MATHEMATICS: ESSENTIAL RESEARCH, ESSENTIAL PRACTICE

volume 1

Proceedings of the 30th annual conference of the
Mathematics Education Research Group of Australasia

Edited by
Jane Watson & Kim Beswick

MERGA

Mathematics: Essential Research, Essential Practice, Volume 1

Proceedings of the 30th annual conference of the
Mathematics Education Group of Australasia

Edited by Jane Watson & Kim Beswick

ISBN 978-1-920846-13-8

Published by
MERGA Inc.
GPO Box 2747
Adelaide SA 5001
www.merga.net.au

The 30th annual conference of the
Mathematics Education Research Group of Australasia
was held at Wrest Point Hotel Casino, Hobart, Tasmania. Australia
2–6 July 2007

Conference Organisers

Convenor:	Kim Beswick
Conference Committee:	Kim Beswick, Noleine Fitzallen, Tracey Muir Jane Skalicky, Jane Watson
Proceedings Editors:	Jane Watson, Kim Beswick
Editorial Assistants:	Judith Deans, Noleine Fitzallen, Lynda Kidd, Jane Skalicky, Suzie Wright
Conference Program:	Noleine Fitzallen, Jane Skalicky, Suzie Wright

Conference Sponsors

University of Tasmania

CONTENTS

VOLUME 2

RESEARCH PAPERS

SYMPOSIA

ROUND TABLE DISCUSSIONS

SHORT COMMUNICATIONS

Preface

This is a record of the proceedings of the 30th annual conference of the Mathematics Education Research Group of Australasia (MERGA). The theme of the conference is *Mathematics: Essential research, essential practice*. The theme draws attention to the importance of developing and maintaining links between research and practice and ties in with the joint day of presentations with the 21st biennial conference of the Australian Association of Mathematics Teachers (AAMT). This special feature highlights the benefits of collaboration between researchers, practising classroom teachers, and curriculum developers.

We are pleased to welcome conference participants who are attending MERGA for the first time. We hope you will make yourselves known so you can be made welcome and introduced to others who share your research interests. Authors from nine countries are represented in these proceedings, as well as from nearly every university in Australia and New Zealand with education programs. There are also participants from state and private school systems and government ministries of education. We look forward to the dialog that will emerge from the varying perspectives brought by participants, especially through the forums that will take place on the joint day shared with the AAMT.

All research papers and symposia submitted were blind peer-reviewed (without the author/s being identified), by two experienced mathematics education researchers who followed strict guideline that have been honed over a number of years. Where the two reviewers, who did not know the identity of the other reviewer, disagreed about the acceptability of a paper, another blind review was carried out by a third reviewer. For consistency, a small panel of highly experienced reviewers undertook the task of reviewing papers in this category. Only those research papers that were accepted by two reviewers have been included in these conference proceedings. The abstracts for short communications and round table discussions were read by two reviewers, who provided feedback and advice to authors on the MERGA guidelines for these types of presentation.

We would like to thank the University of Tasmania, Faculty of Education, for the financial support provided to complete the publication of these proceedings, as well as the hardy team of PhD students and research assistants who helped the academic staff with the conference program.

Kim Beswick
Chair, Conference Organising Committee
Editor

Jane Watson
Editor

The Beginnings of MERGA

Preamble to the Annual Clements/Foyster Lecture

In the middle of 1976 John Foyster, who was then based at the Australian Council for Educational Research (ACER), came to see me at Monash University, where I was in charge of the Mathematics Education program. John talked about how the Australian Science Education Research Association (ASERA) had recently been established, with Professor Richard Tisher (then of Monash University) as the prime mover. John wondered whether the time was ripe for a similar national group interested in *mathematics* education research to be established, and asked whether he and I might take steps to establish such a group.

My immediate reaction was yes, we should do it. Then came the doubts and reservations. How would the Australian Association of Mathematics Teachers (AATM) react to such an initiative? After all, AAMT already had a "Research Committee." In any case, would there be enough mathematics educators in Australia, interested in such a group to make it a viable proposition? Who would provide the funds likely to be needed for the establishment of such a group?

It was John's and my opinion that the AAMT Research Committee had not reached out to embrace most of the people lecturing in mathematics education in Australia at teachers colleges or in universities at the time. Intuitively, I thought Australia needed a group like the one John was proposing. My intuition told me that AAMT was not the organisation to move towards the establishment of such a group.

John assured me that he would put up any funds needed to get the group going (and, hopefully, any group that was established would be able to pay him back within a few years). Hence we decided to proceed with the idea of establishing the group and to strike while the iron was hot, so to speak, by conducting a national conference at Monash University in the middle of 1977. I came up with the name "Mathematics Education Research Group of Australia" which John liked because of the acronym MERGA, which suggested a "merging together." We sent out notices of our intention to form MERGA late in 1976. Neither of us knew many of the people who might be interested in joining such a group, so the notices were addressed to the "Mathematics Lecturers at…"

Soon after we had decided to go ahead, I heard of the existence of a group, based in New South Wales, called the Mathematics Education Lecturers' Association (MELA). John and I talked about whether MERGA and MELA might become one from the outset, but we decided that the aims of MELA seemed to be sufficiently different from those that we envisaged for MERGA, focused far more on research than lecturing, that we should proceed with the MERGA idea.

And so it came to be that in May 1977, the first of what was to become the annual conference of MERGA took place. About 100 people attended, with papers frenetically being read from 9 am to about 10 pm, for three days, in a Rotunda Theatre at Monash University. Professor Richard Tisher was present at the start of the Conference, and talked of his experiences in establishing ASERA. Frank Lester, of Indiana University, was among those present. In the event, two volumes of papers read at the Conference were produced (the first volume being available on the first day of the Conference, and the second several months later).

At a post-Conference meeting it was decided that, yes, MERGA should be formed, that the second meeting would be at Macquarie University in May 1978, and that an annual conferences should be held each year at a different academic institution. At that second conference it was decided by those present that MERGA should continue and a constitution and election of offices would be decided on at the third conference to be held at the then Brisbane College of Education. And so MERGA was born.

Ken Clements

Teaching and Learning by Example

The Annual Clements/Foyster Lecture

Helen L. Chick
University of Melbourne
<h.chick@unimelb.edu.au>

The mathematical problems, tasks, demonstrations, and exercises that teachers and students engage with in classrooms are, in general, specific instantiations of general principles. Indeed, the usual purpose of such examples is to illustrate those principles and thus facilitate their learning. With this in mind, it is clearly important for teachers to be able to choose or design suitable examples, to recognise what is offered (or afforded) by particular examples, and to know how to adapt an already existing example to better suit an intended purpose. Although writers of textbooks and other teaching resources also need these skills, it is ultimately the teacher who puts the examples to work in the classroom. Teachers' choice and use of examples is indicative of their pedagogical content knowledge (PCK)—the complex amalgam of mathematical and pedagogical knowledge fundamental to teaching and learning—and reflects their understanding of the mathematics to be taught and how students can be helped to learn it. This paper examines some of the issues associated with example use and how it is informed by and can inform us about PCK.

When a mathematics teacher asks a class to find the solutions of $x^2 - 5x + 6 = 0$, an observer may already have an idea about the point of the exercise. The task appears to be about solving equations—more specifically, quadratic equations. Beyond this, however, some contextual information is needed in order to understand fully the teacher's purpose in choosing that particular example. What if the next problem assigned is to find the solutions of $x^2 - 2x + 5 = 0$? Does this tell us anything? The two problems do not appear very different structurally, so why assign both? How are the two problems the same and different? What more does the second example tell us about the teacher's learning intentions?

This scenario highlights a number of issues. First, the teacher's purpose in using the tasks most likely is *not* to solve the specific problems but to teach more general principles. The actual solutions to the specific equations $x^2 - 5x + 6 = 0$ and $x^2 - 2x + 5 = 0$ are not of interest, but the teacher is likely *very* interested in highlighting conceptual issues such as equation-solving methods and the nature of solutions. Second, the purpose of an example is always context dependent. In this case, the presence of the second problem suggests that the focus of the learning activity might be on the fact that some equations do not have real solutions. Third, a particular example may be used to exemplify different things. For instance, the intended purpose for solving the equation $x^2 - 5x + 6 = 0$ might be factorising, completing the square, using the quadratic formula, or highlighting the fact that an equation can have more than one solution. Finally, for an observer to determine (or hypothesise about) the purpose of the examples requires mathematical knowledge. More significantly, however, the *teacher* had to know what mathematical ideas she wanted to convey and, with this knowledge, needed to be able to design or choose examples to suit her purpose.

Although this illustration comes from the secondary mathematics curriculum, the principles apply more broadly, including to primary mathematics teaching, the focus for the research reported here. Investigating these issues closely involves a consideration of what constitutes appropriate teacher knowledge, how to examine opportunities inherent in classroom activities, what is meant by "example", and how examples can be used.

Background

Pedagogical Content Knowledge

Before narrowing the focus to that part of teaching that involves example choice and use it is useful to briefly examine the broader domain of *pedagogical content knowledge* (PCK). Shulman's 1986 introduction of the term highlighted the fact that teacher knowledge—and resultant teacher effectiveness—depends on more than discipline content knowledge alone. He identified many of the facets of knowledge that contribute to PCK, including knowing what models and explanations support learning, understanding typical student conceptions, and recognising what makes a task complex or easy. These have now gained the attention of many researchers who have examined the nature of this knowledge in more detail. Other aspects of PCK include knowledge of connections among and within topics (e.g., Askew, Brown, Rhodes, Johnson, & Wiliam, 1997), deconstructing knowledge into key components (e.g., Ball, 2000), content knowledge (e.g., Kahan, Cooper, & Bethea, 2003), knowledge of representations (e.g., Leinhardt, Putnam, Stein, & Baxter, 1991), and Profound Understanding of Fundamental Mathematics (PUFM) (e.g., Ma, 1999). Lampert (2001) highlights the complex interplay among aspects of PCK in the classroom milieu. Drawing on this work, Chick, Baker, Pham, and Cheng (2006) developed a framework for pedagogical content knowledge (see Appendix 1). The framework attempts to identify the key components of PCK, how they are evident in teaching, and the degree to which both pedagogical and content knowledge are intertwined (see also Marks, 1990).

Everything that a teacher does—planning lessons, implementing them, responding to what arises in the classroom, interacting with students—involves one or more aspects of PCK. A lesson on the numeration of decimals, for example, might involve the decision to use a particular model to illustrate the concepts. This requires knowledge of different models and what they offer, recognising that their strengths and weaknesses depend on their *epistemic fidelity* (see Stacey, Helme, Archer, & Condon, 2001), that is, the capacity of the model to represent the mathematical attributes of the concept effectively. Having chosen the model, the teacher then has to use it appropriately in the classroom, recognising the students' present levels of understanding, developing appropriate explanations, and finding ways to respond to students' uncertainties and questions. The tasks that are then set in order to consolidate understanding or to foster its further development also reflect the teachers' PCK, since they should match the lesson's learning objectives.

Affordances and Didactic Objects

Considering tasks and how useful they might be in the classroom requires an evaluation of what they have to offer. Gibson (1977) introduced the term *affordances* to refer to the uses perceived for an object by a potential user. So, for example, a chair affords uses as a seat or a bookshelf but, at first, may not seem to afford a use as an umbrella. That said, however, observing a gorilla holding an upturned chair over its head in the rain reveals that, in the gorilla's perception, "rain shelter" *is* one of the affordances of a chair, and, thus, becomes an affordance of the chair for the observer now that the observer has perceived it too. This emphasis on the "perceived" uses is problematic, especially for some of the issues considered here, because in teaching there are many opportunities and examples that have the potential to be applied in pedagogically useful ways, and yet are not because the teacher does not perceive the opportunity. As a consequence, the term *potential affordances* is used to refer to the opportunities that are inherent in a task or lesson. A teacher may well be aware of some of them—indeed, awareness of these potential affordances is usually evident in how the task is used—but the teacher may not necessarily be aware of all of them, or even the "best" of them. Furthermore, in the unscripted world of the classroom, some of these opportunities may not come to fruition because of other interfering factors; as Anne Watson writes, learning environments involve "a complex interplay between what *could be* possible, what *is* possible, and what *is seen as* possible" (Watson, 2003, p.37). A teacher's PCK influences the degree to which she identifies the potential affordances in tasks and activities, makes pedagogical choices that allow her to offer desirable affordances in the classroom, and then finds ways of making those affordances give rise to effective learning.

Thomson (2002) talks more specifically about the role of discussion and usage in the learning process, and uses the phrase *didactic object*

> … to refer to "a thing to talk about" that is designed with the intention of supporting reflective mathematical discourse. … [O]bjects cannot be didactic in and of themselves. Rather, they are didactic because of the conversations that are enabled by someone having conceptualized them as such. (p.198)

This has relevance to models and representations, and, of course, examples. To illustrate this for models, note that although multi-base arithmetic blocks (MAB) are conventionally used to model base 10 numbers—especially units, tens, hundreds, and thousands—they can also be used to model decimal numbers. To do so, however, requires a reconceptualisation not only for the teacher, but also for the students. The MAB blocks afford the opportunity to model decimal fractions, but the reconceptualisation is needed to turn them into a didactic object. A whole new set of conversations must be evoked by the teacher in order to use MAB in this way, at the same time taking account of the epistemic fidelity issues (again, see Stacey et al., 2001). An example has the same capacity, potentially affording many things but delivering none until conceived as a didactic object. "Find the solutions of $x^2 - 5x + 6 = 0$" could illuminate many concepts, but its purpose must be identified by the user and then utilised in such a way that the desired concepts become apparent.

Examples

The meaning of "example" has, so far, been assumed as understood. It is necessary, however, to define it. For the purposes of this paper an example is a specific instantiation of a general principle, chosen in order to illustrate or explore that principle. This covers the usual sense of "example", such as a teacher making a point by giving a specific illustration (e.g., "eight is an even number because it can be written as two times a whole number") or demonstrating a solution procedure (e.g., a calculation using the long multiplication algorithm). It also covers assigned exercises and extended tasks.

Bills, Mason, Watson, and Zaslavsky (2006) give an extensive overview of the history of example use and the role of examples in learning theories. Ball (2000) highlights how a particular task needs to be examined by the teacher to determine what it offers students, and then discusses the issue of deciding how to modify the task to make it easier or simpler, or to make it illuminate particular concepts. Watson and Mason (2005, 2006) highlight the way in which changes to examples can highlight different concepts, and also show that getting learners to construct examples provides rich learning experiences. In fact, the situations discussed in the early chapters of their 2005 book show two significant aspects of examples. Although their primary thesis concerns examples constructed by students and how these develop mathematical understanding, in most cases these examples would not be generated without an appropriate task assigned by the teacher. Some of these tasks are quite open (e.g., “Construct a data set of seven numbers for which the mode is 5, the median is 6 and the mean is 7”, p.2). If the teacher’s intention with the task is to have it illustrate a general principle, notwithstanding that the students develop the specific instantiations, then it is argued that this makes the task an example too—perhaps in a “meta” sense, but an example nevertheless. Indeed a task may reflect more than one level of example-hood. A teacher may, for instance, select the “pizza” model for fractions—with the pizza exemplifying a fraction—and then ask students to show that $^1/_4$ and $^2/_8$ are the same—with the choice of $^1/_4$ and $^2/_8$ intended to exemplify general issues associated with equivalent fractions.

For all that a specific example may be an instantiation of a general principle, one of the key concerns in example use is to ensure that the general is revealed out of the particular. This requires teachers to identify the important and unimportant components of the example that illustrate the generality. Bills et al. (2006) cite a case from the work of Rowland and Zaslavsky illustrating how variation in some digits in the subtraction problem 62-38 still allows regrouping to feature, but that other choices “ruin” the problem for that purpose. Watson and Mason have adapted an idea of Marton (cited in Watson & Mason, 2005), *dimensions of possible variation*, to discuss ways in which an example’s scope can be varied. Skemp (1971, pp. 29-30) talks about the role of *noise* in examples, and that identifying the general principle requires the learner to distinguish the salient features from the extraneous. One key implication of this is that teachers’ example choices must allow the relevant features to be detected through the noise (although Skemp points out that some noise is important). Since there are often many variables and features in an example, choosing the appropriate instantiations is critical, and requires adequate PCK.

Returning to the framework for PCK (Appendix 1), all aspects of PCK can influence example choice and use. Of particular significance are (i) the underlying content-related aspects—such as PUFM and knowledge of connections and representations; (ii) knowledge of student thinking—both current and anticipated, together with knowledge of likely misconceptions; and (iii) the capacity to assess the cognitive demand of a task.

Bills et al. (2006, p.138) suggest that there is a scarcity of research on teachers’ choice of examples. Zazkis and Chernoff (2006) describe a situation where a researcher taught a student about prime numbers through choosing strategic examples, with the teaching situation such that examples had to be generated spontaneously rather than being planned in advance. This clearly relied on the researcher’s deep understanding of prime and composite numbers and the ability to construct examples that were appropriate for the student’s needs. Zaslavsky, Harel, and Manaster (2006) examined the mathematical knowledge brought into play by a teacher introducing Pythagoras’ Theorem to students on two different occasions. On the first occasion the cases chosen were intended to build up to the general result and reflected the teacher’s understanding of geometrical configurations that are useful for Pythagoras’ Theorem. On the second occasion the physical constraints of the way she had set up the examples—needing all sides to be integers—reduced the number of examples that could be given and may have affected the students' capacity to see the entire generalisation. Little has been done to investigate more specific aspects of PCK; this is part of the purpose of the present study.

Finally, it should be noted that many researchers have actually used examples to probe PCK. Hill and colleagues (Hill, Rowan, & Ball, 2005; Hill, Schilling, & Ball, 2004) have used multiple-choice questions that require teachers to examine a situation—a specific instantiation of a general scenario, involving a particular mathematical problem—and identify appropriate content- or pedagogically-based responses. Watson, Beswick, and Brown (2006) used a particular fraction/ratio problem to probe teachers' content knowledge, with follow-up questions investigating teachers' knowledge of students' likely thinking, including misconceptions, and their possible approaches for teaching the topic or remediating difficulties. The project from which the present research is drawn also used teaching situations based on specific examples to probe different aspects of teachers' PCK (see Chick & Baker, 2005a; Chick, Baker, et al., 2006; Chick, Pham, & Baker, 2006). In all cases the examples used were designed carefully in order to reveal general rather than specific aspects of the levels of PCK held by the teachers.

The Focus of this Paper and the MPCK Project

The current study considers some of the examples used by upper primary teachers. The intention is to examine the affordances inherent in the examples, and the way in which the teachers implement them to turn them into didactic objects. This examination provides insights into the teachers' PCK, and what needs they may have for developing it, particularly in regard to example choice. Although the examples are from the primary curriculum, it is anticipated that there are general principles that apply for teachers of other age groups.

The data for this study were collected as part of the ARC-funded Mathematical Pedagogical Content Knowledge project. This project involved fourteen Grade 5 and 6 teachers who volunteered to participate over a one- to two-year period. Part of the project's purpose was to examine teachers' PCK and how it is enacted in the classroom. A questionnaire and follow-up interview were used to gather initial data, and then pairs of lessons were observed and video-taped. The two lessons were on the same topic and conducted consecutively, with the teacher nominating the topic for observation. Up to four such pairs of lessons were recorded for each teacher. During the lessons the video-camera focused on the teacher, and the teacher's words were recorded via a wireless microphone that was sensitive enough also to record some student utterances. Field notes were also made. Following each pair of lessons, the teacher was interviewed about the original plans for the lessons, perceptions of successes and difficulties, changes and adaptations made, and future follow-up plans.

Several of the pairs of lessons involved fractions, and these lessons were subjected to a "content analysis" approach (Bryman, 2004), in which individual examples that arose in the classroom were identified, according to the definition of "example", and then categorized according to the way in which the teacher used it. This identified, for instance, whether the example was used as a teacher demonstration, or as a student task; or whether the example focussed on conceptual or procedural matters. From this data, and from data from three other pairs of lessons on other topics (probability, and measurement) several illustrative cases were selected to allow comparisons among the ways in which tasks were used, the affordances they offered, and the PCK involved. The purposeful selection of these cases makes them what Bryman (2004, p. 51) calls *exemplifying cases*, which are used for the purposes of a multiple-case comparative study.

There are, of course, some caveats about what can be learned from such a research design. Although information about teachers' intentions was obtained from the post-lesson interviews, these interviews were wide-ranging and did not always focus on examples per se. Consequently the teachers' purposes have, at times, been inferred from their implementations and classroom actions. Furthermore, it is easy, as an outside observer with the benefit of repeated video viewings, to see alternative options that teachers might have utilised to good effect. It is, however, important to acknowledge the complex milieu of the classroom, the speed with which some decisions must be made, and that, in these cases, mathematics is not the only area of the curriculum that primary teachers must teach.

Three Sets of Examples

This section describes three sets of examples that highlight important issues associated with example choice, affordances, and PCK. As explained earlier, the examples were purposefully selected from the lessons of eight of the MPCK teachers (names are pseudonyms), from nine of their pairs of lessons. The examples were chosen for what they illustrate qualitatively rather than to reflect any quantitative assessment about either the types of examples used in general or by a particular teacher. The scale of the examples varies, ranging from an assigned computational exercise through to an extended problem that the teacher utilised to illustrate a wide range of mathematical concepts. The pedagogical implications—such as the affordances offered by the examples described, and the PCK evident or missing in the choice and implementation of the examples in the classroom—are also examined.

Fractions

Six of the teachers presented pairs of lessons on fractions. In some cases their focus was on the meaning of a fraction, whereas in others they addressed fraction operations. In the majority of these lessons the teachers used many "small" examples, usually illustrations of particular fractions or exercises for students to solve. A range of these are presented here to show what examples were chosen and how the teachers used them, with discussion on what the examples might have afforded and what PCK was evident.

Cake halving. Meg used a square cake and repeatedly halved it, emphasising that the cake is the "whole" and remains the same quantity, but that the pieces were getting smaller. She also clarified the terms numerator and denominator. A student wrote the associated unit fractions on the board, finishing with $^1/_{32}$, and Meg emphasised that as the pieces get smaller the denominator gets bigger.

The idea of "cake cutting" has the potential to model almost any fraction, not just those with a power of two for a denominator nor just unit fractions. Meg's repeated halving allowed students to see some atypical primary school fractions, such as $^1/_{16}$ and $^1/_{32}$, but omitted many other unit fractions. Furthermore, her emphasis on unit fractions allowed a focus on the relationship of the denominator to the size of the piece, but prevented a deep examination of the meaning of the numerator. Although there is no evidence that this caused problems for these students, a well-known misconception is that students will, for example, regard $^2/_5$ as bigger than $^6/_7$ because fifths are bigger than sevenths. Meg may not have been aware of this particular misconception, or, if she was, may not have seen that although her emphasis on the relationship between the denominator and the size of the pieces was important it had the potential to lead to such a problem. Finally, the emphasis on halving appeared to interfere with later examples involving thirds and fifths.

Aero bar. Irene began her introduction to fractions with a KitKat chocolate bar, which allowed her to talk about quarters and emphasise the meaning of numerator and denominator. She also used a piece of paper torn into four pieces to illustrate the importance of having equal parts. Her next example used an Aero chocolate bar, which has seven pieces. She broke off three pieces and asked what fraction would represent how much she had. This example allowed her to illustrate sevenths, a denominator different from the familiar halves, quarters, and thirds. She also pointed out that sevenths are difficult to show with the "pizza" model of fractions.

Irene's choice of chocolate to model fractions suggests knowledge of how to "get and maintain student focus". In addition, by beginning with the four-piece KitKat she could model the familiar quarters, and then use torn paper to emphasise the importance of equal pieces, which had been implicit rather than explicit in the chocolate bar. The KitKat example provided an appropriate segue into the Aero bar, which allowed a "real world" example of sevenths, and Irene also emphasised the role of the numerator. There is a disadvantage in using the two different chocolate bars in that they are not suitable for making comparisons of quarters and sevenths; nevertheless, the chocolate bar models were suitable for the purposes to which Irene put them.

Smarties. After an initial review of fraction terminology and the use of a circle divided into three unequal pieces to emphasise the importance of equal parts, Jill used discrete materials rather than continuous materials to reinforce fraction notation. Students counted the numbers of each colour in small boxes of Smarties, and expressed this as a fraction of the total number of Smarties in the box. They also had to create a fraction strip on grid paper to show the fractions obtained, by dividing the strip into equal parts representing the total number of Smarties and then colouring in the relevant proportions. Unfortunately this model then caused problems when Jill tried to illustrate addition of fractions with the same denominators. She used an example of one person having 12 out of 14 orange Smarties and a second person having three out of 14 orange Smarties and added these as fractions to get ${}^{15}/_{14}$ (since there is a "common denominator"), before she turned this improper fraction into a mixed number. The problem here, however, was that the situation implies that there were, in fact, 28 Smarties involved. Jill acknowledged that there were actually two boxes of Smarties but told students to treat them as one box.

In theory, at least, the box of Smarties can be used to model fractions, but great care needs to be taken about identifying the "whole". Jill did not give this concept enough emphasis, with the added difficulty that the number of Smarties per box can vary. Jill knew about the latter problem and attempted to address it, but the former issue made modelling fraction addition difficult. In this case, the model/example was inadequate or did not have the level of epistemic fidelity needed to deal successfully with addition of fractions, despite the fact that it was suitable for simply representing fractions.

Fraction wall. Meg used the well-known "fraction wall" idea, and asked students to fold equal length strips into different numbers of parts. Obtaining halves, quarters, and eighths was easy, especially after the earlier cake-halving demonstration. Thirds were a little harder to fold (and some students anticipated that she would ask for sixteenths next), and then when Meg asked them what fraction they could find next, many students suggested fifths, whereas Meg had been thinking of halving again to get sixths. Fifths required even more adeptness at folding, and in the end Meg and some of the students resorted to measuring and calculating the lengths, a task made easier by the fact that the strip was 20cm long. Students did tenths next, and Meg made a conscious decision not to tackle sevenths because of the challenge of finding a strategy for folding the paper into seven. This meant that the students' fraction walls had all the fractions up to eighths and tenths, with the exception of sevenths and ninths.

Since the fraction wall model for fractions uses strips of equal width to build up a wall, the fractional parts are represented both by area and by length. It is a powerful model for comparing fractions, and can also highlight equivalent fractions. Meg's chosen sequence of fractions to make (halves, quarters, then eighths; thirds, then sixths, fifths and finally tenths) echoed her focus on halving as implemented with the cake-cutting activity earlier in the same lesson. There was no detailed discussion, however, of how halving the thirds gives sixths, thus missing an opportunity to strengthen connections between the ideas of halving and doubling. The omission of sevenths and ninths, which Meg acknowledged as being a consequence of time constraints and the difficulties of folding, may have reduced the students' capacity to generalise the fraction concept from the examples given.

Comparing fractions. Lisa had previously done work on equivalent fractions, which provided a foundation for her two lessons on comparing fractions. She began with a pizza comparison, asking students to decide who ate more if one person ate half a pizza and the second person ate four pieces of a pizza that had been cut into ten pieces. Students then had to generate fractions using a deck of cards, by selecting pairs of cards to generate the numerator and denominator of a proper fraction, and then comparing two fractions thus obtained. This led to some challenging problems, in one case involving twelfths and sevenths, which caused difficulty for some students. Prior to the second lesson she asked students to compare $^2/_5$ and $^1/_3$ for homework, and in the second lesson had students show how they had used equivalent fractions to make the comparison. She also showed how the equivalent fractions could be modelled on a fraction bar, giving a very careful discussion of how the fifths on a fraction bar could be turned into fifteenths by dividing each part into three.

Lisa's pizza consumption example provided a relatively simple context for looking at comparison of fractions and equivalent fractions, where one denominator was a multiple of the other. Her use of a deck of cards for generating fraction comparison problems introduced a random element to the tasks, and meant that she lost control of what kind of denominator relationships would arise. It is not clear that this was because she did not realise that denominator relationships might be important, or that the task, as designed, would affect them. The consequence was that some students had to grapple with quite difficult comparisons (such as twelfths and sevenths), which may have been too cognitively demanding for them. On the other hand, the choice of $^2/_5$ and $^1/_3$ for the homework task was more manageable, and afforded the opportunity to relate the problem situation to both the equivalent fraction calculations and to a model used to represent them. The choice of values is particularly good for this purpose: the two fractions are sufficiently close that comparing them demands an equivalent fractions strategy, rather than being obvious through visualisation; the values for the denominators make the calculation and representation of the equivalent fractions achievable yet still suitably cognitively demanding for the students; and the conceptual connections can be highlighted.

Exercises with fraction operations. The lessons that focused on fraction operations had a strongly procedural rather than conceptual orientation. Frank's lesson was purportedly a revision lesson, focusing on all four of the fraction operations. He used the example $^1/_6 + {}^3/_6$ to illustrate addition of fractions with the same denominator, without commenting that $^3/_6$ is, in fact, $^1/_2$, or that the final answer of $^4/_6$ can be simplified as $^2/_3$. A later exercise for students was $5^1/_2 - 2^7/_{12}$ which Frank expected students to solve by converting the mixed numbers to improper fractions and then finding common denominators if necessary. When one student explained that she had subtracted the whole numbers first, found an appropriate equivalent fraction for the half, and successfully regrouped after realising that $^7/_{12}$ could not be subtracted from $^6/_{12}$, Frank's response was to suggest $5^1/_2 - 2^7/_{19}$ as an example that might be difficult to attempt using such a strategy, implicitly privileging the "convert to improper fractions" method.

A second teacher, Brian, provided students with some exercises for converting from mixed numbers to improper fractions. There were four examples written on the board, $1^4/_{10}$, $7^3/_4$, $5^3/_6$, and $8^9/_{12}$, with three not in their simplest form. His emphasis was on the procedure for converting to improper fractions. The non-simplified nature of the fractions was not discussed, either before or after the conversion.

Both Frank and Brian demonstrated sound procedural knowledge. The focus, however, seemed to be on one concept at a time, ignoring other concepts that were evident in the example, as evidenced in Frank's $^1/_6 + {}^3/_6$ addition problem and three of Brian's mixed numbers problems, where the concept of equivalent fractions was overlooked. Here connections among concepts were *not* being established or reinforced; each process—equivalent fractions, operations with fractions, converting among forms—appears to exist in isolation.

Frank's impromptu construction of the example $5^1/_2 - 2^7/_{12}$ was intended to illustrate a situation where it might be difficult to subtract using the fractions in their mixed form rather than converting to improper fractions. Although it made the denominators harder to work with, the resulting example was, in fact, easier to solve using mixed numbers, given that the new choice of numerators actually eliminated the need to regroup. This suggests that whereas Frank could determine some of the cognitive demand of a problem, he could not quickly work his way through the consequences for the example in its equivalent form. In particular, he could not identify which were the salient pieces of the example to vary.

Probability

The next example, first discussed by Chick and Baker (2005b), comes from the topic of probability. Irene, an experienced teacher, and Greg, who was in only his second year of teaching, were Grade 5 teachers in the same school. They had chosen to use a spinner game worksheet activity suggested in a teacher resource book (Feely, 2003). The spinner game used two spinners divided into nine equal sectors, labelled with the numbers 1-9. The worksheet instructed students to spin both spinners, and add the resulting two numbers together. If the sum was odd, player 1 won a point, whereas player 2 won a point if the sum was even. The first player to 10 points was deemed the winner. Students were further instructed to play the game a few times to "see what happens", and then decide if the game is fair, who has a better chance of winning, and why (Feely, 2003, p. 173). The teacher instructions (Feely, 2003, p. 116) included a brief suggestion about focusing on how many combinations of numbers add to make even and odd numbers but did not provide any additional direction. The "example" in this case is the spinner game in its particular configuration.

Before examining what the teachers did in the classroom, it is informative to look at the affordances of this example. Careful consideration reveals that it affords worthwhile learning opportunities associated with sample space, fairness, long-term probability, likelihood, and reasoning about sums of odd and even numbers. The significant issue here, especially in the absence of explicit guidance from the resource book about *how* these issues can be brought out, concerns the choices that teachers make when implementing this activity; especially in terms of what they allow it to exemplify. To add to the complexity of what is already a conceptually rich example, the configuration of the spinners generates an interesting difficulty that could undermine the activity or could be turned to advantage, depending on how it is addressed. This difficulty arises because the chances of Player 2 (even) winning a point is ${}^{41}/_{81}$ compared to ${}^{40}/_{81}$ for Player 1 (odd), as revealed by analysis of the sample space. This miniscule difference in likelihood implies that the game's unfairness is unlikely to be convincingly evident when playing "first to ten points".

The interest is in how the teachers implemented the activity in the classroom, and in what they allowed it to exemplify and what students might have learned from it. Irene preceded her use of the game by getting students to toss a coin 100 times and record the number of heads and tails, with pairs of students starting to play the spinner game as soon as they had completed their 100 tosses. This meant that some students had more time to engage with the game than others, and that some of the important teaching moments occurred for small groups of students rather than the whole class. Most students had played the game for a few minutes before Irene interrupted them for a discussion of the coin tossing results and then the spinner game. Her focus here was really on the coin tossing results, and time constraints limited the attention given to the spinner game. Nevertheless, some of its attributes were addressed. She asked the class if they thought it was a fair game. Discussion ensued, as students posed various ideas without any of them being completely resolved. For instance, there was a brief discussion about how the "structure" of the game needed to be fair, implying that fairness means that as long as the two players play by the rules of the game then they should have an equal chance of winning. Most of the arguments about fairness were associated with the number of odds and evens, both in terms of the individual numbers on the spinners (there are more odds than evens on each spinner) and in terms of the sums. One student neatly articulated the erroneous parity argument, that since "odd + odd = even and even + even = even but odd + even = odd, therefore Player 2 has two out of three chances to win". Irene said she was not convinced about the "two out of three", but she agreed the game was unfair. Irene then allowed one of the students to present his argument. At the start of the whole class discussion this student had indicated that he had not played the game at all but had "mathsed it" instead, and at that time Irene made a deliberate decision to delay the details of his contribution until the other students had had their say. He proceeded to explain that he had counted up all the possibilities, to get 38 for even and 35 for odd. Although this was actually incorrect Irene seemed to believe that he was right and continued by pointing out that this meant that "it's not *terribly* weighted but it *is slightly* weighted to the evens". Irene then asked the class if their results bore this out, and highlighted that although the game was biased toward Player 2 this did not mean that Player 2 would always win.

Greg spent a much longer time on the spinner game. The students played it at the end of the first of the two observed lessons, and during the course of their exploration of the game a few pairs came up with the parity argument, accompanied by the observation that there are more odd numbers on the spinners. That lesson concluded with an extensive discussion of whether or not the game was fair. Greg did not indicate whether or not he thought the students' suggestions were correct; he seemed to want to hear all the contributions. He later asked if any of the students had considered all the possible outcomes, and suggested that this would something they would look in the next lesson. In the post-lesson interview Greg told the researchers that the decision to explore sample space was made only during the first lesson while students were already working on the task. He also acknowledged that when he chose the activity he was not sure of all that it offered.

Greg then devoted nearly half of his second lesson to an exploration of the sample space. As reported in Chick and Baker (2005b) he tightly guided the students in recording all the outcomes and could not deal with alternative approaches. He asked the students to calculate the probabilities of particular outcomes, which was helpful in highlighting the value of enumerating the sample space, but detracted from the problem of ascertaining whether even or odd outcomes were more likely. Students eventually obtained the "40 odds and 41 evens" conclusion, at which point Greg stated that because the "evens" outcome was more likely the game was unfair. There was, however, no discussion of the narrowness of the margin.

It must be noted that in both classes the students did not—could not—play the game long enough for the unfairness to be genuinely evident in practice, yet most students claimed that the game was biased towards even. This may have occurred because the incorrect parity argument made them more aware of the even outcomes than the odd ones.

As suggested earlier, the spinner game provides the opportunity to examine sample space, likelihood, and fairness. Given the impact of time constraints on Irene's lesson, sample space was not covered well, although she believed that the student who had "mathsed it" had considered all the possibilities. This highlights a contrast between her knowledge of his capabilities and the details of the content with which he was engaged. On the other hand, her content knowledge was sufficient for her to recognise the significance of the small difference between the number of odd and even outcomes and its impact on fairness. Greg was much more thorough in his consideration of sample space, but also very directive. He seemed constrained by his content knowledge, having only one way to think of the sample space—via exhaustive enumeration—and was unable to recognise the possibility of an alternative approach in one of his students' erroneous suggestions.

Neither teacher seemed aware of all that the game afforded in advance of using it, as evidenced by the way it was used, although Greg recognised the scope for examining sample space part way through the first lesson. Both teachers were, however, able to bring out some of the concepts in their use of the game, with Irene having a good discussion of the meaning of fairness and the magnitude of the bias, and Greg illustrating sample space and the probability of certain outcomes.

An important observation needs to be made here. The teacher guide that was the source of the activity gave too little guidance about what it afforded and how to bring it out. Even if such guidance had been provided, there is also still the miniscule bias problem inherent in the game's structure that affects what the activity can afford. It is very difficult to convincingly make some of the points about sample space, likelihood, and fairness with the example as it stands. It can be done, but the activity probably needs to be supplemented with other examples that make some of the concepts more obvious (see, e.g., Baker &

Chick, 2007). This highlights the crucial question of how can teachers be helped to recognise what an example affords and then adapt it, if necessary, so that it *better* illustrates the concepts that it is intended to convey.

Area and Perimeter

The final case involves Clare, a Grade 6 teacher with five years' experience. She conducted two lessons focussing on area and perimeter simultaneously, having done work in the past on each separately. Part of her first lesson is presented here in detail, to highlight the way the actual implementation of an example in the classroom may develop in unanticipated ways and to indicate how important PCK is in dealing with this.

Clare began by reviewing the concept of area, where she emphasised that "Area measures the *space* inside a shape, so what that actually is, is the number of squares inside the shape". She then asked students to draw a rectangle with an area of 20cm^2 on grid paper and cut it out. Her choice of what might be called an open "reversed" task was appropriate given that the students had worked with area before, including the area formula for rectangles. Shortly after this instruction the following exchange took place between Clare and a student.

S: Can I do a square?
Clare: Is a square a rectangle? […] What's a rectangle? […] How do you get something to be a rectangle? What's the definition of a rectangle?
S: Two parallel lines
Clare: Two sets of parallel lines … and …
S: Four right angles.
Clare: So is that [points to square] a rectangle?
S: Yes.
Clare: Excellent. [Pause] But has that got an area of 20?
S: [Thinks] Er, no.
Clare: [Nods and winks]

It is not clear whether Clare's original choice of 20 was made with any awareness of geometrical implications, but the fluency with which Clare moved from area measurement to spatial issues—addressed with clear attention to geometrical properties—and back again required ready access to the PCK of both the measurement and spatial domains. She also exhibited effective use of questioning to elicit understanding from the student. Shortly after this she discussed rectangle properties with the class.

Clare then invited a student to bring his 4×5 cut-out rectangle to the front of the class, recorded it on an overhead transparency, and confirmed that its area was 20cm^2. She led a class discussion on how multiplying length × width is the same as counting squares and hence gives the area. Clare thus used the concrete example to highlight the link between the conceptual meaning of area and the procedural calculation. She did not stop there, however; in the following exchange it can be seen that Clare knew that students need to know that the area formula L×W only applies to certain shapes.

Clare: When [S1] said that's how you find the area of a shape, is he *completely* correct?
S2: That's what you do with a 2D shape.

Clare: Yes, for *this* kind of shape. […] What kind of shape would it *not* actually work for?
S3: Triangles. […]
S4: A circle.
Clare: [With further questioning, teases out that L×W only applies to rectangles.]

A student then suggested 2×10 as a second example of a rectangle with area $20cm^2$, at which point Clare confirmed that all the students had chosen either this one or the 4×5 case. When she asked for other possibilities the students suggested the original examples but oriented at 90°, together with 1×20, which had not been suggested earlier. With all the integer-sided rectangles on display Clare asked the students to look for a pattern in the examples found, which led into a discussion of factors of 20. She continued:

Clare: Are there any other numbers that are going to give an area of 20? [She paused, with an attitude of uncertainty. There was no response from the students at first.]
Clare: No? How do we know that there's not?
S: You could put 40 by 0.5.
Clare: Ah! You've gone into decimals. If we go into decimals we're going to have *heaps*, aren't we?

It appeared that she was targeting only whole numbers—and, as a consequence, some argument about exhausting the factors of 20—but she clearly understood the significance of the student's unexpected answer, and to what degree it would apply. The open scope of her questions allowed this extension to arise, even though it had not been her original intention; however, she made a decision not to pursue this aspect—even though it would have been a valuable use of the $20cm^2$ example—because she wanted to move on to different examples that would highlight other relationships. Instead she used the $20cm^2$ example to focus on the search for all factors of 20.

