this study to clearly articulate their knowledge of important concepts and processes relating to length, area, and volume. It also points to the importance of professional development that focuses on the measurement strands of primary curriculum, including content knowledge and knowledge of how children learn measurement.

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Use of the Internet for Teacher Professional Development and for Teaching Mathematics: Supports and Inhibitors

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This paper reports on a part of my ethnographic study about uses of the Internet for teacher professional development and teaching mathematics. Drawing on two case studies of teachers in Queensland, I analyse what factors (personal and contextual) support or inhibit mathematics teachers in making use of the Internet for teacher professional development and for teaching mathematics. The findings support the notion that teachers' knowledge and beliefs are key determinants in embracing technology as a tool for teaching and learning.

Introduction

Numerous studies have been conducted related to the educational potential of the Internet, including in relation to enhancing mathematics teaching (e.g., Moor & Zazkis, 2000) and teacher professional development (e.g., Gibson & Bonnie, 2004; Timmerman, 2004). However, the potential of the Internet for teachers to improve instructional practices has remained largely unexplored. Very little is known about how mathematics teachers use the Internet for their own professional learning and how this learning with the Internet would impact on the qualities of students' learning processes in the classroom. This issue is addressed by the ethnographic study reported in this paper.

The study firstly investigated a *High Use Internet (HUI)* teacher, that is, a teacher who intensively uses the Internet to sustain his/her professional growth as a mathematics teacher. It explored ways in which the HUI teacher used the Internet in teaching mathematics, reasons to integrate the Internet in teaching mathematics, the impacts of mathematics teaching using the Internet on students' learning and philosophical beliefs about teaching and learning using technology (Patahuddin & Dole, 2006a, 2006b).

In the second phase of the study I worked with a *Low Use Internet (LUI)* teacher, a teacher who had not made use of the Internet for those main goals but had a willingness to do so. This involved identifying the theoretical framework to work with the LUI teacher, developing a website and Blogs, exploring many websites recommended by the HUI, and familiarising myself with the potential of the Internet as a source of information, a means for communication, and a site for collaboration. My aim was to assist the LUI teacher to use the Internet for his professional development and for his mathematics teaching. The first phase of the study with the HUI teacher helped me in anticipating strategies to encourage the LUI teacher to use the Internet as a tool for learning and teaching mathematics.

My work with the LUI teacher was guided by five characteristics of effective professional development that were generated from analysis of relevant literature (Abdal-Haqq, 1995; Little, 1993; Putnam & Borko, 1997; Wilson & Berne, 1999): Effective professional development:

- is on-going;
- is collaborative and aims to promote and connect participants in learning communities;
- is student-oriented, focusing on student-centred approaches to teaching;
- takes into consideration the individual teacher and his/her context; and
- has as its prime focus the enhancement of pedagogical content knowledge.

In the process of analysing the data using this effective professional development framework, I found it difficult to explain a number of noticeable factors that facilitated or supported the LUI's attempts to use the Internet in his teaching and learning and a number of factors that interfered with or hindered change. Consequently, I turned to Goos's (2005a, 2005b) adaptation of zone theory in order to address the main research question that drove my study:

- What factors support and inhibit effective professional development for using the Internet as a tool for teaching and learning mathematics?

Three Zones of Influence in Teacher Professional Learning

The theoretical framework employed in this paper refers to the three zones of influence in teacher professional learning developed by Goos (2005a). This theoretical model adopts Valsiner's zone theory, an expansion of Vygotsky's Zone of Proximal Development (ZPD). The ZPD symbolises a space where a child's potential for learning will occur. It is the distance between what a person can do with and without help. In Valsiner's studies of child development, he introduces two additional zones, the Zone of Free Movement (ZFM) and the Zone of Promoted Action. The ZFM represents what actions the child is allowed, while the ZPA represents actions that the adult promotes as an attempt to influence the child's behaviour (Valsiner, 1997).

Goos extended the use of Valsiner's zone theory for several purposes: to understand the complexities of teachers' construction of identity (Goos, 2005a), to study interaction between students, technology and the teaching-learning environment (Goos, 2006), to analyse the pre-service and initial professional experiences of a novice teacher in integrating technologies into his classroom (Goos, 2005b), and to evaluate the effectiveness of the professional development (Goos, Dole, & Makar, 2007).

Goos (2007) proposed the zone theoretical model of teacher learning and development as follows.

> The first Zone [ZPD] represents teacher knowledge and beliefs, and represents the potential of development. This zone includes teacher's disciplinary knowledge and pedagogical content of knowledge.., and beliefs about their discipline and how it is best taught and learned. The second zone [ZFM] represents the professional context which defines the teaching action allowed. Elements of the context may include curriculum and assessment requirements, access to resources, organisational structures and cultures, and teacher perceptions of student background, ability, and motivation. The third zone [ZPA] represents the sources of assistance available to teachers in promoting specific teaching actions, such as that offered by a pre-service teacher education course, supervised practicum experience, professional colleagues and mentors in the school, or formal professional activities. (p.416-417)

Goos (2006) also stated that the relationship of the three zones "provides a useful way of analysing the extent to which teachers adopt innovative practices involving technology". She theorised that the development of teachers is determined by the relationship between all the elements of the three zones.

Design of the Study

The HUI and LUI teachers in this study were given the pseudonyms of Ann and Jack. Ann was observed in her normal school setting for two weeks and interviewed on several occasions over a four week period. Continued e-mail conversations between the teacher and the researcher occurred after the observational period. The study with Jack in his classroom was from February-August and November-December 2006.

An ethnographic approach was chosen for this study. It employed multiple data gathering methods: participant-observation, interviews, questionnaires, and written and non-written sources. In this study, the teachers completed several questionnaires, providing information about their professional background, how they use the Internet for professional development and for teaching and learning mathematics. Informal interviews with the teachers were held on several occasions, to clarify responses to questionnaire items and to verify my perceptions of what I had observed. I also accessed the Internet to find the teachers' favourite websites, as listed in their computers' bookmarks. E-mail communication was a part of the data collection method, and this occurred both during and after the fieldwork. Videotaped lessons also assisted in completion of the field notes.

Ann's Case Study

Ann was an experienced teacher who had been teaching at different grade levels from pre-school to Grade 7 at several different schools across Australia. She had many opportunities to attend various ICT workshops or conferences. Her professional learning about the uses of technology, including the Internet, started during the early years of the availability of the Internet for education in Australia. Ann stated that many PD programs she has attended have continued to inspire her to use the Internet in her teaching. They have triggered her curiosity to explore further uses of the Internet as a learning tool through using the Internet itself. The most exciting feature of the Internet, as identified by Ann, was how it easily it can be used as a communicative device. This has allowed Ann to learn from other teachers without the need to leave the school. Ann regards the Internet as a rich living resource. One memorable comment that sums up Ann's feeling towards the Internet was: "If I didn't have access to the Internet, it would be like having my hand cut off".

Ann has a less than favourable environment since her classroom was a very crowded space and had limited resources. Even though she had six computers (including one very old computer that cannot save any work) in her classroom, only two were connected to the Internet. Things were difficult when her classroom had problems with the computers and there was not ICT support available in her school. Ann's students were very passive at the beginning of the year and often disengaged in learning for many reasons. However, Ann's perceptions of her students were very positive and she believed that her students had potential to think and to control their own learning.

Ann has a broad view about mathematics. She believes that mathematics is more than right or wrong, that it exists everywhere and as a tool to solve problems. Ann's belief about learning is that it is a process of constructing ideas by the learners and therefore teaching should engage students in their own learning. Because she stated her belief that "students are at the centre of learning" and every student has different levels of understanding and interest, her instructional practices cater for this diversity to make sure that every individual student is learning.

Ann's belief about ICT, including the Internet, is that ICT skills are important for a student's life and it is a powerful tool to enhance teaching and learning. Through compilation of interview data and classroom observations, I discovered that Ann's existing knowledge and beliefs about mathematics, teaching and learning mathematics are compatible with her beliefs about the Internet. Ann was already a *student-centred* teacher when she first used the Internet. She realised that the Internet is a powerful way to place the student at the centre of the learning process. Therefore Ann regards technology and including the Internet as vital to her learning and teaching style.

Interestingly for Ann, when the Internet first became available at her school (one Internet-connected computer available at the library in 1996), she immediately started using it in her teaching. In 2000, the Internet became available in her classroom and since that time, the Internet has become a part of her mathematics teaching. She particularly was keen to exploit the academic potential of the Internet as a communication tool to connect her students with other students from other schools in Australia or other countries. She has been continually experimenting with her students in using the Internet as a learning tool (Patahuddin & Dole, 2006b).

At the time of this research, Ann was implementing the new Queensland Mathematics Syllabus. This syllabus has a high emphasis on thinking, reasoning and working mathematically and emphasises the uses of communication technologies in teaching mathematics (Queensland Studies Authority, 2004).

Jack's Case Study

Jack is a beginning teacher who had commenced his second year of full-time teaching at the time of this study. In his first year, Jack taught Year 3 and in the second year he was teaching Year 2. During his first year of teaching, Jack attended several teacher professional development programs, but none of these related specifically to using the Internet for mathematics teaching and learning. He is a competent user of Microsoft Word, PowerPoint, Excel, and Kid Pix. He also knows how to make a website with FrontPage. Jack has access to the Internet both at school and at home, and uses the Internet for e-mail, banking, and shopping.

Jack's school was well-resourced. His classroom has a spacious feel. The desks for the 26 students are organised into five groups, leaving a large space for students to sit comfortably in a big circle on the carpet. His classroom has good access to four computers with a fast Internet connection. His school espouses support for use of ICT. The students were mostly from wealthy and well-educated families representing many cultures, and they were motivated and well-behaved. All these presented favourable opportunities to use the Internet for mathematics teaching and learning.

His words – "I want the students to have a good understanding of addition strategies, and the number facts" – and my observations suggest that Jack placed more emphasis on computational skills and less on mathematics thinking and problem solving than is suggested in the Queensland Mathematics Syllabus. Even though in some ways he said that he wanted to be student-centred, I have seen through my work with him that he preferred a teacher-centred approach. This is clearly indicated by the reason he gave for using the data projector instead of giving opportunities to students for more individualised instruction:

> … because I find it is easy with a data projector because you can do it for a whole class at once. And I think, they can all sit down, they can all see it rather than have them to use the computers. And then I think I prefer to do it that way then have the children back to their desks and doing an individual activity from there, like maths textbook or something from the board, as a consolidation because we only have four computers and I don't have time to supervise them all.

Unlike Ann, Jack's beliefs about the Internet for learning are very limited. He appeared to see the Internet as an "add-on". His claim that students look happy without the Internet reflects his beliefs that the Internet is just another tool and teachers do not necessarily have to use it.

While Ann sees the new Queensland Mathematics Syllabus as an opportunity to integrate technology into learning and teaching, Jack sees it as constraint. He did not use this new syllabus because he preferred to use the school program. At the beginning of the year, he told me that the school program was more practical for him than the new mathematics syllabus. Hence, he had not familiarised himself with the new mathematics syllabus and instead relied solely on the school program. Jack as a beginning teacher had rudimentary pedagogical knowledge as well as pedagogical content knowledge. Through my work with Jack, I discovered his difficulties in managing the students' learning. Jack's perception of the ability of his students was quite low; for example, I could see that he tended to pose simple problems to his students. He was very surprised when he found that one of his students could answer a non-routine and complex problem of the school enrichment test.

Time seemed a problem for Jack. Through interviews, I discovered that Jack often saw lack of time as a constraint to use the Internet. For example, he had not enough time to read emails. He lacked time to do some rotational activities that involved computer activities for one group. He gave up providing opportunities for students to access the Internet in their free time in the morning as it was the busiest time of day for him in managing his teaching plan. This might be because it was his first experience in teaching a Year 2 class. He worried about lack of time for teaching if students were involved in many group activities.

Analysis Using Zone Theory

Relationship between Ann's ZPD, ZFM, and ZPA

Ann attempted to enact her pedagogical beliefs (ZPD) by optimising the potential of the Internet. For example, she managed the classroom so that over time, students rotate through an activity to use the Internet. She took advantage of the limited resources to help her in assisting student's learning and to provide more individualised instructions.

The relationship between the curriculum element of Ann's ZFM (the new Queensland Mathematics Syllabus) and her ZPD (knowledge and beliefs) is very positive. This part of the ZFM appeared to afford teaching actions consistent with her beliefs about mathematics teaching and learning. The new syllabus is an affordance for Ann because it supports her preferred teaching approaches. From the interview, I discovered that she interpreted the recent syllabus as allowing her to teach investigation and problem solving and opening opportunities to incorporate the Internet into mathematics teaching.

There is a strong overlap between Ann's knowledge and beliefs (ZPD) and the kinds of professional development that she has sought out (ZPA). Her initial reaction to Internet related PD was positive because she could see how to use the Internet in ways consistent with her pedagogical beliefs (student-centred, investigative). The affordances of the Internet allowed her to continue her learning through virtual PD in her own time. Her beliefs about the Internet and her interest in its communicative potential led her to explore many learning and teaching resources as well as to involve herself in online professional forums. Also, the ZPA offered through the Internet (e.g., multiple representations) is compatible with Ann's beliefs about student-centred approaches. This positive relationship allows Ann to cater for the diversity of students' interests and abilities. For example, Ann used the Internet for remediation or extension or for "disengaged learners". As a result, Ann regards the Internet as an extension of herself as a teacher.

In summary, Ann looks at her professional context through the filter of her ZPD/ZPA relationship. The configuration of Ann's knowledge and beliefs, professional contexts, and sources of support came together to shape opportunities for her professional learning using the Internet.

Relationship between Jack's ZPD, ZFM, and ZPA

I was present in Jack's classroom as a participant observer to promote uses of the Internet for his learning and for his mathematics teaching without explicitly asking Jack to change his teaching direction. Ideas to support Jack had emerged as a result of my daily interactions with him. In this case, the researcher is one of the elements of Jack's ZPA.

All the approaches I used with Jack emerged, developed and proceeded at different times throughout my work with him. For example, I used e-mail communication, organised educational websites into Blogs, planned and designed mathematical investigations with Jack, showed him mathematical websites and worked with his students, and linked selected websites to his mathematics teaching program (see details in Patahuddin, 2007). Even though I continuously evaluated what I had done and thought carefully about the next type of strategy I could use, I often found it difficult to promote the uses of the Internet for Jack's professional development. Zone theory is useful for analysing matches and mismatches between influential factors in the three domains of knowledge and beliefs (ZPD), professional context (ZFM), and sources of assistance (ZPA). Two critical events from my work with Jack are selected to exemplify this analysis.

Because I discovered that Jack was unfamiliar with the new mathematics syllabus I decided to provide a table that linked a variety of websites (e.g., for remedial work, extension, hands-on activity, individualised instruction) directly to every topic in his mathematics teaching program. This strategy yielded positive outcomes in that it gave Jack more time to explore mathematical websites. He also began to ask: "How can we use all these great websites without using rotational activities?", which suggested that he had started to realise that his existing preferences and beliefs about teaching with data projector presentations (mentioned previously in this paper) was not compatible with his new notions about teaching with the Internet. Analysing this event through the lens of zone theory, we note that this aspect of the ZPA I offered was a good match for Jack's existing ZPD. Here, my strategy seemed successful in challenging Jack's ZPD. As Borko et al. (1997) explain, "… efforts to help teachers make significant changes in their teaching practices must also help them to acquire new knowledge and beliefs. At the same time, teachers come to understand new practices through their existing knowledge and beliefs" (p.272).

I took advantage of Jack's emerging dissatisfaction with his instructional practice by trying to change the way he organised classroom use of computers (Jack's ZFM). After observing that the computers were used very rarely by students, I suggested they might use the Internet as a part of their regular free morning activities. Jack was initially enthusiastic, but quickly gave up applying this idea because of other elements of his ZFM (only two computers were available as he kept two computers for teachers) and his ZPD (beliefs that students could not manage their learning using the Internet and that his teaching was effective without the Internet).

My attempt to bring Jack's ZPD and ZFM closer together was not entirely successful as I did not fully address Jack's existing ZPD. Apparent enthusiasm does not necessarily translate into confidence in use. Previous researchers (Borko et al., 1997; Borko & Putnam, 1996) inform us that changing beliefs usually takes a long time. Also, what teacher states as his/her belief does not necessarily mean that they own this belief.

Conclusion and Implications

Perhaps, most teachers in Australia would look at Ann's ZFM (poor resources, cramped space, and passive students) as constraints in achieving rich learning using the Internet. However, my study shows that Ann has agency to act on her environment. She works on shaping a classroom environment that will allow her to put her beliefs into practice. I argue that what really drives Ann in her teaching is her knowledge and beliefs (ZPD). This means that the Zone of Free Movement is not necessarily the real environment around teachers, but a way teacher see and interpret the environment (Goos, 2005). Thus one person can see a classroom like Ann's as a constraint, while another with different knowledge and beliefs might see the same classroom as an opportunity.

Jack's case study illuminated issues for a beginning teacher who was familiar with technology in his daily life but not yet integrating it into his own learning and teaching. While his ZFM was favourable, his preferred teaching approach and pedagogical beliefs (his ZPD) hindered optimal use of the Internet for learning. Jack views professional development through the filter of his ZPD/ZFM relationship. His beliefs and his professional context orient him towards seeking PD that will allow him to function more effectively in that context, for example, by providing ready-to-use resources he can use in whole class, teacher-centred lessons.

This core finding demonstrates that resources alone do not guarantee rich learning and successful teaching and supports previous findings that Internet availability in schools has not been optimised in teaching (Becker, 2000; Gibson & Oberg, 2004; Wallace, 2004). Governments in many developed countries (e.g., Canada, USA, England, Japan) have invested substantial amounts of money in technology and yet the potential for this technology to improve educational outcomes has not been achieved. The present study suggests that teaching resources are not the key to a technology-rich mathematical learning environment, as computers and the Internet are only tools that could either enrich or restrict learning.

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A Situated Perspective on Learning to Teach Secondary Mathematics

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This paper applies a situated perspective on learning to investigate the process of becoming a secondary mathematics teacher. We followed a group of beginning teachers into their early teaching careers. Each participant was interviewed on three separate occasions during their university studies in order to examine how they interpreted their practicum experiences. In the following year, we again interviewed some of the participants to investigate their sense of self as mathematics teachers and to document the issues which they identified as significant in shaping their classroom practices. The beginning teachers saw the culture of the school and pedagogies of more experienced colleagues as the most important factors affecting their professional growth and ability to implement Working Mathematically in the classroom.

Introduction

Calls to reform the learning and teaching of secondary mathematics have been around for over a decade and their message is quite consistent. They call for a seismic shift away from a view of mathematics as the accumulation of rules and formulae which are drilled and practised to one where mathematics is a sense-making activity and learners are actively engaged in their lessons. However, despite a growing tradition of reform-oriented documents and syllabuses, research suggests that little has changed (e.g., Hollingsworth, Lokan, & McCrae, 2003). There is, in many mathematics classrooms, a reliance on a "shallow teaching syndrome" (Stacey, 2003) where students complete a large number of repetitive, low complexity problems often by blindly copying procedures at the direction of their teachers.

How then can the message of the reform movement start to take root in schools and begin to bring about the reformist vision? One possibility often proposed is through the work of beginning teachers who are trained according to constructivist principles in their university studies and encouraged to experiment with student-centred approaches in their school-based practica. In doing so, it is hoped that these novice teachers might act as agents of change who can carry the message of reform into schools and provide an example of constructivist pedagogy for their more experienced counterparts. Yet this too can be a problematic path due to the ever-present influence of the school culture and the likely impact it may have in shaping how beginning teachers learn their craft.

This paper examines the experiences of a group of secondary mathematics teachers in their first year of employment in schools. Our research is underpinned by a socio-cultural framework which interprets the process of becoming a teacher from a "situated perspective" (Greeno, 1998) as we investigate the extent to which the school culture influences the ideas and practices of beginning teachers.

Situated Learning

Fullan and Hargreaves (1996, p. 37) define the school culture as "the guiding beliefs and expectations evident in the way a school operates particularly in reference to how people relate (or fail to relate) to each other. In simple terms, culture is the way we do things and relate to each other".

Cognitive learning theories consider the acquisition of knowledge as a change in conceptual structures within the mind of an individual. As such, what is learned is largely independent of the context in which the learning takes place. From a cognitive perspective, knowledge is viewed as an entity which is acquired in one setting and can be readily transferred to other situations. Situated theories, on the other hand, regard learning as located in particular forms of experience and not simply in the mind. From a situated perspective, the learner and what is being learned are always situated in activities, processes and contexts. Situated paradigms therefore focus primarily on how individuals interact in socially organised activities or practices that are oriented towards specific goals and are always contextualised because the nature of what is learned is shaped and influenced by the environment in which learning takes place.

Learning is distributed across the individual, interactions with others, and different kinds of artifacts such as language, symbols and tools (Wenger, 1998). So, while cognitive approaches focus on knowledge, situated perspectives view learning as authentic participation in the discourse and practices of a community i.e. within the culture of the school. Learning is therefore regarded as a social activity that involves individuals making sense of their experiences as they increase the range and level of participation in the norms and practices of a community (Lave & Wenger, 1998).

Participation is described in terms of negotiating new meanings through the accumulation of experiences that arise from active engagement in "communities of practice" (Wenger, 1998). Communities are critical because they help to shape the knowledge and skills that are learned and how this learning occurs (Franke & Kazemi, 2001). At the same time, the practices of the community are themselves always evolving since individuals both participate in and contribute to the development of the practices of their community based on their prior experiences and the unique perspectives which they bring with them (Rogoff, 1995). There is merit, therefore, in analysing not only the practices of the community but also the diversity of each individual's mode of participation within it (Cobb & Bowers, 1999).

An important aspect of situated theories of learning concerns the consistency of patterns of participation across different settings and the extent to which engagement in one context can facilitate successful participation in another. Ensor (2001) refers to the notion of "recontextualizing" as a "process of disembedding, reembedding and change" (p. 297) to describe how resources, tools and discourses can be transformed across contexts. These ideas are useful in the present study which seeks to understand whether or not the practices learned by pre-service teachers in the practicum are recontextualised in their first year of teaching and the extent to which immersion in the practice of teaching can sometimes create a barrier for beginning teachers who wish to implement a reform agenda.

Situated perspectives can offer a useful means of analysing the professional growth of teachers (Lerman, 2000) because they focus on schools where beginning teachers work with their more experienced counterparts (Putnam & Borko, 2000). Situated approaches consider classroom social practices and examine patterns of classroom discourse and the kinds of activities in which teachers and students are engaged. Learning to teach is therefore viewed as a process in which beginning teachers increase their participation in the activity of teaching and, through this participation, gain knowledge and insights about the practices of teachers (Adler, 1998). In other words, beginning teachers learn to teach by interacting directly with their colleagues in schools, by talking with them, and by taking part in specific activities associated with teaching. They become part of the culture of the school by adopting its norms and practices and by observing other teachers as they work (Stein & Brown, 1997) and they often learn what is valued and practised by their colleagues (Stein, Silver, & Smith, 1998).

Peressini, Borko, Romagnano, Knuth and Willis (2004) identified a focus on individuals and their contexts of practice as a promising way forward for developing ideas about the process of learning to teach secondary mathematics. In this paper we investigate how the context of the school of employment (Year Two of the study) affects beginning teachers' perceptions of themselves as teachers and their classroom practices. The study took place during implementation of a new secondary mathematics syllabus which emphasised Working Mathematically, so we were particularly interested in the participants' reports on how they were implementing the Working Mathematically strand of the syllabus and the aspects which they identified as influencing their decisions. For a discussion of the pre-service teachers' practicum experience (Year One of the study), see Cavanagh and Prescott (2007).

Method

Participants

The participants in Year One of this study were ten pre-service secondary mathematics teachers who were enrolled in one-year Graduate Diploma of Education courses taught by the authors in two universities. The beginning teachers who obtained full-time employment in metropolitan schools were invited to participate in Year Two of the project. Four teachers agreed to take part and this paper reports on Year Two.

Data Collection

The four first-year teachers were interviewed individually for approximately 30 minutes in the middle of the school year. All of the interviews were semi-structured and designed to probe the participants' views of themselves as developing teachers and the particular impact of the school culture in their schools of employment.

Data Analysis

The analysis was conducted by firstly reading the interview transcripts and noting common responses. From this initial reading, three broad categories were identified in terms of the likely impact of the school culture: self-perception, implementation of working mathematically, and general classroom practices. Specific themes began to emerge as recurring phrases and sub-categories were identified within each of these broad research foci. The presentation of the findings of the research is organised according to these three key components.

Results

Self Perceptions

Many studies (e.g., Adams & Krockover, 1998; Kardos & Johnson, 2007) have found that beginning teachers find their first year of teaching stressful, chaotic, a roller coaster ride, and emotionally draining as they find themselves in a situation where they move from one 'crisis' to the next. The school culture can increase or minimise the acculturation shock felt by so many beginning teachers.

It was very easy for the beginning teacher to obsess about the relatively small number of difficult students they dealt with and lose sight of their achievements.

> Low points. Every day, one out of 60 kids doesn't do it. Or, you know, has been rude or whatever and I was concentrating more on that one person than the whole lot. It took me a while to sort of think: You know, there were 60 today and just one was out of whack. But it was very draining. [Neroli]

Beginning teachers are torn between wanting others to perceive them as effective teachers and wanting support – far more than the other teachers around them. Teaching is one of the few professions that 'throws' new practitioners in at the deep end, allowing them to function in the classroom on their own in exactly the same way as experienced teachers. This can be a very solitary experience and may even be a terrifying prospect if the beginning teachers perceive they are without support (Kardos & Johnson, 2007). In an attempt to fit in, or at least not stand out as a new teacher, beginning teachers believe that 'toeing the line' will help them show the other teachers that they are competent. To this end they find it hard to ask for help because they do not want to appear to be floundering and besides, the other teachers all look so busy. Unfortunately, many experienced teachers believe that beginning teachers are not assisted in their development as teachers by being mollycoddled with too much help.

> I guess one thing would be good if the mentor came in and observed my lessons … Not necessarily to grade me but to say I can see some difficulties, here are some things you can do. [Stephen]

The beginning teachers' perceptions of themselves as teachers were also coloured by the feeling that they never had a spare moment. The administrative details of their job and activities such as playground duty took them away from their work as teachers. They were keen to produce excellent lessons but spent so much time on details outside the classroom that the idea of being a reflective teacher could not happen.

> I've got about eighteen lessons a week but in addition to that I've got things like sports… so that's time I can't use for anything else. I've got assembly … pastoral care, those kind of other things. [John]

Implementing Working Mathematically

The study also sought to determine the ability of the beginning teachers to implement the Working Mathematically strand in the classroom. The emphasis in the syllabus requires a less textbook oriented approach to teaching mathematics and the emphasis in the university courses supports this. However, many teachers are predominantly textbook oriented and so the culture of the school does not support the beginning teachers as they seek to incorporate a working mathematically approach to their teaching. Beginning teachers also see 'fitting in' as conforming to the style of teaching exhibited by their more experienced colleagues so working mathematically becomes problematic (Boomer & Torr, 1987; Schuck, Brady, & Griffin, 2005).

The beginning teachers recognised the need for a balance between the traditional textbook approach and the working mathematically approach to mathematics teaching but were also sure they were not yet getting the balance right. There is an unresolved tension in that the beginning teachers saw the value of working mathematically but were fearful that teaching that way would take longer and probably create classroom management issues. They said they would postpone working mathematically until they felt more confident in the classroom.

> I know [less able students] need [working mathematically] the most but I just fear that if I do this that they won't listen or they'll muck up. [Stephen]

> Once I establish a good relationship and ... good communications with them ... then I'll be popping up interesting questions. [John]

Many of the beginning teachers found resources were limited to textbooks. Of course, if experienced teachers see little point in spending money on resources, the textbook will dominate the classroom. One school used the textbook rather than the syllabus as its programming document making anything but a textbook oriented approach much harder.

Because the beginning teachers were on probation, they felt they had to keep a tight rein on their students and they felt that a textbook oriented style of mathematics teaching made this easier. They were fearful of trying something new in case it did not work, especially as the students were not used to working mathematically.

> You're on probation and you've got a teaching certificate to get so you don't want to be taking too many risks. The teachers often walk past my classroom so you want to keep the class reasonably quiet. [Stephen]

The beginning teachers' lack of experience in the classroom and (perceived) lack of support in the staffroom also led to a belief that the textbook would help their students because the examples were all pitched at the right level. It allowed them to concentrate on their explanations and classroom management.

> I have the support of the textbooks so I'll be focussing on my ability to explain things, try and keep the right level, use the right words etc. [John]

One of the beginning teachers was keen to undertake working mathematically in the classroom. During the practicum his supervising teacher made it very clear that he must prepare textbook oriented lessons but now he felt he had support from the school and the freedom of his own classes and was enjoying the experience. The support had come from the availability of resources and from a variety of people, including a mentor, the head of department and the principal.

> I guess I had a number of philosophical differences with my supervising teacher [last year in the practicum]; just, totally different approach. So [this year] I was finally able to do what I wanted to do. I didn't have to worry so much, you know, I could, if I wanted to do a lesson and have a discussion for the most of, you know, most of it or whatever then that was my decision to do, and I didn't feel like I was having to please somebody else. [Peter]

Despite the encouragement of his mentors, this beginning teacher was still using the more traditional approach because he saw this as more closely conforming to the culture of the school.

The beginning teachers also felt pressure from their colleagues to keep up with parallel classes so that all material had been covered in time for the examinations. While they knew that pushing students through the work in this way was not effective teaching practice, showing their colleagues that they were competent teachers had much greater impact on their teaching.

> You've got to try to teach the material so that they can have some opportunity to do well in the exam … I'm trying to push them through the work so at least they've seen it. … So long as you've taught it, that's OK. But whether they've learned it or not is immaterial. [Stephen]

It was not a case of making sure the students understood the work, rather it was a case of ‘covering’ the material in class so you could sign off the register. The beginning teachers also knew that the examination questions were usually procedural and that they would be unlikely to test conceptual understanding (despite working mathematically being central to the syllabus).

Many students’ experience of mathematics teaching was almost exclusively instrumentalist so they were used to being told how to do their mathematics by a rule and then practice. A working mathematically approach to teaching requires more from the students and so there was conflict between beginning teachers who wanted to develop conceptual understanding and students who just wanted to know how to do the examination questions. The resistance from students discouraged the beginning teachers from pursuing anything but a traditional lesson, and those who tried and felt their lessons were poor were fearful of trying again.

> They said why are we doing this? We don’t need this. They rebelled. [John]

Classroom Practices

The beginning teachers’ ideas about being effective in the classroom were limited to being able to deal with classroom management issues and being a good communicator so that explanations were clear. The problem for them was that they saw the discipline issues as emanating from their inability to cater for the range of abilities in their classes. They saw the main problems as adapting to the needs of different classes and determining the amount of work to be covered in each class. Added to this, the full teaching load made lesson preparation and reflection a luxury rather than an essential component of good teaching practice.

Classroom management problems meant that many hours were needed to follow up recalcitrant students and in dealing with students who needed extra help. The inability to deal with the various levels of ability in the classroom meant that lunchtimes were used for helping students to catch up with their work and to improve understanding.

> I see a lot of students benefiting from individual attention which I can’t give them during a normal lesson and I tell them ‘Look you have to come for a lunchtime because I want to cover this area in detail with you’ [John].

To the beginning teacher, experienced teachers appear to prepare lessons with little time and energy (particularly if it is straight from the textbook). This ability to prepare lessons at will only served to emphasise the time pressure experienced by the beginning teachers. This meant that the time-consuming creation of worksheets and development of activities gave the beginning teachers the idea that they were always catching up rather than working as competent teachers.

> I often don’t have time to reflect on what worked and what didn’t. I do sometimes but not half as much as I would like to … it’s basically trying to survive to the next lesson. [John]

Discussion and Conclusion

We investigated the beginning teachers’ sense of themselves as teachers, and the issues that they saw as influencing their classroom practice, particularly their implementation of working mathematically. The situated learning framework allowed us to interpret the comments made by the beginning teachers about their classroom practice and identify the factors that they regarded as impinging on their ability to become more effective practitioners.

While we separated the results into three discrete categories because we saw them as common themes in the interviews, in reality many of the discussions showed links across the three domains of self-perception, implementing working mathematically and classroom management. In other words, the beginning teachers did not always see these as separate issues, so the examples they used and the stories they told during the interviews demonstrated an interconnectedness between them.

The beginning teachers sometimes found themselves receiving mixed messages about what constituted being an effective teacher. On the one hand, more senior colleagues encouraged them to experiment with student-centred activities in their lessons, but the culture of the school and the example of other teachers was very traditional and did not appear to support a working mathematically approach because the textbook was the dominant resource. The beginning teachers were not confident in their pedagogy and chose to resolve this dilemma by conforming to the dominant practices of the school in the hope of being seen as effective in the classroom. Clearly the culture of the school was a powerful influence on their decision-making, counteracting ideas they professed during their university year (Prescott & Cavanagh, 2006).

The beginning teachers were also confronted by the relative ease with which their more experienced colleagues prepared and delivered traditional style lessons which appeared to be successful – success being predominantly measured by the school in terms of quiet classrooms and acceptable scores on common tests. The contrast between the relative calm of other teachers and their own highly anxious state forced the beginning teachers to acknowledge the impact of the daily struggles in learning to teach. Faced with this situation, they chose to adopt the style of their colleagues in the hope that their lessons would become easier to prepare, classroom management would improve and their students would see them in a positive light. Again, we see the culture of the school as a dominant force in shaping the practices of beginning teachers.

Woods and Weasmer (2004) suggest that there are reciprocal benefits when experienced teachers and beginning teachers share their ideas with each other – including a clearer understanding of the school culture and a stronger sense of what is expected. Only then is it possible that beginning teachers might be agents of change within the school.

Our work indicates the pervasiveness of the context in which teachers work. Mathematics teacher educators, and colleagues and mentors of beginning teachers should all be aware of this influence and provide beginning teachers with the skills necessary to identify and deal with the impact of the culture of the school. The participants in our study claimed that they would adopt a more working mathematically approach when they saw themselves as competent teachers who had earned the respect of their colleagues and their students. Further research could be undertaken to test this assertion and determine whether the culture of the school maintains its influence as the teachers become more experienced.

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The Hospital Problem Revisited. Tertiary Student's Perceptions of a Problem Involving the Binomial Distribution

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This paper considers the intuitive solutions of 26 first year tertiary students to a binomial probability problem on entry to a statistics unit. For this problem a successful solution requires consideration of the sample size. On the basis of a Rasch analysis, students were classified into three groups according to their ability, and the reasoning they used compared. The problem was again posed at the end of the unit and the answers and reasoning used by the students were compared with their earlier responses.

Half of all newborns are girls and half are boys. Hospital A records an average of 50 births per day. Hospital B records an average of 10 births per day. On a particular day, which hospital is more likely to record 80% or more of female births?

A variation of this problem was first reported by Tversky and Kahneman (1982) in a study of the heuristic rules used by undergraduate students to judge events that are uncertain in outcome. Since then this same problem, or variations of it, have been used in studies of school students (Fischbein & Schnarch, 1997, Watson & Moritz, 2000), and pre-service teachers (Watson, 2000). This study describes the types of reasoning used by tertiary students to answer this question on entry to a statistics unit, and after the unit was completed. This question was used as part of a wider study to investigate students' intuitive reasoning in statistical inference.

To answer the Hospital Problem successfully, it is necessary to look beyond the proportions and to appreciate the effect of sample size. In the studies cited, the most common answer given was that the likelihood of recording more than 80% of female births was equal for both hospitals. Tversky and Kahneman (1982) refer to this as an example of the representativeness heuristic, where samples are assumed to be more like the overall population than sampling theory suggests. In the hospital problem, this heuristic leads to the conclusion that sample size is not relevant, that as the two events are described by the same statistic they will be equally representative of the general population. Sampling theory, however, suggests that the smaller sample is more likely to deviate from the 50% rate of births for each gender (Tversky & Kahneman, 1982). As a sample increases in size, the sampling statistic (here the proportion of girls born) is more likely to approach the theoretical value for the entire population (Fischbein & Schnarch, 1997).

In Tversky and Kahneman's study, 53 out of the 95 undergraduate students answered that the two hospitals were equally likely to record an uneven proportion of births. Fischbein and Schnarch gave a similar question to students in grades 5, 7, 9, 11 and to college students who were prospective teachers specialising in mathematics, none of whom had previously studied probability. The students in the lower grades had a high number of non-responses. When the question was answered by the younger students, the most common response was that of the largest hospital. As the age of the students increased, the number of responses also increased, and whereas the likelihood of choosing the larger hospital decreased, the likelihood of answering that the events were equally likely increased. Out of the 18 college students 16 gave the answer of equal likelihood (the other two did not respond). Fischbein and Schnarch suggested that as the understanding of ratio improved with age, this understanding became dominant at the expense of an understanding of the effect of sample size. Out of the whole study only one grade 9 student gave the smaller hospital as the answer.

Watson and Moritz (2000) interviewed 62 students from grades 3, 6 and 9 from a variety of school regions, including suburban and rural schools in Tasmania. There were equal numbers of males and females. During these interviews students were asked about the size of a sample needed to study the weights of grade 5 children, and for the grade 6 and 9 students were then asked a variation of the hospital problem. The question was:

> The researchers went to two schools: One school in the centre of the city and one school in the country. Each school had about half girls and half boys. The researchers took a random sample from each school: 50 children from the city school, 20 children from the country school. One of these samples was unusual; it had more than 80% boys. Is it more likely to have come from:
>
> - The large sample of 50 from the city school, or
> - The small sample of 20 from the country school, or
> - Are both samples equally likely to have been the unusual sample?
>
> Please explain your answer.

Out of the 41 respondents, only 8 chose the small sample, with only 6 of these being able to give adequate reasons. Those students who picked the larger sample suggested that as there were more children to pick from, there were more children to get the higher number of boys. The most common response was that of equal likelihood (61%), and the proportion of this response did not vary between the grade 6 and grade 9 students. The reasons given were either that the process was random or because each school population from which the samples were taken had a 50% occurrence of each gender. It is apparent from this study that the context of the question is of importance. In an earlier question students had been asked about the number of students needed to study the weights of grade 5 children. Eighty percent of the students who had stated that larger samples were needed to study the children's weights did not recognize that a smaller sample was more likely to give extreme results in this question.

Watson (2000) gave the hospital problem to 33 preservice students who were all in a post-graduate teaching program. There was wide variation in the mathematics background of these students; 23 had at least studied mathematics up to the second year as part of their previous university courses (one of these was on leave from a PhD enrolment in mathematics), and 10 had less than this. They were given the hospital problem to work on overnight and asked to complete it on their own.

This study recorded the reasoning used by the students. The students were divided into those who used intuitive reasoning only, mathematical reasoning only, or a combination of the two. The mathematical reasoning was divided into whether the binomial distribution was used (formal), or more elementary mathematics such as percentages were used (basic). The results are summarised in Table 1.

It is apparent that the students who used mathematical arguments alone were more successful than those who used intuition alone. It is also apparent that mathematics alone is not entirely successful, as those who made errors in their formal mathematical calculations were unaware of their error. However those who used both mathematical and intuitive reasoning were only 50% successful.

Table 1

Responses to the Hospital Problem (Watson, 2000)

	Strategy		
Correctness of conclusion	*Intuition*	*Mathematics*	*Intuition and Mathematics*
Correct (n = 18)	7	Formal: 2 Basic maths: 6	Formal: 3
Incorrect (n = 15)	8	Formal: 4	Basic maths: 3
Total	15	12	6

Method

The study of the Hospital Problem described in this paper is being carried out at as part of a wider study of students' intuitive statistical reasoning and inference at an Australian university. On entry to the unit the students were given a questionnaire where they were required to interpret probability statements, to recognise independence and sampling variation, and to make simple inferences. The Hospital Problem, as described in the opening paragraph, was part of this questionnaire. At the end of the unit the students were given another questionnaire that required them to make statistical inferences and explain their reasoning. The Hospital Problem was one of three questions that were repeated from the first questionnaire.

Participants

The participants were volunteers who were enrolled in a first year statistics unit. This unit is a service course for students who are studying Biomedical Science, Aquaculture and Environmental Science. The unit is also taken as an elective by students studying Health Science, Computing, and Education. The initial questionnaire was given to 26 students. Of these, one had studied mathematics at year 11, 20 had studied mathematics at year 12, one at TAFE, and four at University. Nineteen of these 26 reported that they had studied some form of statistics in their last mathematics course. Due to circumstances beyond the researcher's control, the second questionnaire was completed by nine of these students.

Data Collection and Analysis

Answers to the questionnaires were rated according to the SOLO taxonomy. With this taxonomy the answers were scored so that answers that showed more sophisticated levels of statistical thinking were given higher scores. A Rasch analysis (Bond & Fox, 2007) which simultaneously gives a score for both the items and individuals, was used to rank the students and items on all the items on the first questionnaire. Based on this analysis, the students were divided into three groups, above average, average and below average. These groups were then examined to see if there was any pattern in the type of response according to ability. After the second questionnaire, the answers to the Hospital Problem were then examined to see how these answers may or may not have changed.

Results and Discussion

With the Rasch analysis the Hospital Item showed misfit ($z = 5.4$), suggesting that the students were using a different form of reasoning for this question than for the other items in the questionnaire. Using the Rasch rankings, the students were divided into three groups, above average (0.9 logits or above), average (-0.13 to 0.75 logits), and below average (-0.27 logits and below). The overall responses to the Hospital Problem in the first questionnaire are summarised in Table 2. There was one non-response to this question.

Table 2

Pattern of Response to the Hospital Problem

	Previous study of statistics	
	Yes	No
Hospital A (incorrect), n = 3	3	0
Hospital B (correct), n = 12	5	7
Equally likely (Incorrect -main inappropriate conception), n = 10	9	1
Total	17	8

When asked to explain their answers, all the students who chose Hospital B used a form of reasoning that showed that they recognized that it was more likely, or as some stated, "easier", for Hospital B to deviate from the 1:1 ratio. For example: It is more likely that 8/10 will be female than 40/50, as it only requires that 3 births are female instead of the probable male births, instead of 15 that is needed to be female.

All the students who chose Hospital A used a form of reasoning that suggested that as there are more births to choose from, it is more likely that there will be more female births. These answers are similar to those noted by Watson and Moritz (2000) for the grade 6 and 9 students. For example: Because Hospital A has more births each day than Hospital B, it is likely that there will be more female births too.

Two of the students who chose the equally likely option used reasoning that involved proportions or ratios. For example: Percentage is independent of the total number of births, it is a proportion.

The other students who chose this option used reasoning that involved the constant probability for each individual outcome. For example: The likelihood of gender is individual to the delivery not on the hospital and the number the hospitals deliver.

The responses according to student ability and the form of reasoning used are summarised in Table 3.

None of the above ability students chose Hospital A or used proportional reasoning. The least sophisticated reasoning, that is there are more births in Hospital A therefore more girls would be born, was evenly spread between the average and below average groups. All of these students who chose Hospital A had stated that they had been exposed to statistics in their school mathematics. Of all the students who chose Hospital B, only one, who was in the average group, specifically mentioned the effect of increasing sample size: A large amount of births will allow the average of boys to girls to even out. Hospital B has a lower amount of births and has a higher chance of reaching 80% female.

Table 3

Types of Responses Used in the Hospital Problem Grouped by Students' Ability

		Type of reasoning used			
Grouping	Answer	More likely for Hospital B	Independence of each single birth	Proportional reasoning	More births in hospital A
Above average	Hospital A				
(n = 9, 1 no answer)	Hospital B	4			
	Equally likely		4		
Average	Hospital A				2
(n = 12, 1 no answer)	Hospital B	6			
	Equally likely		2	1	
Below average	Hospital A				1
(n = 5)	Hospital B	2			
	Equally likely		1	1	
	Total	12	7	2	3

In the last week of semester the students were then given a second questionnaire that also included the Hospital Problem. Responses were available for nine students. The responses and types of reasoning used by these students in both questionnaires are described in Table 4.

Four out of the five students who had initially chosen the equally likely option had now chosen Hospital B so now all except one student now gave the answer of Hospital B. Of interest is that the ability of the student who gave this exceptional response was rated as above average. The student quoted earlier, who acknowledged that the expected statistic will be approached with a higher sample size, stated a similar argument in the second questionnaire, whereas one student (in the above average group) also now acknowledged the effect of an increasing sample size.

There was no specific intervention to address the inappropriate reasoning displayed in this question, but during the statistics unit the students did study probability. During this module the students were introduced to the definition of probability in terms of long term frequencies, and the length of run it might take for coin tosses to reach a 1:1 ratio was discussed. The students had also used the binomial distribution for the calculation of probabilities. While no definite conclusions can be drawn from this small number of students, it is encouraging that some students were able to make the transition from purely proportional reasoning to consideration of sample size.

Table 4

Comparison of Reasoning Used in the Hospital Problem in Questionnaires 1 and 2

		Questionnaire 1		Questionnaire 2	
Person	Grouping	Answer	Reasoning	Answer	Reasoning
1	A. average	Equal	Constant probability	Equal	Constant probability
2	A. average	B	More likely for B	B	Effect of larger sample size
3	Average	A	More births, more girls	B	More likely for B
4	Average	Equal	Constant probability	B	More likely for B
5	Average	Equal	Constant probability	B	More likely for B
6	Average	B	More likely for B – effect of larger sample size	B	More likely for B – effect of larger sample size
7	Average	A	More births, more girls	B	More likely for B
8	Average	B	More likely for B	B	More likely for B
9	Average	B	More likely for B	B	More likely for B

Conclusions and Recommendations

It is apparent that with the right experience, students can move from proportional reasoning only to reasoning that allows for the effect of sample size. It is of interest that in the initial questionnaire all three students who gave the larger hospital as their response, and nine out of the ten who said the two hospitals are equally likely, had all studied statistics in a previous course. In contrast, seven out of the 12 correct responses came from students who had not studied statistics previously. It could be inferred that the previous studies in statistics had, at the least, not helped in their reasoning. It would be of interest to determine the details of the students' previous mathematical experience to see if the dominance of proportional reasoning comes about by the students misconstruing course content, or whether this dominance has come about purely by lack of experience with sampling.

The unit completed by the participants in this study was an applied statistics unit, and required a lower level of mathematical ability than statistics units that are theoretically based. It would also be of interest to see how students in theoretical statistics courses which require this higher level' of mathematical ability, can apply their theory to practical problems such as this.

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The Identification of Partially Correct Constructs

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We show how the RBC model for abstraction in context can be used to follow the emergence of a learner's knowledge constructs and to identify in detail the learner's partially correct constructs (PaCCs). These PaCCs are used to explain the learner's inconsistent answers and provide added insight into processes of knowledge construction. The research process is illustrated by means of an example from elementary probability. We thus demonstrate the analytic power of the RBC model for abstraction in context.

Abstraction has been a central issue in mathematics and science education for many years. The classic work by Piaget, Davydov, Skemp and others has in recent years been succeeded by research fora, symposia and discussion groups at various conferences, as well as several special issues of research journals, most recently the *Mathematics Education Research Journal* (Mitchelmore & White, 2007).

One of the approaches to research on abstraction presented on these occasions is *abstraction in context,* or AiC (Hershkowitz, Schwarz & Dreyfus, 2001). This approach considers abstraction as a process of emergence of knowledge constructs that are new to the learner. In order to describe such processes at a fine-grained level, abstraction in context makes use of a model, the RBC model, which is based on three epistemic actions to be described below. The RBC model has been used for this purpose by different research teams with students of different ages learning about different mathematical topics (including square roots, algebra, probability, rate of change, function transformations, and dynamical systems), in a variety of social and learning contexts (see e.g., Hershkowitz, Hadas, Dreyfus, & Schwarz, 2007, and references therein).

Unsurprisingly, students' emerging knowledge constructs may be less complete or less correct than required in the particular situation, and the model has been used to describe the emergence of such *partially correct constructs*, or PaCCs (Ron, Dreyfus & Hershkowitz, 2006). The aim of this paper is to show the analytic power of the RBC model by demonstrating that, and how, it can be used for identifying PaCCs.

Abstraction in Context

Freudenthal has brought forward some of the most important insights to mathematics education in general, and to mathematical abstraction in particular, and this has led his collaborators to the idea of "vertical mathematization" (Treffers & Goffree, 1985). Vertical mathematization points to a process of constructing by learners that typically consists of the reorganization of previous mathematical constructs within mathematics and by mathematical means. This process interweaves previous constructs and leads to a new construct.

AiC adopts this view and defines abstraction as a process of vertically reorganizing previous mathematical constructs within mathematics and by mathematical means so as to lead to a construct that is new to the learner. The genesis of an abstraction passes through a three stage process, which includes the arising of the need for a new construct, the emergence of the new construct, and its consolidation. The need may arise from the design of a learning activity, from the student's interest in the topic or problem under consideration, or from a combination of both; without such a need, however, no process of abstraction will be initiated.

We note that this view of abstraction follows van Oers (2001) in negating the role of decontextualisation in abstraction, and embraces Davydov's dialectic approach (1990) in that it proceeds from an initial unrefined first form to a final coherent construct in a dialectic two way relationship between the concrete and the abstract (see Hershkowitz et al., 2001; Ozmantar & Monaghan, 2007).

Furthermore, we found that activity theory (Leont'ev, 1981) proposes an adequate framework to consider processes that are fundamentally cognitive while taking into account the mathematical, historical, social and learning contexts in which these processes occur. In this, we follow Giest (2005), who considers activity theory as a theoretical basis, which has an underlying constructivist philosophy but allows avoiding a number of problems presented by constructivism.

According to activity theory, outcomes of previous activities naturally turn to artefacts in further ones, a feature which is crucial to trace the genesis and the development of abstraction throughout a succession of activities. The kinds of actions that are relevant to abstraction are *epistemic actions* – actions that pertain to the knowing of the participants and that are observable by participants and researchers. Pontecorvo & Girardet (1993) have used this term to describe how children developed their knowledge on a historical issue during a discussion. The observability is crucial since other participants (teacher or peers) may challenge, share or construct on what is made public.

The RBC model

For the above reasons, Hershkowitz et al. (2001) have chosen to use epistemic actions in order to model the central second stage of the process of abstraction. The three epistemic actions they have found relevant and useful for their purposes are Recognizing (R), Building with (B) and Constructing (C). *Recognizing* takes place when the learner recognizes that a specific previous knowledge construct is relevant to the problem he or she is dealing with. *Building with* is an action comprising the combination of recognized constructs, in order to achieve a localized goal, such as the actualization of a strategy or a justification or the solution of a problem. The model suggests constructing as the central epistemic action of mathematical abstraction. *Constructing* consists of assembling and integrating previous constructs by vertical mathematization to produce a new construct. It refers to the first time the new construct is expressed by the learner either through verbalization or through action. In the case of action, the learner may but need not be fully aware of the new construct. Constructing does not refer to the construct becoming freely and flexibly available to the learner. Becoming freely and flexibly available pertains to the third stage of the genesis of an abstraction, consolidation. Examples for the three epistemic actions will be given below, in the subsection entitled “Data and Interpretation”.

This description of the model raises the question what it is that is being constructed, what constructs the researchers focus on. In some cases, the answer to this question is rather immediate and obvious. In other cases, it requires an in depth analysis of both, the design of the curriculum and the learners’ actions. We usually begin by an a priori analysis of the instructional design (in which the designers may participate), which aims at identifying the intended constructs. These constructs are thus not absolute but relative to an instructional activity, a learning unit, or a curriculum. They can be mathematical concepts, methods, strategies, and so on. We call them mathematical *principles*. The RBC analysis of the transcripts of the learning events then focuses mainly on these principles, with the researcher’s task being to identify specific learner actions as R- or B-actions with these principles and to identify sequences of R- and B-actions as C-actions of these principles. This analysis may lead to a modification of the list of principles, for example by including alternative constructs made by the learners.

The RBC model constitutes a methodological tool used for realizing the ideas of abstraction in context. In this sense, it has a somewhat technical nature that serves to identify learner actions at the micro-level. On the other hand, the model also has a definite theoretical significance; Hershkowitz (2007) has discussed the theoretical aspects of the model, its tool aspects, and the relationships between them.

Partially Correct Constructs

Ron, Dreyfus and Hershkowitz (2006) have studied cases where students’ incorrect answers overshadow meaningful knowledge they have constructed, or cases in which students’ correct answers hide knowledge gaps. Both cases may indicate knowledge constructs that are partly but not fully appropriate to the mathematical problem situation at hand. Thus, students’ constructs that only partially fit the mathematical principles underlying the learning context have been called *Partially Correct Constructs*, or PaCCs. We stress that PaCCs refer to the principles and thus are relative to the problem situation, the activity, and the curriculum at hand, as well as to the researchers’ decision to focus on some knowledge aspects rather than others. PaCCs can be used as tools for interpreting situations in which some answers or actions of a student are inconsistent with others.

Consider, for example, the task in Figure 1. It is expected (see e.g., Nilsson, 2007) that many students predict Ruti to win in question 1b, basing their analysis on a 21 event sample space. In a sample of 120 grade 8 (age 13) students, it turned out that 57 students indeed answered thus. However, almost as many students (35) predicted Ruti to win in spite of answering in 1c that there are 36 possible outcomes. The answers “Ruti wins” and “the sample space has 36 events” appear to be inconsistent.

Such inconsistencies call for explanations. The RBC model for abstraction in context allows the researcher to follow a student’s knowledge constructing process in order to identify the student’s constructs with respect to the principles underlying the learning context. Thus, the micro analytic nature of the RBC analysis can help us answer the question, which student constructs are partially correct and how they led to the inconsistent answers.

Identification of Partially Correct Constructs: A Case Study

In this section, we will use the RBC model in order to follow the knowledge construction process of one grade 8 student, Roni, working with his peer, Yam, on the task in Figure 1.

1a	Yossi and Ruti roll two white dice. They decide that Ruti wins if the numbers of points on the two dice are equal, and Yossi wins if the numbers are different. Do you think that the game is fair? Explain!
1b	The rule of the game is changed. Yossi wins if the dice show <u>consecutive</u> numbers. 2 and 3 are consecutive numbers, or 23, 24 and 25 are consecutive numbers. Do you think the game is fair?
1ci	How many possible outcomes are there when rolling two dice? An outcome of rolling two dice is, for example, .
1cii	Reconsider your answers to tasks 1a and 1b: Are the games fair?
1d	If Yossi and Ruti play with one red die and one white die, does this change the answers to 1a, 1b and 1c?

Figure 1. A probability task (translated from Hebrew).

Methodological Considerations

Because of space limitations, we keep this subsection short.

A ten lesson elementary probability unit was designed for grade 8 (age 13) and experimentally taught to six student pairs in laboratory conditions and to seven classes. In each of these classes, one or two focus groups were video-taped throughout the unit, Roni and Yam among them. We refer the reader to Hershkowitz et al. (2007) for more details about the choice of topics and the design of the unit. Here we only stress that the activity based on the task in Figure 1 was the first activity in which the students had an opportunity to deal with a two dimensional sample space.

Principles

We recall that the principles are relative to the design, in our case to the design of the task in Figure 1. Our analysis led to four principles needed to solve the task, namely simple event, sample space, compound event and probability value. While these principles may seem natural and almost trivial to the reader, they are not operational for the researcher who wants to trace the construction of knowledge. For this purpose, we need to decompose the principles into elements that are defined operationally so that we can tell from students’ actions whether or not they are using these elements. A full treatment of the principles would need more space than available here; a somewhat partial version follows.

The *simple event* principle: A simple event in two dimensional sample space is an ordered pair of one dimensional simple events, one for each dimension. The simple event principle thus has two elements *pair* and *order*, which are operationally defined as follows: We shall say that students have constructed the pair element when they relate to the occurrence of the two one dimensional simple events as to a single event. We shall say that students have constructed the order element if they relate to pairs like (,Δ) and (Δ,) as to two distinct events. The order element can be constructed only if the pair element has been constructed.

The *sample space* principle has three elements, one of which is *systematic counting*; another one is the *product* element: The number of events in a two dimensional sample space is the product of the numbers of the events in the two relevant one dimensional sample spaces. We shall say that students have constructed this element if they calculate the number of all the events in the sample space by the appropriate multiplication.

The *compound event* principle has two elements, one of which it *systematic counting*.

The *probability value* principle has a single element, *ratio*. We shall say that students have constructed the ratio element if they calculate the probability value as the ratio between the number of simple events in the appropriate compound event and the number of simple events in the sample space.

The principles are connected as follows: the probability value principle contains the compound event and the sample space principles as elements, while the compound event and the sample space principles each contain the simple event principle as an element. Thus each principle in fact contains up to two more elements than enumerated above.

Data and Interpretation

Roni is one of the 35 students in the above sample who correctly answered that there are 36 possible outcomes when rolling two dice. On the other hand he claimed, regarding questions 1b and 1cii (Figure 1), that Ruti has a better chance than Yossi to win the game because she wins with 6 possible outcomes, while Yossi wins only with 5 possible outcomes.

Already while working on question 1a, Roni declared

Roni	7	Each number can come out five [times], true? In order for him to win [with] two: five more; three: five more; four: five more; five: five more; six: five more. Multiply six by five. And six times, six possibilities for Ruti.
Roni	9	One with one, two with two, three with three, four with four, five with five, six with six. That's six out of 36. So let's write it.
Roni	10	[Dictating to Yam] The game is not fair because out of 36 possible outcomes, 30 times Yossi will win and only six times Ruti will win. And in brackets write "on average".

In the interpretation of these (and the following) data, we will rely on the three epistemic actions of the RBC model, recognizing, building with, and constructing. In his answers to question 1a Roni shows evidence for having constructed several of the knowledge elements listed above, in the context of the specific task he is working on, including the pair element: In R9 he explicitly refers a pair of one dimensional outcomes as a simple event to be counted. In his further work, Roni consistently recognizes the pair element and builds with it. In R7 we also have evidence of having constructed the systematic counting elements both, for the sample space size and for the relevant events that are included in the compound event. We note that Roni found the number of possible outcomes by calculating a product (R7). In R9 and in the written answer he refers to the probability to win the game as the ratio between the number of simple events of the compound event and the number of the events in the sample space. Roni had constructed the ratio element at an earlier stage, when dealing with one dimensional sample spaces. Now he recognizes this knowledge element as relevant for the new context of two dimensional sample space, and builds his calculations with it. However, so far we have no evidence about Roni constructing the order element.

Roni and Yam proceed to question 1b. Yam speaks first, spelling out five pairs of consecutive numbers. Roni soon dictates the common answer.

Roni	29	Write: The game is not fair because out of …
Roni	31	… eleven possible outcomes, six times …
Roni	33	Six times …, write in parentheses "on average", Ruti will win, versus, versus five times, that Yossi will win.

Roni did not use the order element. While Yam was the first to count only five of the ten consecutive pairs, Roni did not correct him, and dictated their common answer. We further note that, for the second time, Roni stressed the meaning of probability as an average behaviour, thus showing a deep understanding of the probability concept.

The students turn to question 1c. Despite the fact that Roni has talked about 36 possible outcomes earlier, he now stops to think again, but when Yam starts enumerating events, Roni stops him and declares

Roni	38	Every one has six. Thirty six possibilities. Every number six times.

We can see that Roni has constructed the product element. He correctly determines the number of possible outcomes without referring to each of them.

The next question asks them to reconsider their decisions whether the games are fair. Roni immediately says that the games are not fair. He explains why to the teacher who happens to approach their desk and asks for clarification. Roni's explanation is a repetition of his earlier claim. When approaching question 1d, Roni soon stops to reflect.

Roni	69	The same.
Yam	70	The same.
Roni	71	The same but here there is a red die. Here [points to the white pair of dice] if you get five, two, it is impossible to then get two, five. This is considered the same thing. But here [points to the red/white pair of dice] it is possible. If you get, say, five red and two white, maybe it's possible five white and two red.

The written answer that Roni dictates indicates that he may now have constructed the order element in one context:

Roni	79	There will be a different answer because it can be 5 on the white die and 2 on the red one; and the opposite is considered a different answer.

However, even now that Roni is aware of the existence of pairs of symmetric events, at least in the context of dice with different colours, he does not connect this awareness to the game of Yossi and Ruti who play with a pair of white dice, and does not change his mind regarding the fairness of the game in question 1b.

We note in passing that Roni's explanation in 71 and 79 clearly shows that his count of 5 consecutive pairs is not due to the fact that he considered only increasing pairs like (2,3) as consecutive and not decreasing ones like (3,2); rather his count of 5 is due to explicit identification of symmetric pairs in the case of dice of the same colour. We have analogous evidence about other students' reasons for counting 5 rather than 10 consecutive pairs; and we have no evidence of any students having discounted decreasing pairs.

We further note that Roni's construction of the order element is an example for the role of context in processes of abstraction. Here, the context is mathematical, but social context or learning context such as technological tools have been shown to be equally crucial to processes of abstraction (see e.g., Hershkowitz et al., 2007).

Analysis and Discussion

We defined PaCCs as constructs that only partially match the relevant underlying mathematical principles and their elements. Thus constructs are considered to be partial with respect to the mathematical principles that underlie a specific learning unit, design, or activity. A mathematical principle is then considered a PaCC if only some of its elements have been constructed and are recognized as relevant when appropriate, whereas others may be lacking or may themselves be PaCCs. The analysis leading to the identification of PaCCs now consists of tracing every knowledge element from the time it was first constructed through recognizing it as relevant for other contexts, and through building with it further. The RBC analysis is thus used as a tracer for identifying cases of partial match between a student's construct and the corresponding mathematical principle.

In order to carry out the analysis in the specific case at hand, we consider Roni's constructions concerning the four principles underlying the current stage of the learning design. Concerning the simple event principle, the data show that at an early stage of his learning process Roni constructed the pair element without the order element. While he worked on question 1d, he constructed the order element for the context of different coloured dice. However, even then, he claimed that when playing with two white dice, one should not consider (2,5) and (5,2) as different outcomes. We thus consider Roni's construct for simple event as a PaCC since the order element remained unconnected in the context of question 1b.

Since there is no space here for a more detailed analysis, we summarize the discussion of the other three principles as follows: the other three principles all inherit the property of being PaCCs from the simple event principle. Since simple event is an element of sample space and simple element is a PaCC, sample space is a PaCC as well; the same argument holds for compound event, and a similar but less direct argument holds for the probability value principle.

As one can see, the only knowledge element that was not constructed by Roni until a rather late stage is the order element. This element remained disconnected until the end of the work on Task 1. The order element is an element of the simple event principle. However, not only the simple event principle is considered as a PaCC, but also all those principles that are built on it are considered as PaCCs because they inherit the PaCC property from it. An example of the expression of the lack of the order element in a higher principle is Roni's determination of five consecutive pairs in the compound event "Yossi wins".

Conclusion

By following students' knowledge construction processes at a micro-level, the RBC model for abstraction in context can be used to explain students' inconsistent answers via the notion of partially correct constructs. Roni's case has been presented above. Similar analyses of other students' constructs for the order element as well as of Roni's constructs for more sophisticated elements of probabilistic knowledge, specifically for the area model, have met with analogous success in explaining inconsistent answers.

These results have been achieved by using the epistemic actions of the RBC model as tracers. The use of the epistemic actions as tracers is thus an efficient tool for identifying the nature of PaCCs. Using the epistemic actions as tracers requires using them in the order CRB: Once there is evidence of construction, the construct is traced to see whether the student recognizes and builds with it in later situations, where it is relevant. The student's failure to recognize and build with the construct where it is relevant indicates a PaCC. In the specific example described above, Roni builds with the product principle without being bothered whether he counts ordered or non-ordered pairs. He did construct the order principle when working on question 1d, but only for a narrow context in which one can distinguish between the two dice and hence assign an order to the pair of dice. In the game of question 1b, however, the dice are not distinguishable. Roni does not recognize the order principle as relevant to that context and, therefore, does not experience any need to reconsider the fairness of the games, even when asked to do so.

The RBC model of abstraction in context has thus been proved its power as an analytic tool that allows the researcher not only to identify a student's PaCCs but also to determine the precise nature of these PaCCs in terms of knowledge elements that have been constructed and others that have not.

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Making Connections: Promoting Connectedness in Early Mathematics Education

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Prominent policy guidelines suggest that effective mathematics pedagogy assists students to "make connections" between various types of mathematical knowledge and between mathematical knowledge and real-life phenomena. This paper describes the perspectives and practices of two early years teachers involved in a reform that encourages teachers to help students make connections between forms of disciplinary knowledge and between disciplinary knowledge and real-life experience. The purpose of this description is to reveal strategies that assist students learn to make mathematical connections.

Helping students learn to make connections between various forms of mathematical knowledge, as well as between mathematics and real-life experience, is increasingly recognised as integral to effective mathematics learning and teaching. Internationally, the *Principles and Standards for School Mathematics* (National Council of Teachers of Mathematics, 2000) state that mathematics instruction should enable students to:

- recognise and use connections among mathematical ideas;
- understand how mathematical ideas interconnect and build on one another to produce a coherent whole;
- recognise and apply mathematics in contexts outside of mathematics.

The *Connected Mathematics Project*, a prominent North American mathematics curriculum based on the Principles and Standards, includes among its fundamental themes an emphasis on "significant connections, meaningful to students, among various mathematical topics and between mathematics and problems in other disciplines" (Lappan, Fey, Fitzgerald, Friel, & Phillips, 2002, p. 1). In Australia also, the requirement that effective mathematics pedagogy should help students learn to make connections between various forms of disciplinary knowledge and between disciplinary knowledge and real-life experience is embedded in policy guidelines. For example, the Queensland Government's Framework for Action 2007-2010, *Numeracy: Lifelong confidence with mathematics,* states that teachers of mathematics should support students to develop "the confidence to choose and use mathematics skills they learn at school in everyday life" (Education Queensland, 2007, p.1). In other words, there is broad consensus that in order to become numerate, students must become competent in perceiving the connections between mathematics and other forms of knowledge and between mathematics and their lived experience, as well as competent in applying the mathematical knowledge necessary to maximise the productivity of such connections.

The consensus that students should learn how to make mathematical connections does not extend to agreement about the ways in which students should be supported to do this. While current policy guidelines and reform-oriented curricula advocate the use of problem solving and discussion-oriented pedagogies, the ability of such pedagogies to support all students to become numerate is contested (Lerman & Zevenbergen, 2004). Lerman and Zevenbergen (2004) describe a small but important body of research (e.g., Cooper & Dunne, 2000; Lubienski, 2000) that suggests working class students tend to experience greater difficulty completing real-life mathematical tasks than their middle class peers. Various styles of questioning have also been shown to mediate student participation in mathematics education and ultimate access to mathematical knowledge in line with social class (Lubienski, 2004; Zevenbergen, 2000). Rather than dismiss the worth of problem solving and discussion-oriented pedagogies to students from working class backgrounds, more work is needed to establish how teachers can enact such pedagogies so that students from a range of backgrounds can both learn mathematics and learn how to connect mathematics learned with learning in other subject areas and with their real lives.

A recent Queensland education reform, the *New Basics* (Education Queensland, 2000) advocates the utilisation of a set of 'Productive Pedagogies' designed to help students make connections between various forms of disciplinary knowledge and between disciplinary knowledge and real life. These 'Productive Pedagogies' are presented as necessary enablers underpinning students' effective participation in curriculum and assessment oriented towards the solution of complex real-life problems. Four of the model's 20 Productive Pedagogies, those grouped under the category of "connectedness", are specifically concerned with helping students learn to make connections (Department of Education, 2002, p. 20).

The New Basics trial concluded in 2003. However, most of the 38 schools involved in the trial proper as well as another 21 schools involved in a second phase of the model's implementation continue to espouse a New Basics philosophy. Investigation of the practices of teachers in these schools has the potential to reveal how students can be supported to make connections between mathematics and other forms of disciplinary knowledge and between mathematical knowledge and real-life experience when engaged in curricula oriented towards problem solving. This paper describes research examining the perspectives and practices of two teachers in a New Basics school, drawing on concepts from Bernstein's (2000) theorisation of pedagogic discourse to reveal how each teacher conceptualises the importance of learning to make connections within mathematics education, as well as the ways in which they assist students to make connections via their teaching of mathematics.

Theoretical Framework

The concept of pedagogic discourse is at the heart of the theoretical model developed by Bernstein (1975; 1990) to explain how schools are implicated in the reproduction of inequitable social relations. Pedagogic discourse is the principle, or "rule", by which knowledge is translated into a form useable for transmission by a teacher to a learner (Bernstein, 1990, p. 183). Such translation or 'recontextualisation' is achieved via pedagogic discourse's two constituent parts: Regulative discourse and instructional discourse (Bernstein, 2000). Regulative discourse establishes the social order within the classroom, by setting norms of conduct, character and manner (Bernstein, 2000). Instructional discourse sets the parameters for particular knowledge discourses by determining the selection, sequencing, pacing and evaluation of what will be considered legitimate knowledge (Bernstein, 2000). Regulative discourse is the dominant of the two discourses, as messages about what constitutes legitimate knowledge are always carried on contained within messages about what constitutes appropriate behaviour. Utilisation of the concept of pedagogic discourse, particularly its reliance on regulative discourse to transmit instructional discourse, allowed the research described here to examine how teachers went about setting norms for what would be considered legitimate knowledge in their classrooms, and how these norms were communicated through the social practices of their classes.

More in-depth description of the features of instructional and regulative discourses circulating within the classes studied is supported by the utilisation of Bernstein's (1975, 1990) concepts of classification and framing. Classification refers to the degree to which boundaries between knowledge discourses are maintained (Bernstein, 1990, 2000). When describing the degree to which a discourse is separated from others, Bernstein (1990) refers to the 'strength' of its classification. A discourse that is strongly classified is one well insulated from others, and therefore viewed as highly specialised. A weakly classified discourse is one whose boundaries are fuzzier and less clearly defined. Framing is the principle through which classificatory relations between knowledge discourses are communicated to students (Bernstein, 1990, 2000). Framing determines both the social and discursive information that is communicated within a particular pedagogy. Like degrees of classification, the nature of the framing within a particular discourse is also typed according to its strength (Bernstein, 1990, 2000). A strongly framed pedagogy is one in which the selection, sequencing, pacing, and evaluation of instructional material is kept under strict control by the teacher. A more weakly framed pedagogy is marked by more relaxed communicative practices. Selection, sequencing, pacing and evaluation are negotiated with learners in an attitude of exchange. The concepts of instructional and regulative discourse, along with classification and framing, are used here to facilitate interrogation of research data and facilitate explanation of how observed classroom practices supported students to make mathematical connections.

Research Design and Approach

Research data described in this paper were collected during an explanatory case study (Yin, 2003) of two Year One/Two classes in a school that identifies itself as a New Basics school, Mirabelle State School. Mirabelle is a small inner-city school with approximately 200 students from mostly middle class and low SES students, with a considerable number of students from culturally and linguistically diverse backgrounds. The two teacher participants had formed a co-operative arrangement, whereby on two days each week their classes were divided. All Year Ones worked with Mrs Kelly on those days, while all Year Twos worked with Mrs Roberts. This had the effect of ensuring that all students in Years One and Two worked extensively with both teachers over a two year period. During the study, both teachers' mathematics teaching was observed and recorded over a period of 14 weeks, first with ethnographic field notes and later using video-tape. Teachers were observed while teaching their own Year One/Two classes as well as when they taught the single year level group made up of students from both classes. Class teachers were also individually interviewed on three or four occasions for at least one hour. During each individual interview, teacher participants responded to a key question related to their perspectives on the purpose and/or appropriate conduct of mathematics education. The main ideas from the teachers' responses were recorded on chart paper to form concept maps (Leiken, Chazan, & Yerushalmy, 2001; Novak, 1998). The ultimate content and design of each concept map was negotiated with teacher participants until they felt that maps produced accurately represented their views. The concept mapping technique enabled the interviewer to prompt and probe the teachers' responses to produce rich conversations which were transcribed as interview data and analysed along with the concept maps. The scope of this paper is limited to certain preliminary conclusions drawn from the inductive analysis of interview and observational data, which revealed patterns in how the teacher participants, Mrs Kelly and Mrs Roberts, helped students learn to make mathematical connections.

Making Connections: Teachers' Perspectives

Both of the teacher participants in the study viewed helping students learn to make connections, or connectedness, as fundamental to mathematics education. When asked "What should children learn in maths at school?" Mrs Kelly explicitly named connectedness as the primary purpose of mathematics education, situating her belief in the importance of connectedness within her commitment to the Productive Pedagogies in general.

Researcher: If you were explaining to a student teacher, what is it that children have to learn in maths at school, what would you say?

Mrs Kelly: Well, the first thing would be, going back to the Productive Pedagogies, is the connectedness to the world. So it's um, they um, they should learn that maths is um…not something isolated, but something, um, that is a part of everyday living. And then, learning how to use maths as part of their everyday living.

The second teacher, Mrs Roberts, also talked about how mathematics education should equip students to see and understand how mathematics is related to everyday tasks.

Mrs Roberts: Can you see what I'm saying? They've got to see that counting out the forks and knives, and putting one-to-one correspondence and stuff, to see that that is maths. Because, you know, they'll come back and say "But that's not maths, that's just setting the table."

Researcher: So they've got to see that maths is useful and they've got to see the maths …

Mrs Roberts: … in everyday things. Yes. Yes … …That it's not a separate entity. That's it's not something you just pack away in the corner. Oh, we've got to learn [it] at school but it's got no function.

Concept maps produced during individual interviews illustrated the teacher participants' view that learning to make connections is fundamental to mathematics education, as evidenced by the example of Mrs Roberts' concept map provided in Figure 1.

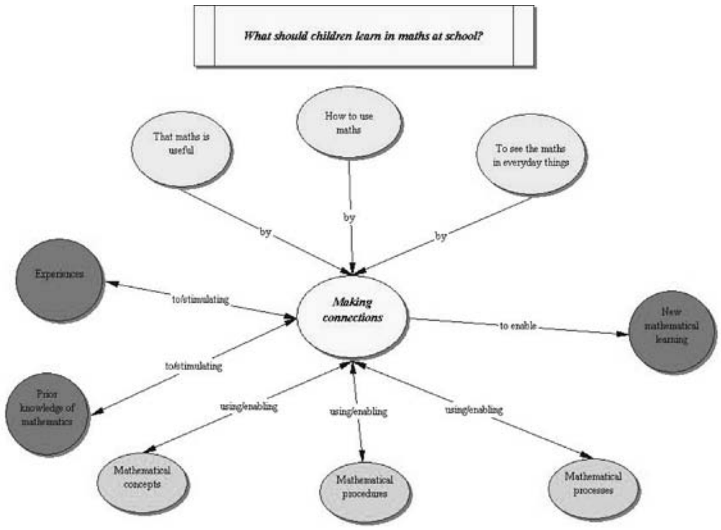

Figure 1. Mrs Roberts' concept map: What should children learn in maths at school?

Making Connections: Teachers' Practices

Observation and analysis of Mrs Kelly's and Mrs Roberts' teaching revealed a number of ways in which they supported students learn to make mathematical connections. In this excerpt from a lesson transcript, Mrs Kelly sets students the task of creating a symmetrical representation of a bat. Her presentation of the task and the way in which she responds to student suggestions creates the conditions for students to identify and capitalise upon connections between the mathematics task at hand and learning in other subject areas.

Mrs Kelly: Now, think back to what symmetry is. I want you to draw or cut out a bat and make sure it is symmetrical. How do you think you might be able to do that?

Robin: When you do one side, and you're up to the other side, you'd just copy.

Mrs Kelly: So, you'd do one side at a time.

Robbie: I would fold it in half so I could cut out half a bat … .We did it in art before.

Michael: I would draw an ear, then the other ear.

Mrs Kelly: So, you'd go from side to side. Anyone know how you could use your grid book to make sure your bat was symmetrical?

Will: Drawing a wing, count how much squares.

Mrs Kelly: Oh, looking at the area covered? Your challenge now is how are you going to go about producing a bat that is symmetrical?

This transcript shows that one student, Robbie, made a connection between the mathematics task at hand and a skill learned in Art. Soon after this exchange, Robbie again discussed his plan to utilise learning from another subject area to enhance his ability to complete a mathematical task:

Mrs Kelly: Can you do it in your grid book first, then on black paper? Some people sometimes like to try two different things to see which one works for them.

Robbie: I'm going to do it like in Art.

Mrs Kelly: Can you see how other subjects sometimes help us? What we've learned in Art is now helping us in maths.

By endorsing Robbie's plan to draw on a skill learned in Art to help complete a mathematics task, Mrs Kelly made it clear to students that they could profitably work across disciplinary boundaries without compromising the boundary strength (classification) of each discipline, and that such inter-disciplinary work could enhance their effectiveness as mathematicians. The opportunity for this learning arose because Mrs Kelly had not dictated the procedures by which students should create their bat, but indicated that they must select an appropriate procedure themselves. This is an example of a weakly framed pedagogy, in that the teacher has relaxed selection of available procedures and also relaxed the lesson's pacing to allow students time to select and test a number of procedures. However, it was certainly not a case of 'anything goes'. At the beginning of this lesson, Mrs Kelly told students that they would later be required to explain and justify their choice of method to others in the context of a sharing circle. By asking students to suggest possible ways of approaching the task early in the lesson, and then pointing out the mathematical aspects of each method, Mrs Kelly modelled how methods selected could be justified within the specialised language of mathematics. When the sharing circle was convened later in the lesson, students' ways of working were evaluated by the teacher and others against mathematical criteria (e.g., "So, is your bat symmetrical?"), acting to reinforce the boundary strength of mathematics. The regulative discourse circulating within this class was one which valued choice, reflection, and discussion. As a consequence, the instructional discourse conveyed the message that mathematical problem-solving can proceed through hypothesis testing, developing justifications and proofs that draw on a wide range of ideas - provided those ideas are re-framed for communication within the specialised language of mathematics.

Mrs Roberts also occasionally enacted a weakly framed pedagogy to allow students to explore connections identified between mathematical learning and real life. Here, in a lesson during which Mrs Roberts had planned to cover how times that are "half past" the hour are represented, Mrs Roberts relaxed her pacing to pursue an alternate mathematical topic introduced by students related to their real life experience.

Mrs Roberts: How does that sound? So, it's halfway. So, if my big hand was here. Where must that big hand be, to have a half past time, Alan?

Alan: On the six.

Mrs Roberts: Which numbers tell you that it's halfway?

Lex: Something, then the number thirty.

Mrs Roberts: Yeah, but why's the 30 underneath the six? (Mrs Roberts holds up her arm so that the students can see her watch) I don't have…I do have numbers, but they're special numbers.

Kristy: From a different country, or from a Grandpa Clock.

Alan: My dad has those on his watch.

… Mrs Roberts stands and goes to the chalkboard. She writes the series of Roman numerals from I to X vertically on the board.

As she writes the children count aloud in unison from one to ten.

Mrs Roberts points to the I and asks the children what that numeral says. The children reply "One" . Mrs Roberts asks what makes them think that numeral means "One".

Fiona: 'Cause we're just counting on. 'Cause it's only got one line.

Mrs Roberts: So, what do you think this one means? (pointing to the V symbol).

Fiona: Five … …

Mrs Roberts: So, nobody knows what they are? (No one responds) No, I'm going to get you to go home and ask your mum and dad. Sometimes on a clock, you have these types of numbers and do you know what they are? We got a bit side-tracked from our halfway discussion.

While Mrs Roberts frequently relaxed the pacing of lessons to elaborate on connections made by students to prior learning or to real-life experience, she retained strong control over the selection and evaluation of material. Mrs Roberts' teaching included numerous episodes of 'drill and practice'. This reflects her view as represented in her concept map (Figure 1) that children must learn mathematical processes, procedures and concepts in school to enable them to make connections to real life experience. She sought to help students learn a repertoire of mathematical skills so that when confronted with a problem situation, they would have a number of ways of working 'at their fingertips' from which to choose. The regulative discourse circulating in this class valued conforming to established protocols, which carried the message (instructional discourse) that mathematicians used a number of established and accepted procedures to respond to problem situations.

Conclusion

The conviction expressed by Mrs Kelly and Mrs Roberts that making connections was fundamental to mathematics education influenced their teaching practice in a number of important ways. The effectiveness of their practice was evidenced by examples of students in classes taught by these teachers who demonstrated the ability to make connections between mathematical knowledge and other forms of disciplinary knowledge, and between mathematical knowledge and real life. The data presented in this paper suggest a number of strategies that, while not forming a complete practice guide, can be used by teachers to help students learn to make mathematical connections when engaged in problem solving:

1. Assist students to become competent in using a range of mathematical procedures.
2. Require students to select the knowledge and procedures that will assist with the solution of mathematical problems. Don't always tell students what procedures to use.
3. Expect students to explain and justify methods selected for working out problems.
4. Encourage students to draw on ideas from other disciplines or from their own experience when solving problems or recording their thinking.
5. Assist students to re-frame ideas or information from other disciplines or their own experience so that they are expressed using the specialised language of mathematics. Model how to do this when evaluating students' suggestions and chosen methods.
6. Respond positively when students themselves identify connections between diverse bodies of disciplinary knowledge, or between mathematical knowledge and real life.

Not all of these strategies were practised by both teachers to the same degree. While Mrs Kelly and Mrs Roberts utilised different styles of pedagogy, the effect of their co-operative arrangement was that students benefited from receiving explicit instruction about the boundaries and contents of mathematics as a result of Mrs Roberts' strongly classified and framed pedagogy, and were able to utilise this knowledge in the more weakly framed pedagogy utilised in Mrs Kelly's room.

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Engagement versus Deep Mathematical Understanding: An Early Career Teacher's Use of ICT in a Lesson

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This paper describes how an early career teacher used technology in a lesson about capacity utilising calculators, a bank of PCs and an Interactive White Board. Data analyses of the digital recordings of the lesson using a framework for classroom practice indicated mostly student-centred learning and opportunities to experiment and talk about their activities. However, there was limited evidence of deepened understandings of mathematics. Findings from this study warrant further investigation into the usefulness of a tool to refine classroom practice.

Since the late 1990s there has been increasing support for the integration of technology in the curriculum in policy documents. Yet, in 2005 many completing a preservice teaching course at an Australian tertiary institution indicated that they had rarely observed innovative uses of computers during their practicum. Indeed, many of the examples described were drill and practice type games for early finishers or lessons involving rotations of activities, one of which was a computer game, which was often unrelated to the topic being taught. None of these practices were consistent with the advice offered in their mathematics education coursework.

The case reported in this paper is from a small study investigating two research questions:

How is this novice teacher, who completed an Australian undergraduate primary teacher education degree, using technology in his/her teaching of mathematics?

- How effective is the teaching?
- ICT in School Teaching and Learning Contexts

In recent policy documents such as the *Victorian Essential Learning Standards* (VELS) (Victorian Curriculum Assessment Authority, 2006) greater recognition has been given to *Information Communication Technologies* (ICT). The domain is classified within the *Interdisciplinary Learning Strand* which encourages *ICT for Visualising Thinking, Creating and Communicating* (VCAA, 2006). There are various ICT applications and types of computer software available for teaching and learning contexts across disciplines. These applications range from programs which provide practice for a skill that has been previously taught to those which develop conceptual understandings through problem solving.

More specifically, there are several reasons for incorporating ICT into mathematics instruction. According to the National Council of Teachers of Mathematics "Electronic technologies …furnish visual images of mathematical ideas, they facilitate organizing and analysing data, and they compute efficiently and accurately. They can support investigations by students in every area of mathematics. When technology tools are available, students can focus on decision making, reflection, reasoning, and problem solving" (NCTM, 2000, p. 24).

Studies such as Clements and McMillen (1996) reported positive outcomes for students who used computer programs that were open ended, encouraged discussion and solving problems, and supported the development of conceptual knowledge. Similarly, Wall, Higgins and Smith (2005) used a template of a classroom scene with blank speech and thought bubbles to collect students' views of how ICT can be used to aid learning. The templates were used with groups of four to six students much like a focus group; however, students could complete their template in their own way: for example, some added extra bubbles or detail to the scene to illustrate the meaning. Wall et al. (2005) reported that students held positive views of using *interactive white boards* (IWB) for learning and identified eight subcategories for facilitating learning which included: understanding, concentration, students' use of IWB, present information, games, assists remembering, easier, and thinking process.

Wall et al. (2005) noted that the majority of students "commented on how the visual and verbal elements complemented each other and promoted effective learning" (p. 860) and that "many of the positive comments linked IWB use and mathematics with fun and games" (p. 861). Hence, Wall et al. suggested that IWBs may be useful tools for initiating and facilitating the learning process especially given students' views.

Reynolds, Treharne and Tripp (2003) challenged findings of studies which imply that the use of ICT causes higher levels of student achievement. Instead, they argued that often increased student outcomes are evident in schools where subject review and assessment procedures drive curriculum and ICT is integrated across teaching and learning areas; and, in cases, where ICTs are used by staff who have been provided with effective in-service training.

Similarly, Zevenbergen (2004) described two cases in which teachers from different socio-economic schools used ICT differently. In one example, the teacher, who worked in a school with students from socially disadvantaged backgrounds, clearly modelled the steps needed to complete the ICT related skills or activity however; there was no evidence of discussions between teacher and students linking the ICT activity with the conceptual understanding of the mathematics. In contrast, another teacher, who worked in a school with students from middle socio-economic backgrounds, modelled his thinking, used various conceptual representations and engaged students in meaningful discussions about their uses of ICT and mathematics.

Sutherland, Armstrong, Barnes, Brawn, Breeze, Gall et al. (2004) noted that many year 4 students already knew how to manipulate programs such as MS Excel even though they had never been taught at school. Hence, a mathematics lesson modelling the use of the MS Excel program did not address the students' needs. Their findings also indicated similarities in everyday classroom practices leading to the successful integration of ICT. In many cases, students were "engaged for sustained periods of time in activities that related to what the teacher intended to teach" (p. 422). In contrast many of the less successful cases of embedding ICT into subjects involved teachers who believed that simply using ICT and appropriate software would be sufficient. Sutherland et al. (2004) concluded that collective and critical discussions are essential for students to learn how integrating ICT tools contributes to the development of subject knowledge and subject culture, in other words, what is discipline-specific valued knowledge.

It seems that although in many cases schools are equipped with computers, they are not utilised to their full potential. Lerman (2004) reported that teachers were reluctant to use innovative activities with students who displayed behavioural issues. Furthermore, that "one does not often see innovative work using technology in any area of mathematics, with any groups of students" (p. 622). Instead, students who were perceived to have poor mathematics skills were often given more of the same activities of the kind that they had failed before.

Zevenbergen and Lerman (2006) reported findings from a three-year study involving six schools representing the diversity of Australian communities. The focus of the project was to "identify the ways in which ICTs were being used in the classroom" (p. 594). Zevenbergen and Lerman reported that there were varied levels of ICTs usage between the schools; however, there was relatively limited use of programs to support numeracy learning and that generally teachers indicated greater confidence in using ICT for literacy than for numeracy learning. They concluded that teachers need to feel more supported if they are able to use ICT in their classrooms. They argued that this was particularly important for students who enter schools with less ICT-experience than their peers.

Zevenbergen and Lerman (2007) used the *Productive Pedagogy* (PP) (Gore, Griffiths, & Ladwig, 2004) framework as a basis for data analysis for the use of ICT in upper primary classrooms. Data indicated that when teachers used ICT to support numeracy learning there were very low levels of quality learning potential. However, results were even lower when teachers used interactive whiteboards. Zevenbergen and Lerman (2007) concluded that "the use of interactive whiteboards actually reduces the quality of mathematical learning opportunities, provides fewer opportunities for connecting the world beyond schools, and offers little autonomous/independent learning opportunities for students" (p. 859).

Zevenbergen and Lerman (2007) also posited that as a consequence of teachers using the pre-prepared lessons for the interactive whiteboard and/or the in-built tools and features of the interactive whiteboard itself, the pace and focus of lessons may attend less to the specific needs of students as they arise during the delivery of the lesson.

In summary, findings from studies reviewed indicate that some experienced teachers find it challenging to integrate ICT into lessons and deepen mathematical understandings. Hence, it seems important for teacher educators to investigate this issue.

Data Collection, Analysis tool and Approaches

The 60-minute mathematics lesson reported here was digitally recorded during Term 3, 2007. The footage was later analysed by researchers using an existing classroom observation schedule and protocol (Gore et al., 2004; Zevenbergen & Lerman, 2007).

Digital recordings of classroom practice have been widely used to capture ways in which teachers transform theories and policy directives into their own classroom contexts. These recordings may be used for various purposes and audiences. Mousley, Lambdin and Koc (2003) argued that providing technology within teacher education programs brought theory and practice closer together and allowed preservice teachers to consider personal theories about pedagogy. They offered three different purposes for using technology in teacher education: first, to study teaching practices by viewing resources such as, videotapes and multimedia resources; second, to enable preservice and in service teachers' opportunities for professional development and communication; and, third, to use electronic resources such as calculators and computers for doing mathematics.

On one hand digital recordings of classroom practice provide opportunities for repeated viewings, on the other hand there are known limitations using this source of data. There are issues affecting reliability and validity both in the data collection and analysis phases such as having additional people in the classroom observing the lesson and then using an appropriate tool and consistent protocol to analyse the data.

Several steps were taken to address these known issues and to reduce their effects. Three of the four researchers met with each teacher and class prior to the day of recording and explained the purpose and process of the exercise to minimise the impact of foreign agents to the context. The fourth researcher did not have any contact with the participants at all. This step was planned to add another layer of critical objectivity in the data analysis stage.

Gore et al. (2004) provide items and key questions that address each of the PP framework dimensions. The framework identifies *intellectual quality, relevance, supportive classroom environment,* and *recognition of difference* as four dimensions of classroom practice that are essential for student learning. The schedule includes items which elaborate each dimension, for example, *Intellectual Quality* includes items *higher order thinking, deep knowledge, deep understanding, substantive conversation, knowledge as problematic, and metalanguage*.

The PP framework was used by each researcher to ensure consistency in data analyses. At the first data analysis meeting the research team discussed each item and dimension to gain a shared understanding of the schedule. This also involved a critical and collaborative review of the footage of one lesson to become more accustomed to the schedule. Following this session, each of the researchers independently viewed and completed the PP schedule with the scoring system ranging from 0 to 5. A value of 1 indicated minimal evidence of the dimension and 5 indicated a strong presence of the dimension throughout the entire lesson. At the third data analysis meeting, researchers shared their scores and debated differences until consensus of scores was achieved for each dimension.

Results and Discussion

This section presents data from the digital recordings of one lesson in two ways: first, as a descriptive snapshot of the events; second, as an analysis of the classroom practice using the PP framework. Pseudonyms are used in the paper.

Julia, a novice teacher in her mid-twenties, has three years teaching experience with Preparatory classes in an Australian school situated in a predominantly middle-class suburb. In 2007, Julia's class comprised 18 students all of whom were native speakers of English. Everyone, including the teacher, appeared happy, calm and settled. The classroom environment was inviting, stimulating and well-organised. There were four computers in the classroom specifically for student-use situated along one wall. Students worked at tables clustered together and sat in a space at the side of the room for whole-class activities. The IWB was located in another room in the same building which was available for class bookings.

An Overview of the Lesson

Figure 1. Story read and discussed.

Figure 2. Explanation of Task A.

The 60-minute lesson followed the *whole-small-whole* lesson format. As depicted in figure 1, for the whole-class introduction, Julia read *Alexander's Outing* (Allen, 1993) a story about a duck that strays from his brood, falls down a hole and helpful passers-by rescue him. Next, she led a class discussion about how Alexander was helped out of the hole. The introduction was interesting and the discussion was focussed. However, at no stage did the teacher mention the words *volume* or *capacity*. The key point Julia made was "that the hole had to be filled with water all the way to the top; otherwise, Alexander would not have been able to get out."

To commence the small format of the lesson, the teacher explained two tasks which were to be completed in pairs as shown in figure 2. Task A required students to estimate first and then use plastic containers, scoops, water and plastic ducks to imitate Alexander's rescue. While one child filled the container with scoopfuls of water, the partner used the constant function on the calculator to keep record of the number of scoops used to float the plastic duck to the top of the container (figure 3). These counts were then recorded on the class recording sheet (figure 4).

Three points were emphasised in Julia's explanation and demonstration: the difference between a full and partially full scoop of water; the need for careful recording of each scoop using the calculator's constant function, i.e. one press of the equals key for one scoop; and, to estimate and record the number of scoops of water required to fill the container before the measuring commenced. The explanation was clear and students were on-task, yet, there was still no mention of the terms to describe the mathematics topic.

Figure 3. Measuring and counting

Figure 4. Recording estimates and final count

Figure 5. Galaxy Maths capacity acitivity

For Task B students worked at the four computers and took turns with selected activities from the software program *Galaxy Maths* (Sunshine Multimedia, 2000). It was evident that students had prior experience with computers and these short activities which explored the concepts of volume and capacity. The commentary from one activity included the following instructions: 'Click on the containers in order, from those which hold the most to those which hold the least.' In another game: 'How many cups do you think will fill the container? You have a guess and then Number Cruncher will have a guess.' It seemed that students were having success with the activities which sought an estimate given an informal measuring unit and various shaped vessels.

Following the paired activities, the whole class gathered again on the floor and some students shared their experiences of completing Task A. The following excerpt is between Child A and Julia:

Julia: Show us the container that you used to measure, to save Alexander from.

Child: (Holds up a small plastic jug).

Julia: How many scoops did you estimate it would take to save Alexander, to fill the container?

Child: Ten.

Julia: And when you measured, how many was it?

Child: Ten.

Julia: And was that more, or less, or the same?

Child: The same.

Julia: The same. Good boy.

The same questions were used with several students. In each case, the focus seemed to be on the number. The relationship between the number of scoops and the differing sized containers was never mentioned. Similarly, students were not asked to compare the results from the measuring exercise with another nor to make generalisations.

For the remaining 15 minutes of the lesson, the class moved to another room where they completed *The Mud Cake* story and three activities from *Galaxy Maths* (Sunshine Multimedia, 2000). The class looked on as various students interacted in turn with the IWB.

Figure 6. Child using IWB.

Figure 7. Classmates and teacher observe.

It was obvious that the students enjoyed the activities on the IWB even though the same programs had been used in their classroom. As mentioned earlier, these activities provided opportunities to estimate and check given various scenarios which appealed to children. It was disappointing but understandable that when Julia asked the class, "what maths have we been learning?" that one or two students mentioned counting yet none mentioned capacity or volume.

To address the second research question, these data are examined a second way. Table 1 presents an analysis of the lesson using the PP framework. The final column of the table indicates the duration the items were evident in the lesson. Scores of 3 - 4 indicate that the particular item was evident throughout the lesson for approx. 30 or 40 minutes respectively, whereas 0 - 1 indicate that the item was not apparent or only for approximately 10 minutes in the digital recording of this lesson. These data suggest the teaching practices contributed to developing a positive and *supportive school environment* which is an important ingredient for learning and that the activities were relevant to an extent. However, amongst other points, the intellectual quality of the mathematics learning was limited.

Table 1

Presence of Items From the Productive Pedagogy Framework in Julia's 60-Minute Lesson

PP Dimension	Item	Key question	Score
Intellectual quality	Higher order thinking	Are higher order thinking and critical analysis occurring?	0
	Deep knowledge	Does the lesson cover operational fields in any depth detail or level of specificity?	2
	Deep understanding	Do the work and response of the students provide evidence of understanding concepts and ideas?	2
	Substantive conversation	Does the classroom talk break out of the initiation/response/evaluation pattern and lead to sustained dialogue between students, and between students and teachers?	2
	Knowledge as problematic	Are students critiquing and second guessing texts, ideas, and problematic knowledge?	1
	Metalanguage	Are aspects *of* language, grammar and technical vocabulary being foregrounded?	0
Relevance	Knowledge integration	Does the lesson range across diverse fields, disciplines and paradigms?	2
	Background knowledge	Is there an attempt to connect with students' background knowledge?	1
	Connectedness to the world	Do lessons and assigned work have any resemblance or connection to real life contexts?	4
	Problem based curriculum	Is there a focus on identifying and solving intellectual and/or real world problems?	1
Supportive school envt	Student control	Do students have any say in the pace, direction or outcome of the lesson?	0
	Social support	Is the classroom a socially supportive, positive environment?	3
	Engagement	Are students engaged and on-task?	4
	Explicit criteria	Are criteria for student performance made explicit?	3
	Self-regulation	Is the direction of students' behaviour implicit and self-regulatory?	4
Recognition of difference	Cultural knowledges	Are diverse knowledges brought into play?	0
	Inclusivity	Are deliberate attempts made to increase participation of all students from different backgrounds?	2
	Group identity	Does teaching build a sense of community and identity?	2
	Citizenship	Are attempts made to foster active citizenship?	0

Conclusion

This case illustrates how on the surface even an interesting, engaging, student-centred lesson which integrates ICT well does not necessarily result in deepened mathematical understandings. Often another question from the teacher would have helped the student to make links between understandings. Similarly, more critical appraisal of the software being used, asking oneself, "What will the students gain from completing this activity?" Of course, it is to be expected that novice teachers will refine their skills over time with continued critical reflection on practice and ongoing professional development. Nonetheless, it is useful for teacher educators to choose snippets of classroom practice and to use them as discussion starters to draw out opportunities for richer teaching and learning episodes which focus on discipline-specific language and understandings in the future. Authors of this study are keen to develop the PP framework into an observation tool for viewing snippets of classroom practice to emphasise that student engagement alone will not necessarily lead to deepened mathematical understandings.

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Investigating a Phase Approach to Using Technology as a Teaching Tool

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Technology, such as dynamic geometry software (DGS), is available in many secondary mathematics classrooms. Whilst studies have highlighted the motivational aspects of DGS to explore geometric concepts, there exists a need to explore specific strategies for using technology in the mathematics classroom as a teaching tool, as opposed to a display tool. The project presented explored a phase approach to incorporate technology into the teaching/learning cycle to facilitate developmental progression in a secondary mathematics classroom. This paper presents the findings of a project which linked theory and practice within a technological environment.

Over the past thirty years there have been numerous studies that have characterised the nature of levels of understandings in Geometry (Burger & Shaughnessy, 1986; Currie & Pegg, 1998; Fuys, Geddes, & Tischler, 1985; van Hiele, 1986; Serow, 2002). In particular, the van Hiele Theory (van Hiele, 1986) is comprised of a five level framework of development in Geometry from which to view students' understandings. Some studies have extended this work and focused upon developmental pathways and identified hurdles leading to higher-order thinking (Serow, 2002, 2007a). There is an urgent need, however, to explore teaching/learning practices that facilitate student developmental progression. This exploratory study, is part of a larger study which explores the effectiveness of dynamic geometry software as a teaching tool. The activities are structured within the van Hiele teaching phases framework.

The project presented extends previous work which identified developmental pathways associated with class inclusion concepts (Serow, 2002). These pathways highlighted the difficulties associated with students' attempts to understand and utilise networks of relationships in geometry. Serow (2002) highlighted the reasons students find class inclusion concepts in Geometry difficult to grasp, and detailed the hurdles encountered by many students through the characterisation of the development of relationships among figures and relationships among properties. The philosophical stance taken by van Hiele in regards to teaching Geometry is grounded within the notion of insight. The opportunity to exhibit and develop insight is described by van Hiele (1986) as the aim of teaching mathematics. Thus, for the promotion of growth in understanding, learners require geometrical tasks that allow them to control their individual problem-solving environment (Hoffer, 1983, p. 205). Essentially, Dynamic Geometry Software (DGS) provides the potential for student-centred problem-solving tasks that remain in the control of the individual student. DGS allows the "continuous real-time transformation often called 'dragging'. This feature allows users, after a construction is made, to move certain elements of a drawing freely and to observe other elements respond dynamically to other altered conditions" (Goldenberg & Cuoco, 1998, p. 351).

The tools within The Geometer's Sketchpad, Version 4.0 (Jackiv, 2001) a form of DGS, provide teachers with the opportunity to explore the relationships of quadrilateral figures and properties both intuitively and inductively. Goldenberg and Cuoco (1998, p. 396) found the "dynamic nature of these tools makes them both exciting and accessible, even to elementary students" (p. 396). In addition, dynamic geometric investigations are possible when students have time to consider their mathematics ideas as opposed to concentrating on the technicalities of pen and paper constructions (Tikoo, 1998).

It has been contended by McGehee and Griffith (2004) and Coffland and Strickland (2004) that teachers need to focus on the ways that technology may enhance mathematical thinking and enhance conceptual understandings. Many teachers are comfortable using technology to display material but often lack confidence in sequencing technological tasks as an integral component of a teaching/learning sequence. A teaching framework that has the potential to address this need is the basis of the work of Dina van Hiele-Geldof (van Hiele, 1986). The five teaching phases represent a framework to facilitate the cognitive development of a student through the transition between one level and the next. The van Hiele phases are centred on the notions that progress is easier for students with careful teacher guidance, the opportunity to discuss relevant issues, and the gradual development of more technical language. The phases are organised in such a way that they acknowledge the assumptions underpinning the van Hiele levels, while providing students with the opportunity to exhibit insight. The van Hiele teaching phases address the concern that "teachers often feel reluctant or uncomfortable because their pedagogical knowledge perhaps does not include a framework for conducting technology-based activities in their lessons" (Chua & Wu, 2005, p. 387). A description of each of the five van Hiele teaching phases is summarised in Table 1.

Table 1

Descriptions of the van Hiele Teaching Phases

Phase	Description of Phase Focus
1. Information	For students to become familiar with the working domain through discussion and exploration. Discussions take place between teacher and students that stresses the content to be used.
2. Directed Orientation	For students to identify the focus of the topic through a series of teacher-guided tasks. At this stage, students are given the opportunity to exchange views. Through this discussion there is a gradual implicit introduction of more formal language.
3. Explicitation	For students to become conscious of the new ideas and express these in accepted mathematical language. The concepts now need to be made explicit using accepted language. Care is taken to develop the technical language with understanding through the exchange of ideas.
4. Free Orientation	For students to complete activities in which they are required to find their own way in the network of relations. The students are now familiar with the domain and are ready to explore it. Through their problem solving, the students' language develops further as they begin to identify cues to assist them.
5. Integration	For the students to build an overview of the material investigated. Summaries concern the new understandings of the concepts involved and incorporate language of the new level. While the purpose of the instruction is now clear to the students, it is still necessary for the teacher to assist during this phase.

(Serow, 2002, p. 10)

The five-phase teaching approach provides a structure on which to base a program of instruction. As can be seen, the phase approach begins with clear teacher direction involving exploration through simple tasks, and moves to activities that require student initiative in the form of problem solving.

Serow (2002) identified a generic developmental pathway leading to an understanding of class inclusion concepts, incorporating networks of relationships, in Geometry. The cognitive processes undertaken by learners, and hurdles met along that path have been articulated. Whilst it is essential to have a framework as a content analysis tool, the framework is not the focus of this project. Table 2 outlines the categories of responses, in ascending order of complexity, concerning relationships among quadrilateral figures which provided the basis for analysis of student responses in the reported project.

Table 2

Categories of Responses Concerning Relationships Among Quadrilateral Figures

Category	Characteristics of Responses
A	A single property or feature is identified to link the figures. The focus of the response is upon the identification of an observed single quantifiable aspect, which places figures into spontaneous groups. There is a strong reliance on visual cues.
B	Classes of figures are known by name and are characterised by a single property. The class represents an identifiable unit. Links do not exist between classes, unless supported by visual cues. Observed differences in traditional shapes, such as right angles, play a significant role.
C	Similar to the Category B response above, the Category C responses incorporate classes of figures, which are known by name. These classes are characterised by more than one property. Links are not made between classes where differences in properties are accentuated by visual differences.
D	Relationships exist between classes of figures, which are based upon similar properties. Inclusive language is used to describe the classes of figures; hence, property descriptions allow for similarities to be acknowledged.
E	When prompted, tentative statements are made concerning the possibility of subsets within a class of figures. There is no acceptance of this notion, however, it is able to be discussed tentatively.
F	There is an unprompted acceptance of a class of figures containing subsets. While this notion of class inclusion is accepted and utilised, it is not justified adequately.
G	The notion of class inclusion is an integrating feature of the response. A class of figures incorporates subsets, which are inclusive of generic categories identified by other names. Each class maintains a workable identity while the focus is upon the network of relationships based upon the properties of each class.
H	The notion of class inclusion acquires further development. Conditions are placed upon the classes of figures, which acknowledge more than one system of relationships. This requires an overview of the interrelationships among classes and their subsets, which utilises subsets within subsets, and precludes inappropriate examples of figures.

(Adapted from Serow, 2002, p. 214)

In consideration of the background provided, the natural progression presented in this paper is a focus on suitable teaching strategies to assist students in meeting and rising above the identified cognitive hurdles within a structured technological environment. The research questions for this study were:

1. Is the van Hiele teaching phases framework an effective structure for designing teaching sequences involving dynamic geometry software?
2. To what extent does the implementation of student-centred tasks, which utilise dynamic geometry software, facilitate student growth in understandings of relationships among quadrilateral figures?

Method

This study uses a pre-experimental design with one group (23 students), involving pre-tests and post-tests (Cohen & Manion, 1994, p. 165). In addition, the design incorporated a delayed post-test to assess longitudinal retention of demonstrated understandings. The teaching sequence was designed with two main elements; the teaching phases as a design framework, and the embedding of dynamic geometry software, in conjunction with spreadsheets (Excel) and concept mapping software (Inspiration). The intervention involved a two-week teaching sequence (eight sessions of forty minute duration) and was delivered to a Year 9 (ages 14–15) secondary mathematics class. The focus content strand of the teaching sequence from the K-10 Mathematics Syllabus (Board of Studies, 2002) was Space and Geometry and the target outcomes addressed by the teaching sequence were "classify, construct, and determine properties of triangles and quadrilaterals" and "verify the properties of special quadrilaterals" (Board of Studies, 2002, p. 39). The teaching sequence aimed to integrate dynamic geometry software using the van Hiele teaching phases as a framework (van Hiele, 1986) for maintaining student ownership of their mathematical ideas. This teaching sequence included student-centred tasks that aimed to acknowledge the progression from informal to formal language use. The pre and post-tests were in the form of written tasks. These tasks comprised of open-ended items (Cohen & Manion, 1994, p. 277) to elicit qualitative student responses at the commencement and completion of the teaching sequence. A written delayed post-test, delivered to each student in the sample, was implemented to determine the retention rate of conceptual understanding. All responses to open-ended items were categorised via the identified developmental pathways associated with understandings of relationships among figures (Serow, 2002).

Teaching Intervention

The two-week teaching sequence (Serow, 2007b), which is detailed below in Table 3 is sequenced using the van Hiele teaching phases. Each activity described includes the target teaching phase.

Table 3

Teaching Sequence

Activity and Phase	Activity Description
Information Phase and Directed Orientation Phases Activities 1: Mechanics and Recall	1. Students work through simple constructions in Sketchpad and brainstorm known quadrilaterals. Constructions involve: a) Write your name using sketchpad. b) Create a person and reflect the person. Measure a selection of corresponding sides and angles. What do you notice when you drag one of your people? c) Create a house design using the six quadrilaterals, namely, kite, trapezium, square, rectangle, rhombus, and parallelogram. At this stage, the students, in most cases, will construct their figures using the line tool. This will be extended in later phases. When the students are asked to drag (drag test) the quadrilaterals they have formed this way, they will notice that the constructions are not robust (do not remain the intended figures).
Explicitation Phase Activities 2: Robust Templates and Recording	2. Students create robust templates for each of the six quadrilaterals on separate Sketchpad pages. If the drag test allows the figure to remain as intended, the construction will involve known properties of each figure. Discussions will begin to occur concerning relationships among figures. For example, comments such as 'this is really strange, when I drag the parallelogram it is sometimes a rectangle, square or rhombus'. This activity will involve constructions such parallel lines, perpendicular lines, and transformations. It is essential for the students at this phase to describe their construction within a text box on Sketchpad and to record the properties for each quadrilateral on a teacher-designed table.

Directed Orientation Phase Activity 3: Irregular Quadrilateral and Midpoint Construction	3. Students are instructed to: a) create any irregular quadrilateral using the line tool; b) construct the midpoints; c) join the midpoints to construct another quadrilateral; d) answer the question, What do you notice?; and, e) investigate the properties off this shape to justify what you have found, and record your justification in a textbox.
Explicitation Phase Activity 4: Further Exploration of Properties and Figures	4. Students design a spreadsheet where the six quadrilaterals are contained in the first column, and the first row contains all possible properties of quadrilaterals. Particular care needs to be taken to include diagonal properties such as ‘diagonals meet at right angles’. The students record the properties of each figure by ticking the appropriate cell. There is an element of surprise in the classroom when the students notice that the square has the maximum number of ticks.
Free Orientation Phase Activity 5: Diagonal Starters Game Design	5. This activity is designed to reinforce diagonals as a property and not merely a feature of the quadrilaterals. Students are given the challenge to create the diagonal formation needed for each of the quadrilaterals. The aim is for the students to construct templates for younger students to complete the figure and explore the properties.
Free Orientation Phase Activity 6: Concept Maps and Flow Charts	6. Students create; a) a concept map b) a flow chart, to summarise their known relationships among quadrilateral figures.
Integration Phase Activity 7: Information Booklet Design	7. Students organise the constructions that they have made, justifications, tables, spreadsheets, concept maps, and flowcharts to produce an information booklet to explain what they know about the relationships among quadrilaterals and relationships among quadrilaterals figures. Students are instructed to include an overall summary of their findings.
Integration Phase Activity 8: Sharing and Routine Questions	8. Class sharing of booklet designs. Routine questions involving known properties and relationships.

The teaching sequence was presented to the class using a team-teaching approach involving their classroom teacher and researcher. Each student had individual access to a computer and relevant software.

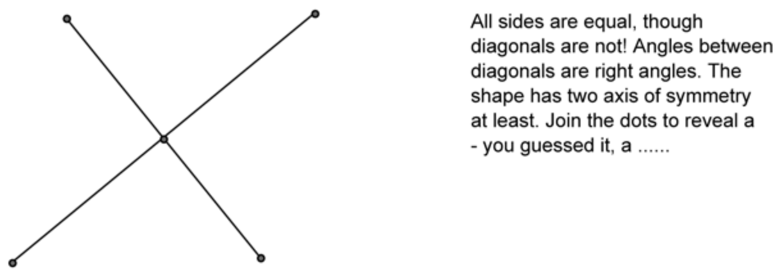
Instrument

The pre-test, post-test, and delayed post-test included the same set of items. The items included are summarised below in script form:

1. Int: Write a list of all the quadrilaterals that you know above the line.
2. Int: We will now check to see if we have them all. Did you include rectangle, square, parallelogram, rhombus, kite, and trapezium?
3. Int: Below the line, write the shapes that you may have missed.
4. Int: We are going to create a concept map with each of these quadrilaterals. You will have enough time to include all the information that you feel is important.
5. Int: Draw a diagram to illustrate each quadrilateral. Make sure your drawings clearly indicate each quadrilateral. Draw lines to indicate relationships among the quadrilaterals. Use circles if you would like to show groups. Write your reasons for the groups you have identified. Write one paragraph justifying the manner in which quadrilaterals are related to one another.
6. Students were asked to comment (in written form) on the following two scenarios.
   Scenario 1: John states to the class "The square is a rectangle". Do you agree or disagree? How could he justify this statement if he was asked to explain it?
   Scenario 2: Megan writes on her paper that "The rhombus is a parallelogram".

Results and Discussion

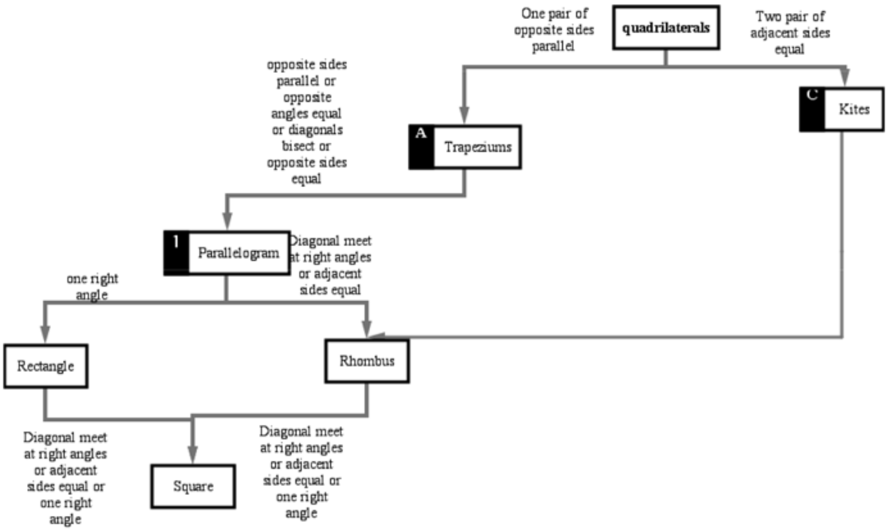
In relation to research question 2, the van Hiele teaching phases was an effective design framework for sequencing activities that involved dynamic geometry software. In addition to the overall increase in complexity of the student responses at the completion of the intervention, the students remained on-task through each of the activities and conversed with one another as they gradually moved from informal to formal language use. The following student samples typify the tasks completed by the students when immersed in the activities. Figure 1 below is a student sample of one section of the diagonal starter's game (free orientation phase) where the task focussed the students constructions on diagonal properties and the resultant figures and properties.



(Serow, 2007b, p.386)

Figure 1. Sample of student's diagonal starters diagram.

It was evident that the students were implicitly placed in a situation where they were required to use the properties of the figures if their constructions were to remain the intended figure when 'dragged'. The phase approach provided an avenue to design and implement activities which assisted in making the properties and relationships among the properties the explicit focus of the student activities. Hence, the use of spreadsheets for recording, text boxes for student recording of findings, and concept maps/flow charts played an important role in 'making the most' of the DGS activities. Whilst the concept mapping activity (free orientation phase) included typical venn diagrams, the flow chart design facilitated an environment where students considered the properties and figures that they had explored in the DGS environment and organised them into an hierarchical structure. A typical student response to this task is contained in Figure 2.



Serow (2007b, p.387)

Figure 2. Sample of student's flow chart diagram.

Table 4 details the coding for each of the students' responses to the pre, post, and delayed post-tests. When comparing the results of the pre and post-tests it is evident that the students' understandings of the relationships among quadrilateral figures did change after the teaching intervention. In reference to research question 2, using dynamic geometry software as an integral component of the student-centred activities did result in overall growth in understanding when considering the group as a whole.

Table 4

Relationships Among Figures

Category	A	B	C	D	E	F	G	H	Total
Pre-test	11 (48)	4 (17)	4 (17)	3 (13)			1 (4)		23 (100)
Post test		5 (22)	4 (17)	7 (30)	2 (9)		1 (4)	4 (17)	23 (100)
Delayed post test		5 (22)	4 (17)	7 (30)	2 (9)		1 (4)	4 (17)	23 (100)

Percentages of the sample for each category in each test are included in brackets.

In the pre-test, 48% of responses focussed on a single feature or property with a reliance on visual cue when attempting to describe relationships among quadrilateral figures In the post-tests, none of the responses were of this nature. The pre-test also indicated that 17% were characterising a class of quadrilaterals by a single property, and 17% were focussing on more than one property. Respectively, in the post tests, these figures were 17% and 30%. In the post-test a larger percentage of students were focussing on the relationships among classes of quadrilaterals based on similar properties. Overall, in the pre-test, only 4% of responses focused on the notion of class of inclusion (Categories G and H) and in the post-tests this has risen to 21%. Of this 21% of responses, 17% focussed on the placement of conditions upon the class of figures which enabled subsets within subsets. It is particularly interesting to note that the coding for the post-test and delayed post-test were consistent across each individual student.

Conclusion

This project aimed to undertake an exploratory teaching experiment to provide base line data on the effectiveness of dynamic geometry software to facilitate student growth in understandings of networks of relationships in geometry. A fundamental aspect of the project was the melding of cognitive frameworks, phases of teaching, and the embedding of Information and Communication Technology within a teaching sequence. This study highlights the importance of embedding technology within a pedagogical framework. In terms of mathematics education research as a whole, it raises interest in exploring the melding of existing theoretical frameworks with emerging technological tools that are currently available to secondary students.

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The Introduction of Interactive Whiteboard Technology in the Primary Mathematics Classroom: Three Case Studies

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Interactive whiteboard (IWB) technology is present in a large number of Australian primary schools. Despite the rapid increase in availability of this technology over the last five years, previous research suggests that the technology is being used for sophisticated transmission style teaching as opposed to constructivist approaches. This paper presents findings of a project which considered the implementation of IWB technology in three Victorian primary mathematics classrooms. The case studies explored the teaching strategies adopted by three teachers as they embarked on the use of IWB technology as an integral component of mathematical activities with the support of professional development.

It is evident that "few studies have been published on the effectiveness of technology within geometry classrooms" (Coffland & Strickland, 2004, p.347). Many classroom teachers confidently use technology as a presentation or display tool, but remain unaware of the potential for Information and Communication Technology (ICT) to promote concept development in the mathematics classroom. This was emphasised by Fitzallen (2005) through the recognition of "a need for teachers to gain an understanding of how Information and Communication Technology (ICT) can be used to extend students' thinking and problem-solving skills, rather than just a publication and research tool" (p. 253).

An Interactive Whiteboard (IWB) is a touch sensitive display board, sensitive to finger or pen-like devices, used in conjunction with a computer and a digital projector. The IWB technology used as a medium in this project was a student pad. The student pad is a portable A4 size template, which is touch-sensitive, to allow students to manipulate the IWB technology from any position in the classroom. Although the number of classrooms in Australia that have IWB technology installed is unknown, there are a large number of primary and secondary schools that have purchased and/or installed the technology and intend using IWBs during mathematics lessons. It is evident that both teachers and students are enthusiastic about using IWB technology in the classroom, but it is not clear if this excitement and enthusiasm is transformed into effective teaching strategies and meaningful mathematical experiences (Tanner & Jones, 2007).

Deaney, Ruthuen, and Henessy (2003) stated that the supply of technology is of limited value without an understanding of "the interactions and processes engendered by using technology in different settings, and how pedagogical strategies to enhance students' learning might be developed effectively through them" (p. 142). This suggestion echoed McGehen and Griffith's (2004) contention that teachers must develop an appreciation of the ways in which technology can enhance and encourage mathematical thinking. It has been claimed, however, that IWB technology can reinforce teacher-centred pedagogy, leading to students becoming passive recipients of information rather than active, engaged learners (Moss et al., 2007).

A number of themes have been identified in relation to students' perceptions of problems when using IWB technology. These included technical problems with the equipment, varied levels of ICT skills of teachers and students, and lack of student access to the IWB technology during the classroom activity. This study found that

> … while the technology is clearly engaging from the students' perspective there is a concern that any gains in this direction will be lost if the technology is not reliable, if teachers are not adequately trained to use it, and perhaps more importantly, if the educational climate militates against increased pupil access to the technology. (Hall & Higgins, 2005, p. 114)

This finding suggested that both teachers and students needed experience in "playing" with new technology, and teachers should have professional development that addresses both skill-based aspects of technology use and effective pedagogical approaches when technology is used. In relation to this issue, Smith, Hardman, and Higgins (2006) stated that "more extensive research needs to be carried out into ways of effectively supporting teachers in their professional development in order to promote more reciprocal forms of teaching to increase the opportunities for extended teacher-pupil interaction" (p. 456).

The study reported in this paper involved three teachers who were implementing IWB technology in their primary classrooms for the first time. They were supported through professional development sessions in which they were introduced to developmental models of learning with a teaching framework provided by the pedagogical phases developed by Dina van Hiele-Geldof. The five teaching phases aim to facilitate the cognitive development of a student through the transition between one level of development and the next. The phases originate from the idea that "help from other people is necessary for so many learning processes" (van Hiele, 1986, p. 181). Learning is deemed a social process and stems from the notion that students find it very difficult to move unassisted from one thought level to the next. This idea, however, is exactly what IWB technology appears to inhibit. The use of van Hiele teaching phases addressed the concern that "teachers often feel reluctant or uncomfortable because their pedagogical knowledge perhaps does not include a framework for conducting technology-based activities in their lessons" (Chua & Wu, 2005, p. 387).

The second framework used in the professional development sessions was the SOLO model. The SOLO model grew from Biggs and Collis' (1982) desire to explore and describe students' understandings in the light of the criticisms of the work of Piaget. Rather than focus on the level of thinking of the student, the emphasis in the SOLO model is on the structure of students' responses. Of particular interest to this study were investigations concerning geometry concepts (e.g., Davey & Pegg, 1992; Olive, 1991; Pegg & Davey, 1998; Serow, 2002, 2006). The framework is composed of two main components, these being: modes of functioning and cycles of developmental levels. Application of the SOLO model was part of the professional development of this project to assist teachers in determining effective teaching practices that could make use of the potential of technological tools within the framework provided by the teaching phases.

The first professional development meeting introduced the teachers to the school pad and its associated software. Teachers were able to use the school pads and to explore the range of possibilities offered. The later professional learning sessions were explicit about linking technology use to concept and cognitive development. Teachers saw use of the IWB technology modelled in presentations, were able to experience using the school pads and to interact with computer programs in ways that promoted developmental ideas.

With this background, the research questions for this study were:

1. In what ways does using IWB technology support students' active involvement in learning mathematics?
2. What issues emerge in introducing school pad technology into primary mathematics classrooms?

Method

The study took place over one school year in 2007. Teachers were supplied with a school pad and undertook to use it in mathematics lessons wherever possible. The particular focus was the space strand of mathematics, because this had been identified as an area of need by the schools involved. Apart from these commitments, no restrictions were placed on teachers as to how they implemented IWB technology within their different contexts.

Sample

The teachers involved came from three different primary schools in Victoria. School A was situated in a suburban area. It had an enrolment of approximately 450 students, of whom more than 80 percent spoke English as a second language. The teacher, Mark, taught a year 6 class. He was an experienced teacher, having taught for about 10 years. The classroom was equipped with an interactive whiteboard, as well as a small school pad supplied through the project. School B was also suburban. It had 600 students, mainly of Anglo-Celtic origins. The teacher, David, taught a year 5 class. He was the least experienced classroom teacher, having come into teaching as a mature-age entrant after a varied career, including some time teaching in the TAFE sector. The school pad supplied by the project was the only access to IWB technology, but he also made use of a computer laboratory. School C was situated in a country town. This school was a small school of approximately 133 students. The teacher, Agnes, was very experienced, having taught for well over 20 years, and she also held a Masters degree. She taught a year 4 class, and the school pad provided the only access to IWB technology in the school.

Data Collection and Analysis

Data were collected from three sources. Classroom visits were made to all three schools, and lessons using the school pads observed. Particular attention was paid to the ways in which the students used the school pad, and the teachers' questioning. Three meetings were held with the teachers. At these meetings, teachers reported their classroom experiences and activities, received training in the use of the school pads and input about developmental approaches to teaching the space strand, with particular reference to the use of IWBs. The final data source was student work samples provided by the teachers or collected during school visits.

A case study approach was taken with each school providing a case (Stake, 1995). In addition, records of the meetings and classroom observation data were analysed qualitatively using a clustering approach (Miles & Huberman, 1994) to identify common themes as well as differences among schools. The work samples were used to triangulate teachers' comments about their students' progress.

Results

The results are presented as three short case studies, followed by an identification of common themes. Each school is discussed separately.

In school A, Mark made less use of the school pad than the large wall-mounted IWB. He used some features of the IWB software effectively to develop students understanding. In particular he used the spotlight or curtain feature to reveal gradually a 2-dimensional shape, providing a virtual equivalent of a well known activity included in the Early Years Numeracy Research Project assessment interview (Clarke et al., 2002). Students were interested and engaged, but they tended to consider single features of a shape rather than integrating these into a deeper understanding of the properties and relationships among properties. Students produced a PowerPoint report about angles, but the focus tended to be a description of single features rather than links to properties of shapes, or angle as a measure of rotation. Mark was also making use of support materials provided electronically by the education system through the Internet, as well as other public Internet sites, including *TeacherTube* (http://www.teachertube.com/). These motivated and interested the students in mathematics.

David, in school B, also made extensive use of technology, including the school pad. He took longer to implement technology use in the classroom saying "I needed to get my head around it first". He also found some difficulty working with the technology until he was able to set up the necessary computer and data projector permanently in his classroom. By third term, however, David had planned and implemented a fully online unit of work around properties of shapes. Students were making extensive use of technology in a number of ways, including exploring shape, creating a PowerPoint and submitting this for assessment electronically through the school intranet. Students were able to act autonomously in their choice of shapes to explore, and it was noticeable than many had chosen relatively unusual shapes, such as nonagons. Use of the school pad appeared to be restricted to individuals taking it in turns to practice using the tool, although David did say that he was using this in other ways in subjects outside mathematics.

The younger students in Agnes' class were also impressively fluent in their use of technology. In this school, as in School B, infrastructure was an issue. Access to data projectors and screens was limited and Agnes had to rely on makeshift arrangements using window blinds to project the image. The focus in this classroom was on group work with several activities all addressing the same general topic of 3D objects and their properties. Agnes had laid down some rules for use of the school pad. Each student had to take a turn so that collaboratively the group created a representation of a 3D object. In the observed lesson, a group was working cooperatively using isomorphic dot "paper" and the school pad to draw a square-based pyramid. This created much discussion among the students in the group about where to draw the lines and whether the representation was accurate. The rest of the class was undertaking a variety of activities, including building with concrete materials, writing about a chosen shape in their mathematics books and using drawing packages on computers to explore 3D shapes. Agnes stated that she felt that the most powerful aspect of the IWB technology was the conversations it created among the students. During the school visits, it appeared that Agnes was using the van Hiele phases of teaching to design her lessons, moving from the students' development of mathematical ideas and informal language, to more formal language to communicate mathematically.

Student work samples were revealing. All teachers expressed the view that their students had been more engaged and motivated by the use of the school pads, and that their learning had benefited. Certainly the students observed were confident and engaged by the technology. Despite the students' fluency in using the technology however, the focus of this use tended to be on presentation rather than the mathematical ideas.

In some instances, the level of thinking displayed was somewhat disappointing. For example, a year 5 student wrote "For my shape I chose a triangle. A triangle has 3 corners and three sides. The triangle that I did was a 2-d (sic) shape." Another student chose a more adventurous shape and wrote "The decahedron (sic) has 10 sides, 10 vertices and 10 obtuse angles. The decahedron is the shape of a 50c coin." In both of these examples, students were focussing on a list of disconnected features, relying solely on visual cues. The lack of use of geometrical language and the incorrect use of terms in their final published work could suggest that the interactions between teacher and students did not assist them to develop their thinking. Nevertheless, the students' confidence in choosing "cool" shapes such as octagons was encouraging in its motivation.

A year 4 student provided several representations of a pyramid using a computer drawing package. This student had previously used the school pad to draw 2D representations of 3D objects. The representations are shown in Figure 1, and include a birds-eye view of a rectangular pyramid as well as a tetrahedron and an attempt at drawing a square-based pyramid. This example indicated a more sophisticated outcome. Representing accurately 3D objects in a 2D form is not a trivial task for young students. Agnes, the teacher involved, explicitly linked concrete objects, skeleton models of these using play dough and matchsticks, virtual representations using the school pad and isomorphic dot "paper", and the language of geometry in her questioning and review at the start and end of the lesson. Technology use is not independent of the teacher, although Agnes reported that once the students were familiar with the technology she left them alone using the school pad while she worked with other groups.

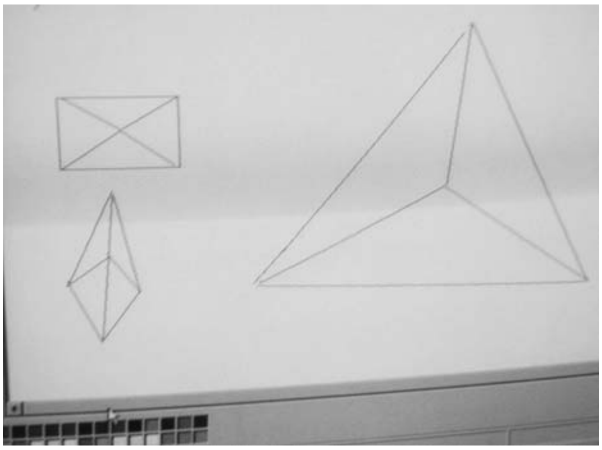

Figure 1. Computer representations of pyramids.

Three common themes emerged at the teacher meetings. The first was the difficulty of getting underway with the technology. Apart from Mark in School A, who had a large IWB fixed in his classroom, the other schools had no experience in IWB use and did not have appropriate hardware easily available. Although all schools possessed data projectors, and teachers had laptop computers, accessing the projector was not easy. It often involved planning ahead and booking the projector for a given time. These difficulties, on top of coming to terms with new teaching approaches and a new tool, proved challenging.

The second issue was teachers' relative lack of understanding of geometry. There were often comments such as "I hadn't realised that" or "I have never been taught this" when they were undertaking tasks designed to focus on geometrical understanding. Although all the teachers involved were widely experienced they readily acknowledged gaps in their mathematical knowledge.

Another issue that became evident in the meetings was the perceived motivational power of IWB technology. The teachers reported that the students repeatedly commented on the "fun" nature of the equipment and their desire to use it again. There were multiple comments on the manner in which the IWB enhanced the students' interest level during mathematical activities. The students' work samples endorsed this impression with comments such as: "Using the interactive whiteboard was a bit challenging but was fun when you put your mind to it. It was a very fun activity to do" and "I enjoyed using the Interactive Whiteboard. At first it was hard but after a while it became easy. It was lots of fun! ☺".

Discussion

Research Question 1

The reason for using school pad technology, as opposed to large fixed IWBs, was to encourage interaction among students. Of the three teachers observed, only Agnes was using this pedagogical approach. It is interesting to note that although Agnes did not have access to a fixed IWB and initially described this as a limitation, the collaborative small group tasks she designed which utilised the student pad were constructivist in nature and allowed the students to formulate and discuss their mathematical ideas. It was evident that the initial use of IWB technology as a large display in the classroom led to a tendency for the teacher to adopt transmission style teaching approaches, with the focus of the activity concerning the development of new technological skills. There were benefits in students having direct access to the student pads during geometrical activities in association with other concrete materials. A combination of small group constructions, electronic geoboard constructions, and recording of known properties and relationships was observed as an effective strategy. In addition, the time taken to bring the groups of students together as a class and discuss the ideas formed and language used was invaluable. This finding is consistent with those reported by Moss et al. (2007), and emphasises the comments of Deaney et al. (2003).

There were positive factors in relation to the motivational aspects of IWBs, but the "fun" and "pretty" side of the mathematical activities appeared to impinge on the development of mathematical concepts. Although the students enjoyed presenting mathematical concepts in a variety of technological forms, the display emphasis lost sight of the potential of the equipment as a teaching tool. The effective lessons engaged the students in mathematical investigations and problem-based learning tasks, without an emphasis on producing a display such as slides or booklets, in line with Fitzallen's (2005) thinking.

In terms of geometry specifically, although teachers did use aspects of the provided software, this use appeared to be limited to drawing, or interactive games. David, for example, used the geometric shapes to produce tessellations but this provided a digital mimicry of the use of pattern blocks and did not seem to lead to a deeper understanding of the conditions under which a shape would tessellate. Such use locks students into lower levels of thinking, but this may also have been reinforced by the teachers' own perceived lack of knowledge. Complex dynamic geometry software is not appropriate for use in primary schools, and many of the interactive activities and games available can be undertaken successfully with an inefficient "trial-and-error" approach, encouraged by the speed of the IWB interface. A new approach is called for to utilise the power of the IWB interface.

It appeared that IWB technology was motivational and encouraged students' interest. Unless careful thought was given to the pedagogy, however, student learning was not greatly enhanced and the potential gains from their increased interest not realised.

Research Question 2

The issues that emerged during this small, initial study were not unexpected. Access to appropriate infrastructure was critical and the time teachers took to get underway with the project was increased when this support was not available. The technology of itself was not effective unless teachers carefully planned and thought through their teaching approaches. The professional development sessions were important to support teachers' growing understanding of appropriate pedagogical approaches, in line with research findings from elsewhere (Hall & Higgins, 2005).

The finding that the large wall-mounted IWB was more consistent with teacher-centred approaches was also indicated in the literature (Moss et al., 2007). The small school pads, used appropriately, did appear to have potential for encouraging student-student and student-teacher interaction, and this may be a fruitful area for further research.

The call for a framework for conducting technology-rich lessons (Chua & Wu, 2005) was also borne out by the findings. Teachers appreciated the value of the van Hiele (1986) teaching phases and the SOLO model (Biggs & Collis, 1982), but in this short study were able only to develop a surface understanding of the possibilities. Nevertheless, the use of such frameworks to develop lessons using technology appears to have potential.

Conclusion

Through these case studies, it appeared that student-centred mathematical activities that used IWB technology as a teaching tool, required the teacher to facilitate tasks considered to be the most appropriate place to use the IWB within a developmental teaching sequence. The students enjoyed direct access to the equipment regardless of the task, but gained most benefit when actively involved in tasks which challenged their mathematical thinking and allowed for communication with their peers. The technology tasks were embedded in a range of activities that explicitly addressed different phases of the van Hiele teaching sequence. There was a tendency to rely on the display nature of IWBs in the classroom, leading to lessons where the final result of the activity was the display of pre-existing student ideas. The case studies also emphasised the need to provide sustained professional development which focused on the place of IWB technology in the light of known developmental frameworks in mathematics education, providing a useful platform for further research.

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School Readiness: What Do Teachers Expect of Children in Mathematics on School Entry?

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This paper explores the perceptions of teachers of five-year-old children and the expectations these teachers have of the children when they begin school in terms of their level of mathematics knowledge. Teachers were asked about their beliefs in relation to mathematics teaching and learning and how they thought children best learned mathematics. This was matched with classroom observations to compare teacher practice with teacher beliefs. A significant finding was the general lack of attention that teachers paid to the knowledge and skills that children had when they started school. There was a lack of consistency between what teachers said they believed in compared with what they actually did in the classroom.

This paper is a study on teacher beliefs and practices regarding the nature of the mathematical knowledge that five-year-old children enter school with. These beliefs have the potential to impact on the nature of the mathematics programme that the teachers present to the children and the factors that contribute to this programme. Schooling differs from children's prior experiences at home and/or at pre-school in that schooling in general has a focus on more structured procedures specific to the school environment. Peters (1998b) reported on the importance of primary teachers understanding the knowledge and skills that children already possess when they start school. She emphasised the fact that "children need to learn to make sense of school" (Peters, 1998a, p. 107) in that they develop awareness of the school situation and shape their learning to the confines of the classroom (Jackson, 1987). While there are a range of tools that primary teachers have available to use with children when they start school, Dockett and Perry (2001) indicated that teachers were more concerned about the child's ability to socialise rather than their academic ability. The following research looks at teachers' beliefs and practices relating to the mathematical knowledge of children when they begin school (at 5 years old in New Zealand).

Rationale

Clay (1991) described the child's entry to school as "not the beginning of development or of education in its' broadest sense; but it is the beginning of society's formal attempt to instruct all children, in groups" (p.19). The New Zealand system is one of the very few in which children begin school on or about their fifth birthday. The new entrant (emergent) class teacher has a continuous supply of individual children joining the class throughout the year from a variety of pre-school environments. Teachers within the primary sector were found to have insufficient knowledge of the children's achievements in the early childhood area and were thus unable to effectively support children by building on their prior knowledge (Timperley, McNaughton, Howie, & Robinson, 2003). Primary teachers held the view that it was important when children settled into school, they behaved in a way that was acceptable to the teacher (Renwick, 1987), implying that the social aspect of the new entrant classroom was of primary importance. The children are required to fit into the school system.

New entrant teachers are able to accurately assess where children are in terms of their mathematical content knowledge within days of the children starting school and yet the same teachers then spent the rest of the year teaching 80% of the children what they already knew along with giving multiple opportunities to practice their existing skills (Young-Loveridge, 1998). Results from the EMI-4 study (Young-Loveridge, Carr, & Peters, 1995) reported that when children in their study were ready to start school, 80% of children could rote count to 10, 42% could accurately count a group of 5 objects and 55% of these same could join a set of 2 and a set of 3 to make a set of 5 objects, indicating a significant level of mathematical competency.

The term 'school readiness' is used to describe a variety of understandings of what children need to know in order to be successful at school entry. The social promotion policy in place in New Zealand schools contributes to the notion that at particular stages/ages children have mastery of particular skills, strategies and knowledge. Evidence from the EMI-4 study (Young-Loveridge et al., 1995) demonstrated that children have relatively stable numeracy skills at age 5 and a considerable range of achievement levels and yet some teachers commented that they were only familiar with a small section of the mathematics curriculum as their children passed on to the next class level relatively quickly. This would suggest that these teachers had little awareness of the range of abilities these children possessed, and in fact they assumed a readiness for children to move on after a short period of time in the new entrant classroom adjusting to the ways of the school. Many teachers had a pre-determined view of the mathematics skills of five-year-olds when they started at school followed by firm ideas on what these same children needed to acquire before they moved to the next level, irrespective of the children's current skill levels.

Mathematics teaching and learning in New Zealand is currently guided by the New Zealand Curriculum Framework (Ministry of Education, 1993) and Mathematics in New Zealand Curriculum (Ministry of Education, 1992). Neither document specifies the level of achievement nor the level of mathematics knowledge expected of children when they begin their formal schooling.

At mathematics time in the new entrant classroom the teacher is primarily working with small groups of children while the other children are working on independent or practice activities for mathematics. The teacher is usually working alone, sometimes with the support of a teacher aide who may take responsibility for the supervision of a group working on a practice or an independent activity. The direction of the learning is generally determined by the teacher.

Many of the theories relating to mathematics learning, held by teachers of 5 year old children, which underpin classroom practices, are based on management procedures and classroom activities to support these. When teachers make decisions in their classrooms relating to how children learn, they appear to be based on teachers' personal beliefs rather than on some particular education theory based on specific knowledge (Bell, 1990; Farquhar, 1991; Spodek, 1988). McCaslin and Good (1992) found that many primary teachers' expectations of children when they [the children] commenced school were focussed on socialisation and direct instruction. "Work is seen as something they [the children] must do, while play is seen as something they [the children] can do" (Spodek, 1988, p. 163). This view reflects the idea that when children begin school it is the children who need to conform to the school expectations and fit in. The underlying message conveyed to the children is to obey (Biber, 1988), described by McCaslin and Good (1992) as "compliant cognition". Management practices in the classroom focus on conformity and order rather than on the intention of the curriculum to produce self motivated and active learners (Ministry of Education, 1992).

Methodology

A purposive sampling approach was used to select five schools from a similar geographic area, with decile rankings[10] ranging from 1 to 10. The schools were selected due to a prior relationship with the author, having already participated in the Numeracy Project professional development programme. The research involved semi-formal interviewing of the five new entrant teachers regarding their beliefs about how they decided what was important for them to teach children when they started school in mathematics followed by two observations of their classroom practice. Teachers were asked how they thought children best learned mathematics and what they believed was important to develop in their mathematics programme.

The questions in this semi-formal interview were adapted from a study by Miller and Smith (2004) that enabled themes to be gathered from the information from the interviews. The characteristics of an interviewer inevitably have an effect on the interviewee and the interview (Denscombe, 2001) and every effort was made to overcome these by attempting to maintain a passive stance while supporting the interviewee to be frank.

Two three-hour observations were conducted in each classroom focussing on the kinds of mathematical interactions that occurred. Data was recorded at five minute intervals as to the event that was occurring and full details were recorded of mathematical events. The observer was positioned as discretely as possible in each classroom to minimise the possible observer effects.

The interviews were conducted first, followed by the two, three hour observations over a two week period. The data was gathered during the first half of the school year.

The ethical guidelines of the New Zealand Association for Research in Education were followed in this

10 The decile rating of a school is based on a Government assessment of the school in terms of the nature of the school community, particularly regarding the predominant socio-economic make up of that community with 10 being the highest.

study and no teacher or school is able to be identified. Interviews were not recorded by dictaphone but were transcribed and given back to the participants for verification as the participants felt more comfortable with this form of interviewing.

The interview transcripts were analysed and categorised by major themes related to teacher beliefs about mathematics education to look for similarities and differences. The major themes decided on were the "units" used for analysis. This process is referred to by Denscombe (2001) as unitising the data. The observations were similarly categorised to elicit major themes relating to teacher practice in a mathematics classroom.

The major units chosen for analysis were related to:

- Teacher beliefs regarding children's learning of mathematics;
- Teacher perceptions of 5 year old children's mathematical knowledge on school entry;
- Teacher practice as observed relating to the following categories: directing learning, supporting learning, organising learning and evidence of specific mathematical interactions; and
- Teacher usage of curriculum.

Results

Teacher Beliefs Regarding How Children (Age 5) Learn Mathematics

All the teachers in this study believed that children learned mathematics by being exposed to hands on problem solving experiences that were fun. Children learned best by physically doing the problems and the teachers' role was to provide activities that supported the children as they learned. The teachers stated that it was important to cater for the children's own interests as this contributed to successful learning. Teachers believed that they provided a safe environment that allowed the children to develop the ability to take risks and move forward in their learning.

The Knowledge of 5 Year Olds on School Entry as Described by the Teachers

Without exception the teachers had a clear set of expectations relating to children when they started school, not just in mathematics but in every aspect of learning. The teachers felt that children should have some sound academic skills in place before they arrived at school. That is, children should know their letters and some numbers (Aorangi School), they should have the ability to sit for longer than five minutes (Cascade School) and they should have the ability to count and recognise sets of numbers (Dundass School). All the teachers mentioned the importance of the children being socially able to function at school in terms of being able to take turns, sit on the mat and be more focussed. The expectation of the teachers was that these skills should be put in place during their early childhood years.

These expectations were specific in all cases and clearly outlined the behavioural and knowledge levels the children should have when they begin.

> When children are nearly ready for school they need less play and more time preparing for school, like learning how to count and know their letters. (Aorangi School)

The Mathematics curriculum (Mathematics in New Zealand Curriculum – MiNZC) states an achievement objectives for level one (approximately 5 -7 years of age) as "make up, tell and record number stories up to 9 about given objects and sequence pictures" (Ministry of Education, 1992, p. 32). But the teachers already had clear expectations of what they expected the children to have mastered when they commenced school.

The MiNZC states that by the end of level one children should be able to "form a set of up to 20 objects" (Ministry of Education, 1992, p. 32). But on school entry, project teachers stated that:

> I want them to know their basic number concepts and how to make small sets of things. (Cascade School)

> Children should come to school being able to write their names and know some letters of the alphabet and be able to interact nicely. (Tongariro School).

Teacher Practices

The teachers had a clear vision of where they wanted the children to go in their mathematics learning. The classrooms were warm and caring with a variety of good mathematics displays and the teachers were concerned with children's learning. The teachers were in control in the classroom directing the learning of the children to ensure that all the gaps were filled. The teachers unanimously agreed that because the children came to school with limited skill levels, they (the teachers) had to plug these gaps before they could teach the children how to problem solve.

In terms of classroom practices, the teachers believed that it was important to provide meaningful and fun activities, to engage the children in problem solving with mathematics integrated into the whole day in order to develop life-long learners.

Observations of classroom practices portrayed a different story. While the teachers advocated a hands on problem solving approach to mathematics teaching and learning, in reality there was very little interaction in these classrooms. Teachers were firmly in control in the classroom and mathematics lessons followed a tight lesson plan. The format of the teaching was clear and structured and followed the teacher's plans with very little observed adaptations to the children's needs, interest or academic level. Four of the five teachers believed that they knew what 5 year old children needed to know so that is what they taught them.

Curriculum Usage by Teachers

Four of the 5 teachers either did not know where their copy of the relevant curriculum was or did not find it either relevant or useful for their teaching.

> I don't need the curriculum documents, I use the numeracy project books instead. I know what children of this age need so that is what I teach them. (Angela at Aorangi School)

> I don't find either document useful or relevant. (Donna at Dundass School)

Only one teacher (Tanya from Tongariro School) said that in her team they pulled out specific objectives from the curriculum and planned from these. She did not mention looking at the needs of the children while planning.

Discussion

Teacher beliefs regarding mathematics learning were centred round ensuring that the children had fun and had the opportunity to actively problem solve. In all cases the observations of classroom practice contradicted this more constructivist approach. In the classroom the teachers had a more traditional transmission approach to teaching. The teachers followed their plan and directed the children's learning according to their predetermined intentions (the teachers). There were few opportunities for the children to direct either the pace, the direction of the learning or to actively problem solve. The lessons were delivered with pre-determined outcomes with little or no opportunities for the children to follow their own direction or extend their own learning.

The classrooms were safe and welcoming for children and teachers but within this environment the children are facing few challenges. The teachers had a fixed idea as to the particular needs of children of this age and thus taught to pre-determined plans with little regard to the existing knowledge and skill levels of the children. This combined with four out of five teachers explicitly disregarding the use of the curriculum is a cause for concern.

Lack of specific knowledge relating to the curriculum, suggests that these teachers would have limited knowledge of the curriculum for early childhood, Te Whariki (Ministry of Education, 1996). The teachers, by holding firm views on the knowledge of 5 year old children, do not then need to attend to the knowledge and skills the children already have when they start school, they already know what they need. Teachers, by not attending carefully to children's prior knowledge and skill, run the risk of spending too much time teaching the children things that they already know. By focussing predominantly on behavioural expectations, opportunities for learning may not necessarily be capitalised on.

This is a small study and difficult to generalise from. The lack of knowledge relating to the primary school curriculum and possibly the early childhood curriculum would suggest that there needs to be some significant professional development in this area. The failure of these teachers to take into account what the children already knew is in direct contradiction to the curriculum which requires teachers to take into account children's prior knowledge and understanding in order to meet the specific learning needs of each child.

Teachers do have clear beliefs about the effective teaching of mathematics for children when they start school but in most instances their classroom practice contradicts their expressed beliefs. More work needs to be done in the area of teacher beliefs and practices to more fully understand these and how they can be shifted for the benefit of the children.

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Gaining Insight into Alice's Pedagogy with Respect to Five Dimensions of Numeracy

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Numeracy is a complex construct and has become an essential capability for the twenty-first century. This paper reports the case of one teacher, Alice, with respect to her positioning of numeracy in a reform curriculum. By analysing her conversations, Alice's numeracy pedagogy was described and visually represented according to five dimensions of practice. The case of Alice, as presented in this paper, demonstrates the potential for gaining insight into the numeracy beliefs and practices of individual teachers and for further understanding of the complex nature of numeracy teaching.

Curriculum reforms occurring in Tasmania over the past five years have brought a focus on pedagogy within the mathematics classroom. The importance of being numerate has been emphasised rather than solely knowing and doing mathematics. Tasmania's Essential Learnings curriculum framework (Department *of* Education, Tasmania [D*o*ET], 2002) put thinking skills and strategies at the centre of a values-based curriculum where the connection of knowledge and concepts across the curriculum was encouraged. The reforms have matched those occurring nationally and internationally in mathematics education in terms of a shift from procedural teaching approaches to a conceptual approach focusing on student thinking and reasoning (Anderson & Bobis, 2005).

The pedagogical shift required of teachers in reform environments has required that teachers extend their knowledge of mathematics to include what Shulman (1987) described as pedagogical content knowledge (PCK). PCK goes beyond knowledge of the subject matter itself, to knowledge of how to teach the subject, knowledge of curriculum and resources, and knowledge of how students learn. Teachers' knowledge, beliefs, and practices significantly influence student learning (Hill, Rowan, & Ball, 2005) and it is important to examine the pedagogy of teachers with respect to the teaching of numeracy.

The aim of this study is to describe the numeracy pedagogy of one teacher, Alice, to gain insight into the idiosyncratic nature of numeracy pedagogy, and to consider the implications for the teaching of numeracy.

Theoretical Framework

Numeracy is accepted as having its foundations in mathematics (Australian Education Council, 1990). Beyond this there are many and varied definitions of numeracy, each with its particular theoretical underpinnings. Just as Green (2002) advocated a synthesis of the operational, cultural, and critical dimensions that play a role in the development of literacy, consideration of the different dimensions that are necessary for the development of competent and effective numeracy practice is important. Mathematical language, skills, and functions are required for students to make sense of, and critically evaluate, the contexts in which the mathematics is embedded. The socio-cultural and critical aspects of knowledge construction enable the selection of appropriate mathematical tools and informed critique of both mathematics and society. This study is informed by social constructivist theory (Prawat, 1996; Shepard, 2001), drawing from contemporary cognitive, constructivist, and socio-cultural theories and acknowledging the important contribution each element brings to a comprehensive definition of numeracy.

A comprehensive review of the literature has resulted in the development of a view of numeracy, incorporating five dimensions of practice, presented in Table 1 (Skalicky, 2007). In particular, the work of AAMT (1998), Steen (2001) and Queensland School Curriculum Council (1999), in presenting comprehensive balanced views of numeracy extending across foundational mathematical concepts and skills, strategic thinking, disposition, recognition of context, and critical practice, informed the description of the five dimensions. Essential to all these conceptions of numeracy is the view that mathematics is a vital tool in today's society, a tool that should be accessible to all members of society. The conceptions also acknowledge the complexity involved in numeracy and the many aspects, beyond mathematical skill, that contribute to a high level of numerate behaviour.