This exploration of the $20cm^2$ example took the first 15 minutes of the lesson. Clare then had students repeat the search for rectangles with area $16cm^2$. She used this example to highlight the process of systematically searching for factors, and to highlight the set inclusion property "a square is a rectangle". She recapped that they had been working on areas, and then reminded students about perimeter, how to work it out for rectangles, and that linear rather than square units are involved. She guided the class to work out the perimeters of the different $16cm^2$ rectangles they had found, and indicated that although shapes might have the same area they do not have to have the same perimeter. She revisited the $20cm^2$ examples they had, and calculated the perimeters to focus again on the variation in perimeter.

The final example/task for the lesson was for students to work in groups to find as many shapes—not just rectangles, but constrained by being made of contiguous squares—with an of area 12 cm^2 and determine the perimeters. She wrote "What is the relationship between area and perimeter?" on the board as a learning objective for this activity. She allowed students to explore the task for about five minutes, then interrupted their work to help them develop strategies to work systematically and instruct them to record the perimeters of each shape. About 20 minutes later, she held another class discussion that acknowledged that there were "heaps" of possible shapes, looked at one group's systematic work and discussed some symmetry implications, and then asked students to focus on finding a shape with the greatest perimeter and one with the smallest perimeter. The 90-minute lesson concluded with a ten-minute discussion of the students' results, which emphasised the use of linear units for perimeter, that shapes with small perimeters were more "compact", and that moving one of the squares on a shape without changing the number of joining edges will not change the perimeter.

Clare's conclusion re-emphasised the points of her lesson: that area and perimeter can have the same or different numerical values, that two shapes with the same area can have different perimeters, and that systematic work can help find all the possibilities in a problem. These learning outcomes were achieved through the use of just three examples that had been carefully chosen to illustrate these points.

Clare seemed to have a very clear idea about what she wanted her examples to achieve. They were effective as didactic objects for two reasons: Clare's careful choice of the examples themselves and then the way she facilitated conversations about them. It is not clear that there was a purposeful reason for considering rectangles of area $20cm^2$ first, followed by those of area $16cm^2$; in particular, it is uncertain that there was an intention to allow discussion of "squares are rectangles" in the second case after just focussing on non-square rectangles. However, whether it was an intended focus or an opportunity that arose fortuitously, Clare was able to address this geometric concept fluently, demonstrating her capacity to make connections across topics. The final extension considered shapes of area $12cm^2$. If only considering rectangles this would have been no more difficult than what students had already done—and potentially redundant—but because she wanted students to consider other shapes as well, it was appropriate to pick this "simpler" number. Interestingly, given the magnitude of the enumeration task, there is potential to debate whether 12 is, in fact, simple enough. One of the researchers observing the lesson at the time wondered if she had chosen wisely. As the lesson progressed, however, it was clear that although she wanted to address the issue of enumerating all possible shapes systematically, her main focus was still associated with area and perimeter, and the choice of 12 allowed enough variety of shapes to make it a non-trivial task to find those with the greatest and least perimeter.

There was an interesting decision point that arose in the lesson when a student gave the 40×0.5 rectangle example. It seemed that Clare's focus on factors influenced her decision to acknowledge this response, briefly recognise its implications, but then continue with whole number dimensions. It is not clear whether she weighed up (a) what concepts could have been developed if she had detoured with an exploration of non-integer dimensions, (b) how such a detour might have interfered with her goals for the lesson, and (c) whether or not all her students would have been capable of following the detour. Certainly such an exploration could have given more extreme perimeter values than the students obtained, but the importance of identifying factors of numbers might have been obscured.

The strength of Clare's PCK was evident as the lesson progressed, as well as in her responses to the questionnaire and interviews (see Baker & Chick, 2006). She appeared to have a deep understanding of concepts, the rich connections among them, and the links between concepts and procedures. Her conceptual fluency was evident in the ease with which she responded to unanticipated events in the classroom. In addition to specific content knowledge she advocated general mathematical principles, such as the need to work systematically, and to justify and explain results. Her knowledge of student thinking was evident in her identification of likely misconceptions, and in knowing how to ask questions and respond to students' difficulties. Finally, her choice of examples had appropriate cognitive demand for her students, led to conceptual understanding, and afforded exploration of a range of mathematical concepts.

Conclusions

For most of these cases, the teachers selected the example's structure and specific values prior to implementing it in the classroom, strongly influenced by their PCK and what affordances they thought the example offered. At times, though, teachers had to develop or respond to an example on the spot; but again their capacity to do so was affected by their PCK and their ability to construct or recognise examples with the affordances required. It is worth making some observations about the source of the examples and the PCK for some of these situations, in order to highlight the complexity associated with this critical issue.

- A teacher's current level of PCK can allow him/her to recognise a situation that could be turned into a useful example, as evident in the use of the Aero bar.
- A teacher's current level of PCK may allow him/her to devise a partly appropriate example, but deeper PCK would reveal that it has limitations. This occurred with the Smarties and with Frank's fraction subtraction example.
- Professional development (PD) can enhance PCK and a teacher's repertoire of examples. The fraction wall and the paper strip folding activities conducted by Meg had their origins in PD and reflected, in her paraphrased words, part of a change in her teaching style from a procedural focus to a conceptual one. That said, however, a teacher's implementation of an example demonstrated to him/her in PD may not always reflect the potential affordances identified by the PD designers. The omission of the sevenths and ninths was Meg's choice; most advocates of the fraction wall would include these examples.
- External sources of examples do not always indicate the affordances of the example and how to implement them. This was strikingly evident in the case of the spinner game. It cannot be assumed that teachers do not or should not need this support.
- A teacher's current level of PCK and his/her identification of affordances can develop in the process of implementing an example. This occurred for Greg as he used the spinner game. Moreover, he recognised this development as such.
- A teacher with rich PCK can devise examples that illustrate a range of concepts, can highlight connections among topics, and identify which are the central ideas and which are peripheral. This was evident in Clare's area and perimeter examples.

The complexity of mathematical concepts, together with the limited opportunities that teachers have to master all these concepts and their pedagogical implications before entering the classroom, highlight how difficult it is to ensure that teachers have the depth of PCK required to identify and draw out the affordances of an example. Recognising the ways in which "Compare $^2/_5$ and $^1/_3$" is different from "Compare $^3/_7$ and $^5/_8$" and the consequent implications for what might be learned, for instance, requires attention to a range of fraction issues followed by a decision about which aspects are regarded as more important for the day's teaching objectives.

These observations raise the question of how to prepare future teachers so that they develop adequate PCK and can successfully choose, use, and modify examples. Clearly there must be an endeavour to ensure that teachers have a deep conceptual understanding of mathematics, and rich PCK for its teaching. Given the centrality of examples to the teaching and learning process, however, time also needs to be spent applying this understanding to an investigation of examples and their pedagogical implications. We need to develop ways to help teachers identify more potential affordances in examples, to recognise an example's salient and non-salient features, and to ascertain the implications of any interrelationships that exist.

This suggests that teacher education and professional development opportunities must be more explicit about the issues associated with example use. In particular, the affordances of the examples used in teacher education and professional development should be identified and discussed, so that teachers learn to realise that an example has many potential affordances and to discriminate between the productive and the unproductive. There is a need to identify the dimensions of possible variation for an example, so that the impact of changes to the particular values and structure can be considered, and the significant and extraneous components of the example can be identified. This is essential if teachers are to learn how to change examples to make them conceptually harder or easier, to produce counterexamples, or to emphasise a different principle. Indeed, teachers and potential teachers need opportunities to engage with examples, to trial them, and to learn how to adapt them successfully to meet different needs. It would be valuable to have teachers contrast examples, attending to affordances and what varies between the examples (the earlier illustration of examining the ways in which "Compare ${}^{2}/_{5}$ and ${}^{1}/_{3}$" is a different example from "Compare ${}^{3}/_{7}$ and ${}^{5}/_{8}$" is a case in point). In all of this, there needs to be deeper discussion of the connections among mathematical topics and how an example illuminates these connections. Finally, there must be discussion of how to implement the examples in the classroom, so that the examples become successful didactic objects that illustrate the desired general principle. Without this, the opportunities for learning afforded by examples may go unfulfilled.

Acknowledgements: This research was supported by Australian Research Council grant DP0344229. My thanks go to Monica Baker and Kiri Harris who were research assistants on this project, and to Robyn Pierce and Jane Watson who provided feedback on drafts of this paper. I also greatly appreciate the involvement of the teachers in the MPCK project who allowed me into their world. I learned much from them and their examples.

References

Askew, M., Brown, M., Rhodes, V., Johnson, D., & Wiliam, D. (1997). *Effective teachers of numeracy. Final report*. London: King's College.

Baker, M., & Chick, H. L. (2006). Pedagogical content knowledge for teaching primary mathematics: A case study of two teachers. In P. Grootenboer, R. Zevenbergen, & M. Chinnappan (Eds.), *Identities, cultures and learning spaces* (Proceedings of the 29th annual conference of the Mathematics Education Research Group of Australasia, pp. 60-67). Sydney: MERGA.

Baker, M., & Chick, H. L. (2007). Making the most of chance. *Australian Primary Mathematics Classroom, 12*(1), 8-13.

Ball, D. L. (2000). Bridging practices: Intertwining content and pedagogy in teaching and learning to teach. *Journal of Teacher Education, 51*(3), 241-247.

Bills, L., Mason, J., Watson, A., & Zaslavsky, O. (2006). *RF02:* Exemplification: The use of examples in teaching and learning mathematics. In J. Novotná, H. Moraová, M. Krátká, & N. Stehlíková (Eds.), *Proceedings of the 30th annual conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 125-154). Prague: PME.

Bryman, A. (2004). *Social research methods* (2nd ed.). Oxford: Oxford University Press.

Chick, H. L., & Baker, M. (2005a). Investigating teachers' responses to student misconceptions. In H. L. Chick & J. L. Vincent (Eds.), *Proceedings of the 29th annual conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 249-256). Melbourne: PME.

Chick, H. L., & Baker, M. (2005b). Teaching elementary probability: Not leaving it to chance. In P. C. Clarkson, A. Downton, D. Gronn, M. Horne, A. McDonough, R. Pierce, & A. Roche (Eds.), *Building connections: Theory, research and practice* (Proceedings of the 28th annual conference of the Mathematics Education Research Group of Australasia, pp. 233-240). Sydney: MERGA.

Chick, H. L., Baker, M., Pham, T., & Cheng, H. (2006). Aspects of teachers' pedagogical content knowledge for decimals. In J. Novotná, H. Moraová, M. Krátká, & N. Stehlíková (Eds.), *Proceedings of the 30th annual conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 297-304). Prague: PME.

Chick, H. L., Pham, T., & Baker, M. (2006). Probing teachers' pedagogical content knowledge: Lessons from the case of the subtraction algorithm. In P. Grootenboer, R. Zevenbergen, & M. Chinnappan (Eds.), *Identities, cultures and learning spaces* (Proceedings of the 29th annual conference of the Mathematics Education Research Group of Australasia, pp. 139-146). Sydney: MERGA.

Feely, J. (2003). *Nelson maths for Victoria: Teacher's resource Year 5.* Melbourne: Thomson Nelson.

Gibson, J. J. (1977). The theory of affordances. In R. Shaw & J. Bransford, (Eds.), *Perceiving, acting and knowing: Toward an ecological psychology* (pp. 67-82). Hillsdale, NJ: Lawrence Erlbaum.

Hill, H. C., Rowan, R., & Ball, D. L. (2005). Effects of teachers' mathematical knowledge for teaching on student achievement. *American Educational Research Journal, 42*(2), 371-406.

Hill, H. C., Schilling, S. G., & Ball, D. L. (2004). Developing measures of teachers' mathematical knowledge for teaching. *The Elementary School Journal, 105*(1), 11-30.

Kahan, J. A., Cooper, D. A., & Bethea, K. A. (2003). The role of mathematics teachers' content knowledge in their teaching: A framework for research applied to a study of student teachers. *Journal of Mathematics Teacher Education, 6*(3), 223-252.

Lampert, M. (2001). *Teaching problems and the problems of teaching*. New Haven, CT: Yale University Press.

Leinhardt, G., Putnam, R. T., Stein, M. K., & Baxter, J. (1991). Where subject knowledge matters. In J. Brophy (Ed.), *Advances in research on teaching: Teachers' knowledge of subject matter as it relates to their teaching practice* (Vol. 2, pp. 87-113). Greenwich, CT: Jai Press, Inc.

Ma, L. (1999). *Knowing and teaching elementary mathematics: Teachers' understanding of fundamental mathematics in China and the United States.* Mahwah, NJ: Lawrence Erlbaum Associates.

Marks, R. (1990). Pedagogical content knowledge: From a mathematical case to a modified conception. *Journal of Teacher Education, 41*(3), 3-11.

Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher*, *15*(2), 4-14.

Skemp, R. R. (1971). *The psychology of learning mathematics.* Harmondsworth, UK: Penguin.

Stacey, K., Helme, S., Archer, S., & Condon, C. (2001). The effect of epistemic fidelity and accessibility on teaching with physical materials: A comparison of two models for teaching decimal numeration. *Educational Studies in Mathematics, 47*(2), 199-221.

Thompson, P. W. (2002). Didactic objects and didactic models in radical constructivism. In K. Gravemeijer, R. Lehrer, B. van Oers, & L. Verschaffel (Eds.), *Symbolizing, modeling, and tool use in mathematics education* (pp. 191-212). Dordrecht, The Netherlands: Kluwer.

Watson, A. (2003). Opportunities to learn mathematics. In L. Bragg, C. Campbell, G. Herbert, & J. Mousley (Eds.), *Mathematics education research: Innovation, networking, opportunity* (Proceedings of the 26th annual conference of the Mathematics Education Research Group of Australasia, pp. 29-38). Sydney: MERGA.

Watson, A., & Mason, J. (2005). *Mathematics as a constructive activity: Learners generating examples.* Mahwah, NJ: Lawrence Erlbaum Associates.

Watson, A., & Mason, J. (2006). Seeing an exercise as a single mathematical object: Using variation to structure sense-making. *Mathematical Thinking and Learning, 8*(2), 91-111.

Watson, J., Beswick, K., & Brown, N. (2006). Teachers' knowledge of their students as learners and how to intervene. In P. Grootenboer, R. Zevenbergen, & M. Chinnappan (Eds.), *Identities, cultures and learning spaces* (Proceedings of the 29th annual conference of the Mathematics Education Research Group of Australasia, pp. 551-558). Sydney: MERGA.

Zaslavsky, O., Harel, G., & Manaster, A. (2006). A teacher's treatment of examples as a reflection of her knowledge-base. In J. Novotná, H. Moraová, M. Krátká, & N. Stehlíková (Eds.), *Proceedings of the 30th annual conference of the International Group for the Psychology of Mathematics Education* (Vol. 5, pp. 457-464). Prague: PME.

Zazkis, R., & Chernoff, E. (2006). Cognitive conflict and its resolution via pivotal/bridging example. In J. Novotná, H. Moraová, M. Krátká, & N. Stehlíková (Eds.), *Proceedings of the 30th annual conference of the International Group for the Psychology of Mathematics Education* (Vol. 5, pp. 465-472). Prague: PME.

Appendix 1.

A Framework for Pedagogical Content Knowledge (after Chick, Baker, et al., 2006).

PCK Category	*Evident when the teacher …*
Clearly PCK	
Teaching Strategies	Discusses or uses general or specific strategies or approaches for teaching a mathematical concept or skill
Student Thinking	Discusses or addresses student ways of thinking about a concept, or recognises typical levels of understanding
Student Thinking - Misconceptions	Discusses or addresses student misconceptions about a concept
Cognitive Demands of Task	Identifies aspects of the task that affect its complexity
Appropriate and Detailed Representations of Concepts	Describes or demonstrates ways to model or illustrate a concept (can include materials or diagrams)
Explanations	Explains a topic, concept or procedure
Knowledge of Examples	Uses an example that highlights a concept or procedure
Knowledge of Resources	Discusses/uses resources available to support teaching
Curriculum Knowledge	Discusses how topics fit into the curriculum
Purpose of Content Knowledge	Discusses reasons for content being included in the curriculum or how it might be used
Content Knowledge in a Pedagogical Context	
Profound Understanding of Fundamental Mathematics (PUFM)	Exhibits deep and thorough conceptual understanding of identified aspects of mathematics
Deconstructing Content to Key Components	Identifies critical mathematical components within a concept that are fundamental for understanding and applying that concept
Mathematical Structure and Connections	Makes connections between concepts and topics, including interdependence of concepts
Procedural Knowledge	Displays skills for solving mathematical problems (conceptual understanding need not be evident)
Methods of Solution	Demonstrates a method for solving a mathematical problem
Pedagogical Knowledge in a Content Context	
Goals for Learning	Describes a goal for students' learning
Getting and Maintaining Student Focus	Discusses or uses strategies for engaging students
Classroom Techniques	Discusses or uses generic classroom practices

Introducing Students to Data Representation and Statistics

Richard Lehrer
Vanderbilt Univeristy
<rich.lehrer@vanderbilt.edu>

I describe the design and iterative implementation of a learning progression for supporting statistical reasoning as students construct data and model chance. From a disciplinary perspective, the learning trajectory is informed by the history of statistics, in which concepts of distribution and variation first arose as accounts of the structure inherent in the variability of measurements. Hence, students were introduced to variability as they repeatedly measured an attribute (most often, length), and then developed statistics as ways of describing "true" measure and precision. The design of the learning progression was guided by several related principles: (a) posing a series of tasks and situations that students perceived as problematic, thus creating a need for developing mathematical understanding as a means of resolving prospective impasses; (b) creating opportunities for developing representational fluency and meta-representational competence as constituents of conceptual development; (c) introducing statistics as invented measures of the qualities of distribution; and (d) adopting an agentive perspective for orienting student activity, according to which distribution of measures emerged as a result of the collective activity of measurer-agents. Instructional design and assessment design were developed in tandem, so that what we took as evidence for the instructional design was subjected to test as a model of assessment, resulting in revision to each. I conclude with a look at ongoing work to design an assessment system to measure students' understandings of data and statistics, and with some thoughts about prospective synergies between mathematics and science education.

The discipline of statistics originated in problems of modeling variability (Porter, 1986; Stigler, 1986). History has not changed all that much: Professional practices of statisticians invariably involve modeling variability (Wild & Pfannkuch, 1999), and as in other sciences (e.g., Giere, 1992), it is through model contest that statistical concepts become more widespread and stable (Hall, Wright, & Wieckert, 2007). Another lesson of history is of particular importance: Reasoning about variability was initially most prominently pursued in contexts of measurement error. Astronomers, for example, suggested that distances between stars were fixed, but that measurements varied, just by chance. Mathematical efforts to characterize the form and structure of chance gave rise to concepts and models still in use today, such as least squares fit.

Our research program follows in this historic tradition: Contexts of measure afford children entrée to a series of core conceptual structures or "big ideas" in the discipline and also, to the core disciplinary practice of inventing and revising models. Accordingly, I outline a design of instruction that features repeated measure for introducing students to practices and related concepts of data representation, statistics, chance, and modeling. These practices and concepts are all developed by students to account for observed variability in measurements. As I describe components of the design, I characterize some of the recurrent patterns of student reasoning that we observed during successive iterations of the design in fifth- and sixth-grade (10, 11 years of age) urban classrooms in the United States. These collectively establish a sense of "lessons learned". Our efforts to account for emerging patterns of student reasoning were accompanied by corresponding efforts to encapsulate these patterns of reasoning in the form of an assessment system, which is sketched in the second section of the paper. I conclude with some prospects for integrating

Proceedings of the 30th annual conference of the Mathematics Education Research Group of Australasia
J. Watson & K. Beswick (Eds), © MERGA Inc. 2007

mathematics and science education via a shared interest in constructing and revising models of variability.

Designing Instruction to Support a Learning Progression

The instructional design was guided by an image of statistical reasoning as emerging from and enmeshed within a larger system of activity that we refer to as data modeling (Lehrer & Romberg, 1996; Lehrer & Schauble, in press). As Figure 1 suggests, data modeling is composed of two coupled systems of activity. The upper triangular region in the figure depicts the learning challenges and resources associated with the *design* of research. Designers confront challenges such as posing questions and identifying the nature of variables and their measures.

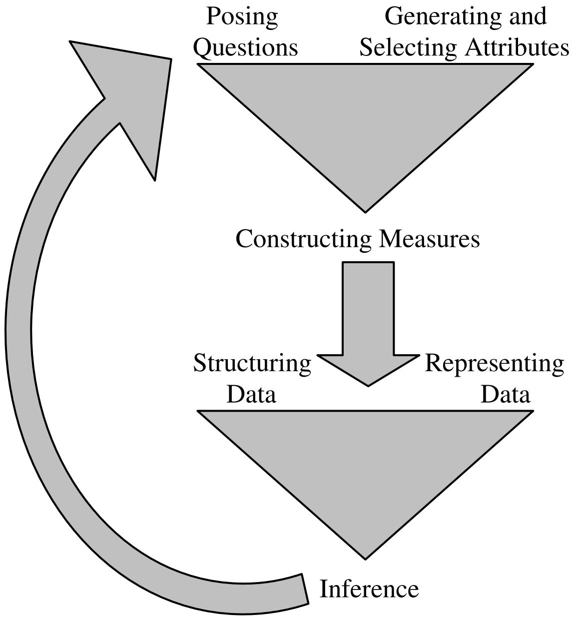


Figure 1. Schematic of data modelling.

The lower triangular region encompasses *analysis*, depicted as an interaction among data structures, representations, and models of inference. Analysts confront challenges of imposing structure on data, of choosing displays to highlight aspects of structure, and of making judgments about phenomena in light of variability and uncertainty. Although the cycle as illustrated invites inference of linear progression, in practice, these components of data modeling are typically interactive. For example, attempting to develop a measure of an attribute often profoundly alters one's conception of that attribute.

To initiate students into practices of data modeling, we designed a hypothetical learning progression – a sequence of tasks, tools, activities, and forms of argument – aimed at supporting students' development of mathematical accounts of the inherent variability of measure. The learning progression was envisioned to unfold in three coordinated phases in the classroom. In the first, students all repeatedly measured the same object and designed a representation intended to communicate trends in the collection of measurements that they noticed. In the second, students used these displays to invent statistics. One invented statistic indicated the "best guess" of the measure of the attribute of the object and the precision of the measurements. Students explored the qualities of their invented statistics with new samples of measurements of the same object conducted with a better tool. The latter resulted in distributions that were less variable but that had approximately the same centre. The third, modeling phase included investigation by students of the behavior of chance devices and the subsequent harnessing of these devices to construct models of measurement error. In the sections that follow, I describe the rationale for each of these three phases and also suggest recurrent patterns in student reasoning that we observed as we implemented the design over several iterations in fifth- and sixth-grade classrooms in an urban school in the United States. Participating students were from under-represented groups in the United States. Their families were of lower socioeconomic status.

Inventing Representation

Students measured an attribute of a familiar object, such as the arm-span of their teacher. To measure arm-span, each student first used a 15-cm ruler and then a metre stick. Each time, students recorded the value of the measure. The aim of this initial activity was to provide students with a context in which collective properties of the data, especially distribution, could be viewed as emerging from the actions of individual agents. We anticipated that students' prior history with measurement would serve as a resource for making sense of the variability of the measurements. For example, the 15-cm ruler had to be iterated more often than did the meter stick to span the same distance. (The former resulted in greater error and hence greater variability among the measurements.)

We presented students with an unstructured collection of their measurements and challenged them to create a display (of the more variable measurements) that communicated what they noticed about the batch of data. After students created their displays, other students presented the display to the class and described what the display tended to "show and hide" about the data. This tactic was intended to foster representational fluency (Greeno & Hall, 1998). With instructor support, students compared and contrasted their invented displays. We anticipated that comparing and contrasting different displays would clarify relations between the choices made by designers and the resulting "shape" of the data. This tactic was also intended to foster meta-representational capacity (diSessa, 2004) – the capacity to view a data display as representing a trade-off. Different choices resulted in different perceptions of the shape of the same data. We were especially interested in helping students understand how displays that grouped data and counted cases within each group produced a symmetric, bell-shaped distribution. Students considered possible reasons for the bell-shape of grouped data in light of the process of measure. We concluded this phase of instruction by soliciting students' conjectures about what might happen if "we measured again".

Recurrent Patterns of Representation

The most striking feature of the displays generated by the students was their variability. Despite years of education emphasizing conventional graphs, students often found this task challenging and even daunting.

Highlighting order. The most common solution to the problem of display was to structure the data by ordering the magnitude of the cases. Some solutions were lists, such as that displayed in Figure 2.

Figure 2. Ordering data as a list.

Others relied on space to convey a visual sense of order. The student solution displayed in Figure 3, a type of array graph (Snecedor & Cochran, 1968), exemplifies the latter. Bars or lines represented magnitudes of measurements. The designers, but not typically other members of the class, indicated that plateaus showed modes or clusters of values.

Figure 3. Invented array graph.

Elaborating order. A second class of solutions appeared to elaborate on order by highlighting relative frequency. Figure 4 illustrates this propensity. Students ordered the cases and displayed their relative frequency as a square icon. Note that the interval between case values is not represented. When the teacher asked the students which values would not be likely to recur if they measured again, students pointed to the lowest value. The display made the multi-modal nature of these data visible. The statistics represented on the display are remembrance of past classes – things that one did to batches of data. But after computing them (some incorrectly), they never referred to the statistics again.

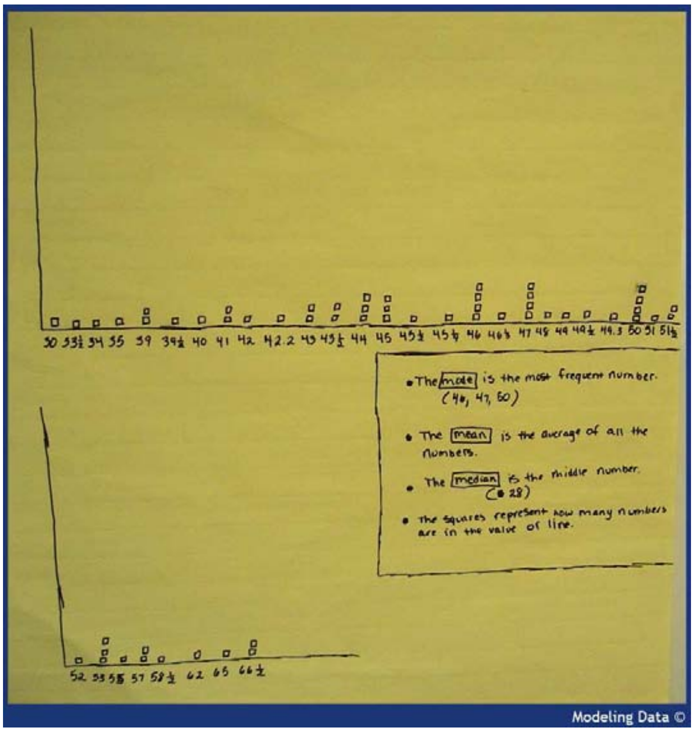


Figure 4. Ordered case frequency display.

Grouping and ordering. Solutions that involved grouping similar values into “bins” or equal-interval groups were relatively infrequent. The designers of the display depicted in Figure 5 grouped measurements in 10s, and they ordered the bins not by magnitude of the measurements but instead by relative frequency. Another pair of designers in the same class rendered their display to coordinate the order of the magnitude of the observed measurements with the relative frequency of each interval class (Figure 6). The corresponding difference in the shape of the data is striking.

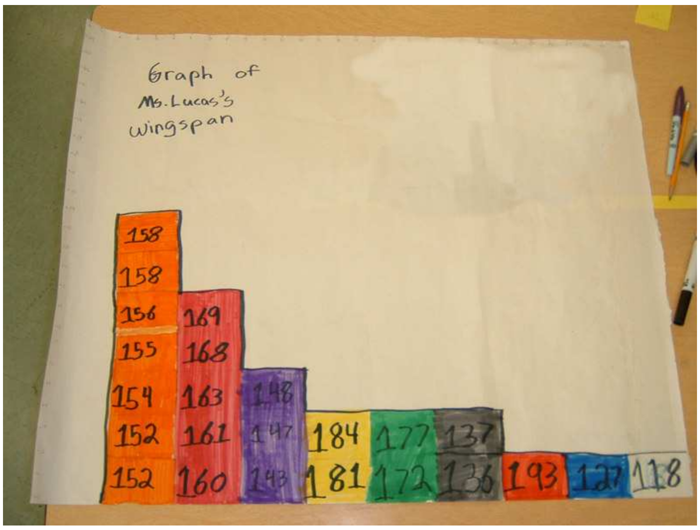


Figure 5. Bin display ordered by relative frequency.

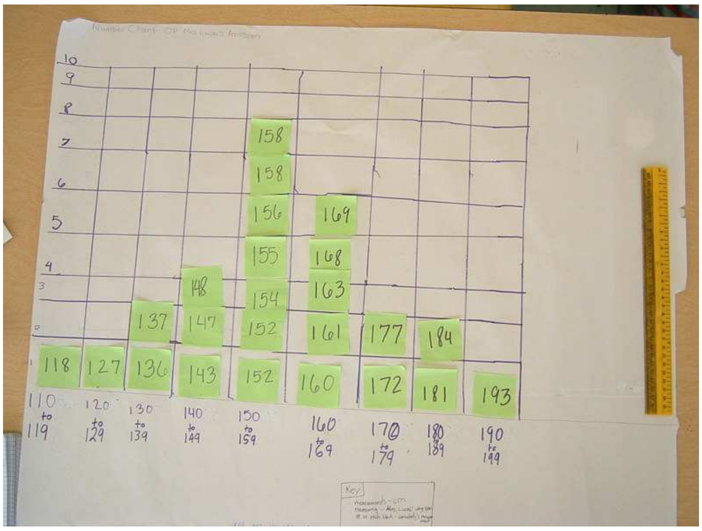


Figure 6. Bin display coordinating case magnitude and frequency.

Interval displays. The least common form of recurrent display was that of interval. These were developed by students who wanted to represent both what was missing as well as what was present in the data, so that holes and clumps could be viewed simultaneously. For example, in Figure 7, a pair of sixth-grade students listed relative frequencies where zero indicated missing values in the interval described by the observed measurements. Hence, 0 = 14 refers to the number of values in the interval between 30 feet and 66 feet for which there was no missing case. The 1 = 9 refers to the number of values in the interval for which there was only 1 case missing. Figure 8, a display designed by a pair of fifth-grade students, illustrates similar attention to interval but in a manner that is more conventional.

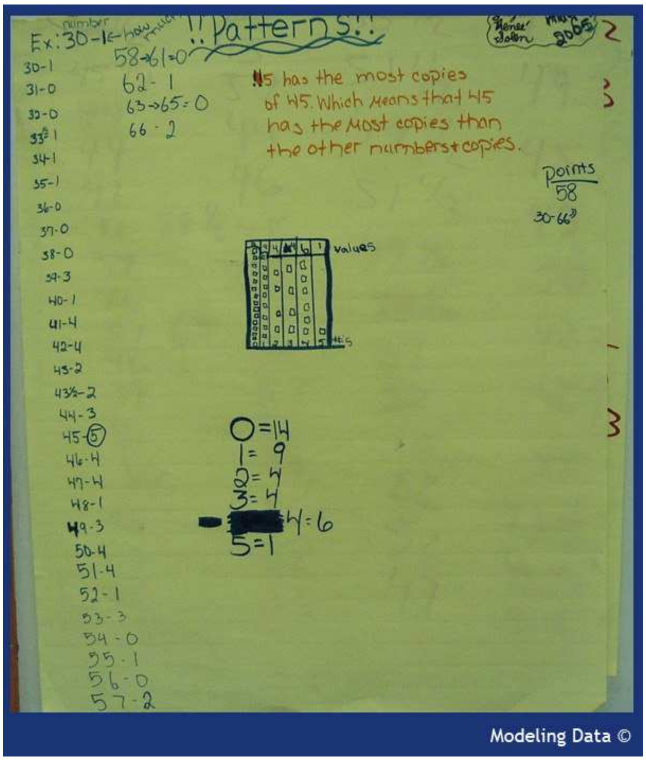


Figure 7. Representing what is missing.

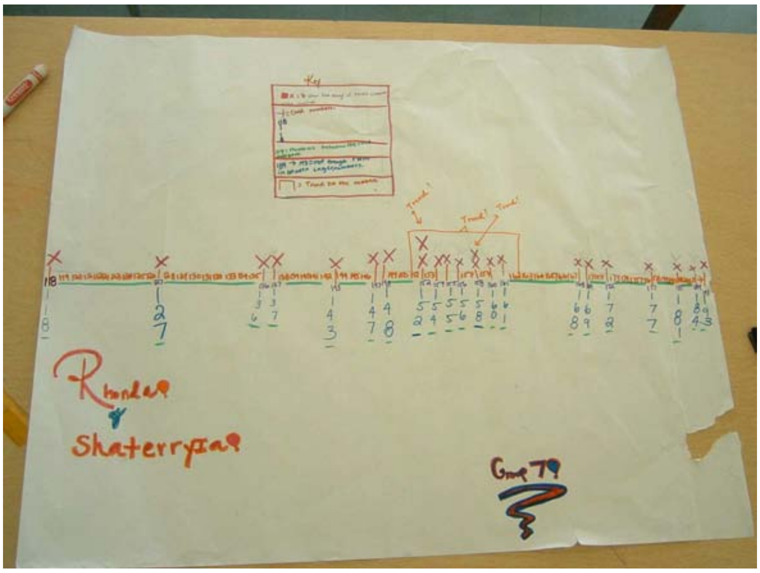


Figure 8. Interval display of relative frequencies.

Comparing representations. Discussions about the variations in design helped students develop an appreciation of different senses of the “shape” of the data. However, students typically focused on individual displays and did not spontaneously engage in comparative analysis. When prompted to compare two different kinds of displays, they often referred to qualities such as icons employed by the designers. For example, students said that they could see squares in one display (to show number of cases) but these were not used in another display. Students often mentioned that a certain display was easy to be seen because it had larger text size. More rarely, a student looked at a display that listed all possible measurements on a number line and said, “They put numbers in between, so you can see how far they went.” Hence, I often took a more active role, drawing student attention to trade-offs among displays by asking them to translate a cluster of cases from one representational scheme into another. I also asked students to develop and test conjectures about the relation between the size of a bin (interval) and the resulting shape of the data. These scaffolds appeared to raise students’ awareness of relations between design decisions and shape.

Inventing Statistics

Following the invention of a representation of the data, students were challenged to invent a measure of the “best guess” of the length of the attribute (e.g., the height of the school’s flag pole). At this point in the learning progression, we anticipated that students could draw on resources of representation and on their knowledge of how the measures were produced. By considering how to develop a measure, we aimed to engage students in deeper consideration of the nature of distribution. What might be worth attending to about the data? Students could use any of the invented displays to help answer this question. We later engaged students in a similar process to develop a measure of the precision of measurements. The definition of precision was intentionally left up to the imagination of the students, so that we could engage students in the relation between measure and qualities of attributes noted in the upper triangular region in the data modeling cycle displayed in

Figure 1. During this period of time, we introduced students to *TinkerPlots* (Konold & Miller, 2005), so that *TinkerPlots* capabilities for dividing and re-organizing the data could be used to construct a measure of precision.

After inventing measures, other students attempted to make use of them. The pedagogical intention was to help students consider the communicative uses of algorithm. Students tried out their methods with other batches of data (to promote generalization), including the measurements of the same attribute with a better tool. For the latter, students noted a reduction in the spread of the data, and I asked if their measure corresponded in meaningful ways to what they could readily perceive in the displays.

Recurrent Patterns of Invented Statistics

Many students struggled with the very idea of inventing of a measure. Some suggested that the only reasonable approach was to ask an authority – a member of the custodial staff or the manufacturer – to find the height of a flagpole. Others found the notion of representing many measurements by a single value implausible. We seized these challenges as opportunities to conduct conversations about qualities of good measures and of the need to be explicit about one's method, so that others could find the same measure.

Measuring centre. Students' invented solutions to estimate the true measure of the attribute generally focused either on repeated values or on the location of the centre clump. Because the data were often multimodal, modal solutions were perceived as less useful, because the inventors typically failed to justify one choice of mode rather than another. Most solutions involving the centre clump used a graphical method to identify the centre clump, and then found the middle value of this centre bin. Many students found this persuasive, but others pointed out that it left out many of the other measurements. A few student teams (at least one in every iteration of the design studies) invented the median, although they did not know this convention at the time of invention. Their reasoning was guided by a sense of splitting the data "in half" and they used bin displays of the data to count an equal number of cases from the tails of the distribution toward the centre. In some data sets, the number of cases was even and the choice for median did not correspond to any observed value. Classmates objected when the median value was not instantiated by an actual measurement, but were persuaded by appeal to the measurement process: The median represented a value that might have easily been someone's actual measurement. It was a "possible measurement". This form of student reasoning signalled a shift away from considering only cases toward considering the aggregate.

Measuring precision. Students' efforts to develop measures of precision most often generated a focus on the "closeness" of the data. More precise measures were those that were closer. We supported this intuition by asking students to predict the value of the measurements if the measures were "absolutely" precise. The three most common solutions to the problem of precision were (a) focus on extreme cases (the range), (b) focus on closeness as distance between a case and other cases or a common point, such as the median, and (c) centre clump solutions, motivated by considerations such as "where the precision was where most people had their numbers".

The range corresponds to convention and thus requires no further explication. The activity of a pair of fifth-grade students exemplifies the second class of solution methods. Their method was spurred by consideration of potentially perfect agreement among the measures, which they suggested would result in no spread or a measure of 0. I asked how they might define their measure so that zero would result. Their response was to consider differences between each case and the median (which they had invented in the previous portion of this phase of the design study). On the basis of previous work with integers, they decided that they would first find the absolute value of each difference. Then, they proposed finding the sum of these absolute values. Their confidence in this measure was bolstered by its ability to differentiate between distributions of measurements where students employed more precise and less precise tools (e.g., 15-cm rulers vs. metre stick for arm-span). I asked students what they might expect if the number of measurers using the more precise tool increased to 100 (about 3 times the original sample) and this precision was compared to the less precise tool used by fewer measurers. The students noticed that use of their measure would mislead: 'People will think that the more precise tool is worse than the less precise tool' (' denotes paraphrase). To solve this problem, one suggested the modal difference and the other, the median. They settled on the median but had difficulty maintaining the relation between the medians for the distribution of measures and of differences (Figure 9). My suggestion to consider the median of these differences as representing "typical closeness" appeared to stabilize this distinction (meaning that when presenting to classmates, they were able to clearly articulate the distinctions).

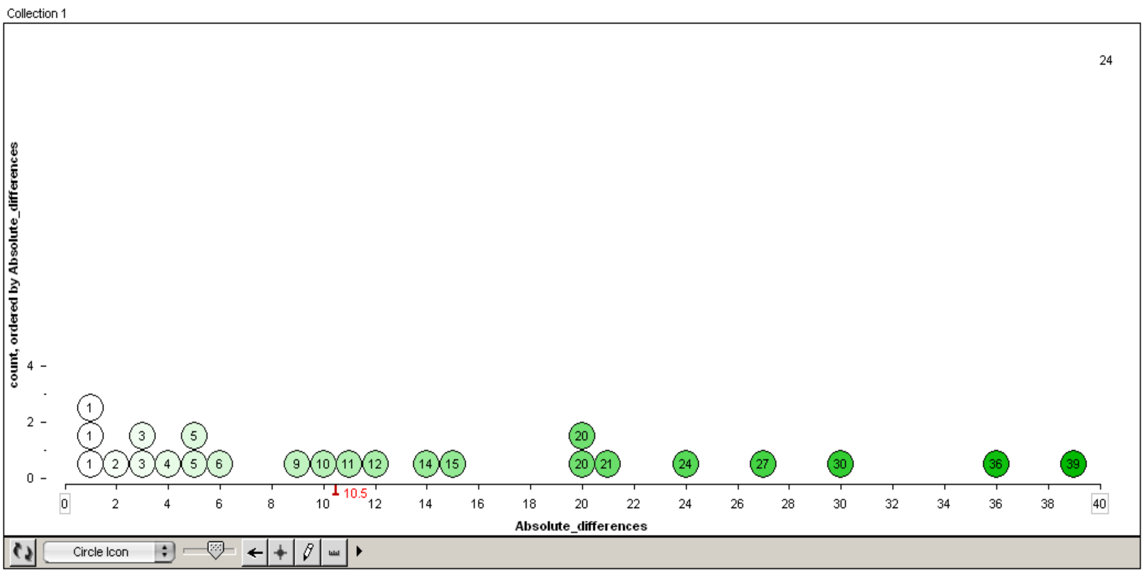


Figure 9. TinkerPlots graph of absolute values of differences with indication of median difference.