Table 1

Dimensions of Numeracy (Skalicky, 2007, p. 3)

Aspects of knowledge construction	Dimensions of numeracy	Description
FOUNDATIONAL	Mathematics	The foundational understanding and use of the language, concepts, and skills of mathematics, as they relate to number, measurement, space, data and chance, and pattern and algebra.
PSYCHOLOGICAL	Reasoning	The use of (i) mathematical thinking strategies to question, identify, represent, explain, and justify mathematical approaches relevant to a given context, and (ii) general thinking strategies to support the problem solving process, from lower level cognitive processes, such as recall and application, to higher level critical thinking processes involved in evaluation, judgment, decision making, and creativity.
AFFECTIVE	Attitude	The confidence and disposition to choose and use mathematical understandings wherever required. Willingness to take risks and persevere in approaching new mathematics and new contexts.
SOCIO-CULTURAL	Context	The ability to select and apply the appropriate mathematical tools for sense-making in a given context and understanding how the context impacts on the mathematics. Contexts related to school and everyday life, public and social issues, and an awareness of mathematics connected to history and culture.
CRITICAL	Equity	Awareness that mathematics can be used inappropriately, can be represented to promote bias, and can therefore promote inequities in society. The ability to question assumptions and use mathematics in an analytical and critical manner to make decisions and resolve problems and investigations.

Method

The research reported in this paper forms part of a larger PhD project, investigating the positioning of numeracy by teachers of middle grade classrooms (Grades 5-8) in Tasmania's reform environment and student experiences of numeracy in these classrooms. The larger research project adopted a collective case study approach (Yin, 2003), with four case studies involving five participant teachers and their students. All teachers planned and implemented units of work informed by Tasmania's Essential Learnings curriculum framework (D*o*ET, 2002). These teachers were positive toward the broader curriculum reforms, and in particular had an interest in the teaching of numeracy. In addition, a representation of middle years grades was sought across a range of schools, both government and independent. The larger study used a combination of interview, observation, document, and photographic data to provide insight into the positioning of numeracy as enacted in the classroom by each teacher and the experiences of her students.

In this study the numeracy pedagogy of one teacher, Alice, is reported, based upon interview data. The teacher interviews were semi-structured and lasted approximately 40-50 minutes. They were designed to gain an insight into teacher beliefs and practices with respect to current curriculum reforms, views concerning the place of numeracy within these reforms, and the teachers' planning, teaching, and assessment practices.

Cluster analysis (Miles & Huberman, 1994, p. 248) was used to code the interviews. Segments were clustered according to the five dimensions of numeracy as detailed in Table 1 of the theoretical framework. Further analysis was undertaken, informed by Bloom's revised taxonomy (Anderson & Krathwohl, 2001), to categorise each dimension of numeracy as being low, moderate, or high in terms of the level it was displayed by the teacher. Table 2 presents an overview of the levels of performance that informed the second level analysis of the teacher interviews.

Table 2

Exhibited Levels of Numeracy Pedagogy

Coding Level	Degree to which dimension is exhibited	Key aspects illustrating the levels of numeracy pedagogy
0	N/A	Not demonstrated.
1	Low	Awareness demonstrated by describing and explaining concepts and ideas.
2	Moderate	Application in the classroom is described, using specific examples. Includes evidence of examination of practice.
3	High	Multiple examples used and present consistent demonstration of the dimension, with justification supported by an underlying philosophy. Critical reflection and evaluation of practice are evident.

Radar charts are used as a means of visually representing the numeracy pedagogy of each case at the time of the study and in context of the unit of work being taught. The radar chart was chosen due to the opportunity it provides to observe deviations in the degree to which each dimension of numeracy was evidenced.

Context of the Study

Alice was a Grade 5/6 primary school teacher. In 2004, Alice had participated in a D*o*ET targeted professional learning program for teachers of middle years students, Grades 5 – 8. The program, *Being numerate in the middle years*, occurred over six days and brought together 48 teachers from across Tasmania. It explored the role of numeracy, planning and structuring numeracy learning, thinking and working mathematically, mental computation, and the important middle years concepts of proportional reasoning; average; and fractions, decimals, and percents (Watson, Beswick, Caney, & Skalicky, 2005). Alice experienced a major shift in her beliefs and practices concerning numeracy teaching and learning as a result of participating in this program. Following this, Alice worked toward changing her classroom environment and her numeracy teaching.

Alice participated in the research project approximately six months after her involvement in the program. Alice's beliefs and practices concerning numeracy and its place in the curriculum were explored through the teacher interviews, documentation, and observational data. This study is concerned with the interview data only.

During the research, Alice taught a three-week numeracy unit on graphing. This learning took place during three of the four weekly numeracy sessions that were part of Alice's usual program. Alice usually ran her numeracy program alongside an integrated unit of work or learning sequence. It was Alice's numeracy unit that formed the focus of the study. The motivation for the numeracy unit came from a concern that arose about the students' difficulties in interpreting and representing information during an integrated unit of work designed to explore the unique features of the Australian continent. Alice wanted her students to gain the understanding that there are a variety of ways that information can be represented and to have the knowledge and skills to make and implement those choices from an informed perspective.

Results

Alice was working on her numeracy pedagogy and bringing those aspects of her teaching that came naturally to her in other curriculum areas, particularly literacy, into her numeracy teaching. Since the introduction of the Essential Learnings, Alice was aware of and described some significant changes that were occurring in her practice. Alice had begun to challenge her own ideas of mathematics teaching and was working with students to help them see the connections among concepts, for example, fractions, decimals, and percentages, rather than teaching them separately. Although Alice still found numeracy to be an area that did not naturally sit in her integrated learning sequences, she had made a significant shift in her focused numeracy time.

I was teaching the stuff, or some stuff, but I wasn't teaching them for understanding … my planning [now] reflects key concepts and ideas, not just well today we'd better do some addition but thinking about what are the big ideas that I want these kids to know.

The following subsections detail how Alice's conversation about her teaching could be described according to the five dimensions of numeracy as stated earlier in the Theoretical Framework in Table 1.

Mathematics. Alice expressed a sense that numeracy was more than the mathematical skills that she had previously focused upon. Alice revealed some uncertainties in her own mind about the distinctions between mathematics and numeracy, and also whether such distinctions were important. At the same time, Alice described numeracy as "that whole notion of using mathematics and transference of that kind of knowledge".

When Alice was describing the structure of her numeracy time she referred to focused time at the beginning of each numeracy block, for the development of number sense, and the inclusion of "explicit teaching time" where the strands of mathematics, including measurement, chance and data, pattern and algebra, and space where explicitly taught, with her overall objective being to plan "for understanding of key mathematical concepts and ideas". In sharing her discontent with seeing her mathematics teaching as disconnected from her other work, Alice evidenced her belief that mathematics has a role to play in understanding ideas and concepts planned for in other areas of the curriculum. She wanted to do this in a "positive way that [was] not a contrived way".

Alice sought to equip her students with the "language of mathematics" so that they could share effectively their strategies and solutions. The emphasis she placed on student understanding of important mathematics concepts was evident in Alice's conversations. For example, she discussed the concept of "doubling", and wanting to support her students in making connections between mathematics concepts, by considering the meaning and application of doubling in terms not only of number, but also in considering measurement and pattern relationships.

For Alice, mathematical skills also remained important: "I see these skills that you have to teach, data and how do we read and how do we collect and all those sorts of things … so many skills that we have to logically work through". With respect to the foundational mathematics in her thinking and program, Alice displayed a *moderate* level of absorption in terms of the criteria described in Table 2.

Reasoning. The development of mathematical thinking and reasoning played a very important role in Alice's classroom. She was very interested in seeing how the language of thinking applied to mathematics, with students often asked to "justify" their position or choice of a particular strategy or to "elaborate" on their thinking. Students in Alice's classroom were given the freedom to select and apply problem solving strategies. This had driven the shift in the culture of the classroom with students "beginning to see themselves as problem-solvers".

Alice had participated in and led many professional learning sessions in the field of Thinking, as a result of the curriculum reforms. She was very interested in the application of this within the numeracy classroom. Alice worked with her students to use tools to support their thinking within mathematical settings. For example, cooperative learning strategies, such as jigsaw techniques (Aronson & Patnoe, 1997) and think-pair-share activities (Kagan, 1994) were specifically taught to provide students with tools to share and describe their own thinking. Time was allocated at the end of every numeracy session for the whole class to share and to reflect upon and articulate their learning. Alice was clear about the outcomes that this shift in her teaching practice brought to student learning.

> They are actually making connections and seeing a range of possibilities … they are making those connections for themselves. ... They are moving away from that memorisation … that rote learning being their only strategy.

In terms of the psychological reasoning dimension of numeracy, Alice displayed a *high* level of absorption and implementation.

Attitude. In sharing the changes in her numeracy pedagogy, Alice identified a distinct connection between these changes and the changing attitude of her students toward their mathematics learning. She reflected upon the previous general view of her students in "[not] seeing themselves as capable in the area of mathematics". Alice had transformed not only the structure of her numeracy time and the classroom learning environment but she had also found that her own attitude to mathematics teaching had undergone a shift wherein numeracy was highly valued and seen as an important component of the curriculum. She was no longer accepting disruptions from outside during this time, "I had to change what was going on and let the kids see that this is really important to me and make it really important".

Alice wanted to build a "community of learners" where students shared and reflected upon their learning and she described these times as "exciting" for both her and her students. Her efforts to "create a learning environment … so that the kids will take risks" were resulting in the students "taking chances in the classroom". Alice believed that there was "real potential for these kids to be self-motivated, confident, articulate users of mathematics".

> Students are now free to manipulate numbers, they were frightened my kids, of numbers because I was going to ask them something really hard and they never put their hands up. Now they play around with numbers and they talk about numbers and the conversations in my classroom are really exciting.

The increasing development of a positive attitude toward mathematics for Alice and for her students was very entrenched in the changed culture of the classroom and the embedding of thinking within mathematical learning. With respect to the affective dimension of numeracy, Alice displayed a *high* level of absorption.

Context. In describing numeracy, Alice felt that "everything [had] to be in context to be meaningful". She was "trying to make [mathematics] meaningful and connected with the real-world". Her initial focus was on the transfer of number calculations to real-world situations. Alice also gave an example of using cooking to further develop students' understanding of the concept of doubling, and discussed the relevance of doubling in terms of measurement and use of recipes.

She still felt that "things [were] a bit contrived" and although she valued the transference of students' mathematical knowledge to new contexts she expressed a desire to work on the implementation of this area in more meaningful ways. Alice described her biggest challenge as incorporating numeracy into her planning of transdisciplinary units of work informed by the Essential Learnings framework and her desire to "drag mathematics out there into my learning sequences in a more "positive way that is not a contrived way". Alice displayed a *moderate* level of the context dimension of numeracy.

Equity. With Alice focusing on the development of understanding of mathematical concepts and building a classroom community that would enable students to begin to explore mathematical strategies this higher level element of critical engagement was not yet a part of Alice's pedagogy.

Alice's overall numeracy pedagogy at the time of the study is represented for the five dimensions, in the radar chart in Figure 1.

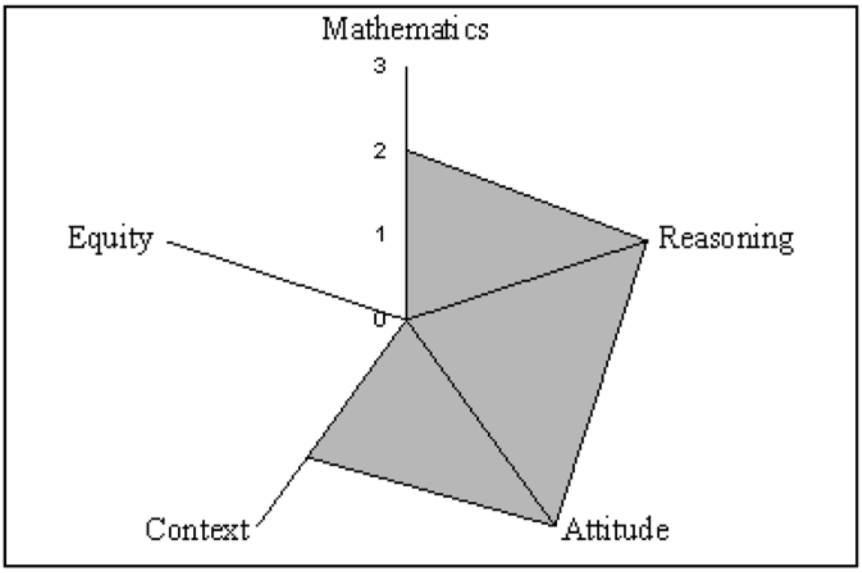

Figure 1. Alice's numeracy pedagogy.

Discussion and Conclusion

Alice was aware of and described some significant changes that were occurring in her practice. She was equipping her students with the language of both mathematics and thinking to support their participation in the learning environment. Alice was encouraging students to use and share their own strategies for problem solving and to develop understanding of mathematical concepts.

Alice was exploring ways that she could naturally connect mathematics to her integrated learning sequences, but found this challenging. She was therefore focusing on her dedicated numeracy block and the learning experiences she was providing for her students during this time. Alice felt that explicitly teaching thinking strategies and related language was important for students as it enabled them to approach tasks in a purposeful and meaningful way. The classroom learning environment was important to Alice, as she worked hard to establish a community of learners. The opportunity to work continually on this aspect of her teaching was a feature of Alice's practice.

The high levels exhibited by Alice in the dimensions of *Attitude* and *Reasoning* were reflections of her focus in these areas as she worked hard to develop her numeracy teaching within the reform environment. Alice's numeracy pedagogy, as presented in this study, provides a snapshot of her pedagogy at a point in time and in the context of a particular unit of work.

This study has examined the numeracy pedagogy of one teacher; distinguishing levels for each of the five dimensions, and when reported graphically, giving a visual representation of numeracy pedagogy at a point in time. The example case of Alice demonstrates the possibilities for considering a holistic view of numeracy teaching and the potential it provides to gain an in-depth understanding of the complex nature of numeracy teaching.

Alice's case is one example of the four case studies that formed part of the larger research project. All the teachers who participated in the larger project were recognised by their respective schools as being effective teachers of numeracy and all presented with distinctively different numeracy pedagogies from that of Alice. The four cases will be reported together at a later date.

Researchers in both Australia and the United Kingdom have identified key practices that are indicative of highly effective teachers of numeracy (Askew, Brown, Rhodes, Johnson, & Wiliam, 1997; Clarke & Clarke, 2002). This study seeks to acknowledge that, within the breadth of the many effective teachers of numeracy in today's classrooms, each teacher has their own idiosyncratic numeracy pedagogy with different strengths and areas of focus.

Recognising that teacher pedagogy is a changing construct, the results presented enable the distinction to be made within and across the different dimensions of numeracy both descriptively and visually. It is hoped that in describing numeracy pedagogy in this way it may help mathematics educators and teachers to continue to evaluate, reflect upon, and improve the teaching and learning of numeracy.

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Modes of Reasoning in Explanations in Year 8 Textbooks

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Understanding that mathematics is not just an arbitrary collection of rules to follow is basic to good mathematics learning, but studies show that many classrooms exhibit little mathematical reasoning. In order to better understand the nature of reasoning in schools, this study examined the modes of explicit reasoning in the explanations, justification and proofs of several topics in four textbooks. Eight distinct modes of reasoning are identified, illustrations of these are given, and their characteristics are discussed.

Introduction

Understanding that mathematics is based on reasons and is not just an arbitrary collection of rules to follow is basic to any good mathematical education. Reasoning, explanation and proof at an appropriate level should therefore be a prominent part of learning mathematics. Unfortunately, this is often not the case. For example, TIMSS Video Study (Hiebert et al., 2003) looked for evidence of mathematical reasoning in lessons in a random sample of lessons from 8 countries. In the 87 Australian lessons, they found almost no lessons which explicitly contained formal or informal proof or verification. They also identified 'making connections' problems where there was some linking between mathematical concepts, facts or procedures. In total, 15 per cent of Australian problems were in this making connections category, a low figure but similar to that in three other countries. When the actual solutions presented in the class (by teachers or students) were analysed, only 2 per cent of the total number of problems exhibited evidence of *making connections*. More commonly, the public solution was to state a concept, use a procedure or just give the result. Together these results point to an absence of mathematical reasoning in the average Australian Year 8 mathematics class.

Given the importance of encouraging reasoning, this paper aims to understand better the modes of reasoning that are available in Australian mathematics classrooms. We present a preliminary study of the reasoning that is evident in mathematics textbooks. The purpose is to examine the type of explanations, justifications and proofs that are, or might be, presented to Year 8 students, thereby aligning with the Video Study age group.

Harel and Sowder (2007) provide an important recent resource on proof in mathematics teaching and learning. They use the term 'proof' to include both formal and informal justifications, verifications and explanations and qualify it as 'mathematical proof' for that acceptable to professional mathematicians. Proving is seen as "the process employed by an individual (or community) to remove doubts about the truth of an assertion" (p. 808) and consists of two parts: Ascertaining (removing one's own doubts) and persuading others. In the case of textbook explanations, we note that the three processes of finding an assertion to prove (usually discovering a rule), ascertaining its truth, and persuading others of its truth are all involved, often in just a couple of lines of the textbook.

Harel and Sowder report on extensive studies of students' attempts to prove, which generally highlight the difficulties observed around the world both with student learning outcomes and with formally teaching about proof. Many studies are in the context of teaching students to prove in Euclidean geometry, but there are also reports of widespread attempts to imbue all mathematics teaching with a spirit of mathematical enquiry. Through analysing students' proofs, Harel and Sowder developed the construct of 'proof schemes'. A proof scheme is "what constitutes ascertaining and persuading for that person (or community)" (p.809). They group proof schemes in three classes: External conviction (e.g., depending on the authority of a teacher or book), empirical (e.g., depending on evidence from examples) and deductive, which includes mathematical proof. They cite many studies which show that students' proof schemes frequently belong to the external authority or empirical classes, rather than the desired deductive class.

The Video Study reported that at least 90 percent of lessons in all the countries made use of a textbook or worksheet (Hiebert et al., 2003), so we decided to examine explicit reasoning in textbooks, whilst appreciating the important role of the teacher as a mediator between the text and the student. There are a few studies of the nature of proof, justification and explanation in textbooks. Reys, Reys, and Chávez (2004) compared traditional US textbooks with recent 'standards-based' textbooks. They note that rather than merely "covering" topics, standards-based textbooks emphasise teachers helping students to "uncover" important

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mathematical ideas. A recent study by Stylianides (2008) examined how proof is promoted in a popular US standards-based curriculum for middle grades. He found that about 5% of student tasks involved proof in Harel and Sowder's sense. Key concerns raised were the need to increase students' understanding of what constitutes a mathematically legitimate proof; how to reconcile the often competing considerations of students developmental trajectory and mathematical integrity (see also Ball, 1993) and how to provide teachers with sufficient support.

Methodology

For this preliminary study, we examined the 2006 best-selling Year 8 textbooks (textbooks A, B, C and D) in four states. Each was a clear market-leader in its state, selected to give a picture of the mathematical explanations presented to many Australian students. Textbooks A, B, C and D are a subset of those that the present authors used in an earlier study and are coded in the same way (Vincent & Stacey, in press).

We examined several topics from Number and Measurement. State differences mean that not every topic was in every textbook. For each topic, we examined all the explanations, justifications and reasoning presented explicitly. We did not examine implicit reasoning, either within worked examples, or required when solving exercises. In almost every case, we found that texts only presented reasoning in the process of deriving a formula or rule which was immediately illustrated with worked examples and practised.

Each explanation was examined very carefully to identify the nature of the reasoning that supported the critical steps of the argument. The mode of reasoning was identified, and from these examples, a list of the modes of reasoning was created. These modes of reasoning include the proof schemes above, but we do not use this label because they do not necessarily constitute formal or informal proving for the textbook author. Some may only be explanatory pedagogical devices. The purpose of this paper is to illustrate these modes of reasoning. Four topics are selected for presentation below. It is not yet known whether these modes of reasoning are typical of the modes of reasoning for other topics, in other textbooks or at other year levels, nor how comprehensive this list is.

Eight Modes of Reasoning Illustrated

Modes of Reasoning

The following sections illustrate the eight modes of reasoning that were identified:

- Logical deduction (in Harel & Sowder's deductive proof scheme)
- Deduction through guided discovery (in H&S deductive proof scheme)
- Deduction of a rule from a model (in H&S deductive proof scheme)
- Concordance of a rule with a model (in H&S empirical proof scheme)
- Property extension (in H&S empirical proof scheme)
- Empirical demonstration (in H&S empirical proof scheme)
- Analogy (unclear if in any H&S scheme)
- Appeal to authority (in H&S external conviction proof scheme).

Division of Fractions – Concordance of a Rule with a Model

Two of the textbooks, A and B, introduced division of fractions. The other 2 textbooks only provided revision exercises without any explanation. Textbooks A and B began with the definition of reciprocal of a fraction, in one book as the fraction obtained by inverting the initial fraction, and the other gave the mathematical definition of the number by which the initial fraction is multiplied to give 1. In both texts the single aim of the section appeared to be to introduce and justify the 'invert and multiply' algorithm, and to practise its use in a variety of cases (e.g., with mixed numbers). Additional aims are possible; for example to give meaning to the concept of division by a fraction. As described below, this received a little attention in the explanation and in the subsequent exercises, which each included less than 5 word problems.

The derivation of the rule in both texts proceeded in a similar way,

- first, drawing on a model to solve a carefully chosen division problem,
- second, demonstrating that the invert and multiply rule produces the same answer,
- third, stating the invert and multiply rule
- fourth, giving several 'naked number' worked examples and much practice.

We call this mode of reasoning 'concordance of a rule with a model'. Meaning for division of fractions derives from a model of division. In both cases, quotition division was chosen as the model, presented in one case as the number of pizza quarters in half a pizza (giving the answer 2) and in the other as the number of half circles in 3 circles (giving the answer 6). These answers are then shown to correspond to the answers obtained by 'invert and multiply", which is then stated to be the rule to apply in future.

$$3 \div \frac{1}{2} = 6 \text{ and } 3 \times 2 = 6 \qquad \frac{1}{2} \div \frac{1}{4} = 2 \text{ and } \frac{1}{2} \times \frac{4}{1} = 2$$

There are variations between the textbook presentations. For example, textbook A presented a partition division example to supplement the quotition example, which certainly extends the meanings for division of fractions that students need to understand in order to identify situations where division of fractions is applicable. Textbook A also presented the rule as multiplication by the reciprocal, rather than invert and multiply, which has greater re-use potential in topics such as solving equations and ratio and proportion problems involving fractions or decimals.

In this 'concordance of a rule with a model' mode of reasoning the alignment of the answers obtained in the two ways is the essence of the explanation of why the result is true. It is possible to derive the rules by logical deduction based on the model, but this is not what the concordance mode does. For example, it is not difficult to argue that any number of pizzas can be cut into 4 times that number of quarter-pizzas; so that to divide by a quarter is always to multiply by 4; and that this argument applies for any whole number, not just 4. It is also simple to argue that to multiply by a quarter is to divide by 4 (both are taking a quarter of a quantity). In both cases, we see that division by a number (admittedly limited to whole numbers and unit fractions) is multiplication by its reciprocal. This different and more mathematical mode of reasoning we call 'deduction of a rule from a model'.

Another characteristic of the 'concordance of a rule with a model' mode of reasoning is that the real world or diagrammatic model is used to provide some initial meaning for the operation of division. However, there is no attempt to use the model as a tool for thinking, problem solving or for understanding the meaning of the answers to the many practice examples following. Gravemeijer and Stephan (2002) note that "Usually, something is symbolized ('model of'), and the symbolization is used to reason with ('model for')" (p. 159). In these fraction explanations, the model remains a 'model *of* division', but it never becomes a 'model *for* division'. The rule has been shown to be in concordance with one or two instances of the model, but it is not used to give meaning to the operation of division beyond this, even though students often find the meaning problematic. This seems to be a lost opportunity, although a Ball (1993) illustrates, thinking with models is often not easy.

One reason for moving quickly to the division algorithm is that using models to give meaning to division of fractions is not straightforward. The quotition meaning of division works well for some divisions (e.g., those above), and the partition meaning (sharing) works well for division of a fraction by a whole number, but for a randomly chosen fraction division (e.g., 2/7 divided by 4/9) neither meaning is really satisfactory and it seems best to think of such divisions simply as the inverse of multiplication. This is a serious pedagogical limitation of the models available for fraction division, but unlike both textbooks A and B, we think that a stronger development of meaning for division is required.

Multiplication of Two Negative Integers – Five Modes of Reasoning

Ogden Nash (1902 – 1971), an American poet best known for light verse, wrote:

Minus times minus results in a plus,
The reason for this, we needn't discuss.

The three textbooks that covered negative numbers in Year 8 (A, B and C) fortunately disagreed with Mr Nash about this notorious result. Five different modes of reasoning were evident. Textbook A used two modes. The first was 'appeal to authority'. Students create a spreadsheet that multiplies numbers, use it to multiply directed numbers and observe the results of certain calculations. We call this 'appeal to authority' because the spreadsheet here is providing answers (e.g., -5 × -6 = + 30, -5 × + 6 = -30) which students believe because they trust the spreadsheet. As an aside, we note that this is a common use of technology, for example, in calculator supported investigations in both primary and secondary schools. It seems that the technology is better able to act as the authority than a teacher.

Textbook A also used a mode of reasoning that we call 'property extension', as did Textbook B. These arguments extend, to negative whole numbers, the observation made on positive numbers that as the multiplicand in a multiplication table changes uniformly, the product also changes uniformly. This initial observation is illustrated in column 1 of Figure 1. The subsequent columns outline the steps involved in demonstrating successively multiplication of the three combinations of positive and negative numbers. The information upon which this 'property extension' mode relies can also be seen in the spreadsheets and multiplication grids which the textbooks show. However, the distinction that we make with the 'appeal to authority' is that in those cases, one or a few individual results were considered separately without highlighting the numerical patterns which reflect the properties of number operations. Textbook A presented the argument in some detail (comparable to Fig. 1). However, the Textbook B argument was much abbreviated and possibly most students and some teachers would miss the reasoning that lies behind it.

Step 1	**Step 2**	**Step 3**	**Step 4**	**Step 5**
Observe that when multiplicand decreases by 1, the product decreases by 5	Extend this observation to negative multiplicands, to infer that - × += -	Extend the property of commutativity, and start a new table	Observe that as multiplicand decreases by 1, product increases by 3	Extend this observation to negative multiplicands, to infer that - × - = +
5 × 3 = 15	5 × 2 = 10	-3 × 5 = -15	-3 × 4 = -12	-3 × 2 = -6
5 × 2 = 10	5 × 1 = 5	-3 × 4 = -12	-3 × 3 = -9	-3× 1 = -3
5 × 1 = 5	5 × 0 = 0	-3 × 3 = -9	-3 × 2 = -6	-3 × 0 = 0
5 × 0 = 0	5 × -1 = -5	-3× 2 = -6	-3× 1 = -3	-3 × -1 = +3
	5 × -2 = -10	-3 × 1 = -3	-3 × 0 = 0	-3 × -2 = +6
	5 × -3 = -15	-3 × 0 = 0		-3 × -3 = +9

Figure 1. Extending properties of whole numbers to directed numbers.

Textbook B also used reasoning by analogy. Directed numbers were modelled by movement east or west on a film running backwards or forwards. The analogy was drawn between multiplication of directed numbers and apparent movement across the screen. We have called this reasoning by analogy, rather than from a model because it was only presented at a qualitative level and because the mathematical correspondence between quantities and the operations was not made clear. In particular, no numbers were multiplied, only signs as in 'positive × negative = negative'.

Textbook C adopted a different approach, drawing on an interpretation of – as 'taking the opposite'. Multiplication by a positive number is established as repeated addition. (In this textbook as in all the others, directed number arithmetic is almost exclusively done with integers.) So +2 × +3 is interpreted as 2 lots of +3 (+2 × + 3 = 3 + 3 = 6) and +2 × -3 is interpreted as 2 lots of -3 (+2 × -3 = -3 + -3 = -6). To this point the argument is using logical deduction, from a special case that is intended to be general. 'Seeing the particular in the general' is a common feature of teaching and learning mathematics, discussed by authors such as Mason and Pimm (1984). When the multiplier is negative (as in -2 × +3), textbook C draws on the interpretation of the – sign as "the opposite of". Students are instructed to calculate 2 × +3 = 6 (2 lots of +3) and then they take 'the opposite', in this case -6. 'Doing the opposite' was introduced as part of the explanation of addition and subtraction of directed numbers, using the model of journeys along a number line. Since this is a valid model of addition and subtraction of negative numbers (although for various reasons, we believe it to be a difficult model for students to use), we classify the mode of reasoning here as 'deduction of a rule from a model'. The meaning of subtraction was given as "the subtraction sign between two numbers means *do the opposite of*" (e.g., move left instead of right). Ball (1993) discusses the difficulties of using a similar model to develop students' argumentation. Note that the argument presented in textbook C has two subtle mathematical flaws. First, it does not actually show that -2 × -3 = + 6 but instead shows that - (2 × –3) = + 6 which does not involve the multiplication of two negative numbers at all. Second, it confuses the subtraction operation (the binary operation) with the 'negative' operation (the unary operation taking the additive inverse). This example illustrates that as far as possible the classification of modes of reasoning is not concerned with mathematical correctness or completeness.

All the Australian textbooks tend not to use the presented model as a thinking tool but to replace it immediately by use of the rule in worked examples. This relates to a study by Mayer, Sims, and Tajika (1995). They compared teaching directed numbers in Japanese and US textbooks, noting that Japanese books made stronger connections between the models, words and calculations. This supported the Video Study findings (Hiebert et al., 2003), that found large inter-country differences in 'making connections' in lessons.

Area of a Trapezium – Logical Deduction and Guided Discovery

Textbooks A and C derived the rule by placing two congruent trapezia to form a parallelogram (Figure 2a), and then using the rule for the area of a parallelogram that had already been presented. This mode of reasoning is 'logical deduction'. The much abbreviated style of explanation (diagrams with few words) is common in textbooks, and is reminiscent of the 'proofs without words' enjoyed by mathematicians, as in Nelson (1993).

Textbook D used a different approach. Instead of deriving the rule and then setting exercises, this textbook placed two multi-step exercises leading to the area of a trapezium area within a problem set. Each exercise guided students to dissect a trapezium of specific dimensions and rearrange into shapes of known area (see Figures 2b and 2c). Textbook D further used this guided discovery approach for areas of kites and rhombuses. Only in a final section were the rules explicitly stated and practice exercises provided. This approach foregrounded the importance of students being able to find areas of polygonal figures of varied shapes by dissecting into areas of known shapes, rather than relying on memorised rules. We call this mode of reasoning 'logical deduction through guided discovery'. Although specific measurements are used in the Textbook D exercises, we judge that students are intended to see the generality in the particular, so do not class this as reasoning only from specific examples, as in Harel and Sowder's (2007) empirical proof scheme.

Area of a Circle – Deduction, Guided Discovery and Empirical Demonstration

Textbooks A, C and D all derived the formula for the area of a circle by dissecting a circle into sectors, rearranging them to form an approximate rectangle, and calculating the area of the rectangle, and hence the area of the circle (see Figure 3). We classify this mode of reasoning as logical deduction even though it requires very considerable refinement (especially related to the limit processes) to become a mathematically acceptable proof of the formula for the area of a circle. However, we judge that it functions well as a justification of the formula in this simplified version at Year 8 level.

Textbook C presented this argument alone, followed by exercises using the formula. Textbook A prepared students for this argument by preceding it with practical version of the dissection in Figure 3, where students cut a photocopy of a circular protractor into sectors, construct the 'rectangle' and hence find the area of the protractor. They were asked what would happen if the sectors were narrower, thereby acknowledging the limiting processes involved in the mathematical proof. We classified this activity as logical deduction, rather than empirical measurement, since we judged that it was to prepare students for the general argument.

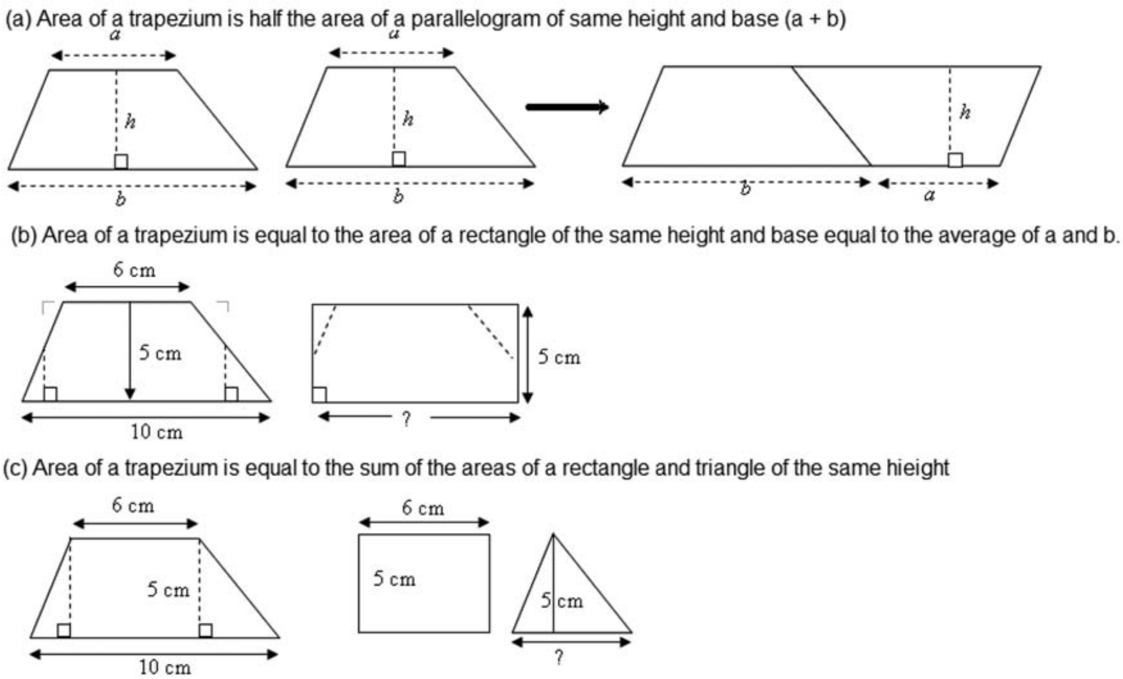

Figure 2. Three derivations of the area of a trapezium.

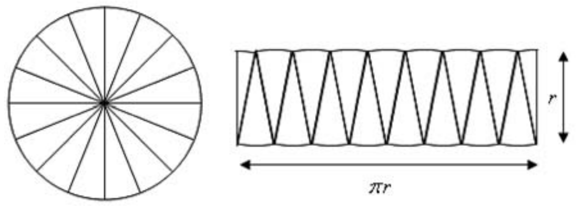

Figure 3. Area of a circle represented as area of a rectangle of equivalent area.

Textbook D presented a variety of approaches to justifying the area of a circle, approaching all through guided discovery. First, students find that the area of the circle must be between $2r^2$ and $4r^2$ (so approximately $3r^2$), by drawing circles inside and outside squares. This is followed by an *empirical* approach of placing a grid over the circle and showing 316 of 400 grid squares fall inside the circle (hence obtaining an area of $3.16r^2$.). Then students were guided through the explanation in Figure 3, supplying length, breadth and area of the rectangle themselves. Then, another dissection proof of the formula was also presented in guided discovery mode. Only then was the formula stated.

Conclusion

All of the textbooks made some attempt to explain each rule. Pleasingly, no textbook simply presented "rules without reason". On the other hand, almost always the sole purpose appeared to be to derive the rule in preparation for the practice exercises, rather than to use the explanations as a thinking tool. The explanations are, in general, very short with essential aspects of the reasoning unstated. Hence they are unlikely to stand alone, so students must rely on teachers to elaborate on the explanation provided. It is unlikely that all teachers can present these elaborations from the material provided, so this study further highlights the need for teachers' deep mathematical pedagogical content knowledge. Some of the electronic resources now being added to textbooks, including geometric dynamic demonstrations and templates for construction, are filling gaps. Nearly all of the explanations we examined were correct (rather than incorrect) although they were generally very curtailed and omitted basic reasoning (e.g., to state that a finding about one case also applies in general) as well as omitting difficult cases and subtle points.

Students encounter a considerable variety of the modes of reasoning in these explanations. At this stage of our investigation, it appears that the variety is between topics, more than between textbooks. An exception is that only one textbook used ‘deduction through guided discovery’ involving students actively in the deduction process. Four of the eight modes of reasoning are unacceptable from a logical and mathematical point of view: Reasoning by analogy, appeal to authority, empirical demonstration and concordance of a rule with a model. However, as illustrated above, all of these modes of reasoning might have a place in assisting students’ learning. The critical point for developing students’ mathematical reasoning is whether students understand that some modes of reasoning are indeed part of the acceptable range of modes of reasoning in mathematics, whilst others simply serve a local pedagogical purpose, such as helping them remember a rule.

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What Does Three-quarters Look Like? Students' Representations of Three-quarters

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Forty-one students from Years 3 to 10 completed a clinical interview on the topic of fractions. Four of the questions from the 20-30 minute interview involved the fraction $^3/_4$; students' responses to these questions are analysed in this paper. When asked to illustrate $^3/_4$, the most popular model chosen by students was a circle, yet fewer than 25% of the students knew that a circle divided into 4 parts with unequal areas (using 3 vertical lines) did not represent $^3/_4$. Such students are unable to identify the appropriate attribute which is relevant to the given model.

Understanding fraction concepts is clearly difficult for many students. For example, 69% of the Year 8 Australian sample in TIMSS (2003) correctly chose $^{16}/_{30}$ on this multiple choice item: *In a group of children, 16 have birthdays during the first half of the year, and 14 have birthdays during the second half of the year. What fraction of the group have birthdays during the first half of the year?* (Item Number M012041). Although this result compares favourably with the international average (52%) it is clear that about one in three Australian Year 8 students in this sample lacked fundamental fraction understanding after about four years of the topic being included in the school curriculum. This paper will further the research into students' understandings of fractions by investigating the models they use to represent fractions. In addition, we wish to determine if students understand the essential features of the models they use.

The interviews upon which this paper is based are part of a larger research project investigating the strategies that students use to compare fractions and how their choice of strategy is influenced by their understanding of fractions. The interviews included a mixture of tasks: for example, a fraction symbol was provided and the student was required to draw a representation either on a blank card or on the diagram/shape provided; the student needed to select which of the given representations matched the spoken fraction word name; and students needed to reconstruct the whole, given the part. The interviews also included a range of representations: discrete models and continuous models in various dimensions (linear, area, and drawings of volume), as well as ratio, operator, and quotient interpretations of fractions. The items considered in this paper all involved the fraction ¾.

Literature

Lamon (2007) provides a comprehensive discussion of research into rational numbers conducted over the past two decades, and the various interpretations which have been used: that is, part-whole, measure, operator, quotient, ratio/rate. Lamon notes the difficulties in teaching and learning topics involving multiplicative structures:

Of all the topics in the school curriculum, fractions, ratios and proportions arguably hold the distinction of being the most protracted in terms of development, the most difficult to teach, the most mathematically complex, the most cognitively challenging, the most essential to success in higher mathematics and science, and one of the most compelling research sites. (p. 629)

The teacher of the Grade 6 students in the study of Olive and Vomvoridi (2006) assumed that students were aware of the need for equal sized partitions of the whole (8 pizzas shared among 10 people) and did not attempt to draw this or mention it explicitly. The teacher's "use of approximate representations throughout her instruction, coupled with the lack of any explicitly spoken intention to draw equal parts, may have unintentionally supported some students' lack of an equipartitioning scheme for unit fractions" (p24).

Various researchers have noted the relative difficulty that many students experience with number line tasks. For example, Hannula (2003) reported on two tasks involving $^3/_4$ with over 1000 students in each of Grades 5 and 7. In the bar task, students were provided with a rectangle/bar (already marked in eighths) and asked to shade ¾. In the number line task, students were provided with a number line where 0 and 1 were marked, but were not the endpoints. As well as noting improvements in performance with grade, he found that the (overall) success rate for the number line task was 38% in contrast with 71% success in the bar task. Clarke, Roche, and Mitchell (2007) also found students experienced difficulty with tasks involving number lines. They report on clinical interviews with over 300 Grade 6 students and found that only around half of the students could draw an appropriate number line that showed $^{2}/_{3}$. Bright, Behr, Post, and Wachsmuth (1988) noted that students in their study found the 0 to 1 number lines easier than the 0 to 2 number lines.

Baturo (2004) devised a Cognitive Diagnostic Common Fraction Test to probe students' fraction understanding. The first item contained various two-dimensional drawings, some of which were equipartitioned but unusual shapes (e.g., a rhombus) while others were typical fraction shapes (e.g., a circle) that were not equipartitioned. Responses to such diagrams allow us to glimpse into students' thinking. Students who accept (incorrectly) all familiar shapes are not attending to need for equipartitioning. In contrast, students who reject (incorrectly) all unfamiliar shapes appear to be making decisions based on their familiarity with the shape; that is, they recognise only prototypical or iconic images.

Out study continues the tradition of clinical fraction interviews and includes tasks to probe students' understanding. We also note the cautions expressed by various researchers: that correct answers to one fraction task does not imply success on others and, that students can have difficulty verbalising their thinking. For these reasons we included complementary tasks to probe students' thinking with an expectation that some of these could be adapted to other modes of assessment (for example, pen-and-paper or online tests) having a strong diagnostic emphasis.

The sample

The 41 interviews reported in this paper were conducted as part of a larger project that involved testing approximately 800 students from Years 3 to 10 in four schools to determine the prevalence of various incorrect strategies for ordering fractions. The researchers compiled a list of students who made at least one error on the test as potential interviewees. Schools were provided with these lists and approximately 50% of these students returned signed consent forms; return rates were much lower for the older students, compared with the younger students. The 41 students who were interviewed do not constitute a random sample at each year level; rather, they have been chosen from the consenting students to create a diverse group of "non-expert" students. This procedure was used to ensure that interview time was spent only on non-expert students who could contribute to our knowledge of incomplete understanding. Table 1 contains the number of students interviewed at each year level.

Table 1

Number of Students Interviewed by Year Level

Year level	Yr 3	Yr 4	Yr 5	Yr 6	Yr 7	Yr 8	Yr 9	Yr 10	Total
Numb er of students	9	9	6	5	6	1	3	2	41

The Interview

The audio-taped interviews lasted 20 to 30 minutes and were conducted by the researchers in a quiet location in each school. Each student completed approximately 25 tasks, but only four tasks are discussed here. These tasks involved the fraction three-quarters, a non-unit fraction which students were expected to be familiar with, and which is easy to represent by repeated halving (hence avoiding the issues associated with partitioning into three or five equal parts). We wanted to learn more about students' conceptions of fractions, including the flexibility to move between various representations. As indicated in Table 2, in three of these tasks the students were handed a single card and the written instruction was read aloud by the researcher. In the remaining task, the students were handed a set of 8 cards and given a verbal instruction only. The four tasks are described in more detail below.

Task 6, *Model of choice*, is an unpublished item used in the interviews reported by Pearn (Pearn et al., 2003). The symbol $^{3}/_{4}$ was written on a card and students were asked to "draw a picture or diagram to show what it means" as well as "explain" their drawing. Student's drawings were judged to be correct or not, as well as analysed according to their chosen representation. This task was given early in the interview so that students' responses to this task would not be affected by the representations contained in other interview tasks.

In Task 8d, *Unit square* (Hollis, 1984), an unmarked square printed on a card and students were asked to "shade the shape to show the fraction $^3/_4$". Students' drawings were analysed for correctness as well as according to the way they subdivided the square.

In Task 9, *Sorting* (Willis, 2004), students were handed a set of 8 cards (labelled A to H) and given this verbal instruction: "Some of these cards show the fraction three-quarters and some don't. Sort them into two piles; put the cards showing three-quarters here and the other cards there". Details of Cards A to H are provided later in Figure 1, accompanied by the results. Cards A and D were prototypical representations of $^3/_4$ using the circular (area) and discrete models, respectively. Cards B and H were included to determine whether students appreciate the need for portions of equal area. Cards F and G were included to determine whether students had over-specialised their interpretation of $^3/_4$; they would reject F if they felt that the portions need to be adjacent and reject G if they did not understand equivalent fractions.

An additional task that also included the fraction $^3/_4$ was Task 16, *Number line*, (Newstead & Olivier, 1999). Students were handed a card with a number line with marks on 0, 1, 2, 3, and 4, and a list of four numbers; (A) $^3/_4$, (B) $1^1/_3$, (C) $^{12}/_6$, and (D) $^2/_3$. Students were asked to "Show the following numbers on the number line".

Table 2

Details of the Four Interview Tasks

Task Number & Name		Number of cards	Instructions	Symbol $^3/_4$ provided?	Representation provided?
6	Model of choice	1	Written & verbal	yes	no
8d	Unit square	1	Written & verbal	yes	Unit square
9	Sorting	8	Verbal only	no	8 models
16A	Number line	1	Written & verbal	yes	Number line

Results

The number of students completing each of the four tasks is provided in Table 3, as well as the number and percentage of students correct on each task. Clearly the first two tasks (where students were required to draw their own model to show $^3/_4$, and then to show $^3/_4$ of a unit square) were much easier for students than Task 9 (involving sorting a set of 8 cards) and Task 16A (marking $^3/_4$ on a 0 to 4 number line).

Table 3

Number and Percentage of Students Correct on Each of the Four Interview Tasks

Task Number & Name		Number of students who completed task	Number of students correct	Percentage correct given completed
6	Model of choice	40	32	80%
8d	Unit square	40	36	90%
9	Sorting	41	4	10%
16A	Number line	35	16	46%

Task 6: Model of Choice

Task 6 was answered correctly by 80% of the 40 students who completed the task. The most common correct response (17 students out 32) was a circle partitioned into four parts which appeared to be roughly equal, using one vertical and one horizontal line to create four sectors, where three parts are shaded. Another 7 students used the same partitioning on a square. Of the remaining 8 correct students, 4 students drew a rectangle (with base longer than height) which was partitioned with three vertical lines, and the other 4 students used a discrete model and shaded three of four objects. There were only 8 incorrect responses to this task; almost all of the incorrect responses came from students in Years 3 and 4. The most common incorrect response was to show 3 groups of 4.

Task 8d: Unit Square

Of the 36 students who gave correct answers to Task 8d, 32 students (89%) used the prototypical partitioning of the square (with one vertical and one horizontal line); three students used vertical partitioning (using three lines) and only one student used two diagonal lines.

Task 9: Sorting

All 41 students completed this task. Whereas only 4 students (10%) completed the full task correctly, over 80% of the students correctly placed 4 of the 8 cards (Cards A, D, F, and C). Clearly some of the cards were more difficult for students to sort correctly. The number and percentage of students who answered correctly on each of the 8 cards is provided in Figure 1. The asterisk on some cards indicates that they do *not* represent $^3/_4$ and so the correct response is to *reject* the card. The cards are presented in decreasing order of student performance. Card A was the easiest (40 students recognised that Card A does represent $^3/_4$) and Card B the hardest (only 10 students recognised that Card B did *not* represent $^3/_4$).

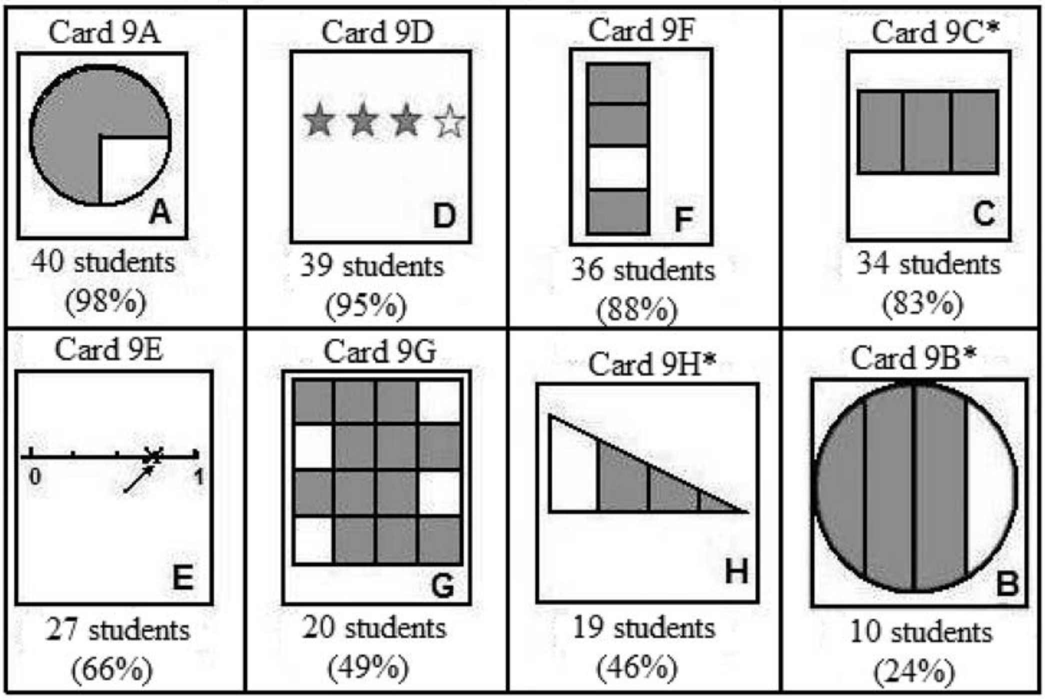

Figure 1: Number (%) of students correct on each card in Task 9 (*Sorting*), ranked from easiest to hardest. (* indicates correct answer is not $^3/_4$)

This group of non-experts was able to demonstrate the following fraction understandings: that the number of portions need not match the numerator (there is only 1 shaded portion in Card A); that fractions can be represented with a discrete model (Card D, although this is a special case where the denominator matches the number of elements in the set); that shaded portions do not need to be contiguous (Card F), and that matching the numerator with the number of pieces is not sufficient (Card C). In comparison with the discrete model (Card D), only two-thirds of the 41 students were able to recognise the number line model (Card E).

About half of the students accepted Card G as showing $^3/_4$. Of those who rejected this card, some might have made "minor" errors, such as miscounting or miscalculating when determining equivalent fractions, but it is likely that for many of these students, the rejection of this card indicates that the concept of equivalent fractions is not understood. All the students who talked about rearranging the squares in some of the rows (e.g., the second and fourth rows) successfully identified this card as showing three-quarters from the fact that the three shaded squares in each row could be seen more clearly.

Cards H and B were the most difficult cards for students. Note that the correct response for both cards is that neither represent $^3/_4$, and hence should be rejected. (While it might appear that this could be the reason that students found these more difficult than the other cards, consideration of Card C disproves this hypothesis as it also does not represent $^3/_4$, yet was answered correctly by over 80% of the students.) To further investigate students' responses to Cards H and B, Table 4 provides a cross-tabulation of the 41 students who completed this task. Only 8 students were correct on both cards; their correct rejection indicates that they appreciate the requirement for equal-sized portions when using the area model.

Table 4

Cross-tabulation of Responses to Cards H and B in Task 9 (Sorting)

Card H*	Card B*		Total
	Correct	Incorrect	
Correct	8	11	19
Incorrect	2	20	22
	10	31	41

* The correct answer is to reject these cards as showing $^3/_4$

Another 20 students were incorrectly accepted both cards as representing $^3/_4$. These students may be focussing on the number of shaded parts (3) and the total number of parts (4) without attending to the need for equal-sized portions, or they may focus their attention on the equal-sized *lengths* (on the base of the triangle in Card H, and the diameter of the circle in Card B). In retrospect, additional cards should have been included which had unequal partitions in length (i.e., one-dimensional) as well as the unequal partitions in area (two-dimensional). Table 4 shows that another 11 students were correct on Card H but incorrect on Card B; so whereas they rejected Card H, this was not done by a careful consideration of the unequal areas (as this reasoning should have resulted in correctly rejecting Card B). Their reason for rejecting Card H and accepting Card B might be that the triangle is not a familiar model for fractions, while the circle is a familiar model. In other words, for some students, their familiarity with the shape in the context of fractions seems to be an important issue, rather than a consideration of the essential features of fractions.

To further contrast the success rate of 80% on Task 6 (Model of choice) with the much lower rate of 24% on Card B in Task 9, students' responses to the two tasks were examined. Of interest are the 11 students who chose to use a circular model in Task 6 and correctly drew a prototypical representation of $^3/_4$ but were then unable to reject the circle with four equally spaced vertical partitions on Card B in Task 9. These 11 students were spread across Years 3 to 9 indicating that this is not solely a difficulty of younger students.

Task 16A: Number Line

Table 3 shows that only 16 students (46% of the 35 students who completed this task) were able to correctly locate $^3/_4$ on a number line from 0 to 4. There were two common wrong answers; 7 students marked the point 3 (they were clearly finding $^3/_4$ of the given line, so seeing $^3/_4$ as an operator rather than as a point which is less than one) and another 7 students who marked $^3/_4$ somewhere between 3 and 4. Figure 2 is a sample from a Year 5 student who marked $^3/_4$ at 3 and then correctly marked the next three points; inconsistently marking $^2/_3$ as a point (measure) rather than as an operator. Perhaps, after marking both B ($1^1/_3$) and C ($^{12}/_6$) as measures rather than as operators, this student continued with this interpretation for D ($^2/_3$). These results confirm Hannula's (2003) findings. He also found students who correctly marked certain numbers on the number line (e.g., 1.5 and $2^1/_5$) but were unable to mark $^3/_4$. He noted that some students could not make any mark as they thought that $^3/_4$ was not really a number. For those that did make a mark, some placed $^3/_4$ at 3.4, and others used the operator interpretation and attempted to find $^3/_4$ of some unit, which may have been the full length of the line provided, or some sub-interval.

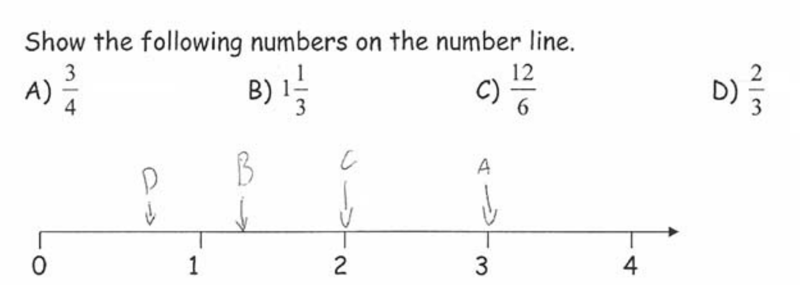

Figure 2: Sample of Task 16 (Number line) by a Year 5 student. Note the placement of ¾ at 3.

It is interesting to compare students' responses to the two tasks with number lines. Of the 25 students who could recognise $^{3}/_{4}$ on a 0 to 1 number line (Card E in Task 9), and who also completed Task 16A, only 12 were able to reproduce this point on the longer number line. We also note that two groups of students (i.e., students with measure or operator interpretations of fractions) will answer correctly on the 0 to 1 number lines, but only the students with the measure interpretation will answer correctly on longer lines.

Conclusion

When asked to illustrate $^{3}/_{4,}$ the most popular model chosen by students was a circle, yet fewer than 25% of the students knew that a circle divided into 4 parts with unequal areas (using 3 vertical lines) did not represent $^{3}/_{4.}$ Hence, we have found evidence of students who recognise only the prototypical or iconic representations of fractions. These students might initially appear to demonstrate fraction understanding, however their tendency to accept incorrect models means that they are not paying attention to the essential features of fractions. When students were asked to explain their thinking during the interview (for example, on the sorting task) students typically mentioned "three out of four parts are shaded" and did not discuss equal lengths or equal area. This is an instance where it might be possible to diagnose their thinking from their actions rather than their words.

The confusion between *3 groups of 4* and *3 out of 4* was found to be limited to students in Years 3 and 4. About two-thirds of the students recognised the point $^{3}/_{4}$ on a 0 to 1 number line, and about half of this group was later able to mark a longer number line with the point $^{3}/_{4}$. Clearly some students confuse the operator construct with the measure construct, and further research might confirm that asking students to mark numbers greater than 1 (such as $1^{1}/_{3}$ and $^{12}/_{6}$) helps them to focus on the measure construct.

The focus of this paper has been an analysis of students' representations of the fraction $^{3}/_{4}$. The sample of 41 students from Years 3 to 10 is not intended to be a representative sample, but rather a collection of non-expert students whose differential abilities to complete various tasks has confirmed results of other research and included some surprises. On the other hand, these non-expert students were not hard to find, so we can expect that there will be some students in most classes from Years 7 to 10 with fraction knowledge that is confused, incomplete, or incorrect and who exhibit basic misunderstandings.

We concur with Clarke et al. (2007),

> Students need more opportunities to solve problems where not all parts are of the same area and shape … it is clear that fraction as a measure requires greater emphasis in curriculum documents and professional development programs, as many students are clearly not viewing fractions as numbers in their own right" (p. 215).

Students need to have their attention directed to the *attribute* on which the model is based: relative number (for the set model), relative length (for linear or one-dimensional models), relative area (for two-dimensional models), and relative volume (for three-dimensional models). Without this attention, students are merely trying to recognise familiar pictures or icons.

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Some Key Junctures in Relational Thinking

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This study uses number sentences involving one and two unknown numbers to identify some key junctures between relational thinking on number sentences and an ability to deal with sentences involving literal symbols. Number sentences involving two unknown numbers distinguish between students who are restricted to computational approaches and those who can genuinely engage in relational thinking. Furthermore, such sentences allow the identification of different stages of relational thinking.

Rationale for the Study

Irwin and Britt (2005) argue that the methods of compensating and equivalence that some students use in solving number sentences may provide a foundation for algebraic thinking (p. 169). Carpenter and Franke (2001) refer to the thinking underpinning this kind of strategy as relational thinking. There is, however, still a debate about whether relational thinking when applied to number sentences can be properly described as algebraic. Here, the view of Jacobs, Franke, Carpenter, Levi and Battey (2007) seems very appropriate:

> One could debate whether our characterization of relational thinking in arithmetic represents a way of thinking about arithmetic that provides a foundation for learning algebra or is itself a form of algebraic reasoning. A case could be made either way. One fundamental goal of integrating relational thinking into the elementary curriculum is to facilitate students' transition to the formal study of algebra in the later grades so that no distinct boundary exists between arithmetic and algebra. (p. 261)

A clearer picture is needed of "how these ideas … develop in children's thinking, and the critical junctures in this development" (Katz, 2007, p. 10). Understanding these critical junctures is important in identifying the development of relational thinking. For example, if children are able to apply successfully ideas of equivalence and compensation to solve number sentences involving all four arithmetical operations, can they also use ideas of equivalence and compensation to deal successfully with sentences involving literal symbols? Moreover, is it possible to identify in students' justifications of their relational thinking any linguistic markers of development of their algebraic thinking?

Methodology

Design of questionnaire

Previous research (Stephens, 2007) used missing number sentences involving only addition and subtraction. Given the central role of the ideas of equivalence and compensation in relational thinking, it is important to get a picture of students' relational thinking across the four operations as they near the end of primary school or begin secondary school. In this study involving students in Year 6 and Year 7, number sentences involving all four arithmetical operations were therefore included. These took the form of sentences with one missing number where the value of that number can be found either by relational thinking or by computation. Examples of items used in the study were as follows:

$43 + \square = 48 + 76$, $\quad 39 - 15 = 41 - \square$, $\quad \square \times 5 = 20 \times 15$, $\quad 21 \div 56 = \square \div 8$

Students were also asked to write briefly how they found the value of the missing number.

Previous studies (Stephens, Isoda, & Inprashita, 2007; Stephens, 2007) identified a small group of students who successfully used computational methods to solve missing number sentences of this type, but who were also able to deal, more or less successfully, with expressions involving literal symbols. This group of students appeared to opt for computational methods to deal with missing number sentences, but were quite capable of using ideas of equivalence and compensation to solve sentences involving literal symbols; and were clearly different from those students who could use only computational methods.

The study therefore needed to include a type of numbers sentence where students are "pushed" to think relationally. Number sentences involving two unknown numbers, such as 18 + (Box A) = 20 + (Box B), seem to have this potential. While it is possible to use computational methods to give particular instances of correct sentences taking this form, identifying a general structural relationship requires students to move beyond computational thinking. For example, a clear relational explanation might say that the above sentence will always be true *as long as the number in Box A is two more than the number in Box B*. Being able to derive a correct mathematical generalisation from numerical examples is key element of algebraic reasoning (Carpenter and Franke, 2001; Zazkis and Liljedahl, 2002).

Number sentences involving two unknown numbers – across all four operations – were included in the study. These allowed some scope for computational approaches, but, following Fujii (2003), identifying the critical numbers and the relational elements embodied in these expressions required that students move beyond computation and focus especially on expressing the underlying mathematical structure.

Finally, to determine if, as Jacobs et al. (2007) suggest, equivalence and compensation provide a foundation for algebraic thinking, some questions involving literal symbols were included. Therefore, several questions modelled after the research programme, *Concepts in Secondary Mathematics and Science* (CSMS, see Hart, 1981), asked students: What can you say about c and d in the following mathematical sentence? $c + 2 = d + 10$

This third type of question allowed students to say that this sentence will be true for *any* values of c and d provided c is 8 more than d. But other students may fall short of this simply giving several values of c and d for which the sentence is true. Other students may say "c is more than d" but cannot specify the relationship. The value of questions such as these is that they can be given partial or complete relational interpretations.

A questionnaire, consisting of eight pages, was comprised of these three types of questions with each type distributed across the four operations. A first type comprised missing number sentences involving one unknown number. A second type comprised arithmetical sentence with two unknown numbers. A third type was structurally similar to the second type but explicitly included literal symbols.

Type 1: Missing Number Sentences with One Unknown Number

The position of the box denoting a missing number was varied for each item. The given numbers were chosen so as to provide numbers either side of the equal sign such that a relational approach to finding the missing number was attractive. Of course, it was also possible for students to solve each question by computation. After each item, space was provided for students to write how they had found the missing number. Similar Type 1 questions (Questions 3, 5, and 7) were used for subtraction, multiplication and division, with following sample items: $104 - 45 = \square - 46$; $36 \times 25 = 9 \times \square$; and $18 \div \square = 6 \div 10$. Figure 1 shows Type 1 questions used for addition used on the first page:

1. For each of the following number sentences, write a number in the box to make a true statement. Explain your working briefly.

$23 + 15 = 26 + \square$	$73 + 49 = \square + 47$
$43 + \square = 48 + 76$	$\square + 17 = 15 + 24$

Figure 1. Question 1 involving addition and Type 1 questions.

Types 2 and 3: Sentences Involving Two Related But Unknown Numbers

Each even-numbered page opened with a Type 2 question using two boxes, denoted by Box A and Box B, and employing one arithmetical operation. Type 2 questions are exemplified in parts (a) to (d) in Figure 2. These were then followed by a related Type 3 question, shown in part (e), involving the same arithmetical operation and literal symbols.

2. Can you think about the following mathematical sentence:

$18 + \square = 20 + \square$

Box A Box B

(a) In each of the sentences below, can you put numbers in Box A and Box B to make each sentence correct?

$18 + \square = 20 + \square$

Box A Box B

$18 + \square = 20 + \square$

Box A Box B

$18 + \square = 20 + \square$

Box A Box B

(b) When you make a correct sentence, what is the relationship between the numbers in Box A and Box B?

(c) If instead of 18 and 20, the first number was 226 and the second number was 231 what would be the relationship between the numbers in Box A and Box B?

(d) If you put any number in Box A, can you still make a correct sentence? Please explain your thinking clearly.

(e) What can you say about c and d in this mathematical sentence? c + 2 = d + 10

Figure 2. Question 2 involving addition and Type 2 and Type 3 questions.

This same format was used for questions involving the other three operations. Question 4 for subtraction was: 72 – (Box A) = 75 – (Box B). Question 6 for multiplication was: 5 × (Box A) = 10 × (Box B). Question 8 for division was: 3 ÷ (Box A) = 15 ÷ (Box B). Corresponding Type 3 items were: $c - 7 = d - 10$; $c \times 2 = d \times 14$; and $c \div 8 = d \div 24$.

The Sample

The sample was drawn from Year 6 and Year 7 students in two schools, one in Australia and one in China. The Chinese sample consisted of two intact classes consisting of 32 students in Year 6 and 36 students in Year 7. In the Australian school, one Year 6 class of 25 students was involved and three Year 7 classes consisting of 71 students altogether. The sample was a convenience sample. The performances of students are therefore not presented as being normative of schools in each country, and may reflect the teaching they have received. It is, however, possible to examine students' performances on the three types of sentences, and to track what students do over certain junctures. Translation of the questionnaire into Chinese was prepared by faculty members at an Eastern Chinese university. Graduate students at the same university and two Chinese speaking graduates in Australia assisted with the translation of students' responses. Each student's written responses were read independently by two markers. A very high degree of consistency of classification was evident across markers in both countries.

Key Questions to Be Investigated

Several questions guided an analysis of students' responses. First, what evidence was there of computational and relational approaches to Type 1 sentences and what forms did this thinking take? Then, what forms of thinking were students able to use on Type 2 sentences? Were Type 2 sentences able to discriminate between computational and relational thinkers, and among relational thinkers? Were particular forms of explanation –

written descriptions and/or mathematical representations – able to distinguish between students? And finally, were those who showed sound relational thinking on Type 2 sentences able to deal successfully with Type 3 sentences involving literal symbols?

Results and Discussion

All students attempted the addition and subtraction questions involving Type 1, 2 and 3 sentences Some Year 6 students in the Chinese school had difficulty going any further, but this provided sufficient evidence. Year 6 students in the Australian school and Year 7 students in both schools generally completed all, or most of, the questionnaire.

Results on Type 1 Sentences

On Type 1 sentences, relational thinking was evident when, for example, written descriptions, arrows or diagrams were used to compare the size of numbers either side of the equal sign; and where these descriptions, arrows or diagrams were used in chain of argument, based on uncalculated pairs, using compensation and equivalence, to find the value of a missing number. When arrows were used, sometimes they went in the same direction; sometimes in opposite directions on a given item, with the direction of compensation varying accordingly. By contrast, in computational responses, students always completed the calculation on the opposite side to where the number denoted by ☐ is shown, and then used this result to calculate the value of the missing number.

In solutions to Type 1 sentences, similar, if not always identical, patterns of relational thinking were evident in both countries. Students in the Australian school who used relational thinking generally preferred to use arrows to solve questions involving all four operations. This may have been a reflection of how they have been taught. In the Chinese school, written justifications of relational thinking were more prevalent on questions involving addition and subtraction, but directional arrows were almost universally used by in solving questions involving multiplication and division.

Students were classified as Relational if they successfully gave relational responses on at least one of the four operations. Students were also classified as Relational if they gave a mix of relational and computational responses. Only those students who gave computational responses on all attempted questions were classified as Computational. Even when occasional errors were present among responses, these mistakes had no bearing on the overall classification. Table 1 shows performance across the two schools according to whether students had used computational or relational thinking on Type 1 sentences:

Table 1

Performance on Type 1 Sentences

Schools and numbers at each Year level	Computational		Relational	
Chinese School Year 6 (N = 32)	20	63%	12	37%
Chinese School Year 7 (N = 36)	8	22%	28	78%
Australian School Year 6 (N = 25)	1	4%	24	96%
Australian School Year 7 (N = 71)	17	24%	54	76%

Results on Type 2 and 3 Sentences

Type 2 and Type 3 sentences had been deliberately crafted to "push" students into relational responses even if it was possible for them to complete parts (a) of these questions by computation. Almost all students without exception were able to place numbers correctly in Box A and Box B to make a correct sentence. Some students admittedly chose quite small numbers to place in the boxes to give correct sentences.

Responses to Type 2 sentences

Having constructed several correct sentences in this way, all students attempted to describe the relationship between the numbers in Box A and Box B. However, there was a clear difference between those who simply commented on the numbers used in the boxes, and those whose responses clearly focussed on the conditions needed to make a given sentence correct. For example, in answering Question 2b, some students merely said “Two more” or “Two difference” or “Two between A and B”. By contrast, other students used precise relational expressions to answer the same question, such as “Box A needs to be two more than Box B”, or wrote “Box B + 2 = Box A”. Other clear instances of relational thinking were evident when students used literal symbols in Box A and B such as in Question 4a, $72 - m = 75 - (m + 3)$, or where students used purposefully large numbers, such as for Question 8a*,* $3 \div 1{,}000{,}000 = 15 \div 5{,}000{,}000$ (Australian Year 7 student); or for question 2a, $18 + 1{,}000{,}000 = 20 + 999{,}998$ (Chinese Year 7 student).

Responses that simply compared the numbers used in Box A and Box B were certainly not wrong, but they appeared to fall short of responses, such as given immediately above. Satisfactory responses to parts a, b and c did not always predict a successful response to part d items which asked “If you put any number in Box A, can you still make a correct sentence?” These part d items distinguished between emergent and clear relational responses. Some students who used emergent relational thinking in parts b and c either rejected the possibility that any number could be used in Box A, or offered incomplete explanations such as “Only if B is correct”. Clear relational responses are evident in the following: A Chinese Year 6 student answered Question 2d involving addition by saying, “As long as B is two less than A”; and an Australian Year 7 student answered Question 4d involving subtraction by saying, “Any number can be in Box A so long as Box A is 3 less than Box B, otherwise the result will be disrupted”.

Responses to Type 3 sentences

Students who could not give a satisfactory answer to the question, “If you put any number in Box A, can you still make a correct sentence?”, could not deal with Type 3 sentences involving *c* and *d* as literal symbols. Many of those whose responses showed emergent relational thinking in parts b and c did not attempt part e items. A few students gave specific values of *c* and *d* that made the accompanying mathematical sentence correct. More frequent instances of incomplete or emergent relational thinking were, “Just put the right numbers in both sides and let the equation be correct”, or “*c* is bigger than *d*” (in Questions 2e and 6e) or the reverse (in Questions 4e and 8e). These incomplete responses are contrasted with the following exemplars of relational thinking: “The *d* is worth 8 less than the *c”* (Australian Year 6 student in answer to Question 2e), and “Can! No matter what is A, so long as B = 5 × A (Chinese Year 7 student in answer to Question 8e).

What these exemplars also show is the importance of logical qualifiers in explaining relational thinking. These logical expressions were also evident in responses to Type 1 sentences. However, in the case of Type 2 and 3 sentences, the use of these logical expressions was more varied and prevalent. Mini-arguments, described by Vergnaud (1979), in the form “Because ……therefore…..” were consistently used by relational thinkers in their responses to Type 2 and Type 3 sentences, as were logical qualifiers such as “should be”, “must be”, “has to be”, “needs to be” (and their Chinese equivalents). Emergent relational responses rarely included any logical qualifying expressions. Two contrasting responses to item 8e show this. Simply saying that “*d* is bigger than *c*” falls a long way short of saying that “*d* must be three times the value of *c*”.

Relational and Emergent Relational Thinkers

Emergent Relational thinkers typically completed the three replicas of the Type 2 sentence by using numbers that were small and/or easy to calculate. Their descriptions of the relationship between the numbers in Box A and Box B in parts b and c generally took the form of a commentary on the numbers used, such as “A is 2 more” [item 2b] instead of a clear statement of what needs to be the case if both sides are to be equivalent. These students did see a pattern between the numbers denoted by Box A and Box B, but whether they understood the general conservation principle that would make such a sentence true was clearly determined by their responses to parts d and e.

In their responses to part d items, Emergent Relational thinkers could not formulate a general statement which would allow *any* number to be used in Box A. Some did say that different values of A and B were possible, or that these values needed to be balanced. They were likely to offer similar incomplete statements in regard to the values of c and d in part e. Sometimes they gave an incorrect relationship between c and d, such as $d = 7c$ [item 6e]. Students who gave incomplete relational responses to all attempted items for Type 2 and Type 3 sentences were classified as Emergent Relational. If any mathematically complete explanations were given to these same items, students were classified as Relational. All students could be classified as either Emergent Relational or Relational.

Among Relational thinkers, there was a striking association between making a clear and correct response to part d *and* describing the relationship between the values of c and d to make the corresponding Type 3 sentence true. Among all Year 6 students, 70% of those who gave a correct response to a part (d) item also correctly described the relationship between c and d in the corresponding part (e) item. For Year 7 students, in both schools, a successful response to a part d item was followed in 90% of cases by a successful description of the relationship in part e between c and d. In no case, was a successful response to a part e item preceded by an inadequate response to its related part d

Questions involving two unknowns clearly met their purpose of pushing students beyond computation. In addition, there were students in the Australian school and in the Chinese school whose responses were consistently and clearly relational in regard to *all* attempted questions relating Type 2 and 3 sentences. These students' responses are referred to in the notes to Table 2 which shows the relationship between computational and relationship responses to Type 1 sentences and responses to Types 2 and 3 sentences.

Table 2

Comparing Performances on Type 1 with Performances on Types 2 and 3

	Type 1	Types 2 & 3	
Chinese	Year 6 Computational (n =20)	15	5
	Year 6 Relational (n = 12)	4	8
	Year 7 Computational (n = 8)	5	3
	Year 7 Relational (n =28)	0	28[a]
Australian	Year 6 Computational (n = 1)	1	0
	Year 6 Relational (n = 24)	11	13[b]
	Year 7 Computational (n = 17)	13	4
	Year 7 Relational (n =54)	0	54[c]
		Emergent	**Relational**

Note a: 6 out these 28 students successfully completed all questions; **Note b**: 3 out these 13 students successfully completed all questions; **Note c**: 28 out these 54 students successfully completed all questions

Analysing the performances of the four different groups does allow several important conclusions to be drawn concerning some key junctures in relational thinking.

Conclusions

From Table 2, it can be seen that fifteen Chinese Year 6 students used computational approaches on Type 1 sentences *and* were unable to deal successfully with Type 2 or 3 sentences. Their difficulties with Type 2 and Type 3 sentences suggest strongly that these students are restricted to using computational methods. Five students who chose to work computationally on Type 1 sentences were able to shift into relational thinking when confronted with sentences involving two unknowns. By contrast, eight students out of the twelve who used relational approaches to deal with Type 1 sentences were able to deal successfully with Type 2 or Type 3 sentences, even if they did not cover all questions.

The Year 6 Australian group showed clear evidence of relational thinking on Type 1 sentences, even if some students were not able to complete all questions on the questionnaire. This may well be attributed to the explicit emphasis that their teacher places on relational methods to solve Type 1 sentences. This Australian Year 6 group demonstrates very clearly that successful use of relational approaches to solve sentences involving one unknown does not translate automatically into success in dealing with sentences involving two unknowns. Almost half of those who had used relational approaches Type 1 sentences had some difficulty with Type 2 sentences. Transition across this juncture can by no means be assumed, possibly because the second type of sentence is structurally more complex and mathematically more demanding for students. This also suggests that specific attention needs to be given to Type 2 sentences.

By contrast, the majority of students in Year 7 in both the Chinese and Australian school employed relational approaches to deal with Type 1 sentences, *and* were generally capable of dealing successfully with Type 2 and 3 sentences. The juncture between relational thinking on numbers sentences involving one missing number and those involving two unknown numbers appears more assured with these Year 7 students. This is not to suggest that relational thinking can be assumed among Year 7 students. Previous studies show the presence of relational thinking is subject to wide variation between schools. The careful use logically qualifying expressions by these Year 7 students, and by some students in Year 6, is a further area ripe for investigation.