Student focus on difference often led to unexpected consequences. For example, one sixth-grader, Robert first focused on the distance between the extreme values of the distribution and the mean. I asked him how he might characterize the precision of the group of measurers rather than just two of them. He decided that he would average the differences, because this would result in a method that would indicate how close the measurements were, "on average". When he attempted to find the mean of the differences, he was surprised that the sum was zero. Robert was puzzled, but he reiterated that he thought his method was good for finding the distances between each score and the mean. He plotted each difference with *TinkerPlots*, and wondered what might have gone wrong (Figure 10).

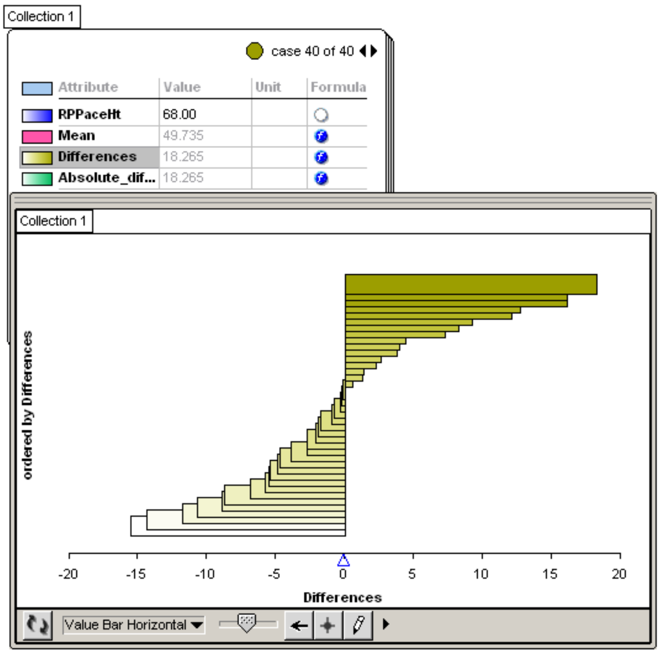


Figure 10. TinkerPlots display of Robert's signed differences.

In light of class discussions about some estimates being over and some under the real height of the flagpole, I asked if Robert were more concerned about the direction, or the magnitude, of each difference. Robert mentioned that the direction of the difference was not that important – some measures *must* be greater than the mean and others less. Hence, what mattered was how far each measure was from the mean. I built on Robert's insight to introduce the absolute value function. Robert used the absolute value function to generate the average deviation. He then plotted the absolute values of the differences, and located their average value – the average deviation (Figure 11), although Robert did not know this convention.

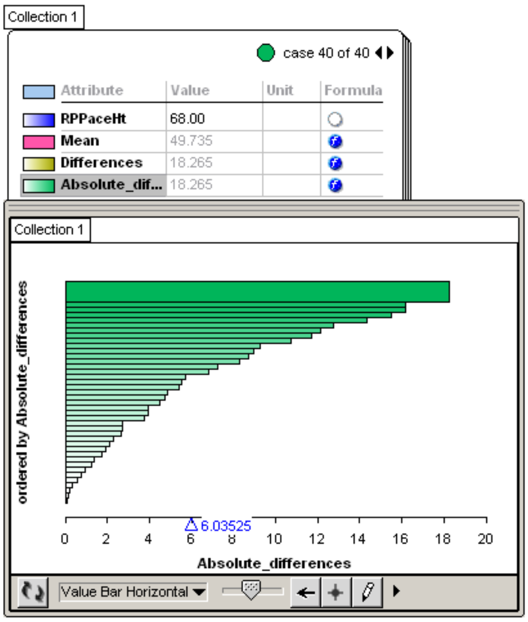


Figure 11. Plot of absolute values of differences and average deviation.

In contrast to close attention to difference, some students defined precision by attending to the relative compaction of the centre clump. Attention to the centre clump typically resulted in measures of precision that corresponded to the inter-quartile range. This definition was supported by the *TinkerPlots* function of "hat plot", but students often used this function only after developing a very similar measure. For example, the solution developed by one sixth-grade student for measuring precision found the lower and upper bounds of the decade-interval that contained the mean. I capitalized on this intuition to introduce the hat plot function, to which the student responded by adding the reference lines to indicate the lower and upper bounds of the mid-50, as displayed in Figure 12.

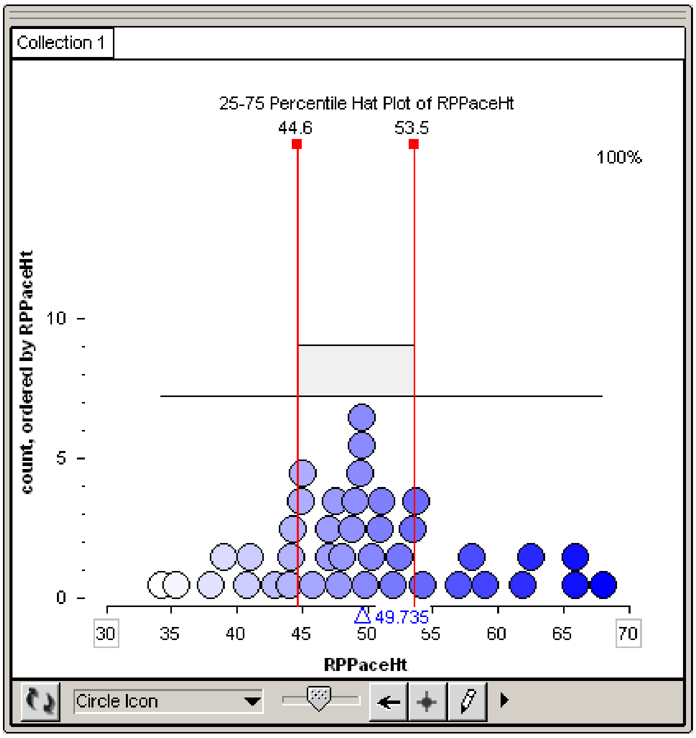


Figure 12. A 25-75 percentile hat plot with reference lines.

Modeling Measure

Following invention of representations and statistics to describe observed trends in variability across different measurement contexts and tools (e.g., arm-span and head circumference, with lower and higher quality tools), the third phase of the learning trajectory is designed to introduce students to the pragmatics and epistemology of modeling chance. We begin with explorations of the conduct of chance devices, starting with hand-held spinners and graduating to a new version of *TinkerPlots* that supports this type of simulation. For example, Figure 13 displays the results of a simulation of a 50-50 spinner with a sample size of 10. Students conducted investigations such as these with varying sample sizes, and we asked students to account for observed differences in departure from expectation as they ran each simulation repeatedly. The line in the Figure 13 was invented by a sixth grade student who thought that changes in slope were a good indicator of departure from expectation as she repeatedly ran the simulation.

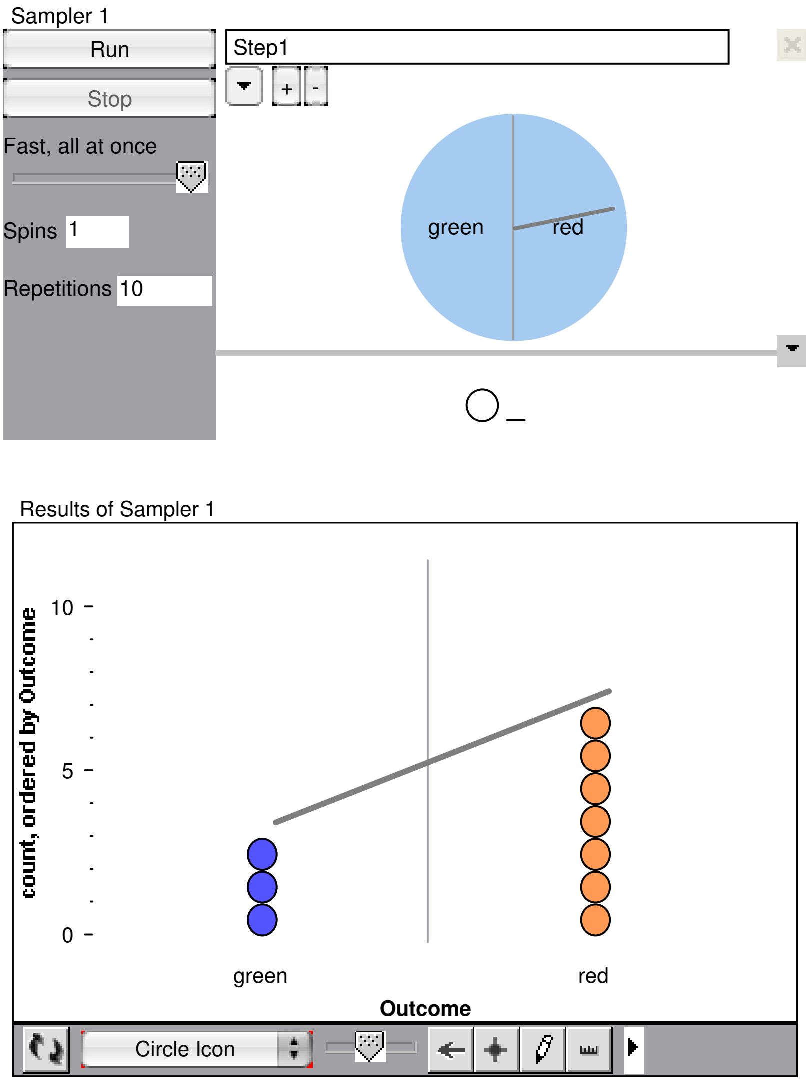


Figure 13. Exploring chance with *TinkerPlots.*

Modelling observed measure. Following investigations of chance, we introduced a prospective model to students of observed measurement as constituted by two sources. Both were familiar to the students. The true measure of the attribute was not directly accessible, but could be reasonably approximated by a centre statistic. The differences among measurements could not be attributed to change in the attribute. (One fifth grader thoughtfully noted that her teacher's head circumference would not change during the interval of measure but she could not say what might happen in the future!) Hence, differences in measure were due to errors of measure. Because students were familiar with processes of measure, we expected that they would be capable of generating conjectures about sources of error. For each source of error, students constructed spinner models that used area to represent relative likelihoods. Relative magnitude and direction of error were also represented as positive and negative values, in the original units of the measure employed by the students. After students constructed and ran simulations of their models, they revised them, as needed. During the final portion of the activity, students constructed "bad" models – models that were designed to employ the same model structure but produce results that would be judged as poor fits to the observed values. This concluding activity provided a window to students' conceptions of model fit and their skill in using the behavior of chance to create the intended structure of outcomes.

Recurrent Patterns of Modeling

Our approaches and technologies for modeling have been revised during successive iterations, so we are least confident of the stability of results. However, during three iterations of the design studies, students appeared to be capable of readily identifying sources of error. For example, when measuring the arm-span of the teacher, students noticed that use of the 15-cm ruler produced much larger spreads (and less precision) when contrasted to the use of the metre stick. They attributed this difference to needing to iterate with the shorter ruler more often. Each iteration provided an opportunity to produce either over-estimates of the true length or under-estimates. Students attributed the former to "laps", instances where the beginning of one measure and the end of another overlapped, resulting in repeated measure of the same distance. The latter were attributed to "gaps", instances where the end of one iteration and the beginning of another were not aligned, resulting in an unmeasured distance.

To illustrate, I consider the efforts of one pair of fifth-grade students to model the batch of measurements of the circumference of their teacher's head. They designed spinners to correspond to three sources of error, which they termed ruler error, slippage error, and reading error. The first two sources of error referred to potential difficulties using tools to measure the circumference of the teacher's head. For example, slippage referred to the tape slipping or stretching as they wound it around the head. Ruler error referred to the difficulty of establishing a common beginning and ending point for the measurements and for measuring the circumference in exactly the same imagined path around the head. Reading error referred to perceptual difficulties, for example, a measurer might have difficulty judging the number of cm. to the nearest whole number. Each observed measurement was represented by the sum of random error (the sum of the 3 spinners) and the median of the observed measurements, representing an estimate of the true length of the circumference. These spinners are displayed in Figure 14. After running this simulation, the students noticed that it tended to overestimate the centre of the distribution and to produce spreads that were not aligned with the observed values. Hence, they re-designed the spinner depicting ruler error (the far left of Figure 14) to eliminate unrealistically large magnitudes and likelihoods. The resulting simulation was a better match to the shape and centre of the observed values. During the conduct of this simulation, the students noticed that net errors were occasionally zero and that unlikely events nonetheless occurred.

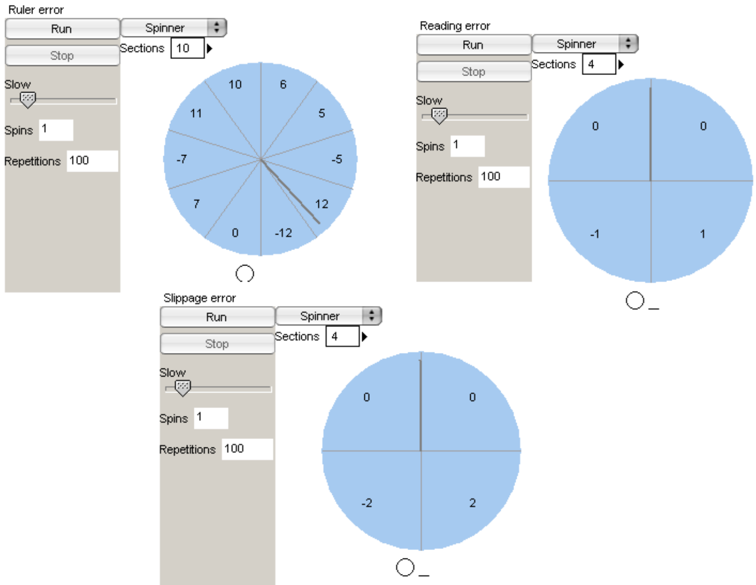


Figure 14. Simulation of sources of random error.

Bad models were a playful way for students to investigate further relations between model design and outcomes. For example, in Figure 15, a fifth-grade student managed to invert the shape of the observed distribution and to produce a skew as well.

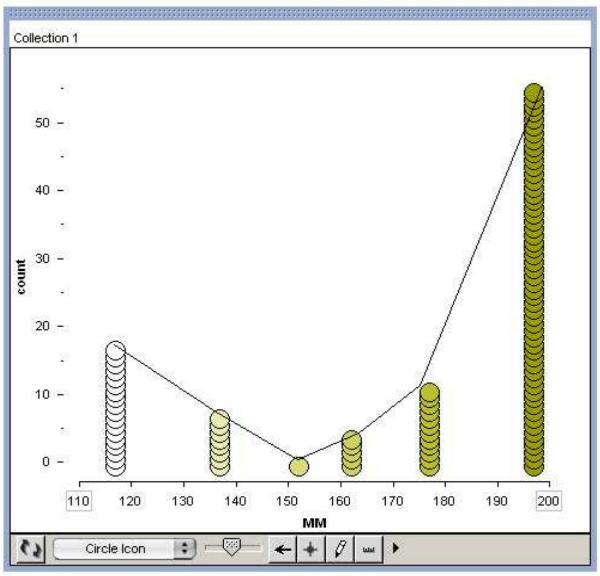


Figure 15. Results of a simulated bad model of normally distributed measures.

Designing Assessment to Support Instructional Design

In most design studies, day-to-day decisions are made in light of evidence about student thinking, most often obtained from inferences based on students' discourse and artifacts that they produce. Much of the previous presentation of recurrent patterns of student reasoning follows in this tradition. In design research, assessment is often considered after the fact, as summative evidence of more widespread patterns of individual performance. However, in our design studies assessment played a central role, both in the conduct of the studies and in the interpretation of the results. In fact, one of the anticipated outcomes is the creation of an assessment system.

To create an assessment model, our conjectures about the forms of knowledge and the nature of conceptual change underpinning learning about variability were expressed as *progress variables* (Wilson, 2005). Progress variables model trajectories of development. They demand that designers of learning progressions make their commitments about conceptual growth explicit. We constructed progress variables in seven conceptual strands: (a) theory of measure (conceptual landmarks for understanding the nature of units and scales of measurement, which are prerequisite understandings for the learning progression), (b) modeling measurement, (c) data display, (d) meta-representational competence, (e) concepts of statistics, (f) probability/chance and (g) informal inference. Table 1 illustrates a summary of the Data Display progress map, which lays out our conjectures about prospective transitions in students' conceptions, from case-based to aggregate-based ways of constructing and interpreting data displays. The full version of each construct contains examples of each performance in both text and video formats.

Although progress maps may appear to have a preordained character, in fact, they are negotiated as the design study unfolds, so that progress maps take several design iterations to "settle". Hence, they serve as a visible trace of prospective conceptual landmarks for the design team.

Based on the construct maps, we designed items to support instruction and to index student progress over longer periods of time. To support instruction, some items were designed as formative tools to diagnose student conceptions. These were administered as weekly quizzes, and the results were employed to re-design the learning progression. For example, during one design study, the results of a formative assessment indicated that many students interpreted their classmates' invented statistic, the median, to be a half-spit of the data located in the "middle" of a string of data. They apparently did not consider the order of the data as critical, relying instead on the spatial centre of the data presented by the inventors of the statistic. Consequently, we decided to problematize "half" by contrasting the distance-based image of the mid-range with the count-based definition of the median. Students thought that any estimate of the best guess of the length of the arm-span should be located in the centre clump. Their image for mid-range was a paper strip folded into two congruent lengths, an image familiar to them from class work earlier in the year finding partial-units of length measure. The fold line of this strip located ½; but, what was the relation of this distance-based sense of half to the half demarked by the median? If the mid-range was "halfway", how could the median also be considered half? How could counting result in a location in the centre clump? We constructed several small sets of imagined measurements with the lowest or highest values located in the centre. By simply counting, the extreme values were considered best guesses of the true measure. Yet, this contradicted children's sense. This contradiction was resolved by re-examining the role of order in determining the median, and by juxtaposing two different senses of "1/2-split" – one based in distance and the other in position within an ordered sequence. We also took this opportunity to investigate robustness of the statistics proposed – by investigating the effects of "one bad measurer" on the estimate of true measure. (The mid-range declined in popularity when students considered that just one student-measurer could shift the value of the mid-range out of the centre clump.) These modifications were incorporated into subsequent iterations of the design.

Table 1
The Data Display Construct

Level	Performances
DaD6. Integrate case with aggregate perspectives	DaD6(a) Discuss how well individual values or regions represent the patterns seen in the whole distribution, or vice versa.
DaD5. Consider the data in aggregate when interpreting or creating displays	DaD5(b) Quantify aggregate property of the display using one or more: ratio or proportion or percent. DaD5(a) Recognize that a display provides information about the data as a whole that goes beyond any of the cases by themselves.
DaD4. Recognize or apply scale properties to the data	DaD4(b) Recognize the effects of changing bin size on the shape of the distribution DaD4(a) Display data in ways that use its continuous scale (when appropriate) to see holes and clumps in the data.
DaD3. Create categories of cases based on relationship among them	DaD3(c) Identify data points that are dissimilar to the rest. DaD3(b) Identify grouping of similar values (e.g., high, medium, low values). DaD3(a) Note similar values or "clumps" in the data set.
DaD2. Concentrate on cases when working with data	DaD2(b) Manipulate data attending only to its ordinal properties. DaD2(a) Concentrate on specific data points (minimum, maximum, median, mode), without relating these to any structure in the data.
DaD1. Treat data as collection of individual numbers or attributes	DaD1(a) Manipulate, notice and explore qualities or relations of data values, without relating to the goals of the inquiry.

Although this effort is still a work in progress, we are currently working to articulate an assessment system that will span both instruction and accountability. From the perspective of conducting studies of learning, the formative assessments standardize our commitments about what counts as evidence of student reasoning. The summative assessments provide a less fine grained but broader spectrum to track conceptual change. This provides an opportunity to engage in design experiments, in which the implications for learning of different instructional designs can be contrasted in a common metric.

Discussion

The links between data analysis, chance, and modeling have often been severed in school mathematics. Yet, in a wide variety of professions, data modeling is integral to practice. The epistemology in professions is one of model building and competition, not one of "descriptive" statistics, followed by "inferential" statistics, which is the standard practice in schools. I propose restoration of the link between data modeling and statistical reasoning in schooling, not merely because it is what professionals do, but more importantly, because it is a viable and fruitful approach for supporting the growth and development of student reasoning about variability. Variability is ubiquitous and it is critical for thinking in the 21st century that we equip students with ways to reason about it.

The learning progression outlined in this paper rests on several general principles of learning and on the potential affordances of measurement as a context for investigating variability. The first is that of agency. If measure is framed as activity, rather than as a product, students can mentally simulate the role of agents and/or they can literally enact measurement process. Agency mediates student apprehension of variability by making process transparent (e.g., individual measurers can recall qualities of method and measure that might lead to "mistakes" in measurement), and it grounds symbolic expression, in that students can readily relate presentational qualities (e.g., hills in graphs) and measures thereof (e.g., medians as measures of centre) to specific forms of activity. A related virtue of agency is that qualities of distribution can be viewed as emerging from the collective activity of agent-measurers. Hence, a statistic, such as the median or mean, can be viewed readily as a measure of central tendency (Konold & Pollatsek, 2002), and the explanation for such a tendency can be attributed to the notion of a true or fixed value.

Second, developing representational and meta-representational competencies have important conceptual consequences. The diversity of representations invented by students supports the concept that the shape of the distribution is not a Platonic ideal, but rather, a result of a particular set of choices made about what to attend to, and what to obliterate, in a system of representation. Not all students fully grasp the idea of representational trade-off, but supporting comparisons among representations provokes mathematically fruitful consideration of different meanings of the "shape" of the data. Seeing hills and valleys is one thing, knowing how they are produced and how they might be magnified or even eliminated is another. We strive for the latter, and it appears that this is a consistent outcome when we deliberately instigate comparisons among representations.

Third, inventing measures of what students can readily "see" in a set of data invites closer inspection of the qualities of the data that contribute to the perception. Students' invention of measures of centre and spread support consideration of just what one might mean by each. Thus, there is an intimate relation between conceiving of the "centredness" or "spreadness" of the distribution and its measure. What students see after inventing measures is often different than what they saw before such invention. Thus, measure is an important cornerstone to quantification (Lehrer, Carpenter, Schauble, & Putz, 2000; Thompson, 1994). Inventing measures supports a meta-conceptual development: What does it mean to measure and what are qualities of good measurements? These developments are supported when students employ their inventions to measure the attributes of new distributions that were formed when measurers used different methods or tools. For example, students' experience suggests that measuring the arm-span of a person with a 15-cm ruler is more error prone than the same measure employing a metre stick (fewer iterations lead to less error). Hence, it makes sense that the distributions have different precisions and that the measure ought to reflect these differences. Measure allows too for a new form of inquiry not as readily sustained by the eyes: How much more (or less)?

Fourth, the conceptual landscape of modeling is altered by the technologies deployed for modeling. When we first began, students used hand-held spinners to construct models, and these were certainly adequate tools for engaging in the process of modeling chance. However, we cannot help but notice that the introduction of *TinkerPlots* alters this landscape. One form of alteration is in ease of model design and revision. Although we wish for more capability from *TinkerPlots*, and we are confident that we will soon see it, the current implementation allows for much more rapid prototyping and running of models. We believe that this has a conceptual consequence: Models that are run more often invite attention to sample-to-sample variation in outcomes. This embarks students on the road to sampling distribution, an unintended consequence from the point of view of our initial conception of the learning progression.

Last, although we often hear that cognitively guided assessment is a virtue, it is difficult to find many examples. Of course, virtue is always distributed more like Poisson than Gauss, but our work with colleagues at the Berkeley Evaluation and Assessment Research Center and the work of Jane Watson and her colleagues (e.g., Watson, Callingham, & Kelly, 2007) suggest that linking assessment to models of learning statistics is not a trivial pursuit. When we work collaboratively with assessment experts during the design of instruction, we find that both of our professional worlds are enriched, and we hope, so too are those of the students.

I conclude with a lamentation. The opportunities for supporting student reasoning about variability are often confined to mathematics education. Yet the origins of the mathematics of variability arose in contexts of modeling nature, and these contexts are still a primary arena for modeling variability. Unfortunately, school science works full time to hide this variability from students, especially in pursuit of laboratory exercises with gargantuan effect sizes that render inference moot. This is a lost opportunity. A science education that encouraged student inquiry and model development would be a natural site for grappling with issues of variability.

Acknowledgement. The research described in this paper is collaborative. Thanks to Min-joung Kim, Charles Munter, Leona Schauble, and Wenyan Zhou for their valuable contributions to the design and implementation of instruction. Mark Wilson and Tzur Karletiz of the University of California-Berkeley lead our effort to design the assessment system. Cliff Konold of the University of Massachusetts-Amherst supports student modeling of chance with his re-design(s) of TinkerPlots.

References

diSessa, A. (2004). Metarepresentation: Native competence and targets for instruction. *Cognition and Instruction*, *22*(3), 293-331.

Giere, R. N. (1992). *Cognitive models of science.* Minneapolis, MN: University of Minnesota Press.

Greeno, J. G., & Hall, R. (1997). Practicing representation: Learning with and about representational forms. *Phi Delta Kappan, 78*(5), 361-367.

Hall, R., Wright, K., & Wieckert, K. (2007). Interactive and historical processes of distributing statistical concepts through work organization. *Mind, Culture, and Activity, 14*(1&2), 103-127.

Konold, C., & Miller, C.D. (2005). *TinkerPlots: Dynamic data exploration*. [Computer software] Emeryville, CA: Key Curriculum Press.

Konold, C., & Pollatsek, A. (2002). Data analysis as the search for signals in noisy processes. *Journal for Research in Mathematics Education, 33*(4), 259-289.

Lehrer, R., & Romberg, T. (1996). Exploring children's data modeling. *Cognition and Instruction, 14*(1), 69-108.

Lehrer, R., & Schauble, L. (in press). Contrasting emerging conceptions of distribution in contexts of error and natural variation. In M. Lovett & P. Shah (Eds.), *Carnegie Symposium on Cognition: Thinking with data.*

Lehrer, R., Carpenter, S., Schauble, L., & Putz, A. (2000). Designing classrooms that support inquiry. In J. Minstrell & E. H. van Zee (Eds.), *Inquiring into inquiry: Learning and teaching in science* (pp. 80-99). Washington, DC: American Association for the Advancement of Science.

Porter, T. M. (1986). *The rise of statistical thinking 1820-1900.* Princeton, NJ: Princeton University Press.

Snedecor, G. W., & Cochran, W. G. (1967). *Statistical methods*. Ames, IA: University of Iowa Press.

Stigler, S. M. (1986). *The history of statistics.* Cambridge, MA: Harvard University Press.

Thompson, P. W. (1994). The development of the concept of speed and its relationship to concepts of rate. In G. Harel & J. Confrey (Eds.), *The development of multiplicative reasoning in the learning of mathematics* (pp. 179-234). Albany, NY: SUNY Press.

Watson, J. M., Callingham, R. A., & Kelly, B. A. (2007). Students' appreciation of expectation and variation as a foundation for statistical reasoning. *Mathematical Thinking and Learning, 9*(2), *83-130.*

Wild, C. J., & Pfannkuch, M. (1999). Statistical thinking in empirical enquiry (with discussion). *International Statistical Review*, *67*(3), 223-265.

Wilson, M. (2005). *Constructing measures: An item response modeling approach*. Mahwah, NJ: Lawrence Erlbaum Associates.

Studies in the Zone of Proximal Awareness

John Mason
Open University & University of Oxford
<j.h.mason@open.ac.uk>

Helen Drury
Open University

Liz Bills
University of Oxford

I have long promoted the conjecture that expressing generality lies at the heart of school algebra. Indeed, I have gone further to suggest that "a lesson without the opportunity for learners to generalise is not a mathematics lesson". It seems beyond doubt that experiencing and expressing generality is natural to human beings. The pedagogic issue is why there is so much resistance amongst teachers and learners to using this power in mathematics lessons. The notion of generalisation here includes both abstraction from context and generalisation within context. Pondering this question has led us to wonder why generalisation happens sometimes and not others, what can be done to prompt useful mathematical generalisation, and under what sorts of circumstances: in short, what are the conditions for and evidence of imminent or proximal generalisation?

The present paper arises from reflections on lessons involving the expression of generality, sometimes by learners and sometimes by teachers. Our reflections led us to try to organise and inter-relate a variety of forms of mathematical generalisation: empirical, structural, and generic; syntactic and semantic; metonymic and metaphoric; enactive, affective, and cognitive. The idea is to prepare the ground for further studies. Our reflections also led to the notion of a *Zone of Proximal Generalisation* as a particular case of a *Zone of Proximal Awareness*, in order to try to describe and distinguish differences in the imminence of appreciation and competent expression of generality in and by different learners at different times. This in turn opened up a domain of further investigation into various proximal zones as projections of Vygotsky's original intention with the *zone of proximal development* into the three classic dimensions of the psyche.

Introduction

Historically, algebra is usually seen as arising through a desire to be able to solve problems involving some unknown number or numbers. As Mary Boole (Tahta, 1972) put it, by "acknowledging your ignorance" you can denote what you do not know with a letter, and then manipulate that letter as if it were a number in order to express relationships and constraints arising from the problem. Support for this view can be found in the use in early algebra texts of the term *cosse* ("thing") as the "as-yet-unknown".

At the same time however, there is a pervasive historical thread by authors wanting to solve every problem, or trying to indicate that the solution to a particular problem is to be seen generically as a method for solving a whole class of similar problems (Gillings, 1972; Cardano, 1545; Viete, 1581). Authors used a variety of means for informing the reader of the "general rule", in words, and through the use of examples. Newton (1683) may have been one of the first to use letters to denote as-yet-unspecified parameters so as to solve a problem "in general".

There is however a conceptual commonality between the use of a letter to stand for an as-yet-unknown and the use of a letter to stand for an as-yet-unspecified parameter: both depend on the person to be stressing the letter as label rather than as the value of the label.

This means treating the letter as a slot (we now call it a variable) without attending to its contents, trusting that the contents will look after themselves. Flexible movement between attending to the label and attending to the content (syntax and semantics of expressions) is the essence of working effectively with expressions of generality.

Every learner who arrives at school walking and talking has displayed the power to perceive and express generality and the case has been argued in one form or another by many (Whitehead, 1932; Gattegno, 1970). The issue for teachers is therefore not whether learners are capable, nor even whether learners will use those powers in lessons, but how to foster and support the use of those powers in mathematical ways, not only within mathematics but also so as to use mathematics to make sense of the world.

Expressing Generality

It is worth mentioning in passing that stressing *expressing generality* as a root of and route to algebra (Mason, Graham, Pimm, & Gowar, 1984; Mason, 1996; Mason, Johnston-Wilder, & Graham, 2005) does not mean that working on patterns in number sequences and matchstick pictures is either necessary, or sufficient. A much more comprehensive view is implied, as articulated in the abstract.

Every mathematics lesson involves generalisation, and the sooner and more frequently that learners are invited to try to express those generalities through actions and then words, the more likely they are to appreciate what algebra can do for them. This corresponds closely with Vygotsky's notion that teaching converts ability to do something into ability to do something knowingly, that is, to "transform an ability 'in itself' into an ability 'for himself'" (van der Veer & Valsiner, 1991, p. 334), and to Gattegno's notion of schooling as educating "awareness of awareness" (Gattegno, 1987; see also Mason, 1998).

Kieran (2004) refers to work with patterned sequences in which learners express generalisations as *generational* aspects of algebra, contrasted with manipulation and use (Kieran's *transformational* and *global/meta*). Without the generational there is no purpose in the transformational. Furthermore, both the transformational and the global/meta emerge directly out of multiple and rich experiences of expressing generality as the range and sophistication of expressing generality merges with core ubiquitous mathematical themes such as freedom and constraint and doing and undoing (Mason et al., 2005).

The aim of this paper is to try to discern what might be going on for learners who are on the edge of expressing generality in some situation. In particular, what might be the effects on different learners of being in the presence of expressed generality, and what can be done as a teacher to try to maximise the effectiveness of enculturation into perceiving and expressing generality.

Generality in this paper includes both the usual uses such as recognition of a feature in a situation as a parameter that could be varied, the abstraction that takes place when people focus attention on their actions rather than on the objects on which they are acting, and mathematical abstraction in which context is back-grounded and structural relationships are put forward as properties that are treated as axioms.

For example, adopting the practice of counting-on, when guided, can lead, through the natural process of reflective abstraction (in Piagetian terms) to focus on the action, creating counting-on as a new awareness (Mitchelmore & White, 2004). This in turn can lead to a change of level through interiorisation of a higher psychological practice (in Vygotskian terms).

Ways of Working

My way of working starts with recent experience reminding me of past experience. It involves identifying phenomena and issues that seem to deserve elucidation so as to inform my future action and that of others. I seek accounts (in my own and in other people's data) that highlight or illustrate phenomena and issues, and I test these for resonance with colleagues. I also construct tasks through which it is possible to have a taste of what it might be like for people in similar situations to those observed. Sometimes the mathematical sophistication of the task is appropriate for learners, and sometimes appropriate for colleagues with more extensive mathematical experience. Then I try out those tasks on others, modifying them and honing them so that most people report not only recognition of the phenomena, but recognition of the distinctions that have proved fruitful, linked with possible actions to take when working with learners. I do not try to prove that specific strategies are guaranteed to produce specified results, either statistically or through observational data. I seek educated awareness, not mechanical reproduction. For me what matters is awareness in the moment of planning and of teaching. This means what is being attended to, how it is being attended to, and what possibilities for action come to mind. I am content to offer experiences that might sensitise others to notice opportunities (through making pertinent distinctions) to act freshly and effectively in situations.

Phenomena

We begin with some characteristic phenomena that highlight a need to clarify and precise what is meant by generalisation in the mathematics classroom. Some situations are described that are likely to be recognisable in experience and some tasks are offered through which you may experience directly some significant aspects of generalisation. After each example there are comments which inform and illustrate distinctions.

Observed Phenomena

The story begins with a report by the second author of a repeated phenomenon in her classroom.

> ΦA: in one lesson, a learner asserted that "anything times zero is zero!". Her voice tones suggested surprise, as if it was a new thought. In actual fact she had uttered this same generality several lessons previously, with similar surprise.

Comment. The activity of the "maths fairy", which intervenes to wipe learners' memories (Houssart, 2001; 2004), is one way to account for this phenomenon. Does the learner really not recognise the same generality again, or is there an element of giving the teacher what the learner thinks is valued, namely conjectures and surprised voice tones?

As Rowland (1995) observes, learners may be very tentative in expressing half-formed thoughts and partially formulated ideas. He draws attention to the fragility of self-esteem and the use of linguistic hedges, presumably in order to distance the conjecturer from the conjecture in case it is seen as silly or wrong. He recommends creating a zone of conjectural neutrality, in which what is said is considered independent of the person who says it, and treated as something which may need to be modified or augmented. Establishing an atmosphere in which people are expected to and are supported in expressing half-formed thoughts makes a vital contribution to mathematical development, and particularly to expressing and appreciating generality (Mason, 1988, p. 9). The whole point of a conjecturing atmosphere is to overcome such sensibilities.

Since generalisations are being perceived and expressed, but apparently forgotten, expressing generality is not in itself a guarantee of "learning". It may however be a sign of cognitive development, of the use of powers that can be evoked or even called upon explicitly in future lessons.

Often generalisations are appropriate, if some what ambiguously expressed.

ΦB: Asked what was 3 + 5, and then 5 + 3, and again, 2 + 6 and 6 + 2, Q (aged about 6) sat thoughtfully for a few moments and then said "anything plus anything is anything plus anything".

Comment. This illustrates a (possible) awareness of a generality, expressed spontaneously in response to attention being directed to a few facts. However the expression of that generality is highly ambiguous and reminiscent of "alge-babble" (Malara & Navarra, 2003) or "emergent algebra" (Ainley, 1999), as an example of attempts to express something without a firm grasp of the grammar and syntax used by others. It suggests a sensitivity to generalisation through the fact that attention drawn to a few "facts" made them exemplary of a more general fact. Because of the ambiguous multiple use of "anything", it is impossible to tell from the account whether the learner's attention was on commutativity, or on the fact that different pairs of numbers can add up to the same thing. Although there is a taste of the empirical, the learner's attention seems to be on structure.

Sometimes learners generalise inappropriately from partial or even incorrect data:

ΦC: Learners, invited to generalise the following observations
$3 + 4 + 5 = 3 \times 4$; $6 + 7 + 8 + 9 + 10 = 5 \times 8$; $7 + 8 + 9 + 10 + 11 + 12 + 13 = 7 \times 10$
focus on the first expression and propose that $n + n+1 + n+2 = n\ (n + 1)$.

Comment. A task similar to this is mentioned in Rowland (2001), and the same thing has happened in other places, with the same results: someone looks only at the first statement and tries to generalise. It seems as though the person treats both the 3s the same without recognising the structural role of the second 3. Perhaps the statement is read simply as a succession of numbers with an equality sign, rather than as a structural statement about arithmetic. At the same time, the learner ignores the other proffered statements, perhaps because one is in the habit of dealing with one thing at a time, perhaps because one single statement occupies the attention fully, or perhaps because it all looks too difficult (which it might do if you were not attending to the structural detail of the statements but simply seeing them as strings of symbols). The *relational* thinking necessary to make structural sense has been studied in arithmetic by Molina (2007), (Molina, Castro, & Mason, 2007) and in equations using algebra by Alexandrou-Leonidou and Philippou (2007). It involves a subtle but important transformation of the structure of learners' attention (Mason, 2003).

ΦD1: Teacher's account: I am talking to the whole class about the way in which they derived the equation of a circle with radius 2 and centre (3, 5). I have written the equations $\sqrt{(x-3)^2+(y-5)^2}=2$ and $(x-3)^2+(y-5)^2=4$ on the board. I ask "where did the 3 the 4 and the 5 come from in this (the second) equation?" Trevor replies that the 4 is the diameter of the circle. (Bills & Rowland, 1999, p. 110)

ΦD2: In a lesson on right-angled triangles, the first two examples were a 6–8–10 and a 5–12–13 triangle. A learner observed that the area and perimeter were (numerically) the same and conjectured that "this happens every time". (Bills & Rowland, 1999, p. 104)

Comment. In ΦD1 the learner has discerned a 4 but treated it as a structural 2 times a particular 2, rather than as a particular 2 squared. The teacher was attempting to shift students' attention from the process (of deriving the equation) to surface but structural relationships (for example that the term on the right hand side is the radius squared). But it seems that Trevor's attention was already on the surface relationships, and what is more, was triggered into seeing 4 as double two rather than as 2 squared. This has the flavour of a mixture of empirical and structural awareness at the mercy of metonymies.

ΦD2 is typical of what happens when examples have unintended and unanticipated features to which a learner attends. From the learner's perspective, some aspect was stressed (and so seen as a dimension of possible variation) while other aspects were ignored (and so seen, at least for the moment, as invariant). Fortunately the learner expressed the conjecture out loud; often there are relationships that are perceived implicitly, below the surface of awareness, but which pervade future thinking. Fischbein (1993) coined the term *figural concept* to describe the way in which unintended features inappropriately become part of the concept as constructed by the learner. In a conjecturing atmosphere, it is possible to praise the act of conjecturing while at the same time critiquing the conjecture itself. Faced with a conjecture that seems implausible, it is natural to seek a counter-example, or even to characterise all objects with the stated property. Zaskis and Lilejdahl (2002) point to several similar ways in which learners inappropriately generalise.

By contrast, some generalisations require considerable effort. Rowland (2001) reports on learners struggling to see how a process of reasoning used in a particular case in number theory could be applied to a general case involving any prime number. The indeterminate nature of the general was hard to pin down, because the reasoning seemed to require knowledge of the particular numbers. Consequently, some learners stuck with particulars. In a more elementary setting, learners unaccustomed to expressing generality and then asked to "say how to do it" very often revert to "well if you had …" and use a particular. Sometimes it is possible to get a flavour of how they are seeing through the particular to the general; sometimes it is difficult to tell whether their "method" is seen as particular, but able to be carried out in many different particular cases. In these situations, learners seem poised somewhere between experiencing the possibility of a generality, experiencing a generality but not being able to articulate it, and expressing a generality.

Considerations like these raise questions about how generalities expressed publicly in lessons contribute to or obstruct the perception and expression of generality by learners. It all came to head in the following incident in the second author's classroom.

> ΦE: In a lesson with 15 yr olds about rationals and irrationals, learners proposed $\sqrt{17}, \sqrt{5}, \sqrt{18}$ as examples of irrationals. The teacher then asked for a rational "one" and was given $\sqrt{9}$, then $\sqrt{16}$ and $\sqrt{4}$ She then said to the class "so all square roots of square numbers are rational".

Comment. The overall aim of the teacher was to work on the definitions of rational and irrational. It took 25 minutes of example construction to reach the point of formulating a definition, and 8 minutes to formulate definitions. There were other opportunities like the square-root example, for generalising along the way. The teacher had choices to make as each possibility came up: whether to engage the learners in expressing a generality for themselves, or whether, judging by the fluency of construction and the flow of examples, most learners appreciated the particular generality being exemplified at the moment. The teacher could also have chosen to ignore byway generalities altogether, or as an intermediate strategy, she could have suggested in passing that there was a generality to pursue some other time.

It is tempting for the teacher to utter "the generality" that she assumes everyone has experienced. But what is the effect of her statement on learners who have not yet become consciously aware of the property, or on learners who are vaguely aware but have not yet isolated it as a phenomenon and expressed it to themselves, or on learners for whom it is an almost unnecessary statement? For these latter, it may act as confirmation of what they were already well aware of, an important role for teachers to play. For learners who had not yet expressed it for themselves, it could act as a crystallisation, a bringing to the surface what they now recognised they had been aware of, albeit not explicitly. However, it could also serve to take away that moment of "things falling into place", when the act of generalisation releases just a little bit of energy in the form of pleasure or surprise as several particulars are subsumed under one label, as in "aha! that's what's going on!". For some learners the statement may simply pass them by because it does not speak to their current thinking, does not fit with, amplify, or summarise nascent awareness.

There is a direct analogy between the states being described here and the forms of noticing identified in Mason (2002): not noticing at all (below the surface of awarenesses that can be resonated); noticing but not marking (able to be resonated, in the sense that when someone else draws attention to it you recognise what is being described but you could not have initiated such a description yourself); marking (where you could initiate a description yourself); and recording (making some sort of external note about what is noticed).

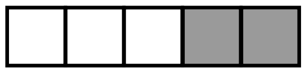
Before developing the notion of "being poised to generalise", here are some tasks which may provide an explicit taste of the same phenomena.

Experiential Phenomena

The following tasks may afford some opportunity to experience freshly some of the perceptions, states and issues arising from the previous phenomena.

Task A: Can You See …?

While gazing at the following diagram, can you "see"…



two-fifths of something? three fifths of something? two-thirds of something? one third of something? three-halves of something? five thirds of something? two thirds of three-halves of something? What other fraction calculations can you "see" directly? (Thompson, 2002)

Comment. The prompt to "try to see" signals a shift in how you attend to the figure, what you stress and consequently what you ignore, which Gattegno (1987) proposed as the mechanism of generalisation. What is available for generalisation is the particular fractional parts used, the particular diagram, and the way of working (stressing seeing rather than writing down).

Task B: Differing Products

Extend and generalise the following facts

$3{\times}2 - 3 - 2 = 2{\times}1 - 1$			
$4{\times}3 - 4 - 3 = 3{\times}2 - 1$	$4{\times}2 - 4 - 2 = 3{\times}1 - 1$		
$5{\times}4 - 5 - 4 = 4{\times}3 - 1$	$5{\times}3 - 5 - 3 = 4{\times}2 - 1$	$5{\times}2 - 5 - 2 = 4{\times}1 - 1$	
$6{\times}5 - 6 - 5 = 5{\times}4 - 1$	$6{\times}4 - 6 - 4 = 5{\times}3 - 1$	$6{\times}3 - 6 - 3 = 5{\times}2 - 1$	$6{\times}2 - 6 - 2 = 5{\times}1 - 1$

Comment. There are various ways to "see" structure. Watson (2000) used the metaphor of splitting wood to capture the natural propensity to recognise and pick up on a flow of familiarity, referring to it as "going with the grain": following or making use of familiar structure such as here with the flow of natural numbers. She uses the metaphor as a reminder that in order to make sense and to learn from the recognition of such patterns, it is necessary to "go across the grain", exposing the structure of the wood, and by analogy, the structure responsible for the perceived patterns. Going with the grain has been called recursive, or iterative, because it focuses on how the next term is obtained from previous terms. Going across the grain involves structural generalisation.

Here, going with the grain detects the flow of natural numbers both horizontally and vertically. Stressing what is invariant provides a skeleton with which to flesh out descriptions of what is changing, and how. Having more than one thing changing at a time often causes particular difficulties for people trying to generalise (Mason, 1996). Note that the presence of the equals sign emphasises the structural over the empirical, for it is the relationship that is being studied, not arrays of numbers from some unknown source.

Task C: Pólya Crosses Out

Write out the natural numbers in sequence for at least 10 terms. Now cross out the third, the sixth, the ninth, etc. and in a second line, record for each term the "sum so far" of the terms that are left. Repeat, but this time crossing out the second, fourth, sixth, … before forming the next line of "sums so far". You might recognise the final sequence.

1, 2, ~~3~~, 4, 5, ~~6~~, 7, 8, ~~9~~, 10, … 1, ~~3~~, 7, ~~12~~, 19, ~~27~~, 37, … 1, 8, 27, 64, …

Comment. This task appeared in Mason et al. (1982) but was taken from Pólya (1962). Many years later it showed up again in Conway and Guy (1996, pp. 63-65), who revealed its origins in Moessner (1952). The phenomenon of interest here is that apart from generalising to more rows and hence to bigger gaps in the first crossings-out, I was unable to see any other ways it might generalise. However, Conway and Guy displayed a whole world of intriguing and fascinatingly complex ways in which it generalises. For me the task is a reminder of the phenomenon that there are often dimensions of possible variation undiscovered in even the simplest of situations.

Questions Arising. These and other phenomena raise puzzling questions about why it is that, despite having displayed the power to generalise (and to particularise) in many different contexts, learners display a range of responses from no recall of previous generalisations (as in ΦA), through reticence, to downright refusal to make use of those powers in mathematics. What can teachers do to foster and sustain generalisation as a regular feature of mathematics lessons? It is sensible to start addressing this by distinguishing different forms of generalisation.

Forms of Generalisation

A distinction is often drawn, though mainly implicitly and with considerable variation by different authors, between *generalisation* and *abstraction*. This is related to a distinction between generalising the result of an action on objects as properties of the objects, and generalising (abstracting) that action away from the objects themselves. The abstracted action is then available to carry out on other (presumably similar) objects in other contexts. Then there is a distinction between *empirical*, also known as *generalisation from cases*, and *generic* (also sometimes referred to as *structural*) generalisation. It too is rather slippery as a distinction, and is employed differently by different authors. Distinctions are sometimes aimed at the mathematics, and sometimes at the activity of learners. For example, a distinction can be drawn between *syntactic* and *semantic* generalisation, and between *metonymic* and *metaphoric* generalisation, in an attempt to distinguish between the activity of someone satisfied with a *surface* approach to learning, and a *deep* approach. In a paper of this length there is not space to elaborate fully on all of these in detail, but it may be useful to attempt to relate them together as part of a web in which to try to catch a variety of generalisation experiences. Most generalisation, especially in algebra, is seen as a *cognitive* process, but sometimes generalisation happens without conscious awareness and thus could be considered as an *enactive* generalisation. The second author has observed that there is often an associated *affective* component of generalisation either supporting or inhibiting disposition to generalise mathematically in the future.

Elaboration. Piaget saw reflective abstraction as arising from a shift of attention from objects being acted upon, to the action itself. Thus mentally imagining linear transformations of the number line and the plane can involve calculating images of individual points and sets of points, but when attention shifts to the fact of the transformations themselves, then an algebra of transformations emerges. Some authors see this as *reification* of a process (Sfard, 1991), or as *proceptual* development (Gray & Tall, 1994).

Vygotsky stressed the need for ability "in itself" to be transformed into ability "for himself" (van der Veer & Valsiner, 1991 p. 331). This means shifting from carrying out an action, perhaps with considerable fluency but only when prompted or guided, to internalising it as an integral part of behavioural functioning, and so "knowing to" act in the moment, to use the ability "for oneself". In a sense then, abstraction can be seen as a change of level, a shift in both the object and the structure of attention (what is attended to, and how).

Starting from an observation of the mathematician MacLane (1986), and Bills and Rowland (1999) studied *generalisation from cases* meaning generalisation that subsumes several particular cases such as in Task B, or as in seeing the expansion of $(a + b)^n$ as a generalisation of $(a + b)^2$, $(a + b)^3$ etc., which involves three parameters (a, b, and n) any of which can be seen as a variable. In this example the general rule summarises some features of the specific cases. It also asserts the plausibility of the generalisation in cases beyond those which have been examined, bridging the "epistemic gap" between known and unknown. They go on to quote a number of other mathematicians expressing similar sentiments. Generalisation from cases can take different forms:

- one case can be used generically to expose and articulate s*tructural generality*;
- a few cases can be used to reveal dimensions of possible variation;

- several cases can be used empirically or inductively/recursively.

ΦB, ΦC, ΦD2, and Task C illustrate aspects of empirical generalisation from cases. Sometimes however, it is possible to proceed directly from a single instance to the general simply by recognising one or more features that could be generalised (for example, as in Task A and ΦD1). *Generic* generalisation occurs when a single example is seen as generic, as illustrating relationships that are perceived as properties holding in a class of similar instances or examples. Balacheff (1988, p. 219) puts it clearly and elegantly:

> The generic example involves making explicit the reasons for the truth of an assertion by means of operations or transformations on an object that is not there in its own right, but as a characteristic representative of the class.

The mathematician David Hilbert advocated a generic approach to difficult problems (Courant 1981, see also Mason & Pimm, 1984), and Krutetskii (1976, pp. 261-262) found high-achieving learners generalising through a single generic example. To see something generically is to look through the particulars to a generality, that is, to stress certain relationships and to treat them as properties common to all examples in the class being exemplified (Task A can be experienced this way). Task C is a reminder that seeing through a particular to a generality is no trivial matter. Watson and Mason (2005) refer to these particulars as *dimensions of possible variation* based on Marton's notion of *dimensions of variation* (Marton & Booth, 1997). The word *possible* was inserted because at any moment the teacher and the learners may be aware of different features that could be generalised. Furthermore, even when they are aware of the same dimension that could be varied, they may not be aware of the same *range of permissible* change of or by that variable. Typically in algebraic contexts it results in a parameter inserted for a constant.

Empirical generalisation always has some semantic content, in the sense that for the learner, surface relationships *are* the meaning, but they may not contact deep (mathematical) structure. ΦB offers an example of this. Task B is typical of tasks that can often be carried out with a surface rather than a deep approach, with the learner content to "get the answers" rather than to appreciate what is going on. Some intervention may be necessary by the teacher in order to provoke or promote a shift to appreciating structural relationships. (See later section for strategies.)

The terms *syntactic* and *semantic* generalisation provide another way to try to express the difference between what happens when learners use a predominantly *surface* or predominantly *deep* approach to learning (Marton & Säljö, 1976, 1984). Learners may be content with syntactic variation, a version of Watson's "going with the grain", which is evidenced most clearly when learners undertake a "copy and complete" type of exercise from a text by paying attention only to surface features. Again, in language stemming from grammar and linguistics, it may be possible to distinguish between *metonymic* generalisation that arises from surface associations triggered by personal and collective idiosyncrasies, homonyms, and the like (as in ΦD1), and *metaphoric* generalisation based on resonance with a sense of structure. The more experienced you are, the more structures you have encountered, and so the more likely you are to recognise an instance of a structure, not through calculated use of analogy, but through metaphoric resonance. Lakoff and Nunez (2000) argue that this can always be traced back to bodily sensation of some kind. Freudenthal (1983) advanced the case that a structure can be taught by locating a phenomenon that requires that structure in order to explain it.

The term *structural* generalisation applies to overlapping ground between empirical and generic generalisation, for the aim of empirical generalisation is (usually) to reveal structural properties that make the examples exemplary of some generality. Similarly, working with a generic example aims to locate structural invariance that is neither particular nor peculiar to the "example" being used. However, the real issue is that learners can act superficially to engage in empirical generalisation without reference to underpinning structure that generates the objects, or even structure within the objects themselves. Empirical is sometimes used to refer to "pattern spotting" in numbers without reference to what is generating those patterns. Without making use of the source of the "numbers" there can be no justification and hence no contact with underlying structure. ΦB, ΦC, and ΦD illustrate attempts and failures to contact structural generalisation. Hewitt (1992) highlights the natural tendency for teachers to try to short cut the lengthy process of learners using their own powers to generalise, by promoting a strategy of "draw up a table and look for a pattern in the numbers". Once reduced to mechanical behaviour, such activity becomes what he called "train spotting".

Pólya (1945) distinguished four states during empirical generalisation from cases: Observation of that particular case; Generalization; Conjecture formulation based on previous particular cases; and Conjecture verification with new particular cases (prior of course to justification of the generality). Cañadas and Castro (2007) find it useful to discriminate even more finely, inserting a further four: Organization of particular cases; Conjecture generalization; Search and prediction of patterns; and Justification of general conjecture. When observational data are organized in different ways, different patterns emerge, and cases can be organised more or less systematically or intentionally without prior detection of pattern. Cañadas and Castro describe "search for patterns" in terms of the learner seeking the "next" term in a sequence or in an array, without the pattern being thought of as applying to other cases. This is usefully seen as a structured form of attention in which the focus is on recognising local relationships in the specific situation, as distinct from perceiving properties that might apply in other cases (Mason, 2003; see also Pirie & Kieren, 1989; 1994). ΦD provides an illustration, and tasks B and C can be experienced in this way. When a relationship between terms is evident, learners may find their attention attracted to iterative relationships (often referred to as inductive or recursive) rather than to expressing the generality as a function of the position in the sequence or array. Stacey and MacGregor (1999) report considerable difficulty in provoking learners to shift from iterative/recursive generalisations to direct formulae, but then there may be little evident purpose or utility (Ainley, 1997) in making such a shift. A rich seam of mathematics has developed around methods of converting iterative/recursive generalisations into formulae (e.g., finite differences, formal power series).

Enactive generalisation is a description of situations in which the body perceives a generality before the intellect becomes aware of it, as when a learner shifts from copying each term in its entirety and starts repeating the invariant items before inserting the items that are changing. Most research has focused on *cognitive* generalisation, but the second author has observed indications that *affective* generalisation may accompany cognitive and enactive generalisation, in the form of altered dispositions either to engage, or not to engage in generalisation in the future.

What Generalisation is Like

How empirical generalisation comes about is obscured by the fact that even to see different objects as (potential) cases runs into the *exemplification paradox*: to come to see something as an example of something more general you already need to have a sense of that generality; to come to appreciate a generality you (usually) need to have some examples. One feature of learning to think and act mathematically is learning to cope with generality through particularising (in Pólya's language, specialising) in order to get a sense of underlying relationships, which when expressed as properties re-emerge as your own version of the original generality or an extension.

Gattegno suggested that generalisation (he included abstraction) comes about from *stressing* or fore-grounding some features and consequently *ignoring* or back-grounding others. This is manifested in the pedagogic strategy promulgated by Brown and Walter (1983) which they call *what if not*, in which some feature or aspect is interrogated for other possibilities. A particular version of this strategy is to read the statement out loud and to put special stress on one or another word. It is amazing how the stress attracts attention and invites asking why this word rather than some other word, and consideration of the possibility of changing it, of treating it as a dimension of possible variation.

The phrases *seeing the general through the particular* and *seeing the particular in the general* (Mason et al., 1985; see also Whitehead, 1911, pp. 4-5, 57) were formulated to try to capture that moment when a particular aspect is stressed and becomes the fore-ground of attention, so that other features fall away, and in that moment of stressing, other possibilities arise. It literally becomes a *dimension of possible variation,* almost as if a new world or new dimension opens up, however minor. There is often a corresponding sense of freedom, partly to do with subsuming previously different objects or actions under one heading, and partly to do with the freedom to choose different examples or instances. The person's example space is enriched (Watson & Mason, 2005). Sometimes the dimension was already part of the person's awareness, but the *range of permissible change* is extended, as in realising that the binomial theorem could apply to fractional as well as integer exponents, or more simply, that the a and the b could be not just integers, or fractions, or indeed any real number, but also algebraic expressions etc.

Although, as many have pointed out, generalisation is a power possessed by anyone who has learned to speak and to function in the material world, there are subtleties in evoking that power appropriately in mathematics classrooms. It is of course very tempting to try to do both the specialising and the generalising for learners: constructing tasks and suites of exercises that display particulars intended to be seen as instances, cases, or examples of a general class of such objects. It is then assumed that if learners "work through" the particular cases, they will emerge with a sense of the generalised whole. This assumption is contradicted by the observation that "one thing we do not seem to learn from experience, is that we rarely learn from experience alone". Something more is required.

That "something" is, presumably, what Piaget was trying to get at with his term *reflective abstraction*, what Vygotsky referred to as *internalisation of higher psychological functioning* through being in the presence of a relative expert displaying that functioning, and what Gattegno described as *integration through subordination*. Piaget and Gattegno stressed the natural and individual nature of such a transformation, of course supported and promoted by careful choices of teaching; Vygotsky stressed the importance of the social, including the incorporation of cultural tools and engaging in social interaction. Both dimensions are of course vital to a full appreciation.

Teachers' Influence

ΦE introduces the issue of how teacher intervention and articulation might influence learners' appreciation of generality. By uttering a generality herself, the teacher's utterance might be experienced by some learners as

- a crystallisation of a semi-focused awareness;

- a restating of the obvious;
- bridging or filling in an awareness of which they were not yet aware, and so taking away a generative experience; or
- nothing at all because it passes them by.

For some learners the utterance in ΦE might be part of the wallpaper of the lesson, for others it has a transformative action, and for others it confirms an awareness.

Just because some or several examples have been given by learners, and even when the flow of examples accelerates, awareness of the generality itself may not be present. It has to do with not just what learners are attending to, but precisely how they are attending.

These reflections on ΦD led the second author to the question of when it might be useful, effective, and appropriate for a teacher to utter a generality that generalises examples that learners have encountered or constructed for themselves. It was a short jump to the notion of a *Zone of Proximal Generality*. The idea was to describe and draw attention to various states of learner sensitivity to the possibility of generalisations in a particular setting, states in which learners are beginning to be aware of their subconscious awareness of a mathematical generalisation. For some learners, the generality was already present, perhaps explicitly, perhaps ready to be crystallised by someone expressing it in terms that linked to learners' experience. Such affirmation is often important for learners. For others however, the expression might displace and even block personal realisation that was underway and or imminent. But for some, any expression of that generality might have been ignored or even simply not heard: it was not in their current zone of proximal generality, perhaps because their attention was absorbed elsewhere.

The notion of a zone of proximal generality was soon recognised as a particular case of a *Zone of Proximal Awareness*. A generality is just one kind of awareness that can come to someone as a result of engaging in activity with cultural tools and using practices encouraged and displayed by a relative expert. The idea was to use the term to describe awarenesses that are imminent or available to learners, but which might not come to their attention or consciousness without specific interactions with mathematical tasks, cultural tools, colleagues, teacher, or some combination of these.

Since Vygotsky's original conception of the ZPD was as a dynamically emergent metaphoric space of possibilities describing potential development of conscious use of already familiar but all-engrossing behaviour, we recognised that his oft quoted definition has led researchers to a very truncated perception, a projection of the original idea into the behavioural aspects of the human psyche. What most people use is really a *zone of proximal behaviour*: what behaviour patterns might learners soon adopt for themselves?

Coining the term *zone of proximal generality* does more than provide a useful axis around which to accumulate classroom strategies and ways of analysing tasks. It also offers an opportunity to explore the implications of a zone of proximal awareness and a *zone of proximal affect*, which includes, for example, the *zone o*f *proximal relevance* proposed by Mason and Watson (2005). It also links with *zones of promoted action* and *free movement* (Valsiner, 1988; Goos, 2004). Once conceived, the general notion of zones gave access to consideration of various zones in relation to the ZPD, which as a language might help us to articulate finer distinctions in the states and experiences of learners, including ourselves. These ideas require further study in order to elucidate their usefulness and interconnections.

Methodological Issues

There are tricky methodological issues however. It is tempting to describe learner behaviour in terms of "making a generalisation", when in fact all you have observed is behaviour consistent with making a generalisation. Learners may be immersed in action but only "going through the motions"; they may be immersed in action and reproducing behaviour patterns detected in the teacher and in peers, but without any underlying sense of meaning or larger story about why they are doing what they are doing. On the other hand, they may be immersed in action but generating or re-constructing that behaviour from themselves or with the support of peers. They may even be "acting for themselves", that is, "acting because", in the sense that they are able, when questioned, to articulate an appropriate mathematical justification.

ΦB underlines the teacher's predicament in seeking evidence that learners have appreciated, or integrated a generality into their functioning. When a teacher encounters a learner who is "acting as if" they are in possession of or aware of mathematical theorems and properties there is a dilemma:

- is the learner behaving as if he/she knows a theorem (or fact) without having explicit awareness of it?
- is the learner reproducing socially enculturated patterns of behaviour without being aware of them consciously, and/or without appreciating a larger picture and underlying principles?
- is the learner producing or reconstructing actions which are not only principled, but articulable at some level of sophistication without being prompted?

The first two may sometimes be distinguished by offering learners tasks in which the setting or situation is very different from those used to demonstrate or display the expected practices. A learner working from awareness, however subconsciously, may be able to use that awareness where someone working from socially induced behaviour will not. The third can arise spontaneously when one learner asks another for assistance or when there is some impetus to bring the reasoning to articulation.

It is very difficult to distinguish between "acting as if" and "acting because", or as Vygotsky put it, between a learner's quasi-concepts and true concepts (Confrey, 1994). The first situation was referred to by Vergnaud (1981) as *theorems-in-action*, to capture the sense of action that appears principled but which the learner may not be aware of explicitly, nor be able to articulate. The second and third are different responses of learners to what Brousseau (1984, 1997) called the *didactic contract*, from which arises the *didactic tension.* Since learners look to the teacher for the behaviour being sought, the more clearly the teacher indicates the behaviour being sought, the easier it is for learners to act "as if", to display that behaviour without actually generating it from themselves.

Distinguishing between the three responses is at best difficult, and at worst, delicate because excessive probing may have a negative rather than a positive impact. The act of probing for explicit description of the actions, for principles guiding those actions, or for justifications for those actions may have the effect of prompting the learner into a new level of awareness (and hence justify the use of the term Zone of Proximal Generalisation), but it may also have the effect of creating obstacles for the learner who may not recognise what is being asked, and so may develop affective, cognitive or even enactive blocks to further progress.

Pedagogic Strategies

How might a fistful of distinctions inform a teachers' future practice and so enhance the possibilities for learning? If distinctions remain at the intellectually academic level, so that teachers only know *about* them, then at best some ground may have been prepared. Where distinctions have become significant either because they help make sense of past experiences, or because they provide a label for sharpening noticing and a vocabulary for describing and analysing with colleagues, they begin to function, to inform choices of actions whether in planning or in the moment during a lesson. Distinctions become richer and more integral to a people's functioning when they are enriched by relevant personal experience. Significant choices involve choosing to act in some way that might not otherwise have come to mind, so that teachers find themselves "knowing to" act in the moment. Some strategies that have proved helpful in this respect regarding generalising include the following.

Guided Exploration. There are many difficulties with being a "guide on the side". Terms such as *scaffolding,* introduced by Wood et al. (1976), attempted to describe ways in which teachers could act as "consciousness for two" (Bruner, 1986, pp. 75-76) in supporting behaviour that would ultimately be taken up and directed by learners themselves, a process known as *fading* (Brown, Collins, & Duguid, 1989), through a process of progressively more and more indirect prompts until learners are adopting the behaviour spontaneously. The process is highly problematic, because a teacher acts in the moment; it is only later that the learners' behaviour makes it possible to describe the whole process as scaffolding and fading. Put another way, "teaching takes place in time; learning takes place over time".

What if Not (Stress and Ignore). This strategy was described in an earlier section. It involves stressing single words in an assertion, or some feature of an expression, and then asking what happens if that is allowed to change in some way.

Directing Attention (Stress and Ignore). More generally, learner attention can be directed towards something (as in Task A), with the consequence of back-grounding something else, but you cannot intentionally direct people to ignore something. Much of what teachers do in classrooms is to direct attention towards pertinent features. By directing attention in a structured manner it is possible to provoke awareness of sameness and difference, and so promote generalisation.

Watch What You Do. Encouraging learners not only to work on or construct an example but also to pay attention to how their bodies go about it, affords a perception that can often be translated into a generality. It can be used to enrich enactive generalisation, and to build links with structure (Mason et al., 2005).

Same and Different. Enculturating learners into the practice of looking for similarities and differences between objects under investigation leads quite directly to the perception and expression of salient features. Becoming aware of similarities and differences results in stressing or fore-grounding and consequently ignoring or back-grounding, which is the basis for both generalisation and abstraction. Brown and Coles (2000) report learners taking over the strategy and internalised it, integrating it into their everyday functioning in the mathematics classroom.

Say What You See. Getting learners to say something of what they see, and to listen to what others say they see often has the effect of re-directing learner attention, with the possibility of, at the very least, extending their awareness of possible interpretations or ways of seeing, and sometimes of detecting similarities that can emerge as generalities.

Predicting What is not Present. When learners are confronted with a possible pattern, it can help them express vaguely sensed generalities to try to predict other examples that are not present. They can then use *Tracking* to bring their awareness to expression.

Tracking Arithmetic. By deciding to forego closure on arithmetic operations concerning one number in a calculation, the progress of the value through the various steps in a calculation can be tracked. It is then a simple matter to ask what calculation would be needed if that tracked number changes: only its value changes, while the rest of the calculation remains the same. This is particularly useful in a difficult problem in which it is possible to check whether a proposed answer is correct, but hard to see how to find such a number. The arithmetic can be tracked on a "guess", the calculation generalised, and then an equation established whose solution(s) provide the desired answer. This is the method that Mary Boole called "acknowledging ignorance" (Tahta, 1972).

Why Does Generalisation Happen Sometimes and Not Others?

Changes in how people attend to something are not simple transformations. They involve a complex of experience, reflection, and perceived effectiveness (according to one's own criteria). To become robust they need integrated contributions from all three aspects of the psyche: cognition, enaction, and affect. Models or metaphors of learning that imply simple levels or steps to be ascended, or a few obstacles to be overcome, fail to account for the wide variation in learner experience and learner dispositions.

Teaching is not simply a matter of guiding or driving learners into appropriate patterns of behaviour, nor is it simply a matter of waiting for learners to display "readiness". Provoking generalisation is more about releasing learners' natural powers than it is about trying to force feed. Because promoting mathematical generalisation lies at the core of all mathematics teaching, at all ages, and because it concerns the development of higher psychological processes that are most likely to be accessible to learners if they are in the presence of someone more expert displaying disposition to and techniques for generalising, it is important for teachers to be seen to generalise, to value learners' attempts at generalisation, and to get out of learners' way so that they can generalise for themselves.

Acknowledgements. The authors are very grateful for useful comments made by the reviewer.

References

Ainley, J. (1997). Constructing purpose in mathematical activity. In E. Pehkonen (Ed.), *Proceedings of the 21st annual conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp.17-24). Lahti, Finland: PME.

Ainley, J. (1999). Doing algebra-type stuff: Emergent algebra in the primary school. In O. Zaslavsky (Ed.), *Proceedings of the 23rd annual conference of the International Group for the Psychology of Mathematics* (Vol. 2, pp. 9-16). Haifa, Israel: PME.

Alexandrou-Leonidou, V., & Philippou, G. (2007, February). *Elementary school students' understanding and use of the equal sign*. Paper presented at CERME 5 Working Group 6, Lanarka, Cyprus.

Balacheff, N. (1988). Aspects of proof in pupils' practice of school mathematics. In D. Pimm (Ed.), *Mathematics, teachers and children* (pp. 216-235). London: Hodder and Stoughton.

Bills, L., & Rowland, T. (1999). Examples, generalisation and proof. In L. Brown (Ed.), *Making meanings in mathematics: Refereed proceedings of the British Society for Research into Learning Mathematics* (pp. 103-116). York: QED Publications.

Brousseau, G. (1984). The crucial role of the didactical contract in the analysis and construction of situations in teaching and learning mathematics. In H. Steiner (Ed.), *Theory of mathematics education* (pp.110-119). Bielefeld: Institut fur Didaktik der Mathematik der Universitat Bielefeld.

Brousseau, G. (1997). *Theory of didactical situations in mathematics: Didactiques des mathématiques, 1970-1990* (N. Balacheff, M. Cooper, R. Sutherland, & V. Warfield, Trans.). Dordrecht: Kluwer.

Brown, L., & Coles, A. (2000). Same/different: a 'natural' way of learning mathematics. In T. Nakahara & M. Koyama (Eds.), *Proceedings of the 24th annual conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, 153-160). Hiroshima, Japan: PME.

Brown, S., Collins, A., & Duguid, P. (1989). Situated cognition and the culture of learning. *Educational Researcher, 18*(1), 32-41.

Brown, S., & Walter, M. (1983). *The art of problem posing*. Philadelphia: Franklin Press.

Bruner, J. (1986). *Actual minds, possible worlds*, Cambridge: Harvard University Press.

Cañadas, M., & Castro, E. (2007). A proposal of categorization for analysing inductive reasoning. *Revista de Investigacíon en Didáctica de la Matemática*, 1(2), 67-78.

Cardano, G. (1545/1969). *Ars Magna or the rules of algebra* (T. Witmer, Trans.). New York: Dover.

Confrey, J. (1994). A theory of intellectual development. *For the Learning of Mathematics, 14*(3), 2-8.

Conway, J., & Guy, R. (1996). *The book of numbers*. New York: Copernicus, Springer-Verlag.

Courant, R. (1981). Reminiscences from Hilbert's Gottingen. *Mathematical Intelligencer*, *3*(4), 154-164.

Fischbein, E. (1993). The theory of figural concepts. *Educational Studies in Mathematics*, *24*(2), 139-162.

Freudenthal, H. (1983). *Didactical phenomenology of mathematical structures.* Dordrecht: Reidel.

Gattegno, C. (1970). *What we owe children: The subordination of teaching to learning*. London: Routledge & Kegan Paul.

Gattegno, C. (1987). *The science of education Part I: Theoretical considerations.* New York: Educational Solutions.

Gillings, R. (1972). *Mathematics in the time of the pharaohs*. New York: Dover.

Goos, M. (2004). Learning mathematics in a classroom community of inquiry. *Journal for Research in Mathematics Education*, *35*, 258-285.

Gray, E., & Tall, D. (1994). Duality, ambiguity, and flexibility: A proceptual view of simple arithmetic. *Journal for Research in Mathematics Education*, *25*(2), 116-140.

Hewitt, D. (1992). Train spotters' paradise. *Mathematics Teaching*, *140*, 6-8.

Houssart, J. (2001). Rival classroom discourses and inquiry mathematics: The whisperers. *For The Learning of Mathematics*, *21*(3), 2–8.

Houssart, J. (2004). *Low attainers in primary mathematics: The whisperers and the maths fairy*. London: Routledge.

Kieran, C. (2004). The core of algebra: Reflections on its main activities. In K. Stacey, H. Chick, & M. Kendal (Eds.), *The future of the teaching and learning of algebra: The12th ICME Study* (pp. 21-33). Boston, MA: Kluwer..

Krutetskii, V. A. (1976). *The psychology of mathematical abilities in school children* (J.Teller, Trans.; J. Kilpatrick & I. Wirszup, Eds.). Chicago: University of Chicago Press.

Lakoff, G., & Nunez, R. (2000). *Where mathematics comes from: How the embodied mind brings mathematics into being*. New York: Basic Books.

MacLane, S. (1986). *Mathematics: Form and function*. New York: Springer-Verlag.

Malara, N. & Navarra, G. (2003) *ArAl Project. Arithmetic pathways towards favouring pre-algebraic thinking*. Bolgona: Pitagora.

Marton, F., & Booth, S. (1997). *Learning and awareness.* Hillsdale, NJ: Lawrence Erlbaum.

Marton, F., & Säljö, R. (1976). On qualitative differences in learning 2: Outcome as a function of the learner's conception of the task. *British Journal of Educational Psychology, 46*, 115-27.

Marton, F., & Saljo, R. (1984). Approaches to learning. In F. Marton, D. Hounsell, & N. Entwistle (Eds.), *The experience of learning* (pp. 36-55). Edinburgh: Scottish Academic Press.

Mason , J. (1988, 2nd revised edition 1998), *Learning & doing mathematics*. York: QED Books.

Mason, J. (1996). Expressing generality and roots of algebra. In N. Bednarz, C. Kieran, & L. Lee(Eds.), *Approaches to algebra: Perspectives for research and teaching*. (pp. 65-86). Dordrecht: Kluwer.

Mason J. (1998). Enabling teachers to be real teachers: Necessary levels of awareness and structure of attention *Journal of Mathematics Teacher Education*, *1*(3), 243-267.

Mason, J. (2002). *Researching Your own Practice: The Discipline of Noticing*, RoutledgeFalmer, London.

Mason, J. (2003). On the structure of attention in the learning of mathematics. *Australian Mathematics Teacher*, *59*(4), 17-25.

Mason, J., Graham, A., Pimm, D., & Gowar, N. (1985). *Routes to, roots of algebra.* Milton Keynes: The Open University.

Mason, J., Johnston-Wilder, S., & Graham, A. (2005). *Developing thinking in algebra*. London: Sage (Paul Chapman).

Mason, J., & Pimm, D. (1984). Generic examples: Seeing the general in the particular. *Educational Studies in Mathematics, 15*(3), 277-290.

Mitchelmore, M., & White, P. (2004, July). *Teaching mathematical concepts: Instruction for abstraction*. Invited regular lecture at the International Congress on Mathematical Education, Copenhagen.

Moessner, A. (1952). Ein bemerkung über die potenzen der natürlichen zahlen. *S.–B. Math.-Nat. Kl. Bayer. Akad. Wiss., 29* (MR 14, 353b).

Molina, M. (2007). Desarrollo de pensiamento relacional y comprehensón del signo igual por alumnos de tercero de educación primaria. Unpublished doctoral dissertation, University of Granada, Granada.

Molina, M., Castro, E., & Mason, J. (2007, February). *Distinguishing approaches to solving true/false number sentences*. Paper presented at CERME 5, Working Group 6, Lanarka, Cyprus.

Newton, I. (1683). *The Mathematical Papers of Isaac Newton, Vol V* (D. Whiteside, Ed., 1964). Cambridge: Cambridge University Press.

Pirie, S., & Kieren, T. (1994). Growth in mathematical understanding: How can we characterise it and how can we represent it?, *Educational Studies in Mathematics*, *26*(2-3), 165-190.

Pirie, S., & Kieren, T. (1989). A recursive theory of mathematical understanding. *For the Learning of Mathematics*, *9*(4), 7-11.

Pólya, G. (1945). *How to Solve it: a new aspect of mathematical method*. Princeton, USA: Princeton University Press.

Pólya, G. (1962). *Mathematical discovery: On understanding, learning, and teaching problem solving*. New York: Wiley.

Rowland, T. (1995). Hedges in mathematical talk: linguistic pointers to uncertainty. *Educational Studies in Mathematics*, *29*(4), 327-353.

Rowland, T. (2001). Generic proofs in number theory. In S. Campbell & R. Zazkis (Eds.), *Learning and teaching number theory: Research in cognition and instruction*. (pp. 157-184). Westport, CT: Ablex Publishing.

Sfard, A. (1991). On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the same coin. *Educational Studies in Mathematics,* 22, 1-36.

Stacey, K., & MacGregor, M. (1999). Taking the algebraic thinking out of algebra. *Mathematics Education Research Journal*, *11*(1), 25-38.

Tahta, D. (1972). *A Boolean anthology: selected writings of Mary Boole on mathematics education/* Derby: Association of Teachers of Mathematics.

Thompson, P. (2002). Didactic objects and didactic models in radical constructivism. In K. Gravemeijer, R. Lehrer, B. van Oers, & L. Verschaffel (Eds.), *Symbolizing, modeling, and tool use in mathematics education* (pp. 191-212). Dordrecht: Kluwer.

Valsiner, J. (1988). *Developmental psychology in the Soviet Union*. Brighton: Harvester.

van der Veer, R., & Valsiner, J. (1991). *Understanding Vygotsky*. London: Blackwell.

Vergnaud, G. (1981). Quelques orientations théoriques et méthodologiques des recherches Françaises en didactique des mathématiques. *Actes du Vième Colloque de PME* (vol. 2 pp. 7-17). Grenoble: Edition IMAG.

Viète, F. (1591/1983). *The analytic art: Nine studies in algebra, geometry, and trigonometry from the Opus restitutae mathematicae analyseos, seu, Algebra nova / by Francois Viète* (T. Witmer, Trans.). Kent, OH: Kent State University Press.

Watson, A. (2000). Going across the grain: Mathematical generalisation in a group of low attainers. *Nordisk Matematikk Didaktikk, 8*(1), 7–22.

Watson, A., & Mason, J. (2005). *Mathematics as a constructive activity: Learners generating examples*. Mahwah, NJ: Lawrence Erlbaum.

Whitehead, A. (1911). *An introduction to mathematic*s. Oxford: Oxford University Press.

Whitehead, A. (1932). *The aims of education and other essays* London: Williams and Norgate.

Wood, P., Bruner, J., & Ross, G. (1976). The role of tutoring in problem solving. *Journal of Child Psychology and Psychiatry*, *17,* 89-100.

Zaskis, R., & Liljedahl, P. (2002). Repeating patterns as a gateway. In A. Cockburn & E. Nardi (Eds.), *Proceedings of the 26th annual conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 212-216). Norwich, UK: PME.

Empowered to Teach: A Practice-based Model of Teacher Education

Janette Bobis
University of Sydney
<j.bobis@edfac.usyd.edu.au>

This paper examines a practice-based component of a primary teacher education program to gain insight into the type of experiences which assist beginning teachers translate theory-based knowledge to their teaching practices. Eighty-six prospective teachers participated in the study. Data were collected from (a) weekly lesson plans; (b) researcher field notes; (c) reflective journals; and (d) interviews with four participants. A theoretical rationale for various aspects of the practice-based component is provided and the implications for teacher education programs are discussed.

A persistent problem in mathematics teacher education is the general inability of beginning teachers to translate theory-based knowledge of the university context into their own teaching practice once in the classroom (Korthagen & Kessels, 1999; Moore, 2003; Zeichner & Tabachnick, 1981). A major concern for teacher educators has been the need to find more effective ways to address this apparent theory/practice gap and better prepare our teachers to master the realities of teaching (Bobis & Aldridge, 2002; Tobin & Roth, 2006).

The field experience (or "practicum") is overwhelmingly considered by experienced and prospective teachers as one of the most powerful – if not *the* most powerful – component of their teacher education programs (Wilson, Floden, & Ferrini-Mundy, 2002). Yet, the nature of this potentially powerful experience can determine whether teacher preparation is enhanced or hindered. Study after study confirms that for the majority of practice teachers, the focus of field experiences is on procedural and management concerns such as behaviour management and whether expected lesson content is covered (Liston, Whitcomb, & Borko, 2006; Moore, 2003). Although such procedural matters are important, beginning teachers' preoccupations with them generally means that they are unable to consider new, more cognitively demanding, teaching approaches advocated in key policy documents (e.g., Australian Education Council, 1990; National Council for Teachers of Mathematics (NCTM), 2000) during their teacher preparation programs. A challenge facing teacher educators is to design teaching-learning environments that will empower beginning teachers to translate theory into their practice more effectively.

The aim of this inquiry was to gain greater insight into the type of experiences that will assist beginning teachers translate the theory-based knowledge of mathematics teacher education courses to their teaching practices. This paper examines a particular component of a primary mathematics methods course in an attempt to reflect, improve it and share what has been learnt. Multiple cohorts of prospective and beginning teachers have resoundingly confirmed that a "practice-based" component of this methods course provided *the* most influential experience in their mathematics teacher education preparation (Bobis & Aldridge, 2002).

A Practice-based Model of Teacher Education

The focus of this study is a 3 to 4 week in-school component of a semester-long mathematics methods course for prospective primary teachers. This practised-based component arose from the need to address an apparent gap between their university-based knowledge of theory and what they did in the classroom. It has evolved to its current form over many years and is based on "what works" best in practice. Through a continuing process of design, implementation, evaluation, and refinement, the component and the methods course in which it is nested, has taken on three distinct characteristics – alternating situated learning contexts, co-teaching, and embedded assessment. Although derived from practice, each characteristic has a well-developed theoretical rationale for its usefulness in a teacher education program. Taken together, they form a practical and theoretical framework for the current study.