A relatively small number of students in both schools used computational approaches on Type 1 sentences and were able to shift gear, so to speak, when they needed to think relationally on Type 2 and Type 3 sentences. It is, however, sobering to note that among Year 7 students in both the Chinese and the Australian school there are still students who seem to be restricted to using only computation approaches. Thirteen of the seventeen computational thinkers in the Australian school fell into this category as did five out of eight computational thinkers in the Chinese school. It can be anticipated that these Year 7 students are likely to experience serious difficulties in the learning of algebra.

The strikingly close association between successfully identifying conditions under which "any number could be used in Box A" and a successful response to the corresponding Type 3 sentence points to a key juncture between relational thinking on Type 2 number sentences and an ability to explain relationships between literal symbols. Greater attention should be given to using Type 2 number sentences as a bridge to algebra.

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Chinese Young Children's Strategies on Basic Addition Facts

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Kindergartens in China offer structured full-day programs for children aged 3-6. Although formal schooling does not commence until age 7, the mathematics program in kindergartens is specifically focused on developing young children's facility with simple addition and subtraction. This study explored young Chinese children's strategies for solving basic addition facts as well as their intuitive understanding of addition via interview methods. Results indicate a strong impact that teacher-directed teaching methods have on young children's cognitions in relation to addition.

Introduction

In ancient China, mathematics was called "Suan Xue", which means "the knowledge of computations" (Zhang, 1998). Thus the Chinese mathematics curriculum predominantly focuses on mathematics as calculation. As a result, Chinese teachers are very interested in teaching faster and easier ways to do given computations (Ma, 1999). Therefore, performing calculations mentally is strongly emphasized in Chinese mathematics classes due to the speed at which calculations can be performed over other methods.

The abacus is a tool to help students with mental calculation (Zhang, 1998). It becomes a fashion that many kindergarten classrooms in China begin to teach children Mathematics with the help of an abacus. An abacus consists of columns of beads. The beads are separated as upper part and lower part by a crossbar. Each column has one bead above the crossbar and four beads below it. Each lower bead represents one unit and each bead above represents five units.

The abacus facilitates speed of calculation. According to Zhang (1998), Chinese students spent 5 seconds to complete an addition calculation for two whole numbers with two digits, and 14.5 seconds for multiplication (Zhang, 1998).

Literature Review

Developing Addition

Research into the development of addition. Addition is an important concept in mathematics education. It is the first algorithm learned by children and it is also fundamental to children's later learning in mathematics (Braten, 1998; Siegler, 1987). An understanding of the principles of additive composition by which parts are combined to form a whole is the conceptual core of the mathematics curriculum in the early elementary school period (Resnick, 1989).

The development of addition has been shown in sequential stages, and different levels of strategies have been identified either by chronometric research (Groen & Parkman, 1972; Suppes & Groen, 1967) or interview research (Baroody, 1987; Carpenter & Moser, 1984; Siegler & Jenkins, 1989), as children have solved single digit addition facts with two addends.

When computing basic addition facts, counting strategies and non counting strategies are found to be employed by young children. Counting strategies are identified by many researchers (e.g., Groen & Parkman, 1972; Suppes & Groen, 1967) as counting all by sum, counting all from the first addend, counting all from the second addend, counting on from the smaller addend and counting on from the larger addend (termed as Min strategy by Parkman & Groen, 1971). Carpenter and Moser (1982) further classified those counting strategies into counting with models such as fingers or other physical objects, which were also indentified as concrete counting strategies in Baroody's (1987) study; and counting without models.

Non counting strategies include guessing (Siegler, 1987), known fact and derived facts (Carpenter & Moser, 1982). Known facts and derived facts strategy were termed as retrieval and decomposition strategies in Siegler's (1987) study respectively. Known fact referred to the number facts that children already knew and could recall instantly, while derived facts were the facts children derived by linking to a known number fact (Carpenter & Moser, 1982).

Baroody's (1987) study suggests that there is a developmental sequence based on efficiency, and possibly greater understanding, and familiarity with numbers and number properties. Counting on from the first is the shortcut of counting all starting with the first addend. Counting all starting with the larger addend and counting on from the larger addend minimises the keeping-track process. Counting on starting from the first is the shortcut of counting all starting with the first addend. Counting on starting from the larger addend short cuts the procedure of counting all starting with the larger addend.

Siegler (1987) also found that there were clear developmental trends in children's strategies. He contended that, as children became more proficient with addition, the use of retrieval and decomposition strategies increased and the use of counting all and guessing decreased.

In studies on young children's early addition strategies, three models appear to be the most frequently identified: counting all by sum (Sum model), counting on by holding the first addend constant, and counting on from the larger addend (Min model) (Carpenter & Moser, 1984; Houlihan & Ginsburg, 1981). The Min model is considered to be the most economical and advanced counting strategy in addition (Braten, 1998).

Strategy choice and variability. Although many researchers have suggested that addition models/strategies used by children to solve addition problems are hierarchical, children do not use the most efficient strategy at all times (Carpenter & Moser, 1984; Christensen & Cooper, 1991; Houlihan & Ginsburg, 1981). For example, Carpenter and Mose found that children did not always use the most efficient strategy, though they had known several strategies. Even when children mastered several strategies, they did not use the most efficient one consistently but would use them interchangeably, and in some circumstances, they would often revert to a less efficient strategy.

Many researchers agree that addition strategies used by children will vary according to their ages and background knowledge (e.g., Siegler, 1987; 2003; Siegler & Campbell, 1989; Siegler & Jenkins, 1989). In other words the strategies that a child may use will depend not only on the child's age, but also on how the child builds new strategies on their existing knowledge. Strategy use has also been shown to depend on the type of addition problem. Siegler (1987) summarized the conditional probabilities of strategy use, stating that children used retrieval to solve the problems with small sums and "ties" or "doubles" (e.g., 5+5) because these problems could be easily executed to obtain correct answers. The Min strategy (Counting on from the large) is often used when children solve problems containing small addends or where there are large differences between addends, or both. On solving problems with small differences between addends, children use the Counting-All strategy. Guessing is used most often on problems with large sums. More recently, Carr and Jessup (1994) have suggested that children's metacognitive knowledge is significantly correlated with their use of correct strategies. And further, current research indicates that strategy use/choices is facilitated by number sense (Griffin, 2003).

Developing Basic Facts

According to Isaacs and Carroll (1999) one of the objectives of elementary mathematics instruction is to teach basic number facts. Basic number facts underpin mental computation facility (McIntosh & Dole, 2000).

Isaacs and Carroll have proposed a "strategies-based approach" for teaching basic facts that differs from a traditional rote approach that may discourage the development of children's mathematical thinking. Their approach is based on building children's informal mathematical knowledge, and helping them develop their immature addition strategies into more mature strategies.

Many researchers (e.g., Isaacs & Carroll, 1999; Johnson & Siegler, 1998; Postlewait, Adams, & Shih, 2003) have been trying to find a meaningful way to teach children addition and subtraction in the early childhood classroom. All these approaches highlight that instruction should be based on the informal mathematical knowledge children bring to school, and build on children's number sense. However, Chinese kindergarten children appear to learn mathematics in a very formal environment and they do well in achievement tests. As there is a large body of literature on elementary mathematics that focuses on children who are aged 6 and above, and many cross-nation studies conducted detailing the reasons why Chinese students outperform their peers, the aims of the study are to investigate young (aged 3-6 years) Chinese children's basic addition fact strategies during their three years of formal kindergarten education, and to examine the impact of formal teaching upon kindergarten children's choice of addition strategies.

Methodology

Participants

Seventy-two children, aged from 3 to 6 years, enrolled in a public Kindergarten in Baoding, a middle sized city in China, participated in this study. The children were grouped by year level: Group K1 consisted of children in their first year of kindergarten; Group K2 consisted of children in their second year of kindergarten, and Group K3 consisted of children in their third year of kindergarten. The specific number, gender and mean age of the children within each of these three groups are displayed in Table 1.

Table 1

Number, Gender and Mean Age of the Children within Each of Three Groups in the Study

Kindergarten Year	Number of Boys	Number of Girls	Mean Age
K1	11	11	3 years, 5 months
K2	13	9	4 years
K3	15	13	5 years, 5 months

Instruments

The basic addition facts test was used to assess children's basic addition fact knowledge and the strategies they used. The test contained 40 addition problems, grouped into 6 categories as follows: Adding on 1, 2, and 3, Adding 0, Bridging 10, Near Doubles, Doubles, and Tens facts. Children were required to give the answer orally and to explain the strategy they used to arrive at the answer to each item.

Procedure

Step 1 – Interviews with children. Each child was tested on the Basic Addition Facts Test. All children were taken individually to a separate room that was attached to their classroom to do the test.

The addition problems were presented one at a time on cards, and the children were encouraged to use whatever strategy made it easiest for them to arrive at the answer. They were encouraged to speak aloud their thinking and were told to keep talking as long as they could. The strategies were classified as counting all by sum, counting on from the first addend, counting on from the larger addend, min, abacus, recall, visualising abacus, and guessing.

Step 2 – Classroom observation. Observations took place over a period of two weeks for each year level of kindergarten as the teacher implemented the planned curriculum for addition. The researcher and one trained observer observed each classroom. During these observations, teacher and children's interactions were video taped, and notes were taken, which described what children were doing and general impressions of the class. The researcher watched and reviewed the tapes of the teachers, adding information from field notes until fairly complete records were compiled.

Findings

Mathematics Class Observation

The abacus was first introduced when children were in K1; K2 children started to have more experience with the abacus and children were overtly given instructions and lots of drills on using the abacus. It seemed that most of children had become very proficient with the abacus in K3.

The Mathematics learning outcomes for K1 Children is developing one to one correspondence and number combinations up to 5. Children were also introduced to the abacus in K1. They were expected to learn how to present numbers from 1 to 5 on the abacus by arranging the beads in the appropriate form.

The Mathematics outcome for K2 children is mastering combinations up to 10. Additional lessons on the abacus were conducted. Children learned the rules on how to use the abacus for doing simple addition (See Figure 1 & 2).

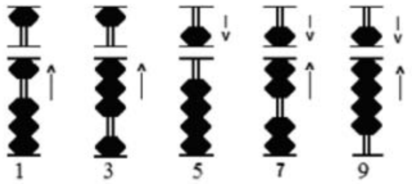

Figure 1. Some examples of numbers shown on an abacus. (Adapted from Heffelfinger & Flom, 2004)

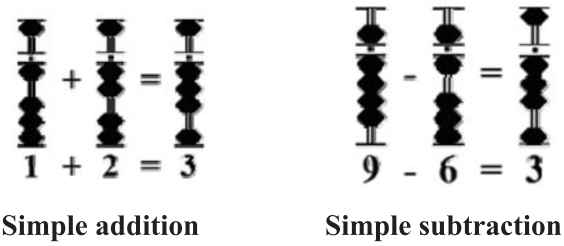

Figure 2. Simple addition and subtraction by using the abacus. (Adapted from Heffelfinger & Flom, 2004)

When there are not enough beads on the column to complete addition or subtraction, the technique employed is the use of combinations (complementary numbers) of 5 and 10 (Heffelfinger & Flom, 2004). Therefore, children have to be proficient with combinations of 5 and 10. In doing addition, one always subtracts the complementary numbers. Two groups of complementary numbers (4 & 1 and 3 & 2) are used for number 5. Five groups of complementary numbers are for number 10 (9 & 1, 8 & 2, 7 & 3, 6 & 4, and 5 & 5). For example, in the case of 4+8, one has to push up 4 beads on the column B first. As there are not enough beads on Column B to represent 8, the complementary number is used. Two is 8's complementary number in making 10. Then add 8 by subtracting the complementary number 2 and carry one to Column A (See Figure 3).

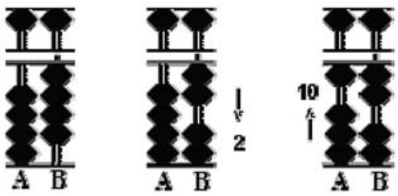

Figure 3. The action on an abacus for 4+8=12. (Adapted from Heffelfinger & Flom, 2004)

At K3, children were very familiar with abacus. They even learned how to compute addition and subtraction with three digits. Many practices on abacus were given. In the observed lessons, the children were usually given a group of similar problems to solve until they mastered them. In the case of learning 4+8, children would give similar problems such as 4+7, 3+8, 2+8, 9+7 and so on, to practice until they mastered them.

The idea of using the abacus was to eventually help children to transform mechanically adding and subtracting beads into visualised abacus in their minds. At K3 stage, children were continually encouraged to visualise the abacus in their mind when doing computation. Children were found to physically use the abacus only when the numbers were bigger and were beyond their ability to reach the answer automatically. Counting with fingers was actively discouraged by the teacher.

Strategy Use to Solve Basic Addition Facts by Each Year Level

At K1 level, where children were 3 to 4 years old and in their first year of kindergarten, only a few children attempted to compute all the addition tasks in the test. In total the overall success on the test was 20%. The majority of K1 children used a guessing strategy, or made no attempt. This finding echoes level 1 strategy in Griffin's (2003) study with kindergarten children. Griffin found most of the children in the sample at the age of 3 to 4, responded with blank expression or said they did not know the answer instead of making any attempts to solve the problem 4+3. However, results of this study also show that K1 children used a range of strategies including sum, min, guessing strategies to solve basic addition facts. There was evidence of recall, and visualisation of the abacus. In this study, many K1 children easily recalled 1+1 (86%), and some recalled 2+2 (32%) but not other facts. These K1 children relied on Sum strategies even sometimes they could not arrive at the right answers. Only one child was found to visualise the abacus.

At 4 to 5 years of age, children in K2 were in their second year of kindergarten. Like K1 children, results of the basic facts test show that K2 children experience difficulties when they compute basic addition facts where the addends are bigger than 5. Yet overall, they could fluently compute (recall) the facts with addends smaller than 5. The mean of correct answers for the basic facts test was 79.6%.

Nine different addition strategies were used by K2 children in this study: recall, derived fact, represented or visualised addends and recalled fact, counting on, min, sum by materials, sum by fingers, guessing, and visualising the abacus. However, the strategy most used by the children was the sum strategy either with the help of fingers or materials. This supports prior research findings that children don't use the most efficient strategies all the time (Carpenter & Moser, 1984; Christensen & Cooper, 1991; Houlihan & Ginsburg, 1981). Although K2 children used a range of strategies, most of them had many facts automated (i.e., they used a recall strategy), and there was 79.6% mastery from these very young children. The other developing strategy identified was the visualising abacus strategy that was used by many more K2 children than K1 children.

At 5 to 6 years of age, children in K3 were found to use six strategies for addition facts: recall, Derived facts, Represented or visualised addends and recalled fact, visualised abacus, min (count in mind), and abacus. Children in K3 had a high level of mastery of the basic facts, with a group mean of 99% on the basic facts test. Recall strategy (48%) was found to be the dominant strategy used by K3 children and the second dominant strategy was Visualised Abacus (31%). Children frequently used visualised abacus or abacus strategy when they computed facts especially, with larger addends (e.g., Bridging 10, Near Doubles or Doubles).

In the basic facts test, there are six categories of addition facts. They are (1) Adding on 1, 2, 3 facts, (2) Adding 0 facts, (3) Bridging 10 facts, (4) Near double facts, (5) Doubles facts, and (6) Tens facts. In this study, K1 children spent the least time on Doubles facts and the most time on Bridging 10 Facts. For the Adding on 1, 2, 3 facts, six main strategies were found to be used by K1 children, and 23% of the children could successfully reach the answers. Children were more likely to attempt to calculate Doubles facts and Tens facts, and more children used recall strategy for these facts.

For K2 children, Adding 0 was the easiest group of facts, as the average time spent on these facts was 3.27 seconds, and 91% of K2 children could successfully arrive at the correct answers. Eighty-seven percent of the K2 children could correctly calculate the Adding 1, 2, 3 facts, while 81% of them could reach the right answers for Doubles. The most difficult ones were Bridging ten facts, as there were only 64% of K2 children who could correctly compute these facts. Children seemed to be confident in doing Doubles with addends no bigger than 5. There were more than 50% of K2 children who could give the answers within 2 seconds (86% for 1+1, 2+2, and 5+5; 68% for 3+3; 50% for 4+4). Comparatively, 1+1 and 5+5 are the easiest facts to K2 children, whereas with K1children, they were 1+1 and 2+2.

All children in K3 could reach correct answers when solved Adding 0 facts and Tens facts. Bridging 10 and Near double facts were facts with which K3 children made the most errors. Children used the derived facts strategy most frequently to solve Bridging 10 and Near Double facts.

Overall, Bridging 10 was found to be the most difficult group of facts for all three year levels of children in this study (K1, K2, and K3).

Discussion and Conclusion

The results of this study appear to suggest that formal instruction has a strong impact on children's strategy choice for basic addition fact computation. Six strategies identified in the literature, namely, Sum, Min, Counting on, Recall, Derived facts and guessing were all evident as used by the kindergarten children in this study. However, one further strategy not discussed in the literature, but used by children in this study, was the abacus as the children had been taught how to use an abacus since their first year in kindergarten.

The findings also indicate that there is a developmental sequence in addition strategy use. Children who made attempts to calculate the facts in this study predominantly used guessing and the sum strategy in their first year of kindergarten. In K2, children began to use diverse strategies to compute different facts; however, they are efficient in sum strategy and use recall strategy very often. Use of the abacus strategy was also employed more by K2 children than K1, but at the same time, they still relied on Sum. When children entered K3 level, they could recall most of the facts. However, they depended on Visualised abacus strategy most often to calculate the facts that they were not confident to recall. It was also noteworthy that the K3children did not use their fingers or other concrete materials to compute addition facts as they also felt ashamed to do so.

It is interesting that only few children frequently used the min strategy in this kindergarten. It seems reasonable to suggest that the Min strategy period may not be an important transition period for those children to develop more efficient addition strategies. The visualised abacus strategy was a very dominant strategy in this study. These data strongly suggest that instruction dominated by use of the abacus has supported the development of number sense and some understanding of number relationships.

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Self-Efficacy in Mathematics: Affective, Cognitive, and Conative Domains of Functioning

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Self-efficacy has the potential to facilitate or hinder our mathematics learner's motivation, use of knowledge, and disposition to learn. This paper examines the use of a questionnaire and classroom discussions to access the self efficacy of 64, year four to six students. The questionnaire and classroom discussions gathered data on the students affective, cognitive, and conative psychological domains of functioning. The findings from the questionnaire and classroom discussions are presented and discussed in regard to their relationships with the students' self-efficacy.

Self-efficacy is the judgements we make about our potential to learn successfully and the belief in our own capabilities. The choices we make, the effort we put forth, and how long we persist are influenced by self-efficacy (Bandura, 1997; Schunk, 1996).

Perceptions of self-efficacy come from personal accomplishments, vicarious learning experiences, verbal persuasions, and physiological states (Bandura, 1986; Ingvarson, Meiers, & Beavis 2005; Tanner & Jones, 2003). Self-efficacy impacts on a learner's potential to succeed (Bandura, 1977). An insight into the self-efficacy of their learners is a valuable tool for mathematics educators.

> It is important for educators to know how their learners feel, think, and act, about, within, and toward mathematics. The influence of attitudes, values and personality characteristics on achievement outcomes and later participation in the learning of mathematics are important considerations for mathematics educators. (Yates, 2002, p. 4)

One way to gain insight into how their learners feel, think, and act, about and toward mathematics is to examine their psychological domains of functioning: the affective, the cognitive, and the conative (Huitt, 1996; Tallon, 1997). It is important to examine each domain as a student may feel efficacious within the affective domain but less confident within the cognitive domain.

Affect is a student's internal belief system (Fennema, 1989). The affective domain includes students' "beliefs about themselves and their capacity to learn mathematics; their self esteem and their perceived status as learners; their beliefs about the nature of mathematical understanding; and their potential to succeed in the subject" (Tanner & Jones, 2003, p. 277).

The cognitive domain considers students' awareness of their own mathematical knowledge: their strengths and weaknesses; their abstraction and reification of processes; and their development of links between aspects of the subject (Tanner & Jones, 2000). Cognition refers to the process of coming to know and understand; the process of storing, processing, and retrieving information. The cognitive factor describes thinking processes and the use of knowledge, such as, associating, reasoning, or evaluating.

Conation refers to the act of striving, of focusing attention and energy, and purposeful actions. Conation is about staying power, and survival. The conative domain includes students' intentions and dispositions to learn, their approach to monitoring their own learning and to self-assessment. Conation includes students' dispositions to strive to learn and the strategies they employ in support of their learning. It includes their inclination to plan, monitor, and evaluate their work and their predilection to mindfulness and reflection.

This research examined the self-efficacy of 64 year 4 to 8 students (aged 8 to 11) towards their mathematics learning by analysing their responses to affective, cognitive, and conative statements.

Methodology

The teacher participants in this research were selected because of their existing relationship with the researcher. Within this relationship each teacher had discussed their concerns regarding the impact of their student's self-efficacy on their potential to succeed. The following tables outline the teachers' demographic data and their students' year levels.

Participants

Table 1

Teachers

	Students (N = 64)	Years Teaching	Current Teaching Level	Years at Current Teaching Level	Highest Qualification
Teacher One	17	31	Year 3/4	3	B.Ed
Teacher Two	26	20	Year 5/6	7	B.Ed
Teacher Three	21	25	Year 6	4	B.Ed

Table 2

Students

Name	Year 4 (n = 17)	Year 5 (n = 15)	Year 6 (n = 32)	Total
Teacher One	17			17
Teacher Two		15	11	26
Teacher Three			21	21
Total	17	15	32	64

Method

Each student (N = 64) was given a questionnaire (adapted from Tanner & Jones, 2003) containing twenty statements sorted into three domains: affirmative, cognitive, and conative. The students responded to each statement by selecting either: I agree, I do not know, or I disagree. The findings from the questionnaire were collated and graphed by the researcher. The participating teachers met with the researcher to discuss the findings from the self-efficacy questionnaire and formulated six questions for each teacher to ask their class. The researcher recorded the discussion and noted the students who contributed.

Materials

The self-efficacy questionnaire asked the students to respond to the following statements.

Affective Domain Statements

Statement 1: Working hard leads to success in mathematics.

Statement 2: Some people just cannot do mathematics.

Statement 3: Some people are naturally good at mathematics.

Statement 4: You cannot change how good you are at mathematics.

Statement 5: There is no point in me trying in mathematics.

Statement 6: I know when I have got a mathematics question wrong.

Statement 7: I often get a mathematics question wrong but I do not understand why.

Statement 8: I know if I am going to get a mathematics question right.

Cognitive Domain Statements

Statement 9: If I do well in mathematics it is because the questions are easy.

Statement 10: If I do well in mathematics it is because I work hard.

Statement 11: If I do badly in mathematics it is because my memory let me down.

Statement 12: If I do badly in mathematics it is because I have no natural ability.

Statement 13: I know which parts of mathematics I do not understand.

Statement 14: Mathematics does not make sense to me.

Statement 15: I like finding bits of mathematics which go together.

Conative Domain Statements

Statement 16: Mathematics is about working together with others to solve problems.

Statement 17: When I am stuck it is useful to talk to others.

Statement 18: I could do better in mathematics if I worked with others.

Statement 19: If I make a mistake I try to find out where I went wrong.

Statement 20: I make sure that I understand mistakes that I have made.

Findings

The first set of findings discussed are from the self-efficacy questionnaire that each student (*N* = 64) completed and the second set are the findings from the questions posed by each teacher (*n* = 3) to their class.

Affective Domain Questionnaire

The affective statements asked the students to consider what they believed and examined their perceived status as a learner. Figure 1 shows the percentile results of the students' responses to statements 1 to 8.

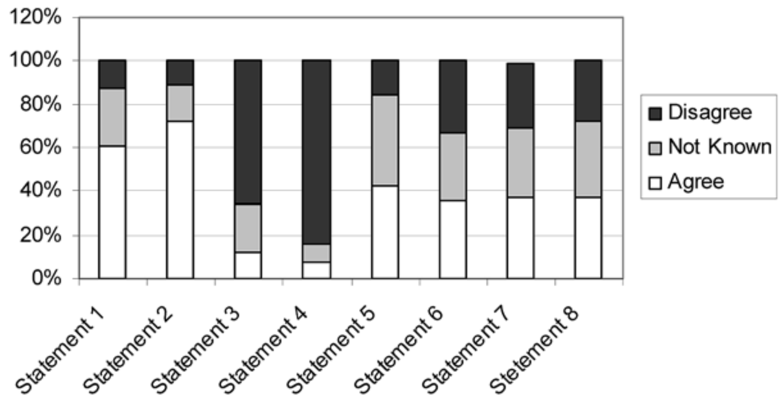

Figure 1. Student (N = 64) affective domain questionnaire results.

As a group the students agreed that hard work led to success in mathematics (S1 61%). The students disagreed that some people are naturally good at mathematics (S3 66%) and disagreed that you cannot change how good you are at mathematics (S4 84%). From these three findings it would appear that the students believe that maths is not something you are born good at, that anyone could get better, and that success could be attributed to hard work. However, as with the findings of Tanner and Jones (2003), a "worryingly hard core" (p. 280) 72% of the participants agreed with statement 2 some people just cannot do Maths.

Forty-two percent of the students either agreed or were unsure if there was any point in them trying in mathematics (S5). This finding correlated with the students lack of certainty in their responses to statements 6, 7 and 8 which were about knowing if you have a mathematics question wrong and why, and knowing if you have a mathematics question right. Each option of agree, do not know, and disagree was in the 30% range for each statement. This would imply that the students were less sure about knowing if they were right or wrong (Tanner & Jones, 2003) and that some students lacked confidence in self-regulating.

Cognitive Domain Questionnaire

The cognitive domain is the students' awareness of their mathematical knowledge, their strengths and weaknesses, and their ability to make connections with, and within the curriculum. Figure 2 shows the percentile results of the students' responses to statements 9 to 15.

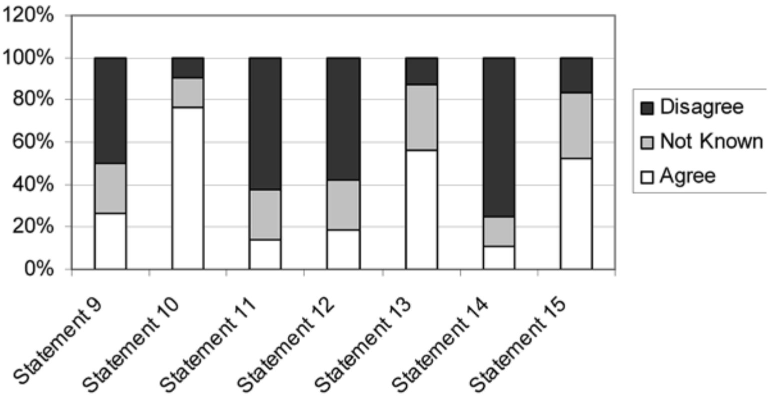

Figure 2. Student ($N = 64$) cognitive domain questionnaire results.

The students reiterated their affective belief that success and failure in mathematics related to working hard. Seventy-seven percent of the students agreed with statement 10 if I do well in mathematics it is because I work hard. The students agreed that they knew which parts of maths they didn't understand (S13) and they liked finding bits of mathematics which went together. This was endorsed by the students' disagreement with statements 9, 11, 12, and 14 which included reasons for doing badly in mathematics such as easy questions, no memory, or natural ability, and mathematics not making sense. The students were not attributing their successes to uncontrollable factors such as easy questions or a good memory and appear to have a detailed knowledge of their strengths and weaknesses. This is a positive finding as students who attribute their success or failure in mathematics to uncontrollable factors are unlikely to apply effective learning strategies (Tanner & Jones, 2003)

Conative Domain Questionnaire

The conative domain includes students' dispositions to learn, their approach to monitoring their own learning and to self-assessment. Figure Three shows the percentile results of the students' ($N = 64$) responses to statements 16 to 20.

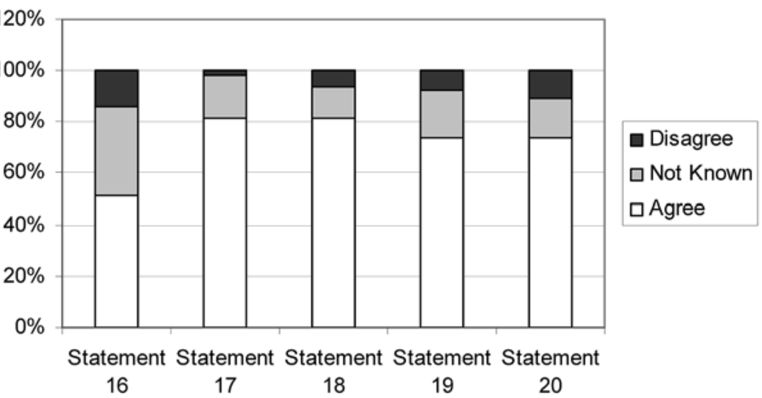

Figure 3. Student (N = 64) conative domain questionnaire results.

As a group the students were in stronger agreement in their responses to four of the five conative domain statements. There was a strong sense that it was useful to talk with others if they got stuck (S17 81%) and that they could improve by working with others (S18 81%). The students agreed that it was important to find out where you went wrong if you made a mistake and to try and understand your mistake (S19 74% & S20 74%). Statement 16, related to working with others to solve problems, did not engender as strong a response with 52% agreeing, 30% being unsure, and 18% disagreeing. This could suggest that talking to and working with others could be useful but that mathematics tasks were more of an individual pursuit than a team pursuit.

Discussion Findings

Following the analysis and discussion of the self-efficacy questionnaire the teachers decided to ask their classes the following questions:

1. Why do you believe that some people just cannot do maths?
2. What are the causes for some people not being able to do maths?
3. Why do you think some students have difficulty in knowing if they are right or wrong?
4. What helps you to do well in mathematics?
5. Why is it good to work with others?
6. In what ways is working with others valuable to you?

Affective Domain Discussion. When asked why some people just can't do maths the students discussed how mathematical ability was related to mathematical interest. It seemed that those who were not interested in mathematics were the basis for the group who just can't do maths. The students believed that you are good at what you like and you like what you are good at. So whilst some people were not naturally good at maths and you could change how good you were for those who did not like mathematics failure was both expected and accepted.

> It's not an interest and so they choose not to do it. And if you don't do it then you won't get better at it. (Student Year 4, Teacher 1)

> If maths isn't one of your favourites [subjects] then you aren't going to be very good at it, like if art was a favourite [subject] you would be good at that. (Student Year 6, Teacher 2)

> For some people mathematics is just not their subject–and they don't like mathematics because it's boring and they don't try and they don't do well. What you like is kind of what you are good at and if you like something you are more likely to be good at it. (Student Year 6, Teacher 3)

The students believed that for some failure was a self-fulfilling prophecy. They described how some students may have doubts about their mathematical abilities and the doubt could lead to a lack of commitment, stress, and eventually failure.

> Some people don't believe in themselves and they think they are not good at it and so they don't try. They might have not got anything right for a long time and they get frustrated and give up. (Student Year 6, Teacher 3)

> They might get all worried when they make mistakes and then that means they make more mistakes and so they stop trying. (Student Year 6, Teacher 3)

Cognitive domain discussion. The difficulty in knowing if you had a question right or wrong was discussed in terms of time pressure. The pressure to be finished on time was discussed as well as how the potential for errors was increased through rushing to finish. The students identified the need to be finished as more important than the need to be right. Speed was valued over accuracy and it would appear that these students thought it was better to write something down and be thought a fool than to have a blank page and remove all doubt.

> You haven't finished and you don't want to look dumb so sometimes you just speed up and hope it's right. (Student Year 6, Teacher 3)

> Getting finished means that you gave it a go, not finishing means not trying, finished means trying your best. If you haven't finished then you haven't achieved your goal. (Student Year 4, Teacher 1)

One Year Four student disagreed and described a success leads to success philosophy.

> But getting more right and less finished shows you tried your best. So if you go slower and get it right the first time you would have more chance of getting it right the next time. (Student Year 4, Teacher 3,)

The students agreed that working hard and having knowledge was important as well as having the ability to strategise when the answer was not immediately accessible.

> Mathematics is about working hard but you also need to know your stuff so that you have something to work hard with. (Student Year 4, Teacher 1)

> Mathematics is about knowing how to do it but not about knowing all the answers straight away. Strategies help you get the answer if you can't get there straight away. (Student Year 5, Teacher 2)

Conative domain discussion. The students responses to the questions posed at this point were not as clear cut as their responses to the conative statements. Whilst the majority of the students agreed that working with others was valuable, they did have some qualifiers. Working with others was valuable if those working together were in agreement, otherwise it was seen as a waste of time.

> It's good to work with others if you all agree but it takes a lot of time if you have to work with others and you don't agree. (Student Year 4, Teacher 1)

> It's not good working with others if they don't agree with you because you can waste a lot of time. (Student Year 5, Teacher 2)

Discussion and Implications

Bandura (1977) believed that the development of life-long learners of mathematics depended on the interaction of three linked psychological domains of functioning: the affective, the cognitive, and the conative.

The students' responses and comments to the affective domain questions showed a strong correlation between enjoyment, motivation, and success. The students saw liking or not liking as the beginning of a cycle of success or failure. The responses imply that the students sought external confirmation of their answers being right or wrong and suggest that some students are unsure of their own capabilities and capacity where mathematics is concerned. Nearly half of the students looked to someone else, possibly the teacher, for positive affirmations. This could impact negatively on the students in the future as they do not appear to know how to effectively monitor or regulate their responses. "Monitoring is the hub of self-regulated task engagement and the internal feed-back it generates is critical in shaping the evolving pattern of a learners' engagement with a task" (McDonald & Boud, 2003, p.210).

Analysis of the cognitive domain questionnaire and classroom discussions showed that the students related mathematical success to hard work, and recognised the need to have knowledge and strategies to bring to their mathematics learning. The findings from the conative domain questionnaire showed that the students saw the value of talking with others and the potential value of working with others. However analysis from the classroom discussion showed that the students believed that the potential for working successfully with others was conditional on being able to quickly reach agreement. It was only useful to talk with others if everybody agreed; differences and disagreements were not seen as valuable.

The students in this research are efficacious within the cognitive domain. They are confident about what they can do mathematically, but less sure about what they know and can achieve. This signifies a need for mathematics that is accessible and enjoyable for all learners and increased use and expectation of students regulating and monitoring their own learning.

Students' self-efficacy for mathematics may be defined as their judgements about their potential to learn the subject successfully. Students with higher levels of self-efficacy set higher goals, apply more effort, persist longer in the face of difficulty and are more likely to use self-regulated learning strategies (Wolters and Rosenthal, 2000, p.276).

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Neuropsychological Evidence for the Role of Graphical and Algebraic Representations in Understanding Function

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There are difficulties accessing students' thinking about mathematical concepts, although methods such as task observation and interviews provide some useful information. In recent years it has become possible to use functional magnetic resonance imaging (fMRI) techniques to access brain activity while students are thinking about mathematics. In this study we have used this technique to examine brain activity while students were processing graphical and algebraic representations of function. Results show some evidence for increased difficulty of translation between these formats for linear compared with quadratic functions. We also describe regions of the brain that are involved in the translations.

Function is one of the fundamental concepts of school and tertiary mathematics, and yet it is often misunderstood by students and school teachers. This may be due to the number and apparent complexity of its representational manifestations and the concept images these may evoke. In a comparison of the function concept maps of eight professors having PhDs in mathematics with those of twenty-eight university mathematics students, Williams (1998) found the latter had an emphasis on minor detail, such as the variable used, algorithms, and the idea that functions are equations in the student maps. In contrast she found that "none of the experts demonstrated the students' propensity to think of a function as an equation. Instead, they defined it as a correspondence, a mapping, a pairing, or a rule." (*ibid,* p. 420). Some, including teachers, have a tendency to think of functions graphically and in terms of processes, even to the extent of separating algebra from functions (seen as graphical) in their thinking (Chinnappan & Thomas, 2001), rather than seeing function as a concept crossing representational boundaries. Even's (1998) research with college students exemplified the importance of representations in understanding of function, with students having difficulties flexibly linking different representations and finding links between pointwise and global approaches to function problems.

Mathematics is essentially a symbolic practice in which signs are used, invented, and re-created (Saenz-Ludlow & Presmeg, 2006), and in particular one where we have the capacity to substitute some signs for others (Duval, 2006). Hence it is important to have a semiotic perspective on the role of signs in mathematical learning and practice (Radford, 2000). The situation is complicated by the relationship between the external sign and its internal interpretation by the individual, which is necessarily a function of the individual's existing cognitive structure. Thus, the word representation is used in different ways in the literature, either to refer to external signs that are part of a system or for their cognitive analogue. In this paper we will try to maintain a distinction between the use of the word sign for the external stimulus and representation for the internal, cognitive construct. One of the key aspects of mathematics is that it is probably the knowledge domain in which we find the largest range of semiotic representation systems (Duval, 2006), and hence its concepts need to be understood via a multiplicity of signs, (and hence representations) with each sign emphasising or de-emphasising different characteristics of a concept. As Otte (2006) notes, this involves an epistemological triangle between the sign, its referent (and hence representation) and the conceptual object, with epistemology involving the relationship between such entities, objects and signs. Thus developing rich mathematical thinking requires the ability to establish meaningful links between representational forms and translation from one representation of a concept to another, which Thomas (2008), calls *representational versatility,* a construct that includes qualitatively differing cognitive interactions with signs, through representations. Duval (2006) describes translations between systems as *conversions*, and it is the challenging nature of some conversions that makes some mathematics so demanding for learners. For example, the concept of function has associated graphical, algebraic, ordered pair, tabular, and other representation systems, with the links between each of them contributing to overall understanding of the concept.

Since 1991 brain function and cognition has been studied using functional magnetic resonance imaging (fMRI). This employs nuclear magnetic resonance (MR), which is non-invasive and produces images of the human body with excellent soft tissue contrast. Areas of the brain that become active show a temporary increase in blood supply, and the resulting change in the ratio of oxygenated to deoxygentated haemoglobin, can be measured by fMRI using the blood oxygen level dependent (BOLD) contrast. While most mathematical experiments involving imaging techniques have considered the most elementary of concepts, such as number, counting or arithmetic (e.g., Butterworth, 1999; Dehaene, 1997), its use in other mathematical investigations has been relatively rare, probably due to the difficulties inherent in experimentally isolating higher cognitive processes. One exception is the use of fMRI to study learning of elementary algebra, such as equation solving in highly competent college students (Anderson, Qin, Sohn, Stenger, & Carter, 2003), where solutions to linear equations were categorised into three levels of complexity by the number of transformations required to solve them. Results showed that the size of the BOLD response in the parietal and prefrontal regions directly reflected the number of transformations occurring. A later study (Qin, Carter, Silk, Stenger, Fissell, Goode, & Anderson, 2004) compared results using the same kinds of equations with ten 12–15 year old students to those from the previous study with adults. They found that the active areas in children's algebra equation learning were similar to areas active in adults, except that in children the parietal cortex showed a practice effect that was not found in adults. Recent research by Lee et al. (2007) considered the role of problem representation. They compared model (lengths of rectangular boxes represent variable values) and symbolic methods for construction of linear equations from word problems. They concluded that there were more accurate responses in the model than in the symbolic condition. One reason for the efficacy of the model method was its lower demand on attentional resources. There have been virtually no fMRI experiments involving mathematical concepts beyond basic algebra. In this research we were concerned with using fMRI to investigate whether one can identify brain areas active in the process of translation between graphical and algebraic function representation systems, or registers, if this brain area activity is format dependent or independent for graphical and algebraic representational formats, and whether the translation is independent of format direction.

Method

The ten participants in the study comprised five undergraduates ('novices") and five graduates ('experts") at The University of Auckland. There was no significant difference in age between the experts and novices. Gender was balanced (but not across the expertise groups which were eventually collapsed). The study was approved by the local ethics committee. All participants received instructions and training on a subset of stimulus items in an initial screening session. They also filled out a questionnaire at this session, including the question "How often do you think of mathematics in terms of pictures or images?". Several days later they returned for an fMRI session and were scanned for 40 minutes while carrying out simple mathematical tasks involving function representations. These tasks required viewing pairs of mathematical functions (graphs or equations) that were flashed consecutively on a computer screen above their head while they lay in the MRI scanner. For each pair they had to press a button to indicate whether the functions were the same or different. There were four different experimental conditions considered, two "same format" conditions where participants were asked whether the stimuli represented the same function (graph to graph and algebra to algebra), and two "cross format" conditions (graph to algebra, and algebra to graph), with the same question. The functions were presented for 200 ms each, with a blank screen between, with a total of 6000 ms for each trial. Participants responded by pressing a key with their left index finger for "non-matching" or a key with their right index finger for "matching". The stimulus pairs were presented in short blocks of five differentiated by equation type, linear versus quadratic. There were four of these sequences altogether giving a total of 20 stimulus pairs in each condition. Brain activation was compared to a baseline condition where particpants fixated on a small central cross with their eyes open. Experimental software recorded response accuracy and reaction time (RT). Participants completed a post-experimental questionnaire in which they were asked to rate the exent to which they employed various strategies, and were given an open question about strategy use.

Results

Analysis revealed no differences in performance between the novices and experts so results here represent a single group. Reaction times and accuracies[11] showed that the cross-format conditions (graph to algebra, algebra to graph) were significantly more difficult than the non cross-format conditions (graph to graph, algebra to algebra). Overall, participants were 543 ms slower responding to the cross-format questions, and

11 Some data from one participant is missing in this analysis due to experimenter error.

had an average accuracy of around 84%, compared to 93% for the same format conditions (see Tables 1 and 2 for representative examples of questions with corresponding accuracy). This effect of difficulty was slightly larger for the novices, but was not statistically significant for the small sample size of nine. The two cross-format conditions did not differ significantly in accuracy or RT, nor did the two same format conditions. While there are other, possibly significant, interactions present, notably with the within/cross format effect mentioned above, However, it is instructive to analyse this difference to attempt to understand the relative roles of the graphical and algebraic representations for each function type.

One might expect that translation of quadratic equations would be more difficult than linear ones, but they proved actually slightly easier overall, with a more accurate response by participants to the quadratic than the linear functions in both directions of translation (graph to algebra: 90% quadratic vs. 79% linear, respectively, n=9, Wilcoxon z=2.25, p<0.05; algebra to graph: 89% quadratic vs. 79% linear, respectively, Wilcoxon z=2.18, p<0.05), although there was no significant difference in RT between the two, with a combined mean time of 1376 ms for the graph to algebra condition, and 1317 ms for the algebra to graph condition.

Table 1

Representative Questions for the Blocks Involving Translation Algebra to Graph

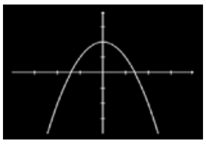
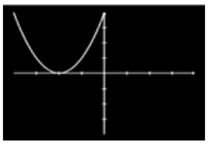
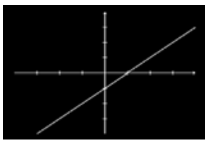
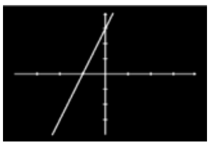
First stimulus	Second stimulus	Item Numbers	Correct Response	Correct
$y = -x^2 + 2$		13, 13	True	100%
$y = (x-2)^2$		8, 10	False (Translation wrong direction)	67%
$y = x - 1$		5, 5	True	88%
$y = -3x + 3$		19, 17	False (Gradient wrong)	50%

Table 2

Representative Questions for the Blocks Involving Translation Graph to Algebra

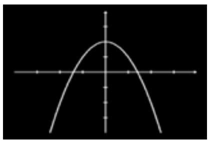
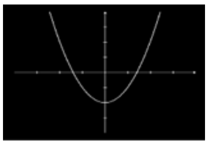
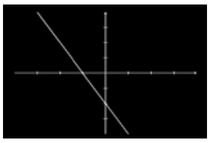
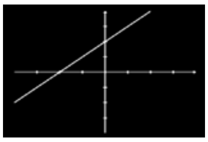
First stimulus	Second stimulus	Item numbers	Correct response	Correct
	$y=-x^2-1$	13, 15	False (Translation wrong direction, size)	100%
	$y=x^2-2$	5, 5	True	88%
	$y=-x+2$	16, 10	False (Wrong slope, intercepts)	63%
	$y=x+2$	3, 3	True	78%

Table 3 shows a breakdown of accuracy based on the individual items, allowing a comparison of direction of translation between linear with quadratic functions. In this analysis we find that the translation from graph to algebra was significantly more difficult for linear than for quadratic functions (t=2.27, p<0.05). However, when the direction of translation was reversed the difference in accuracy was not significant. It is clear from Table 3 that there was no difference in difficulty for linear function items when translating from algebra to graph compared with graph to algebra, and the same was true for quadratic functions. This is contrary to the finding of Duval (2006), who suggests that recognition of linear graphs, measured by conversion of graphical sign (register) was relatively poor (25% success for y=2x) compared with conversion of an algebraic sign (register).

Table 3

Mean (SD) Translation Accuracy Averaged Across Linear and Quadratic Function Items

Direction of Translation	Linear (N=20)	Quadratic (N=20)
Algebra to Graph	81% (20%)	89% (16%)
Graph to Algebra	78% (21%)	90% (12%)

How can we explain the fact that the linear translations in either direction were apparently more difficult? In analysing the results of this condition one must ask: what does one need to notice or pay attention to (Mason, 2003) in order to be able to respond to whether the algebraic equation matches the graph? Two aspects on which attention can be focused are perceiving specific properties and reasoning on the basis of these properties that are taken to characterise objects. For the linear functions, corresponding to $y=\pm mx\pm c$, $m,c\in\{-3,-2,-1,0,1,2,3\}$, working from graph to equation one would expect that the salient features, or properties, of the graphs to remember were: whether it had a positive or negative gradient; the value of the x-intercept; the value of the y-intercept; and hence the value of the gradient (see Tables 1 and 2 for examples). This last piece of data is a calculated value, rather than a perceived one, requiring a procedural interaction with the graphical representation rather than observation (Thomas & Hong, 2001). When working from the algebra to the graph the functions were all given algebraically (see Tables 1 and 2 for examples), and so one might note: the gradient from the first $\pm$; the y-intercept from the c, and the gradient from the m. This analysis, which shows that more data needs to be analysed in translating from graph to algebra than vice-versa, agrees with that of Duval (2006), who provided evidence that conversion from graphical to algebraic format for linear functions proved harder for students than going in the opposite direction.

For quadratic graphs, in the graph to algebra condition one again needs to pay attention to the salient features of the graph of the quadratic function. The participants had all practiced the task and so were familiar with the types of function used, namely simple translations of $f(x)=x^2$, of the type $\pm f(x)\pm k,\ k\in\{1,2,3\}$ or $\pm f(x\pm k),\ k\in\{1,2,3\}$. The use of $f(x)=x^2$ removed the added complication of needing to find the gradient multiplier (as in $f(x)=kx^2$) and hence, one needs to see just four things: the orientation of the graph to obtain the first $\pm$; whether it is translated parallel to the *x*- or *y*-axis; the direction of the translation for the second $\pm$; and its size (see Tables 1 and 2 for examples of the graphs). In the algebra to graph condition the participants were presented with an algebraic equation (see Tables 1 and 2 for examples). Again one needs to note: the first $\pm$ giving the orientation; the kind of translation; the direction of the translation from the second $\pm$; and its size.

Questionnaire Results

Examining the questionnaire data can give a measure of the validity of the above analysis. In answer to the question How often do you think of mathematics in terms of pictures or images?, 70% said quite often or most of the time, compared with 20% responding rarely or not very often. Table 4 shows the percentages responding with Agree (A)/Strongly Agree (SA) or Disagree (D)/Strongly disagree (SD) to the other questions.

Table 4

Strategy Question Facilities (N=10)

Question	% A/SA	% D/SD
Graph to algebra		
As soon as I saw the graph I started to put it into equation form	50%	40%
I focused on trying to keep a picture of the graph in my head	50%	40%
As soon as I saw the graph I tried to pick out key aspects	100%	0%
I didn't need to have a strategy it was obvious	0%	80%
I'm not really sure which strategy I used	10%	80%
Algebra to graph		
As soon as I saw the equation I started to imagine the graph	90%	0%
I focused on trying to repeat the equation to myself	30%	50%
As soon as I saw the equation I tried to remember key aspects	100%	0%
I didn't need to have a strategy, it was obvious	10%	60%
I'm not really sure which strategy I used	0%	80%

This data gives a reasonably clear picture of what the participants' strategies were. The majority (80%) in each case were clear about their strategy, and all of them responded that their method was to try and remember 'key aspects' or properties of either the graph of the equation they were shown, and then match these properties to the next image. While doing this some tried to keep the graph (50%) or the equation (30%) in mind, but the others were clear they did not do this. We note too that while the equation evoked an immediate attempt to see the graphical representation for 90% of the participants, but the graph was treated in this manner by only 50%, giving some evidence of a stronger visual link for algebra to graph than for graph to algebra. The open responses, particularly from the 'experts' confirmed the role of the property search strategy we have described above. For the direction graph to algebra S1 responded "Sometimes I would see the graph, then see the equation and tried to 'see' the graph again. But in general I would say that I looked for key aspects of the graphs first." Other comments were:

> S2. If possible, I worked out the equation for the graph in the gap and then checked it against the equation that came up. I did this by picking out the key features, like intercept, translation, positive or negative slope etc. that determine the equation's form. When the equation came up on the screen I also did the reverse.

S7. When the equation came up I would not take in the whole equation (for example…I wouldn't think about $y = x^2 + 1$ but look for corresponding details i.e. $-x$ or x and number at the end i.e. 1, although I think that often with the adding number I wouldn't pay so much attention to the sign in front of it. So it was more a matter of the form of the equation.)

For the translation from algebra to graph direction they said:

S2. I looked at the graph and worked out key features, which meant I could work out what the graph would look like. I then compared it to the graph shown. I think when I compared the two graphs I didn't so much compare them visually, but I compared there [sic] key features.

S7. Like the other case I'd generally try pick out details i.e. presence of negatives or added constants and then find corresponding bits on graph in a general sense as I had certain expectation of what those parts would combine to give in overall shape and slope.

It is clear that these participants were using the key properties of the signs, there is also a sense from S7 that he was concerned about the overall form or structure of the sign. We will be reporting in more detail about the areas of brain activity seen and conclusions we can draw from these in a further paper but we present here several results. Significantly greater activity for graphs versus equations was seen in right superior parietal cortex (previously associated with mathematics and attentional shifting), in visual areas in the occipital cortex (left mid and right superior), and in the right mid temporal cortex (possibly associated with object processing). Whether this is due to mere perception of the stimulus as opposed to higher-level processing is hard to know. The extent and amount of brain activation was much greater when participants had to translate between formats (see Figure 2), and activation specific to these cross-format conditions was seen in the left inferior frontal gyrus (Broca's area; related to speech production and verbal memory), and two small clusters in the right hippocampus (associated with memory) and right cingulate (associated with cognitive control/attention).

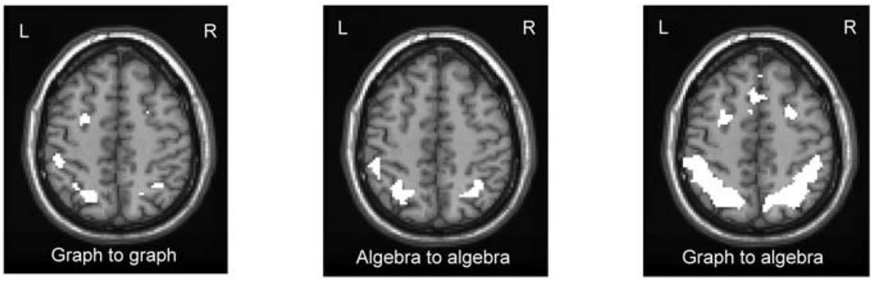


Figure 1. The extent and amount of brain activation for within and between graph and algebra formats.

Thirdly, we considered whether there might be brain areas involved in format independent representation of functions. In this case the intersection of active areas for within format comparisons between the two conditions should include these areas. We found significant conjunction clusters in an area already associated with abstract representation of number, the intraparietal sulcus (IPS), as well as other nearby areas often associated with mathematics, the superior parietal lobule (bilateral), and the left angular gyrus. Smaller clusters were also found in the left mid frontal cortex and right lingual gyrus. Some of these areas (most likely the IPS) may be related to format independent representation of functions, although this requires confirmation in future experiments.

Conclusion

What are the possible implications of these results for the teaching and learning of function? The concept of *representational versatility* (Thomas, 2008, p. 79), includes "*the ability to work seamlessly within and between representations, and to engage in procedural and conceptual interactions with representations*", that is, it encompasses both the treatments and conversions of Duval (2006). One point arising from this research is that one should not assume that translations (or conversions) of linear functions are easier for students to process than quadratic functions, since for this sample they were more difficult, in both directions of translation, probably due to the increased cognitive load of having more properties to pay attention to. In contrast Duval (2006) describes evidence that the conversion from graphical to algebraic format for linear functions proved harder for students than going in the opposite direction. However, we agree that "The true challenge of mathematics education is first to develop the ability to change register." (*ibid,* 2006, p. 128). Thus to learn which changes in function representations are significant students need to work at translating between representations. Unfortunately, there has been a tendency on the party of some to separate the teaching of algebraic, or equation, function signs from their graphical counterparts, making the likelihood of such translations very small. The translation difficulties we have discussed above may suggest that students would find the relationships between the function signs easier to comprehend, and the properties easier to pay attention to, if their teaching was integrated for each function type and tasks were set that encouraged students to make explicit links between the properties of each sign, in the context of an overall structure.

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Speaking with Different Voices: Knowledge Legitimation Codes of Mathematicians and Mathematics Educators

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This paper uses a textual analysis of two documents prepared by the mathematics community and the mathematics education research community to the National Numeracy Review in 2007 to uncover and compare knowledge legitimation within these two fields. The paper shows that knowledge within these disciplines is based on different epistemic devices, and hence that debates surrounding mathematics education arise, at least in part, from differing ways of viewing knowledge.

Curriculum debates rage in the United States between proponents of a "reform" curriculum and those of a "mathematically correct" curriculum. Reformists accuse mathematically correct advocates of a reductionist, back to basics approach that subjugates the process of learning mathematics to a set of well-defined procedures. On the other hand those who claim to be mathematically correct accuse reformists of being "fuzzy", of valuing any method so long as it works, and of allowing students to work everything out for themselves (Klein, 2007).

Similar debates are rising to the surface in Australia. On the one hand mathematics educators, in particular university-based teacher educators and mathematics education researchers, call for a mathematics curriculum that is responsive to a changing society, that values and incorporates the use of technology and that recognises the hesitant way in which students construct knowledge. On the other hand, Donnelly (2007, p. 55) influenced by some mathematicians and mathematics teachers, calls for a more rigorous curriculum, arguing against constructivist approaches, against "outcomes-based and politically correct" education and against "fuzzy maths". This call foregrounds mathematics as a precise discipline, valuing clear definitions and standard procedures.

This paper uses a framework that looks at how knowledge is produced and legitimated within a discipline (Maton, 2000). It shows that knowledge within the disciplines of mathematics and mathematics education relies on different epistemic devices, and hence that debates surrounding mathematics education arise, at least in part, from differing ways of viewing knowledge. I use a textual analysis of two documents prepared by the mathematics community and the mathematics education research community to the National Numeracy Review in 2007 to uncover and compare the epistemic devices in these two fields. The purpose is not to privilege one view of knowledge over another, but rather to promote greater understanding, and hence to promote greater acceptance of a divergence of views and move the debate forward.

Locating the Issue

Mathematicians and mathematics educators in Australia naturally take a keen interest in the school mathematics curriculum. This interest was particularly evident in the early 1990s, with the development and introduction of *A National Statement on Mathematics for Australian Schools* (Australian Education Council, 1991) and its associated document *Mathematics – a Curriculum Profile for Australian schools* (Australian Education Council, 1994). The mathematics education community and the mathematics community were united in their concern over the process by which the documents were produced, citing lack of adequate consultation in their development and the apparent determination of the writing team to pursue a particular agenda. Both groups also expressed over the content of the documents, however these concerns had very different bases (Ellerton & Clements, 1994).

Mathematics educators were concerned that "reductionist behaviourist approaches to teaching and learning mathematics … give rise to atomistic approaches to curriculum development and encourage methods of teaching and learning that fail to assist the development of a holistic view of mathematics" (Ellerton & Clements, 1994, p. 10). A behaviourist approach, it was stated, was contrary to the view of leading national and international educators who, throughout the 1980s, had argued for a curriculum that promoted relational understanding (Skemp, 1976). While also being concerned about atomistic approaches to curriculum, mathematicians condemned the *Statement* and *Profile* for a lack of quality of mathematical thinking. "(I)f the documents do not faithfully reflect the history of mathematics and do not represent quality contemporary mathematical thinking, then the school mathematics programs engendered by these documents will inevitably be less than satisfactory" (Ellerton & Clements, 1994, p. 10). Mathematicians expressed concern at the omission of important topics in mathematics and at the lack of rigour expected of teachers and students in the pointers contained in the *Profile*.

Given the recent development of the national *Statements of Learning for Mathematics* (Curriculum Corporation, 2006) and the subsequent establishment by the Rudd labor government of a National Curriculum Board to develop national curricula in English, history, science and mathematics, it is opportune to examine the philosophical bases of the views of those with an interest in school mathematics.

Theoretical Framework

This paper argues that the debate over what counts in mathematics education and the school curriculum is, in effect, a battle for control of the epistemic device (Moore & Maton, 2001) arising from conflicting beliefs about the production and validation of knowledge. This epistemic device "regulates: who can produce legitimate knowledge; the ways in which antecedent knowledge is selected and transformed in the course of producing new knowledge; and the criteria for adjudicating claims to new knowledge" (Moore & Maton, 2001, p. 30). The epistemic device thus describes the relationship between knowledge and the knower, casting light on why people view the world as they do and in turn shaping the way they respond to new ideas.

Mathematics[12] and mathematics education[13] are horizontal discourses characterised by a set of "specialised languages with specialised modes of interrogation and criteria for the construction and circulation of texts" (Bernstein, 1999, p. 162). In the case of mathematics these specialised languages consists of fields of study such as geometry, number theory or algebra. In the case of mathematics education the languages may consist of different research paradigms or different lenses through which to view theory and practice in mathematics teaching and learning. Within each of these disciplines knowledge is produced by people working within a particular field and validated by others in the academic community within the discipline. However the process of this validation is based on different principles. In the case of mathematics new ideas are knowledge validated while in mathematics education they are knower validated.

Mathematics has a strong internal grammar (Bernstein, 1999) consisting of accepted principles of logic, internal and external consistency and lack of gaps in reasoning. Andrew Wiles' proof of Fermat's Last Theorem is a classic example of the strong internal grammar of mathematics. Although few could comprehend Wiles' proof in its entirety, the grammar of mathematics allowed a gap in the proof to be detected. Wiles was then able to work on this gap to complete a proof that would stand up to rigorous scrutiny according to the logic of mathematics. Although Wiles' proof was evaluated by his peers in the mathematics community, ultimately it was the product rather than the person that mattered.

Mathematics education, on the other hand, has a weak internal grammar (Bernstein, 1999). Journal and conference papers in the mathematics education research literature are reviewed according to relatively flexible criteria such as whether the paper builds on and interrogates published research, the open-endedness and thoughtfulness of the research questions, the clarity of description of methodology, the ethics of the research and the cohesion of the argument (Gordon, 2002). "In an interpretive paradigm individuals construct their own meanings and a researcher cannot persuade practitioners by logical arguments that his or her story about the world is better and should be used" (Gordon, 2002, p. 2). While reviewers make every attempt to be fair, ultimately it is the person rather than the product that matters.

These differences in knowledge legitimation are described by Maton (2000) as knowledge or knower codes respectively. Maton claims that languages of legitimation are more than mere rhetoric; rather, they "represent the basis for competing claims to limited status and material resources" (Maton, 2000, p. 149). Knowledge and knower legitimation codes are based on underlying principles concerning the epistemic relation, that is the relation between educational knowledge and its object of study, and the social relation, that is between educational knowledge and its author. These principles structure both what can be legitimately claimed as knowledge within a given field and who can legitimately claim or validate that knowledge. Maton (2000) uses

12 It is acknowledged that the term mathematics is contested. In this paper no attempt is made to look at ethnomathematics, as school curriculum is dominated by a Western view of mathematics characterised by relatively hierarchical knowledge structures.

13 It is equally acknowledged that the term mathematics education is contested. Again, within this paper mathematics education is used to refer to research into mathematics teaching and learning within the dominant culture of Australian society, and particularly schools.

Bernstein's (2000) concept of classification and framing to discuss the nature of these principles. Classification refers to the strength of boundaries between categories or contexts, while framing refers to the locus of control within a category or context. The epistemic and social relations that determine the knowledge legitimation mode vary according to the relative strength of the classification and framing on each dimension.

In the case of mathematics, the epistemic relation is both strongly classified and strongly framed. That is, it is clear what counts as legitimate mathematics and there is tight control over what is accepted as legitimate mathematics. On the other hand the social relation is relatively weakly classified and framed. Cultural differences and social disadvantage notwithstanding, in the end who develops mathematical knowledge is less important than the knowledge itself. When the Wolfskehl prize of 100,000 marks for a successful proof of Fermat's Last Theorem was announced, the University of Gottingen received a flood of entries. "Regardless of who had sent in a particular proof, every single one of them had to be scrupulously checked just in case an unknown amateur had stumbled upon the most sought after proof in mathematics" (Singh, 1998, p. 143). Mathematics, then, has a knowledge mode of legitimation.

In mathematics education research, the strength of classification and framing of the epistemic and social relation are reversed. Mathematics education is, by its very nature interdisciplinary. It draws upon knowledge from a wide variety of fields such as psychology, sociology and philosophy, as well as mathematics itself (Presmeg, 1998). The epistemic relation is therefore weakly classified in that it permits, and indeed encourages, a wide variety of knowledge paradigms as legitimate knowledge. Furthermore these different paradigms exist in "different cultural traditions in mathematics education, arising from different communities and sub-communities" (Sierpinska & Kilpatrick, 1998, p. 31). Thus the epistemic relation is also weakly framed in that the locus of control is not located within a particular group. However the social relation is strongly framed and classified. Mathematics education research, particularly of an interpretive nature, is frequently culture-dependent, thus the researcher "needs to be part of this world, interpreting its events for an extended period" (Presmeg, 1998, p. 59). Of his list of thirteen critical problems facing mathematics education Freudenthal's (1981), first and most urgent was "Why can Jennifer not do arithmetic?" He distinguished this from the more abstract questions "Why can Johnny not do arithmetic?" and "Why can Mary do arithmetic?" In making the distinction Freudenthal described Jennifer as a living child whom he could describe in detail. Jennifer's experience in mathematics at school was context-dependent, and being able to understand those experiences depended upon being in that context. Mathematics education is thus strongly framed with respect to the social relation - it matters who does the research. It also matters who reviews the research as the reviewer must be able to place herself within that context, which depends on having personally experienced similar situations. As noted by Southwell (2004, p. 540) in her discussion of the reviewing process for articles submitted to the *Mathematics Education Research Journal* "(t)he skill of the reviewers will, in the end, determine the quality of the journal."

I suggest that these claims regarding epistemological differences are at the heart of the debate about school mathematics curriculum. One mode of legitimation is not more acceptable or appropriate than another, yet they compete for credence within the broad mathematics educational community. Each, together with the mathematics teaching community has a legitimate claim to a voice in the debate, and each has something unique to offer. The different voices of mathematicians, mathematics education researchers and mathematics teachers will be examined in the remainder of this paper.

Methodology and Data Analysis

The paper uses text analysis to examine the knowledge legitimation codes in two documents. In selecting the documents I chose to use ones which purported to represent the views of the mathematics and mathematics education research communities regarding numeracy in Australian schools. The papers were prepared by the Australian Mathematical Sciences Institute (AMSI) (Australian Mathematical Sciences Institute, 2007) and the Mathematics Education Research Group of Australasia (MERGA) (Mathematics Education Research Group of Australasia, 2007) respectively in response to the Australian government's National Numeracy Review in 2007. The Review aimed to analyse research about teaching, learning and assessment practices in mathematics, examine mathematics pre-service and practising teachers' pedagogic content knowledge, identify the relationship between teachers' content knowledge, pedagogic content knowledge and practice, and identify effective assessment methods (Monash University, 2007).

The documents were analysed for conceptual content using Leximancer, which allows the researcher to examine large amounts of text using automatic recognition of the main concepts within the text together with their relative strength, relation to each other, and contextual similarity. The results of the analysis are presented as a visual map that enables the researcher to analyse the conceptual structure of the document and to refine the search for concepts and their relationships using further iterations through the text. Leximancer has been used for conceptual modelling of text in areas as diverse as risk management (Martin & Rice, 2007) and analysing the rules of baseball and cricket (Smith & Humphreys, 2006). Conceptual mapping of the responses to the National Numeracy Review enables a comparison to be made of the level of importance afforded to, and the relations between, various concepts in each of the submissions.

Results and Discussion

Each document was analysed, in turn, using Leximancer. The software was used to make an initial pass of 1000 iterations through each document to produce a visual map of the most common concepts. Although it is possible within the software to delete or combine concepts, or to add new ones, the decision was made to retain those identified by the software. Following the initial pass through the documents, a further 2000 iterations were performed, by which time the conceptual map of each document was relatively stable. It should be noted that the results reported are by no means an exhaustive analysis of the documents. Nor is the analysis a detailed text analysis using, for example, systemic functional linguistics or critical discourse analysis. This is potential further research.

The visual maps of the AMSI and MERGA documents are presented in Figure 1.[14] The relative frequencies of the most common concepts in the documents are presented in Table 1.

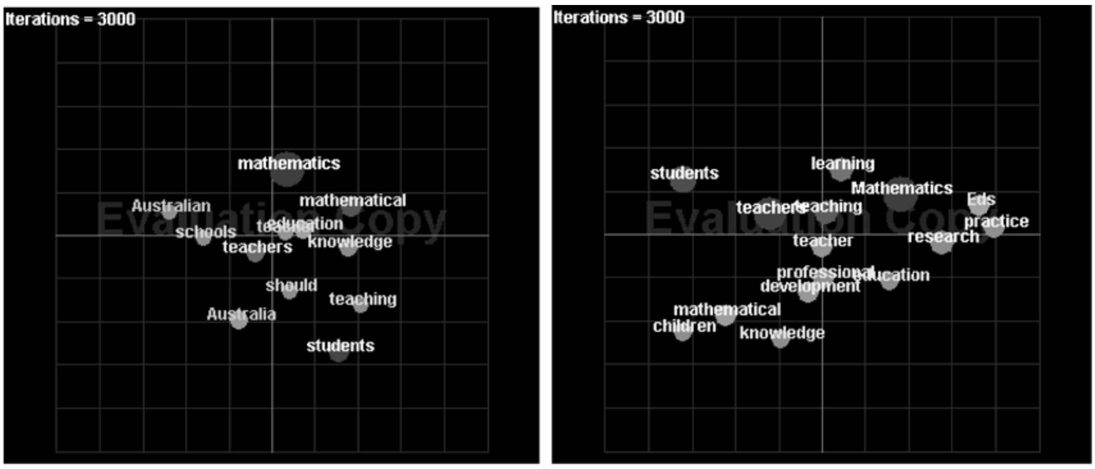

Figure 1. Conceptual maps of AMSI (left) and MERGA (right) responses to National Numeracy Review.

14 The maps were produced using an evaluation version of Leximancer, as the full version was in the process of being purchased at the time of writing the paper. Hence "evaluation copy" appears on the maps. The analysis process is, however, unaffected by whether or not the software is an evaluation copy.

Table 1

Relative Frequencies of Concepts in AMSI and MERGA Responses

Concept	Relative frequency of concept, compared to most common concept	
	AMSI	**MERGA**
mathematics	100%	100%
students	36%	60%
teachers	26%	82%
mathematical	28%	38%
knowledge	20%	24%
Australia	20%	<10%
teaching	19%	44%
education	19%	24%
time	17%	<10%
curriculum	16%	<10%
schools	15%	<10%
should	15%	<10%
learning	<10%	50%
Eds	<10%	49%
research	<10%	49%
practice	<10%	37%
children	<10%	35%
development	<10%	32%
professional	<10%	28%

Some striking similarities and differences emerge when examining the conceptual maps and the table of relative frequencies. The most obvious similarity is that in each of the documents the concept *mathematics* is the most common. In fact, each of the documents emphasises the centrality of mathematics, recommending the abandonment of the term *numeracy*, which is not prominent in either document.

The concepts *students* and *teachers* both appear prominently in each document, however with much greater relative frequency in the MERGA response than in the AMSI response. This may be seen as unsurprising given the nature of the two associations, however it is also suggestive of an emphasis on the person (a knower mode) rather than the content (a knowledge mode). The differences in emphases on concepts such as *teaching*, *development* and *professional* may reflect a similar emphasis on the person rather than the content. The conceptual map for the MERGA document clearly shows the conceptual proximity of the terms *professional* and *development*, suggesting that they could, in fact, be considered as one complex concept.

The concept *curriculum* appears with relative frequency greater than 15% in the AMSI document, but less than 10% in the MERGA document. Indeed, the AMSI document recommends as the first step "clearly defin(ing) the mathematics expectations for each year level in the compulsory years of schooling". It further states that "(w)e do not believe this would be very difficult but it must be done." This statement and the high relative frequency of the term *curriculum* suggest an emphasis on content (a knowledge mode) rather than the person (a knower mode).

The concept *Eds*, which is used in references at the conclusion of the document to papers in conference proceedings, and *research* appear frequently in the MERGA document, but not in the AMSI document. In fact the MERGA document contains 190 references to conference papers or journal articles in the mathematics education research literature. The AMSI submission contains 32 footnotes, of which one is a reference to a published conference paper. The emphasis on research suggests that the MERGA response sees its recommendations as much more dependent on evidence garnered from its members' and other researchers' contributions than does the AMSI document.

The concept *learning* appears frequently in the MERGA document, but not in the AMSI document. This again places an emphasis on people, as learning necessarily depends on interactions between teachers and students. This does not, of course, suggest that the AMSI document devalues concepts such as learning or the people involved in the learning process. It merely suggests that knowledge is seen to be a priority.

The above analysis does not purport to be a complete analysis of the documents. There are other concepts that could be discussed, and further work could be done in identifying other aspects of the text such as its ideational, interpersonal and textual function (Morgan, 2006). Nor does the analysis purport to represent the intentions of the authors of the documents, which requires further research such as interviews, or indeed the views of mathematicians and mathematics educators more generally. However the analysis is indicative that the groups do have different modes of knowledge legitimation, and that these differences are worthy of further investigation.

Conclusions

The analysis of these documents has shown marked differences in the construction of the epistemic device within the mathematics and mathematics education research communities. These differences have implications for the future of mathematics education in schools, in that each group has a legitimate claim for representation and input in the development of curriculum and in setting the agenda for school mathematics.

Debates over the introduction of national curriculum frameworks in Australia in the early 1990s have been well documented (Ellerton & Clements, 1994), as have the arguments promulgated in the so-called US Math Wars (Klein, 2007). The national curriculum frameworks in Australia spurned the formation of the Australian Mathematical Sciences Council (AMSC), which was an attempt to speak with one voice. However the AMSC was beset by internal divisions, resulting in the withdrawal of the Australian Association of Mathematics Teachers. Although the desire to speak with one voice is commendable, perhaps ultimately the knowledge legitimation codes in mathematics and mathematics education research make such a goal not only difficult but epistemologically impossible. Rather it may be more constructive, at a time when debate around national curriculum is set to increase, to see these as different but complementary voices.

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Recognising Different Starting Points in Aboriginal Students' Learning of Number

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Eighteen Aboriginal students, in Years 1 to 11 at a remote community school, were interviewed using standard counting tasks, and a 'counting' task that involved fetching 'maku' (witchetty grubs) to have enough to give a maku to each person in a picture. The tasks were developed with, and the interviews conducted by, an Aboriginal Research Assistant, to ensure appropriate cultural and language contexts. A main finding was that most of the students did not see the need to use counting to make equivalent sets, instead using an 'estimation' strategy, even though they were able to count.

Mathematics is a way for people to "understand and make sense of their environment and their practices through identifying patterns that assist in organization" (Perso, 2003, p. 11). Thus, although Western mathematics is the mathematics generally taught in schools, traditional indigenous cultures have what are often distinctly different ways of making sense of, organising, and acting in their environments. They have different world views and related social practices that impact upon what is valued and used as knowledge, mathematical or otherwise. For example, Australian Aboriginal people use genealogical patterns to make sense of their world; that is, "ordered ways of naming and construing the relationships of natural things according to perceived ancestral or familial linkages" (Watson & Chambers, 1989, p. 30, cited in Perso, 2003). In comparison, many other cultural groups use number patterns based on counting and measurement with a focus on 'quantity' rather than 'relationship' (Perso, 2003).

Malcolm, Haig, Königsberg, Rochecouste, Collard, Hill, and Cahill (1999) suggested that Aboriginal children develop very early an understanding of the sophisticated, complex family and social networks making up their world. For these children, family and kinship relationships are a central component of their cultural background and the ways they interpret and function in their world, including the ways they approach 'mathematical' tasks. The research study reported in this paper arose from observations that some Aboriginal students were able to correctly complete 'counting' tasks without counting.

Treacy (2001) created a task to find out whether students would choose to use counting to solve problems, such as for making an equivalent set. Called the Ice-Cream Task, students were shown a picture of 6 people and asked to go to the 'ice-cream shop' to get enough ice-creams for all the people. The 'ice-cream shop' consisted of a box of cut out paper ice-creams placed across the other side of the room. The process was repeated with a picture of 10 people and then a picture of 14 people.

This task has since been adopted by *First Steps in Mathematics* and used in schools around Western Australia, and in other places across Australia and overseas where the *First Steps* program has been implemented. Two accounts of students' responses are:

> Winona (eight years old) glanced at the picture and after choosing the ice-creams very carefully … brought back six. … For the picture of fourteen I thought she would bring back a handful as she had only glanced at the picture and certainly had not had time to count them. I was amazed when she gave out fourteen ice-creams. She said, 'I didn't count' but could not explain how she knew how many to bring back.

> Victor [pre-primary student] looked at the picture and then went to the ice-cream shop and chose his ice-creams very carefully. He returned to the table with six ice creams and proceeded to hand them out saying 'that's one for the baby, one for the Dad, one for the daughter and so on.'

In both these examples, and in others, there was no indication that the students used any form of 'counting', and yet they were able to complete the tasks correctly. Thus, this research study was designed as a small-scale, pilot study to investigate these phenomena further. More specifically, the study aimed to:

- Examine Aboriginal students' strategies in completing 'counting' tasks similar to the ice-cream tasks; and
- Compare the students' responses for these 'counting' tasks to their counting knowledge and skills.