Alternating Situated Learning Contexts

A situated perspective on learning acknowledges that some types of knowledge are best constructed in one context rather than another and that the more authentic the context, the more effective the interplay between theory and practice (Brown, Collins, & Duguid, 1989; Putnam & Borko, 2000). The mathematics education course in question alternated between 6 weeks of theory-based lectures and tutorials at university, 3 to 4 weeks of practice-based teaching in a local primary school and then 3 more weeks of lectures and tutorials at university. Hence, prospective primary teachers were first introduced to important knowledge for the teaching of mathematics (e.g., mathematics content knowledge, pedagogical content knowledge, and knowledge of curricula) in a traditional university-based learning situation. In the school-based context, student teachers spent their normal tutorial times working with one or two peers to teach a small group of primary-aged children – still under the supervision of their normal mathematics education tutor and a classroom teacher. The final three weeks of university-based learning served as a "debriefing". It focused on issues that had arisen during the school-based teaching and aimed to further contextualise theory-based knowledge drawing on shared experiences from the previous 3 to 4 weeks. Korthagen and Kessels (1999) found that an important factor in determining the extent to which beginning teachers could translate their knowledge into practice was the degree to which teacher education programs integrated and alternated theory and practice in a similar way. Although the focus of this paper is on the in-school component, it is important to note that its impact is made more powerful due to the overall course structure of alternating learning contexts.

Co-teaching

Co-teaching occurs when two or more "persons teach a group of students with a dual purpose: providing more opportunities for students to learn and providing opportunities for the persons to grow as teachers" (Tobin & Roth, 2006, p. 17). Co-teaching is different from "team teaching" in that it involves colleagues working together at all phases of the teaching/learning process, from initial planning to implementation to assessment and evaluation. Team teaching, on the other hand, normally requires the persons involved to divide the work and take on different and clearly defined responsibilities. According to Tobin and Roth (2006), co-teaching helps bridge the gap between theory and practice as it allows two or more individuals (not necessarily peers) to teach and subsequently to discuss, debate, and reflect together about their teaching and their students' learning.

The principles of co-teaching have been implemented in the practice-based component of the mathematics education course in question long before the term was first coined, because, like Tobin and Roth (2006), they have been found to work in practice. Hence, two or three student teachers work together to plan and teach a sequence of lessons based on an initial assessment of a small group of students' mathematical needs. After each lesson, the student teachers reflect on the children's learning and their own teaching. They then use this information to plan subsequent lessons. In this particular co-teaching situation, routine and procedural management concerns are minimised due to the size of the "class" and the fact that the teachers (student teachers, classroom teacher, and mathematics educator) learn from each other how to implement them effectively. In this way, student teachers are able to focus more attention on their own teaching and on the children's learning.

Embedded Assessment

Black and Wiliam (1998) found that formative assessment feedback can enhance student learning when it focuses on what is needed for improvement. Although they concluded that such practices are rarely found in schools, it is probably even rarer in universities. Shavelson (2006, p. 65) outlines a continuum of formative assessment practices for teacher education. He refers to "on-the-fly" formative assessment as that which is unplanned, requiring intuition or wisdom of practice, and very difficult to teach teachers. Towards the other end of the continuum, he refers to "embedded assessment", which is formally planned formative assessment tasks that are integrated into the learning experiences of the students and where feedback on performance and remediation is immediately provided.

Embedded assessment best describes the formative assessment task undertaken by prospective teachers as part of the practice-based component of their course. Teachers in each group submit their co-constructed lesson plans to their tutor who observes teaching "snapshots" and provides immediate written feedback about their plans (e.g., appropriateness of content, clarity of goals, etc.) and their teaching. Brief field notes, in the form of observation notes and reminders about the aspects each group of teachers are asked to attend to, are made by the tutor. It is expected that student teachers take account of the tutor's feedback and their reflective evaluations of their own teaching and the children's learning in subsequent sessions. They are not required to rewrite lesson plans that have already been taught. The field notes help the tutor keep track of student teacher progress and ensure that feedback is considered as they learn to teach. At the end of the practice-based component, a mini-program of work consisting of all original lesson plans, formative comments from the tutor, student teacher responses to the feedback, and their own reflections on their teaching and the students' learning, is submitted along with a summative comment for final assessment.

Method

Previous investigations of mathematics' methods courses that situate prospective teachers' learning in alternating contexts such as those just described, indicate that they offer an effective vehicle for the translation of theory-based university knowledge into practice (Aldridge & Bobis, 2001; Bobis & Aldridge, 2002). The same body of research found that a practice-based component was perceived by multiple cohorts of graduating students and beginning teachers to be *the* most powerful mechanism by which this was achieved. To date, reasons for this perception have not been fully explored and evidence to support this view has not been sought. Hence, there were two main foci of the current investigation. First, it sought evidence to support the hypothesis that the practice-based component provided an effective mechanism for the translation of theory into practice.

Second, it sought to explore prospective teachers' perceptions of their own learning and teaching during this component in an effort to highlight strengths and weaknesses of the methods course.

Participants and Setting

Eighty-six prospective primary teachers (78 female and 8 male) enrolled in a 4-year Bachelor of Education degree participated in the study. The mathematics education course at the centre of the study is nested in the third year of the degree and is the second of three mathematics methods courses, each of 12 weeks duration. Prior to this course, student teachers had undertaken an introductory 8-day (one day a week for eight weeks) field experience and one block field experience of 15 days. Importantly, the practice-based component of the mathematics method course is not linked to the normal field experiences as it occurs totally within university-based tutorial times with the same mathematics educator supervising each of the four tutorial groups.

Prior to the practice-based component commencing, each participant selected to work with one or two other student teachers from the same tutorial group. This resulted in the formation of 40 groups of student teachers across the four tutorials. The methods course focused on the mathematics content area of measurement so it was negotiated with the four cooperating teachers that the weekly practice-based sessions would cover content from the volume and capacity sub-strand of the K-6 Mathematics Syllabus (Board of Studies, New South Wales, 2002). Each tutorial group was informed of the grade level they would be teaching two weeks prior to their first in-school session. They were asked to draw on theory and practical experiences of recent lectures to prepare suitable activities that would enable them to assess children's needs in the target content area. This information was expected to inform their lesson planning and co-teaching for the next 3 weeks. At the start of the first session, each group of student teachers was matched to a group of approximately four children of mixed ability.

Data Collection and Analysis

Data were collected from the following sources: (a) weekly lesson plans produced by each group of student teachers; (b) researcher field notes made while observing groups of student teachers co-teaching and during conversations aimed at providing additional feedback to that which was recorded on their lesson plans; (c) reflective journal entries from student teachers concerning their teaching and learning made after each in-school session; and (d) semi-structured interviews with four participants at the end of the methods course.

The field notes acted like an initial analysis of the lesson plans so these two forms of data were jointly analysed. Together they gave insight into the types of pedagogy prospective teachers sought to employ or did not employ and to changes in pedagogy over the three weeks as a result of formative feedback from the supervising mathematics educator.

Reflective journal entries made by prospective teachers gave insight into their abilities to use theoretical information to analyse and reflect on their teaching practice. Additionally, participants who had indicated their willingness to be interviewed individually were invited to a follow-up interview at the conclusion of the methods course. Four female students accepted the invitation and participated in a 30-minute semi-structured interview. Importantly, the interviews were conducted by an interviewer independent of the methods course and were not part of the assessment for the course. The purpose of the interviews was twofold: to provide further insight on the findings that emerged from other forms of data gathered and to validate these findings via a process of triangulation. The interviews were tape-recorded and later transcribed for analysis. The focus of the questions was on the effectiveness of the practice-based component and its impact on the process of learning to teach. They were also asked to explain their reasoning for their comments. Analysis of data from the various sources involved multiple readings of lesson plans, transcripts and journal entries to pinpoint emerging themes in the data.

Results and Discussion

Field Notes and Lesson Plans

It is beyond the scope of this paper to explore all aspects of the lesson plans and the associated shifts in pedagogy over the three weeks. Hence, the focus of the analysis will be on the most salient features to emerge.

Analysis of the first week of lesson plans and field notes revealed that prospective teachers were experiencing difficulties implementing higher-order questioning. Although the initial plans showed that 75% or 30 groups of student teachers deliberately planned higher-order questions at some point in their lessons, they generally occurred towards the end of a lesson and were often "surrounded" by a much larger number of lower-order type questions (e.g., requiring recall of knowledge). Although the higher-order questions were considered well-designed and appropriate – ranging from open-ended questions to those requiring children to explain their reasoning – there was concern that they may have been omitted altogether if timing of the lesson became an issue or if allowed to be dominated by the lower-order questions. Hence, feedback was given suggesting student teachers integrate the questioning throughout the lesson plans and that they experiment initiating activities with such questions. Analysis of field notes and lesson plans for the subsequent weeks revealed a major shift in the number of higher-order questions integrated into lessons and that five groups actually used open-ended problems to initiate extended investigations.

Analysis of lesson plan tasks indicated a significant change in the nature and focus of tasks across the three weeks. Given the nature of the content being treated (volume and capacity), the use of tasks requiring children to manipulate materials physically was never an issue. However, the preoccupation with providing "busy" or "fun" activities that lacked directionality if children's understandings of difficult concepts were to be enhanced was obvious when prospective teachers were questioned about the purpose of such tasks and why they were not consistent with the stated outcomes for their lessons. Forty-two percent of the lesson plans in the first week did not contain clear statements of purpose for the tasks planned. If goals were stated, they generally referred to an action or behaviour students were expected to perform. For example, a typically cited "goal" in the first week of lesson plans was "measuring and ordering the capacity of containers". Lesson plans for the second and third weeks showed a major shift to tasks that focused on concept or skill development with associated goal statements explicitly referring to strategy development and conceptual understanding. For example, goal statements included: "Students will create a calibrated measuring container to increase their understanding of mL and the need to measure more accurately", "To increase student's understanding that capacity refers to the amount a container will hold", "To make comparisons through accurate measuring and reflecting on the reasons why containers differ in capacity".

Another major change in the nature of the tasks presented over the 3 weeks, was the increased occurrence of tasks requiring children to explain their strategies and to communicate their reasoning orally or in writing. It was also noted in the third week that there was an increased number of tasks and their associated goals that explicitly addressed children's misconceptions of volume and capacity. For instance, six plans referred to tasks designed to address confusion surrounding an object's mass and its displacement. By the third lesson plan, analysis revealed that 37 of the 40 plans explicitly planned for the enhancement of conceptual understanding and integrated working mathematically processes such as applying strategies, communication, and reasoning, as indicated by their goals and the nature of the learning experiences presented.

A final notable shift in lesson plans over the 3 weeks, was the increased occurrence of detailed explanations of the concepts prospective teachers considered more difficult to relate to children. Although only explicitly occurring in five lesson plans, field notes also made reference to conversations with another four groups of students about their need to "rehearse" or "script" detailed explanations and complex instructions to assist the flow of their lessons. It was perceived that such scripting raised prospective teachers' confidence levels to teach complex mathematical concepts and when considered successful, provided powerful memories that became useful sources for reflection during subsequent debriefing sessions.

Reflective Journal Entries and Interviews

Reflective journal entries and the interviews provided evidence of two main aspects of prospective teachers' knowledge: (1) their ability to use theoretical information to interpret and analyse the teaching of mathematics in practice; and (2) perceptions of their own learning, strengths, and weaknesses during this component of the methods course.

Ability to use theoretical information. Although many aspects of the prospective teachers' plans and teaching indirectly provide insight into their abilities to analyse theoretical knowledge in terms of their practice and vice versa, some journal entries and interview data explicitly referred to theory and practice relationships. For instance, when interviewees were questioned about the benefits of the practice-based component, all four considered that the "experience enabled us to place our theoretical knowledge into practice". To illustrate how this was achieved, Lauren explained that she and her partner built "upon the mind map idea from tutorials, we were able to see how the students' knowledge developed. We will definitely use this strategy in the future" and Rebecca referred to the "whole teaching and learning process" because it enabled "us to try out ideas rather than just write about them."

Three different groups of prospective teachers commented in their journals on the way "we structured a sequence of learning activities that reflected the stages of the measurement framework". One group considered that "this allowed us to clarify not only the stages of student understanding but also our own understandings of the concepts" learnt about in tutorials (Andrew and Lucy, journal entry). Another group reflected on their ability to "sequence individual lessons that scaffolded student learning through initial engagement, the introduction of new concepts and concluding with a reflection upon both prior and new knowledge" as a real "strength" of their teaching. These comments validated what was evident in student teachers' lesson plans as being "deliberate" and "successful" translations of their theory-based knowledge to their practice. They also indicated that a small number of prospective teachers were not only able to integrate theory and practice, but were able to theorise about their own practice when given the opportunity to critically reflect on it.

Perceptions of their own learning, strengths and weaknesses. The two most commonly discussed aspects of their teaching in both journal entries and by interviewees was the use of explanations and higher-order questioning. Higher-order questioning was considered a shortcoming in more than 60% of journal entries for the first two weeks. Student teachers regularly conceded: “in our eagerness to ask the students questions we were consistently asking directed questions focused on producing the correct answer”. However, by the final week of the practice-based component, journal entries referred to how their questioning had “improved” with one prospective teacher indicating that she learnt to ask better questions from her co-teacher. “N… showed herself to be an excellent questioner”, asking questions that required “a deeper and higher order of understanding. We organised our lesson plan with a “questioning” column, and this enabled me to really think about what I wanted the students to achieve…”. This comment also illustrates the benefits of co-teaching, when prospective teachers can not only jointly share and reflect on their experiences, but also learn from the strengths of each other.

Time and behaviour management issues, as found by Moore (2003), remained an important consideration for student teachers as they were mentioned in 37% of the reflective journals. However, unlike Moore, who found that the comments related to unresolved issues, a large number of the reflective journal entries outlined what the student teachers had learnt that would help them in the future. For example, a group of three prospective teachers wrote about their need to be flexible with their time management:

> We concluded that it is better to spend a little extra time to ensure that students comprehensively understand the concepts of one activity, than abide by a time frame at the expense of having students with little or no understanding of 2 or 3 activities. (Brian, Kim and Sue, journal entry)

Another group of prospective teachers discovered that “since the students were always engaged in the activities there were only a few behaviour issues. This is definitely something to think about when working with a whole class”. Given that the children were regularly working with water and were located outdoors or in “wet areas” for their lessons, even small groups of children required careful behaviour management skills. Hence, many prospective teachers learnt after the first lesson “not to leave any aspect of management to chance – we needed to have clearly thought-out instructions for all procedures” (Emma, interviewee).

Over the 3 weeks in which reflective journal entries were made, 75% of the prospective teachers considered “catering for different abilities” as one of the most challenging and “frustrating” aspects of their teaching. However, as one group confided, it also “proved to be a very worthwhile lesson to learn and something which we will be more prepared for in the future”. In their interviews, both Emma and Lauren mentioned a need to modify the planned activities and to use more open-ended questions after their initial assessment because they had not anticipated the “variation in the children’s understandings”.

In her interview, Rebecca commented that her group “became more explicit in what we wanted the students to do … we clearly know the purpose of each activity in the lessons. Without this, the activities looked pointless”. Her comments reflect similar sentiments in a growing number of journal entries by the third week and indicate an increasing concern for directing student learning according to a perceived research-based trajectory.

The interviewees were asked to comment on aspects of the practice-based component that were considered of most and least benefit to prospective teachers and to explain their reasoning. The only suggestions for improvement referred to extending the "time in the school". Reasons for their positive perceptions varied, but Emma and Lauren considered the "cumulative assessment" very helpful as it "helped us learn step by step and target the areas of our teaching in most need of improvement". Rebecca emphasised the importance of "sharing the experience, and learning from my" co-teacher. This sentiment was echoed in a number of reflective journal entries. For instance, one group of three co-teachers wrote in the third week that they "found working together as a group of three teachers very helpful. We got ideas from each other, and thought it was valuable as well for the students as they were presented with similar lesson content through different approaches".

Journal entries indicated that the co-teaching arrangement also helped address perceived weaknesses in content knowledge and the confidence levels of prospective teachers.

> Initially, our main fear was our lack of content knowledge…after a little research it did not take long for the ideas to flow between us. In our first session we were quite nervous and our questioning fumbled several times … For the next lesson, we made sure we used more open-ended questions. (Clare, Anna and Sohpie, journal entry)

Another aspect of the practice-based component mentioned by a quarter of prospective teachers in their journals was the benefit of focusing "on one content area for three lessons" as this "enabled me to fine-tune my teaching strategies – particularly questioning and explanations and to deepen my own knowledge" (Andrew and Lucy, journal entry). The ability to "fine-tune" and "reflect on" strategies, skills and knowledge was repeatedly mentioned as a benefit of the practice-based component in journal entries, as was the "ability to teach our own way without worrying if we were teaching the way another teacher wanted us to". A sense of "empowerment" was conveyed by many prospective teachers as a result of the practice-based component:

> This in-school experience was my most successful practical experience to date in terms of the achievement of intended outcomes for my students. I feel really empowered to have such a positive feeling about the children's learning and the activities I designed … (Renee, journal entry)

Learning to teach is a complex process. To understand that process better we need to examine the impact of teacher education programs and courses on prospective and beginning teachers. Previous research has shown that alternating theory and practice-based contexts in teacher education programs can assist the translation of theory-based knowledge into the practice of beginning teachers. This study sought further understanding of why and how a practice-based component of a teacher education program might achieve this. It also sought to explore prospective teachers' perceptions of their own learning and teaching during this component in an effort to highlight strengths and weaknesses of the methods course. Results confirm that prospective teachers were able to use theoretical-based knowledge to interpret, analyse, reflect on and improve their teaching of mathematics in practice. Evidence indicated that particular elements of the component – the situated learning context, co-teaching, and embedded formative assessment – empowered them to do this. As a result of undertaking the practice-based component, an overall shift in teaching towards the use of higher order questions, the increased use of "scripting" explanations, the use of tasks explicitly designed to enhance the conceptual development of children and to address perceived misconceptions in their mathematical understanding were among the most notable shifts in practice. Such teaching strategies are consistent with current visions of teaching mathematics (Australian Association of Mathematics Teachers (AAMT), 2002; NCTM, 2000; NSWDET, 2003). Importantly, "context" plays a major role in the success of this component and the mathematics methods course in which it is nested. As mentioned earlier, the component has evolved over many years and is based on what works in this situation for the type of student teachers attracted to this institution and primary education program. Although aspects can be adapted, the simple transfer of some or all elements to another context may not yield the same successes.

Practical Implications for Teacher Education

Informed by a growing body of research literature, current views of quality teaching reflected in policy documents and key professional literature from around the world emphasise the importance of teachers' professional knowledge and their knowledge of practice (AAMT, 2002; NCTM, 2000; NSW Institute of Teachers, 2006). For instance, the *Standards for Excellence in Teaching Mathematics in Australian Schools* (AAMT, 2002) recognises the importance of teachers possessing professional knowledge of "current theories relevant to the learning of mathematics", of content, of students and of how students learn mathematics best (p. 2). It also states that excellent teachers of mathematics possess strong practical knowledge so they can carefully plan learning experiences that "enable students to develop new mathematical understandings … engage them actively in learning" and allow teachers to plan appropriate future learning (AAMT, 2002, p. 4). Besides conforming to research findings of quality teaching, such views also form the basis for teacher accreditation criteria (e.g., NSW Institute of Teachers, 2006). Hence, it is imperative that teacher preparation programs include such outcomes for their graduate teachers, and for the sake of their credibility, should provide research-based evidence to verify their effectiveness in achieving them and in the ability of their graduates to translate such knowledge to their teaching. Importantly, although these documents suggest or even "mandate" outcomes for graduating teacher education students, *how* they are achieved is rightly left to individual teacher education programs to determine.

The results of this study illustrate how one teacher education program is addressing this challenge by providing practical suggestions for reshaping traditionally-structured teacher education courses, especially those attached to field experiences. In particular, the following elements have greatest implications for assisting the translation of theory to practice.

- Alternating the learning context from university-based tutorials to one situated in a school provides prospective teachers with rich opportunities to examine and reflect on their practice in terms of the theories behind their pedagogical decisions and vice versa;
- The situated learning context removes the power of the mentor teacher often noted in traditional field experiences and provides a secure environment in which prospective teachers can rehearse teaching (pedagogical) strategies and develop heuristics or "scripts" (e.g., explanations for complex mathematical concepts) that can be used in whole-class field experiences and eventually in their own classrooms;
- Co-teaching provides prospective teachers opportunities to learn from each other and encourages them to "take risks" and experiment with novel teaching strategies;
- Co-teaching enables prospective teachers explore what to teach, how to teach it and how students learn best before being placed in the added stress of a whole class situation; and
- Embedded formative assessment allows shortcomings in planning and teaching to be addressed immediately. Hence, it can refocus prospective teachers' attentions on more pressing concerns of teacher quality such as higher order thinking and conceptual understanding rather than allow them to become preoccupied with more overt lower order and procedural concerns.

In summary, the findings suggest a change in thinking about structure and focus of teacher education courses by looking for opportunities for prospective teachers to discuss, interpret and reflect on the relationship between theory and practice. Teacher education programs are regularly criticised for the inability of their graduates to cope with the realities of the classroom. At times they have been criticised for teaching too much theory at the expense of practical experience, for not incorporating effective mechanisms that encourage the transfer of theory to practice, and for even teaching the *wrong* theory (Wilson et al., 2006). Perhaps the most important practical implication of this research is the need to provide an evidence-base to redress such unsubstantiated criticisms.

References

Aldridge, S., & Bobis, J. (2001). Multiple learning contexts: A vehicle for changing preservice teachers' mathematical beliefs, knowledge and practices. In J. Bobis, B. Perry & M. Mitchelmore (Eds.), *Numeracy and Beyond*, (Proceedings of the Twenty-Fourth Annual Conference of the Mathematics Education Research Group of Australasia, Sydney, pp. 43-49), Sydney: MERGA.

Australian Association of Mathematics Teachers. (2002). *Standards for excellence in teaching mathematics in Australian schools,* Adelaide: AAMT.

Australian Education Council. (1990). *A national statement on mathematics for Australian schools*. Canberra: Curriculum Corporation.

Black, P., & Wiliam, D. (1998). *Inside the black box: Raising standards through classroom assessment*. London: King's College London School of Education.

Board of Studies, New South Wales. (2002). *Mathematics K-6*, Sydney: Board of Studies.

Bobis, J., & Aldridge, S. (2002). Authentic learning contexts as an interface for theory & practice. In A. Cockburn & E. Nardi (Eds.), *Proceedings of the 26th Conference of the International group for the Psychology of Mathematics Education* (pp. 121-127).PME: University of East Anglia, Norwich.

Brown, J., Collins, A., & Duguid, P. (1989). Situated cognition and the culture of learning. *Educational Researcher, 18*(1), 32-42.

Korthagen, F., & Kessels, J. (1999). Linking theory and practice: Changing the pedagogy of teacher education. *Educational Researcher*, *28*(4), 4-17.

Liston, D., Whitcomb, J., & Borko, H. (2006). Too little or too much: Teacher preparation and the first years of teaching. *Journal of Teacher Education, 57* (4), 351-358.

Moore, R. (2003). Reexamining the field experiences of preservice teachers, *Journal of Teacher Education*, *54* (1), 31-42.

National Council for Teachers of Mathematics. (2000). *Principles and standards for school mathematics*. Reston, VA: NCTM.

New South Wales Department of Education and Training. (2003). *Quality teaching: Discussion paper*, Sydney: NSWDET.

New South Wales Institute of Teachers. (2006). *Professional teaching standards.* Sydney: NSW Institute.

Putnam, R., & Borko, H. (2000). What do new views of knowledge and thinking have to say about research on teacher learning? *Educational Researcher, 29*(1). 4-15.

Shavelson, R. (2006). On the integration of formative assessment in teaching and learning: Implications for new pathways in teacher education. In F. Oser, F. Achtenhagen, & U. Renold (Eds.), *Competence Oriented Teacher Training* (pp. 63-78). Rotterdam: Sense Publishers.

Tobin, K., & Roth, W. (2006). *Teaching to learn: A view from the field*. Rotterdam: Sense Publishers.

Wilson, S., Floden, R., & Ferrini-Mundy, J. (2002). Teacher preparation research: An insider's view from the outside. *Journal of Teacher Education*, *53*(3), 190-204.

Zeichner, K., & Tabachnick, B. (1981). Are the effects of university teacher education washed out by school experiences? *Journal of Teacher Education, 32*, 7-11.

Mathematical Investigations: A Primary Teacher Educator's Narrative Journey of Professional Awareness

Judy Bailey
University of Waikato
<jlbailey@waikato.ac.nz>

As a teacher educator, I used narrative inquiry to investigate my professional practice in working alongside pre-service primary teachers in mathematics education. One theme that emerged from this research was the exploration of narrative as a powerful means with which to pursue professional development. In this process I encountered, and subsequently changed, previously unknown personal beliefs about learning mathematics. A second theme focused on the value of mathematical investigations, for myself as a mathematical learner and for supporting pre-service teachers to develop their understandings of what it means to learn and teach mathematics.

Introduction

Over a period of 20 months I used narrative inquiry to investigate my professional practice as a pre-service teacher in mathematics education. Undertaking this inquiry was motivated by a desire to more closely examine my professional practice whilst simultaneously meeting the requirements for a Master of Education. Reflection, albeit informally done, has always been an integral part of my teaching practice. However, narrative inquiry provided a more structured vehicle for the ongoing critical reflection of my professional role, and resulted in significant personal learning.

Narrative inquiry, which in essence is a form of story-telling, has become a recognised form of educational research, and is regarded as a powerful means with which learners can reflect on and develop their own professional practice (Chambers, 2003; McCormack, 2002; Rushton, 2001; Smith, 2006). A goal of narrative inquiry is for participants to learn, and possibly change their thinking as a result of this learning (Clandinin & Connelly, 2000). One example of such learning that occurred for me was the discovery of changing beliefs about the nature of mathematics.

The other main theme to emerge during my research centred on the use of mathematical investigations to support pre-service primary teachers to consider what the learning and teaching of mathematics may entail. As part of this process I personally did several mathematical investigations, embedding myself in the position of mathematical learner. My conception of a mathematical investigation is of an open-ended problem or statement that lends itself to the possibility of multiple mathematical pathways being explored, leading to a variety of mathematical ideas and/or solutions. Such investigations tend to take more time than usually encountered in more traditional mathematics problems frequently used in schools.

Narrative Inquiry

Over the past two decades the practice of reflection has been recognised as a legitimate aspect of action research in education (Adler, 1993; Francis, 1995; Schön, 1983). More recently, narrative inquiry has also become a valued form of research (Chambers, 2003; Luwisch, 2001; O'Connell Rust, 1999; Smith, 2006) and can be regarded as a journey

Proceedings of the 30th annual conference of the Mathematics Education Research Group of Australasia
J. Watson & K. Beswick (Eds), © MERGA Inc. 2007

during which researchers come to know more deeply about their lives and who they are as people. Beattie (1995) makes a particularly strong case for the use of narrative inquiry within educational research, writing, “at the heart of meaningful educational reform and change, lie the narratives” (p. 66).

A central tenet of narrative inquiry is that of change (Clandinin & Connelly, 2000). Winter (2003), a writer who draws parallels between basic principles of action research (and thus narrative inquiry) and some key Buddhist doctrines also refers to the condition of impermanence, i.e., change. Although Winter also refers to the human tendency of wishing to avoid change, Mason (2002) cautions that we cannot change others, but rather, we can work at changing ourselves.

One might suppose that because there is an inherent focus on change that there is a supposition of an initial deficit position. However, this is not necessarily the case. An alternative model of emancipatory practitioner research, based on the work of Jacques Lacan, is offered by Brown and Jones (2001). These authors suggest that rather than seeking resolution or an end-point, the research process can be regarded as the building of a narrative layer that supports and grows alongside the writer’s life as it occurs. Thus, perfection or an ideal is not sought, but a greater awareness of one’s professional practice with the possibility of instituting change if that is deemed worthwhile.

In narrative “the subject is never given at the beginning, but it unfolds as the story is told” (Ricoeur, 1986, as cited in McCormack, 2002, p. 337). McLaughlin writes that this is a part of narrative and suggests that one needs to be able to “live with the ambiguity and lack of clarity long enough to formulate a specific focus to research” (McLaughlin, 2003, p. 70). This can be an unsettling process.

Reflection is an integral part of narrative inquiry and is linked to the gaining of new understandings. Reflection can also lead to the discovery of contradictions in one’s writing. Winter (2003) suggests that seeking out such contradictions is a part of the process of narrative inquiry. A useful model for reflection is offered by Korthagen (2004). His model consists of a series of layers which seek to deepen one’s reflection, with the innermost layers including an examination of one’s beliefs.

When writing narrative, different perspectives or interpretations of situations, and the writing, are always possible. Chambers (2003, p. 412) writes, “different perspectives further open up possibilities for engaging in the process of reflection in that they offer specific and sometimes comparable or contrasting points of view”. Wilber (1998) however, warns against the extremes of post-modernism whereby all interpretations would be considered to be equally valid.

The results of narrative research are not definitive statements or generalisations about an aspect of that which is being researched (Adler, 1993; Beattie, 1995; Brown, 2001; Brown & Jones, 2001; Winkler, 2003; Winter, 2002, 2003). McCormack (2002) refers to such research as not providing a “map” but allowing “the reader to witness the process of the story’s construction and its meaning for the storyteller” (p. 337). The readers of such research might then be in a position to tell stories about how the research may connect with their own practice.

The second theme encountered in this research revolves around my story of encountering mathematical investigations, both as a learner and as a pre-service teacher. In line with the literature, I do not have a definitive statement about how to be an educator of pre-service teachers learning mathematics. Rather I share my story and what it means to me, and this may create the opportunity for readers to make connections with their own stories and/or practice.

Procedure

During the first semester of the 20-month research period, our mathematics education team met weekly to look at what and how we taught in our two compulsory mathematics education papers. During this process it was decided that we would use mathematical investigations with our first year pre-service primary teachers (a cohort of approximately 200 students) to support them to:

1. explore and learn/re-learn some mathematical ideas; and
2. provide a means with which they could look at their attitudes and beliefs about mathematics learning.

All students were informed at the beginning of the semester that their mathematics education lecturers were doing some research that was looking at ways to improve pre-service teacher education in mathematics. They were invited to participate voluntarily in the research, which could involve a number of aspects including: being observed during class sessions at university and when working with a young child in a school; making their journals (which were an essential requirement for one of their assignments) available to form part of the research data; and being informally interviewed. Sixty-one of the 75 pre-service primary teachers (81%) in my classes agreed to be involved.

An investigative approach was a new approach for me. I had never previously taught or learned mathematics in this manner, so I initially had a lot of questions and some concerns. I began teaching three groups with this approach in the following semester. During this time I kept a journal in which I recorded my thoughts, feelings and questions. This continued a writing process (i.e., a narrative) begun in the previous semester.

The pre-service teachers also kept journals of their experiences, thoughts and mathematical thinking as they completed their investigative work. Audio-tapes of five pairs of pre-service teachers' conversations were collected as they worked on mathematical investigations during class time in the second week of the semester at the beginning of undertaking a two-hour investigation. An observer organised and instructed each student pair in the use of the tape-recorder and then withdrew to a corner of the classroom where she observed the student pairs and the class as a whole.

An informal discussion with a pre-service primary teacher who was struggling with this investigative approach to learning mathematics was also audio-taped and transcribed. This discussion took place after I had become aware of this student's discomfort both during class and as recorded by the student in a concurrently occurring online-discussion.

I also participated in collegial observation. One colleague observed my teaching of one class during this investigative approach, and I observed two colleagues whilst they were teaching. Notes were recorded during each observation and informal discussions took place after each observation. Reflections on these observations were recorded in my personal journal.

The pre-service teachers' journals, transcripts of pre-service teacher conversations, observations and the transcript of my discussion with one student were written and collected during the first five teaching weeks of the first year paper. They were analysed shortly after this five-week period and this analysis became part of my ongoing narrative.

After using this investigative teaching approach I explored several mathematical investigations myself. The first of these involved looking at a hypothetical trajectory of a billiard ball on different-sized billiard tables. Following this experience, as a learner using mathematical investigations, I was keen to re-explore the approach in my role of pre-service educator, this time with one class of second year pre-service primary teachers.

To begin I created a number of "stations" where the pre-service teachers were asked to engage in an activity and identify questions that they had about geometry and/or measurement which were stimulated by the activities. I then selected five different investigations that linked to the students' questions. The pre-service teachers were asked to choose one of these investigations to pursue over a period of approximately 6 hours in class time. They were free to work on their own or with others. The majority of pre-service teachers chose to work with one other person or within a small group.

At the end of that semester I asked if some individuals would be willing to be informally interviewed about their experiences and thoughts regarding participating in mathematical investigations. Francis (1995) suggests that the high profile of reflection in teacher education (such as my narrative inquiry) is only warranted if it impacts on more equitable and just outcomes for pre-service teachers, and ultimately on children's learning. As such, although I had been informally monitoring the pre-service teacher's progress and reactions, I believed it was also necessary to hear, in the pre-service teacher's own words, how they were experiencing this process. This would ultimately also impact on my development as a pre-service mathematics educator. Four pre-service teachers, who had now experienced an investigative approach twice, volunteered to be interviewed and I proceeded to conduct the interviews over the few weeks following our investigative work in class.

Results and Discussion

Re-thinking Mathematics and What it Means to Learn Mathematics

This narrative journey resulted in significant, multi-faceted learning, including reforming my ideas about the nature of mathematics; thinking more deeply about mathematics teaching and learning in general; and more specifically, learning about my own professional practice as a pre-service mathematics educator with particular reference to the use of mathematical investigations.

As Ricoeur (1986, cited in McCormack, 2002) describes, the process of narrative is an unfolding one. As my narrative unfolded I was somewhat surprised to find myself, early in the process, deliberating about the nature of mathematics. Even after eighteen months I was still thinking and struggling with ideas of "what is mathematics?" and do "mathematical truths exist?" I wrote in my journal:

> I still struggle with the notion of 'mathematical correctness or truth'. How do my newer beliefs that mathematics is about "doing" fit with the existence of mathematical rules and proofs? Is it, that a rule or proof only exists in the "doing" or "discovering". That is, it does not exist without or outside the mathematician, and thus must only be found in the doing? (13/08/04)

Following a discussion with a colleague regarding the issue of the validity of multiple answers in response to a mathematical problem I wrote:

> My thoughts are that the answers were all correct (referring to a problem in class) given the differing sets of assumptions or interpretations that each person/group made. Usually these interpretations

> have to be the teacher's and thus teachers (and the children who think in the same way as the teacher) have been the ones who hold the power. Thus, mathematics has not been accessible to many people. Learners justifying their answers with their own reasoning relocates the power to the learner (this does not allow for "shoddy" thinking however). I propose that always defining problems so tightly as to create only one correct answer does not lead to useful life or problem-solving skills. Nor does it lead to "real" learning, rather the "game" of "let's guess what the teacher wants us to do/say now", i.e., it is the teacher's interpretation that matters. Thus accepting multiple interpretations supports the learner to "really" learn, and creates an expectation of learners making sense of contradictions and a range of perspectives. (13/08/04)

These excerpts of writing represented considerable changes in my thinking about the nature of mathematics. As Clandinin and Connelly (2000) suggest, one's thinking can be changed by narrative inquiry. I later wrote that I believed that there is not an absolute body of mathematical truth that exists somewhere as a separate body of knowledge. Rather, that one's interpretation and understanding of the context of a mathematical problem determines the "truth" that may or may not exist within any given context. I went on to describe mathematics as a sense-making activity (involving discovering and doing) involving numbers, pattern, shape and space, rather than existing as a predetermined body of knowledge.

Changes in my Teaching Practice and Beliefs

There were several other changes that also occurred for me during this research. For example, I initially held concerns regarding whether or not mathematical investigations would result in mathematical learning. Following my experiences I later embraced the use of mathematical investigations as one means with which to hopefully initiate and encourage mathematical learning and reflection with pre-service teachers. There also appeared to be change in what I "expected" within a mathematics lesson. Whereas I previously would have wished for a definitive statement of learning about some mathematical idea, there was more room for students to explore, conjecture and think.

Whilst engaged in the mathematical investigations as a learner, during which time I was also continuing my journal writing, I discovered I held several subconscious beliefs, all of which were contrary to what I espoused in the classroom. This discovery and examining of beliefs illustrates Korthagen's (2004) suggestion that deepening one's reflection is a worthwhile practice. I found that I believed that "real" mathematicians solve problems quickly, do so on their own, do not get stuck and that there is only one correct interpretation of a problem. In direct contrast to this, in my teaching I promoted social constructivist and enactivist theories of learning (Barker, 2001), both of which propose that learning occurs with other people. One of the readings given to our first year pre-service teachers stated that it is "honourable" to be stuck (Collier, 1999) and we encouraged the acceptance of multiple interpretations. To find that I did not *really* believe any of these things was an eye-opener to say the least!

Changes in my teaching practice occurred as a result of discovering these unconscious assumptions. For example, having personally experienced being "stuck" I now believe that this really is a worthwhile part of the learning process. My practice in the classroom, with respect to this issue, is now more congruent with what I have espoused for a number of years. An example of this occurred whilst working alongside first year pre-service teachers working on an algebra investigation. When they became stuck, rather than rushing in to "relieve" them I was able to stand back if I judged that to be most helpful, or ask questions and/or provide hints.

I found the idea, offered by Brown and Jones (2001), that a researcher is not necessarily seeking an "ideal" (e.g., becoming the perfect mathematics educator) welcome as I engaged in the narrative inquiry. However, the process of letting go of reaching for an "ideal" was not a smooth one. Interestingly, when I was first asked to write about "what makes an effective mathematics educator?", I had no problem with setting out what I thought. It was certainly evident that I had a fixed notion of what constituted an effective mathematics educator, and what was needed to reach such an ideal. If I were asked to write in response to the same question now, I am not sure I could. Like Korthagen (2004) who refers to the complexities of what makes a good teacher, I am now much more attuned to the variety and changing range of influences and factors operating in a classroom at any one moment in time.

In narrative "the subject is never given at the beginning, but it unfolds as the story is told" (Ricoeur, 1986, as cited in McCormack, 2002, p. 337). I initially found this aspect of narrative research to be very unsettling. I was sure I should have some predetermined goal or question to be researching. However, the story did unfold, despite my worst fears and enduring resistance that it would not. I now trust the narrative process, and perceive it to be a powerful and liberating one. It was certainly in the ongoing reflection and writing that I came to understand more fully the journey, with the predicting of an outcome being less important – an idea proposed by McCormack (2002). This also links with the writing of McLaughlin (2003) who suggests that the practitioner researcher needs to be able to, "live with the ambiguity and lack of clarity long enough to formulate a specific focus to research" (p. 70). Having done this I relate with McLaughlin's suggested feelings of confusion, anxiety, frustration, doubt, feelings of inadequacy, and a desire for clarity as the research process unfolds.

Pre-service Teachers' Experiences of Mathematical Investigations

I used an investigative approach with classes of pre-service teachers twice during the research period. On the second occasion I was particularly delighted by most of the students' engagement. Indeed *they chose* to present what they had learned to the class (writing the mathematical ideas they had learned onto an overhead transparency and presenting this to the class with demonstrations and models as appropriate) at the end of the six hours. Although their teaching/presenting skills in such situations are still developing it was evident that they certainly had understood various mathematical ideas. This was also apparent when working alongside the pre-service teachers during the 6 hours. For some of the pre-service teachers, some of the mathematics ideas had been encountered for the first time whilst others found they developed an understanding of a particular procedure or idea for the first time. For example, one group of pre-service teachers developed an understanding of why π is equal to approximately 3. One student with whom I had an informal discussion, showed particular pleasure at coming to understand the meaning of π, and appeared to have a greater appreciation of mathematics as a sense-making experience rather than an arbitrary set of rules.

Comparable and contrasting points of view provide opportunities for engaging in further reflection (Chambers, 2003). With this in mind, at the end of this semester I interviewed four pre-service teachers. As Chambers (2003) suggests, having different perspectives creates new opportunities for reflection. I found that although some of the pre-service teachers' experiences resonated with mine, others were different and, I was able to gain new insights and perspectives about this teaching and learning approach. These included:

1. the need for discussing, in more depth, pre-service teacher beliefs about the learning and nature of mathematics;
2. discussing the place of traditional skill teaching that might occur alongside this approach; and
3. continuing to observe carefully the learning (of mathematics) that is hopefully occurring.

The pre-service teachers were mostly positive about the investigative approach. However it would appear, for two of the four interviewees, that there was an initial period where the process of participating in an investigation was an unfamiliar experience, and created some feelings of discomfort. This was particularly evident for one pre-service teacher who later recognised that he was initially creating "barriers" to the process. This finding, of feelings of discomfort, was comparable with feelings that I too had initially experienced. I wonder what part I may have played in creating these initial feelings of unease, because of my own concerns and discomfort.

Because I now have more experience with this investigative approach as a learner and teacher I believe that I can take this awareness of possible feelings of discomfort into my teaching, and as a beginning point, be able to empathise with students who experience this. Having also experienced the learning that can result from this approach, I believe it to be pertinent to highlight the possibilities of learning that can occur if the student can be encouraged to persevere through these initial feelings of discomfort. Pre-service teacher beliefs about the learning and/or nature of mathematics could also be openly acknowledged and discussed within a supportive environment. It would also appear that using this approach more than once is productive and enables the students to make deeper connections with the issues that arise.

One of the four pre-service teachers that I interviewed described how she found investigations to be less threatening and experienced them as being less pressured compared with her previous experiences in mathematics learning situations. Another described how coming to understand why π is equal to "3 and a bit more" was an "a-ha" moment. This appeared to be quite a pivotal experience for him in developing a new enthusiasm for the investigative process. He said:

> … like that activity (referring to a practical activity where the value of π is discovered) , when we went outside. My thinking was 'if you want us to go outside, I'll enjoy some sunshine and that's about it'. Little did I know that I was going to have an a-ha moment and that was great.

Receiving such positive feedback is certainly encouraging when considering whether or not to continue using this approach with future cohorts of pre-service primary teachers.

All four pre-service teachers stated that they learned some mathematical ideas, or understood a previously learned concept for the first time, by participating in the investigations. This is congruent with my own personal experience. The four interviewees also alluded to a deeper level of learning using this approach. This level of learning could be contrasted with a more traditional approach where a teacher might impart some knowledge (e.g., telling students a piece of information, finding the value of π, or showing a particular procedure) followed by students practicing numerous examples. One pre-service teacher described her experience of the deeper learning saying:

> With the traditional method, I can sometimes see there is [sic] good points to it, but then when we did that 'one', obviously I would've been told what π was (referring to her past), but I never remembered it. So when we started, I thought 'really, what is it?' and when I found out, it'll be in my head for the rest of my life. I found out for myself.

This pre-service teacher seemed to link this deeper learning with "doing it herself" rather than being told something. Another described the difference in learning as follows, "it is learned today, but it was taught in the old days". This same pre-service teacher stated that the investigations had "reignited the flame" with respect to her enjoyment of mathematics, and also referred to the importance of being able to relate previously learned mathematical ideas to a context. It would certainly seem that for these four that an investigative mathematical approach had been worthwhile.

One of the pre-service teachers expressed concern about the time taken to learn mathematical ideas by using an investigative approach. She stated that for her, "it is more time consuming" and asked the question, "have we got more hours in the day to spend on maths…?" I too have had that concern. However, based on my experiences both as a learner and teacher using this investigative approach, I believe that the learning is deeper and more meaningful and thus warrants the required time. Also, when considering my new ideas about what the learning of mathematics may entail, I now believe that this approach more closely captures the essence of what mathematics is actually about, i.e., a process of making sense of situations involving number, patterns, shape and space, rather than the finding of a particular answer using a set procedure that someone else has previously discovered.

The pre-service teachers also perceived their mathematical behaviours to have changed, For example, they became more thorough in their investigating, and open to the idea that perhaps multiple interpretations are valid in the learning of mathematics. Some of the pre-service teachers' beliefs and/or ideas about teaching mathematics also appeared to change. Three of the interviewees indicated they would try using an investigative approach when they begin to teach. One pre-service teacher stated that her thinking:

> … has shifted from being formula based mathematics [to] social constructivism … you are interacting with others, you are using your previous knowledge and ideas and you are experimenting with it. I had never been allowed to do that with maths before and I enjoyed it.

One pre-service teacher indicated she felt "frightened" that she would be unable to deal with the possibilities that children might raise within the course of an investigation. She stated however, that, "I believe I am now preparing myself to work through whatever their ideas are, which I think is really positive". Once again, I can empathise with this experience. I too, found this investigative approach to be initially somewhat unsettling with respect to possibly not knowing the mathematics that might be encountered during the course of the investigation. Perhaps this is part of a process of moving from viewing mathematics as a discipline where it is important to know the answer, to an alternative view of seeing mathematics as a process of doing and discovery. In this alternative view not knowing the answer would be seen as an exciting and natural part of doing mathematics and an opportunity for new learning.

It gave me a great deal of pleasure when after the second year practicum one pre-service teacher returned to show me the results of children's work done during an investigation that she had used whilst on practicum. She spoke of children wanting to do mathematics and being disappointed when it was not scheduled for a particular day.

Although I value the experiences and insights that these interviewees shared, I am also aware that there will be other experiences and interpretations of the mathematical investigative approach that are not represented by these four pre-service teachers. It was not my intention in this research to gather quantitative data that is representative of all students, but I believe it is nevertheless important to remain open to the ideas and insights that others may hold. This remains a possibility for further research.

Conclusion

Narrative research led to a number of changes in both my beliefs and teaching practice. Personally working on a mathematical investigation was a pivotal point in the journey that led to the discovery, and subsequent change, of previously unrecognised beliefs about learning in mathematics and changes in my thinking about the nature of mathematics. I am now comfortable with the notion that mathematics learning takes time and can be aided by collaboration between students and between teacher and student. I also accept that being "stuck" can be an acceptable and helpful part of learning mathematics; and that multiple interpretations are a valid part of the learning process. Mathematics is now viewed as a sense-making activity, involving discovering and doing.

I believe this has led to changes in my teaching practice. The changes included using this approach knowing that the approach results in mathematical learning, giving students "space" to be stuck, and providing more in-depth interactions to support their mathematical learning. Earlier uncertainty about whether investigations are a useful approach to support the learning of mathematics ideas were at least partially resolved with the positive experiences encountered whilst using mathematical investigations, as a teaching approach and as a vehicle for personal mathematical learning. It was also evident from talking with some pre-service teachers that this approach is a valuable one to engage them to think more deeply about the learning and teaching of mathematics.

Although I have undergone valuable personal learning I do not wish to become a crusader advocating that using mathematical investigations will solve all challenges involved in supporting our pre-service teachers to become more skilled at teaching and learning mathematics. Rather it has been a personal journey that at this point has found mathematical investigations to be a useful learning and teaching tool. A next step would be to explore in more detail the experiences of a greater number of students and to follow their professional progress in an effort to ascertain the value of engaging in mathematical investigations during their pre-service teacher education.

References

Adler, S. A. (1993). Teacher education: research as reflective practice. *Teacher and Teacher Education, 9*(2), 159-167.

Barker, M. (2001). How do people learn?: Understanding the learning process. In C. McGee & D. Fraser (Eds.), *The Professional Practice of Teaching* (2nd ed., pp. 35-66). Palmerston North, N.Z.: Dunmore Press.

Beattie, M. (1995). New prospects for teacher education: narrative ways of knowing teaching and teacher learning. *Educational Research, 37*(1), 53-70.

Brown, T. (2001). *Mathematics education and language: Interpreting hermeneutics and post-structuralism*. Dordecht: Kluwer.

Brown, T., & Jones, L. (2001). *Action research and postmodernism: Congruence and critique*. Buckingham: Open University Press.
Chambers, P. (2003). Narrative and reflective practice: recording and understanding experience. *Educational Action Research, 11*(3), 403-414.
Clandinin, D. J., & Connelly, F. M. (2000). *Narrative inquiry: Experience and story in qualitative research.* San Francisco: Jossey-Bass Publishers.
Collier, C. P. (1999). Much of what I needed to know I learned by teaching mathematics. *Mathematics Teaching in the Middle School, 4*(8), 500-503.
Francis, D. (1995). The reflective journal: A window to preservice teachers' practical knowledge. *Teaching and Teacher Education, 11*(3), 229-241.
Korthagen, F. A. J. (2004). In search of the essence of a good teacher: Towards a more holistic approach in teacher education. *Teaching and Teacher Education, 20*(1), 77-97.
Luwisch, F. E. (2001). Understanding what goes on in the heart and the mind: Learning about diversity and co-existence through story telling. *Teaching and Teacher Education, 17* (2), 133-146.
Mason, J. (2002). *Researching your own practice: The discipline of noticing*. London: Routledge Falmer.
McLaughlin, C. (2003). The feeling of finding out: the role of emotions in research. *Educational Action Research, 11*(1), 65-77.
McCormack, B. (2002). 'Removing the chaos from the narrative': Preparing clinical leaders for practice development. *Educational Action Research*, *10* (3), 335-351.
O'Connell Rust, F. (1999). Professional conversations: new teachers explore teaching through conversation, story, and narrative. *Teaching and Teacher Education, 15*(4), 367-380.
Rushton, S. P. (2001). Cultural assimilation: A narrative case study of student-teaching in an inner-city school. *Teaching and Teacher Education, 17*(2), 147-160.
Schön, D. (1983). *The reflective practitioner: How professionals think in action*. New York: Basic Books.
Smith, T. (2006). Self-study through narrative inquiry: Fostering identity in mathematics teacher education. In P. Grootenboer, R. Zevenbergen & M. Chinnappan (Eds.), *Identities, Cultures and Learning Spaces*. (Proceedings of the 29th annual conference of the Mathematics Research Group of Australasia, Canberra, pp. 471-478). Sydney: MERGA.
Wilber, K. (1998). *The marriage of sense and soul: Integrating science and religion*. New York: Random House.
Winkler, G. (2003). Are we nice or are we real? Ethical issues emerging from collaborative narrative research. *Educational Action Research, 11* (3), 389-402.
Winter, R. (2002). Truth or fiction: Problems of validity and authenticity in narratives of action research. *Educational Action Research, 10* (1), 143-154.
Winter, R. (2003). Buddhism and action research: Towards an appropriate model of inquiry for the caring professions. *Educational Action Research, 11*(1), 141-155.

Describing Mathematics Departments: The Strengths and Limitations of Complexity Theory and Activity Theory

Kim Beswick
University of Tasmania
<Kim.Beswick@utas.edu.au>

Anne Watson
University of Oxford
<anne.watson@edstud.ox.ac.uk>

Els De Geest
University of Oxford
<els.degeest@edstud.ox.ac.uk>

This paper draws on two studies of mathematics departments in 11-18 comprehensive maintained schools in England to compare and contrast the insights provided and questions raised by differing theoretical perspectives. In one study a mathematics department was viewed as a complex system and analysed accordingly. In the other activity theory was used to describe and analyse features of the departments involved. In both cases the departments involved were considered to be systems and it was the learning of the system rather than of individuals that was of interest. The affordances and limitations of the analytical perspectives are discussed.

In this paper, mathematics departments are seen as identifiable systems, operating with a purpose that distinguishes them from other groups of people within their respective schools. Although mathematics teachers may have other roles, such as being form tutors, teaching other subjects, or undertaking management responsibilities outside the teaching of mathematics, they belong to the mathematics department with respect to their work of teaching the subject. Departments concerned with teaching different subjects may operate in similar ways for many purposes, such as putting school policies into practice, responding to timetable designs, preparing reports, reporting assessment information and so on, but might also be distinguishable through characteristic epistemic cultures (Knorr-Cetina 1999), in that the concerns of mathematics departments might have some things in common with other groups of people concerned with mathematics just as art departments might have some things in common with other groups of people concerned with art. For the purposes of the studies reported in this paper, it was assumed that they would be distinctive in ways which might be epistemic. We also assume that they would be distinctive in ways that relate to current trends in school mathematics teaching in England such as the possible shortage of mathematics teachers (so that 25% of classes at this level have to be taught by people not qualified in the subject); high turnover of mathematics teachers; pressure for results as schools are compared using test results in core subjects; the high political focus on mathematics; and the inherent difficulties of teaching and learning the subject. The departments on which this paper is based were also distinctive in being subjects of research.

Complexity theory and activity theory offer two different ways of describing and analysing systems. In this paper we briefly describe salient features of each, outline their respective use in two studies of mathematics departments, and compare what each offers as a theoretical perspective through which to analyse school mathematics departments.

Davis and Simmt (2003) explained how complexity theory has developed in recognition of the fact that some systems cannot be understood using conventional analytic tools. That is, the behaviour of some systems cannot be predicted by analysing the actions of individual elements of the system. This is not simply a problem related to the difficulty of analysing large numbers of such interactions but to qualitative differences between systems that are complicated by virtue of the numbers of interactions, and systems that are complex. Complex systems typically comprise living agents who are autonomous, at least to some extent, and are characterised by features that are emergent in that they arise from the interactions of agents but cannot be directly attributed to particular agents (Davis & Simmt, 2003).

Complex systems are also adaptive in that their response to a given stimulus is dependent not only on the stimulus but also on the history of the system. Complex systems thus embody their histories as they adapt to their environment and hence can be described as learning. Applied to human systems, learning can be seen as an emergent feature of the collective, and knowledge as residing with the collective rather than with individuals (Davis & Simmt, 2003). This is not to deny the existence of individual learning because individuals too can be described as complex systems nested within others. Indeed, Davis and Simmt (2003) illustrated the nestedness of complex systems by referring to the relationships between cells, organs, individuals, and society, all of which learn in the sense of adapting to their environment.

Davis and colleagues (e.g., Davis, 2004; Davis & Simmt, 2003; Davis & Sumara, 2005) have described educational settings in terms of complexity theory and have described five necessary but not sufficient conditions for emergence to occur. These are: *Diversity* among agents (typically students in a class), which allows for novel responses; *Redundancy* in the sense that agents have sufficient in common to allow meaningful interaction and to compensate for each other's weaknesses; *Enabling constraints* that balance order and focus in the collective's activity with the expression of its diversity; *Decentralised control* that recognises that outcomes, including the emergence of complexity, can not be predicted but instead emerge from the collective activities of agents; and *Neighbour interactions* between ideas rather than simply between agents.

Although these conditions have proved useful in describing educational settings (e.g., Sinclair, 2004) there is necessarily intentionality on the part of a teacher whose conception of teaching is essentially one of engineering an environment to include these conditions. (Towers & Davis, 2002). Davis (2005) attempts to deal with the dual role of the teacher as one of many agents in a classroom in which purpose is an emergent feature, and the teacher's intentionality by likening the teacher to the consciousness of the collective whose role is to direct and focus attention and to choose among possible interpretations and actions open to the collective. Although helpful, this falls short of recognising the capacity for intentionality characteristic of all agents in a collective of human beings. Unlike other living agents that comprise complex systems, humans are not obliged to act according to rules (although their may be powerful forces that encourage them to do so) and hence any agent in a human system has the capacity to disrupt or alter the system through the exercise of choice (Kurtz & Snowden, 2003). A skilled teacher is able to notice emerging patterns, intervene to stabilise those that are helpful (in terms of his/her intentions) and destabilise those that are not, and to structure the environment by *seeding* it or creating *attractors* around which patterns of interaction emerge, so that desired purposes and outcomes are likely to emerge (Kurtz & Snowden, 2003).

Activity theory focuses analysis on structured features of a department's work and the ways in which they interrelate. Activity consists of a group of people engaged in activity (the *subject*: in this case the teachers, student-teachers and classroom assistants), the direction of their work (the *object* or motive: in this case the mathematical learning of the target students), the *goal-directed actions* that are needed to achieve the object, and the *operations*, or routines, which keep the system working fluently (Leontiev, 1974). These operations can be subcategorised as *rules*, *community* characteristics, and *division of labour*. All these features are in balance, so that if one changes, other changes will take place to adjust the whole system. The object might change as a result of activity, and the activity might change as the object changes. This inherent instability is recognition of the nature of human agency within a system, and that the object is dependent on how it is understood by the people concerned. Despite this instability, patterns of behaviour within the system are often fluent, well-practised, and by-and-large replicate patterns of school subject departments in general.

The role of *mediating tools* in learning is multi-layered: Teaching and learning in classrooms can be seen as a knowledge-creating process of interaction between teacher, learner, and mediating artefacts. In mathematics, these include concrete tools such as boardpens, textbooks, and computers and also less transparent tools such as language, symbols, analogies, and examples.

Study A

Study A, Development of a Mathematics Departmental Culture (DMDC), concerned a department which had recently undergone significant staff changes. There was a new Head of Department, (HoD), a new teacher with responsibility for Key Stage 3 (lower secondary) and essentially “third in department”, and two newly qualified teachers. The school had specialist mathematics status, and the extra funding which derived from this meant that the HoD had been appointed at Assistant Head Teacher level with a brief that included teacher development, community engagement, and dissemination of good practice. The existing team comprised six teachers, including two other Assistant Head Teachers who taught 50% of a full load, and two heads of year. One of the assistant heads and one of the heads of year were not mathematics specialists but had trained in physical education and music respectively, with the latter dividing her teaching equally between mathematics and music. Both had taught mathematics for many years and the other teachers all had strong backgrounds in mathematics. Three of the team were studying, or had recently pursued academic professional development courses at a nearby university. The study was conducted in the first term of the school year and aimed to describe how the department developed as an entity. Particular foci were the development of shared beliefs and the ways in which individuals adapted to one another and influenced the department as a whole. Data comprised: individual interviews with each of the 10 department members at the beginning and end of the term; additional interviews with the HoD, the new third in department, a newly qualified teacher, and a teacher who had been at the school for a number of years; and audio-tapes and observations of departmental meetings.

Complexity theory was considered an appropriate theoretical tool in this context because the new HoD’s brief included change, or learning, at the departmental level. In addition, although an established department may have norms of practice and interaction that have been implicitly or explicitly agreed to and hence not be complex, the influx of new staff would necessarily require the renegotiation of roles, relationships, procedures, and patterns of interaction such that the outcomes would be unpredictable. Emergent phenomena included: an increasingly shared understanding of the meaning and importance of mathematical thinking in improving students’ attainment; consensus around the idea of providing access to higher levels of attainment for all students; a long term view of improving attainment; and a shared sense that the department was supportive. Although it is possible to identify contributions made to each of these by individuals their emergence is not entirely explicable in terms of direct causal links. Rather, they appeared to arise from interactions among the teachers in a form that was not precisely represented by any individual contribution.

The particular focus in this paper is the use of complexity theory to analyse retrospectively the HoD's attempts to influence mathematics teaching practices in the department. Because emergent phenomena can be perceived but not predicted (Kurtz & Snowden, 2003) such retrospectivity would have been necessary even if she had been consciously attempting to create the conditions for complexity (Davis & Simmt, 2003), The extent to which each of the five conditions for complexity were present in the department and the purposeful use and management of attractors by the HoD are described below.

The HoD had clear purposes in mind, which she articulated throughout the term in the context of interviews, staff meetings, and in informal contexts. These related to enhancing students' opportunities to achieve, and focussing on students' thinking and how that could be moved forward in such a way that they achieved deep understanding of mathematical structures. She saw the two as related in that deep thinking and understanding would contribute to long term gains in achievement. She also likened the department's learning to that of students and compared the way she would like the department to operate to the way in which she wanted classes to operate, that is, characterised by deep, independent thinking, sharing of perspectives, and both individual and collective construction of understanding.

The ingredients for complex emergence (e.g., Davis & Simmt, 2003) appear to have been present in the department partly as a result of the HoD's choices and partly as a result of outside influences upon it. The diversity of views and approaches to mathematics teaching represented by the ten teachers was mentioned by several teachers when prompted to describe the department's strengths. The HoD also acknowledged the diversity represented by the teachers when she described the professional learning needs of the department as follows.

> … it's a question of people really building up their own areas of expertise and following those rather than one size fits all. In terms of one size fits all that's more of our working together rather than using people from outside. Take for instance, how to introduce algebra, I think we've got the skills between us to work together on that, and where it's a question of people following their own levels of expertise and areas of expertise, there are people that they need to work with perhaps on a national level …

Much of the redundancy evident was a consequence of the teachers' familiarity with the English National Curriculum, examination procedures, and usual school organisational practices that included setting on the basis of prior attainment. The overriding importance of ensuring that the school's examination results were satisfactory was taken as a given and enhanced opportunity was understood in terms of making higher grades accessible to all students. The strong mathematics background of eight of the teachers, and extensive experience of mathematics teaching of all ten, enabled all to participate in conversations of a mathematical nature. Interestingly, the externally imposed constraints of curriculum and examinations not only contributed to redundancy but also appeared, by virtue of their familiarity, to act as enabling constraints for some teachers. It seemed that the system requirements had been internalised by all of the experienced teachers to such an extent that they felt some degree of freedom to experiment with teaching approaches. The HoD expressed a similar view of school level policies, explaining that, "We really do have quite a lot of freedom, that's the sort of feeling I have".

Enabling constraints were also provided by the HoD as she worked to encourage conversations about students' thinking. These included asking teachers to bring examples of students' books to a departmental meeting so that the ways of providing feedback could be discussed. Initially only the HoD herself had examples to share but at a subsequent meeting a few other teachers also brought examples. On another occasion teachers were asked to bring examples of how they had incorporated the idea of equivalence into their mathematics teaching of any topic with any class and the request included a brainstorm of opportunities in which the idea might arise. Most teachers did report examples of highlighting equivalence in their teaching. The purpose of enabling constraints is to balance order and the expression of diversity (Davis & Simmt, 2003) but, since the unit of analysis is the system as a whole, complexity theory does not offer an explanation of why the same constraints appear to be enabling of some individuals but not others. Other perspectives that take account of social relationships might be better able to do this. From Kurtz and Snowden's (2003) perspective, enabling constraints can be thought of as attractors that establish a degree of order around them. The unpredictability of the impact or effectiveness of attractors, or even whether an influence on a system acts as an attractor at all, is inherent in the nature of complex systems (Kurtz & Snowden, 2003).

Other attractors included the HoD's enthusiasm for mathematics and for teaching, her constant references to students' thinking and the need to move it forward, and the fact that most of the teachers in the department had desk space in a team room. The HoD's references to thinking included an A4 poster she created with the slogan, "Learning to Think, Thinking to Learn" that was displayed in several of the mathematics classrooms and the team room, and was referred to by several teachers when they were asked about the department's ethos. The energy that the HoD devoted to teaching was evident to her colleagues who saw her as having high standards.

The team room's function as an attractor was due to its role in facilitating neighbour interactions. The HoD, the two newly qualified teachers, the new "second in charge", and two teachers who had been in the school for a number of years all spent most of their non-teaching time in that space and informally shared their practice. The usefulness of these conversations was described by the HoD.

> Sometimes we're working and talking at the same time, there's lots of it, and somebody else comes in and they join in. People seem to be much more ready for that than if you were to convene another formal meeting because they don't feel they have to be there, they're drawn in by interest, and then they make a contribution and they don't have to do exclusively that, they might be sorting through a few tests while contributing to the conversation …

Others who did not work in the team room because they had office space elsewhere (i.e. the Assistant Heads and one head of year) or who chose to work in their classrooms still made regular visits to the room to collect and return resources stored there or to seek out advice. The HoD recognised the value of such interaction and, in Kurtz and Snowden's (2003) terms, acted to stabilise this emergent pattern by proactively ensuring that she regularly visited the teachers who primarily worked elsewhere.

The department was necessarily constrained by school and system requirements but in other ways the teachers were autonomous and hence control was largely decentralised. The HoD was aware of the need to provide a safe environment in which people could take risks as they tried to change their practice. To this end she avoided directly observing her colleagues' teaching but instead monitored practice principally through conversations with them and also by listening to classes as she walked through the corridors. In her words:

> I'm not keen on doing things which I think leave the person feeling insecure and on the hop. What I want to do is … get somebody to take risks and work outside their comfort zone. They're much less likely to do that if they think you're about to barge in any second and I think what you need is just somebody to say well okay, the students are here, we want them to be here and we need to take risks to get them from here to here and if they think that the game is that any second you're about to walk in, I think for most us that's very risky, … I probably do a bit more from the corridor than people realise I do.

Study B

Study B is a three-year funded ethnographic study designed to tell the story of three mathematics departments as they set about making significant changes to the ways in which they teach mathematics to low-attaining students. Two of the schools serve inner-city areas of social deprivation, one of them highly multicultural, the other predominantly white working class. The third school serves a wide rural area. In England it is usual to teach students in different groups according to prior attainment, and the study focuses on those who would end up in "low" groups under this system. Such groups typically include students from the most disadvantaged socio-economic groups, even in comparatively well-off areas. A range of data has been collected: teacher interviews, lesson observations and videos, notes and audio-recordings from department meetings, schemes of work, lesson ideas, student interviews, test scripts, national test scores, students' work, and background data about past achievements and school statistical predictions. The units of analysis are: (a) a sample of students from one cohort as it passes through the first three years of secondary education, and (b) the department as it organises their mathematical experiences. The academic task is to connect the departments' activity to the achievement of the students, to identify factors that contribute to success or otherwise; and to tell plausible stories about how the departments operated.

The capacity of activity theory to describe the interplay between stable practices and instability in the departments made it a suitable frame for our analysis. For this paper we are interested in the structures that enable the department to pursue its purpose, in particular the tools, including teaching tools and also department tools such as meeting agendas, resource banks, emails, and memos that enable the activity to take place. It seems, in our analysis, that there are other features that are not usually described as artefacts but which also have this role in departments: individual knowledge is one of these and the nature of meetings is another. One of the outcomes of this study is more understanding about the nature of "tools" that mediate knowledge within mathematics departments.

During the analysis we noticed that the object of the system was also the object of individual classrooms, and that these too could be seem as activity systems, albeit with different subjects and communities, so the third generation activity theory developed by Engeström (1998) seemed an appropriate way to continue. In fact, Engström (1998) used this to lay out the behaviour of a school mathematics department undergoing deliberate change, with the same distinction between departmental activity and classroom activity.

In this paper, we refer to semi-structured interviews with teachers in the three schools who were teaching year 7, the entry cohort to the study. These interviews were undertaken at the start of the study, after decisions had been made about how year 7 was to be taught, and again towards the end of the first year. Interview data are, of course, highly subjective but are appropriate for this analysis because activity systems depend on human consciousness and agency and hence affective self-report is informative. Other data will inform us about enacted intentions and learners' experience, but the analysis of these is beyond the scope of this paper.

Content identification was used to confirm that the categories associated with activity theory would enable us to sort and categorise what was said for each separate interview, and then enable further comparisons, such as between teachers, between schools, and between interviews with individual teachers, to be made. We could thus construct shared understandings and contradictions within schools, similarities and differences between the three schools, and changes during the first year.

The process of analysis threw up many interesting observations, before such comparisons were carried out. Having decided that activity theory was the most appropriate framework, what followed was an exercise in: fitting the data to the structure and seeing what did not fit; seeing whether the structure could be interpreted to accommodate the data; and questioning the structure and the data. These processes embody the way in which structures are used as tools to mediate meanings in data, and can symbiotically imbue data with meaning. The analytical questions are: "What can these data tell me if I look at them with this perspective?" and "What do I learn about this perspective from these data?" The following examples are illustrative.

Many teachers talked of contributing ideas to the department resource bank in their school. This action seems to describe a division of labour. However, by contributing an idea to the bank, they were also contributing *their ways of seeing the teaching of mathematics*, either through the bank or through discussions about their suggestions. Thus, their knowledge was more than something they did individually, but became available to be used by others – a potential pedagogical tool. In this sense, individuals' knowledge can be seen as a mediating tool within department teams to learn more about pedagogy. Further, department meetings could be described as a feature of the way the community operates, or as part of the rule-structure of the department, but the discussions that take place in them can be seen as mediating devices for pedagogical learning. When interviewees mentioned department meetings it was always in the latter sense, rather than in the sense of a departmental structure or rules of behaviour. This description of individual knowledge acting as a tool within a department, to be taken up and used by others, seems more useful in this context than to see it as merely part of more generally distributed knowledge.

There were interesting differences between what people said was supposed to happen and what actually happened. The most common was that they were all supposed to contribute ideas, but in the schools where this meant "put some lesson plans into the file" most claimed not to have done that. Thus "division of labour" was that some did and some did not, whereas "rules" included the expectation that all would do so. We expanded "rules" to include "expectations" so that "division of labour" could be left to describe what people said actually happened.

For Engeström (1998), the interesting thing about systems is how they learn, where learning is understood as the constant flux between internal inconsistencies and their resolution. Asked about priorities for year 7, the teachers in one of the schools began the year with the shared aim, articulated by all teachers, that students should "enjoy" mathematics. By the end of the year many teachers were saying that they were concerned about students' basic knowledge and that "skills" were one of their priorities. This was not a stated aim through departmental communication channels but had emerged from the grounded experience of the teachers. The object, in Leontiev's terms, had been transformed through activity. For these teachers, their classroom aim incorporated "basic skills" but the department rhetoric was still about "enjoyment" and not about curriculum coverage. This could be seen as a rupture between the department and individual classrooms, or could be seen as transformation of the object of the department. Resolution had to involve restructuring of a tool, the scheme of work, but also negotiation of priorities and individuals' ways of seeing their work.

A more dramatic finding was in the interpretation different teachers made when they imagined they were talking about the same thing. In one school, some teachers talked about open-ended tasks and investigating mathematics whereas the HoD talked about learning mathematical structures, as if they were all aiming at that. Meanwhile, in formal and informal interactions, everyone appeared to believe they were talking about the same thing apart from a few teachers who were known to be adhering to a transmission form of teaching. The latter difference was overt and seen as a training need; the former was not recognised by anyone except the researchers. Here again, there are queries about interpretation of the shared object. For some teachers this was shown in the very different uses they make of "the same" artefacts, that is the meanings with which they were imbued by individual teachers in classrooms were different, and knowledge of pedagogy was not unambiguously mediated through the resources. Some teachers did not use the resource bank at all: there was no shared object, and no common tools, although the teachers were actors in the same system because they taught the target groups, or because they were in our research project!

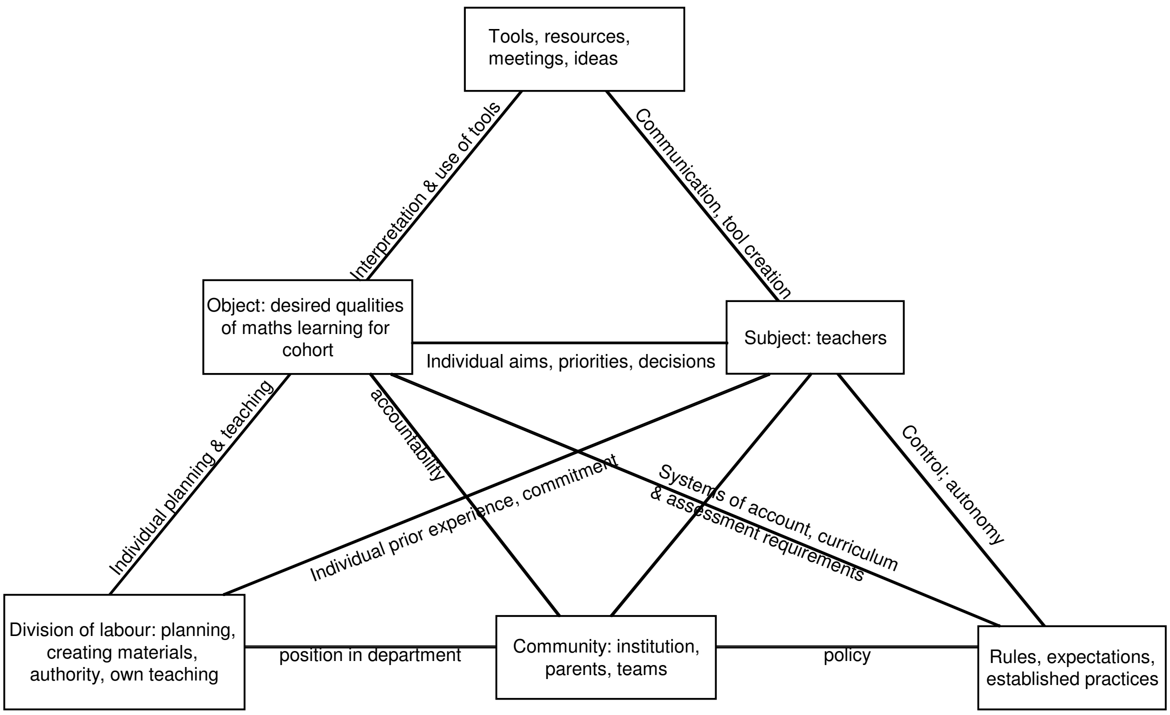


Figure 1. The work of the mathematics departments seen from an activity theoretic perspective (after Engestrom 1998).

The triangle in Figure 1 gives more detail about how the interview contents were interpreted and structured in our analysis. In this diagram we have been able to represent all aspects of department activity as described by the teachers, except, as we said earlier, values, and reasons for individualisation of interpretations, objects, and actions. We were able to describe systemic influences on relationships between the points on the triangle. There were highly individual differences in dealing with external requirements, such as accountability. HoDs in two schools gave guidance that was much less prescriptive than several teachers chose to adopt. For this reason "accountability" does not appear under "rules" or "community" but edges more towards individual interpretation of the object.

Activity theory has helped us to make sense of most features of departmental activity, with respect to the target students, and has also enabled us to connect classrooms with departments as systems which *may* have common purpose. From these linkages, and attempts at linkage, we found some conflicting aspects for which resolution was likely to change the system. This analysis did not, however, enable us to make sense of different teachers' interpretations of goals and artefacts in their action, and how these related to the department's work. Nor did it enable us to deal with ruptures that depended on interpretations of the object (what it means for the target group to learn more mathematics) rather than changes in the stated object itself. Indeed, it did not allow us to structure interpretations and value systems into our analysis – but it did reveal them, and showed that these differences were conflicting and that there were splits and potential splits, both known about and unknown.

Comparing the Affordances of the Different Theoretical Perspectives

The overarching question in choosing between complexity theory and activity theory is, "Is this department a complex system (characterised by emergence and adaptation) or is it more like an activity system, in that it is totally structured?" The choice necessarily influences what is looked for and noticed. In the four departments considered in these studies, there were aspects of their functions that were known, predictable, and governed by agreed procedures and allocated responsibilities. In Study A these aspects included the compliance with examination entry procedures and setting, but the aim of improving students' attainment was a shared goal in relation to which each teacher acted autonomously albeit influenced by their interactions with one another and particularly by the intentional interventions of the HoD. In Study B important aspects of the departments' efforts to achieve their aim of raising attainment for a particular group of students were much more structured. This difference can be attributed to the facts that the aim in this case is more tightly defined (i.e. it was a condition of involvement in the research and was subject to timelines and measurement), and that the aim was not necessarily in tune with the aims of each small grouping within the system. For this reason it needed to be managed centrally with questions like, "Who will take responsibility for this necessary task or role?" (division of labour) and, "What common tools do we need to carry this out?" It thus seems that choices made by leaders in relation to bringing about change, particularly whether they attempt to facilitate the emergence of the desired aim or seek to devise and impose systems that will further the aim, are highly relevant to whether the system is best thought of as complex system or as an activity system.

A further difference between the two approaches is how each perspective deals with change. Both claim to show how systems might continually change and learn. Activity theory, however, seems to see change as structural disruption, in that systems necessarily contain, within their ways of functioning, relationships that might break down, or might be in conflict with other relationships. Thus change is manifested as a crisis that requires reorientation of parts of the system, renegotiation of roles and rules; introduction of new mediating tools and meanings; and redefinition of objects. Activity theory predicts and models the reorganisation that precedes and follows a change in heads of department, and also shows up the potential problems arising from a lack of shared objectives, or from contradictory interpretations of objectives. Complexity theory embraces change as a necessary characteristic of systems, recognises that change to one part of a system triggers adjustments throughout, and sees "adjusting" as part of the overall dynamic functioning of the system. Complexity theory is therefore better at describing fluid systems in which related members take a large number of autonomous decisions (decentralised control); members work in parallel and might influence each other through neighbourhood. We also found that activity theory allowed us to incorporate some institutional requirements directly as rules, which may have been alien to the department, whereas incorporating institutional requirements in study A as aspects of complexity did not show whether they had an alien and contradictory quality.

Just as the Study A department included aspects that were highly structured, aspects of the Study B departments' functions, for example the teaching of mathematics in classrooms, were less structured and arguably less amenable to analysis using activity theory. It was for the specific task of the departments' teaching of one cohort that activity theory, and the attempt to describe the activity as a structure, were useful in showing up conflicts, gaps, and differences in interpretation. Complexity theory tells us about diversity and unpredictability that are inherent in human systems, whereas activity theory offers a tool to analyse activities that at least for a time seem structured and predictable. Neither is capable of adequately dealing with the role of individual differences of action and interpretation within the system nor claims to be.

References

Davis, B. (2004). *Inventions of teaching: A genealogy*. Mahwah, NJ: Lawrence Erlbaum.

Davis, B. (2005). Teacher as "consciousness of the collective". *Complicity: An international journal of complexity and education, 2*(1), 85-88.

Davis, B., & Simmt, E. (2003). Understanding learning systems: Mathematics education and complexity science. *Journal for Research in Mathematics Education*, *34*(2), 137-167.

Davis, B., & Sumara, D. (2005). Complexity science and educational action research: Towards a pragmatics of transformation. *Educational Action Research, 13*(3), 453-464.

Engestrom, Y. (1998) Reorganising the motivational sphere of classroom culture: An activity theoretical analysis of panning in a teacher team. In F. Seeger, J. Voigt, U. Waschescio (Eds.), *The Culture of the Mathematics Classroom* (pp. 76-103). Cambridge: Cambridge University Press.

Knorr-Cetina, K. (1999), *Epistemic cultures: How the sciences make knowledge*. Cambridge, MA: Harvard University Press.

Kurtz, C. F., & Snowden, D. J. (2003). The new dynamics of strategy: Sense-making in a complex and complicated world. *IBM systems journal, 42*(3), 462-483.

Leont'ev, A. (1974). The problem of activity in psychology. *Soviet Psychology 13*(2), 4–33.

Sinclair, M. (2004). Complexity theory and the mathematics lab-classroom. *Complicity: An international journal of complexity and education, 1*(1), 57-71.

Towers, J., & Davis, B. (2002). Structuring occasions. *Educational Studies in Mathematics, 49*(3), 313-340.

Three Student Tasks in a Study of Distribution in a "Best Practice" Statistics Classroom

Anthony Bill
University of Tasmania
<afbill@utas.edu.au>

Jane Watson
University of Tasmania
<jane.watson@utas.edu.au>

Three selected student tasks from a 2-week study of the statistical concept of *distribution* in Year 9 class are examined. The tasks considered the exclusion of outliers, analysis of data using a semi-formal framework (GICS) developed for this study, and comparing two distributions. The pedagogy was modelled on current statistics education research best practice, with an emphasis on the cultivation of classroom dialogue where students explain and justify their positions. *Fathom*™ software was used by the students in a computer laboratory, and as a teaching aid in the classroom to support learning.

Distribution is a statistical concept that considers a data set as entire aggregate, with its own characteristics of measures of centre, such as mean and median; of measures of spread, such a density; and the shape of the distribution such as that known, for example, as a normal distribution. This comprehensive conceptual entity requires simultaneous consideration and integration of all aspects of the data set. This is a demanding task for students. Sophisticated statistical tools such as standard deviation, taught normally at senior high school, might support analysis, but at the expense of developing more intuitive notions of the data set. Current education research considers whether the use of semi-formal analysis in middle school might provide the essential intuitive foundation for formal statistical analysis that students will encounter in the senior school years.

Three tasks from a two-week study program are presented. The "Students' height" task provided an opportunity for a structured discussion of a data set using formal and informal measures; the "Weighing a small mass" task examined students' understanding of data outliers; and the "Reaction time" task extended these two tasks to compare two distributions. The theoretical background, the results, and the discussion are based on the three tasks presented sequentially. The theoretical background begins with a discussion of what current statistical education research considers as best practice teaching, as this best practice teaching philosophy provides the foundation for the teaching unit used in the research. Examples of students' work are included for discussion. Worksheets were evaluated using the SOLO taxonomy.

Theoretical Background

Current statistics education "best-practice" teaching differs from traditional approaches to teaching statistics. Traditional teaching presents statistics as a collection of rules and techniques rather than a process of quantitative reasoning, problem solving, or developing intuitions (Garfield & Ben-Zvi, 2004). Mokros and Russell (1995) argued that traditional teaching actively interfered with students' natural intuitive sense of basic statistical concepts and Garfield and Ben-Zvi found traditional teaching obscured the "big ideas" of statistics. They also observed that students calculated basic statistics, but did not have a sound understanding of what was being constructed or how statistical concepts interrelated. Traditional teaching also over-emphasised measures of centre, such as mean and median,

Proceedings of the 30th annual conference of the Mathematics Education Research Group of Australasia
J. Watson & K. Beswick (Eds), © MERGA Inc. 2007

giving scant regard to variability and, by implication, distribution (Shaughnessy, 2006). Mathematics teaching generally encouraged a quick response to a problem rather than a reflective and thoughtful analysis (Shaughnessy, 2006). Traditional assessment focused on the correct application of formulas, and the accuracy of computations and of graphs, but this provided only limited information on the students' statistical reasoning (Garfield, 2003).

Contemporary statistics education research is remarkably consistent in relation to recommended pedagogy. Five key features of best teaching practice are identified.