In the context of these research aims it is acknowledged that referring to the ice-cream tasks as ‘counting’ tasks is a Western mathematics perspective, since some Aboriginal students are able to correctly complete the tasks without counting.

Theoretical Background

Nunes and Bryant (1996) suggest that children only really understand counting when they know what counting is for and when to use it to solve problems, in particular, when they choose counting to match sets. According to Nunes and Bryant, children initially come to understand number words and counting as a means of quantifying a single set; they then take time to generalise this understanding to the point where they can use it to compare the size of two sets or to construct equivalent sets. Nunes and Bryant suggest that children have to know, not only how to count, but when it is appropriate to count. They suggest that if children do not choose to use counting to solve problems then they have not fully understood the counting system.

The First Steps in Mathematics Diagnostic Map (Department of Education and Training, 2004) supports this view when it suggests that younger students initially ... *do not spontaneously use counting to compare two groups in response to questions such as: Are there enough cups for all students?* when they are in the Matching Phase (3 to 6 years old). However, by the end of the Quantifying phase (5 to 9 years old), they, *without prompting, select counting as a strategy to solve problems such as: Are there enough cups? Who has more? Will it fit?* At this point they trust that the number at the end of the counting sequence will not change no matter how the collection is counted or arranged.

There has been some research (e.g., Ginsburg, 1982) into the idea of using one-to-one matching however, most early number research has focused on using one-to-one correspondence in order to count (e.g., Baroody & Wilkins, 1999; Fuson, 1992; Nunes & Bryant, 1996). More recently, The Model of Early Number Development in Figure 1 developed by Treacy & Willis (2003) includes two pathways into number understanding. However, it does not include one-to-one matching (for example, based on family relationships) as an alternative pathway. It is possible that this should be included in this model, and could be a pathway into number understanding that has been overlooked by educators in general, but particularly teachers of indigenous students. If this is how some indigenous students solve a problem like that presented in the Ice-Cream Task, there are implications for teachers for how they introduce young indigenous students to number learning.

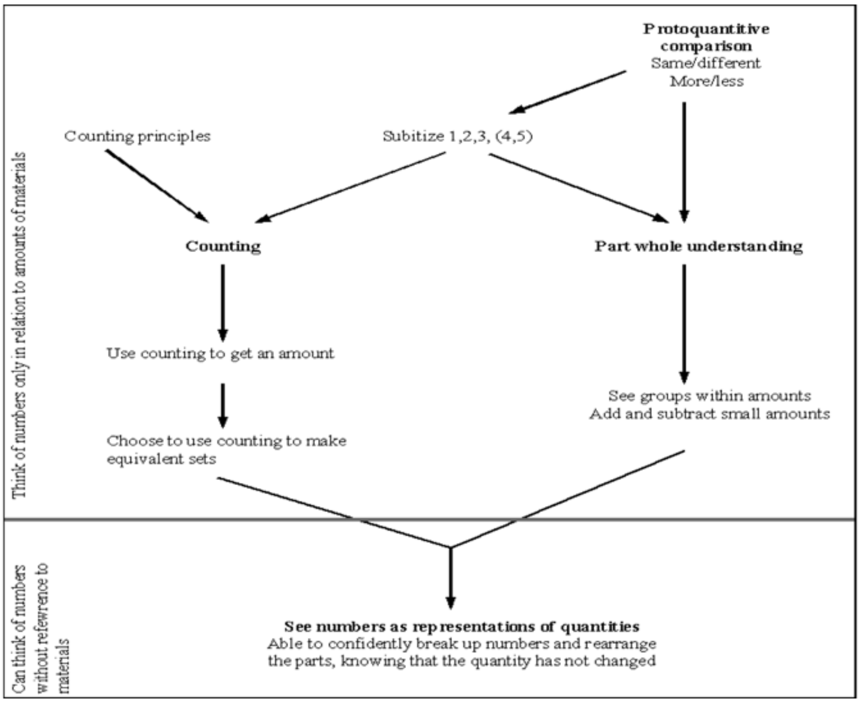

Figure 1. Children learning about number as a representation of quantity – A model.

Thus, in addressing the two research aims, this study was designed to examine if students: choose to use counting; use a matching strategy; use family relationships as a basis; and/or use other strategies in situations like the Ice-Cream Task involving making equivalent sets.

Method

Eighteen Aboriginal students, in Years 1 to 11 at a remote community school in the Goldfields of Western Australia, participated in task-based interviews based on 'counting' tasks. The tasks involved: fetching 'maku' (the local name for witchetty grubs) to give all the people in a picture a maku; identifying a hidden quantity when a part of a collection of maku are covered; and standard counting tasks. This paper does not report on the full findings; for focus, it reports on the maku and standard counting tasks only.

An Aboriginal research assistant who spoke the same language as the students involved in the study (Wangkatha) was engaged to help with the task design and data gathering processes. She grew up in the community and was familiar with all of the students. The ice-cream task was modified to become the Maku Task. The local shop sold lolly versions of the grubs, so these were used instead of pictures of maku. Pictures of the groups of people were constructed from pictures of Aboriginal people that were found in books in the school (see Figure 2). Specifically, the tasks were:

Maku Task. The student was shown pictures with 4, 6, 10, and 16 people and asked to get enough maku for all of the people in the picture. This task was designed to see whether a student would choose to use counting in a situation where it is not obvious to do so.

Oral Count. The younger students were asked to count from one, and the older students were asked what came after a given number, for example, 39, 59, 79, 99, 100, 109, and 199. This task looks at the extent of the student's oral counting sequence.

Get Me Task. The student was asked to get a number of items (maku) and put them in a bag to take home. This task looks at whether the students can use counting when asked.

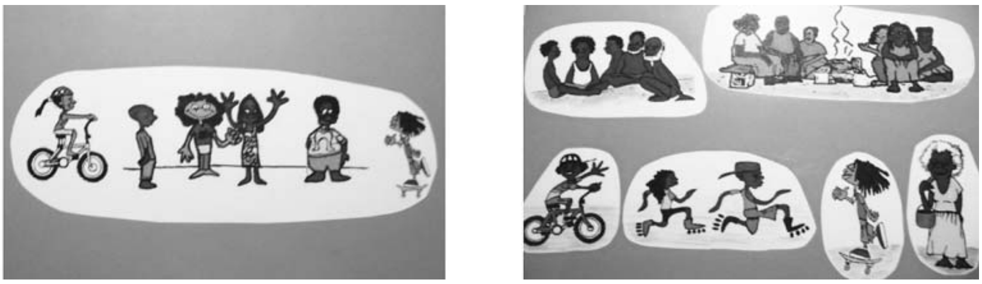

Figure 2. Examples of the pictures of groups of people for the Maku Task.

The interviews were conducted in Aboriginal English and were video taped with permission from the students and their parents. The video tapes were transcribed with the assistance of the research assistant, to ensure a correct interpretation of what the students were saying.

Findings and Discussion

Data from the interviews are summarised in Table 1. It was not always possible to ascertain from the video footage what strategies the students used to complete the Maku Task. Whether they were using a matching strategy or family relationships, the initial foci of this study, it was not possible to say, as the students were not able to articulate enough about how they knew how many maku to collect. However, it was possible to identify if they were or were not using counting, with most students not using counting to complete the Maku Task. In comparison, the students were able to count when asked to get a number of items, and all students, except for the youngest one, could orally count beyond sixteen, which was the largest number within the Maku Task.

Maku Task

Three of the students (see Table 1), Rowena, Justin and Keegan, chose to use counting at some point in the Maku Task to work out the number of people in the picture and then used this count to collect the maku from across the room. One of these students, Rowena, counted the sixteen people in the picture and then collected twenty maku instead of sixteen. When asked how many maku she had, she immediately said twenty. She had put four to the side and knew that sixteen and four made twenty. This suggests that she did not use the count of sixteen people to help her with the number of maku that she needed to collect. Rowena also struggled to use the oral count when she was asked to get seventeen maku. With the oral count sequence, she was able to count to 28, but was not able to continue after this number.

Justin used counting for two parts of the Maku Task, and did not use counting for the largest and smallest part of the task. For the picture of four, he collected two items at a time and brought them back and gave them out. For the picture of 16, he simply grabbed a large handful of maku and then gave them out one at a time. When asked how many maku were in the picture he did not know, and proceeded to count.

Table 1

Summary of the Interview Data

Name	**Year level**	**Maku Task 4**	**Maku Task 6**	**Maku Task 10**	**Maku Task 16**	**Oral Count**	**Get Me Task**
Ainsley	1	Collected one at a time. Afterwards, counted 4 with prompt.	Collected 3 then another 3. Afterwards, counted 6 with prompt.	NA	NA	10	8
Alana	1	Collected 2	✓ No overt count	11 No overt count	3	20	21
Kiara	2	Brought over the container	7 No overt count	Brought over the container	12 No overt count	29	7
Delray	3	✓ No overt count	5 No overt count	✓ No overt count	12 No overt count	20	9
Tosha	4	✓ No overt count	10 No overt count	11 No overt count	15 No overt count	39 needed written then continued. 59 needed written then continued After 100 -didn't know.	When asked to get 20 she counted one item twice so had 19 maku.
Lana	4	5 No overt count	✓ No overt count	17 No overt count	14 No overt count	49	20
Anastasia	4	✓ No overt count	✓ No overt count	✓ No overt count	13 No overt count	After 109 said 300	16
Gracie	5	✓ No overt count	7 No overt count	✓ No overt count	17 No overt count	After 109 said 200	20
Bernard	6	Bought over the container When asked HM : 4	Bought over the container When asked HM : 6	9 No overt count	18 No overt count	20	21
Stuart	6	✓ No overt count	✓ No overt count	✓ No overt count	12 No overt count	39 needed written then continued. Written 59. After 100 said 200.	21 Counted by 2s to 20 then got 1 more
Arnold	7	✓ No overt count	✓ No overt count	✓ No overt count	18 No overt count	After 109 said 2000	20 Counted by 2s
Rowena	7	✓ No overt count	✓ **COUNTED**	✓ **COUNTED**	20 **COUNTED**	28	17 Counted to 12 then missed out 13 … Said 14, 15, missed out 16 said 17
Justin	7	2 then another 2	✓ **COUNTED**	✓ **COUNTED**	19 No overt count	After 39 said 50, self-corrected 40. After 109 said 200.	20

Janey	9	✓ No overt count	✓ No overt count	✓ No overt count	✓ No overt count	200	20
Keegan	10	✓ No overt count	✓ No overt count	✓ No overt count	✓ **COUNTED**	After 99 said, 200	20 Counted by 2s
Wanita	10	✓ No overt count	✓ No overt count	✓ No overt count	13 No overt count	After 109 said 200	20 Counted by 2s
Tyrone	11	✓ No overt count	✓ No overt count	✓ No overt count	17 No overt count	After 109 said 200	20
Sheree	11	✓ No overt count	✓ No overt count	✓ No overt count	✓ No overt count	After 109 said 200	30

Note: Shaded cells show responses that are either correct, or one more or one less than the required number. Diagonal lines across cells show students who chose to use counting for the Maku Task.

Keegan was able to select the correct number of maku for the first three parts of the task without counting and then chose to use counting for the last picture of sixteen people. After getting the right number for the picture of ten, the research assistant asked him several times to explain how he knew to collect ten maku. He was able to reply by saying that "I bin looked at the picture". When asked to complete the section of the task with sixteen people, he said in language to the research assistant, "Do you want me to get it right now?" to which she replied, "Yes". He then said, "Can I count them first?" and she said, "Do what you want to do", so he counted the people first. It is possible that the questions that were asked of him after completing the task with ten people, one of which was, "Did you count?", may have suggested to him that he should be doing something other than 'just look' to work out how many people.

It is interesting to note that most of the children did not choose to count for this task. Of those that did not choose to count, most of them were able to select the exact number of maku needed each time, or to select one more or one less than required. This suggests that for most of these children, counting is not an appropriate strategy for this type of situation.

So how did the students know how many maku to select? Many of the students were not able to articulate exactly how they knew how many maku. When asked, many of them simply said things like "I bin looked," suggesting that they looked at the picture and just knew how many. One student, Tyrone, was not able to say how he knew. After bringing back 17 maku for the 16 people in the picture he was asked how many people in the picture and he was not able to say. Instead he counted the people and then said there were 16 people. This suggests that he did not even think about the number of people in the picture when collecting the maku, and yet he was able to get just one more than the required number. He was very near the required number, without thinking about a number.

Another student, Gracie, looked at the picture of sixteen people for a short time and then went to collect the maku. She showed no signs of counting, she did not touch any of the people in the picture, and she did not nod her head or show any other overt sign of counting. She brought back seventeen maku and placed them on the people in the picture. The research assistant asked, "Did you know how many people from the beginning?" to which Gracie shook her head and said, "When I put them in I thought there was seventeen people but there was sixteen". "Why did you think it was seventeen?" "Because there was lots". The researcher then asked, "Did you count them to start with?" to which Gracie shook her head and said, "No, too quick". "So how did you know to get seventeen? What were you thinking over there?" Gracie struggled to answer this question, so was asked, "Did you have a picture of the people in your head?" Gracie nodded and said, "Five men, six women and five kids".

Oral Count

All of the students in this study, except Ainsley, were able to orally count to at least twenty, which is more than the number required for the largest group of people in the Maku Task. Ainsley did not know what came after ten, which, when considering his age, is not surprising. Interestingly, after Year 4, many of the students knew the counting sequence up to 109, but did not know what came after, with some saying 200, one saying 300 and one saying 2000. The latter student re-thought his response to this question after the task and confessed to the researcher that he knew that it wasn't 2000 as he was walking out the door, to which she asked, "So what is it?" and he replied, "200 unna". Two students did not know what came after 100 and one student did not know what came after 99. When discussing this with the students' teachers, they said that these students had been completing three digit addition, subtraction and multiplication, which means that all but one of these students (Janey, who knew what came after 109) would have been completing calculations with numbers that were not in their counting sequence.

Get Me Task

All of the students within the group were able to use counting to get a number of maku when they were asked to. This task was the last one in the interview and the number of maku they were asked to get varied according to how they had responded on the other sections of the task. The younger students, Ainsley, Kiara and Delray, were asked to get less than ten items, which they succeeded in doing. Alana was one of the last students interviewed and it was decided to try asking her to get one more than her oral count of twenty, to see if she could use her count sequence in a purposeful situation. She succeeded in getting 21 maku. The rest of the students were asked to get 16 or more items to see if they could use counting in this situation, to at least the number of the largest set in the Maku Task. Tosha made a small error when she recounted one item twice, whereas Rowena struggled to keep track of the counting sequence while collecting the requested number of items. She counted to 12 then missed out 13, said 14, 15, missed out 16 and said 17, so instead of seventeen maku she had fifteen. Rowena is an older student who has a learning disability, which may explain why she had trouble recalling the counting sequence.

Conclusions and Implications

Most of the Aboriginal students in this study demonstrated specific counting knowledge and skills in the Oral Count and Get Me Task, yet they chose not to count in the Maku Task. Thus, although it was not always clear what strategies they did use for the Maku Task, the findings suggest that they did not see this situation as one in which counting is required. Some researchers have noted that since Aboriginal languages do not have many counting words, many Aboriginal people do not tend to count in their everyday situations. In particular, Malcolm et al. (1999) suggested that, in general, precision is much more central to Western society than in most Aboriginal contexts. It certainly seems that the students in this study were not concerned about precision or exactness, since being 'close' to the exact number (e.g., one more or one less) was sufficient to complete the task. Many of them were able to collect an appropriate number, suggesting that they were looking at the picture and using an estimation strategy to get a quantity of maku that would be 'about right' for the number of people. Gracie's strategy suggests that she saw groups within the group to help her to work out how many maku to collect.

Rudder (1999, pp. 12-14) also noted that Aboriginal people are not concerned with numbers:

> If you asked how many people went hunting together, you would be told all the names of the people and if you ask what they collected they would tell you the names of the different animals or plants they had gathered. … They were interested in the relationship between the things in the group … they knew everyone and everything by name and by relationships so that is what they saw and they didn't worry about numbers at all.

Rudder goes on to say that in a situation such as when turtle eggs are gathered, "It does not matter if different people have more or less that others, No one will go hungry." The Maku task, in which the children were asked to make an equivalent set, is similar to this scenario about turtle eggs. The students did not appear to see the need to count; they simply collected a suitable number of items and then gave them out.

The findings of this study raise numerous issues for future research and for mathematics curriculum and teaching practices, including the following. (1) Since this study has not been able to clarify the question of whether some Aboriginal students use a matching strategy or whether they use family relationships in situations involving making equivalent sets, more in-depth research needs to be carried out that provides appropriate contexts and opportunities for the students to talk more extensively and to disclose more of what they are thinking and doing.

(2) Additional research needs to be conducted related to the findings of this study that Aboriginal students might have a tendency not to attend to exactness. The implications of these findings for teaching practices also need to be attended to, in that it cannot be assumed that Aboriginal students see a purpose for counting; a purpose needs to be made explicit in activities that a teacher uses to teach counting. In this regard there is

also a need to be explicit about different cultural viewpoints, for example, the value and purpose of precision versus estimation or sharing.

(3) Students need to be provided with purposeful counting experiences with quantities beyond 100, to build their knowledge of the patterns in the number system and to connect quantities to this. Purposeful activities will be a challenge to identify for non-Aboriginal teachers, since what non-Aboriginal teachers might think of as purposeful might not be so for the Aboriginal students.

(4) The students involved in this study are very capable students, however they have not had a curriculum that has been focussed on their learning needs. Without support and information, their teachers were planning lessons based on 'western' assumptions. They were not able to recognise what their students brought with them into the classroom, and were not aware of what they should be looking for. Teachers need to recognise the different starting points, and learning needs of all their students. This is difficult when teachers do not share the same cultural background as their students and there is little available information about what they should be looking for. There is a need to value what students can do and build on it, and a need to recognise and accommodate different starting points in number.

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Deepening the Mathematical Knowledge of Secondary Mathematics Teachers who Lack Tertiary Mathematics Qualifications

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A professional learning program for unqualified practising secondary mathematics teachers regarding senior secondary mathematics teaching is described in this article. Professional learning episodes, artefacts and reflections of three teachers who participated in the program are analysed to identify the development of these teachers' mathematical and pedagogical content knowledge. The findings indicate that a program designed for senior secondary mathematics can enable practising teachers to deepen and broaden their understanding of junior secondary mathematical pedagogy.

In Australian secondary schools significant numbers of teachers of junior secondary mathematics (almost 50%) and even teachers of (usually less advanced) senior mathematics subjects (32%) do not hold the recommended tertiary mathematics qualifications for teaching secondary mathematics (McKenzie, Kos, Walker, & Hong, 2008). In Victoria, it is recommended that teachers of secondary mathematics complete two years of tertiary mathematics (elsewhere in Australia the minimum is three years) and an approved qualification in education (VIT, 2003). The subject of this paper is a professional learning project initiated by one school faced with the prospect of having too few teachers with the knowledge of senior secondary mathematics to be able to provide a full range of senior mathematics options for their students in the new future. This school decided to prepare some of its teachers of junior secondary mathematics to teach advanced senior mathematics. They approached our university to design a professional learning program for teachers from their school and other schools in their region who had not completed the recommended tertiary mathematics study or pre-service training in mathematics. In this paper we describe and analyse the mathematical and pedagogical content knowledge of three teachers who participated in the program.

Theoretical Framework

Professional Development Models

Pedagogical change and innovation has been the main goal of in-service professional development programs of mathematics teachers in Australasia and many practice-based models are structured around designing and/or implementing different teaching approaches and tasks (e.g., Goos, Dole, &Makar, 2007; Watson, Beswick, Caney, & Skalicky, 2005/2006). However it is not clear how teachers may develop their understanding of mathematics in such programs:

> Advocates for practice-based professional development argue that learning experiences that are highly connected to and contextualised in professional practice can better enable mathematics teachers to make the kinds of complex, nuanced judgements required in teaching. Yet, evidence is generally lacking regarding if and how teachers might enhance their knowledge of mathematics through such professional development experiences. (Silver, Clark, Ghousseini, Charalambous, & Sealy, 2007, p. 261)

The importance of teachers' mathematical content knowledge is recognised as critical for improving students' mathematical learning in recently reported Australasian studies (e.g., White, Mitchelmore, Branc, & Maxon, 2004) and engaging teachers in mathematical thinking through working on mathematics-related tasks, and reflecting on these experiences is common to many in-service programs. For example, Biza, Nardi, and Zachariades (2007) used tasks that were situated in teaching practice that involved reflecting upon the learning objectives of a particular task, examining a flawed student solution and describing, in writing, feedback to the student. The professional learning tasks used by Silver et al. (2007) were also situated in teaching practice. These tasks began with an initial non-trivial mathematics problem. Teachers then analysed and discussed a teaching case and finally collaboratively planned and reflected on a mathematics lesson. On the other hand Leikin and Winicki-Landman (2001) used only mathematics tasks that focussed on definitions of mathematical concepts. Teachers were prompted to discuss the logical relationships between different mathematical statements related to a concept. These mathematics tasks entwined both mathematical knowledge and issues of pedagogy. Both kinds of tasks, that is mathematics tasks and pedagogical content knowledge

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tasks imbedded within a practice-based model of professional development would seem to provide optimum conditions for enhancing teachers' mathematical and pedagogical content knowledge.

Pedagogical content knowledge

Shulman (1987) defined pedagogical content knowledge (PCK) as "the blending of content and pedagogy into an understanding of how particular topics, problems, or issues are organised, represented, and adapted to the diverse interests and abilities of learners, and presented for instruction" (p. 8). Ma (1999), who focussed on primary teachers' knowledge of fundamental mathematics, demonstrated the strong relationship between profound understanding of fundamental mathematics and pedagogical knowledge. For her, profound understanding is more than procedural and conceptual knowledge; it is "an understanding that is deep, broad, and thorough" (p. 120), where depth means being able to connect a topic with "more conceptually powerful ideas of the subject" (p.121), breadth being able to "connect it with those of similar or less conceptual power" (p.121), and "thoroughness is the capacity to connect all topics" (p. 124). Such understanding is evident in connected approaches to teaching mathematics that use flexible and multiple representations of mathematics and an attitude that promotes mathematical inquiry and justification.

Chick, Baker, Pham, and Cheng (2006) proposed a framework of analysing PCK. They grouped the elements of PCK into three categories. In the first category, "clearly PCK", pedagogical knowledge of teaching mathematics is entwined with knowledge of mathematics and included knowledge of general and specific teaching strategies, student thinking and misconceptions, knowledge of cognitive demands of tasks, appropriate and detailed representations of concepts, knowledge of resources, curriculum knowledge, and knowledge of the purpose of content. The second category in their framework focuses on mathematics subject knowledge needed for teaching and includes Profound Understanding of Fundamental Mathematics (Ma, 1999), deconstructing content into key components, mathematical structure and connections, procedural knowledge, and methods of solution. The third category concerns general pedagogical knowledge applied to teaching mathematics and includes general and specific goals for learning, getting and maintaining student focus, and classroom techniques.

The Study

The VCE Mathematics Professional Learning Program

The VCE (Victorian Certificate of Education) mathematics professional learning program (PLP) for senior secondary mathematics was designed for practising secondary teachers of mathematics who had no experience of teaching advanced senior secondary mathematics and who had not completed the recommended qualifications. The teachers wanted a program that was situated and practice based and that would enable them to learn the mathematics of grades 11 and 12 (VCE Mathematical Methods and Further Mathematics) and the methods of teaching this mathematics. In this professional learning program, teachers were students of mathematics (Leiken & Winicki-Landman, 2001), and this mathematics included algebra, functions and graphs, rates of change and calculus, probability and data analysis, including statistical modelling.

The PLP involved seminars as well as self-directed practice-based inquiry and portfolio development. Teachers attended 21 three-hour seminars conducted fortnightly on an afternoon during school terms over a school year. We provided each of the teachers with a CAS (computer algebra system) calculator and at least two different textbooks for each of the VCE mathematics subjects, and we encouraged the teachers to purchase a mathematics dictionary of their choice.

We used both mathematics and professional learning tasks during the seminars. A typical session began with finding out what the teachers knew about the mathematics concepts and procedures concerning a mathematics task (e.g., Figure 1). One of us would scaffold discussion of conceptual or procedural ideas relevant to the task. Teachers would then work on mathematics tasks, including closed tasks such as the one in Figure 1, open-ended tasks, tasks designed to explore concepts, tasks with the purpose of deriving or proving rules, or mathematics-related professional learning tasks. These professional learning tasks (Silver et. al., 2007) included analysis of mathematics problems and activities, analysis of teaching materials and teaching strategies, design of mathematics problems and activities, analysis of students' solutions, and review of VCE subject examiners' reports. The session would end with a discussion to summarise the key mathematical concepts and procedures. Teachers reflected on their experience of working on these tasks and the outcomes of this discussion were used to identify potential student misconceptions and key points for teaching.

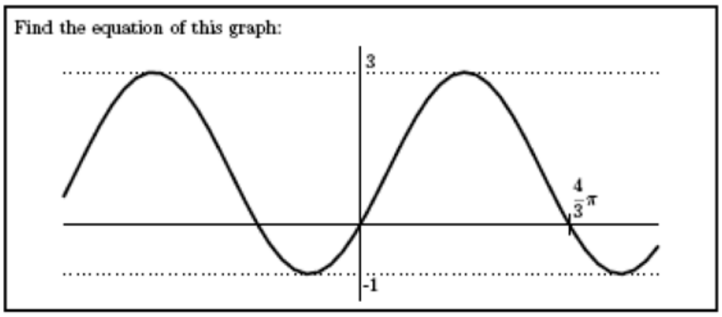

Figure 1. A task on transformation of trigonometric functions.

The sequence of topics for the seminars followed the sequence normally used by teachers of the grade 12 subjects. Hence we included seminars with discussions about the formal assessment tasks of the VCE subjects at roughly the same time that teachers were designing and assessing students with these tasks. Experienced senior secondary mathematics teachers also conducted a few sessions in the program. Their sessions focussed on curriculum knowledge, long-term planning for teaching and assessment, resources for teaching, and tasks used for teaching and assessment.

The practice-based component of the program occurred in the teacher's school between seminars. We encouraged participants to establish a mentor relationship with an experienced teacher of senior secondary mathematics to support their school-based self-directed inquiry. We recommended that they negotiate with their colleagues to observe and/or team-teach grade 11 or 12 mathematics lessons, observe students doing mathematics (in lessons or by tutoring students), reflect on observations, analyse student work, research and critique teaching and assessment resources and materials, and to participate in the moderation processes of student assessment for these subjects. The teachers documented their learning in an annotated portfolio that they presented to their peers in the final session of the program.

Methods

The study used a qualitative design. Eleven teachers (six were women) from five government secondary schools located in a regional city and surrounding towns participated in the program. Ten teachers completed the program (the other teacher had to withdraw early in Term 2 due to family commitments and work responsibilities). Questionnaires, field notes and artefacts from the program and teachers' portfolios were the data used in this study. Data were analysed qualitatively using codes derived from the PCK framework (Chick et al., 2006) described above.

The pedagogical content knowledge with respect to algebra and functions of three teachers in the program from different schools are the focus of this paper. None of these teachers had completed the required tertiary mathematics study or mathematics education training for mathematics teaching. They were selected for this paper because their responses to the initial questionnaire indicated different levels of mathematics knowledge and teaching experience, varied professional learning experiences, and mentoring between sessions. These three teachers also illustrated the kinds of mathematical and pedagogical learning achieved by the teachers in the program.

Teachers' Mathematical and Pedagogical Content Knowledge

Gloria

Gloria, a second year teacher, with one year experience of teaching grade 12 chemistry and grade 10 mathematics, was teaching grade 11 General Mathematics in the year of the program. At the beginning of the course she described her mathematics teaching strengths as "recognising where students are having difficulties with understanding; and patient with explaining skills and formula's (sic) as well as providing alternate explanations where needed" and rated herself as comfortable teaching basic algebra skills and using technology, but thought that she struggled to explain concepts and procedures concerning functions in the VCE mathematics curriculum. Gloria's knowledge of algebraic procedures was demonstrated during one of the early seminars when she demonstrated to the group the procedure of completing the square to derive the quadratic formula for solving quadratic equations. She explained early in the program that she was "getting a deeper understanding of where the rules come from and why they work" indicating development toward thorough understanding of concepts.

When presenting the contents of her portfolio at the end of the program, Gloria described an investigation using Excel that she had used with her grade 10 students. She claimed that this task showed how she had developed during the program since she now understood the importance of investigations and how to implement them in the classroom:

I have included this resource in my portfolio because I think it is an excellent resource for introducing translations, dilations, reflections, etc in graphs. I have successfully used this at the year 10 level and observed students responding positively to it. While being involved in this course I have realised how useful this resource is for students continuing into VCE methods. This resource looks specifically at changes in quadratics graph, but would be easily developed to include other graph types; trigonometry graphs, logarithmic graphs etc, in a similar manner. It provides students with a quick and easy way to visually view the effects of translations in graphs. The questions at the end of each section help students to ensure that they understand what they have observed and consolidate the information.

Gloria's investigation of quadratic functions is an example of a general strategy that is student-centred and engages students in thinking mathematically by exploring cases and paying attention to patterns; in this case, in the relationship between graphic and symbolic representation of functions. It also illustrates her use of multiple representations of concepts facilitated by digital technology. Gloria also indicates through this task that she understands the connections between topics; in this case, transformations of different functions. By generalising the teaching strategy in this context, Gloria believes that students will consolidate their understanding and, by implication, generalise the findings of transforming functions.

Gloria's progress toward deepening and broadening her conceptual understanding was also evident in her analysis of the cognitive demands of tasks and the deconstruction of content to key components. Toward the end of the program, the teachers reviewed the examiners' reports and student results for particular questions on the Mathematics Methods written examination 2 (VCAA, 2006). The two most difficult multiple-choice questions proved to be Q7, a question about the domain of a logarithmic function, and Q19, a question that required students to solve for an unknown in a probability density function, for which only 45% of students had been correct. Gloria argued that students had generally performed worse on the multiple-choice questions in which functions and equations were expressed in what she described as "general form" (Q7: $g(x) = \log_e |x - b|$, where b is constant; Q19: $f(x) = \begin{cases} 1 + e^{\frac{x}{k}} & 0 \le x \le k \\ 0 & \text{otherwise} \end{cases}$). She also commented that perhaps students were also having difficulty with questions about domains. She included these observations among her goals for teaching when documenting what she had learned during the program.

Helen

Helen was also in her second year of teaching. She was a qualified science teacher, who had taught grade 7 mathematics for one year. She had also tutored grade 10 and 11 students from a local private school in mathematics for three years and had established strong professional relationships with mathematics teachers at this school who continued to mentor her. At the beginning of the program Helen wrote that she considered herself to be "a mathematical and scientific thinker" who enjoyed working with structure and "skilled in working with students of different abilities … and good at finding problems and work that challenges all students." While comfortable with teaching some algebra topics, Helen thought that she struggled to explain concepts and procedures concerned with solving exponential and logarithmic equations and graphing and transforming functions.

Early in the year Helen expressed that she was "beginning to see how concepts link up and 'fit in'. I also feel as though I am generating a better understanding of 'how' and 'why', although this still needs further work". At the end of the course Helen reflected on her own experience of learning mathematics, recalling learning the rule and applying it, but not "getting" the conceptual ideas. She observed this in the grade 10 and 11 girls that she tutored throughout the year. Helen admitted that she still looked for rules but that the program had enabled her to focus on understanding and concepts more. In her presentation Helen focussed on the need for students to develop a mathematical attitude toward the discipline of mathematics. She gave particular emphasis to students developing ways of thinking mathematically and developing persistence in problem solving:

> I place a lot of importance and emphasis on students' success in VCE Mathematics from their engagement and problem solving abilities in middle years mathematics. This VCE Mathematics program has therefore assisted me in my teaching of middle years mathematics (as well as VCE Mathematics) by inspiring me to work towards students being engaged, challenged and encouraged when developing understandings of new concepts.

Helen included a curriculum development map (or learning trajectory) on polynomial and modulus functions in her portfolio. She explained that she had based in on one about exponential functions distributed during the program. Helen's map shows the sequence of connected topics from grade 9 and 10 to grade 11 Mathematical Methods and grade 12 Mathematical Methods. It includes symbolic and graphic representations of various functions and lists key skills and concepts to be developed. One thread of ideas through this map is set notation, first included for grades 9 and 10 as an introduction (#4). She listed symbols and examples including representation on a number line. At #7, as part of the grade 11 curriculum she included "develop deeper understanding of set notation, domain & range, and symbols to represent these." Thereafter the map includes set notation when defining functions and their domain. For #13 in grade 12, Helen records that students need to develop "understanding/reading accurately <u>all</u> methods of expressing a function". From this artefact we can see that Helen is developing depth in her mathematical understanding, being able to connect a topic with more conceptually demanding ideas. She thought curriculum maps were really helpful because they showed when the students learned particular concepts and skills and why some students might miss learning some concepts or skills. Helen's curriculum map illustrates the way in which the teachers came to understand the connection between content in junior and senior secondary mathematics and hence the purpose of specific content in junior secondary mathematics for a range of topics.

Helen's interest in student thinking and their misconceptions is also evident in this curriculum map. She included a section on common misconceptions, identifying students' likely "confusion with symbols, & reading set and function notation, difficulties with asymptotes & domain/range of these graphs, translating in wrong direction, what to do when not in TP (turning point) form, modulus functions."

Donna

Donna was the most experienced teacher among these three teachers. She had taught junior secondary mathematics for three years. She was in the fifth year of teaching, having been a health-care professional in her previous career. Donna was one of two people in the program who rated their mathematics knowledge as quite low, claiming that she could only solve simple problems related to VCE mathematics content. She believed her strengths were in her "ability to explain concepts well once I have revised them, enjoy maths myself and succeed in building confidence in my students." At her school the VCE mathematics coordinator mentored the two teachers in the program.

Midway through the program Donna commented on the importance of using the correct terms and their meaning and how this aspect of the professional learning program was impacting on her teaching of junior secondary mathematics. So it was not surprising that definitions and mathematical language figured prominently in her portfolio and presentation of artefacts at the end of the program. Donna included two packs of cards from Barnes (1991) that are resources for a small group activity that she learned about from her mentor. One pack depicts the equation, graph or description of the transformation from the standard form for various quadratic functions on the cards. Donna explained that when students worked in a small group to sort these cards into matching sets, they needed to explain their thinking and present an argument to support their claim of matching cards. She reflected:

> I found these cards very useful when I was revising this work and correcting my own language when describing curves… This activity helps students become familiar with the both the idea and language of transformations.

This task illustrates a general teaching strategy of group problem solving that can be used purposefully for teaching and learning specific content. The task also illustrates the use of multiple representations since using symbolic and graphic representations of concepts, and moving fluidly between them, was important for the topics in the VCE mathematics subjects that we focussed on. At various times during the program examples of appropriate representations that these teachers used in their teaching in junior secondary mathematics were discussed and Donna was usually prominent in these discussions. For example, during the analysis of examiners reports, the teachers identified students' weakness with fraction knowledge and as part of the extended discussion on teaching and learning fractions Donna described using strip models of fractions to develop grade 7 students' understanding of equivalence.

Conclusion

The PCK framework developed by Chick et al. (2006) was useful for analysing teachers' knowledge and the cases of teachers' knowledge presented in this paper illustrate the entwining of knowledge of mathematics and knowledge of teaching and learning (Ball, 2001; Davis & Simmt, 2006; Ma, 1999). While we set out to develop teachers' knowledge of mathematics needed for teaching senior secondary mathematics we discovered that taking this approach deepened and broadened teachers' understanding of junior secondary mathematics content and pedagogy. This was particularly evident through the connections that they made between mathematical concepts, the use and understanding of multiple representations, deconstruction of content into key components, their understanding of students' misconceptions, and their appreciation of the inadequacy of procedural and instructional thinking.

The model of in-service professional development analysed in this article shows promise as an effective means for preparing senior secondary mathematics teachers in the current context of mathematics teacher shortages. Important for deepening teachers' mathematical knowledge and developing their PCK was the '"teachers as learners of mathematics" model used along with the professional leadership provided by mentors. However these findings need to be tested in a follow-up study that analyses the professional paths of these teachers, their pedagogical practices and their students' learning outcomes in both senior and junior secondary mathematics.

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Indigenous Students' Early Engagement with Numeracy: The Case of Widgy and Caddy

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This paper reports on a component of a research project, Young Australian Indigenous students Literacy and Numeracy (YAILN), a longitudinal study investigating learning and teaching activities that support Young Indigenous Australian students as they enter formal schooling. In Queensland students are allowed to attend a non compulsory year of schooling, Preparatory (Prep), if they reach the age of 5 years by June in the year that they enrol. In YAILN one of the participating Indigenous schools' preparatory intake included Indigenous students who had not reached the required age for Prep. Numeracy understandings for two of these students, Widgy and Caddy were tracked during their pre prep year of schooling. Both Widgy and Caddy are from low socio-economic families. Their pre and post test numeracy results and the interview conducted at the beginning of their 'official' preparatory year suggest that this extra year of schooling enhanced their knowledge of mathematics and has (a) put them on an even footing with students from more advantaged backgrounds as they enter Prep, and (b) given them a distinctive advantage over other Indigenous students who have not had equivalent experiences.

Introduction

Young Indigenous Australian students continue to experience difficulties at school, especially in the areas of literacy and numeracy. Results from the National Report on Schooling, National Benchmarks for reading, writing and numeracy in Years 3, 5 and 7 demonstrate a high percentage of Indigenous Australian children performing well below the benchmark (ACER, 2005; MCEETYA, 2008). The latest National Report on Schooling in Australia (MCEETYA, 2008) includes the following results for Indigenous Australian children in Year 7 obtained from testing in 2006. Sixty-three percent are achieving results at the benchmark for reading (p. 27), 73.8% are achieving at the benchmark for writing (p.28) and 47.5% are achieving at the benchmark for numeracy (p. 29). These results have been improving since 1999, but there is still much to be done. Unjustified blame has been laid upon Indigenous students in the past and absenteeism, disadvantaged social background and culture have all be seen as contributing factors (Bourke & Burden, 2000). This paradigm is seen as irresponsible (Cooper, Baturo, Warren, & Doig, 2004; Matthews, Howard, & Perry, 2003). Our longitudinal research project, Young Australian Indigenous students' Literacy and Numeracy (YAILN) aims to improve learning literacy and numeracy outcomes for Indigenous students.

Theoretical Underpinnings

Briefly the research base and design principles that underpinned the development of the Numeracy aspect of YAILN were:

- *Maths Ability*: All children are capable of learning mathematics. Children do not have to be made ready to learn. They freely engage with informal mathematics in everyday life (Greenes, 1999).
- *Role of the teachers:* Play is not enough to assist learning in the early years. Children learn through play but they only discover a certain amount when left to their own devices. Adult guidance is needed to assist them to reach their potential for learning (e.g., Balfanz, Ginsburg, & Greenes, 2003; Vygotsky, 1962). Indigenous students gain less from attending play based programs (Tayler, Thorpe, & Bridgstock 2006).
- *Types of activities:* Hands on activity based learning best supports young Indigenous students engage with mathematics (Cooper, Baturo, Warren, & Grant, 2006).
- *Role of oral language:* A focus on the language of mathematics fosters important language acquisition and assists students acquire meta-cognitive abilities. This focus is even more relevant for students whose first language is not English (Pappas, Ginsburg, & Jiang, 2003).
- *Maths curriculum:* Young students are capable of dealing with a comprehensive mathematics curriculum.

- *Indigenous students' language:* Discourses of Indigenous families often do not match that of the school (Cairney, 2003). Teachers need to create a bridge for young Indigenous students between Aboriginal English (AE) and Standard Australian English (SAE) as these students grapple with new language, new concepts and vocabulary presented for literacy and numeracy.

YAILN is now in its second year. The students who participated in our first year were all from the preparatory classrooms, a non compulsory year of schooling prior to Year 1. Prep classes are conducted 5 days a week and children stay all day. Participants must be 5 by 30 June in the year they start Prep. At the completion of the first year of YAILN our results indicated that although Indigenous Australian students scored significantly lower on the numeracy pre test, intervention focussing on (a) the language of mathematics, and (b) representations that support mathematical thinking in both directed teaching and play based contexts assisted these students to bridge the gaps in their learning (Warren, Young, & deVries, in review). The particular focus of this paper is to investigate the impact that engagement with YAILN had on two Indigenous pre preparatory students aged 4 years 4 months and 4 years 5 months.

One of YAILN schools, a totally Indigenous school (School D), enrolled 2 students in their Prep class that had not reached the age of 5 by 30 June, hence the term pre preparatory students. The class consisted of up to 18 students, many of whom did not attend school on a regular basis. Of the 18 students we managed to consistently track 9 students over the school year, 5 that were the correct age for Prep, 2 Year 1aged students and the 2 pre prep students. The focus of this paper is the 2 pre prep students (Widgy and Caddy). The particular aims are:

1. To ascertain the effect, if any, participating in YAILN had on their understanding of mathematics, in particular their understanding of number, patterning and oral language; and
2. To gauge how these understandings compared to Indigenous students who had not engaged in any numeracy activities before commencing the preparatory year.

The literature suggests that students from low socio-economic backgrounds begin school with many disadvantages. It seems that children who bring to school early mathematical knowledge are advantaged in terms of their mathematical progress through primary school (e.g., Aubrey, Dahl, & Godfrey, 2006), a consequence of this being that students with little mathematical knowledge at the beginning of formal schooling remain low achievers throughout their primary years and probably beyond. Denton and West (2002) showed that low income students usually come to preschool with the same basic readiness to learn as the more advantaged students. The difference lies in how they engaged with advanced concepts and skills. Results from this study indicated 63% of students from high income families and 37% of students from low income families had a strong understanding of the number sequence and could read two digit numbers, identify the ordinal position of an object and solve simple word problems by the end of kindergarten. These differences were seen to reflect the mathematical knowledge each group brought to school. In this instance, by the end of their first year of schooling the gap still remained.

International studies suggest that allowing disadvantaged students and students with lower educated parents to attend school at an early age has a positive effect on their literacy and numeracy scores. Leuvan, Lindahl, Oosterbeek, and Webbink (2004) in a Dutch study involving data from over 16000 students reported that early learning makes subsequent learning easier. They found that increasing enrolment by one month increases the language and maths scores of students from a low socio-economic background or ethnic minorities by 0.06 standard deviation, while for those non-disadvantaged students early enrolment did not make a difference. Both Widgy and Caddy's families were from low socio-economic backgrounds.

Method

In its first year YAILN was a collaboration between researchers and 5 schools in North Queensland. The design of YAILN was a multi-tiered teaching experiment with the 7 preparatory teachers participating in professional dialogue/learning with the researchers on 4 occasions throughout the school year. On each occasion all the teachers were released from their classrooms to participate in a day of professional learning. Subsequent to these days the researchers visited all participating classrooms to continue professional dialogue and assist teachers to trial resources and activities. Discussions during these visits focussed on both mathematics and literacy learning in the early years. From a mathematical perspective the focus of the dialogue was three fold (a) the role of mathematics language in assisting young students engage in mathematical thinking, (b) representations and activities that support mathematical learning in the early years with an emphasis on the language associated with these activities, and (c) how this learning underpins higher levels of mathematical understanding.

In the classroom all activities were situated within the early childhood philosophy of activity based learning with students being encouraged to engage with these activities in a play-based and focussed learning and teaching context. During discussions with their students, teachers promoted the explicit mathematical language embedded within learning activities. They also encouraged students to orally communicate about aspects of each activity and assisted Indigenous students distinguish between AE and SAE in their communications. Initially the focus was not explicitly on number but how various representations worked in a numberless world and the associated mathematical language. For example, each classroom was given a large 5 by 5 grid and the activities involved students playing games while using their whole body. These activities gave students opportunities to talk about 'What is beside you? What is behind you? What comes next? How do you move to that position on the grid? Which row is it in? Which column is it in?' Students were also encouraged to 'act out' positional worlds in their home and school environment, recording these actions digitally, and with their parents writing sentences about their actions. In the later part of the year students then 'mapped' this language onto contexts involving numbers, for example, "What number is beside 9? comes after 9? What number is next? What numbers are between 3 and 8? How do you move from 9 to 11?"

Data Gathering Techniques and Procedures

All schools were a two hour plane flight from the researchers' home town making the sites difficult to visit. Queensland is one of the largest states in Australia. Thus the data tended to be gathered in one week blocks, with the researchers visiting the sites five times during the year. Data gathering comprised four components, namely, pre and post tests, student portfolios, classroom observations, and teacher interviews. In total 120 preparatory students participated in YAILN. All pre and post tests were conducted in a one on one assessment interview. Due to the intensity of the data collection with each assessment interview taking up to one hour, 30 minutes of numeracy and 30 minutes for literacy, only 48 students participated in both the pre and post numeracy tests. This purposely selected sample consisted of all Indigenous students, and a selection of Australian students and students from other cultures representative of a range of abilities. The pre assessment interview (pre test) occurred two months after the school year had commenced. The pre and post tests and teacher interviews occurred in March and November. Insights into the first research question are provided by the results of the tests administered in the pre and post assessment interview.

In order to answer our second research question, at the beginning of the second year when Widgy and Caddy had officially enrolled in preparatory year, a short interview was conducted with Jo, Widgy, Sussi, and Fran. Sussi and Fran were young Indigenous girls who had not attended a pre preparatory year of school. Jo had attended pre preparatory at School D but enrolled midyear and hence did not complete the pre test for numeracy. Unfortunately Caddy was absent the week that the interview occurred. The aim of this interview was to gauge how Widgy's and Jo's understanding of mathematics compared with two students who had not attended a pre preparatory year of school. This interview focussed on their understanding of the number 5.

Pre and Post Tests Results

The pre and post assessment interview consisted of a number test, a patterning test and an oral language test. The number test (School Entry Number Assessment (SENA) consisted of an interview comprising three main sections; number recognition, counting, and early addition and subtraction. This instrument was developed by the researchers and was based on the Mathematics component of School Entry Assessment (SEA), a tool designed by the New Zealand Government. The Patterning test consisted of 11 questions. Students were asked to copy, continue and complete repeating and to identify the repeating part in each. This instrument was also developed by the researchers. BOEHM, the third test, is a commercially produced standardised oral language test. Figure 1 presents samples of typical question from SENA and the Patterning test.

Figure 1. Sample questions from SENA and the Patterning test.

The results of a pre and post test, an interview conducted with 48 students selected from 5 prep school settings (average age 4 years and 11 months) indicated that although the Indigenous Australian students (n = 14) scored significantly lower on the pre test, after one year of school there was no significant difference in their scores as compared with the whole cohort (Warren, Young and DeVries, in press). For both the Patterning test and Oral Language tests, while there was no significant difference for the pre test results and the post test results for Indigenous Australian students and non Indigenous students, both groups exhibited significant improvement in both areas by the completion of the first year of the project.

The Effect Participating in Pre Prep for Widgy and Caddy

The total possible scores for the three tests were: SENA (28), Patterning (11), and Oral language (50). Table 1 presents the pre and post means and standard deviations for the whole sample, 48 students. Fourteen students were Indigenous and 34 students were from non Indigenous backgrounds.

Table 1

Mean Scores and Standard Deviations for All Students (n =48)

Test	Pre test scores		Post test scores	
	Mean	SD	Mean	SD
SENA (28)	14.40	5.34	21.25	3.68
Patterning (11)	3.71	3.06	8.42	1.86
Boehm (50)	27.80	7.74	37.88	7.59

A Wilcoxon Signed Rank Test was performed to ascertain if there were any significant differences between the students' pre and post test scores for the three tests. The Wilcoxon Signed Rank Test revealed a significant difference between the students' pre and post test scores for SENA ($Z = 5.82$, $p < 0.001$), Patterning ($Z = 5.92$, $p < 0.001$), and Boehm ($Z = 5.91$, $p < 0.001$). School D, the school that both Widgy and Caddy attend, is one of the participating schools in YAILN. Pre and post test scores for the three tests were obtained for nine of students from the "preparatory" class at School D. Table 2 presents the mean scores and standard deviations for each test.

Table 2

Mean Scores and Standard Deviations for School D (n =9)

Test	Pre test scores		Post test scores	
	Mean	SD	Mean	SD
SENA (28)	8.67	5.75	19.89	4.37
Patterning (11)	1.67	2.65	7.78	2.05
Boehm (50)	22.0	4.85	32.78	9.32

The Wilcoxon Signed Rank Test revealed a significant difference between the students' pre and post test scores for SENA ($Z = 2.67$, $p < 0.001$), Patterning ($Z = 2.68$, $p < 0.001$), and Boehm ($Z = 2.55$, $p < 0.001$). Table 3 presents the mean scores and standard deviations for Widgy and Caddy for the three tests.

Table 3

Pre and Post Test scores for Widgy and Caddy

Test	Widgy		Caddy	
	Pre test	Post test	Pre test	Post test
SENA (28)	2	15	10	17
Patterning (11)	0	5	5	9
Boehm (50)	21	23	21	34

Widgy and Caddy's pre and post scores indicated a marked improvement in their understanding of number, patterning and oral language after their participation in a pre prep program.

Widgy and Caddy's scores as compared with their cohort and the whole sample indicated that this improvement was similar to the trends exhibited in their cohort and the whole sample. While Widgy and Caddy's post scores for SENA were below the average scores of their cohort and the whole sample, both post scores were within 1 standard deviation from the mean post scores. Widgy was also below the mean score for the cohort and the whole sample for patterning and Boehm, but she still exhibited significant improvement in both scores. Caddy's post patterning score was above the mean patterning score for his cohort and the mean score for the whole Group. His post Boehm score was above the mean score for his cohort and just below the mean score for the whole group. These results would suggest that attending pre preparatory year of schooling did make a significant impact on both of these students' understanding of number concepts, patterning and oral language.

It should also be noted that Widgy and Caddy's post test scores were also above the mean scores for the whole sample pre test scores. This suggests that they now have a very strong foundation on which to build their mathematical understanding as they formally participate in the prep year of schooling. The next section presents the data relating to the second research question, how do these understandings compare to Indigenous students who did not attend pre prep?

Comparing Students Who Attended Pre Prep With Students Who Did Not Attend

The interview consisted of five main components: One-to-one counting to 5: Conversation of 5, Subitising to 5, Counting on to and counting back from 5, and Creating stories about 5 (e.g., 2 and 3 make 5). The preparatory guidelines for Queensland schools (QSA, 2006) indicate that by the end of Prep students should know all about 5, hence the choice of the number five. The interviews were extremely short of approximately 3 minutes' duration. The interview was conducted with four students; Jo and Widgy (both had attended pre prep) and Sussi and Fran (both had not attended pre prep). All four students are presently are in the prep year at School D. All interviews were video-taped. Table 4 presents a summary of the results for the four students. Due to space restrictions these results are presented as dot points.

Table 4

Results Comparing Pre Prep Indigenous Students with Indigenous Students Who Have Not Attended Pre Prep

Students	Understanding of 5
Attended Pre Preparatory year of schooling	
Jo	Recognises all numbers to 5 without counting
	When 2, 3, 4 balls are hidden, recognises how many are left and how many have been taken away without counting
	Can create all the stories about 5 (e.g., 0 and 5, 1 and 4, 2 and 3).
Widgy	Conserve all numbers to 5
	Correctly count different arrangements of numbers to 5
	Subitise to 5
	Can create some of the stories about 5
Did not attend a Pre Preparatory year of schooling	
Sussi	Cannot correctly count different arrangement of numbers to 5
	Can subitise 1 and 2
	Can create 2 and 3 makes 5
Fran	Cannot consistently count to 5.

There was a clear distinction between the understandings that the two groups held about the number 5. Jo was successful on all aspects of the interview. She could answer all questions about 5 and throughout the interview did not use counting to assist her in her responses. Widgy initially had to count the number of objects that the interviewer presented but as the interview progressed switched into discussions about 5 which did not require her to count the objects. Sussi could not consistently count to 5. She had the numbers to 3 under control but experienced difficulties once the interview went beyond 3. Fran could not consistently count to 5. Both Jo and Widgy clearly understood the questions asked in the interview, especially the language of mathematics associated with the questions being asked. The interview was conducted by the students' preparatory teacher, who was also Widgy and Jo's pre prep teacher.

Discussion and Conclusions

Given that this paper shares the results of two Indigenous students who attended a pre preparatory year it is difficult to draw conclusions for the whole Indigenous community. For these two students the results clearly demonstrate that their attendance at school prior to the preparatory year assisted them in obtaining a better understanding of important mathematical concepts. The results also suggest that the understandings that they held at the beginning of their preparatory years is equivalent to the understandings that many students from non Indigenous students hold as they begin school.

The results begin to confirm the theoretical underpinnings of the YAILN project. The role of oral language in developing mathematical understanding especially for students whose first language is not Australian Standard English cannot be underestimated. As indicated by the results of the interview conducted at the beginning of the second year of the project, the students who had participated in pre prep not only possessed a better understanding of numbers to 5 but also the associated mathematical language used to access this understanding. Pappas, Ginsburg, and Jiang (2003) believe a focus on the language of mathematics fosters important language acquisition and assists students acquire meta-cognitive abilities. This research begins to confirm this finding.

For these Indigenous students it appeared that direct teaching together with play based opportunities were also important in learning mathematics at an early age. The results of the SENA component of the pre test for the number and the interview results for the two Indigenous students who had not attended pre prep, indicated that Indigenous students begin school with little knowledge about number. Most did not know the names of the numbers nor could they meaningfully count to 5. We are suggesting that this is not the type of knowledge that emerges solely from play based situations. Adult guidance is needed (Greenes, 1999) and this is especially important for Indigenous students (Tayler et al., 2006).

Both Indigenous students reported in this paper are from a low socio-economic background. Allowing these students to attend school early certainly had a positive effect on their numeracy scores (Leuvan et al., 2004). Their pre test results suggest that they brought to school a paucity of mathematical knowledge, especially knowledge related to understanding white mathematics. Aubrey et al. (2006) claim that, students with little mathematical knowledge at the beginning of formal schooling remain low achievers throughout their primary experience. The results of this research suggest that attendance in a pre prep year of school may be an effective way to address this gap. Widgy and Caddy are now on an equal footing with other students as they begin their prep year. Both students remain part of our longitudinal study. Denton and West (2002) hypothesise that early learning makes subsequent learning easier, is yet to be tested. Our initial conversations with their prep teacher and the distinctions between these students' understanding of 5 as compared to students who had not attended a pre prep year suggests that this is indeed the case.

The question remains, what mix of pre prep students to prep students is most beneficial for learning? In alignment with our theoretical underpinnings we are suggesting that the ratio of pre prep to prep students should be low. Children can only discover so much through play. As the prep students' learning occurs they are in a position to assist pre prep students to higher levels of understanding, assisting pre prep students reach their learning potential (Balfanz, Ginsburg, & Greenes, 2003). We propose that learning from older peers with explicit teacher-directed learning are most effective for pre prep Indigenous students in developing early numeracy understandings.

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Building Informal Inference in Grade 7

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This study reports on the second phase of a design experiment involving classroom implementation of a sequence of four lessons introducing informal inference supported by *TinkerPlots* software to a grade 7 class. A Beginning Inference Framework was used as an implicit foundation for the teachers and as an explicit rubric for assessing students' observed outcomes. Outcomes were judged in relation to saved *TinkerPlots* files annotated with student-completed text boxes and to individual interviews with 12 of the students.

In 2006, as part of a larger professional learning research project, Jenny (pseudonym), a grade 7 teacher in a rural district school (K-10), undertook a case study related to introducing her class to *TinkerPlots* graphing software for middle schools (Konold & Miller, 2005). The case study evolved into a design experiment adapting lessons to cover elements of a Beginning Inference Framework, derived from a model suggested by Pfannkuch (2006). Data collected in the form of *TinkerPlots* files from four sessions were analysed in relation to the Beginning Inference Framework to document students' observed progress in taking up the elements of informal inference (Watson, 2007). The key aspects of the initial intervention included the evaluation of the extent to which the elements of the framework were observed in student output.

The 2006 case study and the subsequent 2007 case study described in this report arose from the desire of the statistics education research community to provide a meaningful bridge to formal inference, which many students will meet at the senior secondary or tertiary level. As well there is the desire to provide students who do not go on to formal statistics with intuitions about the inferential process without the theoretical assumptions and more complex mathematics required in formal statistics courses. The school curriculum provides direction on some of the ingredients required, such as data representation in graphs and data reduction with averages, but often does not signal the purpose of decision making with uncertainty based on samples representing populations. The National Council of Teachers of Mathematics (2000) includes "develop and evaluate inferences and predictions" in its *Standards* at all levels but there is concern on the part of statistics educators about how this is implemented, especially in acknowledging the uncertainty in the evaluation process.

As part of her wider work with senior secondary teachers in New Zealand, Pfannkuch (2006) set up a framework involving eight elements for developing informal inference based on box plots, which were a significant representational form in the New Zealand curriculum. Two other inputs influenced the adaptation of Pfannkuch's model for the study described here. First was the work of Bakker, Biehler, and Konold (2005), which concluded that box plots placed demands on students in terms of proportional reasoning that were beyond the understanding of most middle school students. This was especially true since most representations of box plots appeared without the inclusion of the data that they were summarising. Second was the development of the *TinkerPlots* software and its provision of a tool called the hat plot, which is a simplified version of a box plot. The hat plot (the default form) appears "above" the data (if plotted horizontally) with its crown situated over the middle 50% of the data and its brims over the lowest and highest 25% of the data. The median does not appear in the hat and hence the data are likely to be considered in "thirds," these being the middle cluster and two extremes. The availability of *TinkerPlots* as part of the project hence led to the adaptation of the Pfannkuch framework to the one in Table 1.

Table 1

Beginning Inference Framework (adapted from Pfannkuch, 2006)

Element	Description
Hypothesis Generation	Reasons about trends (e.g., differences)
Summary	Summarizes the data using the graphs and averages produced in *TinkerPlots*
Shift	Compares one hat plot with the other/s referring to change (shift)
Signal/Centre	Refers to (and compares) information from the middle 50% of the data
Spread	Refers to (and compares) spread/densities locally and globally
Sampling	Considers sampling issues such as size and relation to population
Explanatory/Context	Understands context, whether findings make sense, and alternative explanations
Individual Case/s	Considers possible outliers and other interesting individual cases

The results of the 2006 case study (Watson, 2007) were encouraging in that at each of the four data collections, more of the elements of the framework were employed by students, and after a 3½-month break students were able to engage with the elements presented in a structured format including graphs prepared in *TinkerPlots*. Concerning to the research team, however, was the difficulty students had in linking the ideas of sample and population. It was felt that this difficulty may have been related to discussions held with students when their data were collected and questions asked about them and the middle school students in their school. It was not until later that the larger population of "all" middle school students in the state or nation was introduced. The students appeared to have difficulty appreciating why these larger questions were of interest or important. This was one of the main features of instruction that was intended to be amended in the current case study. Within the context described, the research questions for this study are hence the following: What are the observed learning outcomes for grade 7 students in relation to beginning inference in a learning environment supported by a Beginning Inference Framework, a software package for data handling, and revised implementation strategies? How do the learning outcomes suggest further changes to the framework, the implementation, or the interaction with the software?

Methodology

This study is seen as the second phase of a design experiment (e.g., Cobb, Confrey, deSessa, Lehrer, & Schauble, 2003; The Design-Based Research Collective, 2003). The characteristics include the evolving theoretical framework for beginning inference, the interactive nature of the intervention (Jenny, a teacher-researcher (T-R), and the first author), the variety of data sources employed, and the adaptation of the intervention potentially to make suggestions for future research.

Participants. The case study was based in a grade 7 class (12-13 years old) of 15 students (4 other students opted not to be involved with the study and were transferred to other classes for the time of the lessons described here). Jenny, the T-R, and the first author had been involved in the earlier case study (Watson, 2007), which had included professional learning for Jenny and a close classroom collaboration of Jenny and the T-R.

Procedure and data collection. After initial planning, the students had been given time to explore *TinkerPlots*. *Lesson 1* introduced an investigation evolving around the hypothesis of an 81-year-old man that in the population at large males have faster reaction times than females. Students collected data on their right and left hand reaction times as a sample using the Australian Bureau of Statistics *CensusAtSchool* web site. Students entered data in *TinkerPlots* and created graphs to explore the hypothesis. Comments were entered in text boxes. In *Lesson 2*, the T-R led a discussion with Jenny at the computer using a *TinkerPlots* file on homework data. This covered the various tools available in *TinkerPlots* and the students then used these tools to explore their class's data set from the previous session. In *Lesson 3*, students were introduced to random samples of 20 or 200 grade 7 students reaction times collected from the *CensusAtSchool* web site. In *Lesson 4* students had access to a random sample of size 200 grade 5 and 12 students from the *CensusAtSchool* web site. All lessons lasted between 1.5 and 2 hours and videotapes were made of Lessons 2, 3, and 4. These provided audio but not always video records of events. *TinkerPlots* outputs were collected from all students present at each lesson.

The subsequent student interviews several weeks later introduced the students to new data sets already entered into *TinkerPlots* data cards. Students were asked to answer and discuss questions with the interviewer (one of the authors) rather than to write responses in text boxes. Three protocols were used in the interviews. The Comparing Groups protocol (Watson & Moritz, 1999) asked students to compare four pairs of classes on the basis of their spelling scores and decide which class had done better. The first three pairs of classes were of small equal sizes, whereas the fourth pair was not of equal size. The second data set consisted of 16 data cards with the names, ages, weights, eye colours, favourite activities, and numbers of fast food meals eaten per week of 16 students aged between 8 and 18 (Chick & Watson, 2001). Students were asked to explore the cards, suggest interesting hypotheses, and provide plots with evidence to support or refute the hypotheses. The third protocol was based on a *TinkerPlots* data set containing the heights of 136 children at age 2, 9, and 18 (Watson, 2007). Students were asked to form hypotheses about the difference in heights for boys and girls at the three times based on stacked dot plots provided for each year, separated by gender and including hat plots (the scales were different on each of the three plots).

Analysis. Following the method of analysis of the previous case study, the Beginning Inference Framework was the basis for analysis of the three taped class sessions, the student *TinkerPlots* output from each of the four sessions, and the individual interviews. For each lesson and the interviews, matrices were created to document students' work (Framework elements x students). Judgments of outcomes were based on the number of elements employed and the degree to which they were related to each other. The relationships were categorised based on the SOLO Taxonomy (Biggs & Collis, 1982; Pegg, 2002) as employed by Watson (2007). Prestructural responses, reflecting no elements of the Framework, were not observed in this study due to the scaffolding of the sessions and the collection of *TinkerPlots* output from students. Outcomes were judged to be Unistructural (U) if isolated comments or plots were saved, based solely on classroom discussion. Responses that added extra elements of the Framework in a serial fashion to the comments and annotations to plots were judged to be Multistructural (M). Relational (R) responses were those that combined several elements in the text comments to reach integrated conclusions in the *TinkerPlots* output. For the interviews judgments were made based on the overall use made of the elements across the three protocols, suggesting the degree to which the students had internalised the experiences that had taken place across the four teaching sessions, with similar criteria to those above being employed. All quotes have been corrected to fix spelling and grammatical errors.

Results

The results are presented in three parts. First the elements observed in the lessons are documented as evidence for the experiences of the students in the four lessons. Next, the observed outcomes from the students' *TinkerPlots* files are summarised for the four lessons. Finally, the observed outcomes for the individual interviews are presented and a summary given for the students over the five data collections.

Lesson summaries. Table 2 contains annotations in relation to each of the eight elements of the Beginning Inference Framework for the three videotaped sessions. Evidence for Lesson 1 is noted with Lesson 2 and was gleaned from discussions with the T-R and student output. The only elements not addressed specifically in a session were Hypothesis Generation and Explanatory/Context in Lesson 2, which were strongly addressed in Lesson 1, and Explanatory/Context in Lesson 4, as it had not changed substantially over the four sessions.

Table 2

Beginning Inference Framework Elements Addressed Across Lessons

Element	Lessons 1 and 2 – Introduction, Class Data	Lesson 3 – ABS grade 7 data (20, 200)	Lesson 4 – ABS grade 5 and 12 data (200)
Hypothesis Generation	Jim, 81 years old, hypothesis boys have faster reaction times than girls [focus Lesson 1]	Review and reminder of alternatives; general discussion of hypotheses	Suggestions: grade 5 faster than grade 12; some boys faster than some girls
Summary	Ranges, middles, n, percent	S: "dots all over the place"; boys faster left but right about the same	Clusters, stacking, concentrations, reference lines
Shift	More people on left; ranges the same; ranges different	Not much shift in the data	Endpoints of crowns of hats notes; "girls start later"
Signal/Centre	Mean, median, hat: 50%, 25%, 25%	Reminder hat is middle 50%; mean, median	Hat middle 50%; median; "girls end of 50% for boys"
Spread	Range, range of middle 50%; fixing scale to compare spread	Scale, sensible ranges; clumped; spread out; range of middle	Range/scale; girls wider crown, more spread; top & bottom 25%, clusters, pencil
Sampling	What about grade 8, equal numbers, all girls/boys in class?	Samples of 20, 200; equal number of boys/girls?	Missing data; small/large data sets and outliers
Explanatory/ Context	Reasons for possible difference [focus Lesson 1]	Continued as earlier	[Little extra] mainly hypotheses and evidence
Individual Case/s	Outlier [one class member], one left-handed student	Review of outliers; "flukes", 0 as outlier	Outliers, flukes, specific values in data set

Student outcomes – Lesson 1. Of the 15 students in Lesson 1, 10 produced *TinkerPlots* files with at least the scaled data set in stacked format. One produced a plot separated by gender showing 10 males and 9 females. Three produced 2-way plots with four bins showing gender and right hand reaction time in two groups, 0.24 – 0.35 s and 0.36 – 0.48 s. One produced a plot of gender by eight subgroups of right hand reaction time. S4's response was considered typical of Multistructural responses and included five elements in comments with a stacked dot plot: Hypothesis Generation (who is faster), Summary (right hand plot, boys faster by one microsecond), Sampling ("we only did grade 7"), Explanatory/Context (most right-handed, one left-handed), and Individual Cases (two fastest, third fastest named). Unistructural responses were similar to S2, who stated the hypothesis, noted that data had been collected and graphs made, but no conclusion was drawn. Of the responses, 8 were judged to be Unistructural, and 7, Multistructural.

Student outcomes – Lesson 2. Only 10 students were present at the second session. Students analysed their class data with more of the tools available in *TinkerPlots* and saved as many as four new plots. Only newly added text was considered in deciding the level of response. Only one student did not produce a plot separating boys and girls; eight looked at both left and right hand times by gender. Some accounts were mainly descriptive of the procedures followed in creating the plots using the tools. Eight removed the outlier from the left hand times; one kept it "because I thought that it was important to leave her in so it's exact" [S15]. Most of the students used reference lines to detail values and six used hat plots in at least one of their graphs. Comments summarising the graphs displayed a range of uncertainty of language: for example, S13 declared "the boys are faster than the girls," whereas S4 concluded, using reference lines and medians, "some boys are faster than girls sometimes." Three students included an aspect of sampling of "our class" and four described the spread of their plots, for example with ranges of the crowns of the hats [S12]. S1 was however confused that a wider crown meant more data rather than greater spread for a fixed percent of the data. In providing responses such as these, eight of the students were judged to provide Multistructural responses. The other two responses were considered Unistructural because in one case nothing of substance was added to the previous week's work and in the other the student recorded contradictions that precluded understanding the conclusions drawn.

Student outcomes – Lesson 3. In this session 13 students were present. With the choice of considering a randomly selected data set from the ABS *CensusAtSchool* site with either 20 or 200 grade 7 students, 11 chose the set of 200, although later many suggested that is was difficult for them to work with such a large data set. Three students did not delete outliers, despite much class discussion; two deleted high values but not zeroes. Three students did not state new hypotheses or questions to explore. All students considered gender by reaction time, 11 for both hands and two for one hand only. Eleven students produced hat plots but did not mention change in their location for the two sexes. Seven students discussed some other aspect of the middle 50% of the data, either using reference lines or explicitly noting the "middle half" or the "50% majority," whereas four considered spread by giving values for the range of the crowns. Overall the graphs were summarised to the extent of saying "they showed" support for the hypothesis. Generally the students struggled to document evidence in their text boxes to support their conclusions. Although most of the students learned to handle the larger data set with more outliers, in many cases the subtleties of the positions of the hats made decisions difficult. Nine of the students were judged to present Unistructural responses, five of whom had been absent the previous week; of the other four comments were contradictory or did not describe evidence in support of conclusions. Figure 1 shows the two plots produced by S7 who gave values for the ranges of the crowns of the hats and concluded "in the right hand times the girls and boys are about the same in times … but in the left hand times the boys were faster." This was typical of the four Multistructural responses.

Student outcomes – Lesson 4. Of the 13 students present, four had missed Lesson 2 and two had missed Lesson 3. In this session students were aided by a review of the previous session's analysis and most appeared to contribute to the discussion. When given a new randomly selected data set of 200 students in grades 5 and 12, they immediately considered gender with reaction time, without consideration of grade. When reminded of this, students then looked at grade without consideration of gender. They did not have the experience with *TinkerPlots* or time to consider how both attributes could be considered together. The students, however, consolidated the procedures using the *TinkerPlot* tools. All students except one placed hat plots on some or all of their graphs and the one student used reference lines instead.

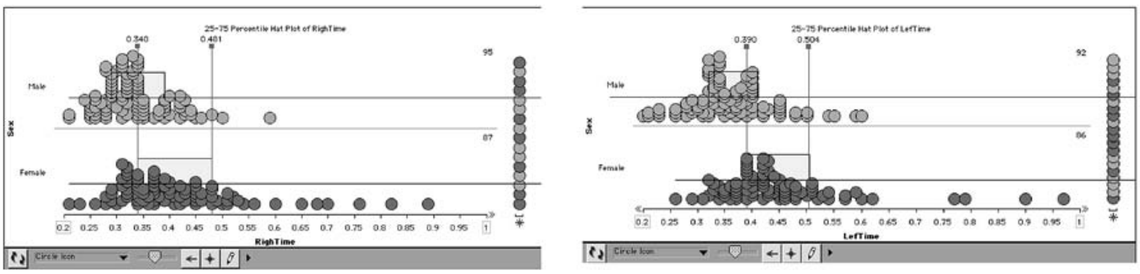

Figure 1. Random sample of grade 7 data for left and right hand reaction time by gender.

The assessment of responses was based on how far students could progress given the scaffolding provided. Three students wrote little text to explain their graphs and did not summarise the information to support a hypothesis. One student based the discussion only on individual values at the extremes of the plots. The responses of these four students were considered to be Unistructural in the context; three of these students had missed one earlier session. Eight students produced responses judged to be Multistructural in that they made reasonable hypotheses about gender and grade separately with respect to reaction time and supported these with reference to their plots. None of the students addressed aspects of sampling, context, or specific shifting of hats, except for S8 who noted for one plot, "boys also have the middle 50% nearer the beginning but the girls are further back." S10 was considered to produce a Relational response in that he made comments reflecting all eight elements of the Beginning Inference Framework, for example, for Shift, "the middle 50% range in the males starts at a lower time than females in the right hand, the end of the 50% range in males and females [is] at the same place"; for Spread, "the slope of the males in the right had has a more jagged slope and the females have a smooth slope"; for Sampling, "year 5 had more people … also more outliers or incomplete data in grade 5." S10 also noted the mixed gender within grades.

Student interviews. The specific elements of the Beginning Inference Framework were not brought to the attention of the 12 students who were interviewed, except that they were asked to generate hypotheses in the second and third protocols. Of interest was the degree to which the elements were integrated into the comments made by the students to the interviewers. Four students showed little awareness of the task of setting hypotheses; three of these had missed one session. The other eight were successful in one or more contexts. Two, including one of the previous four struggled with summarising plots to reach conclusions. Seven students discussed shift, without using the term, for hat plots. There was some confusion on the signal in the central 50% of the data but six could make reasonable comments. All students mentioned spread and looked at individuals (or individual bins with one or two entries) in one or more of the protocols. Seven responses considered Explanatory/Context, for example giving advice about eating fast foods or discussing cases of growth for the final protocol. Only two students discussed the need for larger samples throughout the interview. In considering the overall adoption of the elements of the Beginning Inference Framework, it was judged that four students' responses were Unistructural in focusing overwhelmingly on individual aspects of the data sets rather than aggregate properties represented or representable in plots. Six responses were considered Multistructural in displaying many of the elements at various points of the interview, whereas two were considered Relational in integrating the elements to create meaningful arguments. Table 3 contains the number of elements of the Framework discussed and how they were structured according to SOLO levels for each student for each data collection. The average number of elements included in the *TinkerPlots* output increased from 3.2 in Lesson 1, to 3.9 in Lesson 2, and to 4.8 in Lesson 3, before dropping slightly to 4.6 in Lesson 4. The average number of elements observed in the transcripts of student discussion rose to 5.7 in the interviews.

Table 3

Number of Elements of Framework and SOLO level for Students at Five Times

	Lesson/Interview									
Student	L1 No. Elem.	L1 SOLO	L2 No. Elem.	L2 SOLO	L3 No. Elem.	L3 SOLO	L4 No. Elem.	L4 SOLO	Int No. Elem.	Int SOLO
S	2	U	5	M	–	–	5	M	6	M
S	2	U	–	–	4	U	4*	M	3+	U
S	3	M	–	–	4	U	–	–	3+	U
S	4	M	3	M	2	U	4	M	6+	M
S	5	U	–	–	3	U	4	U	–	–
S	6	U	4	M	–	–	4	U	6+	U
S	7	U	6	M	7	M	5	M	8	M
S	8	M	3	U	6	M	5	M	5+	M
S	9	M	4	U	5	U	4*	M	7	M
S	1	0M	3	M	7	M	8	R	7	R
S 1	2	U	–	–	5	U	4	M	–	–
S	2	2U	3	M	4	U	3	U	5	U
S	1	3U	4	M	6	U	–	–	–	–
S	5	4M	–	–	4	U	5	U	6	M
S	13	5 M	4	M	5	M	5	M	6+	R

*Plus one element that was used inappropriately + Weak use of another element

Discussion

In initiating a second series of lessons in a design experiment framework, one aim was to observe the effect of transposing the introduction of populations and samples from after the collection of student class data to the very beginning of the lesson sequence. This was accomplished by introducing a population-based hypothesis with class discussion of how the sample data from the class might assist in supporting the hypothesis or refuting it. Comments were made about hypotheses not being "right or wrong" but questions to be investigated by collecting evidence. Although appearing to appreciate and participate in the class discussion (for example, observed in audio extracts on the lesson videos), little explicit evidence of this was volunteered in the text boxes or in the interviews. This is probably related to the specific interest in the *TinkerPlots* features and the relative ease with which they could be described. The authors did not ask specific questions about population and sampling in the interviews because of the desire to find out what students would contribute on their own initiative. The second protocol with 16 data cards provided an opportunity for students to make comments on the need for a larger sample or what might happen for the population at large.