1. Engage students with data and concepts – the "big ideas" of statistics – such as variation and distribution (Ben-Zvi, 2000; Franklin & Garfield, 2006).
2. Provide active learning opportunities (Franklin & Garfield, 2006) and authentic data analysis (Groth, 2006) with real or "messy" data sets and meaningful tasks in a context that students can understand and value.
3. Develop a culture and habits of enquiry and statistical process (Franklin & Garfield, 2006); use whole class discussion where students must construct arguments and justify their positions (Groth, 2006). Chance (2002) argued that the mental habits and problem solving skills needed to think statistically should be deliberately taught as it should not be assumed that students would naturally develop these habits through the statistics course. A significant barrier to the enculturation process is that students may lack the vocabulary to express statistical opinions confidently. Teachers should provide students with a working – not necessarily formal – statistical vocabulary. Bakker and Gravemeijer (2004) recommended that students be allowed to use statistical terms loosely, or encouraged to use informal terms, such as "spread out", or "clumped", to describe distributions. Statistical terms would be used with greater precision as students' statistical sense developed.
4. Utilise technology tools that allow students to visualise and explore data by providing different representations of the same data set (Ben-Zvi, 2000; Franklin & Garfield, 2006) and to move back-and-forth between the various representations of the data (Bakker & Gravemeijer, 2004). *Fathom*™ offers these, and other features.
5. Use assessment that genuinely measures student learning and development (Chance, delMas, & Garfield, 2004) and that accurately conveys to the student what is important (Garfield, 1995).

Students' interpretation of data sets was supported in the current study by the GICS (**G**lobal-**I**ndividual-measures of **C**entre-measures of **S**pread) framework. The GICS framework was developed in response to statistics education research that found that middle-high school students perceive data as a collection of individual points rather than as an aggregate (Chance, delMas, & Garfield, 2004). This framework obliges students to examine the information presented from four perspectives – Global, Individual data points, measures of Centre, and measures of Spread – as an interpretation step *before* drawing any conclusions. This process offers a three-fold benefit: it encourages reflection about the data, it develops a culture of enquiry and statistical habits of mind, and it provides a structured multi-faceted foundation for higher level analysis. Classroom discussions are reported in the literature but the dialogue is, often quite deliberately, unstructured. The template used in this study – a single sheet of paper with the four headings – provides a simple framework that is transferable, and the acronym GICS is easily remembered. The iterative nature of this process is designed to reduce the cognitive load on the students.

Students' understanding and use of outliers is not well represented in the current statistics education research literature. Groth (2006) and Ben-Zvi (2000) consider outliers in the relation to context of a statistical problem, and how placing the data set in context was a feature that distinguishes statistics from mathematics. Konold and Pollatsek (2002) argued that to exclude outliers requires an implicit model of the data aggregate; to develop an implicit model implies students must also develop a critical or intuitive sense of the data aggregate. This is consistent with research recommendations that students use authentic data sets (Watson, 2006). In this study outliers are considered to be questionable, rather then extreme, values.

Students' understanding and use of measures of centre to compare two distributions has been examined in the literature (e.g., Watson & Moritz, 1999). Konold and Pollatsek (2002) introduced the concept of average as signal within a "noisy" data set. Gal (cited in Watson & Moritz, 1999) demonstrated that students at Year 9 level were familiar with both the concept and the algorithmic processes to calculate the mean. All three studies reported surprise that the mean was not widely used to compare data sets. Watson and Moritz suggested this may have been a direct consequence of traditional statistics teaching's emphasis on the algorithm to calculate mean, rather than on the development of a deep understanding of the concept of mean.

Method

The sample was a Year 9 class in a metropolitan co-educational high school in Hobart. The classroom component of the research study was taught by the first author as a two-week teaching unit using "best practice" principles identified by statistics education research. These principles emphasise the development of statistical habits of mind through active learning, whole-class discussion and appropriate technology that allows students to explore data sets. The software, *Fathom*™, a product of Key Curriculum Press (Finzer, 2005), was introduced and used throughout the program.

The group was defined as an extended mathematics class, but the colleague teacher believed the group was of mixed ability as students had self-selected to enrol in the course. Of the 29 students enrolled, 8 were female and 21 were male, and the students averaged 14 years old. Not all students completed all the tasks presented here. Students were assigned an identification code based on their birth-date and their initials. Of the 15 tasks examining the statistical concept of distribution assigned to the students, three are presented here.

Task 1: Students' Heights – Introduction to the GICS Framework

The task was students' first exposure to the use of the GICS framework. The task was highly scaffolded and it was conducted in a traditional classroom environment. Data were provided by the students as they had recently measured their height as part of data collection for the *CensusAtSchool* program (Australian Bureau of Statistics, 2006). A graph of students' height was displayed as a *Fathom*™ graph projected as an image onto the whiteboard. Students were provided with a GICS template sheet with a graph of the data (Figure 1) and the four headings of Global, Individual data points, Measures of Centre and Measures of Spread. An extended teacher-led class discussion examined the graph of the data. As students identified an aspect of the distribution e.g., "…most students had a height of 170 cm…" the observation was recorded under the appropriate heading; in this instance, as a Measure of Centre.

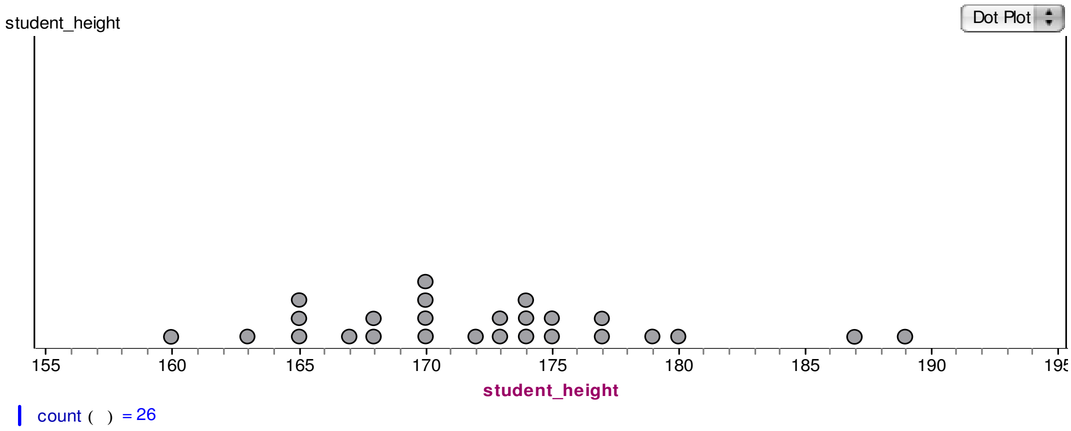


Figure 1. Students' heights.

Task 2: Weighing a Small Mass – Students' Understanding of Outliers

This task was taken from the Statistical Reasoning Assessment (SRA) (Garfield, 2003). Students are asked to consider whether to include, or exclude, an outlier when calculating the mean. The SRA was designed for undergraduate students, but this item is suitable for high school students. The task was given as part of a pre-test and consequently represents students' understanding of outliers before the teaching unit conducted as part of the research study.

> A small object was weighed on the same scale separately by nine students. The mass (in grams) recorded by each student is shown below:
>
> 3.2, 3.0, 3.0, 8.3, 3.1, 3.3, 3.2, 3.15, 3.2
>
> The students want to determine as accurately as they can the actual mass of this object. Of the following methods what would you recommend they use?
>
> a. use the most common number, which is 3.2 grams
> b. use 3.15 because it is the most accurate weighing
> c. add up all the numbers and divide by 9
> d. throw out the 8.3 , add up the other 8 numbers and divide by 8

Task 3: Reaction Times – Comparing Two Distributions

The third, and culminating, task assessed students' development in the use of the GICS framework (Task 1) and an awareness of outliers (Task 2) to compare two distributions. Students compared two distributions to determine whether male or female students had faster reaction times. Students' reaction times were measured by the time taken to respond – by clicking a computer mouse – to the sudden appearance of an image on a computer screen. The data were obtained from the *CensusAtSchool* program web-site (Australian Bureau of Statistics, 2006). The students were familiar with both the data and the method of collection as they had performed the Reaction time test several weeks prior to the research study. Scaffolding for the task was provided by a *Fathom*™ file containing a dot plot of the data and a set of prompting questions. The task was conducted under traditional examination conditions in a computer laboratory using *Fathom*™.

Students needed to complete a sequence of sub-tasks to produce a meaningful analysis for Task 3. Firstly, students were asked to set a filter to accommodate outliers, and to justify setting the filter; secondly, students chose an appropriate scale to display the data effectively; thirdly, students examined the two distributions using the GICS framework; and finally students compared the two distributions using a variety of informal, and formal, statistical measures. The use of the GICS framework provided a structure for the analysis. Shifting the emphasis from analysis to decision making was designed to demonstrate an application beyond the statistics classroom.

The evaluation of students' responses was informed by the SOLO taxonomy (Biggs & Collis, 1982) and the statistical appropriateness of the response. The SOLO taxonomy has been used extensively in the statistics education literature (e.g., Watson & Moritz, 1999) as a means of evaluating students' responses in statistics education by examining how the elements of a task are used and integrated. In this study a simplified three-tiered structure – unistructural, multistructural and relational – was used to code students' responses. A unistructural response employs only one element in the task and does not identify any contradictions; a multistructural response uses at least two elements, often in sequence and identifies but does not resolve any contradictions; and a relational response is distinguished by the effective integration of many elements and resolution of any contradictions to complete the task (Watson, 2006).

Results

Task 1: Students' Heights – Introduction to the GICS Framework

In the context of this task with a high degree of scaffolding it was expected that students would describe several features of the data set as shown in Figure 1. Unistructural responses allowed for several specific and unrelated comments to be made. Multistructural responses added a sequential aspect, whereas relational responses were considered to integrate the information and draw out implications not specifically represented in the graph.

Table 1
SOLO Evaluation of Students' Responses to Task 1

SOLO level	No. of students	%	Criteria
U	4	20%	Provides a limited and incomplete description; does not demonstrate a deep understanding of statistical measures used.
M	10	50%	Uses a variety of statistical measures within the GICS framework to describe the data, but the description is incomplete or repetitive.
R	6	30%	Comprehensively describes the data by selecting and combining all relevant statistical measures within the GICS framework.
Total	20	100%	

Student G2203A provided a unistructural response presenting the information as a series of disconnected facts, as shown by the description of mean and median. The student neglected to provide a global view of the data, and only used the range to describe the spread of the distribution. The student recognised the value of graphical representation. Statistics are quoted to an inappropriate three decimal places.

> From using fathom a lot of data becomes visible. The tallest person is 189 cm and the shortest 160 cm. the graph uses centimeter units. The mode height is 170 cm the median height is 172.5 cm and the mean height is 172.308.

Student K2504B's multistructural response considers the maximum and minimum values, and the range. Two measures of spread are used, but the student does not explicitly consider the spread in relation to a measure of centre such as the mean.

> The tallest height in our class is 189 cm and the shortest is 160 cm. this means that the range of heights is 29 cm. For the measure of centre there is the median : which is 172.5 cm (and) mean: which is 172.308. There is 22 people between 165-180 and there is 12 people between 170-175.

Student Y2206D provided a relational response. The student clearly grasped the essence of the data set by including the student's own height in relation to the data aggregate, gave a global view, considered the extreme values in relation to the main body of the data, and used measures of centre and spread appropriately. The standard of written expression was also very good.

> This graph shows the height of our Maths class. In our class the range is 160 cm (the shortest person) – 189 cm (the tallest). My height is 175 cm and the average height is 172.3 cm. So I am over the average height. The mean is 172.5 cm and if we were to go 5 cm either side of that there would be 16 students heights, mine included. Therefore 88% of the class is 5cm above or below the mean. Only 4 students are shorter than 165 cm or taller than 180 cm.

Task 2: Weighing a Small Mass – Students' Understanding of Outliers

Of the 25 students responding to this task 14 preferred to include the outlier (Task 2, response (c)) when calculating the mean. Two students selected the mode (response (a)), and 9 selected the preferred solution of excluding the outlier (response (d)).

The belief that all data should be included in calculation was fiercely defended by several students in a lively whole-class discussion reviewing the test question. As one student said:

> But if you don't use all the values you can get the answer you want; it's a bit like cheating.

Task 3: Reaction Times – Comparing Two Distributions

Setting the filter was a critical step in the students' task in comparing two distributions. The actual physical test suggested an appropriate filter setting of approximately one second. Students' responses were categorised into fully confident exclusion of outliers, partial exclusion of outliers, or no exclusion of outliers (Table 2). Students who did not set the filter or left the filter at the default setting were considered to give a unistructural response. None of the students explicitly used their own personal experience of the Reaction time test as a method of determining a legitimate reaction time.

Table 2
SOLO Evaluation Reaction Times

SOLO level	No. of students	%	Exemplars or Criteria
U	6	23 %	Does not set filters or leave filter at default setting
M	16	62 %	Sets filter, uses measures of spread and centre, aware of spread of distribution
R	4	15 %	Sets filter < 2 seconds, uses measures of centre and spread and distribution effectively
Total	26	100 %	

Student L2103S provided a unistructural response in Figure 2. The filter was left at the default value of 20 seconds "…because there were no (higher) results…" suggesting the student did not have either a sense of the data or an understanding of the purpose of using the filter. Using the unfiltered data to calculate the mean, the student concluded that females were faster. Statistics were quoted to the default, and inappropriate, six decimal places. The graph scale was adjusted to a finer scale, but all data were displayed.

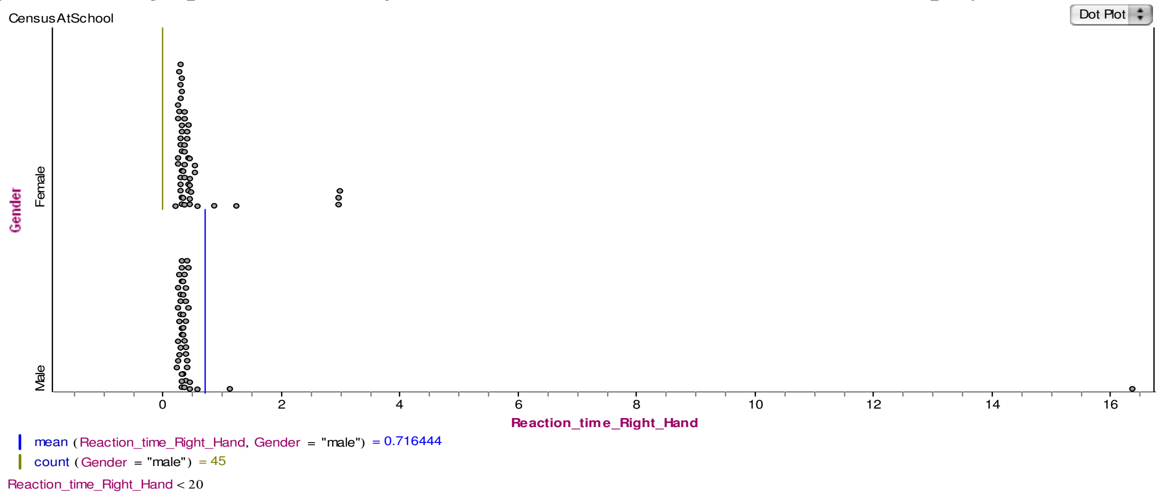


Figure 2. Student L2103S reaction times.

Student N2004E provided a multistructural response in Figure 3. The filter and the range on the graphs were both set at 3 seconds. The filter was set on the basis that only one data point was excluded. The student examined both the mean and the range, and noted that the male reaction times were more consistent than the females. Statistics were quoted appropriately to two decimal places.

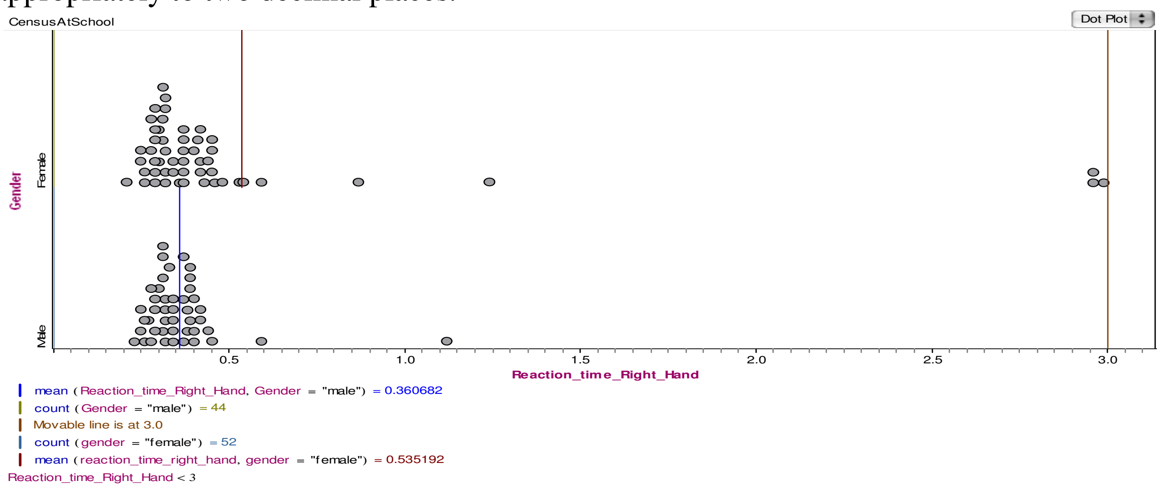


Figure 3. Student N2004E reaction times

Student R2808N provided a rich, relational response (Figure 4) and the filter, set confidently at 0.7 seconds, showed an awareness of an appropriate figure and need to focus on the "…main centres of information…" The student used the GICS framework effectively, describing the distribution using the informal terms of "clumps" and "spread out" and a variety of formal statistics such as range, median and mode were calculated. The student demonstrated a strong sense of the distribution describing the shape as a triangle. Of particular interest were the student's awareness of sample size and the subtle observation that the female distribution had two modes. The information was used appropriately to reach the conclusion "…males are faster, but the times are close…"

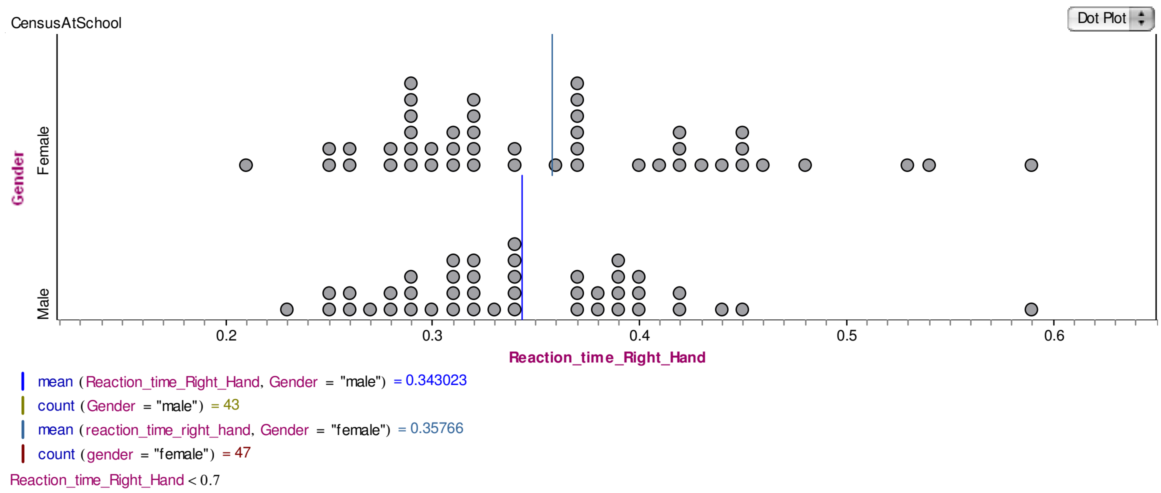


Figure 4. Student R2808N reaction times.

Discussion

The teaching unit that was the foundation of the research study was designed to provide, or refresh, the skills required to complete the complex task of comparing the two distributions. Two of the 15 preliminary tasks in the teaching unit are presented here. These tasks were selected as pre-requites for students to complete the culminating "Reaction time" task; and the three tasks presented here collectively allowed researchers to evaluate individual students' understanding and development. Consistent with statistical education "best practice" described in the theoretical background, the tasks were not designed to evaluate students' computational skills or procedural competence, but to assess students' understanding of the statistical concepts under examination.

Use of the GICS Framework

Students used GICS extensively in the first, highly supported task. Students had little difficulty categorising features of the graph as global, individual, measures of centre, or measures of spread. As a research instrument the value of this task lay in identifying what students selected for inclusion in their written analysis when all the information had been discussed, and notes taken, in the classroom.

Despite prompting, the GICS framework was less well utilised in the final task. There was a sense within the student group that the true objective of the task was the final conclusion, rather than articulating the process of analysis. This could be addressed by providing students with an assessment rubric that emphasised the value of interpretation of the data sets. It could also be argued that students' desire to reach a conclusion is also, to a degree, a product of their experiences of traditional teaching with its emphasis on a "correct" answer rather than thoughtful analysis.

Within the GICS framework, designed to assist "telling the story" of the data, an important aspect of representing the data was how students, in Task 3, modified the graph provided to show appropriate spread (the S in GICS). Many students failed to use scales effectively to display the data; for example, failure to spread out the two sets made visual comparison of the two data sets difficult.

Students' Understanding of Outliers

Students' development of understanding of outliers may be observed by comparing students' responses to the "Weighing a small mass", conducted as part of the pre-test, and responses to the "Reaction times" task, conducted as the final assessment task. Of the 24 students who completed both Tasks 2 and 3, 38% eliminated the outlier in Task 2, whereas 68% did so in Task 3.

Setting the filter, to exclude outliers and include only legitimate data, was a critical step in the analysis of the "Reaction time" data set. None of the students explicitly stated the use of their own personal experiences of the Reaction time test as a means of identifying a legitimate reaction time. Students' interpretation of outliers lay on a continuum of not excluding any data points, excluding only one, or a few, to setting a filter appropriately at a time of one second. To a degree this reflected a student's own confidence. Many students considered an outlier as one, or a few data points, rather than considering what data should legitimately be included in the analysis. In an earlier classroom discussion students were generally reluctant to exclude any data, on the basis that information could be manipulated to achieve any desired result. Two students noted eliminating outliers affected the mean. Several students confused changing the scale with using a filter to remove outliers.

"Messy" data with outliers encourage students to examine critically the raw data. This should not be seen exclusively as a preliminary step, but as an integral part of the analysis process. If students, according to Gal (cited in Watson & Moritz, 1999), must develop an intuitive model of the data aggregate before excluding outliers, it could be argued that failure to do so may indicate that the student has not cultivated that intuitive sense.

Comparing Two Distributions

Students' use of mean and median to compare two distributions in this study was significantly more extensive than that found by Gal (cited in Watson & Moritz, 1999). Two significant differences exist between the two studies: Gal worked with Year 7 students – 2 years junior to this study group – and in this study the mean was provided *Fathom*™ so students did not need to consider both the effort and the value of calculating the statistic.

The responses conveyed a sense that students felt they were expected to give a definitive answer. Students concluded there *was a difference* in the male and female reaction time, but such a conclusion could not be justified by more rigorous statistical analysis. Students' tendency to provide a definitive response may also be a product of traditional statistics teaching.

Students used the difference in the mean of the two distributions as the principal method of comparing the distributions, but it was not used effectively. No student considered whether the *difference* in the means was significant; for example, by calculating the difference as a percentage of the reaction times. This calculation was well within the ability of many students at this level, but the technique had not been introduced in the classroom and they did not use this technique independently. The calculations would also provide a foundation for the development of standard deviation in more senior years. It may also encourage the sense of what is a *meaningful* difference, a concept arguably more important that what is a significant *statistical* difference.

Conclusion

All three tasks were designed to encourage "sense-making" and the development of intuitions. Scenarios – such as the "Students' height" and "Weighing a small mass" – and the use of *Fathom*™ to assist in the calculation of statistics, potentially encourage "sense-making" as students are largely freed of the mechanics of data processing. The tasks collectively provided opportunities to demonstrate all five recommendations of "best-practice" identified in the theoretical background. The GICS framework and the consideration of whether to include, or exclude, particular values (outliers) may also encourage sense-making. When comparing two distributions, calculating the difference as a percentage of the means – a task within the ability of Year 9 students – may provide a foundation for the development of the concept of standard deviation.

Acknowledgement. This research was conducted by the first author as part of a Bachelor of Teaching (Honours) program under the supervision of the second author.

References

Australian Bureau of Statistics. (2006). *Sample reaction times: Australian CensusAtSchool data set*. Retrieved June 12, 2006, from http://www.abs.gov.au/websitedbs/cashome.nsf/home/data%20samples

Bakker, A., & Gravemeijer, K.P.E. (2004). Learning to reason about distribution. In D. Ben-Zvi & J. Garfield (Eds.), *The challenge of developing statistical literacy, reasoning and thinking,* (pp.146-168). Dordrecht, The Netherlands: Kluwer.

Ben-Zvi, D. (2000). Towards understanding the role of technological tools in statistical learning. *Mathematical Thinking and Learning*, *2*(1&2), 127-155.

Biggs, J. B., & Collis, K. F. (1982). *Evaluating the quality of learning. The SOLO taxonomy (Structure of the Observed Learning Outcome)*. New York: Academic Press.

Chance, B. L. (2002). Components of statistical thinking and implications for instruction and assessment. *Journal of Statistics Education*, *10*(3). Retrieved June 10, 2006, from http://www.amstat.org/publications/jse/v10n3/chance.html

Chance, B., delMas, R., & Garfield, J. (2004). Reasoning about sampling distributions. In D. Ben-Zvi & J. Garfield (Eds.), *The challenge of developing statistical literacy, reasoning and thinking* (pp. 295-324). Dordrecht, The Netherlands: Kluwer.

Finzer, W. (2005). *Fathom™: Dynamic data software*. Emeryville, CA: Key Curriculum Press.

Franklin, C. A., & Garfield, J. B. (2006). The Gaise project – Developing statistics education guidelines for grades pre-K-12 and college courses. In G. F. Burrill & P. C. Elliott (Eds.), *Thinking and reasoning with data and chance*: *Sixty-eighth yearbook* (pp. 345-376). Reston, VA: NCTM.

Garfield, J., & Ben-Zvi, D. (2004). Research on statistical literacy, reasoning, and thinking: issues, challenges, and implications. In D. Ben-Zvi & J. Garfield (Eds.), *The challenge of developing statistical literacy, reasoning and thinking* (pp. 397-410). Dordrecht, The Netherlands: Kluwer.

Garfield, J. (1995). How students learn statistics. *International Statistical Review*, *63*(1), 25-34.

Garfield, J. (2003). Assessing statistical reasoning. *Statistics Education Research Journal*, *2*(1), 22-38, Retrieved June 12, 2006, from http://fehps.une.edu.au/serj

Groth R. E. (2006). Engaging students in authentic data analysis. In G. F. Burrill & P. C. Elliott (Eds.), *Thinking and reasoning with data chance*: *Sixty-eighth yearbook* (pp. 41-48). Reston, VA: NCTM.

Konold, C., & Pollatsek, A. (2002). Data analysis as the search for signals in noisy processes. *Journal for Research in Mathematics Education, 33*, 259-289.

Mokros, J., & Russell, S.J. (1995). Children's concept of average and representativeness. *Journal for Research in Mathematics Education, 26*, 20-39.

Shaughnessy, J.M. (2006). Research on students' understanding of some big concepts in statistics. In G. F. Burrill & P. C. Elliott (Eds.), *Thinking and reasoning with data and chance*: *Sixty-eighth yearbook* (pp. 77-98). Reston, VA: NCTM.

Watson, J. M. (2006). *Statistical literacy at school: Growth and goals*. Mahwah, NJ: Lawrence Erlbaum.

Watson, J. M., & Moritz, J. B. (1999). The beginning of statistical inference: Comparing two data sets. *Educational Studies in Mathematics*, *37*, 145-168.

Teacher Researchers Questioning their Practice

Linda Bonne
Redwood School, Wellington
<lindabonne@redwood.school.nz>

Ruth Pritchard
Victoria University of Wellington
<ruth.pritchard@vuw.ac.nz>

Eight teacher researchers examined their own practice to analyse their use of questioning in the context of numeracy, in partnership with two researchers. Each teacher researcher devised their own question categories, from which the research team then developed common categories. Teacher researchers found the most helpful way to categorise questions was according to their purposes for asking them, and that only the teacher could reliably determine this. Dichotomies such as open/closed questions, or lower/higher order questions, did not appear to illuminate the complexity that underpins questioning. The teacher researchers discovered that they had asked more questions than they expected, and were surprised that they asked more questions of students working at higher strategy stages. The importance of context was highlighted as the teacher researchers described the many inter-related factors they considered as they formulated questions and presented questions to students.

Discourse is an important aspect of mathematics classrooms that encourages student inquiry and explanation of solution methods (Cobb, 1994; McClain & Cobb, 2001). Fraivillig, Murphy, and Fuson (1999) highlight the importance of the teacher's role in intervening to advance children's thinking in mathematics. Their framework points to the importance of questions in eliciting, supporting and extending thinking.

Teachers spend much of their time asking questions, reportedly one to two every minute (Gall, 1971; Wragg & Brown, 2001). A number of texts and professional development programmes for teachers in questioning have presented improvement in questioning practices as a technical matter which takes practice: "... good questioning is both a methodology and an art; there are certain rules to follow …" (Ornstein & Lasley, 2000, p. 184). However, it has also been argued that while furnishing teachers with a list of possible questions may give them a starting point, the most effective questions cannot be pre-planned, and must occur in response to a student's action or idea (Jacobs & Ambrose, 2003).

Many writers have suggested that higher-level questions produce deeper levels of learning (Gall, 1984; Marzano, Pickering, & Pollock, 2001; Redfield & Rousseau, 1981). A number of studies (Gall, 1984; Perrot, 1982/2002; Perry, VanderStoep, & Yu, 1993; Stigler & Hiebert, 1999; Wragg, 1993) have highlighted the low proportion of high-level questions to low-level ones when questions are categorised according to taxonomies such as those devised by Bloom (1956). However, Kawanaka and Stigler (1999) found that higher-order teacher questions did not necessarily promote higher-order responses by students.

Several writers have described how patterns of questioning develop within the classroom context (Wood, 1998; van Zee & Minstrell, 1997). Much classroom discourse is thought to be characterized by a pattern of Initiate, Respond/Reply, Evaluation/Feedback (Cazden, 1988; Mehan, 1979) where the teacher initiates, a student responds, then the teacher gives the student evaluative feedback. This pattern places the teacher in a central role and acts to test a student's knowledge, rather than to encourage them to elaborate on their ideas or to extend their thinking. International comparative studies, such as *The Third*

Proceedings of the 30th annual conference of the Mathematics Education Research Group of Australasia
J. Watson & K. Beswick (Eds), © MERGA Inc. 2007

International Mathematics and Science Study (TIMSS) (Stigler & Hiebert, 1999) have suggested that cultural differences exist in pedagogical practices such as questioning.

Much of the recent focus in New Zealand education has been on effective pedagogy (Alton-Lee, 2003; Anthony & Walshaw, 2007; Hattie, 2003; Ministry of Education, 2006a). The synthesis of research by Alton-Lee (2003) described questions and prompts as elements of "quality teaching", forming an important aspect of pedagogy which supports students' task engagement (p. 74), and serving to "provide scaffolds to facilitate student learning" (p. ix). In professional development programmes such as the New Zealand Numeracy Development Projects (NZNDP, Ministry of Education, 2006b), teachers have been encouraged to use questioning to support students' strategic and higher order thinking. Within the New Zealand context of mathematics teaching and learning, research has explored various components of discourse (Thomas, 1994; Higgins, 2003; Irwin & Woodward, 2005).

Up until now, much of the research undertaken to investigate teachers' questioning has been synthesised from data gathered by researchers observing in classrooms. A review of comprehensive research syntheses (Houston, Haberman, & Sikula, 1990; Richardson, 2001; Sikula, Buttery, & Guyton, 1996; Wittrock, 1986) did not reveal any studies deeply grounded in teachers' perspectives. How teachers view the role and formulation of questions within a mathematics lesson, and how questioning might be shaped by contextual factors, have not been a major focus. Furthermore, existing categorisations of teachers' questions have predominantly examined only a selection of the questions asked by teachers during a lesson (Perry, VanderStoep, and Yu, 1993; Vale, 2003).

Methodology

The project had two closely interwoven strands: one strand focused on teachers examining their use of questioning, and the second strand focused on building research capability of teachers. The key objectives that focused on the teachers' use of questioning were to:

- identify the various kinds of questions teachers use in mathematics
- explicate teachers' thinking about the use of questioning during lessons
- describe patterns of teachers' questioning within mathematics lessons

The teacher researchers (TRs) taught at a variety of year levels, and were drawn from urban schools in communities with varied socio-economic backgrounds. Each of them had recently participated in a common in-depth professional development programme: the NZNDP (Ministry of Education, 2006b). The eight TRs were respected members of their teaching communities; several were lead teachers of numeracy in their schools. They had also demonstrated a willingness to share and examine their practices. The research was conducted over the 2006 school year, in five primary schools in the Wellington area.

There were two cycles of data gathering for the TRs, each taking 5 days and occurring in each of the middle two terms of the four-term school year. TRs were released for two days to analyse a transcript of their numeracy lesson, their recollection of which was supported by viewing a videotape of the lesson. A key task was for them to identify every teacher question included, and to sort these into groups of similar questions for which they then devised labels (Miller, Wiley, & Wolfe, 1986). At the end of the second day, they discussed their findings with one of the RTLs in a semi-structured, one-to-one interview

(Denscombe, 1999). In the second cycle, questions were categorised under commonly agreed headings, and TRs also completed a frequency table based on the categories.

Research team discussions formed a key aspect of the analysis and interpretation of findings. Each member of the team brought aspects of their findings to share, and similarities and differences were explored and debated. The Cycle 1 team discussion began the process of establishing common categories with which to analyse the lesson in Cycle 2. The TRs interpreted their findings in light of current research, which they discussed at a team meeting. Also at these meetings, TRs responded to summaries of emerging ideas presented by the RTLs.

Results

Development of Question Categories

The research team devised a working definition of what constitutes a question. For this project, a question was "any form of language that is aimed at eliciting a response". This is perhaps a broader definition than that found in *The Concise Oxford Dictionary* (Allen, 1990), which defines a question as "… a sentence worded or expressed so as to seek information", or "… a problem requiring an answer or solution" (p.980). Utterances such as, "Listen carefully to what Lily is saying" and, "Let's see if we can understand how the mirror, how their hands coming together helped" (Erin, Lesson transcript 2), were counted as questions. Although the definition included "any form of language" the methodology of the project allowed for a focus only on oral questions.

In the first cycle of data gathering and analysis, the TRs worked independently to devise between six and 17 categories for their questions, with three people each devising eight categories. The research team met at the end of this cycle, with the main purpose of developing shared question categories from the TRs' individual ones. This proved to be a complex task that could not be completed with sufficient discussion and debate within the time available. The seven TRs who were at the meeting had varying degrees of input into this process.

Following this meeting, the RTLs met with three of the TRs to further refine/develop the categories. These were subsequently presented at the next team meeting for discussion and feedback. At this point, seven categories of question had been developed, based on the TRs examining a question in terms of the purpose they had in mind when they asked it. The TRs used these seven common category labels when they analysed their second lesson. (Question examples are drawn from TRs' categorised questions.)

Checking understanding

- Okay, but say again, you took the 3 away first you said and then you took away…?
- Do you understand that, David?

Getting a sharp, clear, anticipated response

- Good boy, so that equals…?
- Is there a 3 in the hundreds?

Guiding and supporting (clarifying, repeating, rephrasing, taking another look)

- Excellent, so you would take away the 6 and 3 because you know they actually make 9?
- So you said that you would have 24 and then you would…?

Explaining how and why

- Why is using different colours helpful, do you think?
- How did that make it easy for you?

Making connections and links

- What is the relationship between 4 and 8?
- Is it a "-ty"? Where are some other "-ty" numbers?

Management

- Who is your partner, Victoria?
- Joseph, do you want to roll the dice?

Fostering student interaction

- So what's the number sentence, give me thumbs up if you agree with Trent.
- Ana, why are you shaking your head; do you disagree?

At the second post-analysis meeting of the research team, the TRs further condensed this list by removing the category, "Getting a sharp, clear, anticipated response", which had been categorised according to the students' responses, rather than the teachers' purposes for asking the questions. "Management" and "Fostering student interaction" were merged, as it was agreed that questions in both categories had a strong connection with classroom norms. Consequently, these two categories were combined under the label, "Fostering student interaction in a learning community". By the conclusion of the project, the team had therefore reduced the number of categories to five. For one TR the process of developing common categories meant that their original 17 categories reduced to just five categories by the end of the project.

In the early stages of the research, the TRs often referred to questions as open or closed (25 references in first interviews). Later in the project the TRs reported that their thinking about questions had moved beyond this straightforward dichotomous categorisation. Open and closed questions were referred to less often (11 references in second interviews), and the complexities of these ideas were explored. The TRs suggested that in each of the final categories, there would be examples of questions that might be considered to be open and closed.

Context shaped the TRs' categorisation of their questions. The importance of uncovering teachers' purpose in such research is supported by Erickson (1993): "The teacher comes to know teaching from within the action of it, and a fundamentally important aspect of that action is the teacher's own intentionality" (p. viii). The TRs reported that the actual purpose of a particular question could not be determined by looking at the question in isolation from the context in which it was asked. To identify the purpose of a question, it was necessary to know the conversation that happened before and after the question. Furthermore, even by referring to the full lesson transcripts and viewing the videotapes of lessons, members of the research team felt it was not possible to accurately categorise another person's questions according to purpose. The research team leaders attempted to identify questions that would be illustrative of each category, only to find that they had insufficient information to do so with any degree of reliability. For example, the RTLs thought the question, "How are you going, Jordan, alright?" might have been classified as a Management question. The TR in whose transcript the question appeared considered it fitted best in the "Checking understanding" category, as this was the purpose she had in mind when she posed the question. Similarly, the question, "I have taken away 4. That leaves me with …?" might be perceived by one person to be a "Guiding and supporting" question, but the TR classified it as "Checking understanding". For the questions to be categorised in terms of purpose, rather than form or function, the categorising must be done by the teacher, as only the teacher had the in-depth knowledge of each student's learning needs necessary to identify the specific purpose for which they had asked each question.

Making the categorising of questions still more complex is that questions were asked with varying purposes in mind; similar questions were asked of different students for different purposes, according to the students' needs. For example, the question "So, what do you get if you add three more?" might be asked of one student with the purpose of checking their understanding, while for another student it might be asked in order to guide and support their learning.

Teacher Researchers' Reflections on Questioning

TRs described how they brought together a complex combination of considerations as they formulated questions:

Purpose – What is the purpose of my question? Where am I heading? What is the learning intention? How will I know when the students have achieved it? What will be the next steps?

Student needs – What are the needs of the students – their age, language needs (especially where English is not the student's first language), perceived abilities, established understandings? What do they already know? What pace will best suit them? How attentive are they?

Scaffolding – What will help scaffold their learning in terms of equipment and student interactions? What mathematical language or ideas do I need to include in my question in order to support the students' learning?

Who to ask – To whom will I direct this question – to the whole class or to an individual student, and in this case, which student (for a variety of purposes, e.g., deliberately setting up conflict of ideas, uncovering a suspected misconception, to quickly get the correct answer, or to re-engage a student)?

Timing – When should the question be asked? At what point should the teacher intervene when a student is struggling, for example? How much wait-time should they allow? Is there sufficient time left in the lesson for the discussion this question might elicit?

Predicted responses – What responses do I expect? How am I, in turn, likely to need to respond? What equipment is immediately accessible to support directions in which the discussion might head? (Developed from the Final evaluation meeting)

The TRs talked about how the priorities for formulating questions constantly shifted, depending, for example, on the teacher's stress or tiredness level, or whether other adults were observing the teacher.

Questions were formulated according to students' responses, in the "reflection-in-action" mode (Schön, 1983/2002). The TRs reported difficulty in devising questions when the students did not provide them with responses on which they could readily build:

> …you need the feedback to form your next thought. It's not just one-way communication…you need something to build off, so you need interaction back ... Questions are adapted to the needs of the students in context. (Quentin, Interview 2)

The TRs talked frequently of the need to adapt their questions and be flexible and responsive as a lesson progressed. In a social constructivist classroom, the teacher aims to interact with the students' ideas, rather than be a keeper of knowledge that is handed down to the students (Askew, Brown, Rhodes, Wiliam, & Johnson, 1997). For teachers to yield some of the control to students requires the teacher to have a secure pedagogical content knowledge (Alton-Lee, 2003; Anthony & Walshaw, 2007; Shulman, 1986). But although it may not be possible to predict the exact course a lesson will take, the TRs described the importance of having an endpoint in mind when formulating questions:

> I like to have clear learning intentions and know where I'm going and how I will know that the children have got there, but maybe I'm thinking I need to be a little bit more relaxed about that, so they can take the lesson where they want it to go a little more. … And I think to have less control you have to be more secure in yourself and you also have to be more secure in yourself to guide – not in a pushy way – but to guide as a good teacher. Because it's much easier for us to work out where we want to go and just go our own little way, and do it the way our brains work. (Erin, Interview 2)

Some of the TRs described how the establishing of question categories influenced their practice in the second cycle of data gathering and analysis. Reflection on findings highlighted some potential issues in the TRs' practices, for example, whether teachers might rely too heavily on questions when, sometimes, it might be more helpful to explain something to a student.