It is likely that the drop in SOLO levels, despite the increase in average number of Framework elements observed in Lesson 3, is related to the introduction of a new, and for most students much larger, data set, as well as to the fact that some students had been absent for Lesson 2. The slight drop in number of Framework elements observed in Lesson 4 may be related to the extra cognitive load of considering both gender and grade level. The improved SOLO levels may relate to some students beginning to put together the ideas of combining evidence to reach a conclusion. Increasing numbers of elements employed in the interview may have resulted from the many opportunities provided to students but the continued presence of Unistructural responses suggests that some students still struggled with more than considering single aspects that resulted from employing *TinkerPlots* tools.

Overall the authors conclude that in terms of grade 7 students assimilating concepts of populations and sampling along with the other elements of the Beginning Inference Framework, little is gained by introducing the "big picture" of populations first rather than later in an investigation sequence. The lack of spontaneous intuitive consideration of populations may be associated with students' continued focus at this age on themselves and their immediate environment or it may take much longer with more experiences than were possible in this case study to build appropriate intuitions about sampling.

Although the Beginning Inference Framework was adopted in a context of replacing box plots with hat plots, not all students continued to use hat plots in Lesson 4 or the interview, some preferring data in bins and others using arbitrary reference lines. Whether this was again a function of lack of experience in seeing the usefulness of hats in various contexts is unknown. In Pfannkuch's (2006) study with older students box plots were the only representation provided for interpretations (without actual data values), whereas in this case study hats were one of a number of tools available to summarise a plot of data values. For beginners it seems reasonable to provide a range of tools, such as available with *TinkerPlots*, with the intention of allowing students to build intuitions that will assist in the transition into more formal inference methods and the use of box plots in later years. Further interventions over a number of years will be needed to test these ideas.

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Proportional Reasoning: Student Knowledge and Teachers' Pedagogical Content Knowledge

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This report considers the responses of 1205 students to two chance and data problems involving proportional reasoning and the interventions suggested by 44 teachers for four typical incomplete or inappropriate student responses. Students' responses reflect the relative difficulty of the two items, whereas the teachers' suggested interventions display a wide range of pedagogical content knowledge, including aspects of content knowledge and knowledge of students as learners. Rubrics are provided for both students' and teachers' responses and analysis supports the view that teachers find it difficult to assist students without directly telling them the answer. Suggestions are made for assisting teachers to improve feedback for students.

Following the work of Shulman in the 1980s (e.g., 1987) outlining seven types of teacher knowledge for successful teaching, researchers have focussed on various of the seven and combinations of them (e.g., Kanes & Nisbet, 1996; Mayer & Marland, 1997). Watson (2001) developed a profiling instrument for teachers of chance and data attempting to include all seven of Shulman's types of knowledge, whereas Hill, Rowan, and Ball (2005) appeared to incorporate content knowledge, pedagogical content knowledge, and knowledge of students as learners, into a rich description of "teachers' knowledge for teaching mathematics." This construct was assessed with a series of complex multiple choice items. Chick (2007) included these three types of knowledge, as well as curriculum knowledge, in her expanded framework for pedagogical content knowledge. It appears there is general recognition in the research community of the complexity of the interaction of the types of knowledge leading to teachers' successful implementation of learning programs in mathematics for their students.

Watson, Beswick, and Brown (2006) focussed on teacher content knowledge and knowledge of students as learners in asking teachers to suggest appropriate and inappropriate solutions that would be given by their students to a fraction problem. Further, pedagogical content knowledge was explored in asking teachers how they would use this problem in the classroom and intervene to address the inappropriate responses. Three tasks (8 individual items) of this type were included in the profiling instrument used in the current study but as well four new items were developed based on two items from student surveys. For these items actual student responses were presented to the teachers, rather than asking the teachers to suggest inappropriate student responses. Teachers were then asked how they would respond to the students' answers. A subscale of the teacher profile consisting of the 12 items relating to teachers' responses in the context of students' answers was considered by Watson, Callingham, and Donne (in press) as a measure of the wider interpretation of pedagogical content knowledge (PCK), including content knowledge and knowledge of students as learners. Watson et al. used Rasch (1980) measurement approaches and identified a single dimension of PCK, with few teachers reaching the higher ability levels of the scale.

Proportional reasoning is well-known for causing difficulty for many middle school students (Lamon, 2007; Mitchelmore, White, & McMaster, 2007). It is a necessary prerequisite for performing at the highest level of statistical literacy understanding (Watson & Callingham, 2003) and hence a topic of interest for exploring teachers' PCK in this context. Having data on both student performance and teachers' suggested interventions should provide starting points for professional learning within the larger project in which this study lies. The research questions for the study are hence the following.

What are the distributions of understanding of proportional reasoning shown by students in two chance and data contexts and the association between them?

What are the initial levels of PCK shown by middle school teachers in relation to remediating students' inappropriate responses to proportional reasoning questions in two chance and data contexts?

Methodology

Sample. The sample consisted of 1205 students in grades 5 to 10 across 19 schools in the states of South Australia, Tasmania, and Victoria, whose teachers were involved in the StatSmart project (Callingham & Watson, 2007). Table 1 shows the distribution of the students across grades. The students were from the classes of the 44 teachers who completed the profiling survey. The teachers represented government, Catholic, and independent schools; 23 were male and 21 were female. The highest grades taught by the teachers were primary by 8, middle by 5, junior secondary by 9, and senior secondary by 21 (one unknown), but all teachers taught a class of students in one of grades 5 to 10 as part of the StatSmart project. The years of teaching experience ranged from 1 year to more than 25 years. There were 12 teachers in the longest serving group, with 12 others having taught 16 to 25 years or 6 to 15 years, and 8 having taught 5 or fewer years.

Table 1

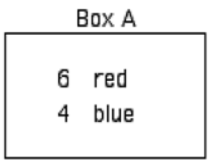
Number of Students by Grade

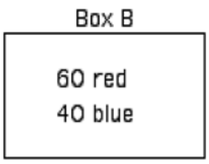
Grade	5	6	7	8	9	10	Total
Number of students	41	175	432	324	137	96	1205

Items. The two problems completed by the students are shown in Figure 1. The top problem, named BOX, was originally adapted from an item used by Konold and Garfield (1992) and analysed for Australian students by Watson, Collis, and Moritz (1997) and Watson and Moritz (1998). It was part of the survey used to define the hierarchy of statistical literacy by Watson and Callingham (2003). The lower item, named SMOKE, was an item employed by Batanero, Estepa, Godino, and Green (1996) and further analysed by Watson and Kelly (2006). Watson and Callingham (2005) used the item as part of a confirmatory study of the statistical literacy construct.

For each of these two problems answered by students, two student incomplete or inappropriate answers were presented to teachers. Items T1 and T2 in Figure 2 are associated with the BOX question and items T3 and T4 are related to SMOKE. The original student questions were in the teacher profile but are not repeated here. The student answers were chosen as typical responses from students in the previous studies.

Box A and Box B are filled with red and blue marbles as follows. Each box is shaken. You want to get a blue marble, but you are only allowed to pick out one marble without looking. Which box should you choose?





(A) Box A (with 6 red and 4 blue).

(B) Box B (with 60 red and 40 blue).

(=) It doesn't matter.

Please explain your answer.

The following information is from a survey about smoking and lung disease among 250 people.

	Lung disease	**No lung disease**	**Total**
Smoking	90	60	150
No smoking	60	40	100
Total	150	100	250

Using this information, do you think that for this sample of people lung disease depended on smoking? Explain your answer.

Figure 1. Two proportional reasoning problems used in student surveys.

	Consider each of the following answers and explanations to the problem and discuss how you would respond to the answers.	
BOX	T1 Student 1: (=) Because you could get red or blue.	T2 Student 2: (A) Because there are only 2 more reds in A and 20 more in B.
SMOKE	T3 Student 3: Yes, 90 who smoked got lung disease.	T4 Student 4: [No] 60 "no smoking lung disease" and 60 "smoking no lung disease" are the same.

Figure 2. Teacher items based on proportional reasoning problems (Figure 1).

Analysis. The rubrics for the student responses were those used in previous research. For the BOX item a four-step rubric as used by Watson and Callingham (2003) is given in Table 2, consolidated from the seven-step rubric used by Watson et al. (1997).

Table 2

Rubric for BOX Item for Students

Code	**0**	**1**	**2**	**3**
Description	No response/reason	Anything can happen	Additive reasoning	Multiplicative/ Proportional reasoning

For the SMOKE item a five-step rubric was devised from the one used by Watson and Callingham (2005), based on a developmental model of considering increasing numbers of elements in the two-way table and combining them in an appropriate fashion. The rubric is summarised in Table 3. The difficulties of the different item-steps were obtained from Rasch measurement so that the relative difficulty of the two items could be considered.

For items presented to teachers, the student answers (see Figure 2) were coded as: BOX items, Code 1 for T1 and Code 2 for T2; SMOKE items, Code 2 for T3 and Code 3 for T4. The rubric in Table 4 was used to code each of the teacher items. The raw scores for students and teachers on each rubric were used as a basis for the analysis. Student items were considered by grade. The association between the two items for both teachers and students was considered using two-way tables.

Table 3

Rubric for SMOKE Item for Students

Code	**0**	**1**	**2**	**3**	**4**
Description	Yes/No without justification; No reason/ response	Yes/No no use of data; knowledge of context	Yes/No single comment on method or single cell data	Yes/No explicit or implicit use of 2 or 3 cells' data	No uses all information with ratio/ percents

Table 4

Rubric for Teacher Responses to T1 to T4

Code	**0**	**1**	**2**	**3**
Description	No/irrelevant response	General strategy; no math. input	Some math. content but vague teaching strategy	Questioning of students; multiple aspects of problem; prop. reasoning strategies; cog. conflict

Results

Research Question 1 – Student Outcomes. The Rasch output showed that SMOKE was, in general, more difficult than BOX for students. Figure 3 shows the relative difficulty in logits of each item-step. The difficulty level of BOX3 was about the same as that of SMOKE2 and a little lower than SMOKE3. It appears that proportional reasoning was more difficult to achieve in the context of SMOKE than BOX. This finding seems consistent with what might be expected, based on the general complexity of presentation of the two items. The jump from BOX2 to BOX3 and for SMOKE3 to SMOKE4 suggests that, regardless of context, it is difficult for students to move to proportional reasoning, relying on relationships between the numbers presented.

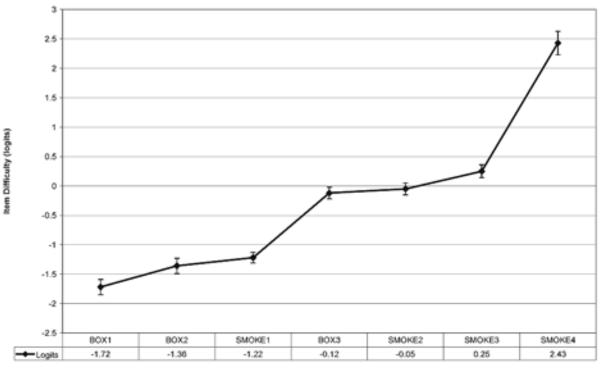


Figure 3. Relative difficulties of item-steps for BOX and SMOKE.

Table 5 contains a summary of the outcomes by grade for each code of the two items and Table 6 contains typical student responses for each code of each item from this data set. For the BOX item, except for grade 9, which was slightly lower and grade 6, which was slightly higher, the frequency of no response or no reason was relatively uniform across grades. Only grade 6 seemed susceptible to "anything can happen" reasoning. Improvement was seen across the pairs of grades 5/6, 7/8, and 9/10 for the appropriate reasoning. Not until grade 9/10 was there a moderate improvement in the appropriate reasoning for the SMOKE item.

Table 5

Percent at each Code by Grade for the Student Items

Code	Gr5 (n=41)	Gr6 (n=175)	Gr7 (n=432)	Gr8 (n=324)	Gr9 (n=137)	Gr10 (n=96)	Total (n=1205)
BOX0	12.1	17.7	9.7	12.3	5.8	10.4	11.3
BOX1	2.4	16.0	5.8	8.6	0.7	2.1	7.1
BOX2	58.5	40.0	39.4	37.3	27.0	25.0	37.0
BOX3	26.8	26.3	45.1	41.7	66.4	62.5	44.6
SMOKE0	29.3	28.0	26.4	18.8	24.1	20.8	24.0
SMOKE1	31.7	48.6	33.6	33.3	21.9	24.0	33.5
SMOKE2	7.3	6.3	10.4	11.4	16.1	9.4	10.5
SMOKE3	29.3	15.4	27.1	33.0	28.5	40.6	28.3
SMOKE4	2.4	1.7	2.5	3.4	9.5	5.2	3.7

Table 6

Examples of Student Responses at each Code Level for the Student Items

Code	Response
BOX0	"A"
BOX1	"=, because it's just a luck you get"
	"=, because it would be a 50-50 pick"
BOX2	"Box A because there is only 2 more red where as Box B has 20 more red"
	"=, they both have more red than blue"
BOX3	"=, because there is an equal chance that you will get a red or blue marble"
	"=, because 60% of the marbles in each box are red"
SMOKE0	"Well I don't really understand because you don't really know"
	"Yes because there are more people"
SMOKE1	"No, smoking is what gave them the disease in the first place"
	"No because it could have been a bushfire or second hand smoke"
SMOKE2	"Yes, because there is 90 people that got it that do smoke"
	"No, not really they need to do the survey on an even amount of people"
SMOKE3	"Yes, there are 60 people with lung disease not smoking and 60 people smoking and no lung disease"
	"Yes, because the people that do smoke had more people getting lung disease than people who don't smoke"
SMOKE4	"No they both had the ratio of 3:2 (smoking : non smoking)"
	"No, because 2/3 of non-smokers have lung disease as well smokers having 2/3"

Table 7 shows the association of the code levels for the two items, again illustrating their comparative difficulty. Achieving Code 3 on the BOX item was no guarantee of even moderate achievement on the SMOKE item, as 107/538 (19.9%) received a Code 0 and only 36/538 (6.7%) received a Code 4. Of those who received a Code 4 on the SMOKE item, 36/44 (81.8%) received a Code 3 on the BOX item.

Table 7

Association of Code Levels for BOX and SMOKE

		BOX				
	Code	**0**	**1**	**2**	**3**	**Total**
SMOKE	**0**	58	14	110	107	289
	1	44	52	159	149	404
	2	5	4	47	71	127
	3	26	15	125	175	341
	4	3	0	5	36	44
	Total	**136**	**85**	**446**	**538**	**1205**

Research Question 2 – Teacher Outcomes. Table 8 shows the percentage of teachers achieving each code for the four items in Figure 2. As can be seen the modal response for all four items was code 2, reflecting limited engagement with the mathematics and potential teaching strategies. The difficulty of the item for students hence did not appear to influence the code achieved by the teachers. Examples of teacher responses at each code level are given in Table 9.

Table 8

Percent of Teacher Responses at each Code Level for each Item

	Teacher survey questions			
	T1(BOX)	**T2(BOX)**	**T3(SMOKE)**	**T4(SMOKE)**
Code 0	20.4% (9)	11.4% (5)	9.1% (4)	20.4% (9)
Code 1	25.0% (11)	9.1% (4)	13.6% (6)	22.7% (10)
Code 2	43.2% (19)	61.3% (27)	56.8% (25)	47.7% (21)
Code 3	11.4% (5)	18.2% (8)	20.4% (9)	9.1% (4)

In considering the association of teachers' responses between items, the strong representation of Code 2 across all items meant that Code 2 was frequently common across pairs of items, ranging from 27.3% to 38.6% for the six pairs (e.g., T1 & T2, T1 & T3, etc.).

Discussion

Several aspects of the results suggest implications for teachers and teaching related to proportional reasoning. These include the relative difficulty of the items for students, the lack of discrimination of teachers over the four student responses considered, and suggestions for improving teacher interaction with student responses.

Although the literature agrees that proportional reasoning is difficult for middle school students, the outcomes of this study suggests that ratios that are multiples of ten are much easier to recognise than those with a non-integer multiple, that is, such as 1.5 in the SMOKE problem. As well the presence of row and column totals in the SMOKE table more than doubles the amount of information that needs to be taken in, in engaging with the question. The cognitive load certainly is greater. For most students there appear to be few similarities in the visual presentation of the problems.

The lack of variability of the level of teachers' responses across the student answers of different codes would appear to reflect a general lack of PCK at the point of matching content knowledge with knowledge of students as learners. Knowing what questions to ask of students or what cognitive conflict to generate, without directly telling them the answer, appears to be a difficulty for these teachers. Teachers are presenting the same types of generic responses regardless of the level of student response and not recognising an appropriate zone of proximal development in which to challenge the students. This may also be influenced by a lack of appreciation for the desired level of response expected in the profiling instrument. It is hoped that professional learning during the StatSmart project will increase teachers' familiarity with their students' understanding and ways of matching that to the appropriate mathematical content. Specific analysis of the levels of student response within a structural model should also assist in increasing the teachers' PCK.

Table 9

Examples of Teachers' Responses at each Code Level for each Item

Code	T1(BOX)	T2(BOX)	T3(SMOKE)	T4(SMOKE)
0		"Correct, but which colour would you more likely get?"	"unsure"	"unsure"
1	"I'd get them to test this not just think about it." "Not sure what question I would ask – need to look at ratios but not sure how to go about it?"	"What chance do you have? More chance of getting either?" "Can you tell me more? This student needs to explain more fully how he arrived at his answer. He may or may not be correct; only a fuller explanation could tell."	"Of what number in total? Ask questions which give meaning to the numbers."	"We would look more closely at the chart and be more accurate with analysing the data under each heading." "I would change the numbers so that doesn't add to the confusion. Then discuss the categories and what they mean."
2	"This student does not understand fractions or proportion Red 6/10 or 60/100 = This student needs more work with concrete aids." "In both cases I would talk about probability and the likelihood of choosing either or percentage and ratio of either blue or red in each box."	"Talk about chance of getting blue over red or vice versa. I'd get them to try both and test their theory." "But there are more blue in Box B?" "In both cases I would talk about probability and the likelihood of choosing either or percentage and ratio of either blue or red in each box."	"Need them to look at 90 out of 150 versus 60 out of 100 – probably look at percentages and ratios." "We would look at the total amount of people being surveyed in each section and try to convert the numbers so we could compare the answers more easily."	"Not a fair answer as totals are different e.g. smokers, 60 out of 150." "May look at ratios – would need to make non smokers = 150 to show clearly."
3	"Are there also more blues in B? If I have less marbles do I have a better chance always?"	"Question: Is it all luck? Would I be more likely to get a red? Why? In both boxes? How could I show my chance of getting a blue in numbers?"	"From which sample? What % of smokers got the disease? What % of no smoking got the disease? Same!! 90/150 = 3/5 = 60% (smokers) and 60/100 (ns)" "I would encourage student to read the whole table and look at all the data. Do high numbers mean a greater chance?"	"I would ask the student to look carefully at the size of the sample groups for smoking & non-smoking and compare the number of %'s of people with & without lung disease. Have they recognised that the sample groups are different sizes, what does this mean when looking at the results?"

The suggestions for professional learning include making teachers aware of the three stages of introducing mathematical concepts in context: terminology, terminology in context, and critical thinking (Watson, 2006). In this case, the terminology associated with proportional reasoning needs to be understood. Then it needs to be related to contexts such as chance (the BOX problem) and data (the SMOKE problem), with the differentiation pointed out. Finally critical thinking to answer questions that create conflict between mathematics and other contextual understanding (e.g., beliefs about smoking and lung disease) needs to be made explicit. In conjunction with this framework, the suggestions of Chick (2007) in terms of the affordances of using examples in the classroom, are also likely to be beneficial.

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Counting On 2007: A Program for Middle Years Students who have Experienced Difficulty with Mathematics

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The Counting On 2007 project was designed to support the professional development of teachers in identifying and addressing the learning needs of middle years students in mathematics. It was based on earlier models and included changes designed to simplify and encourage further and ongoing involvement of schools. One change was a simplified assessment process that provoked initial concerns that teachers would not develop an appreciation of their students' specific difficulties nor a deep understanding of the learning framework in number and thus the students' learning would suffer as a consequence. Implemented in 122 schools across NSW, the findings for the program indicated it was successful in assisting students who had struggled with mathematics with 66% of the students increasing their place value understanding by one or more levels and similarly with 65% for multiplication/division, while providing a vehicle for teacher professional learning.

Background

New South Wales is not the only state concerned with students in the middle years who are struggling with mathematics. Gervasoni, Hadden, and Turkenburg (2007) conducted a large study of number learning in 2006 of over 7000 Victorian children in Ballarat for the purpose of identifying issues that could inform the development of a professional learning plan. A notable number of students (31%) beginning Grade 6 were found not yet able to read, write, order, and interpret four-digit numbers nor use reasoning-based strategies for calculations in addition and subtraction, and multiplication and division.

Are teachers and poor teaching to blame? Vaiyatvutjamai and Clements (2004) analysed the errors made by 231 form three (year nine) Thai students in two Chiang Mai government secondary schools. Students completed tasks before and immediately after a series of 13 lessons. A number of misconceptions were revealed and although some were clarified as a result of the lessons, there were others that remained and seemed to be "fossilised". A "fossilised misconception" was used to denote the situation where a student maintains a faulty conception despite having been specifically taught the "official" defining characteristics of the relevant concept. Associated with this then is the absence of cognitive change over time or even resistance to change over time, so that cognitive inertia persists despite the individual having been taught the "proper" view of the concept. The Counting On 2007 project was designed to support the professional development of teachers in identifying and addressing the learning needs of middle years students who are struggling in mathematics, many of whom may possess "fossilised misconceptions".

The research base for the program was provided through the Counting On Numeracy Framework (Thomas, 1999) which made use of work completed by a number of researchers such as Cobb and Wheatley (1988) who conducted research into children's initial understandings of ten; Beishuizen (1993) who researched the mental strategies, materials, and models used by teachers and students in addition and subtraction computations of numbers up to 100 in Dutch second grade classes; Jones, Thornton, Putt, Hill, Mogill, Rich, and van Zoest (1996) who studied multidigit number sense and developed a framework for instruction and assessment; and the Count Me In Too Learning Framework in Number (Wright, 1998; Wright, Martland, & Stafford, 2000).

The Counting On program has been evaluated a number of times, beginning with a pilot study involving nine schools conducted by Mulligan (1999). The Counting On program began in 2000 involving 40 government schools, more than 600 students, 120 school teachers and 40 district mathematics consultants. Further evaluation reports on the Counting On program were conducted in 2000, 2002, and 2003 (Perry & Howard, 2000, 2001a, 2001b, 2001c, 2002a, 2002b, 2003). During 2001, the Counting On program was implemented in 76 primary, four central, and 75 secondary schools across NSW, involving more than 1400 students, 321 school teachers, and 40 district mathematics consultants. During 2002/2003, the Counting On program involved three high schools per district and two feeder primary schools in each of the 40 districts. It was a feature of all the evaluations that Counting On resulted in an improvement in student learning outcomes in computation and place value.

In 2007 there were significant changes made to the program. Counting On 2007 was implemented in 122 schools across the state and was based on the previous models but included changes designed to simplify and encourage further and ongoing implementation by schools. Features of the revised model included: a simplified assessment instrument; the inclusion of Newman's Error Analysis (Newman, 1977, 1983); a revised Counting On CD using an interactive interface that linked the learning framework to video explanations of the framework and snippets of student responses. It also included additional material and learning objects on fractions, decimals, and percentages; school clusters, the formation of which was intended to strengthen a middle school focus; and a facilitated professional development model for the program which was necessary due to a change from district to regional model by the New South Wales Department of Education and Training. It is beyond the scope of this paper to examine all of the changes and so it will focus mainly on the first mentioned change.

Earlier evaluations had reported teacher concerns with time demands and workload resulting from the program, "the issue of time – an almost universal one with Counting On—raised its head again" (Perry & Howard, 2001, p. 43). The simplified assessment instrument meant that teachers were no longer required to administer a 17 item assessment instrument using individual student interviews that were video-taped for later assessment purposes. The new approach used a whole class approach covering place value, addition, subtraction, multiplication, and division tasks. The class teacher then sorted the student responses into one of three groups: apparent expert, intermediate, and a target group. Only the selected students would then complete the additional assessment items involving two of the original assessment items (one on place value and one on multiplication and division) and two questions involving Newman's Error Analysis. This saved considerable time but there were initial concerns that teachers would not develop an intimate appreciation of their students' specific difficulties nor a deep understanding of the learning framework and thus the students' learning would suffer. This paper concentrates upon these concerns and reports on the evaluation of the student learning outcomes for 2007.

Methodology

The sample consisted of selected middle years students (years 5 to 9) chosen from the 122 schools who were grouped into 30 clusters across nine of the ten New South Wales Department of Education and Training regions. The number of clusters that each region could nominate was fixed, based on an analysis of the system wide Secondary Numeracy Assessment Program (SNAP) results. Most clusters contained from three to five schools, although some contained smaller and larger numbers depending on local circumstances. Primary schools, secondary schools, and central schools were involved in the program.

The revised assessment instrument was administered by the class teacher as a whole class schedule covering place value, addition, subtraction, multiplication, and division tasks. The assessment schedule was closely linked to the learning framework and the data were used by the teacher to identify the student target group.

Table 1

Learning Framework Levels of Conceptual Development in Place Value and Multiplication and Division (Perry & Howard, 2001b, p. 412)

Place value		**Multiplication and division**	
Level	Descriptor	Level	Descriptor
0	Ten as count	0	Unable to form equal groups
1	Ten as unit	1	Forming equal groups
2	Tens and ones	2	Perceptual multiples
3	Hundred as unit	3	Figurative units
4	Hundreds, tens, & units	4	Repeated abstract composite units
5	Decimal place value	5	Multiplication and division as operations
6	System place value	6	Not used

From the target group on two occasions, teachers were asked to conduct a target group assessment process with a minimum of five students per class and to record the student data on an Excel spreadsheet supplied to them. The spreadsheet recorded the initial level on the learning framework (see Table 1) for the targeted students before the Counting On 2007 program was implemented and again following 10 weeks of targeted Counting On 2007 activities. This process was similar to other reported evaluations. These results were compiled and are reported in the next section.

Results

A total of 102 schools from the 122 submitted data to the CA, consisting of 71 primary schools, 26 secondary schools, and 5 central schools. There were 1306 students included on the spreadsheet with 940 primary students (72%) and 366 secondary students (28%). The table below lists the students by their year cohort.

Table 2

Student Numbers By Year Cohort

School Year	Frequency	Percentage Frequency
4	40	3.06%
5	488	37.37%
6	412	31.55%
7	269	20.60%
8	70	5.36%
9	11	0.84%
No Year	16*	1.23%
Total	**1306**	**100.00%**

*Note: There were 16 secondary students missing Year details

The spreadsheet was used to record the initial level before the Counting On 2007 program was implemented and again following 10 weeks of targeted counting on activities.

Place Value

Using a paired samples t-test, the 10 week Counting On 2007 program had a significant effect upon student place value learning outcomes ($t=37.143$, $p<0.001$). The difference graph (see Figure 1) shows diagrammatically the changes that occurred as a result of the program. The graph clearly shows that the majority of students improve by 1 level with a sizeable group improving two levels. There are a small group who improve by 3 and 4 levels as there are some who decline by 1 or 2 levels.

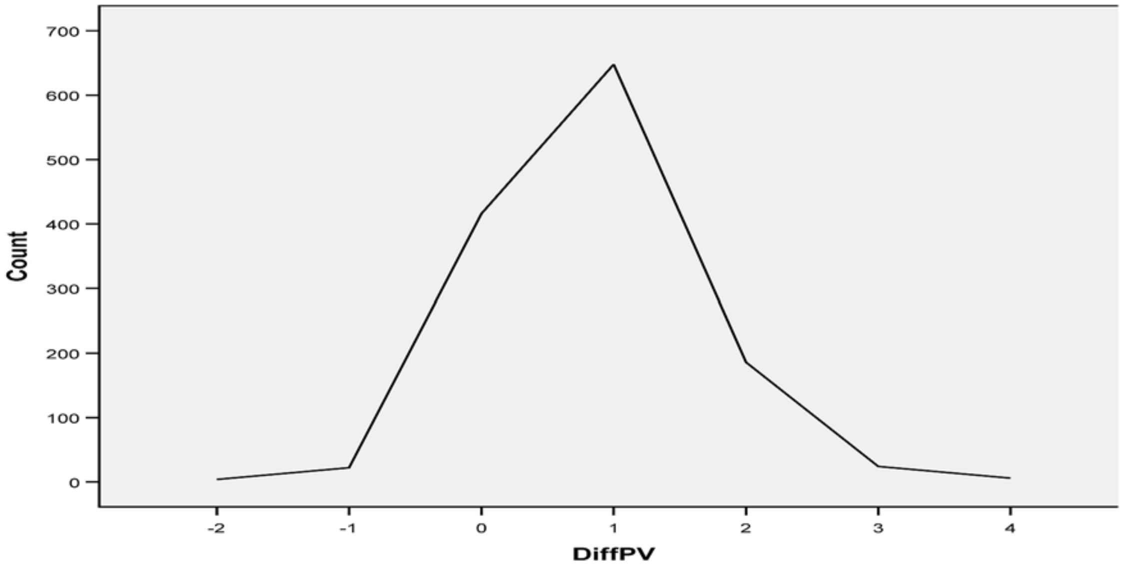

Figure 1. The difference in place value levels.

Multiplication and Division

The results for the students' multiplication and division levels show improvements in student learning outcomes. Using a paired samples t-test, the 10 week Counting On 2007 program had a significant effect upon student multiplication and division learning outcomes ($t=33.754$, $p<0.001$). The difference graph (see Figure 2) shows diagrammatically the changes that occurred as a result of the program. It shows that the majority of students improved by 1 level with the next sizeable group showing no improvement and over 200 students improving two levels.

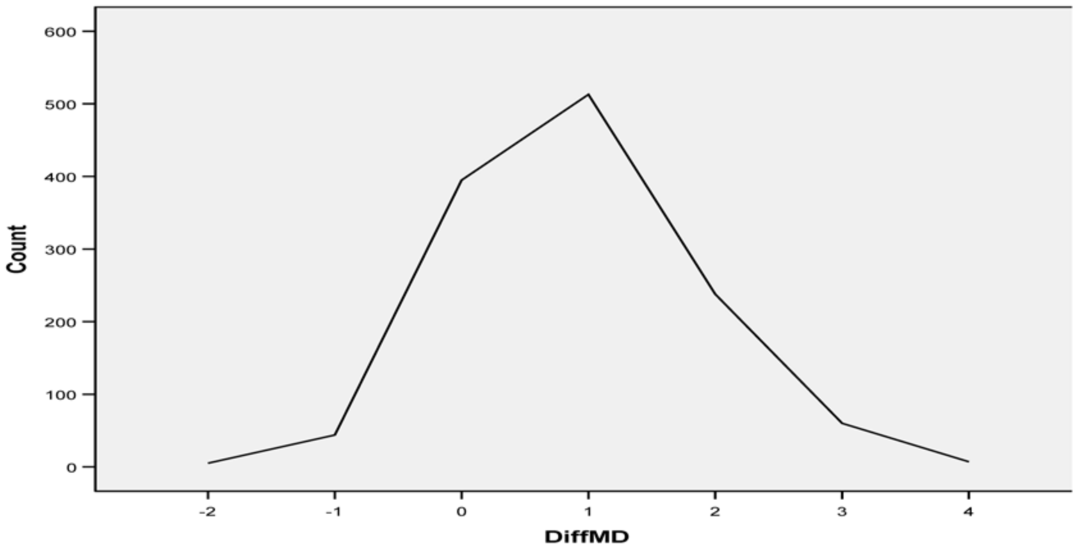

Figure 2. The difference in multiplication and division levels.

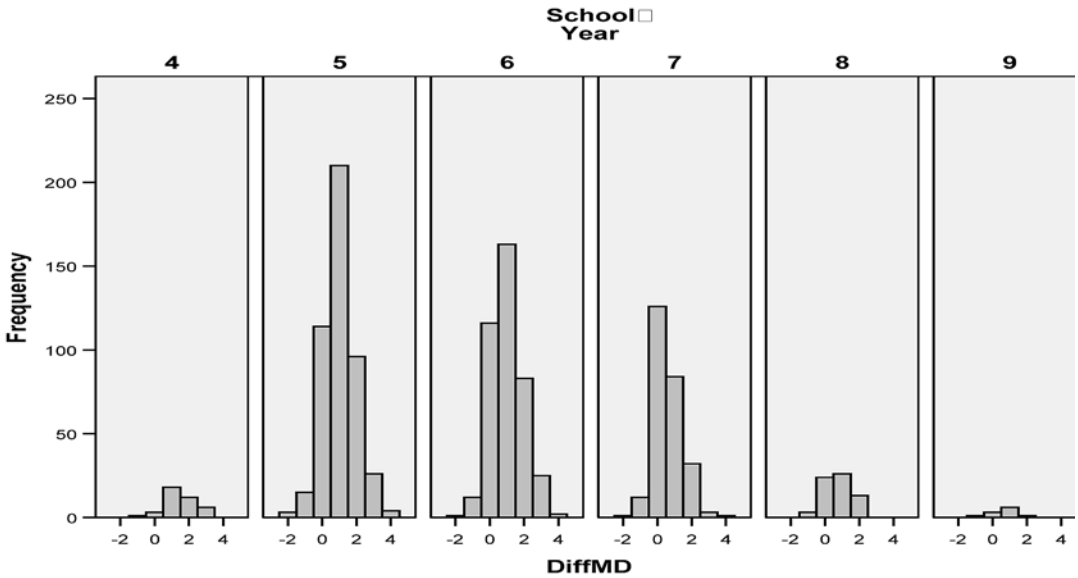

Figure 3. The difference in multiplication and division levels according to student year cohort.

There is a small group of students who improve by three and four levels as there are some who decline by one or two levels. The differences in levels according to student cohort are shown in Figure 3.

Table 3

Frequency Distribution Of Negative Difference Outcomes (Final – Initial Level)

No of negative student outcomes	No of schools	Percentage of Total Negative Difference Outcomes
1	14	18.7
2	4	10.7
3	6	24.0
4	3	16.0
5	3	20.0
6	0	0
7	0	0
8	1	10.7
Total	31	

There were 26 students who received a negative difference outcome (Final level – Initial level) on the place value scale and 49 on the multiplication/division scale. In Table 3, there were 14 schools with only one negative difference outcomes (18.7% of all errors) in their total results, whereas one school accounted for eight (30.7%).

Discussion

In an attempt to reduce the assessment load on teachers a simplified procedure had been adopted. While it saved considerable time, there were concerns that teachers would not develop an intimate appreciation of their students' specific difficulties nor a deep understanding of the learning framework and thus the students' learning would suffer. The results across the Counting On 2007 program indicate an improvement in student learning outcomes by one or more levels in their understanding of place value (66%) and in their understanding of multiplication/division (65%).

However, the concern with teachers understanding of the learning framework is not so easily dismissed. There was an issue regarding the student learning outcomes where some cases indicated the final assessment result was lower than their initial result. There may be a variety of reasons such as the students have done more poorly as a result of the program, that these students were deskilled by the program. Vaiyatvutjamai and Clements (2004) studied students across the range of student abilities, and the results for low performing students challenged the use of the term misconception when associated with many of the errors those students made. "A misconception can be regarded as a fairly stable, but inappropriate, way of thinking ... analysing the errors made by low performers in this study, was that the word 'stable' was not one that could sensibly be used" (p. 181). Students with "unstable" conceptions will give different answers at different times and hence their test scores are not stable and may at times decline.

Another source of possibilities for the decline in some student outcomes could be that teachers became more familiar with the assessment or their students' ability at the time of the final assessment than they were when the initial assessments were made and so the students were not correctly placed initially. An e-survey sought the opinions of regional mathematics consultants (*n=15*) and school program facilitators (*n=40*). Their responses included reasons such as the use of different assessors, poor initial teacher understanding of the learning framework, misdiagnosis, student resistance to assessment, poor student attendance, and transcription errors. Table 3 indicates there were 14 schools with only one negative difference outcomes (18.7% of total) in their results, whereas one school accounted for eight (10.7%). The spread of results suggests a mix of reasons and does not suggest widespread poor initial teacher understanding of the framework.

Conclusion

An important issue for a program such as Counting On which attempts to target students struggling with mathematics in the middle years lies in the difficulty in providing specific assistance for such a group of students without making the task too onerous for the classroom teacher. Previous evaluations of Counting On had highlighted teachers' concerns over time demands and workload. Thus Counting On 2007 sought to adjust the balance by reducing the teachers' load through a number of changes, particularly to the assessment procedure. There was a concern that the adjustments may affect the student learning outcomes and teachers understanding of student difficulties and the learning framework.

This paper has indicated that the adjusted program continued to be successful in assisting the learning of students who had struggled with mathematics in the middle years of school while providing a vehicle for teacher professional learning.

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How Group Composition Can Influence Opportunities for Spontaneous Learning

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Classroom video, and video-stimulated interviews of small group work, in a Grade 5/6 classroom are used to show ways group composition can influence learning opportunities. Vygotsky's (1933/1966; 1978) learning theory on the spontaneous creation of knowledge as compared to the guidance of an expert other frames this group analysis. Illustrations from two groups show how opportunities to spontaneously create new knowledge can be limited or enhanced by psychological factors associated with the inclination to explore that have been linked to resilience in the form of optimism (Seligman, 1995, Williams, 2003). This study contributes to our knowledge on forming groups to promote deep learning. It raises questions about other ways in which learning may be influenced by optimistic orientation and about building this personal characteristic to enable deep learning.

Introduction

During my years as secondary mathematics teacher in a rural area, without access to research findings (1970-1992), I experimented to improve students' learning opportunities. I found students seemed to learn more when new topics commenced with work in groups on unfamiliar problems rather than by initially learning rules and procedures, then memorising and practising. I now know that Skemp (1976) wrote about this deep learning at least ten years previously, and that research into the deep understandings that can develop when students' struggle to develop new mathematical ideas together was well underway (Cobb, Wood, Yackel, & McNeal, 1992; Wood & Yackel, 1990). To improve student opportunities to learn in my classes in the nineteen eighties, I observed group dynamics, and trialled group compositions to find what seemed to work. One of the things I found: "The need for a positive group member who can overcome negative influences in that group" is elaborated and theoretically underpinned through the present study.

I now know, what I described as 'positive' was an 'inclination to explore' (Williams, 2003), and one type of 'negative' was the result of 'not being inclined to explore' or wanting to remain within the confines of what was known rather than explore unfamiliar mathematical territory. Other types of negative influences such as 'inclined to engage in off-task activity' rather than focus mathematically are not explored in this study. My research question is: "Does the relative inclination of group members to explore new mathematical ideas influence group learning opportunities? And, if so, in what ways?

Theoretical Framework

The theory framing this study (Vygotsky, 1933/1966, 1978; Seligman, 1995; Williams, 2007a) is associated with 'spontaneous learning' in comparison to learning under the guidance of an 'expert other' (Vygotsky, 1978; Wood & Yackel, 1990). It includes psychological factors that influence whether a student is likely to undertake spontaneous learning (Seligman, 1995; Williams, 2005). Spontaneous learning occurs when a student or group create their own Zone of Proximal Development (ZPD) or overlapping zones with the assistance of 'cognitive tools'. The ZPD is the distance between what the student presently knows and what they can learn under the guidance of an 'expert other' (Vygotsky, 1978). Vygotsky did not state that this expert other was needed for learning to occur. He used this concept of an expert other to show what a child had the potential to learn at a given time. Vygotsky (1933/1966, 1978) recognized that people could create new knowledge. A personal email communication from James Wertsch articulates this:

> On the one hand, Vygotsky clearly did emphasize the influence of existing cultural and social forces in the development of individual mental functioning in the child. On the other hand … Vygotsky talked about how one uses cultural tools such as toys when playing and in the process creates ONE'S OWN zone of proximal development. (Wertsch in Williams 2005, Appendix 0.3).

Spontaneous learning (Williams, 2007a; Wood, Hjalmarson, & Williams, in press) can occur when students discover mathematical complexities in a task and decide that they want to explore them. In doing so, students ask themselves a question about this complexity, creating a mathematical challenge, overcome by exploring unfamiliar mathematics to develop new conceptual knowledge. Conceptual tools that could be used to support new learning include language, symbols, diagrams, concrete aids and technology, used as tools to think and to assist communication with peers (see for example, Williams, 2005). Deep learning can occur as a result (Williams, 2007a, 2005, 2000). This fits with Davydov's (1990) findings that students who mentally reorganise knowledge can develop new conceptual understandings. Various pedagogies have been developed which have led to instances of spontaneous learning (e.g., Cobb, Wood, Yackel, & McNeal, 1992; Dreyfus & Tsamir, 2004; Wood, Hjalmarson, & Williams, in press).

Some students are willing to explore unfamiliar mathematical ideas to develop new conceptual knowledge (e.g., Dreyfus & Tsamir, 2004; Williams, submitted) whilst others want to remain within the confines of what they have been taught (Anthony, 1996). This inclination to explore (or not) is the psychological characteristic upon which this study focuses.

This inclination to explore fits with Seligman's (1995) construct of explanatory style (Williams, 2003). Explanatory style includes perception of and response to successes and failures. An 'optimistic' child (Seligman, 1995) perceives failure as 'temporary', 'specific', and 'external', and success as 'permanent', 'pervasive', and 'personal'. Inclination to explore is associated with optimism because exploring what is unknown (present failure) is consistent with the perception that 'not knowing' is temporary and 'knowing' can result from personal effort [Failure as Temporary; Success as Personal]. Perceiving success as pervasive is attributing success to a characteristic of self "I succeeded, I am smart". Perceiving failure as specific involves examining failure to see what could be changed to increase the likelihood of future success (instead of perceiving the failure as pervasive or relating to a characteristic of self: "I failed, I am stupid"). Optimistic students look for ways to overcome problems they encounter [failure as temporary] by examining what can be altered to increase chances of succeeding [failure as specific]. They perceive that personal effort can lead to success (Williams, 2003, submitted, 2008).

The interview dialogue and classroom activity of a struggling student (Dean) in a previous study (Williams, 2005) is used to illustrate an optimistic orientation and elaborate on the dimensions of optimism. Dean persevered in spite of the failures he encountered:

> Cause the first time I do stuff um I get a bit *stressed* [quiet laugh at himself] … I always don't get it at *first*. (Dean's interview, p. 285)

His use of "at first" indicates he perceived failure as temporary. His perception of success as personal is illustrated in the following quote:

> I write it down in my book and then when he's talking [about] something that I have already known then I just look over it again. (Dean's interview, p. 285)

Rather than seeing himself as stupid for failing to understand how to juxtapose angles to find their sum [failure as pervasive], he studied the teacher's activity and found how this differed from his own [failure as specific]:

> I didn't know where the corners [angles] went- he [teacher] told me you put [them] facing in but … I was doing it all different- I was facing them out and up (p. 281)

As Dean was still unable to execute the procedure he looked for another way [failure as temporary, success as personal] rather than waiting helplessly. He altered what he attended to and found his own way to achieve success [failure as temporary, specific, personal]. Students like Dean who perceive 'failure' to understand as temporary and able to be overcome through effort by analysing the situation to see what can be changed to increase chances for success (learning more) are 'inclined to explore' / optimistic / resilient. When problem solving in mathematics, they cope with adversities associated with encountering failures before success. This study examines how this influences learning opportunities.

Research Design

This study is part of a broader study of the role of optimism in collaborative problem solving in Grade 5/6 classrooms in an Australian government school. The teaching approach has been demonstrated to elicit creative thinking (see Williams, 2000, 2007b). It involves small group problem solving with interim reports to the class. Three tasks were undertaken across the school year (Tasks 1, 2, 3) for three, two, and one eighty-minute sessions respectively. To enable study of:

Interactions within groups,

Learning outcomes,

Optimism or lack of optimism of group members,

the Learners' Perspective Study (LPS) methodology (Clarke, 2006) was adapted. Three cameras simultaneously focused on six groups in the classroom. The fourth camera captured student reports. The private talk of the three groups in the foreground on the cameras was captured. Video-stimulated interviews were undertaken individually with at least four students after each lesson. To enable students from each group access to both their group and the reporting sessions in the interview, a mixed image of one group with the reporting sessions as an insert in the corner was generated, and a second group had the reporting session in the background on their group video. If members of the third group were interviewed, there was opportunity to use their group video and the video of student reports. Students used the video remote and selected the parts of the lesson that were important to them for any reason and discussed what was happening, what they were thinking, and what they were feeling. In addition, they were asked questions about whether they were good at maths, and how they thought they were going in maths, and how they made those decisions. These questions generated indicators of optimism.

For Task 1, groups were composed by the teacher using my descriptions of what made a 'good group'. The description included:

Gender balance

Similar paces of thinking (as opposed to similar performance)

A student expected to 'positively' influence the most negative member

The iterative changes to group composition from Task 1 to Task 3 were informed by my previous teaching and research knowledge of group composition and my video analysis of group interactions in previous tasks in this study. The intention was to optimise group composition to improve opportunities for spontaneous learning for class members.

Optimism or lack thereof was studied through discourse analysis (Säljö, 1999) and these analyses were triangulated with student enactment of this characteristic on lesson videos. In this study, the intention was to examine how these students enacted optimism or lack thereof and what effects this had on learning opportunities for group members. Video and interview data and photocopies of group work produced in class together provided evidence of what students had learnt, and what had influenced their opportunities to learn.

Purposeful Group Selection

Over the period in which the three tasks were undertaken, Group 1 remained the same because they developed new conceptual understandings during each task. For Task 3, Group 1 (Patrick, Gina, Eliza, Eriz) consisted of three students (Patrick, Gina, and Eliza) because Eriz was absent. All four students in this group displayed optimistic indicators.

Group 2 (Sam, Jarrod, Wesley, Donald) was formed as a result of progressive iterations of group composition intended to compose a group in which a high performing student, Sam, would participate in spontaneous learning. To the surprise of the teacher, he did not do so in Task1. Sam's interview data showed evidence of lack of optimism, so I was not surprised. In his interview after Task 1 and 2, Sam reported that he found the tasks boring and had not learnt anything new. This fitted with evidence on the lesson videos, and with Sam's interview descriptions. Sam had an instrumental understanding of the mathematical ideas associated with Task 1 (Volumes of Cuboids, see Williams, 2007b) prior to and after Task 1. He knew to multiply length by width by height to get volume but his interview showed that unlike other students who had learnt from Task

1, he did not know why this was so. In contrast, Jarrod, Wesley, Gina, Patrick, and Eliza, could explain why this formula was relevant by referring to the rectangular prism and discussing the layers within. In general, these students were not aware of this formula until they generated it during Task 1. Sam was selected for detailed study because he was a high performing student who had not participated in spontaneous learning opportunities. Although any member of Group 1 could have illustrated optimistic activity, Patrick was selected for analysis because like Sam, he was a high performing male student.

Task 3

Make each of the whole numbers from one to twenty inclusive using:

- Four of the digit four and no other digits
- Any or all of the operations and symbols

  + + - - × / ÷ √ . () ^{2}

Think about how to make all the sums as fast as possible

Figure 1. Task 3: The Fours Task.

Task 3 (see Figure 1) was accessible to students with varying understandings of whole number operations because the numbers could be generated with simple operations, or through many permutations and combinations of more complex operations and symbols. During the task, it was anticipated that students would learn more about the operations and symbols and how to use them through their conversations in groups, and during the reporting sessions. Increased familiarity with these symbols was expected to increase their opportunities to create new sums. Trying to find fast ways to generate sums was expected to promote generalisation as it did for Group 1 (see below).

Task Implementation

This task spanned one eighty-minute lesson. The teacher and I team-taught with myself undertaking most of the task implementation and the teacher focusing much of the reporting session. The class undertook the following activities in the order given:

Three minutes commencing the task alone

Shared what they had done with the rest of their group

Approximately ten minutes of small group work

Two minutes preparing their report for the class

Approximately twenty minutes of group reports (1-2 minutes each)
(focused on some aspect of what the group had done or tried to do)

Another cycle of group work and reporting

Groups had a set of tiles with fours, operations, and symbols to assist their thinking and communication. Transparent tiles were used by students on an overhead projector, during reporting sessions, to enabled students to communicate in visual images and language (Ericsson & Simons, 1980). These reports were discussed after each reporting session without the class teacher or myself judging the correctness of the mathematics produced. Rather, we asked questions to stimulate further thinking in groups. Further information on this approach can be found in Williams (2007b) for this study and Williams (2000) and Barnes (2000) for other studies.

Results and Analysis

Non-Optimistic Sam and Optimistic Patrick

One of the indicators of lack of optimism Sam displayed in his interview was Success as External. He described learning for him as listening to the teacher, reading books, and searching the Internet. Unlike Patrick, he did not include self-generated knowledge, which is an indicator of optimism [Success as Personal]. Sam gave some indication that he did not examine situations to see what more he could learn. Sam stated in his interview after Task 1 that he knew everything in the lesson before hand. When asked to identify some of the things he already knew, Sam answered: "Didn't I say I knew it all … Which reports do you think I should be thinking about?" The reasons for this response are not clear-cut. He may not have looked in detail because he considered that he 'knew it all' (as indicated by his boredom), or because he considered his peers would not know more than he did about anything. This response suggests Sam wanted guidance on what to attend to because he was not inclined to go outside his present understandings and identify for himself how the presentations of others matched his own thinking. Whatever the circumstance, there was sufficient evidence to indicate he was not optimistic

In contrast, Patrick described one way he learnt was by thinking about mistakes made by others and how they could be overcome [Success as Personal]. In his interview during Task 1, he discussed a group who had made a twenty-four cube rectangular prism when they had intended to make a twelve-cube prism (Williams, 2007b). This group had not been able to correct their mistake before they reported. Patrick stated in his interview:

> You know how they got it wrong- it made me think about (pause) how they could get it *right* (pause) … it was … 2 2 6 [dimensions of rectangular prism] … they got 24 and they have to get 12 what if they changed the 6 to 3 and that would just halve it and instead of 24 they would have 12.

Although not stated explicitly Patrick appeared to have halved the number of stacks in the height rather than manipulated numbers. This fitted with the way his group had been considering different prisms as made up of flat stacks of cubes. Patrick did not see the failure to make the rectangular prism with 12 cubes as permanent. He examined the situation to see what he could change to gain success through his personal effort. He identified possible variables he could control, and adjusted them [Failure as Temporary; Failure as Specific; Success as Personal], thus demonstrating optimism.

Composition of Sam's Group

Sam was purposefully placed in a group that was expected to optimize his opportunities to undertake spontaneous learning. His group contained all boys because he was a quiet student and it was possible that interacting with girls might have limited his novel contributions on the previous two tasks. Two of the boys placed in Sam's group (Jarrod and Wesley) were high performing students who had demonstrated they could think creatively in Tasks 1 and 2. Although the other boy, Donald had dominated activity in another group and taken their thinking off track, it was considered that Jarrod would be focused enough and sufficiently dominant to keep this group on track. In other words, it was considered that Jarrod would be able to provide the positive influence necessary to focus the group. The email quote below shows an excerpt of my discussion with the teacher of Jarrod's capacity when forming groups for Task 2.

> Callum and Amit played around a lot. I think Amit would contribute if we added a serious eager boy … [like] Jarrod. … [and] Elsa might have more to contribute [in her group] if Jarrod were not dominating (Email communication from researcher to teacher).

This quote captures my analysis of Jarrod's eagerness and capacity to think creatively and my faith in him to entice a good student into creative mathematical thinking, and to bring a group back to a productive direction if ideas with mathematical flaws were presented. Jarrod was considered an ideal group member to entice Sam into creative thinking and to 'overrule' Donald if his thinking was flawed.

Activity in Task 3

Sam and Patrick's activity during individual time showed Patrick's willingness to explore unfamiliar mathematics "I went looking for hard one's first like decimals and stuff and times" and Sam's inclination to remain within the mathematics he knew. Both students generated an equal number of correct sums. Patrick generated most of his sums by retaining the underlying structures and changing the positions of operations. He progressively increased the number of unfamiliar symbols and operations he used (see Williams, submitted, for more information). Sam generated his sums quickly, stopped early, covered his work and waited. Although Sam's number fact recall was faster than Patrick's, the sums Sam generated, and his less sustained use of patterns to generate them, and the way he stopped when there was still more time left suggested he was unable to proceed. Unlike Patrick, Sam did not progressively increase the number of harder operations he used and did not try decimals or brackets. Sam was not inclined to explore.

During group activity, Sam listened to the strategies for finding sums reported by Jarrod and Wesley then instead of engaging in group activity as expected, he spent several minutes extending his list of sums by using the mathematics Jarrod and Wesley had described. He did not extend their ideas. Once he had completed all the sums he could, Sam monopolized the remaining time by explaining to Donald how to find the answers to the sums rather than engaging in exploratory activity. The types of creative thinking previously undertaken by Jarrod and Wesley did not occur in this group. When Wesley stated it was not possible to use the decimal point, this was not discussed; Sam just included it in the presentation without justification. No new ideas were generated.

Patrick contributed to the development of new ideas in his group in various ways. For example, when Gina generated a sum and Eliza queried it, Patrick looked for what could be changed so they did not have to start again "Put something in the middle like a plus or something" [Failure as Specific]. He was the first to begin to package parts of sums as mathematical objects. For example, he put his hand over - 4 + 4 at the end of a sum and moved them away slightly "We don't really need these … they cancel each other out". His ideas formed the foundations of some of the insights developed by this group including:

> -4 + 4 can be used if wanting a small answer;
>
> Brackets can change the size of the answer; and
>
> The order in which operations are applied can change the answer.

These are 'big ideas'.

Discussion and Conclusions

These cases differed in learning outcomes and indicated that the relative optimism of group members influenced available learning opportunities. Where all students were optimistic, rich learning occurred, but limited learning occurred when a non-optimistic student focused group activity on mathematics familiar to optimistic students who had previously created new knowledge. This study raises questions about whether another group composition could have led to Sam creating new ideas. As three different groupings were tried and none were successful, it seems likely that factors other than group composition may need to be changed for a non-optimistic student like Sam to recognise that learning can be self generated. Although further research is required, these findings provide a starting point for identifying what to attend to and what types of group compositions to examine to explore how to form groups to increase learning opportunities. This study did not illustrate a 'positive' student overcoming the influences of a negative student. Study of such group interactions could be fruitful (if they exist and my teaching experience suggests they do). Longitudinal research is required to study students like Sam over numerous tasks to see if he does finally develop optimism and to identify factors that contributed to this. Most importantly though, given this study shows optimism can increase successful problem solving activity, we must build the optimism of students that are not yet optimistic? This is the focus of my present research (Williams, 2007, 2008).

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Success and Consistency in the Use of Heuristics to Solve Mathematics Problems

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The ability to solve mathematics problems is the main goal of mathematics education in many countries. This ability depends on coordinating several types of knowledge and mathematical processes, especially heuristics. Commonly used heuristics include guess and check, draw a diagram, logical argument, and simplifying the problem. This paper describes the heuristics used by a sample of Primary 5 (n = 221) and Secondary 1 (n = 64) Singapore pupils to solve problems like this one: "There are 100 buns to be shared by 100 monks. The senior monks get 3 buns each and 3 junior monks share 1 bun. How many senior monks are there?" The pupils solved two sets of problems; the second set consisted of parallel problems to the first set but was administered a few months later. The pupils' written solutions were analysed according to the heuristics used. A comparison of the heuristics used between the two parallel tests shows that some pupils did not use similar heuristics to solve parallel problems. This issue of consistency in heuristic use should be further researched to unravel its implications for the teaching of heuristics.

Background

Most countries include problem solving as one of the key objectives of school mathematics education. But many pupils at all school levels have difficulty solving unfamiliar or so-called "non-routine" problems. Confronted with undergraduate pupils who did not know how to begin to solve a novel problem, Polya (1957) devised a 4-step model that has impacted enormously on the teaching of problem solving in schools over the past half century. A key element of his problem solving model is the use of heuristics. A heuristic is a generic rule, such as Think of a Similar Problem, Draw a Diagram, and Guess and Check, that can be used to solve different types of problems, though there is no guarantee that it will be successful. In some countries such as Singapore, the teaching of heuristics has become an important step towards developing problem solving skills among the pupils. In Singapore (http://www.moe.gov.sg/cpdd/syllabuses.htm), a dozen of heuristics are taught in primary and secondary schools, and the Primary School Leaving Examination (PSLE) taken by all pupils in Singapore at the end of their primary school education contains "challenge problems" that require the application of heuristics. While the effectiveness of the teaching of heuristics is a controversial pedagogical and research issue, this paper only examines how successful a sample primary and secondary pupils were in solving problems using some of these heuristics. In normal classrooms, the teachers should be able to know how successful their pupils are able to apply heuristics to solve problems, but such information is rarely compiled and shared beyond their classroom settings. This paper fills a gap by providing systematic information about heuristic use by pupils.

The Study

The data in this paper were gathered as part of the research project entitled "Developing the Repertoire of Heuristics for Mathematical Problem Solving" (MPS) conducted in Singapore in 2004. This MPS project had three main components.

The first component examined whether the pedagogical practices of mathematics teachers would support mathematics problem solving. The participating teachers from three primary schools and two secondary schools, with two classes per school, were observed in two rounds. Each round of observations covered several lessons that made up one teaching unit (Ho & Hedberg, 2005). The teachers attended a 3-hour workshop in between the two rounds, separated by about three months.

The second component, the concern of this paper, looked at pupils' solutions to the mathematics problems administered at the end of each round of observations. After each test, the pupils answered a questionnaire about their perceptions of the problem solving process and perceived difficulty of the problems. The aim was to relate pupils' heuristic use to their affective and metacognitive responses. These two rounds of tests are referred to as "pre-test" and "post-test" in this paper, although it is not appropriate to consider this design a rigorous experiment due to the short duration of the intervening workshop. Each test consisted of nine problems. The problems in the post-test were parallel but not equivalent to those in the pre-test, with modifications in the story line and numbers used. One of the problems was identical in both tests.

The pupils were given 40 minutes to solve the problems and 20 minutes to answer the questionnaire. They were instructed to show full workings for the problems. The pupils seemed to react quite differently to the tests. In one school, the post-test was administered after an examination, and the pupils were not motivated to complete the test. After the pre-test, a teacher commented that the statement "This is NOT a test" would affect pupils' attitude towards completing the test to their best ability. Hence, this statement was removed from the post-test. These matters about test administration may affect the pupils' motivation to complete the tests and the results may not reflect their true performance. Due to some contingencies, post-test data could not be collected from one of the two secondary schools, resulting in a sample mortality of two classes. The final sample consisted of 221 Primary 5 pupils (11 years old) and 64 Secondary 1 pupils (13 years old) who took both tests. Since no generalisation to the respective pupil population was intended, this loss was not a serious flaw, though certainly less than ideal.

The third component consisted of videotaping pairs of pupils solving mathematics problems. The aim was to capture their metacognitive processes through talk aloud protocol (Lioe, Ho, & Hedberg, 2006).

Analysis of Pupils' Written Solutions

Teachers and researchers often analyse pupils' written solutions to better understand the problem solving strategies used by the pupils. For example, Covi, Ratcliffe, Lubinski, and Warfield (2006) categorised pupils' written solutions to a given problem and found that among the codeable strategies, Guess and Check was most prevalent.

In this study, the pupils' scripts were analysed according to the types of heuristics used. Five major types of heuristics were first determined based on previous experience and selected scripts: Systematic Listing, Guess and Check, Equations, Logical Argument, and Diagrams. Not all the five heuristics were suitable for every given problem.

The research assistant coded all the scripts. When uncertainty arose, the author and the research assistant discussed the solution and decided upon its code to ensure consistency. Correct solutions or completely wrong ones were relatively easy to code. There was some difficulty in coding partially correct solutions. However, this was not a particularly serious issue with mathematics, in contrast to the more problematic situation of coding affective variables and classroom observations that requires stronger subjective interpretations. The next section describes the findings for only one particular pair of problems, where only the first four strategies were appropriate.

Analysis of One Pair of Problems

Pre-test problem (*Monks*): There are 100 buns to be shared by 100 monks. The senior monks get 3 buns each and 3 junior monks share 1 bun. How many senior monks are there? [Answer: 25 senior monks and 75 junior monks]

Post-test problem (*Monkeys*): The zoo keeper gave 80 bananas to 50 monkeys. The big monkeys ate 2 bananas each, and 3 small monkeys shared 2 bananas. How many big monkeys are there? [Answer: 35 big monkeys and 15 small monkeys]

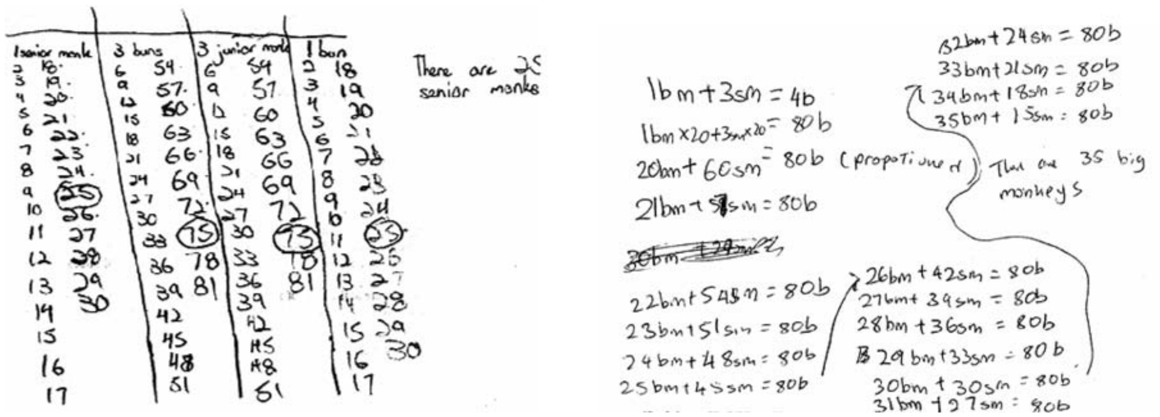
These two problems are similar in structure. They require pupils to coordinate three pieces of information: (a) total number of persons (monks) or animals (monkeys), (b) total number of items (buns/bananas), and (c) ratio of relevant items according to the given conditions. However, the post-test problem is more difficult than the pre-test one because of two factors. First, the numbers of bananas and monkeys are no longer the same, as in the Monks problem. This requires more complex proportional thinking. Second, the Monkeys problem involves a slightly more complicated fraction, $\frac{2}{3}$, compared to $\frac{1}{3}$ in the Monks problem. Nevertheless, the same heuristic can be used to solve either problem.

C1: Systematic Listing (Primary: 2, 2; 1, 8; Secondary: 3,-; 2, 2)

For coding purpose, Systematic Listing must involve at least three consecutive items, to distinguish it from Guess and Check, which may be conducted in an ad hoc fashion including hitting on the correct answer in one lucky or unexplained guess. The figures above show the success rates. For the primary group, two pupils applied this heuristic to get the correct answer for the Monks problem and two got a partially correct solution;

one pupil got the Monkeys problem correctly, while eight got a partially correct solution. For the secondary group, three pupils got a correct solution for the Monks problem and there was no partially correct solution; two pupils got a correct solution and two pupils got a partially correct solution for the Monkeys problem. These values show that this heuristic was used by very few pupils.

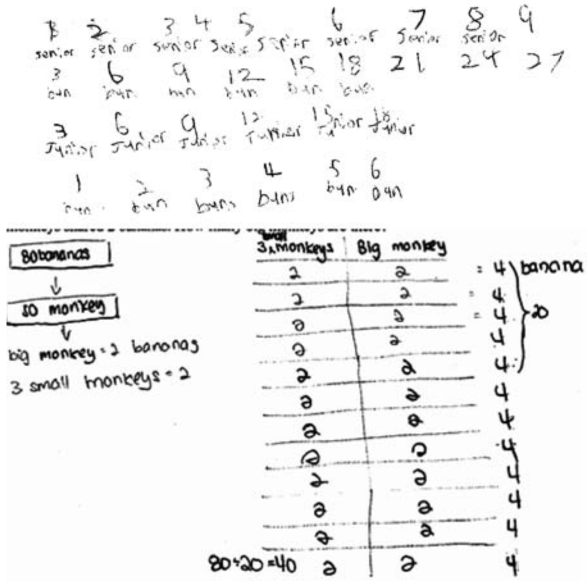
Two correct solutions are shown in Figure 1. In Figure 1(a), the pupil kept the ratio between the number of persons and the number of items constant (1 senior monk : 3 buns; 3 junior monks : 1 bun). In Figure 1(b), the pupil began with the simplest case and varied one of the conditions (number of big monkeys) and adjusted the values accordingly. These two solutions show that Systematic Listing can be carried out in more than one way, depending on which aspect of the problem to work on. Note that the second solution used letters (bm, sm, b) to stand for entities rather than numbers; this is the common *mis*use of algebraic letters to represent objects in the so-called "fruit salad algebra" (MacGregor, 1986).



(a) (b)

Figure 1. Systematic listing. Correct. (a) Monks problem. (b) Monkeys problem.

Errors could arise when the pupils stopped before they reached the correct answer, as in Figure 2(a), or when they did not maintain the correct ratio, as in Figure 2(b).



(a) (b)

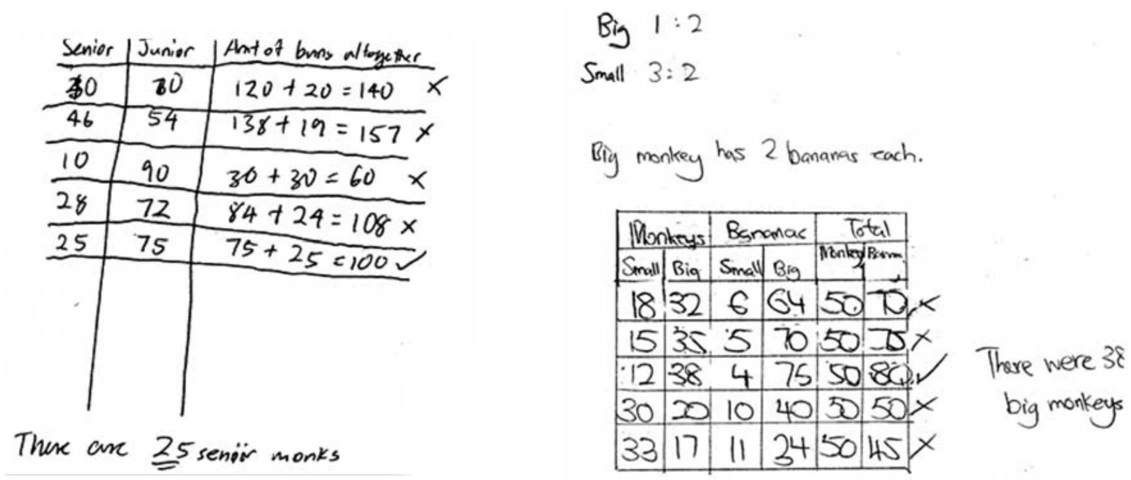
Figure 2. Systematic listing. Partially correct. (a) Monks problem. (b) Monkeys problem.

C2: Guess and Check (Primary: 19, 18; 35, 27; Secondary: 37, 13; 33, 4)

Guess and Check was more popular than Systematic Listing. Secondary pupils were more likely than the primary pupils to get the correct answer with this heuristic. Two examples are given in Figure 3. In Figure 3(b), the pupil divided the number of small monkeys by 3 but did not multiply by 2 (3 small monkeys share 2 bananas). This shows either partial comprehension of the given condition or inability to use proportional thinking.

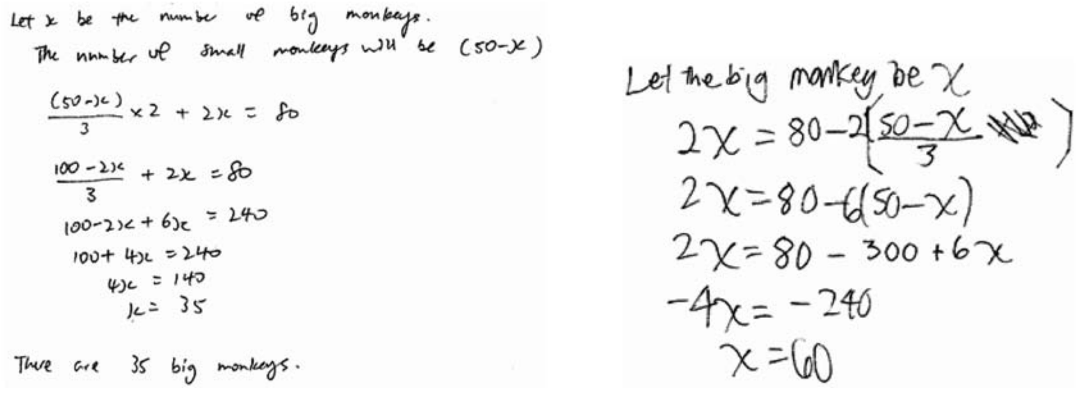
C3: Equations (Primary: Nil; Secondary: 4, 3; 3, 3)

The Singapore primary mathematics curriculum does not include using algebra to solve problems, so none of the primary pupils in the sample used this method. Very few of the Secondary 1 pupils used this method, probably because at the time of the test, they had not learned much algebra. A correct answer is shown in Figure 4(a). The solution in Figure 4(b) contains an arithmetic error about multiplying fraction.



(a) (b)

Figure 3. Guess and check. (a) Monks problem. Correct. (b) Monkeys problem. Partially correct.



(a) (b)

Figure 4. Equations. Monkeys problem. (a) Correct. (b) Partially correct.

C4: Logical Argument (Primary: 26, 7; -, 15; Secondary: 26, 2; 1; 7)

The problem can be solved by forming groups that contain the right ratio. For the Monks problem, a group may consist of one senior monk and three junior monks so that they are assigned to four buns. This is shown in Figure 5, where a diagram was used to illustrate the grouping. Those who tried similar argument were quite successful.

This grouping method does not work readily for the Monkeys problem because the relevant ratio is more complicated, as mentioned earlier. In Figure 6(a), the pupil took “2 bananas” as one share, and the working shows the need to maintain a total of 40 shares in terms of the number of bananas (35 + 5) and 50 monkeys (35 big monkeys and 15 small monkeys). When the ratio is not maintained, the error shown in Figure 6(b) was obtained.

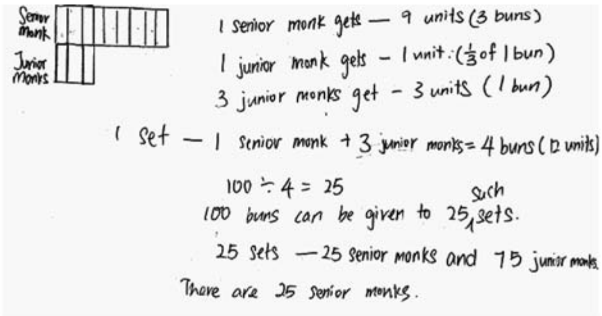
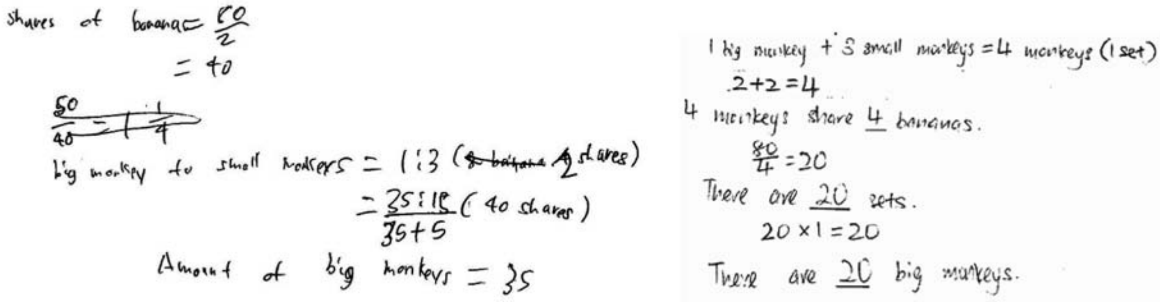


Figure 5. Logical argument. Monks problem. Correct.



(a) (b)

Figure 6. Logical argument. Monkeys problem. (a) Correct. (b) Partially correct.

Other Solutions

The very few attempts to solve these problems using only diagrams were not successful. No pupil wrote down the correct answer without working. About eight pupils in each group re-stated the problems in their own words without making any progress. Many answers could not be coded meaningfully: 23.5% for the Monks problem and 17.2% for the Monkeys problem for the primary group; 13.4% and 2.0% for the secondary group. High percentages of the primary group did not attempt the Monks problem (33.5%) and the Monkeys problem (29.4%), whereas the percentages of no attempts were much lower for the secondary pupils (12.8% and 2.0%).

Inconsistent Use of Heuristics

Table 1 compares the heuristics used by the primary group to answer these two problems. Consider Guess and Check. Forty one pupils used it to solve the Monks problem, 72 used it for the Monkeys problem, but only 27 used it for both problems. This shows that the primary pupils did not use this heuristic consistently to solve parallel problems. However, the success rates were very similar, 46% and 49% respectively. Similarly, inconsistent use of heuristics is also evident for Systematic Listing and Logical Argument.

Table 1

Heuristics Used in Pre-test and Post-test Problem: Primary (N = 221)

	Post-test																		
Pre-test	C1C	C1P	C1W	C2C	C2P	C2W	C3C	C3P	C3W	C4C	C4P	C4W	C5C	C5W	RS	MW	BLK	ABS	TOT
C1C		1		1															2
C1P																	2		2
C1W																			
C2C		1		**8**	3	1								1	2	1	1	1	19
C2P		1		10	**2**						2				1		2		18
C2W					3						1								4
C3C																			
C3P																			
C3W																			
C4C		1		5	1	2					7	1			1	3	4	1	26
C4P				3	1											2	1		7
C4W						1					1						1		3
C5C																			
C5W				1	1	1									1	1	1		5
RS											1				**3**	1	5		8
MW	1	2		3	7	3									5	**16**	13	2	50
BLK		2		4	9	2					3					12	**35**	2	74
ABS																2		**1**	3
TOT	1	8		35	27	10					15	1		1	13	38	65	7	221

Note: C1: Systematic Listing; C2: Guess and Check; C3: Equation; C4: Logical Argument; C5: Diagram. The last letter means: C = correct; P = Partially correct; W = Wrong. RS = Restating problem. MW = Miscellaneous wrong. BL = Blank. ABS = Absent.

Of the 64 Secondary 1 pupils, 24 used Guess and Check for the Monks problem and 36 for the Monkeys problem, with 21 for both tests. The success rates were 71% for the Monks problem and 89% for the Monkeys problem. For both groups, there was a significant shift to the Guess and Check heuristic.

There were six problems whose solutions could be coded under the same set of heuristics. The total frequency for each code across the six problems was obtained. Note that these frequencies do *not* refer to the *total number of pupils* because a pupil may use the same heuristic more than once when he or she solved the six problems. The findings are provided in Table 2.

Table 2:

Comparison of Heuristics Used in Pre-test and Post-test for Six Problems

Heuristics	**Primary**		**Secondary**	
	Frequency	**Success Rate (%)**	**Frequency**	**Success Rate (%)**
Systematic Listing	99 (111) 63	53.6 (71.2) 54.0	23 (50) 16	52.2 (80.0) 56.3
Guess and Check	66 (72) 38	57.6 (66.7) 39.5	45 (55) 27	84.5 (85.5) 70.4
Equations	79 (68) 52	0 (3.0) 0	80 (63) 39	45.0 (54.0) 46.2
Logical Argument	109 (113) 69	43.2 (30.1) 11.6	133 (112) 83	79.0 (67.9) 61.4
Diagram	65 (54) 24	27.7 (42.6) 45.8	16 (17) 8	93.8 (82.4) 100

Consider Systematic Listing. Among the primary group, there were 99 attempts using this heuristic across the six problems for the pre-test (success rate was 53.6%) and 111 attempts for the post-test (success rate was 71.2%), but only 63 attempts for both tests (54.0%). These values show that the primary pupils did not use

Systematic Listing consistently and their success rates also varied. This inconsistency in heuristic use and different success rates are also observed in the other heuristics. For the primary group, the sizeable number of attempts using Equations was quite surprising, but the lack of success was expected. The secondary pupils also did not consistently use the heuristics to solve parallel problems, although the Logical Argument heuristic was most prevalent, followed by Equations.

Discussion and Conclusion

This paper has provided some evidence to show that upper primary and lower secondary pupils had different success in using five of the common heuristics to solve mathematics problems. This difference may be explained by the interaction between nature of the problems, the pupils' mastery of these heuristics, and the nature of the heuristics. The values in Table 2 indicate that for the types of problems given in this study, Systematic Listing and Guess and Check are quite successful for both groups of pupils. Mathematics teachers might wish to give more attention to these two heuristics. On the other hand, exposing pupils to the different heuristics using pupils' work (perhaps not taken from the same class) adds an authentic touch to promote metacognitive processes in problem solving. Partially correct solutions are particularly useful because the pupils can learn from the mistakes and find ways to improve on the solutions, hence deepening their knowledge of mathematics. The pupils' work can be collected and classified into meaningful codes as a joint effort between teachers and researchers, and this collection will be a useful resource for classroom teaching and teachers' professional development.

A sizeable number of the pupils did not use the same heuristic to solve parallel problems, and the reason for this needs to be examined further. A plausible clue, as indicated in the detailed analysis of the given pair of problems, is that there are subtle differences in the parallel problems that might hinder or facilitate the use of particular heuristics. Another reason might be that these pupils were taught certain heuristics in the months between the two test administrations. In the study, the teachers were not given a copy of the pre-test; hence, the pupils did not have the opportunity to reflect on their methods for the pre-test. This is an issue about research design, namely to reduce the learning effect and to be able to use some of the items for the post-test. However, in real teaching, it matters more to discuss the heuristics used soon after the pupils have attempted the problems because providing timely feedback is a critically effective pedagogical move.

Future research should not rely solely on the analysis of written work. The codes need to be validated through the use of complementary methods such as clinical interviews, error analysis, diagnostic tests, and observations and/or videotaping of pupils solving problems.

Acknowledgements

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Fractions as a Measure

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Simultaneous co-ordination of the referent unit, symbolic notation and pictorial representations are one aspect of understanding fractions as a measure. The notion that students must identify the referent unit, fractional parts of the referent unit, and apply it as an accurate method of measurement was investigated. Using different types of pictorial representations, students were required to represent the quantity being measured and conversely, identify the quantity represented. Four questions from a larger study that used fractions as a measure were examined. Nine students were interviewed to gain further insight into their thinking on these questions and their misconceptions identified.

Communication of common fraction ideas and principles relies on a precise system of symbols (inscriptions) and conventions that allows unambiguous and concise communication of mathematical ideas (Thompson & Saldanha, 2003). Fraction inscription comprises of a bipartite symbol in the form $\frac{a}{b}$, where a and b are whole numbers and $b \neq 0$ (Board of Studies NSW, 2002). Although the fraction symbol is devoid of physical meaning and context (Lamon, 2006) it has many interpretations including part/whole, quotient, measure, ratio and operator (Kieren, 1980). Although not independent, each interpretation allows rational numbers to be viewed from a different perspective thus providing a basis for developing an understanding of rational number (Kieren, 1980).

Fractions as a measure can be used to extend the whole number system (Hart, 1981). Fractional units are derived when the standard object or unit of measure is subdivided into smaller equal parts. These fractional units in combination with whole units of measure provide an accurate means of measuring (Skemp, 1986). A measurement task requires the "process of counting the number of whole units usable in 'covering' the region, then equally subdividing a unit to provide the appropriate fit" (Kieren, 1980, p. 236). The fraction inscription represents a quantity resulting from the construction of a reference unit from which situations are redefined in terms of that unit (Lamon, 1993; Olive & Steffe, 1980). The referent unit may be one of the International System of Units (SI) such as kilogram, metre and second or arbitrary unit such as a chocolate bar or pizza. Without the identification of the referent unit by the student, the successful application of fractions in a measure context is unlikely.

Reliable information about students' knowledge and understanding is crucial for teachers. It allows teachers to extend students' current level of understanding and address misconceptions students possess by focusing tasks and lessons on particular areas of concern (National Research Council [NRC], 2001). This paper explores students' understanding of the idea of fractions as a measure by examining the reasoning they employ to solve four fraction measure tasks. Student misconceptions in responding to these tasks were explored and implications for enhancing student learning provided.

Theoretical Perspective

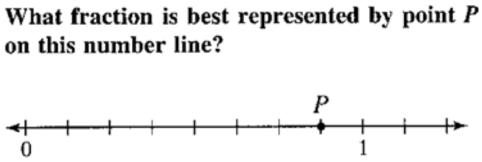
Mathematical models are mathematical representations used to convey, clarify, interpret and understand mathematical ideas (National Council of Teachers of Mathematics [NCTM], 2000). The term is quite nebulous, with descriptions such as representational models (Saxe et al., 2007), representations within contexts (Ball, 1993) or embodiments (Behr, Lesh, Post, & Silver, 1983) used to describe similar contexts. Number-lines and part-whole models are often used by teachers to convey fraction ideas (Ball, 1993; NCTM, 2000; Saxe et al., 2007). These different mathematical models illuminate different aspects of a mathematical concept (Ball, 1993; NCTM, 2000).

The use of mathematical models is supported by a representational system. This system comprises five interrelated elements: written symbols in the form of inscriptions and words, spoken language, hands on materials, pictures and real life situations (Behr et al., 1983). Explaining ideas and concepts often requires students to map different elements onto each other (NCTM, 2000). Words are employed as an intermediary to explain the link between mathematical symbols and mathematical models, typically illustrated using pictures and hands-on materials. Developing an understanding of fractions requires students to: (a) develop an understanding of the conventional models and representations used, and the ideas they capture (NCTM, 2000);

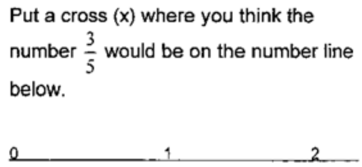
(b) comprehend the mappings across representations, and (c) understand the mappings within representations (Behr et al., 1983).

Influenced by their existing knowledge and experiences, students impart their meaning to teachers' representation of mathematical ideas. Students make sense of what they are taught by creating an internal, cognitive representation or mental model. These mental models provide a "workspace for problem solving and decision making" (Halford, 1993, p. 7). Based on tasks that elicit student thinking, student use of representational elements are the medium through which they demonstrate what they know and understand. Without examining students' work, interpretation of their understanding is impossible (Ball, 1993).

Number-lines are a mathematical model used to exemplify fractions as quantities and the measure interpretation. Conventional examples of number-line questions include Figure 1a, in which students are required to identify the quantity represented on a number-line or, locate a fraction on a number-line as in Figure 1b. Both examples require mapping across symbolic-pictorial representations. Students need to interpret the fraction symbol, recognise the properties of the number-line model, and how fractional quantities are represented.



(a) Identify the fraction (California Department of Education, 2008)

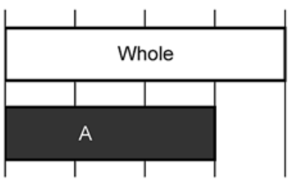


(b) Locate a fraction (Pearn & Stephens, 2005)

Figure 1. Fraction questions using a number-line representation.

Many students experience difficulties in locating fractions on number-lines. Number-lines are constructed with an arbitrary point chosen on a line as the origin. Points on the number line are subsequently labelled by their distance from the origin measured according to a referent unit chosen (NRC, 2001). When asked to place a proper fraction on a number-line, students often view the whole number-line, irrespective of its magnitude as a single unit instead of a scale (Ni, 2001). Placing proper fractions on a number-line labelled from 0 to 1 is also problematic. Fractions are often placed with disregard to any reference point or other known fractions. A lack of accuracy when dividing segments also results in the incorrect location of fractions (Pearn & Stephens, 2004).

Examples focusing on fractions as a unit of measure should not be constrained to number-line models. The idea of measure is closely related to the notion of part-of-the-whole using region and volume models (Kieren, 1980). Figure 2 shows variations of the region model (Watanabe, 2002), characterising mappings within pictorial representations. In the comparison representation (Figure 2a), the part and the referent unit are separate entities. The fraction quantity is constructed from the relationship between the explicit whole and the part to be measured (Watanabe, 2002). There is no confusion as to the referent whole. In contrast, the part-of-the-whole method (Figure 2b), the part is embedded within the whole and the referent unit (entire area) is implied in the diagram (Lamon, 2006). Like number-lines, the use of these representations requires a clear understanding of the referent unit (Ball, 1993; Kieren, 1980) and the peculiarities of each representation.



(a) Comparison representation for $\frac{3}{4}$

(b) Equivalent part-of-the-whole representation for $\frac{3}{4}$

Figure 2. Different pictorial methods of representing fractions using regions.

The fraction shaded represented in Figure 2b is typically considered $\frac{3}{4}$ when the implied referent whole is the entire area. Another view provides the answer $\frac{1}{3}$ represents the ratio of unshaded to shaded parts (Smith, 2002). The various referent units are cognitively different and operations with these units produce different results (Lamon, 2006). Understanding and interpreting the typical representations is one part of developing mathematical understanding.

Students often change their referent unit during problem solving. Olive and Steffe (1980) found that the surface features of pictures often confounded students. The following was observed of Karla a fifth grade student during a fraction lesson using computer software. "She had drawn two candy bars end to end… She partitioned each bar into six pieces… She now had a double candy bar consisting of 12 equal pieces. She issued the command F [6/12] and was surprised when only half of the first bar was filled (three of the 12 pieces)" (p. 61). In the process of constructing two single units which end to end appeared as one physical unit, Karla was not longer able determine the relationship between a fraction and its reference as her perceived referent unit changed.

Students develop an understanding of fractions as a measure when they are able simultaneously co-ordinate the referent unit, symbolic notation and pictorial representations of various fraction models. They are able to make connections between the different conceptual dimensions highlighted by the pictorial representations used to exemplify fractions as a measure (Ball, 1993; NRC, 2001). In this study, students' responses to two types of tasks were investigated. Using different types of pictorial representations of fraction models, students represented the quantity being measured and conversely, identified the quantity represented. By examining students' responses to these tasks, the types of reasoning students' employed, including: (a) the identification of the referent whole, (b) the perceived relationship between symbolic notation and pictorial representations, and (c) issues of interpretation and misconceptions, was explored.

Methodology

Participants

Six hundred and forty-six students in Years 3 to 6 from six co-educational Sydney primary schools participated in the larger study. Participant details appear in Table 1. Students who scored less than the 50th percentile in the Progressive Achievement Test in Mathematics (PATMaths) as compared to the norming sample (Australian Council for Educational Research [ACER], 2005) were identified as potential interview participants. After reviewing their fraction assessments, 45 participants were selected and interviewed from across all grades. Only nine interviews were reported in this paper. The suffix "G" or "B identifies the interviewee as a girl or boy.

Table 1

Details of Participants Completing Fraction Assessments and Interviews

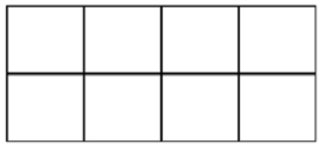
	Average	AFUv1		AFUv2		
Grade level	**Age (years)**	**(*n*)**	**Interview**	**S2 (*n*)**	**S3 (*n*)**	**Interview**
3	8.65	68		104		521G 694B
4	9.72	93		82		
5	10.71	102	358G		84	663G 808G 813G 820B
6	11.61	34			79	625G 781G

Instruments

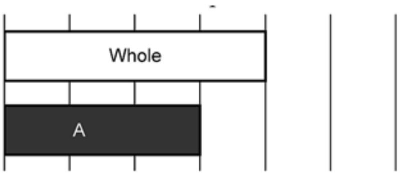
The questions for the Assessment of Fraction Understanding (AFUv1, AFUv2S2 and AFUv2S3) were derived and adapted from various assessment instruments including: California Standards Test (California Department of Education, 2008), and Success in Numeracy Education program (Pearn & Stephens, 2005). AFUv1 was the pilot instrument. It was modified after pilot data analysis, which resulted in the creation of two versions: AFUv2S2 (grades 3 and 4), and AFUv2S3 (grades 5 and 6). Twenty common items were used to link all three assessments. Figure 3 shows the four measure questions analysed from the assessments.

10b. Put a cross (X) where the ½ would be on the numberline below.

14. In the rectangle, shade in enough small squares so that $\frac{3}{4}$ of the rectangle are shaded.



17. The white shape shows a whole bar and another bar is placed below it.bar is placed below it.



What fraction of the whole bar is A? What fraction of the whole bar is A?

26  This rectangle represents one whole.

(a) What do the following represent altogether?

(b) Can you think of another name for the shaded fraction?

Figure 3. Four measure questions analysed.

Question details including representation, type of referent unit exhibited, mapping (s->p: represent quantity being measured given the fraction quantity/symbol; p->s: identify the quantity being measured), and the number of participants completing each question appear in Table 2. Question 10b was completed by participants that were administered AFUv2. A multiple choice version of the appeared in AFUv1 but was not analysed in this paper. Question 14 employed an equivalent, part-of-the-whole representation in which the number of small squares was a multiplicative factor greater than the denominator (Ni, 2001). The referent unit displayed in question 26 comprised of a subdivided referent unit.

Table 2

Question Details

	Representation			Features		AFU	
Question	**Num.-line**	**Comparison**	**P-of-W**	**Ref. unit**	**Mapping**	**v1(*n*=297)**	**v2 (*n*=349)**
10b	X			explicit	s -> p		X
14			X	explicit	s -> p	X	X
17		X		implied	p -> s	X	X
26		X	X	explicit	p -> s	X	X

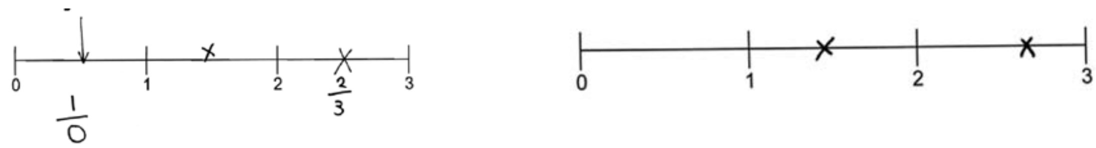
Procedure

All participants were administered a version of the fraction assessment and the third edition of the PATMaths (ACER, 2005), following standardised protocols. Each pencil and paper test was of 45 minutes duration. Calculators were not permitted and participants were asked to show all working in their test booklets and to attempt all questions. Interviews were semi structured with questions drawn from the fraction assessments. The researcher interviewed all participants individually. Participants were provided with concrete materials such as blocks and counters as well as pencils, ruler and paper. The interviews were videotaped to allow further analysis.

Results

The correct and common incorrect responses and percentages are reported for each question along with relevant interview data.

Question 10b: Eighteen percent of assessment participants completed the question correctly, with 57% placing a cross at half way between 1 and 2. Interview data confirmed that one reason for response was the dividing the entire number-line in half. But student 781 used the numerator and denominator as cues for placing the cross between 1 and 2, see Figure 4a. When asked to identify the cross placed between 2 and 3, the fraction $\frac{2}{3}$ was written below it. A similar procedure was employed when asked to identify the fraction shown by the arrow.



(a) Student 781's attempts at identifying fractions *(b)* Student 813's response to locating $\frac{2}{3}$

Figure 4. Student examples of placing fractions on a number-line.

Variations to student 781's strategy were observed. After placing the cross between 1 and 2, student 813 was asked to place $\frac{2}{3}$ on the number-line (Figure 4b). Explaining her method she said, "I'll look at the top number [points to the 2 in 2/3] See how it has the word two so I look… [moves finger along the number-line until she reaches 2] found 2 and look at the 3 as well. So I have to think alright if half is here [points to half way between 2 and 3] so the cross should be a little more higher [to the right of 2 1/2]". Not only did the numbers within the fraction provide cues for the location of the fraction, the fraction quantity assisted in positioning the fraction in the section identified.

Question 14: Forty-six percent of assessment participants answered the question correctly shading six squares, whilst 32% shaded three. Interviews were conducted to determine reasoning used to generate the response of 3. Student 358 placed counters in three squares of the top line of the rectangle. When asked to identify the remaining quarter, she pointed to the blank square on the top line. When asked how many squares make 1 whole, she replied, "um 4". Prior to circling her whole, she asked, "Like even the ones that are blank?" and drew a circle around the top 4 squares. Yet when asked after circling the whole, how many wholes were in the picture, she said, "um, 8 altogether". Confusion about how to interpret the picture, its relationship to a fraction and the referent unit was evident.

Some participants focused on the fraction numerator. Student 521 shaded three squares, but was confused when she looked at the shading and stated, "isn't it three eighths?". When probed whether she would have to shade some more for three quarters, she responded, "Three quarters oh. You do this in quarters [divides the last square shaded is divided into eighths]. Then you shade 1 then that 2 [points to first and second small square] and three quarters [points to the last small square divided into eighths]". Student 694, who also shaded three small squares when probed to examine the number of small squares said, "Ah there's 8. Oh no, it's not three". Using a rectangle divided into quarters (same size as the one divided into eighths), the student was able to shade three quarters of the whole rectangle. When comparing the area shaded in both rectangles, he realised another three small squares needed to be shaded. Three quarters represented using an equivalent pictorial representation was not a mental model student 694 possessed.

Both the numerator and denominator provided cues for student 813, "So I think if I colour three pieces there should be at least 4 left over. But if I colour 3 pieces there'll 5 left over. So I think I should colour in 4 and there should be 4 left over that's a half". When asked whether three quarters was more than a half, she responded, "um, I think no, don't know". Responses from these students suggest their lack of attention to the whole shape but a focus on the fraction symbol.

Question 17: Forty-five percent of participants answered the question correctly, whilst nine percent did not respond. Interview data suggests that two methods were employed to solve the question correctly. Firstly participants recognised the whole and it was divided into four parts. This was followed by: (a) recognising bar

A comprised 3 parts, hence three quarters of the whole, or (b) identifying the difference in size as one quarter missing, hence the shaded part was three quarters.

Student 813 gave the incorrect response of 3, explaining, "I think A is at least [counts number of lines that A crosses] um, 3 bars, like 3 wholes". When 813 was reminded of the size of the whole bar, the following response was given, "Like three thirds. Yep, because here's one missing [points to missing part of A]". This student recognised the significance of the three pieces but did not equate a piece to one quarter or make reference back to the whole. Another variation on this method was exhibited by student 781 who explained, "if you if you count the lines 1 2 3 4 [points to vertical dividing lines along length of A] it stops there. And that counts to 5 [circles the whole bar] the whole bar", hence the answer $\frac{4}{5}$. Reasoning was in part dominated by the surface features of the representation.

Question 26: Thirty-nine percent of participants completed the question correctly by giving the answer $1\frac{3}{4}$ or $\frac{7}{4}$, whilst 15% did not respond. Interview participants who answered the question correctly were able to keep in mind the representation of a whole. Student 781 stated, "Because there is one full shaded [pointing to first rectangle] and that's not full shaded [pointing to second] and it is only three shaded and four there". Identifying the fraction of each part of the referent whole is also necessary for this question.

The response of $\frac{7}{8}$ was provided by 12% of participants. During the interview, student 820 counted the number of parts shaded and altogether in both rectangles by ignoring the referent whole. Some interviewees counted the number of parts shaded (i.e., 7). In some instances the number of shaded parts in each rectangle was used to form a fraction. Student 663 explained, "Four over three… Because there is four [points to the first rectangle] shaded only three [points to the second] there". Using similar reasoning but ordering the rectangles differently, student 808 offered the answer $\frac{3}{4}$.

Discussion

Students' understanding of fractions as a measure is enhanced when they are able to co-ordinate simultaneously the referent unit, symbolic notation and pictorial representations. The interview data suggest that a number of issues impeded students' understanding of fractions as a measure. They generally concentrated on deciphering in isolation the pictorial representations or interpreting fraction notation. Students with some understanding were able to work with the symbolic notation and pictorial representation together, yet were unable to reconcile any discrepancies they encountered.

The interpretation of conventional pictorial representations of number-lines and area models posed a problem for some students. They focused on surface features (e.g., counting the number of parts or number of lines) to identify the fraction quantity. Although the referent unit was explicitly defined in Question 26, the most frequent incorrect response represented the shaded area of the combined rectangles. This was consistent with the findings of Olive and Steffe (1980) and further suggests that the notion of the referent unit is not a key feature of students' understanding of fractions.

Other students were unable to establish the relationship between the fraction symbol and pictorial representation. The numerator and denominator were used as cues for representing the fraction quantity as exemplified by the responses to Question 14. Although some students realised the discrepancy in their representation with the fraction quantity, guidance was required to re-interpret the equivalent part-of-the-whole representation.

The questions and the misconceptions they highlight provide teachers and researchers with a method of assessing students' understanding of the measure interpretation of fractions. This analysis, although limited by the type, number and content of questions selected, provides some suggestions for designing instruction. Firstly, mathematics is based on a system of conventions and symbols. Without a familiarity of the pictorial representations and an understanding of conventions of interpretation, communication of fraction ideas may be compromised. For example, students require a deep understanding of the number-line as a scale prior to its use as a mathematical model applied in the fraction context. Secondly, the notion of the referent unit both implicit and explicit needs consideration. Initial fraction instruction that only addresses proper fractions limits students' exposure to fractions as a measure. Realistic examples should be extended to cakes, pies and pizzas thus providing an avenue to understanding measurement in whole and part units. Finally, fraction symbol notation is one component of understanding fractions. It needs to be carefully linked to other representational elements and mathematical models for students to develop a deep understanding.

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Mixing Colours: An ICT Tool Based on a Semiotic Framework for Mathematical Meaning-Making about Ratio and Fractions

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This paper reports on the research and development of an ICT tool to facilitate the learning of ratio and fractions by adult prisoners. The design of the ICT tool was informed by a semiotic framework for mathematical meaning-making. The ICT tool thus employed multiple semiotic resources including topological, typological, and social-actional resources. The results showed that individual semiotic resource could only represent part of the mathematical concept, while at the same time it might signify something else to create a misconception. When multiple semiotic resources were utilised the mathematical ideas could be better learnt.

Much research has recognised and been advocating the importance of multiple knowledge representations in mathematics education (Adiguzel, Akpinar, & Association for Educational Communications and Technology, 2004; Ainsworth, Bibby, & Wood, 2002; Alagic & Palenz, 2006; Kendal & Stacey, 2003; Patterson & Norwood, 2004; Porzio, 1999; Reed & Jazo, 2002; Schuyten & Dekeyser, 2007; Siegler & Opfer, 2003). ICTs such as computers and graphic calculators have been widely utilised to provide multiple knowledge representations of mathematical concepts. However, among the use of multiple knowledge representations (e.g., text, numbers, icons, objects, graphs, and animations etc.), there has often been a lack of a theoretical framework to inform the use of different representations and to explain the effects these representations have upon the learning of mathematics. Therefore, a semiotic framework for mathematical meaning-making (Lemke, 2001; Yeh & Nason, 2004a) was adopted in this study.

The Semiotic Framework

Representations of mathematical concepts have been classified differently in different contexts. For example, in learning algebra and computer algebra system (CAS) context, representations have been classified into three categories: numerical, graphical, and symbolic (Kendal & Stacey, 2003). Moreno (2002) classified representations into visual, verbal, and symbolic for using interactive multimedia in learning addition and subtraction. For teaching number in early childhood, Payne and Rathmell (1975) developed a teaching model that classified representations into object, language, and symbol. Papert (1993) also categorised knowledge representations into action, image, and symbol. These different types of representations are all essential and central to the learning of mathematics. However, there are other representations such as colour, sound, and animations etc. that could be good representations for certain mathematical ideas yet to be classified and utilised. In searching for a unified and inclusive theoretical framework for knowledge representations, the researchers have developed a semiotic framework.

Originated from the study of linguistics, modern semiotics has evolved to view everything as a "language", where the "language" does not only refer to spoken or written language, but also include dance, gestures, fashion, rituals of primitive tribes, music, sculptures, visual pictorial imageries, dreams, and so forth (Oliveira & Baranauskas, 2000). Semiotics is generally regarded as the study of signs and sign-using behaviour. It is also perceived as a single unified system for meaning-making.

Peirce (1839-1914) developed a semiotic triad consisting of three constructs: sign, object, and interpretant (Figure 1), and argued that all cognition is irreducibly triad. Cunningham (1992) elaborated that a sign mediates between the object and its interpretant. The sign, however, is not the object itself. He pointed out that a sign is only an incomplete representation of the object. A sign can only represent certain aspects of the object and in addition, it has aspects that are not relevant to the object. Therefore, a sign has a certain meaning emission capacity, and will be interpreted differently by different individuals (interpretant).

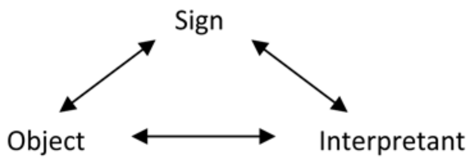

Figure 1. Semiotic triad.

Lemke (2001) made an explicit classification of mathematical signs (representations) into typological and topological resources. *Typological resources* are those which *convey meanings by their distinguishable type*. They are discrete, qualitatively distinctive, and are but not limited to natural languages and symbols. *Topological resources* on the other hand, *convey meanings by their variations in degrees*. They are continuous, quantitatively different, and are but not limited to visual graphics representations. Lemke (2001) stated that in general, mathematical expressions are constructed by typological systems of signs, but the values of mathematical expressions can in general vary by degree within the topology of the real number. He pointed out that students often have a great deal of trouble typically in understanding functional notation (typology) and its meaning in terms of quantitative co-variation (topology). However, they can be greatly aided by employing the topological strategies such as graphs and other visual representations. For example, when asked to order the size or ratio of a given set of fractions 13/19, 11/17, 4/6, 9/13, there is no simple way to tell from these typological representations except by performing the divisions and change to decimal forms. But their relationships can be easily understood if a graph or diagram of those ratios is visually presented.

Lemke (2001) continued to add insights into his typological and topological dichotomy. He argued that mathematics is also a system of related social practices; a system of ways of doing things. He then reworded the dichotomy into *actional-typological* and *actional-topological* resources for mathematical meaning-making. The "actional" modifier imposed a strong and powerful message linking mathematics to social discourse and real-world, authentic applications. Yeh and Nason (2004a) employed Lemke's semiotic framework to inform their design of a computer 3D microworld named VRMath (Yeh & Nason, 2004b) for learning 3D geometry. They found that computer environments are rich in providing these actional-typological and actional-topological resources. Moreover, they identified that the social-actional component of semiotic resources played a critical role in offering opportunities (e.g., negotiation) and legitimacy (i.e., the purpose of learning) in the mathematical meaning-making processes. Therefore, they suggested that a semiotic framework informed ICT learning environment should provide three types of semiotic resource namely: ***typological***, ***topological***, and ***social-actional*** resources.

The Context

Many adult learners in prisons do not have numeracy levels necessary to gain access to many jobs and vocational education programs (Australian National Training Authority, 2001). To assist with the rehabilitation and increase future life skills of adult inmates, the Enhancing Numeracy In Prisons Project (ENIPP) was established. The overall goal of this project was to develop and evaluate an integrated numeracy program utilising latest advances in (1) Critical Numeracy, Mathematics Education, Indigenous Education, and Information and Communication Technologies in Education (ICTE) theory and research, and (2) Information and Communication Technologies (ICTs).

The participants in this research study were a cohort of inmates enrolled in the education program at a male's correctional centre in eastern Australia. These adult inmates were identified to have some basic number facts skills but very little knowledge and understanding about ratio and fractions (National Reporting System Numeracy Level 1-2). In a pre-interview with the participants, the researchers noted that some participants were involved in the vocational training workshops where they were mixing paints. One particular inmate also spoke about his experience in mixing paints for painting his car. This gave the researchers an initial idea of a social-actional and a topological semiotic resource (i.e., colours) that could be designed and utilised for learning ratio and fractions.

Two and four computers were provided in the two classrooms. These adult learners were able to access these computers during class sessions. There was no networking between computers. During teaching and learning sessions, two to three participants were asked to work together on one computer.

Mixing Colours: The ICT Tool

Informed by the semiotic framework, the researcher designed the mixing colours software (Figure 2), which enabled the users to mix five primary colours (i.e., white, red, yellow, blue, and black) in different ratios for a variety of colours.

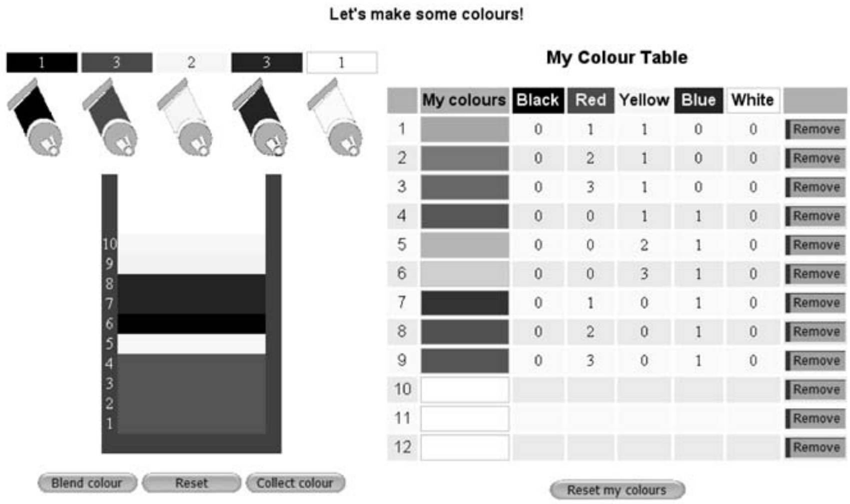

Figure 2. The Mixing Colours software.

Consistent with the semiotic framework, this ICT tool has employed topological, typological, and social-actional resources. For topological resources, the colour spectrum is utilised as the main topological resource. The gradient of colours has the meaning capacity about ratio. For example, the more reddish in the orange series colours means a bigger proportion of red colour in the mix. The container can also be seen as another topological resource because it varies in size to represent the whole. Typological resources include the colour paste icons, texts, numbers, and the colour table. Social-actional resources include the actions of Blend colour, Collect colour, Reset colour, and Remove colour. In regard to the social aspect, a task to mix paints for painting the interior walls of a house was designed to be explored with this ICT tool. Authentic materials such as the colour wheel, colour catalogue and brochure from paint shops were provided with some worksheets.

The Results

The teaching and learning sessions began with the introduction of the task: painting the interior walls of a house. A rainbow colour wheel (Figure 3) was presented to the participants and three colouring schemes were introduced:

1. A monochromatic scheme: Based on one colour, but in many shades and different hues, from dark to light.
2. An analogous scheme: Using the adjacent colours on the colour wheel.
3. A complementary scheme: Using the colours opposite each other on the colour wheel.

Figure 3. The colour wheel.

After the participants had chosen their colour scheme, the colour catalogues consisting of a variety of colours for different colour schemes were given for the participants to practice the colour mixing in the ICT tool. Most participants were very much motivated in this colour mixing activity. They spent a few hours to play colour mixing with adding, blending, collecting, and resetting actions. Some of the participants actually learnt how to derive the secondary colours such as mixing yellow and blue to get green. Few participants didn't bother to remember the formula for secondary colours but kept using trial and error to match the colours in the catalogues. For monochromatic scheme colours, most participants were able to utilise black or white to change the darkness or brightness. During the activity, proportional language such as "more red" and "more blue" was often heard from the participants as they instructed and suggested to each other. In many cases, this ICT tool was unable to create a perfect match of colour to the catalogue colour. The participants (of the same computer) had to negotiate to agree with the closest colour, and then record the colour formula (i.e., the ratio of each primary colour) to the colour table or on a worksheet. It was noted that in this initial activity, most participants' attention was focused on the topological sign of colour variations. It seemed that the participants were able to make sense of different ratios from the colour variations. It was not until the next session that interference of individual signs was revealed.

1:1≠2:2, The Interference of Topological Sign

In the next session, Rob (pseudonym) was proud to show his workbook to the researchers for he had created and recorded 100 different colours. The researchers commended his work and then invited him to recreate some of the results in the ICT tool. Rob went on and recreated those results on screen (Figure 4) as requested by the researchers.

My Colour Table

	My colours	Black	Red	Yellow	Blue	White	
1		0	1	1	0	0	Remove
2		0	2	1	0	0	Remove
3		0	2	2	0	0	Remove
4		0	1	1	0	1	Remove

Figure 4. Colour table recreated.

The researcher asked "Are colour 1 and colour 3 the same?" Rob looked at the screen and replied firmly "No, they are different colours". The researcher then suggested removing the colour 2 from the colour table so colour 1 and colour 3 sat next to each other. When colour 2 was removed, the researcher asked again "Are the two colours the same?" Rob looked and thought for a while, and then he said "They are different colours". The researcher then identified another similar ratio with three colours as 1:1:1 and 2:2:2 in Rob's workbook, and asked Rob to recreate colours again. However, even the two colours were sitting next to each other; Rob still claimed that they looked different. The researcher then asked Rob to look at the numbers showing 1:1 and 2:2, and pointed out that the ratios are the same. Despite the effort to explain, Rob still insisted that the colours looked different to him.

The Mysterious 4th, the Interference of Typological Sign

Luke, Ben, and Cam (pseudonyms) were working together to mix colours for the kitchen walls. They matched a few colours from the colour catalogue and collected those colours to the colour table (Figure 5).

My Colour Table

	My colours	Black	Red	Yellow	Blue	White	
1		0	1	2	2	2	Remove
2		0	1	4	3	3	Remove
3		0	1	3	3	3	Remove
4							Remove

Figure 5. Colours for kitchen.

The researchers questioned "What portion is red colour in colour 1? and how about the other colours? Can you write down the fractions of individual colours in colour 1?" Luke was confident. He wrote down quickly the fractions on a paper as:

	Red	Yellow	Blue	White
Colour 1	1/4	2/4	2/4	2/4

Surprised, the researchers then asked Luke to also write down the fractions for colour 2 and 3. Luke continued to write down fractions as:

Colour 2	1/4	4/4	3/4	3/4
Colour 3	1/4	3/4	3/4	3/4

Ben and Cam seemed to agree with what Luke wrote: all the denominators are four. The researchers soon realised that the whole had been lost. Instead, the whole has been linked to the number of the primary colour used. Because there were four primary colours used (i.e., red, yellow, blue, and white), it gave a wrong sign for the denominator. The researchers then pointed out the container with the number showing the whole of the mix. The three participants were then able to recognise the whole and make sense of the idea about fractions.

Discussion

The two episodes above confirmed and illustrated the incomplete nature of signs to represent the object (mathematical concepts). Depending on different interpretants, a sign may also carry meanings that are not relevant to the object. This applies to both topological and typological signs. Even the social-actional resources, whilst they can reinforce correct ideas to facilitate knowledge building, they might also reinforce incorrect ideas.

Notably, topological resources tend to attract the attention of learners first. Topological resources allow a certain degree of error or uncertainty. In typological resources, accuracy must be met, which could be a source of being perceived as cold, abstract and difficult. However, over-reliance on topological resources can also impede the development of deep understanding of mathematical structures. This is evident in Rob's example, in which he strongly believed in his vision and totally ignored the meaning of the ratio notation.

The importance of multiple representations or the use of multiple semiotic resources is also confirmed in this study. The results showed that when multiple semiotic resources were utilised, the mathematical ideas or concepts could be better learnt. The future design of this ICT tool should also focus on how to minimise the interference of signs, and maximise the meaning emission capacity of signs.

Conclusion

This study has presented a semiotic account of knowledge representations and mathematics as a meaning-making endeavour. This paper will now conclude with two important implications derived from this study:

1. Viewing mathematical knowledge representations from a semiotic perspective is more inclusive than other existing classifications of representations in the research literature. The topological (meaning by degree) and typological (meaning by kind) resources have encompassed most material processes. And with the social-actional resources linking history, culture, and real-world applications, learning mathematics would be most meaningful to learners.
2. The semiotic framework is essential for the design of ICT tools for learning mathematics. ICT tools that are informed by the semiotic framework will seek out to utilise any possible meaning-making resources across typological, topological and social-actional resources. This will also lead to more creative and innovative design of ICT tools.

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Secondary School Students Investigating Mathematics

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This paper describes a research study to find out the ability of Singapore secondary school students in attempting open investigative tasks. The results show that most high-ability students had no experience in open mathematical investigation and they did not even know how to start. Providing sample problems in the tasks for students to investigate did not seem to help them understand the requirements of the tasks. The implication of these findings on research methodology using paper-and-pencil tests will be discussed.

Many school mathematics curricula emphasise the use of mathematical investigation in the teaching and learning of mathematics. For example, the Australian national curriculum states that, "Mathematical investigations can help students to develop mathematical concepts and can also provide them with experience of some of the processes through which mathematical ideas are generated and tested" (Australian Education Council, 1991, p. 14) and the Cockcroft Report in the United Kingdom stipulates that "mathematics teaching at all levels should include opportunities for … investigational work" (Cockcroft, 1982, p. 71). In New Zealand, their Mathematics in the New Zealand Curriculum (Ministry of Education of New Zealand, 1992) also stresses the importance of mathematical investigation.

So what exactly is mathematical investigation and why do many school mathematics curricula place great emphasis on it? There is a long-standing debate about the differences between problem solving and investigation but Pirie (1987) alleged that no fruitful service would be performed by indulging in the 'investigation' versus 'problem solving' dispute. However, Frobisher (1994) believed that this was a crucial issue which would affect how and what educators teach their students. Evans (1987) claimed that Cockcroft (1982) made a distinction although the latter did not say what it was. Some educators (e.g., Orton & Frobisher, 1996) perpetuated the idea that an investigative task must be open, i.e., it should not specify any problem for the students to solve or investigate; students must pose their own problems and thus mathematical investigation involves both problem posing and problem solving. For example, Bastow, Hughes, Kissane and Mortlock (1991) defined mathematical investigation as the "systematic exploration of *open* [emphasis mine] situations that have mathematical features" (p. 1) while Ernest (1991) described problem solving as "trail-blazing to a desired location" (p. 285) and investigation as the exploration of an unknown land where "the journey, not the destination, is the goal" (Pirie, 1987, p. 2). So problem solving is a convergent activity with a well-defined goal and answer while investigation is a divergent activity with an open goal and answer (Evans, 1987).

The use of open investigative tasks helps students to focus on "the process of problem solving and the open-endedness of a problem or investigation" (Hawera, 2006, p. 286) because many genuine problems are open and ill-structured in nature. In real life, no one will tell you exactly what the problem is and what the boundaries of the problem are (Simon, 1973). You have to find the root problem first before you can resolve the issue. Similarly, in mathematical investigation, the task will not tell the students what the problems are. The students will have to think through and pose their own problems to solve or investigate. This helps the students to be more aware of the problem situation and to take charge of the issue at hand. Many educators (e.g., Brenner & Moschkovich, 2002) also favour bringing academic mathematicians' practices into the classroom and this includes letting students engage in a variety of rich mathematical activities that parallel what academic mathematicians do. And what do mathematicians do? They investigate and solve mathematical problems (Civil, 2002). Lampert (1990) believed that such activities encourage students to think mathematically, such as problem posing (Brown & Walter, 2005), conjecturing and generalising (Calder, Brown, Hanley, & Darby, 2006).

In this paper, I will describe a research study on open mathematical investigation performed by some high-ability secondary school students in Singapore. It will begin with the background of the study, followed by the research methodology and findings, and it will end with a discussion of the data collected and some implications for both teaching and research.

Background

The central theme of the Singapore mathematics curriculum is mathematical problem solving (Ministry of Education of Singapore, 1990) and most mathematics teachers in Singapore are familiar with solving mathematical problems. Although the curriculum specifies that "a problem covers a wide range of situations from routine mathematical problems to problems in unfamiliar context and open-ended investigations" (Ministry of Education of Singapore, 2000, p. 10), many teachers are not sure what mathematical investigation involves. Whenever I mention the term 'mathematical investigation', quite a number of teachers will look at me blankly and ask, "What's that?" Some teachers have this vague idea that mathematical investigation has something to do with guided-discovery learning, but, according to Ernest (1991), there are some major differences. Very few teachers actually know what open investigative tasks are, and when faced with such a task, most, if not all, of them do not know what to do (this was gathered from courses conducted by me for teachers). If most teachers are not familiar with open mathematical investigation, then it is unlikely that they will teach their students how to deal with open investigative tasks.

As explained in the previous section, learning how to investigate using open investigative tasks is very important to cultivate important mathematical processes. So there is a need to find out the current state of competency in open mathematical investigation among students in Singapore, and to develop a teaching programme to help students learn how to deal with open investigative tasks. This paper reports a research study to find out the current state of proficiency in open mathematical investigation among high-ability secondary school students and is part of a larger study that researches on the nature and development of thinking processes when students with a wide range of mathematical abilities attempt open investigative tasks.

Methodology

The sample consisted of 29 Secondary One students from an intact class. This class of high-ability students was selected randomly from one of the top schools in Singapore. From the written survey conducted at the end of the written test, all of these students said that they had not seen this kind of open investigative tasks before. The paper-and-pencil test consisted of four open investigative tasks to be completed individually within one hour and thirty minutes. Because of anecdotal evidence that many teachers and students might not know how to begin when faced with such open tasks (see previous section), the first task included sample problems for students to investigate if they did not know how to pose their own problems (see below). Subsequent tasks were completely open with no hints or guidance. The topic for the tasks was arithmetic because Secondary One students were most familiar with arithmetic: If another topic, such as algebra, was chosen, then if the students did not know what to do, it might be because they were unfamiliar with the topic rather than with the investigation itself. The test was administered by the author himself.

Mathematical Investigative Task 1: Powers of a Number

9^5 means 9 multiplied by itself 5 times, i.e., $9^5 = 9 \times 9 \times 9 \times 9 \times 9 = 59\,049$.

Powers of 9 are 9^1, 9^2, 9^3, 9^4, 9^5, 9^6, … etc.

Investigate the powers of 9.

For example, you can investigate the following or you can *pose your own problems to investigate*:

a) Find as many patterns as you can about the powers of 9.

b) Explain why these patterns occur.

c) Do these patterns occur for powers of other numbers?

Findings

Within five minutes from the start of the written test, five students raised their hand and asked me what they were supposed to do for the first investigative task. Some students did not know what to do but they did not ask me. The following are what some of these students (all names are pseudonyms) wrote in their answer scripts or in the written survey.

Albert: I am thinking of asking the teacher what 'investigate' means.

Ben: I find it a bit difficult as I do not understand the meaning of investigate.

In order to analyse how and what the students investigated, two rubrics were developed. Table 1 shows the first rubric which describes *how* students investigated the first task. There are five levels. In Level 1H0 (the first number refers to the task number and the letter H stands for How), the students did not do anything. In Level 1H1, the students wrote something superficial or irrelevant that did not contribute anything to the investigation. I was surprised that some students just concluded, "The pattern is 9^1, 9^2, 9^3, 9^4, 9^5 which will continue." These students did not find the numerical values of the powers, thus suggesting that they did not know how to investigate by examining specific cases. In Level 1H2, the students did list out the numerical values but they did not do anything after that. In Level 1H3, the students tried to find some patterns but there was nothing constructive. In Level 1H4, the students did find some patterns.

The analysis shows that about 35% of the students did not know how to investigate by examining empirical data (Levels 1H0 and 1H1). About 17% of the students did try something but were unable to discover a single pattern (Levels 1H2 and 1H3), not even the simplest patterns such as all the powers of 9 are odd or the last digit alternates between 1 and 9. This suggests that at least half of the students (or 52%) did not know how to investigate. Although the other 48% of the students did find some patterns, many of the discoveries were very trivial and this will be discussed next.

Table 1

A Rubric to Describe How Students Investigated Task 1

Level	Descriptor	No. of Students	Percentage
1H4	Students listed out the numerical values of the powers and found some patterns (quality of discovery discussed in next rubric).	14	48.3%
1H3	Students listed out the numerical values of the powers and tried to observe some patterns but no findings.	3	10.3%
1H2	Students listed out the numerical values of the powers and then did nothing.	2	6.9%
1H1	Students wrote superficial or irrelevant things that did not help in the investigation, and in particular, they did not find the numerical values of the powers but wrote things like: $9^6 = 9 \times 9 \times 9 \times 9 \times 9 \times 9$ $9^1, 9^2, 9^3, 9^4, 9^5$.	4	13.8%
1H0	Students did not do anything.	6	20.7%
	Total	**29**	**100%**

Table 2 shows the second rubric which describes *what* students investigated in the first task. There are five levels. Some of these levels are the same as the levels in the first rubric because there are overlaps: It is difficult to separate how the students investigate from what they investigate. In Level 1W0 (the letter W stands for What), the students did not do anything (the same as Level 1H0). In Level 1W1, the students did something but did not discover anything. This should be the same as the combined levels of 1H1 to 1H3 except for one student who wrote nothing constructive (i.e., Level 1H1) but conjectured some trivial patterns (i.e., Level 1W2). In Level 1W2, the students discovered or conjectured trivial patterns (which might be wrong) such as powers of 9 are divisible by 9. In Level 1W3, the students discovered or conjectured non-trivial patterns (which might be wrong) such as the last digit of the powers of 9 repeats itself after 2 times and the sum of all the digits of 9 is divisible by 9. In Level 1W4, the students discovered or conjectured complicated patterns (which might be wrong) such as the last two digits of the powers of 9 repeat themselves after 10 times. Why wrong conjectures were included in Levels 1W2 to 1W4 was because the process of formulating conjectures was also important, even if the conjectures turned out to be false. If a student discovered more than one pattern, then the level of attainment was the level of the more complicated pattern. It is important to note that *not* all the patterns described in the rubric were discovered by the students.

From the second rubric in Table 2, it was observed that 20 of the 29 students (or 69%) did not know what to investigate (Levels 1W0 to 1W2). Only nine students (or 31%) were able to find some significant patterns although some of these conjectures were false (Levels 1W3 to 1W4). In what follows, I will report some interesting pieces of information on what the non-trivial patterns or hypotheses that these nine students had found or formulated.

Table 2

A Rubric to Describe What Students Investigated for Task 1

Level	Descriptor	No. of Students	Percentage
1W4	Students discovered or conjectured more complicated patterns, which might be wrong, for example: The last two digits of the powers of 9 repeat themselves after 10 times. The last two digits of the powers of other single-digit numbers also repeats itself but after different numbers of times. When multiplying two powers of 9, the indices are added together to give the index of the resulting power of 9 (law of indices).	1	3.4%
1W3	Students discovered or conjectured non-trivial patterns, which might be wrong, for example: Powers of 9 are odd. The last digit of the powers of 9 repeats itself after 2 times. The last digit of the powers of other single-digit numbers also repeats itself but after different numbers of times. The sum of all the digits of 9 is divisible by 9. Powers of 9 are divisible by factors of 9. Odd powers of 9 contain at least one digit 9 and even powers of 9 contain no digit 9 (the latter is false since 9^{12} contains one digit 9). 9^n has n digits (which is false since 9^{22} has 21 digits).	8	27.6%
1W2	Students discovered or conjectured trivial patterns, which might be wrong, for example: Powers of 9 are divisible by 9 or are multiples of 9. When a power of 9 is divided by 9, the result is the preceding power of 9. When the index of a power of 9 is increased by 1, the new number is the same as multiplying the original power of 9 by 9.	6	20.7%
1W1	Students did some investigation but did not discover anything.	8	27.6%
1W0	Students did not do anything.	6	20.7%
	Total	**29**	**100%**

Out of these nine students (or 31%), only four of them (or 14% of the total sample) discovered that the last digit of 9 repeats itself every 2 times, and only two of them (or 7% of the total sample) examined the last digit of other powers. One of these two students, Charles, investigated the powers of 6 and the powers of 2 and concluded that the powers of different numbers have their own patterns although he did not identify the patterns. The other one, Daniel, discovered that the powers of 4 end with 4 but he made no conclusion when discussing about the powers of 5. Instead, he concluded that the pattern for the powers of 9 did not apply to the powers of 4 and the powers of 5! It seems that Daniel interpreted the suggested problem given at the end of the task statement, "Do these patterns occur for powers of other numbers?" to mean whether the exact pattern for the last digit of the powers of 9 (i.e., the repeating pattern of 1 and 9) applies to the powers of other numbers! He was unable to see beyond this specific pattern of 1 and 9 to the general pattern that the last digit of the powers of any number repeats itself. None of the students discovered the pattern in the last two digits (see Level 1W4 in Table 2).

Three of the nine students (or 10% of the total sample), Eden, Frank, and Gilbert, discovered that the sum of the digits of the powers of 9 is divisible by 9. This was rather unexpected because if the powers of 2, 4, 7, and 8 (with a period of 4 for the repeating pattern) were used in Task 1 instead of the powers of 9, then there would not have been a pattern in the sum of the digits. Eden shed some lights as to why he investigated the sum of all the digits:

> Eden: As I have read a book on Maths, I remembered that the sum of the digits in each number adds up to a multiple of 9.

Another interesting observation was made by Frank who concluded that the odd powers of 9 contain at least one digit 9 but the even powers of 9 do not contain any digit 9. Unfortunately, he stopped at the numerical value of 9^{10} because the next even power of 9, 9^{12}, contains a digit 9. It was evident that Frank did not know that the observed pattern was only a conjecture and that it could not be proven just by looking at a few empirical data.

Daniel also made the same mistake of jumping to conclusion too early. He wrote that any power of 9 had one digit more than the preceding power of 9 but he stopped at 9^{10}. If he was to observe carefully from his systematic listing of the numerical values of the powers of 9 from 9^1 to 9^{10}, he would have discovered that the first digit of the powers of 9 had decreased slowly from 9 to 3 because the next power was obtained by multiplying the preceding power by a number that was less than 10. When this list continued, the first digit would have decreased to 0 eventually, meaning that this power of 9 would have the same number of digits as the preceding power of 9, and this would eventually happen for 9^{21} and 9^{22}, both of which have 21 digits each.

One last interesting observation was by Harry. He actually discovered one of the laws of indices! The students had not studied the laws of indices and Harry's working suggests that he did not know this law beforehand because he started by trying to see if the product of two powers was another power. This was what he wrote when describing what he was doing.

> Harry: Trying to multiply the powers to see if they make out another power.

In the end, Harry discovered that $9^m \times 9^n = 9^{m+n}$. Compare this with a research study conducted by Lampert (1990). She taught a class of fifth-grade students using open tasks and mathematical discourse, which was rather similar to an investigative approach to mathematics teaching and learning. In one particular series of lessons, the students were investigating squares and they discovered some patterns in the last digit of the squares. Lampert then made use of the students' discovery to extend the problem to "What is the last digit of 5^4? 6^4? 7^4?" and later to "What is the last digit of 7^5? 7^6? 7^7? 7^8?" The students had some confusion on how to obtain 7^8 from 7^4. This would eventually lead to the discovery that the exponents should be added together when multiplying powers with the same base, which her students were almost on the verge of discovering just before the end of the series of lessons.

To summarise, only nine of the 29 students (or 31%) were able to conjecture some non-trivial patterns for Task 1. Only four out of these nine students (or 14% of the total sample) were able to discover the pattern in the last digit of the powers of 9 and only one of them (or 3% of the total sample), Charles, was able to conclude that the powers of other numbers have their own patterns in the last digit. Nevertheless, there were some interesting or unexpected findings but these were very few: only six conjectures by five students (Daniel, Eden, Frank, Gilbert, and Harry) and some of these hypotheses were even false.

For the other three investigative tasks in the test, the students fared as badly or even worse because the task statements did not provide any sample problems to investigate and so most of the students did not know how and what to investigate. It is beyond the scope of this paper to provide a detailed report of what the students did for these three tasks (these findings will be reported in another paper).

Discussion

The findings suggest that most of the high-ability students did not know how and what to investigate. The first problem is the inability of the students to pose their own problems to investigate. Many students in Singapore have not been exposed to problem posing; they are usually given problems to solve. So when the students in this study were asked to pose their own problems to solve or investigate, many of them were at a loss as to what to do.

The second problem is the failure to understand the task requirement because most students still did not know what to do even when sample problems were provided for them to investigate. Many of them did not understand what it means to find as many patterns as possible about the powers of 9. The students might be able to search for patterns to solve a specific problem but looking for *any* pattern without having a problem to solve had confounded many of them. It seems that the absence of a specific goal and the failure to understand what it means to investigate have caused them much confusion.

The third problem is jumping to conclusion too fast. For the few students who managed to make some non-trivial conjectures and even for those who made trivial conjectures by observing some patterns, many of them did not try to prove their conjectures but concluded that these were the underlying patterns based on a few empirical data, and so as it turned out, some of these hypotheses were false.

However, there were a few students who were able to make some unexpected non-trivial discoveries.

Conclusion and Implications

The current state of competency in open mathematical investigation among Singapore students is very low. If most high-ability students fared badly when given open investigative tasks, it is unlikely that low-ability and average students will know how and what to investigate but there may be some exceptions. Although further research is needed to confirm this, there are some ethical issues to consider. Many of the high-ability students gave very negative comments about the test, even to the extent that they hated it. If this test is to be administered to other classes of students, then this may cause more students to loathe mathematical investigation. This poses a serious dilemma on research methodology: How do researchers assess students' level of competency in open mathematical investigation before implementing any intervention programme? For research on problem solving, a paper-and-pencil pretest can include simpler problems at the start to give the students some confidence, and even when the students cannot solve the more difficult problems, they will at least understand what the tasks require them to do and so they will still try to solve the problems. But for research on open mathematical investigation, there is no way to make the investigative tasks any simpler. If structured tasks with part-questions are given at the start of the pretest, then these tasks are no longer open, and when the students are faced with open investigative tasks at a later part of the test, then they will still not understand the task requirements, even when sample problems are provided for the students to investigate, as the current study has shown. Thus it is necessary to rethink how to conduct research on open investigation that involves testing the subjects' initial level of proficiency.

There is also a need to expose Singapore students to this kind of open investigative tasks but anecdotal evidence (see previous section on Background) suggests that teachers are not ready to teach mathematical investigation because most of them themselves do not know how and what to investigate. Much work needs to be done to study the nature and development of thinking processes in mathematical investigation so that a suitable teaching programme can be designed to guide students to investigate effectively. Then teachers will have to be taught how to implement this programme for their students successfully.

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Teaching Area and Perimeter: Mathematics-Pedagogical-Content Knowledge-in-Action

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This paper examines the influence of teacher's mathematics pedagogical content knowledge (MPCK) in the teaching of area and perimeter to Grade 4 pupils. Lessons of a beginning teacher were studied to determine the activities and teaching strategies used to bring out ideas associated with area and perimeter. Observable MPCK-in-action outcomes are also being studied through video-taping the beginning teacher. In addition, this paper also studies how concept of area and perimeter is developed in class. The complex interplay between area and perimeter concepts was sometimes handled well by the beginning teacher, whereas on other occasions, gaps in mathematics pedagogical content knowledge had the potential to cause misconceptions for pupils.

We know who the good mathematics teachers are and we can recognise good mathematics teaching when we see it; yet, it is not very easy to describe what comprises good teaching. This is often because mathematics teachers often have to apply knowledge from various domains of teacher knowledge to their classroom instructions. Among these domains are knowledge of pedagogy, knowledge of learners, subject matter knowledge, and pedagogical content knowledge (Shulman, 1986). Pedagogical content knowledge (PCK) is that unique knowledge domain of teaching that differentiates the expert teacher in a subject area from the subject expert. PCK is that special professional understanding that teachers have whereby they can incorporate, change and represent subject matter knowledge in ways that are comprehensible to pupils. At the same time, area and perimeter of plane figures have become more important components of the curriculum, particularly at the primary school level. This paper examines the approach of one beginning teacher who uses both content and pedagogical content knowledge to assist Grade 4 pupils to develop key concepts in area and perimeter of plane figure.

Background Issues

Mathematics Pedagogical Content Knowledge and Area and Perimeter

We may have encountered teachers who are very knowledgeable about their content area but are somehow unable to present the content in such a way that their pupils can comprehend. Shulman (1987) considered this to be that very critical part of teacher knowledge where one moves from personal comprehension to preparing for the comprehension of others. A teacher cannot hope to explain mathematical concept if he does not have full comprehension of that mathematical concept. Nevertheless, case study evidence suggests that the influence of teachers' mathematics pedagogical content knowledge (MPCK) has a strong influence on children's learning outcomes (Shulman, 1986). In the teaching of mathematics, Ball (2000) stressed how the depth of teachers' understanding of MPCK is a major determinant of teachers' choice of examples, explanations, exercises, items, and reactions to children's work. Mathematics educators have been constantly urging the mathematics teachers to teach mathematics with relational understanding (Skemp, 1978, Van de Walle, 1994). Only when the mathematics teacher understands something well enough is he able to teach others. He needs to overcome the various obstacles that might otherwise deny his pupils access to knowledge. Studies have shown that beginning teachers often struggle to represent concepts in an understandable manner to their students because they have little or no PCK at their disposal (Kagan, 1992; & Reynolds, 1992). Beginning teachers, for example, do not see much difference in telling and explaining because they have not developed their PCK.

For the area and perimeter content domain, the place of area and perimeter in the curriculum was well established. Many researchers have reported on lower primary pupils' ideas related to the measurement of area (Battista, Clements, Arnoff, Battista, & Van Auken Borrow, 1998; Dickson 1989; Heraud, 1987; Lehrer, Jenkins, & Osana, 1998; Outhred &Mitchelmore, 2000). Although their findings are insightful, their studies were about pupils in grade 4 or below. There were also studies which indicated that pupils in the middle years (Year 5 to 8) confuse the concepts of area and perimeter even though they may give correct answers to standard assessment questions requiring the use of formulae (Kidman & Cooper, 1997). Mulligan, Prescott, Mitchekmore and Outhred (2005) suggest that to avoid such confusion, teaching should focus on the recognition of the structure of an area grid and is likely to lay the foundation for a deeper understanding of area measurement. Other mathematics educators such as Martin and Strutchens (2000) and Van de Walle (1997) recommend the use of squares to make "length×width" understandable. It is very easy for pupils to cover an area empirically with squares and to use multiplication to show how many squares were used. Squares are easy to quantify because they are discrete quantities. Although area and perimeter of plane figures are characterized by procedural techniques, there are deep conceptual issues to address. Significant among these are the ideas that lengths and areas are continuous quantities. It will be extremely difficult for pupils to understand how two lines (the length and the width) can produce an area when they are multiplied. When presented with two lengths perpendicular to one another starting from the same point, the pupil must have a concept of the area as a matrix consisting of "an infinite set of lines infinitesimally close to one another" (Piaget et al., p. 350) to make sense of length x width which is the area of rectangle. In order to provide these, it may require the teachers to have significant content and pedagogical content knowledge of area and perimeter. Unfortunately, a study conducted by Menon (1998) on 54 preservice elementary teachers' understanding of area and perimeter, it was found that they have a procedural understanding of area and perimeter rather than a conceptual and relational understanding. Moreover, the 2003 NCTM Yearbook entitled Learning and Teaching Measurement (Clements & Bright, 2003) has six chapters that deal, at least in part, with the area of a rectangle. However, the authors of all six of these chapters assumed that a square was the unit for area for pupils and did not mention any issue related to the fact that length and area are continuous quantities. Establishing the idea of length and area as continuous quantities requires time. Mathematics teachers in Singapore may thus view it as a costly procedure given the crowded curriculum in Singapore.

Most primary school pupils have a good understanding of perimeter as a special application of length that measures the distance around a figure. Pupils are so accustomed to finding perimeters where the length of every part of a figure is given and they just had to add all the given numbers. Pupils who do not have an adequate understanding of perimeter will find it difficult to deduce the length of the side when it was not stated explicitly. On the other hand, a good relational understanding of perimeter includes reasoning based on relationships among the sides of a given figure. Another common finding from the literature is that teachers confuse the concepts of area and perimeter (Baturo & Nason, 1996; Fuller, 1997; Heaton, 1992), frequently assuming that there is a constant relationship between area and perimeter. Further, teachers often do not use appropriate units when computing area and perimeter, commonly failing to use square units when reporting measures of area (Baturo & Nason, 1996; Simon & Blume, 1994).

The Present Study

In light of the area and perimeter concepts discussed above, and mindful of the challenges inherent in teaching them, it seems timely to look at how a beginning teacher apply his content knowledge and pedagogical content knowledge in teaching area and perimeter. The present study was part of a larger project entitled Knowledge for Teaching Primary Mathematics (or MPCK Project) with the objective of studying the development of beginning primary school teachers' MPCK which involved observing and video-taping five lessons for a beginning teacher. An important part of the research was to determine what the researchers (from MPCK Project) term "MPCK-in-action" outcomes as observed being practised by teachers when teaching mathematics and to ascertain the relative importance of different practices in contributing towards effective pupil learning.

This paper reports on a study of a beginning teacher who presented five of his lessons on area and perimeter to his class of Grade 4 pupils. This paper also look at the area and perimeter concepts and the way in which CK and PCK impact on their development when the teacher delivered the lessons. It seeks to answer the following questions:

1. What key perimeter and area concepts were brought out in the lessons?
2. What are the observable MPCK-in-actions that are present in the teaching of area and perimeter?

Methodology

This paper focuses on this Grade 4 teacher, John (name is pseudonyms) who conducted five lessons on area and perimeter. The beginning teacher, John was from a typical government primary school and was teaching his own Grade 4 class. During each lesson, the video-camera followed the teacher and field notes were made by the author. The video data were analysed by the author to determine key points at which CK and/or PCK were evident in relation to understanding of area and perimeter. In particular, the way in which the chosen activities could be and were used to bring out key area and perimeter concepts were noted, along with other critical moments. The MPCK-in-action practices of this beginning teacher, John are identified through an analysis of video-tapes of five mathematics lessons.

Results

Despite the fact that only five lessons were observed for one beginning teacher, they were rich in their consideration of area and perimeter concepts. John planned the following seven specific instructional objectives for his five lessons:

1. Use the formula correctly to find the area and perimeter of rectangle and square;
2. Find all possible areas rectangle and square given a fixed perimeter;
3. Calculate one dimension of a rectangle and square given its area, perimeter and the other dimension;
4. Solve word problems involving area and perimeter of rectangles and squares;
5. Calculate the area and perimeter of the rectangle and square mentally;
6. Calculate the perimeter and area of the composite figure which made up of rectangles and/or squares.
7. Solve non-routine problems involving area and perimeter of a composite figure.

This beginning teacher, John, involved his pupils extensively in his lessons. Throughout the five lessons, John adopted teacher-centred approach. Only part of the first and fourth lesson, the pupils were involved in some group activities. He had a very effective and seemingly effortless questioning technique in which he was able to draw out from the pupils the ideas that he thought were important. Based on the syllabus, the teacher assumed that the pupils had learnt the concept of area and perimeter of rectangle and square in Grade 3 and proceeds to teach area and perimeter of composite plane figures.

CK, PCK and Area and Perimeter Plane Figures Concepts

In this section we look at the way area and perimeter of plane figures concepts were developed in the lessons. This discussion is not exhaustive, but is intended to highlight some of the critical events during the lessons where concepts were particularly affected by teachers' CK and/or PCK.

Area.

Although the pupils were able to articulate the area of rectangle and square well, it seemed that the pupils were not able to define area. In the first lesson, pupils were asked to define area, promptly answer, "length times breadth!" They understand area as a formula rather than as a concept – the amount of space covered by the boundaries of a two-dimensional figure. From the first lesson, no pupil was able to state that area is the amount of space in an enclosed figure and the teacher did not attempt to assist the pupils to define the term, area. Instead, the teacher showed examples on how the area of rectangle (flag) and area of square (chessboard) were computed using the formula. At the last 30 minutes of the first lesson, pupils were given twenty match sticks to investigate different areas having the same perimeter. He guided the class to work out the area of different rectangles and squares that were found in the worksheet. Finally, the pupils could show that although rectangles and square might have the same perimeter their areas were not equal.

Area of composite figures.

Due to emphasis in the Grade 4 syllabus, the teacher only provides examples of composite figures that made up of rectangles and squares. Opportunities to form composite figures involving rectangle and squares were also implemented in the fourth lesson. The purpose of the activity was to let the pupils experience how composite figures are formed. Pupils used the cut-outs rectangles and squares to form composite figures. The pupils actually measured the dimension of composite figures and made an attempt to form composite figures that were shown on the worksheet. Most of the pupils were able to find the area of the composite figures with ease. The teacher discussed the solutions with the whole class after most of the pupils complete finding the area of composite figures. This discussion included dividing each composite figure into squares and rectangles. He also discussed ways of deducing the length and breadth which were not given explicitly. Unfortunately, perhaps because of time constraints, during the fourth lessons John could not discuss or explore further how to find the area of composite figures in different ways. The examples that were given in the fifth lesson were appropriate but the teacher did not make an attempt to subdivide the composite figure out of the boundary. Pupils were always trying to find and divide the figure within the shape. In the fifth lesson, although there was some discussion of finding the area of composite figures which are more complicated, this received only limited investigation.

Perimeter.

There were two significant occasions at which the idea of perimeter was apparent. The first was at the beginning of the first lesson, in which he devoted five minutes of the lesson time to get pupils to define perimeter. This resulted in the pupils trying to recall the formula for the perimeter of rectangle or square. Although the pupils tend to focus on remembering the correct formula for the perimeter of rectangle and square, teacher emphasized the meaning of a perimeter for a plane figure. Even though the pupils had learnt the concepts of perimeter in Grade 3, at the beginning of the first lesson, John ensured that his pupils were able to define perimeter explicitly.

The second occasion associated with perimeter occurred because of the way in which examples that were showed in the class. A full analysis of the L-shaped figures example which John actually did in his fourth lesson showed that not all the lengths of the L-shaped figure were given. Pupils could not just simply add up all lengths to get perimeter of the L-shaped figure. In the fourth lesson hand-on activity, pupils were given rectangles and squares to form composite figures. Pupils have to match the rectangles and squares with composite figures showed in the worksheet. John made an attempt to lead the pupils to find the perimeters where the lengths of every part of a figure were not given and they just could not add all the given numbers. In the class discussion, pupils were also given the opportunity to reason and based on the relationships among the sides of a given composite figure to deduce the length of a side which was not given explicitly. This had assisted the teacher to tease out any misconception the pupils may have on the understanding of perimeter. In addition, the teacher also gave homework involving composite figures of lengths not stated explicitly. This provided an added experience for the pupils to find perimeter of such composite figures.

Observable MPCK-In-Actions

Sequencing of activities.

Throughout the five lessons, John was able to structure the examples from simple to complex. This was reflected in his fifth lesson where it was intended to solve only non-routine problems after the pupils had enough experience in solving simple problems from the previous lessons. In carrying out the match sticks activity and forming composite figures, teacher adopted the concrete-pictorial-abstract approach. Pupils were given the concrete experience of finding the perimeter before they worked out their computation in their worksheets.

Choice of activities.

John had a good understanding of his pupils' mathematical abilities. He had identified and selected activities that matched his pupils' learning needs. For example, in the fourth lesson pupils worked in pair to form composite figures. This gave the pupils a good concept of composite figures involving rectangles and squares as expected in the Grade 4 syllabus. The pupils responded extremely well to this activity.

Connections between concepts.

At the introduction of each lesson, John always elicited pupils' previous knowledge. He highlighted to his pupils the relevance of learning and applied the area of rectangle concept to a real life problem. For example, John tried to relate area concepts through an example in finding the area of floor plan in a new apartment. In addition, John was able to link the concepts of area and perimeter well. For instance, given a composite figure, pupils need to find the area and perimeter of the composite figure so as to help the pupils to "see" the difference.

Balance between concept development and mathematical procedures.

John consciously emphasised the underlying reasons and explained the mathematical procedures of finding the lengths of a composite figures which are not stated explicitly. When the pupils presented their solutions on the board, he focussed on the essential steps and necessary conditions in the procedures. In fact, he valued computational speed and accuracy so much that he allocated one lesson where the pupils just only worked out the area, perimeter and length of rectangles and squares mentally. Allowing the pupils to explore alternative procedures in solutions was only evident in the fourth lesson. Pupils were asked to find the perimeter of an L-shape figure. A particular pupil was able to find the perimeter of the L-shape in two ways even though length of certain side was not given explicitly.

Questioning techniques.

In most situations, John used structured questioning to establish mathematical procedures. He also exhibited effective use of questioning to elicit understanding from the pupil. John seemed to have a solid knowledge of procedures of finding the area and perimeter of plane figures, of its place in the curriculum and in his own pupils' understanding and of how to teach it. He made no effort to take appropriate actions to rectify errors or correct misconceptions as he circulated in the class to monitor the pupils' class work. Understanding pupils' common misconceptions and strategies for challenging such misconceptions are examples of pedagogical content knowledge which John had shown in most of his lessons.

Discussion and Conclusion

The results depict the complex interplay among concepts (as seen in the fact that some teaching episodes involved two or more area and perimeter concepts), the challenges associated with teaching these concepts and the relationships among them, and the importance of content knowledge and pedagogical content knowledge. The beginning teacher had chosen activities that were rich mathematically, considering the language of area and perimeter, the meaning of composite plane figure, the relationship of area and perimeter and the various ways of finding the area and perimeter of composite figures. The beginning teachers' PCK was evident not just in the choice of activities, but in the ways that he was able to link concepts to pupils' experience. For instance, he gave examples of finding the area of a flag, chessboard and area and perimeter of floor plan of an apartment. The teacher's approaches varied, giving pupils greater freedom to think about concepts and holding rich discussions with groups and individuals. At the initial stage of each lesson, the teacher tended to provide more direct teaching and guidance to pupils about what to do and, perhaps because his belief was that pupils must have a strong basic foundation of the skills and concepts first. From the teacher's direct teaching, class discussions and questioning, it appears that the teacher's CK was strong. He has no much difficulty exploring pupils' ideas when they deviated from his own. Except for one occasion when incorrect content knowledge was evident when the pupils stated that area is equal to length times breadth and he just accepted this definition.. John did not further define the term "area" explicitly for the pupils.

The complexity of pedagogical content knowledge was particularly apparent in the use of the match sticks activity. The match sticks activity provided opportunities to explore areas of different rectangles and squares if the perimeter is constant. The teacher allowed the pupils to do in pair and encouraged his pupils to go through this hand-on activity. In fact, the teacher's decision to do this was made during the first lesson where he planned it in advance. This may well have been the time during which the teacher's pedagogical content knowledge actually developed. However, the teacher did not plan any activity for the same area of different rectangles and squares and deduced if the perimeter of those rectangles and squares could be the same. Teacher was more careful in the presentation of the solutions, using words that would help his pupils understand its meaning. The important role of content knowledge in PCK was also evident in the lessons. On many occasions this was positive, as when the beginning teacher discussed the various ways of finding the area and perimeter of composite figures. The teacher's discussion about finding the area of composite figures showed his content knowledge, but also revealed the complexity of deciding how much to explain to pupils and the difficulty of actually explaining it in a way suitable for Grade 4 pupils. These lessons show that quite deep area and perimeter concepts can be considered in Grade 4. They also highlight the complexity of the concepts, and the importance of having teachers with appropriate content and pedagogical content knowledge. The key area and perimeter concepts identified above need to be understood by teacher himself, plus he must be able to recognise which activities would foster such understanding in his pupils and how to bring this to the fore in his lessons. While a number of primary teachers with weak content knowledge were predisposed to telling pupils rules and explaining algorithmic procedures, John with strong content knowledge appeared to provide conceptual explanation for each example and activity. However, some beginning teachers found it a challenge to provide conceptual explanations for the procedural tasks they performed.

Everything that the beginning teacher John had carried out involved planning lessons, implementing them, responding to what arose in the classroom, interacting with pupils—involved one or more aspects of PCK. Good teaching requires both the mastery of CK as well as PCK. PCK is developed over time. John is a teacher with rich PCK who can devise examples that illustrate a range of concepts, can highlight connections among topics, and identify which are the central ideas and which are peripheral. This was evident in John's teaching of area and perimeter of plane figures.

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Problem Solving Activities in a Constructivist Framework: Exploring how Students Approach Difficult Problems

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The paper describes results of a teaching experiment with five high school (Year 10 and 11) students. Four qualitative characteristics were established: the first step of solution, main information extracted from the problem, generalisation from a problem and completion of solution. From these characteristics the corresponding quantitative indices were introduced and analysed. The structure of two of them, specific S_{FS} and common S_{HP}, are given in detail. Investigation of quantitative indices and their qualitative characteristics gives an opportunity to find out more about interrelations between different stages of the problem-solving process.

Introduction

Problem solving, in various forms and contexts, takes significant time in classroom activities throughout primary and secondary school. This area attracts the interest of both mathematics educators, seeking powerful activities in teaching and learning, and researchers, considering problem solving as a research domain. As a consequence many fundamental research papers on different aspects of problem solving appeared recently. For some of them the concept of a problem was the main focus, in others, students' performance in problem solving and analysis of their thinking strategies were considered. In this paper we attempted to analyse the mathematical behaviour of gifted high school students while they were involved in problem solving activities of non-trivial problems. Indeed, most of students are able to solve standard problems and routine exercises, but with harder ones, the situation looks completely different. It is obvious that even the ablest students often experience difficulties in this kind of activity. However, why does this happen? Which factors were the crucial ones in students' failure to solve a hard problem? How could teachers encourage students in the most effective way, if they fail to address a problem adequately? The answers to these questions and similar ones are extremely important for developing both further theoretical frameworka for the topic and practical implications in work with gifted students. Most high-profile students regularly participate in numerous mathematical competitions. To achieve the best results their training should be grounded on a sound theoretical base. But we have to recognise the fact that the stages of solving hard problems are hidden from researchers in many circumstances and the most talented students find their solutions so natural that explanations are not required. So, how can students' abilities and skills to move toward a solution be investigated, evaluated, and further developed? What is a student's perception of a hard problem? What are the obstacles to finding solutions for such a problem? Our aim was to find out more about the nature of this process. These questions have formed the basis of research the authors focused on for a long period of time (Passmore, 2007; Yevdokimov, 2005a, 2005b, 2006). This paper highlights the linkage between mathematical problems and student cognition. How do they depend and influence each other? Which factors have the most essential influence on students' mathematical thinking and reasoning? In particular, how much helpful information can the statement of a problem provide for students? What part of the problem solving process is the most difficult for them and how we can evaluate students' work on different stages of such activities? The paper attempts to answer these questions.

Theoretical Framework

Mathematical tasks that are different from routine exercises are usually called non-trivial problems. They can be classified in many ways, for example, by level of difficulty. There are other ways to define non-trivial problems in the mathematics education literature. Morton (1927) defined a problem as any mathematical question where the person attempting an answer must select the operations. Krutetskii (1976) used the terminology "task complexity" as equivalent to intellectual complexity of a problem. According to Charles and Lester (1982), problems were classified as standard, non-standard, real-world problems, and puzzles. Hembree (1992) pointed out that "distinctions among definitions of a problem relate to the effort that solvers must make toward solution" (p. 244). Williams and Clarke (1997) identified six dimensions of task complexity – linguistic, contextual, operational, conceptual, intellectual, and representational complexity. We define *a* *hard problem* as one that encourages the use of flexible methods, stimulated guessing, and use of unusual strategies towards a solution. Our conceptual framework is based on the construct of a mathematical problem and its solution implemented by students. It was influenced by the papers of Stein, Grover, and Henningsen (1996) and Henningsen and Stein (1997). The framework, shown in Figure 1, defines a hard mathematical task as a learning activity, the purpose of which is "to focus students' attention on a particular mathematical concept, idea, or skill" (Henningsen & Stein, 1997, p. 528).

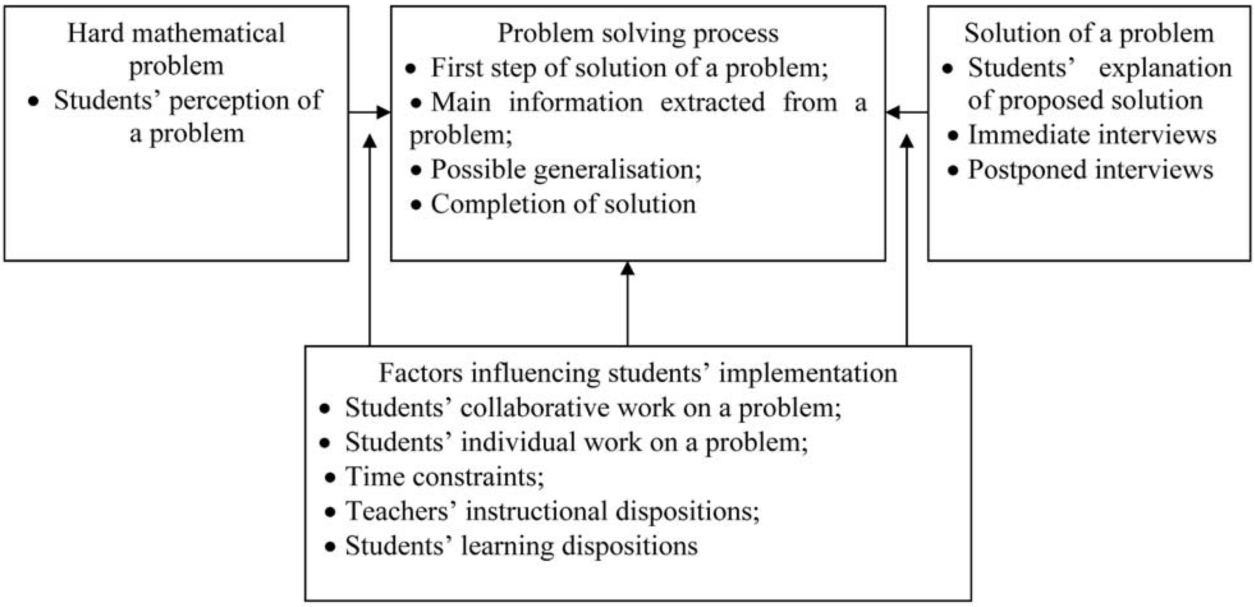

Figure 1. Mathematical task as implemented by students.

More exactly, we consider the modified third phase of the Henningsen and Stein (1997) conceptual framework with respect to hard problems in a constructivist framework. We followed von Glasersfeld's (1995) idea that "learning is not a passive receiving of ready-made knowledge but a process of construction in which the students themselves have to be the primary actors" (p. 120).

In this framework mathematical tasks pass through three stages: as perceived by students in the beginning of their work, as solved by students during their work, and as explained by students on the basis of their work. The first stage sounds similar to the first Polya step (understanding the problem). However, taking into account students' understanding the problem, we paid much attention to their perception of the problem from a psychological point of view. Our hypothesis was that it could have a significant impact on students' performance in problem solving, and, therefore, should be taken into consideration. The second stage consists of four dimensions. Each of them represents a different qualitative characteristic of solving hard problems. All characteristics refer to the thinking processes in which students engage. The aim of the second stage was to analyse different dimensions and establish their role and impact on thinking processes in problem solving, as shown in Figure 2.

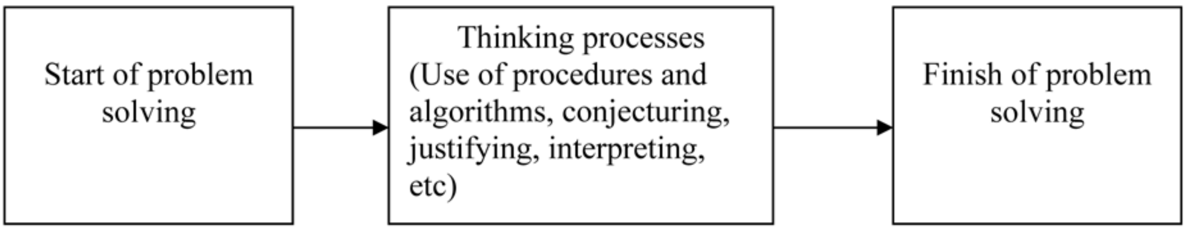

Figure 2. Simplified scheme of problem solving.

The interrelations between these dimensions are complex and need further investigation, which was another aim of the study. They can overlap each other, or even contain one another, influence each other, and, together with factors influencing students' implementation, can change their interrelations with each other. The four dimensions formed a dynamic structure within the framework, as shown in Figure 3.

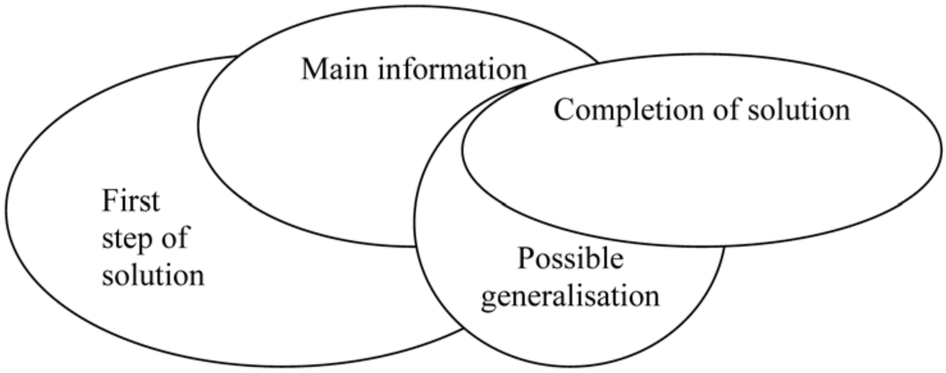

Figure 3. Dynamic structure of interrelations between four dimensions.

Williams (2000) described and categorised students' abilities to solve unfamiliar challenging problems in collaborative work. We focussed mostly on individual student performances, though collaborative work was taken into account. We used two forms of problem solving activities: firstly open problems, and secondly mathematical situations. For open problems proposed for students we followed the Arsac, Germain, and Mante (1988) characterisation:

> The statement of the problem is short, so that it can be easily understood, it fosters discovery and all students are able to start the solution process. The statement of the problem does not suggest the method of solution, or the solution itself, but it creates a situation stimulating the production of conjectures. The problem is set in a conceptual domain, which students are familiar with. Thus, students are able to master the situation rather quickly and to get involved in attempts of conjecturing, planning solution paths and finding counter-examples in a reasonable time.

While solving a certain problem, each student was asked to investigate its "mathematical situation", with his/her own priorities for further inquiry in that problem. Like Brown and Walter (1990), we considered "situation", an issue, which was a localised area of inquiry with features that could be taken as given or challenged and modified. Also, we took into account that current learning perspectives for problem solving activities in a constructivist framework incorporate three important assumptions (Anthony, 1996):

- learning is a process of knowledge construction, not of knowledge recording or absorption;
- learning is knowledge-dependent; people use current knowledge to construct new knowledge;
- the learner is aware of the processes of cognition and can control and regulate them.

Methodology

In order to identify key points of students' performance in problem solving and learn more about their strategies and relation to knowledge construction we distinguished four qualitative problem solving characteristics:

- **FS** – the first step of solution of a problem;
- **MI** – main information extracted from a problem;
- **G** – generalisation possibly required for solution of a problem;
- **C** – completion of the solution of a problem.

We established quantitative indices of students' skills for each of the corresponding qualitative characteristics:

- $\mathbf{S_{FS}}$ – student's skills to find the first step of solution of a problem;
- $\mathbf{S_{MI}}$ – student's skills to find out the main information from a problem;

- S_G – student's skills to make generalisation which possibly could be required for solution of a problem;
- S_C – student's skills to complete solution of a problem and make conclusion.

Finally, we introduced a common index S_{HP} – the level of student's abilities to solve hard problems. We define S_{HP} as a variation of AFKS (Yevdokimov, 2006), being considered in the context of a specific learning environment where hard problems are to be solved by students.

The teaching experiment methodology consisted of long-term interactions between teacher/researcher and individual students. These interactions included interviews and teaching episodes. This methodology concentrates on students' conceptual constructions and their cognitive demands. The main goal was to analyse students' constructions in the problem solving process. Interactions between the teacher/researcher and a student were intended to stimulate the student's mental activity. Interviews and teaching episodes provide for intensive interaction between student and teacher, where a teacher assists the student's developmental constructions.

This teaching experiment was conducted in three parts: an interview part, teaching part, and analysis part. It is important to note that development of students' abilities to solve hard problems is directly connected to the teacher's competence to conduct inquiry activities in a classroom. The teacher has to regulate directions of students' inquiry work into the problem solving process and adapt it to the classroom needs. At the same time, "open problems promote the devolution of responsibility from the teacher to students" (Furinghetti & Paola, 2003, p. 399). The teacher's role in this situation, we feel, should follow Mercer's idea (1995) of "the sensitive, supportive intervention of a teacher in the progress of a learner, who is actively involved in some specific task, but who is not quite able to manage the task alone" (p. 48).

These three parts formed a phase of the research, a full cycle taking two months. This was repeated four times per school academic year to verify students' conceptual constructions and trace the dynamics of the changing qualitative characteristics and their quantitative indices for each student. We calculated these indices at the start of the first phase ($S_{FS,0}$, $S_{MI,0}$, $S_{G,0}$, $S_{C,0}$) and at the end of all phases of the research ($S_{FS,1}$, …, $S_{C,4}$ respectively), as shown in Figure 4.

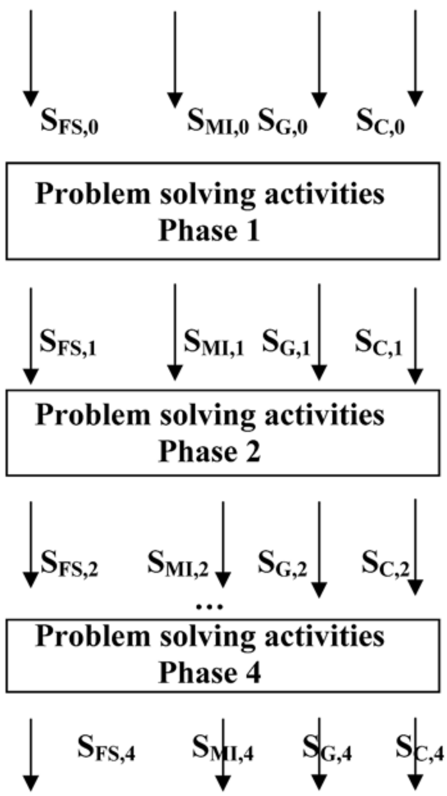

Figure 4. Structure of the teaching experiment.

We compared the common index $\mathbf{S_{HP}}$ with corresponding indices $\mathbf{S_{FS}}$, $\mathbf{S_{MI}}$, $\mathbf{S_G}$, and $\mathbf{S_C}$ to identify their similarities, differences, and mutual influence on each other. During all phases students had been asked to work with testing sheets to solve five problems in the form of tasks. They had to carry out certain problem-solving activities and provide appropriate argumentation for each task (analyse **FS**, **MI, G,** or **C** for a given task, but not all characteristics together for each task). At the same time we had the answers for **FS**, **MI, G,** and **C** in advance for each task on the testing sheets for teachers' use only. Average values of the quantitative indices of students' skills for each qualitative characteristic were calculated on the start and finish of each phase of the experiment. Below are both detailed descriptions of $\mathbf{S_{FS}}$ – the index of the student's skills to find the first step of solution of a problem and $\mathbf{S_{HP}}$ – the common index of the student's abilities to solve hard problems.

Structure of S_{FS}

We follow quantitative methods (Yevdokimov, 2006) on the basis of a formula of elementary probability for the finite number of events (in our terminology – for the finite number of essential levels of student's performance in problem solving):

$$S_{FS} = \frac{\sum_{k=1}^{n} p_k}{n},$$

where either $p_k=1$, if a student's performance was satisfactory, or $p_k=0$, if a student's performance was unsatisfactory, n is a number of levels mentioned above. In the scope of theoretical framework we distinguished six such essential levels, that is, $n=6$ here:

p_1 – a student was able to propose or at least make suggestion about the first step;

p_2 – a student could explain what he or she had done there;

p_3 – this first step leads to the solution of a problem;

p_4 – a student could explain which step of solution should be the next one, in other words, what should be the next step of solution on the basis of the first step proposed by a student;

p_5 – a student could explain (provide) full solution of a problem on the basis of his or her first step;

p_6 – a student provided full solution of a problem, and this solution is the shortest, and can be characterised as one of the best for a problem.

Index $\mathbf{S_{FS}}$ is not a probability value in the proper way, though it has probabilistic sense. We measured changes in $\mathbf{S_{FS}}$ in the range between *0* and *1*, taking into account such factors as studentss experience, constructivism, creativity, and mathematical competence.

Analogous ideas and approaches we used to estimate $\mathbf{S_{MI}}$, $\mathbf{S_G}$, and $\mathbf{S_C}$.

Structure of S_{HP}

To calculate the common index $\mathbf{S_{HP}}$, for each student for a certain task we used the same formula for elementary probability for a finite number of events. However, levels in this case were different from $\mathbf{S_{FS}}$. The solution of any problem was divided into consecutive steps and the step-by-step schemes provided for teacher's use only.

We evaluated student's actual suggestion for each step of the solution with *1*, if he/she could provide clear explanations why he/she did so. Otherwise, a mark for such a step was *0*. The formula was the following

$$S_{HP} = \frac{\sum 1}{N},$$

where N was a number of consecutive steps for a certain task.

Note that N takes different values for each problem and it is also possible for a student to give a different correct solution to a problem from the solution that the teacher has for the $\mathbf{S_{HP}}$ calculation. In such a case, the student's solution is divided into similar consecutive steps and the same scheme is applied to compute $\mathbf{S_{HP}}$.

Findings

Average index evaluations for $\mathbf{S_{FS}}$, $\mathbf{S_{MI}}$, $\mathbf{S_{G}}$, $\mathbf{S_{C}}$, and $\mathbf{S_{HP}}$ are given in the Tables 1 and 2 respectively. At first we calculated average values for each student at each phase. $\Delta S_{FS,k}$ means the difference between two consecutive evaluations $S_{FS,k}$ and $S_{FS,k-1}$, the same notation with other indices. Thus, we could trace dynamics of changes, and which characteristics at a certain phase played more significant role than others. We present average evaluations for five students only to demonstrate the general tendency – how different problem solving characteristics are related each other. The average evaluation for $\mathbf{S_{HP}}$ is in Table 2 due to the different nature of the index.

Table 1

Average Index Evaluations: S_{FS}, S_{MI}, S_{G}, S_{C}

$S_{FS,0}$	0.15	$S_{MI,0}$	0.27	$S_{G,0}$	0.04	$S_{C,0}$	0.38
$S_{FS,1}$	0.21	$S_{MI,1}$	0.22	$S_{G,1}$	0.1	$S_{C,1}$	0.45
$\Delta S_{FS,1}$	0.06	$\Delta S_{MI,1}$	-0.05	$\Delta S_{G,1}$	0.06	$\Delta S_{C,1}$	0.07
$S_{FS,2}$	0.29	$S_{MI,2}$	0.31	$S_{G,2}$	0.18	$S_{C,2}$	0.42
$\Delta S_{FS,2}$	0.08	$\Delta S_{MI,2}$	0.09	$\Delta S_{G,2}$	0.08	$\Delta S_{C,2}$	-0.03
$S_{FS,3}$	0.57	$S_{MI,3}$	0.48	$S_{G,3}$	0.32	$S_{C,3}$	0.58
$\Delta S_{FS,3}$	0.28	$\Delta S_{MI,3}$	0.17	$\Delta S_{G,3}$	0.14	$\Delta S_{C,3}$	0.16
$S_{FS,4}$	0.76	$S_{MI,4}$	0.68	$S_{G,4}$	0.55	$S_{C,4}$	0.84
$\Delta S_{FS,4}$	0.19	$\Delta S_{MI,4}$	0.2	$\Delta S_{G,4}$	0.23	$\Delta S_{C,4}$	0.26

Table 2

Average Index Evaluations: S_{HP}

$S_{HP,0}$	$S_{HP,1}$	$\Delta S_{HP,1}$	$S_{HP,2}$	$\Delta S_{HP,2}$	$S_{HP,3}$	$\Delta S_{HP,3}$	$S_{HP,4}$	$\Delta S_{HP,4}$
0.14	0.18	0.04	0.25	0.07	0.36	0.11	0.52	0.16

It is important to note that the common index of problem solving abilities $\mathbf{S_{HP}}$ and generalising characteristic $\mathbf{S_{G}}$ have similar tendencies and dynamic changes. The easiest from students' point of view was completion of solution, characteristic $\mathbf{S_{C}}$, however, it did not overlap with generalising skills in most tasks and, therefore, this question needs further investigation. We noticed that, after getting some experience, students' performance in $\mathbf{S_{FS}}$ increased significantly but $\mathbf{S_{MI}}$ did not. However, other indices depended strongly on increasing $\mathbf{S_{MI}}$.

We distinguished three basic strategies which were used by students in their attempts to solve different problems. We called them: the "Blind search", the "Going along the fairway" strategy, and the "Conscious search to find main information". With "Blind search", students made stochastic attempts to solve a problem, they were not able to explain their suggestions and preferences. Very often students tried to check all possible situations in a problem. In "Going along the fairway" they tried to apply the last method that they had previously studied. The third strategy took the leading place in students' work during the third and fourth phases. We observed that, in the case of students' successful answers to the question about the **MI** of a problem, all characteristics were correctly specified in most of other problems. Moreover, in some cases, not universally, but quite often, students began their analysis with **MI**, even if the questions were about other characteristics.

Concluding Remarks

Analysis of the qualitative characteristics and their quantitative indices gives an opportunity to develop knowledge of the problem-solving process from a complex-mental-activity point of view. Schoenfeld (1985) has noted a widespread belief that only the brightest students can succeed at problem solving. Hembree (1992.) argued that this belief is not well-founded. Our results support Hembree's conclusion. We noticed that good mathematics students, though not the brightest ones, after gaining more experience in problem-solving, understand that there are few options for the first step in solving any problem. They can distinguish such situations, though not all students are able to explain their understanding clearly. Furthermore, our results show that illumination and insight do not have any significant impact on students' performance despite the fact that students could not always explain why they made a step in a certain direction. We are inclined to think that students and teachers exaggerate the importance of the "Ah ha! moment" in problem-solving activities. This experiment showed that both qualitative characteristics and their quantitative indices may be used as a powerful diagnostic tool in work with gifted and talented students, for the further development their conceptual constructions, and improvement of their problem-solving skills.

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Creating Equitable Practice in Diverse Classrooms: Developing a Tool to Evaluate Pedagogy

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With Australia performing so poorly in terms of equity in mathematical achievement on the PISA scores, there is an increasing recognition for practices that may stem the inequities in education in this country. This paper explores an approach that has been found to be highly successful in the United States and links it to current issues in Australian education. Practical considerations are made regarding the application and implementation of such reform pedagogy when particular nuances of Australian issues are considered. In particular, the development of a tool to evaluate reform pedagogy is the focus of this paper.

Increasingly classrooms are becoming more diverse and with such change, new forms of pedagogy are needed to enable the greatest likelihood for success for all students but most particularly for those students who traditionally have been most at risk of not succeeding in school mathematics. Alarmingly, Australia performed well on international comparisons in terms of overall performance but was one of the poorest performing countries in terms of equity (Lokan, Greenwood, & Cresswell, 2001). These authors contend that the outstanding performance of some students overcompensated for the poorer performing students to allow for a good overall outcome. The concern for us is the large gap between those who perform well and those who do not. Such poor performance is not random but strongly aligned with the social, cultural and geographical location of students. In this paper we discuss these highly differentiated performances and propose an alternate pedagogy that has been found to be highly successful in some contexts outside Australia, but with modifications that appear to be more amenable to the unique situations of Australian education. Further, we discuss the difficulties with the implementation of such a model and the challenges to the implementation of such a successful model.

Differential Outcomes in Australian Education: The Case of Most Disadvantage

MCEETYA (2006) reported the results for students in the national testing schemes from 2005. Comparisons of these figures show that for students who come from Indigenous backgrounds and/or live in geographical remote regions are considerably more at risk of performing poorly on standardised tests than their peers in urban or regional areas, or students who are non-Indigenous. Furthermore, it can be hypothesised that some students may have their disadvantage compounded by the multiple disadvantage caused through the combination of factors. For example, Indigenous students who live in remote areas may be at increased risk of performing poorly in mathematics than their peers who are in different social/cultural or geographical locations. The differences in performance by location can be seen in Table One where there are considerable differences between students who live in Urban areas (and perform higher) than their peers who live in very remote areas (and perform lower).

Table 1

Percentage of Students Performing to Benchmark by Geolocation, 2005 (Source: MCEETYA 2005)

	Year 3			Year 5			Year 7		
State	**Metro**	**Very remote**	***diff***	**Metro**	**Very remote**	***diff***	**Metro**	**Very remote**	***diff***
NSW	95.5	86.1	*9.4*	92.2	74.3	*17.9*	77.3	61.8	*15.5*
Vic	95.5	na		95.2	Na		87.3	Na	
Qld	93.2	76.8	*16.4*	89	63.1	*25.9*	84.5	54.3	*30.2*
SA	93.3	78.7	*14.6*	90	63.1	*26.9*	87	54.1	*32.9*
WA	91.9	67	*24.9*	88.2	57.3	*30.9*	85.9	57.9	*28*
Tas	92.4	86.8	*5.6*	89.8	83.2	*6.6*	82.3	na	
NT	Na	65.5		na	35.6		na	31.9	
ACT	94.6	NA		93.2	Na		88.1	na	
Aust	94.6	73.2	*21.4*	91.8	54.5	*37.3*	83.1	49.4	*33.7*

These data highlight the significant differences in performance by students according to their geographical location. For us, what is alarming is there is relative consistency in the data for students who live in metropolitan areas and their peers in remote areas regardless of state. Furthermore, the decline in performance of students in remote regions over time is a point for noting (and action).

In the following data, we consider the data on Indigenous students (Table Two below). A similar trend to that noted in Table One can be observed for Indigenous students. These data suggest that as Indigenous transit through formal schooling, the difference in performance with non-Indigenous students increases with duration of time. As with the students from remote areas, the gap in performance increases as students move through formal schooling by approximately 10% for each period. State performance varies which we contend may be a factor related to Table One where those states which have considerable numbers of Indigenous students living in very remote areas may have greater likelihood of poorer performance on the state-wide testing schemes.

Table 2

Percentage of Students Achieving the Numeracy Benchmark by State, 2005 (Source: MCEETYA 2005)

	Year 3			**Year 5**			**Year 7**		
State	**All**	**Indig**	***diff***	**All**	**Indig**	**diff**	**All**	**Indig**	**diff**
NSW	95.4	87.6	*7.8*	91.7	75.4	*16.3*	75.8	44.5	*31.3*
Vic	95.5	91.8	*3.7*	95.4	89.5	*5.9*	86.9	66.5	*20.4*
Qld	92.7	78.9	*13.8*	88.1	65.8	*22.3*	83.2	54.5	*28.7*
SA	92.6	74.5	*18.1*	90.1	69.8	*20.3*	85.7	55.8	*29.9*
WA	90.2	64.8	*25.4*	85.9	51.6	*34.3*	84.3	46.8	*37.5*
Tas	91.2	82.4	*8.8*	89.1	78.7	*10.4*	80.5	66.4	*14.1*
NT	86.2	68	*18.2*	69.6	35.1	*34.5*	64.8	24.9	*39.9*
ACT	94.6	92.8	*1.8*	93.2	81.4	*11.8*	88.1	62.6	*25.5*
Aust	94.1	80.4	*13.7*	90.8	66.5	*24.3*	81.8	48.8	*33*

We acknowledge the problematic nature of state-wide tests which have some limitations in what they are able to test, the protocols around the tests, and the limitations imposed by the marking schemes and hence what can be assessed. Such limitations restrict what and how assessment can be developed. However, the tests do alert educators to the considerable differences in performance and the need to redress such performance.

Reform Pedagogy

Drawing heavily on the work of Boaler (2002) who has systematically documented the pedagogy and performance in reform classrooms in the United States over an extended period of time, we take those characteristics that she proposed as instrumental in creating equitable outcomes for students as they come to learn school mathematics. The reform pedagogy that Boaler studied was that of Complex Instruction developed by Cohen and colleagues (Cohen & Latan, 1997). At its basis, it draws on a range of literatures to develop a pedagogy that takes a number of key ideas: group work where students assume responsibility for group learning and action; assigning status to those students who may otherwise be marginalised within a group; complex tasks that are rigorous and foster deep learning; and multidimensionality where students can represent their thinking and processes in ways at suit their unique thinking styles.

Boaler's extensive research in classrooms has highlighted the power of this approach in changing the learning outcomes for students, particularly those students from the most disadvantaged contexts. Her extensive study of Railside has illustrated how the school moved from the poorest performing school in California to 'above state average' over a period of 4 years. This radical transformation was seen to be brought about through the use of the Complex Instruction approach. Boaler (2008) noted that the outcomes of the approach are not limited to cognitive outcomes but also to social outcomes where she found that the students also learned how to resolve social and cultural conflicts outside classrooms as a consequence of their participation in the reform.

Linking Reform Pedagogy with Productive Pedagogies

Boaler's work is overlayed with the extensive research undertaken in Queensland schools through the Productive Pedagogies framework (Education Queensland, 2008). This approach has many of the features of Boaler's reform pedagogy in terms of intellectual quality and supportive learning environments but within a framework for both action and research. We do not intend to expand the Productive Pedagogies Framework in this paper as it has been taken up by most Australian states in some form or another and has been the basis of a considerable number of research papers. There are four dimensions within the framework – Intellectual Quality, Relevance, Supportive School Environment and Recognition of Difference – in which there are a number of pedagogies that are evident of that theme. In total, there are 20 identified pedagogies. These pedagogies have been used as the basis for the Queensland Schools Longitudinal Reform Study (Education Queensland, 2001) where schools across the state were studied in terms of pedagogical quality using the framework. Extensive work was undertaken to break each of the pedagogies into qualitatively different features that explicated the degree of take up so as to form the basis for observational schedules. A scaling system that recognises the degree of implementation for each pedagogy was developed using a 1 to 5 scale that, in simple terms, identified a 1 as being not an integral component of the classroom pedagogy through to 5 which identified the pedagogy being an integral feature of the classroom practice.

The model used within the Productive Pedagogies framework has been a useful tool for analysing mathematics classroom practice (Zevenbergen & Lerman, 2007). It provides a very general framework for deconstructing pedagogy as a whole. However, it also has some limitations. Most particularly for the mathematics classroom, it does not allow for the depth of analysis related to mathematical ideas. We also contend that it does not allow for the depth of analysis that Boaler's equity work has identified specifically as it applies to the teaching and learning of mathematics in diverse classrooms. To this end, we are developing a tool that incorporates the key aspects of the Productive Pedagogies framework and incorporating the aspects identified through Boaler's study of Railside.

Developing a Tool for Analysing Equity in Mathematics Classrooms

In the remainder of this paper we draw on our work where we are seeking to develop a tool for analysing classroom practice in terms of building equitable outcomes for learners. Given the data we highlighted at the commencement of this paper, we see it as critical that pedagogies be explicitly developed to redress these outcomes. To develop a tool, we have combined the work cited above – that is, the equity work of Boaler with the processes and principles used within the Productive Pedagogies framework.

To date, we have been working with a series of video tapes that were part of a previous project and that have been analysed with the Productive Pedagogies framework (Zevenbergen & Lerman, 2006). The videos were known to the research team and were selected on the basis of their breadth in inclusive practices. Some of the videos were highly inclusive through to videos that were very traditional in their approaches to teaching mathematics. These were correlated with the initial analysis of the data (Lerman & Zevenbergen, 2006)

Using the approach adopted with the Productive Pedagogies Framework, the video data has been explored by at least 3 researchers who have negotiated each of the dimensions that were identified in Boaler's work and extended these to specifically address Australian issues. For example, one of these is the use of home language. For many of Indigenous students the language of school instruction is different from that spoken at home. In many remote communities, Indigenous students come to school speaking a Kriol. For these students negotiating meaning becomes complex when there is high demand for translating between the school and home languages, particularly when the home language is relatively "restricted" in a Bernsteinian (1990) sense and does not have the same patterns of signification found in school language.

In working through these videos, the research team has negotiated their understandings of practice in relation to the key dimensions to establish a scoring system that aligns with that of Productive Pedagogies. We have created 4 overarching categories that are then broken into smaller, more identified items. The negotiation process between the research team has created a rich discussion that has enabled the unpacking of what is meant by each criteria and the progressive adoption of each criteria on a 5 point scale which range from 1-5.

In the following sections we provide the criteria that have developed as a consequence of the video analysis and negotiations among the research team. Preliminary application of these criteria to 5 videos indicates that the scoring rubric tends to work across the settings.

The Scoring Rubric for Equitable Pedagogy

In this section we provide the descriptors and scores for each of the identified pedagogies relevant to equitable pedagogy.

Process

Drawing on the process outlined in the University of Queensland Manual, we provide the following description of the scoring process. When scoring for each lesson, observers should carefully consider the explanations given for each dimension, using the descriptors of the scores from 1-5 for each criteria. If any difficulty is encountered in selecting between two scores, the observers should consider whether the minimum criteria for each score have been met. If these criteria have not been met, the lower score should be used. In determining the scores for each dimension, the observers should only consider the evidence seen during the specific period. The observers should complete the criteria sheet at the end of the observation period.

Equitable Pedagogies

Group Work

The group works collectively in resolving the task. People's input is drawn upon to solve the problem.

1. No group work is evident in the lesson
2. Group work is used for a brief activity over a small portion of the lesson
3. Group work is used over about half of the lesson
4. Group work is evident for almost all of the lesson
5. Group work is an essential component of the lesson in its aims and structure.

Multiple Pathways

The teaching approach and/or activity allows for students to draw on their knowledge to construct different pathways to resolving the task or problem. This may be through drawing on different forms of knowledge and knowing.

1. No multiple pathways offered by the teacher for students to solve the task or problem
2. Students given some minor variations or pathways in which to solve the task or problem

3. Students are given different starting points for the task, and some pathway variations, albeit in a limited fashion
4. Students have either different starting points or multiple pathways and some level of choice in representation
5. Students are able to use a variety of representations, multiple pathways and engage at different starting points. All three must be present.

Multiple Entry Points

The task/problem is designed so that students can draw on different entry points when starting the task/problem.

1. Tasks have only one entry point strictly controlled by the teacher
2. Teacher outlines limited variety of entry points with no student control
3. Students have some control over entry points of the task within set parameters
4. Teacher allows a variety of entry points for differing abilities
5. Teacher allows complete student discretion in the undertaking of an open ended task or problem.

Roles within the Group

The social organisation involves the clear expectations that members of a group will have particular roles within that group. The roles are followed so as to enable each person to be an active and instrumental member of the group.

1. Teacher defines roles of the group with no collective responsibility
2. Students have linked roles that are still teacher directed to complete the task
3. Students work as a collective but with predetermined roles and objectives
4. Teacher has limited responsibility in the setting up of the collective roles and responsibilities
5. All students in the group work collectively and take responsibility for each other with no direction from the teacher

Quality Interactions within the Group

The pedagogy allows for quality interactions among peers where peers can discuss and debate mathematical ideas as part of their pathway to resolving the task/problem.

1. Students are arranged in groups but have little or no interaction
2. Students have limited interaction with each other or for a brief period
3. Students are engaged in mostly low level interactions with each other for a substantial portion of the lesson
4. Students are engaged in high quality interactions for a significant portion of the lesson
5. High quality interactions with high order thinking processes evident between students for almost all the lesson.

Teacher as Facilitator

The role of the teacher is to absolve responsibility for learning to the students. Through the careful development of scaffolding techniques and task selection, the students take responsibility for their own learning and the learning of others within the group. The teachers' role is to check that students remain on task, provide quality tasks, and to provide assessment when appropriate to the group's progress.

1. Teacher takes all responsibility for the learning, task design and assessment
2. Teacher absolves very limited responsibility to the students
3. Students take some responsibility for their learning in terms of the collective learning in the group and staying on task

4. Students take responsibility for their learning with the teacher facilitating students' learning
5. Students take full responsibility for their learning with teacher facilitating students' learning through an appropriate task

Use of Home Language

Students are able to draw on their home language to negotiate meanings. When reporting back, the student/s should use standard Australian English.

1. No use of home language allowed within the classroom. Complete reliance on English
2. Teacher allows limited use of home language between students
3. Students and AEWs (Aboriginal Education Workers) often use home language in the classroom
4. Teacher encourages use of home language and also attempts to learn and communicate using students' home language
5. Teacher, AEWs and students use students' home language on a regular basis to facilitate the students' understanding of mathematical concepts and learning

Multi-Representational

Catering for the diversity among learners, the tasks should foster, and allow for, various methods of representation that cater for the different skills and dispositions that learners bring to the task. Provided that the result is reasonable, the pathway and mode of representation is valued.

1. No options given by the teacher for students to represent their work
2. Some limited forms of representation allowed by the teacher but these are strictly teacher controlled
3. Teacher encourages some variance in task reporting but within guidelines set by the teacher
4. Students are given an open ended task with some brief parameters for representational options
5. Teacher supplies abroad, open ended task that allows students to fully decide on ways of reporting back and representing their work.

A Way Forward

As a tool for exploring practice, we have found the Productive Pedagogies method to be a useful but limited tool when exploring equity in mathematics education. The model that we are building towards draws on the extensive work of Boaler, extended and modified for the Australian context, appears to have application for our work in the area of equity and mathematics education. It is our intention to apply this model to a range of projects with which we are currently involved. Preliminary applications indicate that there is considerable scope for success. We anticipate that as the model is implemented it will be further refined for the particular contexts within which we work.

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The Role of Information Graphics in Mathematical Proficiency

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There is scant research on the role of graphics in students' mathematical performance. This paper distinguishes between the contextual and informational roles of graphics and provides an overview of the types of information graphics. It also presents findings from a new mathematics instrument that has been used to quantitatively and qualitatively assess students' performance on information graphics. Key findings using this instrument have provided insights into age, gender and item effects on performance, and difficulties that students experience interpreting graphics.

The development of a mathematically literate populace is a key goal for educators:

> (Mathematical literacy) is an individual's capacity to identify and understand the role that mathematics plays in the world, to make well-founded judgements and to engage in mathematics, in ways that meet the needs of that individual's life as a constructive, concerned and reflective citizen. (OECD, 2003)

Hence, students need to become proficient in interpreting *information graphics* (e.g., graphs, tables, maps) because such graphics are used to manage, communicate, and analyse information (Harris, 1996). Information graphics are distinct from contextual graphics in that they represent mathematical information that supplements rather than complements the text or symbolic expression. For example, in Figure 1, the picture of the scales *complements* the text but contributes no mathematical information. In contrast in Figure 2, the picture of the scales *supplements* the text and is essential to the solution. The purpose of this paper is to provide an overview of information graphics in mathematics and to outline how students' proficiency with graphics can be measured.

Figure 1. A contextual graphic.

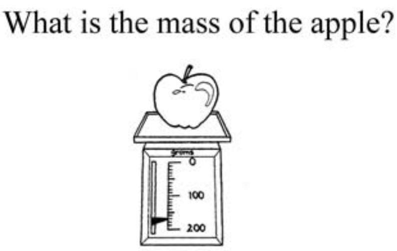

Figure 2. An information graphic (Queensland School Curriculum Council, 2001b, p. 31).

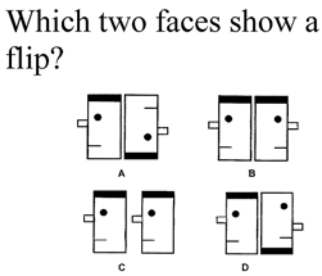

Figure 3. A retinal list graphic (Queensland School Curriculum Council, 2001a, p. 13).

Information Graphics and Mathematics

Information graphics is a burgeoning field with over 4000 graphics in common use (Harris, 1996). In mathematics, these graphics variously convey *quantitative, ordinal* and *nominal* information through a range of perceptual elements (Mackinlay, 1999). These elements are position, length, angle, slope, area, volume, density, colour saturation, colour hue, texture, connection, containment, and shape (Cleveland & McGill, 1984). In mathematics, information graphics can be classified into six graphical languages which have unique spatial structures based on their perceptual elements and the encoding techniques that represent information (Mackinlay, 1999) (See Table 1). For example, Figure 2 is an Axis item because information is encoded by the placement of a mark on a vertical axis. Figure 3 is a Retinal-list item that uses orientation. Further graphic examples can be found elsewhere (Diezmann, Lowrie, & Kozak, 2007; Diezmann & Lowrie, 2006; Lowrie & Diezmann, 2007). Excluding the Miscellaneous languages, the other five languages have substantial similarities within their categories. For example, although there are different kinds of maps, maps typically convey information about the location of landmarks and the distance between locations. In contrast, Miscellaneous languages are broad and undefined and include for example pie charts and calendars.

Table 1

Descriptions of Graphical Languages and Encoding Techniques

Axis Languages (e.g., horizontal and vertical axes	A single-position encodes information by the placement of a mark on an axis.
Opposed-position Languages (e.g., line chart, bar chart, plot chart)	Information is encoded by a marked set that is positioned between two axes.
Retinal-list Languages (e.g., Graphics featuring colour, shape, size, saturation, texture, orientation)	Retinal properties are used to encode information. These marks are not dependent on position.
Map Languages (e.g., road map, topographic map)	Information is encoded through the spatial location of the marks.
Connection Languages (e.g., tree, acyclic graph, network)	Information is encoded by a set of node objects with a set of link objects.
Miscellaneous Languages (e.g., pie chart, Venn diagram)	Information is encoded with additional graphical techniques (e.g., angle, containment).

Assessing Proficiency with Information Graphics

Typically, students' ability to decode particular types of graphics is not the focus in mathematical tests. However because knowledge of graphics impacts on mathematical performance (Baker, 2001), we constructed an instrument to assess students' knowledge of graphics. The Graphical Languages in Mathematics [GLIM] Test is a 36-item multiple choice test that was designed to investigate students' knowledge of each of the six graphical languages. The test was composed of mathematical items containing information graphics that were sourced from state, national and international tests that have been administered to students in their final three years of primary school. The purpose of using previously published items was to ensure that items (a) were representative of what students were expected to be able to do in the upper primary years, and (b) had been subject to rigorous quality control in item development. We compiled a database of possible test items from a large range of published tests, excluded contextual items, and classified remaining items according to the graphical languages. Due to the limited Connection items in existing mathematics tests, content free Connection items were included from published science tests for the same age group. In total, we identified 58 suitable items for trialling. This set of items was variously trialled with primary-aged children (N = 796) in order to select items that: (a) required substantial levels of graphical interpretation, (b) required minimal mathematics content knowledge, (c) had low linguistic demand, (d) conformed to reliability and validity measures, and (e) varied in complexity. A total of six items were selected for each graphical language and arranged according to difficulty based on students' performance on the trial. Our final selection of 36 items was validated by two experienced primary teachers. These items were then arranged so that every sixth item belonged to the same graphical language. For example, Items 1, 7, 13, 19, 25 and 31 are Axis language items in ascending order of difficulty. Figure 2 is Item 31 without the multiple choice answers. We have successfully used the GLIM test over a 3-year period in both mass testing and interview situations. A description of the administration of the GLIM test under these two conditions and some key findings follow.

The GLIM Test in Mass Testing Situations

The multiple-choice GLIM test can be administered in mass testing situations (approximately 45-60 minutes). The items are scored 1 and 0 for correct and incorrect responses respectively. Students' performance is calculated on each language subtest and the overall test. The maximum scores for the subtests and whole test are 6 and 36 respectively. We have used the GLIM test to monitor primary students' performance on interpreting information graphics over a 3-year period through annual administrations of this test. The participants in the 3-year mass testing study were 327 students (Female = 148, Male = 169) from nine primary schools in two states. In the first year of the study, students were approximately 9- 10-year-olds. Three points of interest have emerged from the various analyses of students' results during this project. First, although many of the information graphics are not explicitly taught to primary students, their performance was significantly higher year by year on each graphical language. Second, the analyses reveal gender differences in favour of boys over time. These first two points are discussed in Lowrie (this symposium). Third, after accounting for gender, spatial ability was a contributing factor to students' success (Lowrie & Diezmann, 2007).

The GLIM Test in Interview Situations

We have also interviewed students on items from the GLIM test. The interview students were sourced from different schools to the mass testing cohort. For pragmatic reasons, we interviewed students on 12 different items each year for three years. In the first year of the study, students were interviewed on the easiest pair of items from each graphical language. Students were approximately 9- 10-year-olds. In the second and third years the same students were interviewed on the moderately difficult items and the most difficult items respectively. In each year, the students completed one pair of items at a time from the same graphical language and then explained their responses. Students were encouraged to justify their thinking but no scaffolding was provided by the researcher.

Our qualitative analyses revealed three key issues of interest. First, some students have incorrect conceptions of graphics. For example, some students interpret a structured number line (Axis language) as a counting model rather than a measurement model (Diezmann & Lowrie, 2006). Second, students' conception of a graphic is manifest in how they use it. For example, students who hold a measurement conception of a number line justify their responses to the identity of a missing number in terms of points of reference, relative proximity to given numbers and directionality (Diezmann & Lowrie, 2006). Third, high and low performers interpret graphics differently. For example, high performers used multiple cues from the graphics and were more knowledgeable about everyday graphics (e.g., calendar) than low performers (Diezmann et al., 2007).

Concluding Comments

The GLIM test has provided a useful instrument for gaining insight into students' performance on graphical languages and the issues impacting on their performance. Our experience on this project has increased our resolve that in the Information Age, students' ability to interpret information graphics is fundamental to their mathematical proficiency. That is, just as the mathematical and linguistic demands of an item impact on performance so too does the graphical component. This point has implications for the credibility and interpretation of new national tests (see Diezmann, this symposium). In addition, the extent to which various components of an item impact on performance has been further explored by members of our project team (see Logan & Greenlees, this symposium). (For a full set of papers relating to our research on graphical languages in mathematics see the project website http://www.csu.edu.au/research/glm/index.htm).

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A Longitudinal Study of Student Performance on Items Rich in Graphics

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This investigation examined the performance of students (9–12 year olds) over a three-year period as they solved graphics-based mathematics items which are commonly found in standardised tests. There were statistically significant improvements in the performance of students (across each of the six language types) in each year of the study. Mean scores for boys were higher than girls on most measures with statistically significant differences in Axis (number line), Map and Retinal-list categories.

Interpreting and decoding spatial information (such as the interpretation of graphs, maps, and diagrams) is necessary in educational contexts and essential in everyday life (Åberg-Bengtsson, 1999). In recent years there is renewed debate regarding the approaches that teachers and curriculum developers should take to ensure the development of a numerate populace who can cope effectively with the practical mathematical demands students experience in both school and out-of-school settings (e.g., Australian Association of Mathematics Teachers, 1997). These practical demands include the capacity to collect, handle and interpret data within graphical contexts. Moreover, there has been an increased awareness in the fact that some of the graphics students are required to (increasingly) interpret are not explicitly taught in school contexts (Lowrie & Diezmann, 2005).

Although the focus on representation (and spatial processing) in teaching, learning, and understanding mathematics is widely acknowledged (e.g., Cucuo & Curcio, 2001) the research on the use and understanding of images and graphics is quite limited (Postigo & Pozo, 2004) despite the calls for this essential and increasingly important literacy (Zevenbergen, 2005). Postigo and Pozo (2004) argued that previous research conducted in this field is quite heterogeneous since the study of maps, diagrams and numerical graphs have their own syntax and conventions. It is also the case that student performances across different types of graphics (e.g., number lines and maps) are not overly strong (Lowrie & Diezmann, 2005) and that correlations between items within the same graphic are at best moderate (Lowrie & Diezmann, 2007).

Gender and Performance on Graphics Items

A broad body of literature has examined the differences between students on spatial tasks with gender differences in performance a central concern. Although gender differences are widely acknowledged (Linn & Petersen, 1985) the extent of these differences, the age when these differences occur (and/or diminish) and reasons for these differences have raised considerable debate. Some studies have concluded that differences in the performance of boys and girls across spatial tasks have become less apparent in the past 50 years (across a range of variables) since children's experiences have become less discrete—in the sense that traditional stereotypical roles are less obvious. Lowrie and Kay (2001), for example, argued that 12 year olds were likely to have been exposed to similar spatial experiences—irrespective of their gender. Other researchers (e.g., Spelke, 2005) have argued that the gap between the performance of boys and girls has diminished in the past ten years.

It is certainly the case, however, that few studies have examined gender differences over time. Moreover, researchers have tended to broadly analyse large data sets rather than being more focused and strategic about examining differences between males and females on mathematics tasks. Fennema and Leder (1993) have suggested that rather than examining large data sets of mathematics performance, studies should be purposeful and focused. The present investigation expands upon the research literature by examining students' performance in a specific field of mathematics education—the decoding of spatial tasks that contain information presented in graphics.

Research Design and Methods

The purpose of this investigation was to:

1. Examine the decoding performance of students' solving graphics items over time; and
2. Determine whether there were gender differences in their performance.

The Participants and Procedure

The participants (n = 327: Female = 148, Male = 169) were randomly selected from nine primary schools across different states in rural and metropolitan areas of Australia. These included six non-government and three government schools. The participants were investigated in the last three years of their primary education (age range 9-12 years). They completed the 36 item GLIM Test (see Diezmann & Lowrie, this symposium) in approximately 50 minutes within intact classes on an annual basis. The participants were not involved in any treatment program throughout the study—they continued with the mandatory curriculum of their respective state.

Results and Discussion

1. Performance Differences on Graphics Languages over Time.

The first analysis measured participants' knowledge of graphical languages over a three-year period (Grades 4 to 6). An Analysis of Variance (ANOVA) (year with graphical languages) revealed a statistically significant difference between the performance of students across year [$F(2,1047) = 91.76$, $p<.001$]. Subsequent post hoc analysis revealed statistically significant differences in the performance of students between Grade 4 and Grade 5 [$t(1,351) = 3.28$, $p<.001$] and Grade 5 and Grade 6 [$t(1,324) = 2.07$, $p<.001$].

An ANOVA was then used to determine performance differences of students within each graphical language. There was a statistically significant difference between student performance within language across year [$F(2,1048) = 16.05$, $p<.001$]. Subsequent post-hoc analyses revealed statistically significant differences between students' performance on each of the six graphical languages (see Table 1). Thus, the performance of participants across all six categories of graphs was significantly higher in Grade 6 than in Grade 5 and Grade 4. The results support the findings of other studies showing that the graphic performance of adolescents (Postigo & Pozo, 2004) and mapping skills of primary-aged children (Liben & Downs, 1993) improved over time.

Table 1

Means, Standard Deviations and F Values for Questions 1 and 2

	Grade 4		Grade 5		Grade 6		Year	Gender
							F Ratio	F Ratio
	Male	Female	Male	Female	Male	Female	Df(2,1048)	Df(1,1048)
Axis	4.15	3.37	4.61	3.87	4.95	4.42	42.52***	71.31***
	(1.17)	*(1.33)*	*(1.24)*	*(1.43)*	*(1.13)*	*(1.20)*		
Opposed-position	3.26	3.35	4.05	3.79	4.21	4.10	41.75***	.97
	(1.40)	*(1.27)*	*(1.32)*	*(1.19)*	*(1.26)*	*(1.24)*		
Retinal-list	3.16	2.96	3.86	3.61	4.15	3.98	51.60***	5.2*
	(1.36)	*(1.35)*	*(1.24)*	*(1.40)*	*(1.26)*	*(1.41)*		
Map	4.34	4.10	5.08	4.64	5.25	4.98	60.55***	18.56***
	(1.24)	*(1.31)*	*(.93)*	*(1.09)*	*(.93)*	*(1.06)*		
Connection	3.19	3.23	3.66	3.66	4.20	3.84	35.22***	1.2
	(1.39)	*(1.20)*	*(1.35)*	*(1.16)*	*(1.27)*	*(1.26)*		
Miscellaneous	4.16	4.17	4.61	4.50	5.09	4.95	38.34***	.77
	(1.42)	*(1.48)*	*(1.29)*	*(1.27)*	*(.94)*	*(1.18)*		

Note: * $p < .05$; *** $p < .001$

2. Gender Differences in Student Performance

The second aim of the investigation was to determine whether there were gender differences in the performance of students over time. The mean scores for the male students were higher than that of the female students in all six languages in Grade 4, Grade 5 and Grade 6 (see Table 1). An ANOVA (gender with graphical languages) revealed statistically significant differences between boys and girls [$F(6, 1020) = 15.23$, $d = .08$, $p<.001$] Post-hoc analysis revealed statistically significant differences between boys and girls and *axis*, *retinal* and *map* categories (see Table 1).

Although recent research by Liben and Downs (1993) found no gender differences when Grade 5 children were required to complete a series of axis questions, the Axis language findings are consistent with the results of Lowrie and Diezmann (2005, 2007) and Hannula (2003).

Kitchin (1996) postulated that gender differences in the interpretation and decoding of maps may be a result of females having less access to situations that develop spatial skills or that measuring tasks favour male problem-solving strategies. Boardman (1990) highlighted the fact that gender difference in mapping ability may increase over time and that by adolescence boys demonstrate more highly developed map skills than girls. In the present study, performances differences between boys and girls remained relatively constant over the three-year period. It could be argued that the girls in our study were much more likely to be exposed to maps than students in the earlier studies—especially given the increased attention of maps in the school curriculum (and arguably even more influential, the exposure all students have to maps in everyday life). Despite this, gender differences remain.

Conclusions

There were significant improvements in the students' performances across all graphics categories over the three-year period. The students were not involved in any treatment program during this time and simply continued with the regular mathematics curriculum. I would speculate that the participants' general literacy and quantitative literacy would have improved over this timeframe—and as a result—this increase in mathematics capacity provided an additional knowledge base to call upon when solving the problems. The most marked improvements were across the *connection* language—with similar questions more than likely encountered through science curricula. Consistently, the students found the *retinal-list* tasks most difficult to solve. These items required the students to transform, reflect, rotate and translate objects.

The boys outperformed the girls in each of the six graphical languages. Furthermore, there were statistically significant differences between the performances of boys and girls across the *axis, retinal* and *map* languages.

Acknowledgments. The Graphic Languages in Mathematics project was funded by the Australian Research Council (#DP0453366). Special thanks to my co-investigator, Carmel Diezmann, and our research assistants, Tracy Logan, Jane Greenlees, Lindy Sugars and Nahum Kozak for their contributions.

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Standardised Assessment in Mathematics: The Tale of Two Items

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This paper describes the sense making of 11- 12- year-olds as they interpret two mathematics items which include graphics. In particular, it outlines the changed behaviour (and performance) of students when solving items when slight modifications were made to the graphic or the mathematical language. The results show that performance increased when the graphic was modified but diminished when the language was modified. Implications include the need for test designers to carefully consider the graphic embedded within assessment items.

The capacity to interpret and decode graphs requires the problem solver to use spatial information to make sense of nonspatial relationships and concepts (Gattis & Holyoak, 1996). In mathematics contexts, these relationships and concepts are associated with mathematics literacy and content/context information. Although graphics are often considered "one of the simplest symbolic systems for interpreting information on the relationship between two or more sources" (Parmar & Signer, 2005, p. 250) primary students often find such representations overloaded with information and therefore difficult to decode (Lowrie & Diezmann, 2007). Moreover, graphs can become ineffective if too much information or too little information is presented (Kosslyn, 2006). If the graphics associated with testing items are not well designed, it is unlikely that results (outcomes) will be a reliable reflection of student understanding.

Research Design and Methods

This investigation builds upon the work of a three-year longitudinal study in which we monitored the development of primary students' ability to decode test items with high graphical content. This study focuses on items that were modified in relation to either the graphic or the mathematical language. The aims of this component of the study were to:

1. Ascertain student performance on graphics items and determine which elements of an item influence performance; and
2. Identify the sense-making that led to success on the items.

The Participants

Forty Grade 6 students (aged 11-12 years) from three regional NSW schools (one Government and two non-Government) took part in this study. The participants were from varying socioeconomic and academic backgrounds and were not involved in any treatment program throughout the study. These participants were accustomed to interpreting and solving items of this nature since they were part of the larger study (see Diezmann & Lowrie, this symposium).

Data Collection and Analysis

The following section describes the two phases of the project.

Phase 1. This phase formed part of a larger study which traced these participants' sense making over a 3-year period. The research team conducted structured, in-depth, interviews where students had an opportunity to verbalise and justify the processes they used to complete items from the Graphical Languages in Mathematics test (Diezmann & Lowrie, this symposium).

Phase 2. From an analysis of the interview data, students' responses were coded to ascertain the problem-solving processes students used to solve the respective items. Once these data were collated, consistent patterns in student responses were sought. These patterns were associated with students' interpretation of the item but also elements of the item which included the graphic and the mathematical language embedded within the item. As a result it was evident that these elements had a significant impact on the way in which the participants interpreted and solved the items. These items were slightly redesigned and the students were re-interviewed and asked to solve the modified items.

Results

In this paper, we focus on the two items (see Appendix) that had the highest negative change and the highest positive change between the first and second interviews. We discuss the change in the processing and sense-making students undertook as they solved the modified items.

After initial analysis of the first interview, it was evident that students employed similar (or at least consistent) strategies to make sense of the respective items. Therefore coding was developed in order to capture the patterning of incorrect responses that took place as problem solving occurred. Although the strategies identified were relatively generic it was certainly the case that some of these approaches were more likely to be employed for particular items. Of the 18 incorrect responses to Item 1 (The Whale Item), 15 of the students' responses were coded as—*did not consider **i**nformation on **g**raphic*. With respect to Item 2 (the Line Graph), 31 of the 35 incorrect responses were coded as—***o**verly influenced by **i**rrelevant information or pictures embedded in the graphic*.

Based on students' performance on Test A (original) and Test B (modified), these two items were identified as having the largest change in performance. Although the effect size (measured by Cohen's *d*) for the Whale Item was relatively small, it had the highest negative trend (a diminished performance on the modified test). While the Line Graph (with a large effect size) had the highest positive trend (an increased performance on the modified test). Table 1 highlights these results.

Table 1

Students' Performances across Test A and Test B

	Whale item		Line graph item	
	A	**B**	**A**	**B**
% Correct	60	53	22	60
Effect size (Cohen's *d)*	-.14		.83	

The Whale Item

The majority of students chose one of two "plausible" solutions in Test A (see Appendix). Students who chose the correct response (Fin whale) looked at length as an exact measurement and the mass as an approximate measurement. Those who answered the item incorrectly (Right whale) still used an effective strategy but allocated exact measurements to both the length and the mass. By highlighting the word *approximately* in Test B we hoped to bring to students' attention the second variable, mass and that it was an approximate measurement, thus eliminating one of the two options they were unsure about. However, it was found in Test B that the students who had initially solved the item correctly now became distracted by the bolded word and based their answer on the mass measurement, looking for the exact 80 tonnes and assigning the approximation to the length, the reverse of what occurred in Test A.

> I had a feeling it was the Right [whale] so I looked at mass and it said 80 tonnes. It was closer to 80 to me and the length kind of shows 25m [Tommy].

The modification of one word highlights the ease with which students can misinterpret the language in an assessment item. While the graphic itself did not change, the students' interpretation of the graphic was very closely linked to their understanding of the language and how that information applied to the graph. Adams (2003) found that students often miss important information focusing on key words without reading the entire question. We therefore envisaged that the students' performance would have improved on the modified item as we were drawing attention to an aspect of the question that was initially missed.

The Line Graph Item

On the line graph item in Test A, the performance of students was very poor (22% correct) (see Appendix). It became apparent that the appearance of dots on the line graph at various intervals along the line was being interpreted as a stopping point.

> It has 6am and 7am, so that took 1 hour until she had a rest. Because it's a line graph the circle/dot is like a rest and it tells you how long she rode [Alex].

These dots were removed in Test B (see Appendix) to give the students the opportunity to actually read the line graph without being distracted by the dots. In Test B, students went from having a simplistic understanding of a line graph—being able to read the axes but being unable to interpret the line—to being able to incorporate all elements of the graphic. For example in Test A Rebecca responded, "I chose 1 hour because she started at 6am and she stopped at 7am because here it has a dot where it was a new hour". Whereas in Test B she explained, "I chose two hours because on the graph it keeps on going up until she gets from 10am to 12pm and then it just goes straight so she's not moving any distance which means she must have stopped". This change in Rebecca's thinking suggests that the visual features of the graph can affect children's interpretation of graphical items (Gattis, 2002) and the important role the format of the graph plays in students' comprehension and reasoning processes (Carpenter & Shah, 1998).

Conclusions and Implications

Information graphics have become increasingly important in representing, organising and analysing information and consequently the presence of graphics is now more evident in syllabus documents. The prevalence of such representation in curricula is in turn reflected in assessment practices—and particularly standardised instruments. This study provides insights into the impact that the graphical elements and associated language have on student understandings and performance. These elements are both influential but we found that the design of graphical items can be enhanced (and thus become a more reliable indication of performance) if more attention is paid to the design of the graphic. Such implications are particularly relevant at a time when national testing is becoming increasingly influential in mathematics education research and classroom practice.

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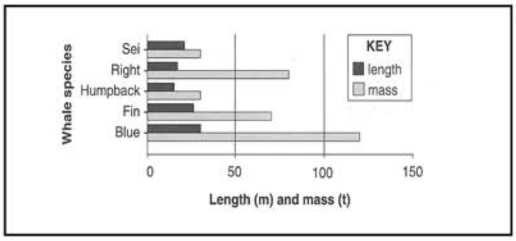
Appendix: Standard and Modified Test Items

<table>
<tr><th>Test A Item</th><th>Test B (Modified) Item</th></tr>
<tr><td>

1 The graph compares the maximum length and mass to which some whales grow.



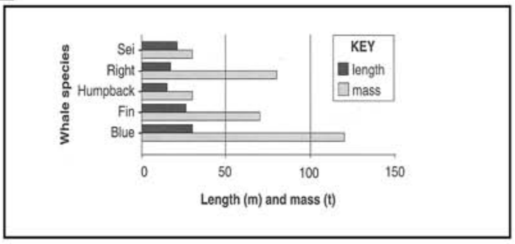
A fisherman reported that a whale 25 metres long and weighing approximately 80 tonnes had beached itself. Which species of whale could this be?

Answer

- ☐ Right whale
- ☐ Humpback whale
- ☐ Fin whale
- ☐ Blue whale

</td><td>

1 The graph compares the maximum length and mass to which some whales grow.



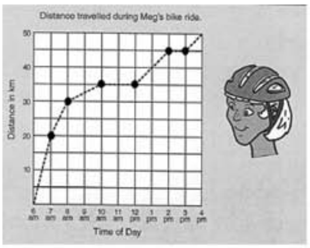
A fisherman reported that a whale 25 metres long and weighing **approximately** 80 tonnes had beached itself. Which species of whale could this be?

Answer

- ☐ Right whale
- ☐ Humpback whale
- ☐ Fin whale
- ☐ Blue whale

</td></tr>
<tr><td>

2 How long was Meg's first rest?

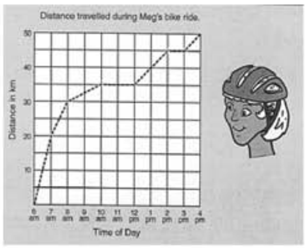


Answer

- ☐ 1 hour
- ☐ 2 hours
- ☐ 3 hours
- ☐ 4 hours

</td><td>

2 How long was Meg's first rest?



Answer

- ☐ 1 hour
- ☐ 2 hours
- ☐ 3 hours
- ☐ 4 hours

</td></tr>
</table>

Graphics and the National Numeracy Tests

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National numeracy tests herald a new era in Australian school assessment. The sets of sample test items suggest that understanding information graphics (e.g., maps) will be an important component of these tests. However, an analysis of test items reveals limited types of graphics in sample sets, poor quality graphics, atypical use of graphics, and a lack of consistency in the graphics depicting a common shape. These findings indicate the importance of repeating the analysis with the actual tests.

Over the past four years, my colleagues and I have been investigating Australian students' performance on test items that include information graphics (see Diezmann & Lowrie, this symposium; Logan & Greenlees, this symposium; Lowrie, this symposium). Consequently, we argue that a students' ability to comprehend a graphic within a test item will strongly influence his or her ability to successfully complete the item. Thus, our team has a particular interest in the graphics within the inaugural Years 3, 5, 7 and 9 national numeracy tests. In this paper, I present an analysis of the sample items for Years 3, 5, 7 and 9 from the national numeracy tests (MCEETYA, 2007a) to ascertain (a) the role of graphics in these tests, and (b) the quality of the graphics that are included. As a background to this analysis, an overview of graphic comprehension is presented, followed by an outline of the various aspects of graphics and their use.

Understanding Graphics

Perceptual and Cognitive Processes in the Comprehension of Graphics

The comprehension of a graphic involves the interaction between a visual symbol system and perceptual and cognitive processes (Winn, 1994). The symbol system is composed of (a) *visual elements*, such as shapes, that represent objects or ideas and (b) the *spatial relationships* among the elements within the graphic (e.g., one shape inside another). Hence, descriptions of graphical languages (i.e., types of graphics) include reference to both the visual elements used and how the elements are spatially related (see Diezmann & Lowrie, this symposium). For example, a map is comprised of information which is encoded through the spatial location (spatial relationship) of marks (visual element) (Mackinlay, 1999)

In mathematics, the selection of a visual element is typically related to how the creator of a graphic wants the element to be perceived. For example, a steep hill may be represented with a steeply sloped line. An alignment between the referent and its representation has perceptual advantages. Other perceptual elements employed in graphics include position, length, angle, area, volume, density, colour saturation, colour hue, texture, connection, containment, and shape (Cleveland & McGill, 1984). In addition to perpetual processes evoked by visual elements, cognitive processes play an important role in the interpretation of graphics (Winn, 1994): "(These processes) involve the *detection* of symbols, the *discrimination* of one symbol from another and the *configuration* of symbols into patterns" (p. 5) (emphasis added). These processes are then followed by the *identification* of the graphic in relation to the individual's existing knowledge. The processes of detection, discrimination, configuration and identification may be revisited as an individual attempts to comprehend a graphic (Winn, 1994).

A Meta-Taxonomy of Graphics and their Use

There has been considerable research into the use of graphics across various disciplines. However, this research is quite disparate and informed by various theories. As a means to further the "science of diagrams" (p. 47), in Blackwell and Engelhardt's (2002) terms, they have proposed a meta-taxonomy for diagram (aka graphics) research from an extensive review of theories. Their meta-taxonomy has four major aspects: (a) *Signs* (i.e., graphic elements, conventions, level of pictorial abstraction); (b) *Graphic structures* (e.g., a tree diagram, a linear diagram); (c) *Meaning* (i.e., correspondence between a representation and its meaning, such as literal or analogical correspondence; and classifications of information by other theorists, such as graphical languages by Mackinlay (1999); and (d) *Context-related aspects* (i.e., the interaction between the person and the graphic in creating or modifying a diagram, the cognitive processes involved in interpreting a diagram, and the cultural context of the graphic). Each of these aspects needs consideration when a graphic is created as a representation of a mathematical situation.

Graphics and the New National Tests

Sample items from the Years 3, 5, 7, and 9 numeracy tests were analysed according to three questions related to the proportion of graphics on the test, the type of graphics employed, and the quality of these graphics. The limitation of using sample items is acknowledged; however final test items were unavailable at the time of writing and the characteristics of sample items are inclusive of the test items (MCEETYA, 2007b).

What proportion of sample items for the new national tests contained graphics?

A total of 49 sample items (53.3%) contained either information (i.e., structural) (n=46) or context graphics (n=3) (Table 1). Hence, students' ability to distinguish between these graphics and use them appropriately will impact on their performance. For example, in Figure 1 students should use the graphic to determine the number of sausages in calculating the length of the sausage string (Figure 1). In contrast, in Figure 2 students should not use the number of wheelbarrows or the size of the sand piles in the context graphic to calculate the amount of sand that was moved. Discrimination is a key process in comprehending a graphic (Blackwell & Engelhardt, 2002).

What types of information graphics are included in the sets of sample items?

An analysis of the sample items revealed that a variety of graphical languages (Mackinlay, 1999) were presented across the tests with the exception of Connection items (Table 1). In addition, Miscellaneous (47.8%) and Retinal-list items (37.0%) seem more likely than other language items to appear on the national numeracy tests. With the exception of Miscellaneous items, information graphics have unique graphical structures based on the visual elements that are represented and the spatial relationships among these items within the graphic. Hence students' understanding of the structural aspects of graphics will influence their comprehension of these graphics (Blackwell & Engelhardt, 2002).

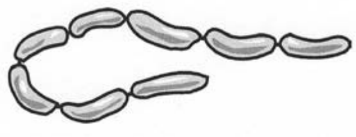

Figure 1. An information graphic (MYCEETYA, 2007b, Sample 2, Item 1)

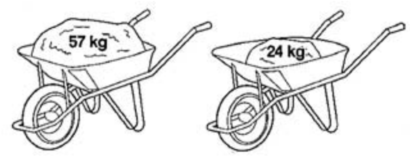

Figure 2. A context graphic (MYCEETYA, 2007b, Sample 1, Item 3)

Table 1

Proportions of Sample Items that Contain Graphics by Year and Type.

Year	Total Items	Graphics Items	Context Graphics	Information Graphics: *Graphical Languages*					
				MI	*RL*	*MA*	*AX*	*OP*	*CO*
3	16	75%(n=12)	1	8	1	2	0	0	0
5	14	64.3% (n=9)	1	4	2	1	1	0	0
7	13	53.9% (n=7)	0	2	4	0	1	0	0
7[C]	12	41.7% (n=5)	0	2	3	0	0	0	0
9	16	50% (n=8)	1	4	2	0	0	1	0
9[C]	21	38.1% (n=8)	0	2	5	0	0	1	0
Total	92	53.3% (N=49)	3	22	17	3	2	2	0

Key: *MI=Miscellaneous; RL=Retinal List; MA=Map; AX= Axis; OP= Opposed Position; CO=Connection* [C] indicates the calculator component of the test in Year 7 or 9.

Do the sample items indicate high quality representation of graphics?

There are three concerns with the quality of the graphics in the sample items. The first concern is the *artistic quality of the graphics*. It would be inexcusable to misspell a word or use mismatched fonts in a numeracy test yet poor quality drawings are included. For example, the drawing of pots in a tray (Figure 3) shows lines inappropriately going through the pots and also the right side of the tray, which affects depth perception. The second concern is the *lack of consistency* in showing how a common three-dimensional shape should be represented graphically. In a Year 3 item, the students are expected to identify a *cylinder* based on its visual characteristics (Figure 5), yet in a Year 5 item a *cylinder* is inappropriately represented with two dimensions (Figure 6). This 3D-2D mismatch has implications for the identification of cylinders and the need to attend to graphical conventions (Blackwell & Engelhardt, 2002). The third concern is the use of *atypical graphics*. For example, a grid is not commonly used to identify large sections of a country (Figure 4). More typically used are states and territories or regions with similar characteristics (e.g., geographic terrain). Thus, there is a lack of attention to the cultural context in which graphics are typically used (Blackwell & Engelhardt, 2002). In each of these examples, there is potential for the quality of the graphics to impact negatively on the students' perceptual or cognitive processes employed in comprehending graphics, which is not ideal in assessment tasks.

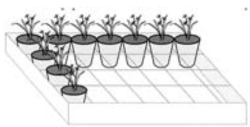

Figure 3. A poor quality graphic (MYCEETYA, 2007b)

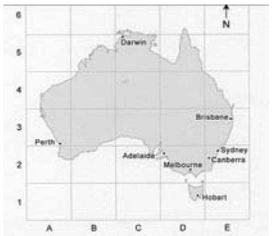

Figure 4. An atypical graphic (MYCEETYA, 2007b)

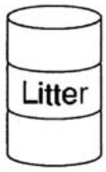

Figure 5. A 3D cylinder (MYCEETYA, 2007b)

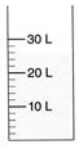

Figure 6. A 2D "cylinder" (MYCEETYA, 2007c)

Concluding Comments

The analysis of sample items in the national numeracy tests has revealed that graphics are likely to play a major role in student' performance on the actual tests. What is within the control of the students (supported by their teachers) is their knowledge of context and information graphics and how to interpret these. What is *beyond* the control of the students is the perceptual or cognitive processing errors and comprehension errors that may result from poor quality graphics. Thus, if the national tests are to have credibility as a performance measure, all graphics need to be of high quality, which is not the case with some of the sample items. In fairness, I will suspend my judgement of the national numeracy test until the tests are available for similar graphical analysis. However, the findings from this study indicate the importance of scrutinising the national tests to provide informed comment on the interpretation of student outcomes from the national tests.

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The Construction of Knowledge: Theoretical Approaches

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Chair: Gloria Stillman *University of Melbourne*

The construction of knowledge by students continues to be a central concern of mathematics education, and research has shown the importance of theoretical approaches for our understanding of construction of knowledge. Tommy Dreyfus has not only made substantial contributions to this field but has forged links between researchers in this area, and encouraged new researchers. This symposium provides an introduction to Tommy's research for some MERGA members, a chance to reflect further on it for others, and opportunities for everyone to develop new insights by connecting ideas in these papers.

Tommy Dreyfus *introduces "Abstraction in Context".*

Building on ideas of the Freudenthal School and of Activity Theory, abstraction is defined as an activity of vertical reorganisation, achieved by means of actions on mental or material objects, through which previous constructs are combined, connected, structured and developed into novel ones. According to this definition, abstraction is not objective and universal but depends on the learning context and the social context in which it takes place. This view of abstraction is consonant with Davydov's view that abstraction proceeds from an undeveloped and often vague initial form to a consistent and elaborate final form, and with van Oers' criticism of decontextualisation as a basis of abstraction. The three epistemic actions of Recognizing, Building-with and Constructing form the basis of a model for describing and analysing processes of abstraction. The model has proved useful to describe processes of abstraction for a wide variety of mathematical contents in both, classroom and laboratory settings. Recent studies on partially correct constructs and on justification have exhibited the analytic nature of the model; these studies will be used to exemplify the approach in separate presentations.

Mike Thomas *discusses "Constructing Versatile Mathematical Conceptions".*

In this talk we will describe the idea of versatile mathematical thinking as comprising three aspects: process/object versatility—the ability to switch at will in any given representational system between a perception of particular mathematical entities that may be seen as a process or an object; visuo/analytic versatility—the ability to exploit the power of visual schemas by linking them to relevant logico/analytic schemas; and representational versatility—the ability to work seamlessly within and between representations, and to engage in procedural and conceptual interactions with representations. Examples, taken from algebra, calculus and linear algebra, will be presented to illustrate the role of these three aspects in building conceptions, and some advantages of versatile mathematical thinking. In addition, ways in which such versatility may be mediated through an informed use of technology, and the implications for teachers' pedagogical technology knowledge (PTK) will be discussed.

Jill Brown *examines "Constructing Knowledge through the Perceiving of Affordances of a Technology-Rich Teaching and Learning Environment (TRTLE)".*

This theoretical approach is intended to find evidence of how mathematical knowledge is constructed in a TRTLE. Gibson's affordances and Valsiner's zone theory are used as lenses on interactions in classrooms and outcomes of those interactions. The affordances of a TRTLE are the offerings of that environment, of which the people are a part, for facilitating or impeding teaching and learning. For learning to occur some of the existing affordances of the TRTLE must be perceived and acted on. What makes this perception possible is analysed using Valsiner's Zone of Free Movement (ZFM) and the Zone of Promoted Action (ZPA). These are ever changing in nature. The ZFM describes what learning is possible and is characterised in a TRTLE by what is currently available (technologies, affordances, allowable actions). The ZPA describes what learning is promoted or encouraged and is characterised by the particular set of activities, artefacts, actions being promoted by the teacher or students. The ZPA is non-binding in nature and hence students may choose to act outside the ZPA.

Gaye Williams *examines "Cognitive artefacts and abstracting: What can we assume?"*

Synthesis of the cognitive elements of Dreyfus, Hershkowitz, and Schwarz's Recognizing, Building-with and Constructing Model or RBC Model, and Krutetskii's 'mental activities' frames this exploration of a surprising finding: insights about the usefulness of the Cartesian Axes System as a tool to interrogate linear functions does not presuppose a connected understanding of variable. Dreyfus' theoretical perspective that "abstraction is not objective and universal but depends on the learning context and the social context in which it takes place" is illustrated in the case described. Data from lesson video, and a video stimulated student interview illuminated the idiosyncratic nature of the constructing process that led to insight even though cognitive artefacts associated with understanding variable were not possessed. This study informs teaching and research by raising questions about what can and cannot be assumed when students construct new knowledge. It makes a theoretical contribution to links between cognitive artefacts assembled and mathematical structures developed.