> I think I've changed my thinking from the initial questions that we did, because this is focused on those particular headings. It might've been symptomatic of knowing what my headings were, so I kind of tailored it towards those types of questions. … Having categories heightens the teacher's awareness of questions and their purposes. I was really aware of asking questions that 'guided and supported' etc – was able to target particular types of questions. I felt my questioning was more focused – avoided trivial questions. (Quentin, Interview 2)

Patterns of Questioning

Completed frequency tables were intended to provide the project team leaders with quantitative data that could yield valid comparisons. However, it became clear that the unit of a question had been interpreted in more than one way. For example, when identifying her questions, one TR had separated every individual question in her transcript so that: "What's 3 and 3?" and the next utterance, "3 and 3?" (Erin, sorted questions, Cycle 2) were counted as separate questions. Others had counted as one question instances when a question was repeated, so that: "You can do 2 plus 5 equals 7. What would you do if you had to change that into a take away? How can you do 2 plus 5 equals 7 as a take away sentence?" (Ingrid, sorted questions, Cycle 2) were classified as one question.

Seven of the eight TRs completed a frequency table as part of Cycle 2. The total number of questions identified in the second lesson ranged from 171 to 344 (see Figure 1), with a mean of 207 questions. There was no apparent pattern to the total questions asked that related to the age group taught, or to the associated strategy stages taught.

A high rate of questioning was evident in the lesson transcripts. Given a maximum lesson time of one hour, the rate of questioning was somewhere between two and six questions per minute; this is considerably higher than the one to two questions every minute reported in the literature (Gall, 1971; Wragg & Brown, 2001). Several TRs remarked in the first interview that they had been surprised to find they had asked so many questions. While throughout the project the TRs indicated their heightened awareness of the number of questions they had asked, none of the TRs commented that this was an issue until the latter stages, when several TRs showed growing concern over this.

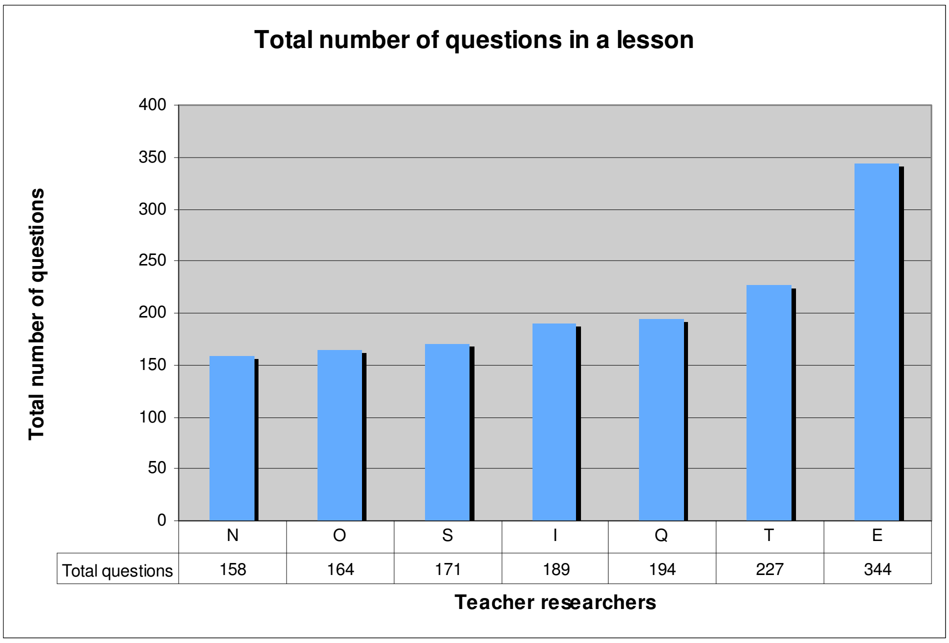


Figure 1. Total number of questions in a lesson – Cycle 2.

The TRs were asked to indicate which of their groups were working at the lower strategy stage and which were at the higher strategy stage. The graph in Figure 2 shows the proportion of the different categories of questions within identified strategy groups. Although there are minor differences between the proportions within each of the categories, the general shape of the graph for each of the groups is very similar. This means that although the number of questions differed for each of the groups, the weighting of the kinds of questions asked was essentially the same. The TRs expressed surprise at this, illustrating the mismatch in teachers' perceptions of their questioning practices, which are often not borne out by research findings (Walsh & Sattes, 2005).

There was a clear difference in the total number of questions the TRs asked the students in their lower strategy stage groups of students compared to those in the higher strategy stages (see Figure 2). A total of 298 questions were asked in six TRs' lessons with students in the lower stage groups, compared to 439 questions asked of their higher strategy stage students – close to 50% more questions.

Possible reasons for the differing numbers of questions for the two groups were offered by the TRs. It was suggested that students in the lower strategy stage groups were more likely to illustrate their strategies with materials, making it unnecessary for the teacher to question them about their thinking. Another suggestion was that teachers would see the higher groups less frequently, so perhaps their session times were of extended duration. Further ideas were: perhaps teachers expected less from this group, expected that "the higher group was going somewhere" and teachers were more active in pursuing this; the less able group tended to be less verbal, so teachers had less to work with; they took longer to work through tasks and wait time needed to be longer. For the higher group, the strategies were more complex, so more guidance was required. All of these conjectures warrant further investigation.

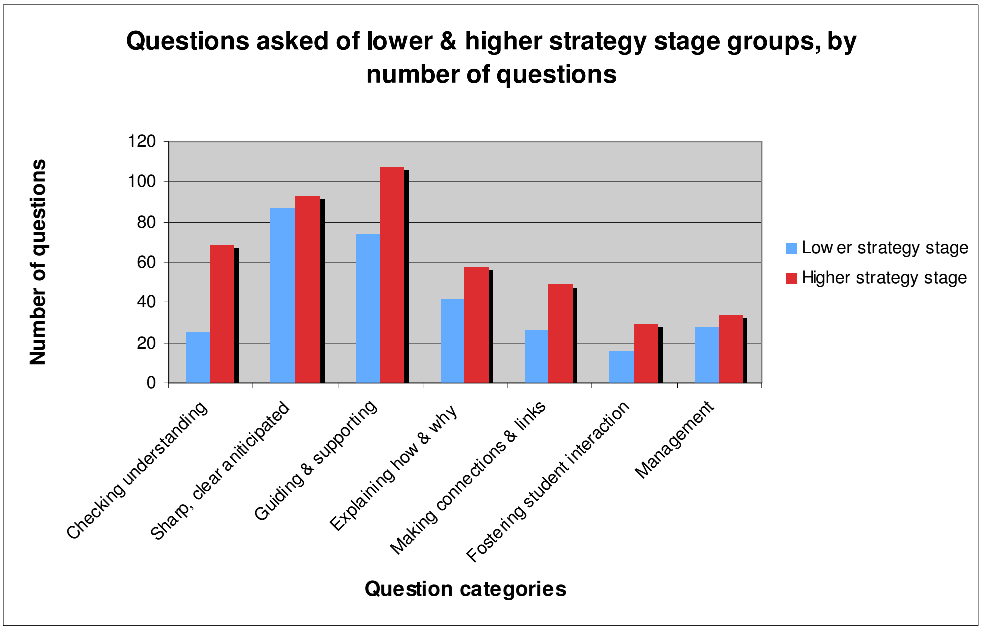


Figure 2. Questions asked of lower and higher strategy stage groups, by number of questions

The TRs were asked to describe any patterns of questioning that they used during a mathematics lesson. The frequency tables helped them to identify the numbers of each category of questions that they asked during different stages of their lesson, and the TRs referred to this data in order to identify patterns in their questioning. However, from the variety of descriptions given by the TRs, no obvious single pattern of questioning over a lesson emerged.

Conclusions

In this project the TRs categorised every question asked in their numeracy lessons. Participants discovered the most useful way to categorise their questions was to reflect on the purpose for which they were asked. This could only be reliably done with the teacher's contextual knowledge, thus it appears that the observation and classification of questions by an outside observer is an unreliable method to uncover the purpose of a teacher's questions. Categorising a question as open or closed, or as lower or higher order, did not prove helpful, as these categories were too broad, and disguised the complexity of teacher questioning. The refined set of categories gave the TRs a common language for discussing the role of questioning in their practice, and for some, helped to sharpen the focus on their purposes for questioning.

Much of the research examining questioning in classrooms has highlighted the high number of questions within a lesson as an issue. The TRs in this study identified at least 158 questions in their hour-long mathematics session and seemed initially to equate the high rate of questioning with effective practice. Also of interest was that the TRs asked close to 50% more questions of students operating at more advanced strategy stages.

Further research is needed to establish:

- the significance – if any – of the number of questions asked;
- the interrelationships between the types of questions used;
- patterns of questions within a lesson;

- relationships between teachers' questions and students' learning.

The unique perspectives of these TRs about questioning provide a valuable contribution to the knowledge base about teaching in this area. The TRs identified many diverse factors that can influence teachers as they formulate and present their questions to students during a numeracy lesson. Their detailed examination of the thinking that underlies the formulation of questions enabled the TRs to examine their metacognitive processes, highlighting some of the intricacies of questioning.

The research team concluded that all question types are important in a lesson; no hierarchy of question types was evident. While there were no common patterns of questioning over a lesson identified during this research, it was clear that the TRs believed it was the combinations of different categories of questions, rather than individual questions, that were powerful in shaping students' learning.

Acknowledgements. The project was reliant on our team of teacher researchers working collaboratively with us and with each other. We are grateful to them for their professional approach throughout the project and to their schools for supporting their participation. This research project was funded by the Teaching and Learning Research Initiative at The New Zealand Council for Educational Research.

References

Allen, R. E. (1990). *The concise Oxford dictionary*. Oxford: Clarendon.

Alton-Lee, A. (2003). *Quality teaching for diverse students in schooling: Best evidence synthesis.* Wellington, NZ: Ministry of Education.

Anthony, G., & Walshaw, M. (2007). *Effective pedagogy in Mathematics/Pangarau: Best evidence synthesis iteration [BES].* Wellington: Ministry of Education.

Askew, M., Brown, M., Rhodes, V., Wiliam, D., & Johnson, D. (1997). *Effective teachers of numeracy in primary schools: Teachers' beliefs, practices and pupils' learning*. London: King's College.

Bloom, B.S. (Ed.). (1956). *Taxonomy of educational objectives: Handbook 1: Cognitive domain.* London: Longman.

Cazden, C. (1988). *Classroom discourse*. Portsmouth, NH: Heinemann.

Cobb, P. (1994). Where is the mind? Constructivist and sociocultural perspectives on mathematical development. *Educational Researcher, 23*, 13-20.

Denscombe, M. (1999). *The good research guide*. Buckingham: Open University Press.

Erickson, F. (1993). Foreword. In M. Cochran-Smith & S.L. Lytle. (Eds.), *Inside outside*. New York: Teachers' College Press.

Fraivillig, J. L., Murphy, L. A., & Fuson, K. C. (1999). Advancing children's mathematical thinking. In J. Sowder & B. Schappelle (Eds.), *Lessons learned from research* (pp.37-38). Reston, VA: National Council of Teachers of Mathematics.

Gall, M. (1971). The use of questioning in teaching. *Review of Educational Research, 40*, 707-721.

Gall, M. (1984). Synthesis of research of teachers' questioning. *Educational Leadership, 42*(3), 40-47.

Hattie, J. (2003, October). *Teachers make a difference: What is the research evidence?* Australian Council for Educational Research Annual Conference: Building Teacher Quality, Melbourne.

Higgins, J. (2003). *An evaluation of the Advanced Numeracy Project 2002.* Wellington: Ministry of Education.

Houston, W. R., Haberman, M., & Sikula, J. (Eds.). (1990). *Handbook of research on teacher education*. New York, Macmillan.

Irwin, K. C., & Woodward, J. (2005). Advancing Pasifika students' mathematical thinking. In F. Ell, J. Higgins, G. Thomas, T. Trinick, & J. Young-Loveridge (Eds.), *Findings from the New Zealand Numeracy Development Projects 2005*, (pp. 80-90). Wellington: Learning Media.

Jacobs, V. R., & Ambrose, R. C. (2003). Individual interviews as a window into teachers' practice: A framework for understanding teacher-student interactions during mathematical problem solving. Proceedings of the 27th International Conference of the International Group for the Psychology of

Mathematics Education, July, Vol. 3, pp.33-38, Honolulu. Nottingham, UK: University of Nottingham.

Kawanaka, T., & Stigler, J. W. (1999). Teachers' use of questions in eighth-grade mathematics classrooms in Germany, Japan, and the United States. *Mathematical Thinking and Learning, 1*(4), 255-278.

Marzano, R. J., Pickering, D. J., & Pollock, J. E. (2001). *Classroom instruction that works: Research-based strategies for increasing student achievement*. Alexandria, VA: Association for Supervision and Curriculum Development.

McClain, K., & Cobb, P. (2001). An analysis of development of sociomathematical norms in one first-grade classroom. *Journal for Research in Mathematics Education, 32*(3), 236-266.

Mehan, H. (1979). *Learning lessons*. Cambridge, MA: Harvard University Press.

Miller, D.M., Wiley, D. E., & Wolfe, R. G. (1986). Categorization methodology: An approach to the collection and analysis of certain classes of qualitative information. *Multivariate Behavioural Research, 21*(2), 135-167.

Ministry of Education. (2006a). *The New Zealand curriculum: Draft for consultation 2006*. Wellington: Learning Media.

Ministry of Education. (2006b). *New Zealand Numeracy Development Projects.* Wellington: Ministry of Education.

Ornstein, A. C., & Lasley, T. J. (2000). *Strategies for effective teaching* (3rd ed.). Boston, USA: McGraw Hill Higher Education.

Perrot, E. (1982/2002). Using questions in classroom discussion. In A. Pollard (Ed.), *Readings for reflective teaching* (pp. 252-255). London/New York: Continuum.

Perry, M., VanderStoep, S. W., & Yu, S. L. (1993) Asking questions in first-grade mathematics classes: Potential influences on mathematical thought. *Journal of Educational Psychology, 85*(1), 31-40.

Redfield, D. L., & Rousseau, E. W. (1981). A meta-analysis of experimental research on teacher questioning behaviour. *Review of Educational Research, 51*, 237-245.

Richardson, V., (Ed.). (2001). *Handbook of research on teaching* (4th ed.). Washington, DC: American Educational Research Association.

Schön, D. (1983/2002). Reflection-in-action. In A. Pollard (Ed.). *Readings for reflective teaching,* (pp. 5-7). London/New York: Continuum.

Shulman, L.S. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher, 15*(2), 4-14.

Sikula, J., Buttery, T. J., & Guyton, E. (Eds.). (1996). *Handbook of research on teacher education*. New York: Macmillan.

Stigler, J. W., & Hiebert, J. (1999). *The teaching gap: Best ideas from the world's teachers for improving education in the classroom*. New York: Free Press.

Thomas, G. (1994). Peer discussion and learning in mathematics. In J. Neyland (Ed.), *Mathematics education: A handbook for teachers. Volume 1* (pp.319-329). Wellington: Wellington College of Education.

Vale, C. (2003). Questions in primary mathematics classrooms. In L. Bragg, C. Campbell, G. Herbert, J. Mousley (Eds.), *Mathematics education research: Innovation, networking, opportunity:* (Proceedings of the annual conference of the Mathematics Education Research Group of Australasia, Geelong, pp. 688-689). Sydney: MERGA.

van Zee, E. H. & Minstrell, J. (1998). Preparing teachers as researchers in courses on methods of teaching science. *Journal of Research in Science Teaching, 35*(7), 791-809.

Walsh J.A., & Sattes, B.D. (2005). *Quality questioning: Research-based practice to engage every learner*. Thousand Oaks, California: Corwin Press.

Wittrock, M. C. (Ed.). (1986). *Handbook of research on teaching* (3rd ed.). New York: Macmillan.

Wood, T. (1998). Alternative patterns of communications in mathematics classes: Funneling or focussing? In H. Steinbring, M. Bartolini Bussi, & A. Sierpinska (Eds.), *Language and communication in the mathematics classroom* (pp. 167-78). Reston, Virginia: NCTM.

Wragg, E.C. (1993). *Primary teaching skills: The report of the research findings of the Leverhulme Primary Project*. London: Routledge.

Wragg, E. C., & Brown, G. (2001). (Rev. ed.). *Questioning in the primary school*. London: Routledge/Falmer.

Imagined Classrooms: Prospective Primary Teachers Visualise their Ideal Mathematics Classroom

Kathy Brady
Flinders University
<kathy.brady@flinders.edu.au>

Research shows that personal experiences have a powerful influence on the views of teaching, learning, and mathematics held by prospective teachers. In this study prospective primary teachers were invited to describe their ideal mathematics classroom in order to explain their views about teaching mathematics. These imagined classrooms provide a valuable insight into their emerging identities as primary mathematics teachers. My analysis of these descriptions addresses the question: What views of the teacher's role, learners and learning, and mathematics are evident in prospective teachers' visualisation of their ideal primary mathematics classroom?

Background

Entering Dispositions

As school children, prospective teachers have already spent long periods of time observing teachers at work. Their beliefs and attitudes about the role of teachers, learning, and curriculum are accumulated and assimilated from the earliest school years. These initial dispositions are subsequently shaped and refined through a variety of formal and informal experiences as prospective teachers prepare to enter the teaching profession (Carter, 1994).

The importance of the relationship between what teachers believe about mathematics and the teaching of mathematics, and the way they actually teach has been well established (Ernest, 1989; Thompson, 1992). Ernest's model for conceptualising teachers' beliefs about mathematics illustrates the importance that is now placed on beliefs and the ways in which they influence the teaching of mathematics. Mathematical beliefs could be defined as "personal judgements about mathematics formulated from experiences in mathematics, including beliefs about the nature of mathematics, learning mathematics, teaching mathematics" (Raymond, 1997, p.551). Artzt (1999) refers to beliefs as the "teachers' integrated system of personalized assumptions regarding the nature of mathematics, of students, and of ways of learning and teaching" (p. 145).

There is a growing body of literature that investigates how prospective teachers make sense of their beliefs (Artzt, 1999; Cooney, Shealy, & Arvold, 1998; Lloyd, 2006a; Mewborn, 1999; Pajares, 1992; Skott, 2001). Brown and Borko (1992) argue that at least some of the prospective teachers' beliefs about mathematics and its teaching are in place before they commence in teacher education programs, suggesting that "they have lenses that dictate, or at least influence, much of what they encounter in teacher education" (p.649). Other studies (Brown & Borko, 1992; Cooney et al., 1998; Raymond, 1997) also indicate that prospective teachers' beliefs about mathematics and how to teach mathematics are influenced in significant ways by their experiences with mathematics and schooling long before they enter the formal world of mathematics education. Although Cooney et al.

(1998) and Raymond (1997) both argue that teacher education programs can only have a limited influence on changing prospective teachers' beliefs, other researchers (Artzt, 1999; Lloyd, 2006b) have investigated ways to change prospective teachers' beliefs about mathematics teaching and learning. Therefore, it is essential to understand not only what prospective teachers believe but also how their beliefs are structured and held for any possibility of developing prospective teachers' beliefs in the teacher education program.

Previous Experiences and Classroom Memories

Previous personal experiences, including experiences as a student in mathematics classrooms, influence the views of teaching, learning and mathematics held by prospective teachers (Brown & Borko, 1992; Carter, 1994; Lloyd, 2006a, 2006b). As prospective teachers have extremely limited, if any, personal experience as teachers, their images of mathematics teaching are based largely on classroom memories. In this respect, prospective teachers have observed and participated in teaching and learning process as students for at least twelve years of their life (Artzt, 1999; Lloyd, 2006b; Pajares, 1992). As Mewborn (1999) observes, when prospective teachers enter the mathematics teacher education courses "they are rich in personal knowledge" (p. 317). However, the stories of previous mathematical experiences that Drake, Spillane, and Hufferd-Ackles (2001, p. 7) describe are unfortunately "dominated by disappointing and discouraging experiences learning mathematics in school. In addition, they all recall losing interest, confidence, or aptitude in mathematics at some time during their elementary or early high school years". It is therefore not surprising that many prospective teachers view mathematics as a closed set of procedures, teaching as telling, and learning as the accumulation of information (Lloyd, 2006a).

Emerging Identities

As the beliefs, attitudes and conceptions of prospective teachers that have been formed by their previous personal experiences as students, and their classroom memories, are shaped and refined through a variety of formal and informal approaches during teacher education courses (Carter, 1994), an emerging identity as a teacher begins to develop. Lloyd (2006b) argues that shifting prospective teachers' perspectives on classroom events from student to teacher is a crucial aspect of teacher education. In addition to developing their emerging identities prospective primary teachers must also confront the issues of teaching and learning that are unique to the teaching and learning of mathematics.

However, identity formation is not a matter of free thinking individuals making rational choices, nor is it about emulating role models (Whitehead, Rossetto, & Lewis, 2005). Rather, identity formation is an ongoing, dynamic process that is open to modification and always occurring in a social context (Britzman, 1986). Other researchers (Lloyd, 2006a; Raymond, 1997) observe that when prospective teachers enter the classroom context, they do not consistently enact their recently developed beliefs about mathematics teaching and learning, as they modify their continually emerging identities.

Storied Identities and Imagined Classrooms

In an effort to create an opportunity for prospective primary teachers to articulate their emerging identities as teachers of mathematics, the prospective teachers in this study were invited to provide a descriptive account of their ideal primary mathematics classroom. The use of stories and narratives are not new in research about the experiences of teachers. Researchers over the last 20 years (Connelly & Clandinin, 1990; Goodson, 2006; Polkinghorne, 1995) have come to appreciate that teachers' stories offer a wealth of information about their individual identities and classroom experiences. Their work builds on the understanding that people live storied lives and share their experiences and identities through stories (Bruner, 1989; Doyle & Carter, 2003; Drake et al., 2001).

For prospective teachers, narrative and biography can be used effectively to understand how previous experiences can paint the portraits of "teacher" that they bring with them into teacher education (Pajares, 1992; Scott, 2005; Sliva & Roddick, 2001, 2002; Wilson & Thornton, 2005). The value of embarking on such an endeavour is corroborated by Doyle and Carter (2003, p.131): "To understand pre-service teachers' development, it is necessary to capture the stories within which this knowledge and understanding are embedded". Rossetto's (2006) research likewise commends the value of visualisation and imagination in the formation of emerging identities in prospective teachers.

Research Significance

In this study, the writing of a descriptive account allowed the prospective teachers to explore classroom situations adopting the identity of a teacher, with the specific intention of encouraging the authors to create images of themselves as a teacher. The importance of research such as this is emphasised by Lloyd (2006b, p. 81): "Teacher educators … may wish to explore ways in which analysis of preservice teachers stories might help to identify preservice teachers' views, to anticipate important aspects of preservice teachers' future development, and to offer opportunities to influence preservice teachers' development in very specific ways". That this research involves prospective *primary* teachers has been identified by Raymond (1997) and Thompson (1992) as an aspect of particular significance, as both note need for further investigations involving prospective primary mathematics teachers.

Research Method

Participants

The participants in this study were 22 prospective primary teachers enrolled in an undergraduate Bachelor of Education (Junior Primary/Primary) or a graduate entry Bachelor of Education (Junior Primary/Primary) at a South Australian university. The undergraduate participants were third-year students and the graduate entry participants were in the first semester of the two year graduate entry program. All students were undertaking the compulsory full year course Curriculum Studies: Mathematics. None of the participants had taken part in any teaching practice experience, or school visits, at the time of the data collection.

Data Collection

At the end of the first 3 weeks of the Semester 1 all students studying this course were required, for assessment purposes, to describe their personal philosophy of teaching primary mathematics, specifically describing their ideal primary mathematics classroom. The written descriptions were between 750 and 1000 words in length. The research participants volunteered to provide their descriptions to the researcher (who was also their workshop teacher) after the assessment process was completed.

The use of assignment work for data collection has advantages from the point of view of expediency and efficiency. It could be argued that the descriptions do not represent genuine beliefs as they may have been constructed to comply with the workshop teacher's point of view (Carter, 1994). However, the framing of the assignment invited the students to construct a personal account that could not be deemed either correct or incorrect thus minimising likelihood of this concern.

Data Analysis

The guiding question for this study is: What views of the teachers' role, learners and learning, and mathematics are evident in prospective teachers' visualisation of their ideal primary mathematics classroom? The analysis was conducted by firstly coding the accounts with regard to the three broad categories that arose from the research question: the views of the role of the teacher, the views of learners and learning and the views of mathematics. Specific sub-categories then became evident as recurring themes were identified within each of these broad categories. The following presentation of the findings of this research, the imagined classrooms of prospective primary mathematics teachers, is organised according to three key components of the research question.

Imagined Classrooms

"I am excited by the prospect of teaching mathematics": The Views of the Role of the Teacher

Almost all of the prospective teachers in this study mentioned the importance of the role of the teacher in the primary mathematics classroom. This finding is consistent with other studies (Lloyd, 2006a; Sliva & Roddick, 2001). In considering the role of the teacher, many of the prospective teachers clearly identified that teachers bring to the role past experiences that may influence their practice.

> A teacher's own experiences and attitudes can affect the way in which they teach mathematics.
> I am aware that there is a possibility that my past experiences could colour the way I teach mathematics.

Some stated they could call upon positive past experiences.

> I would teach in my classroom with the approaches that have made the biggest impact on my learning.
> I could adopt some of the teaching methods which were helpful during my own mathematics education.

However, others were more adamant that their negative experiences of mathematics would not be repeated in their imagined classroom.

> Hopefully I do not use my own negative experience of mathematics to base my teaching.
> In my own classroom I plan to teach maths far differently that I was [taught].

Research by Ball (1990) reveals that teachers are inclined to teach just as they were taught. Some of the prospective teachers seem innately to be aware of how this tendency may impact on the role of the teacher, regardless of their past experiences.

> Teachers often fall back on the way they learnt and use it as a basis for teaching.
> We tend to teach in the way we were taught.
> Many teachers end up teaching in the same way they were taught when they were young.

The role of the teacher unavoidably includes “teaching”. The prospective teachers described a range of orientations to teaching that could be placed on a continuum from traditional direct instruction to teacher-as-facilitator (Sliva & Roddick, 2002). None of the prospective teachers in this study advocated adopting solely a traditional direct teaching approach, although some described how direct teaching may occasionally be part of the teachers’ role.

> There will be times when it is necessary for me to teach information and provide students with answers.
> [Students] need to be provided with direction in their exploration.

Other prospective teachers found the teacher-as-facilitator role more compelling.

> I think it is important to guide students through mathematics, not to get caught in the web of simply telling them how to do it.
> I see my role mainly as a facilitator in the knowledge acquisition of the students.

However, consistent with Whitehead, Rossetto, and Lewis’s research (2005) many of the prospective teachers in this study favoured the understanding that both direct teaching and facilitating would comprise the teachers’ role.

> I would provide a balance between teacher-based instruction and student, peer related tuition.
> I would like to find a balance between instruction and facilitation.

The prospective teachers also described a range of other functions that they considered part of the role of a teacher: knowing the students, having expertise in mathematics, motivating students, and making decisions. The importance of knowing the students as part of the teachers’ role was evident in many of the accounts.

> I must be mindful of individual student’s strengths and weaknesses.
> I must gather a sound awareness of the student’s developmental age level.
> I would need to know the current understandings of each student.

Although the prospective teachers recognised the importance of knowing the students as part of teachers’ roles, their conception of that aspect was limited to teachers knowing the students purely as learners. Only one prospective teacher considered that teachers might need get to know their students more broadly.

> I need to gain insight into the backgrounds and other needs, interests and abilities of individuals within the class.

Having expertise in mathematics, as an aspect of the teachers’ role, featured in several descriptions.

> As a teacher I want to be very knowledgeable and have a clear understanding of how mathematics works.
> A teacher that is well educated on the topic is more beneficial to students’ learning and understanding.
> In order to be able to teach mathematics well [teachers] need to understand it.

There is strong evidence that many prospective primary teachers have mathematical anxiety and see themselves as unable to learn mathematics (Haylock, 2001; Hembree, 1990; Wilson & Thornton, 2005; Wolodko, Willson, & Johnson, 2003). Hence, it may be surmised that the focus some of the prospective teachers in this study placed on the teacher-as-expert component of the teachers' role may be a reflection of their anxiety regarding their ability to fulfil this facet.

Some functional aspects of the teachers' role, such as planning and preparing lessons and behaviour management, have been disregarded by the prospective teachers in this study. The prospective teachers may have overlooked lesson planning and preparation as this aspect of the teacher's role normally occurs "behind the scenes". None of the prospective teachers in this study identified behaviour management as being part of the teacher's role. Yet their classroom memories most certainly would have included observing or participating in this aspect of teachers' work. In the imagined classrooms teachers had a far more idealistic relationship with the students.

"Children learn in different ways and use different strategies": The Views of Learners and Learning

The views of learning and learners dominate the descriptions of the imagined classrooms. Fundamental to the prospective teachers' view was that all learners are individuals.

> Children are unique in the way that they absorb, understand and process information and have preferred learning styles.
> Children learn in different ways.

Dealing with the diverse needs of individual learners was also paramount for many of the prospective teachers in this study.

> I would endeavour to create lessons geared toward many styles of learning.
> I must consider the whole class, aiming to cater for all abilities.
> Because every student is different you need varied learning materials.

The prospective teachers also expressed a strong commitment to providing a safe learning environment where learners are supported and encouraged.

> I want to create a classroom where children feel comfortable and safe.
> I would like the students to feel they are supported and encouraged in mathematics.
> I would like to teach mathematics in a way that children do not feel threatened.

Closely aligned to the view of learners as individual, many of the prospective teachers in this study elucidated a view that learning should build upon existing knowledge. This is confirms research by Scott (2005) detailing the intention of prospective primary teachers to find out and build upon children's experiences. The prospective teachers in this study had been provided with a broad exposure to the term "constructivism" and no doubt the principles of constructivism informed this view of learning. However, it is pertinent to note that very few of the prospective teachers in this study used the term "constructivism" in their descriptions, choosing instead to describe the concept in other ways.

> Students should be able to link the new concepts to their existing knowledge.
> Students are building on from what they already understand and it is a good basis for them to learn and understand new concepts.
> I would seek to provide strategies that allow the children's previously acquired knowledge to be applied in new and unfamiliar situations.
> Students use the knowledge they have previously learned to interpret new information to devise new meaning.

The prospective teachers described a broad range of learning strategies to be employed in the imagined primary mathematics classrooms. These included the active involvement of learners, collaborative learning processes, creating an appropriate physical classroom environment, ensuring learning is relevant, transferable and fun, and the cross curricula integration of mathematics.

The active involvement of learners was deemed to be a high priority for the prospective teachers. The terms "hands on" and "interactive" abound in the accounts and consistent with Scott's (2005) research the prospective teachers in this study had a strong commitment to the use of physical manipulative resources in learning activities.

> I will endeavour to make the use of manipulatives available to students wherever possible.
> [I would] incorporate the use of concrete materials into my mathematics lesson.

The physical environment of the imagined primary mathematics classrooms also played a part in active learning.

> The teacher will need create a physical…environment that is conducive to learning.
> I would like to make my classroom a very visual one…having lots of posters and equipment for hand-on learning.
> My classroom would need to be laid out in such a way as to include floor space where children can spread out.

Many of prospective teachers embraced a range of less traditional learning activities in mathematics such as the use of stories, learning stations, games and technology including software packages and the internet in their imagined mathematics classrooms. Nevertheless, a few still found a place for more traditional approaches to teaching and learning in mathematics.

> Some old practices are as useful as new ones.
> I would always set written homework along with arithmetic homework.
> Some aspects of the curriculum such as multiplication will have to be done by rote learning.
> Include learning tables, the ability to manipulate numbers, such as adding, subtracting, multiplying and dividing numbers.

Others however, questioned the effectiveness of such approaches.

> Many students cannot learn by this method of memorising and repetition.
> Maths can be more about critically thinking, problem solving and logic rather than the more traditional memorizing and focus on finding answers.

Since it has been established that the prospective teachers in this study have recognised that it is likely their teaching practice will be influenced by their previous experiences in school mathematics, these descriptions ought not to be surprising as they quite possibly reflect the bearing that past experiences have had on their views of learning in mathematics.

Both Sliva and Roddick (2002) and Scott (2005) found that cooperative or group learning processes were highly favoured by the pre-service teachers. This view of learning is shared by the prospective teachers in this study. On the basis of her research, Scott (2005) contends that not all prospective teachers share the same understanding of group learning. However, in this study, the prospective teachers shared a more common view of cooperative group work, emphasising the social aspects of learning.

> Students should work together to build understandings and also to learn from each other.
> I would encourage classroom discussion and provide opportunities for the sharing of ideas.
> Students need to learn together cooperatively.
> I would like to incorporate a time for social interaction in mathematics.

In general, learners and learning have been somewhat romanticised in the imagined primary mathematics classrooms. None of the prospective teachers in this study have expressed the view that some students may find learning mathematics a challenge, despite the fact that a number have described the personal difficulties in their past experiences. On the basis of this study, the emerging identities of prospective teachers appear to be heavily influenced by idealism.

"Mathematics … is something that a person does": The Views of Mathematics

There is sound evidence that suggests that a teachers' view of mathematics can have a significant influence on teaching practice (Dossey, 1992; Raymond, 1997). Whilst the prospective teachers in this study furnished a range of views of mathematics, their articulation of this aspect was less pronounced than their views of the teacher's role, or learners and learning. The reluctance of the prospective teachers to describe more expansively their views of mathematics could be directly linked to their limited mathematical backgrounds or a lack of confidence in expressing mathematical understandings.

Ernest (1989) developed three categories the describe teachers' conceptions of the nature of mathematics: the view of mathematics as unified body of knowledge; the view of mathematics as an expanding field of human inquiry; and the view of mathematics as a useful collection of facts, rules, and skills. In this study none of the prospective teachers gave any indication of aligning themselves with Ernest's first view of mathematics as a stable body of knowledge. Some of the prospective teachers approached an association with Ernest's second category, indicating the view that mathematics underpins many aspects of human endeavour.

> Mathematics is not just a subject to be learnt in isolation, but it is found in the world around us: there's mathematics in language, literature, geography, environment, science, art, music and sports. Maths can be found everywhere.

Additionally, a few of the prospective teachers clearly aligned with Ernest's third view of mathematics.

> Students need to have a sound understanding of the rules, the ability and skills of using numbers.

More evident in the prospective teachers' accounts were quite emotive views of mathematics. The negative influence of past experiences in mathematics was evident yet again in some of the prospective teachers' views that mathematics is difficult or frightening.

> Children need to be made aware that sometimes they may find mathematics difficult.
> Mathematics is still seen by some children as a subject to be feared.
> Strike "mathematics" from the key learning areas all together, and replace it with numeracy, a much less threatening word.

Positive views of mathematics were far less prevalent, though it was encouraging to note that for one prospective teacher.

> In my classroom mathematics will be a highly anticipated subject!

Conclusion

In this study the imagined classrooms that were described by the prospective teachers afforded valuable insights into their views of the role of the teacher, learners and learning, and mathematics.

These prospective teachers clearly recognised that their past experiences and classroom memories would influence what they might bring to the role as a teacher. Many appreciated that an aspect of the teacher's role is to utilise a range of appropriate teaching strategies. However, the lesson planning and preparation and behaviour management were totally disregarded by the prospective teachers. It could be anticipated that once they have undertaken periods of teaching practice these more utilitarian functions of the teacher's role will be assimilated into their emerging identities as teachers.

Views of learners and learning dominated the descriptions of the imaginary classroom. However, the prospective teachers idealised their views of learners and learning in mathematics. Their imagined classrooms were going to be "safe havens" where mathematics was "fun and enjoyable" and lessons would be "interactive and relevant", incorporating a broad range of learning activities. In these imagined classrooms learning was never going to be difficult or boring, even though for some of the prospective teachers their view of mathematics was that is difficult and frightening.

This study is the first phase of a proposed longitudinal study. The prospective teachers in this study have graduated and are now first-year teachers. The researcher plans to re-visit these beginning teachers to observe their *real* primary mathematics classrooms. Of interest will be differences between the real and the imaginary classroom, and what has become of the idealism that was found in the prospective teachers' emerging identities.

References

Artzt, A. (1999). A structure to enable preservice teachers of mathematics to reflect on their teaching. *Journal of Mathematics Teacher Education, 2*, 143-166.

Ball, D. (1990). Breaking with experience in learning to teach mathematics: The role of a preservice methods course. *For the Learning of Mathematics, 10*(July), 10-16.

Britzman, D. (1986). Cultural myths in the making of a teacher: Biography and social structure in teacher education. *Harvard Educational Review, 56*(4), 442-456.

Brown, C., & Borko, H. (1992). Becoming a Mathematics Teacher. In D. Grouws (Ed.), *Handbook of Research on Mathematics Learning and Teaching* (pp 209-239). New York: MacMillan.

Bruner, J. (1989). Life as narrative. *Social Research, 54*(1), 11-32.

Carter, K. (1994). Preservice teachers' well-remembered events and the acquisition of event structured knowledge. *Journal of Curriculum Studies, 26*(3), 235-252.

Connelly, M., & Clandinin, J. (1990). Stories of experience and narrative inquiry. *Educational Researcher, 19*(4), 2-14.

Cooney, T., Shealy, B., & Arvold, B. (1998). Conceptualising belief structures of preservice secondary mathematics teachers. *Journal of Research in Mathematics, 29*(3), 306-334.

Dossey, J. (1992). The nature of mathematics. In D. Drouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 39-48). New York: MacMillan.

Doyle, W., & Carter, K. (2003). Narrative and Learning to Teach: Implications for Teacher-Education Curriculum. *Curriculum Studies, 35*(2), 129-137.

Drake, C., Spillane, J., & Hufferd-Ackles, K. (2001). Storied identities: Teacher learning and subject matter context. *Journal of Curriculum Studies, 33*(1), 1-23.

Ernest, P. (1989). The knowledge, beliefs and attitudes of the mathematics teacher: A model. *Journal of Education for Teaching, 15*(10), 13-33.

Goodson, I. (2006). The rise of the life narrative. *Teacher Education Quarterly*, 7-21.

Haylock, D. (2001). *Mathematics explained for primary teachers*. London: Paul Chapman.

Hembree, R. (1990). The nature, effects and relief of mathematics anxiety. *Journal for Research in Mathematics Education, 21*, 33-46.

Lloyd, G. (2006a). Beliefs about the teachers' role in the mathematics classroom: One student teacher's explorations in fiction and in practice. *Journal of Mathematics Teacher Education, 8*, 441-447.

Lloyd, G. (2006b). Preservice teachers' stories of mathematics classrooms: Explorations of practice through fictional accounts. *Educational Studies in Mathematics, 63*, 57-87.

Mewborn, D. (1999). Reflective thinking among preservice elementary mathematics teachers. *Journal for Research in Mathematics Education, 30*(3), 316-342.

Pajares, F. (1992). Teachers' beliefs and educational research: Cleaning up a messy construct. *Review of Educational Research, 62*(3), 307-332.

Polkinghorne, D. (1995). Narrative configuration in qualitative analysis. *Qualitative Studies in Educatoin, 8*(1), 5-23.

Raymond, A. (1997). Inconsistency Between a Beginning Elementary School Teacher's Mathematics Beliefs and Teaching Practice. *Journal for Research in Mathematics Education, 28*(5), 550-577.

Rossetto, M. (2006). *Imagined Communities and Communities of Practice: a study of the impact of visualisation and belonging on prospective teachers' identities.* Paper presented at the AARE Conference, Adelaide.

Scott, A. (2005). Pre-service teachers' experiences the infuences on their intentions for teaching primary school mathematics. *Mathematics Education Research Journal, 17*(3), 62-90.

Skott, J. (2001). The emerging practices of a novice teacher: The role of his school mathematics images. *Journal of Mathematics Teacher Education, 4*, 3-28.

Sliva, J., & Roddick, C. (2001). Mathematics Autobiographies: A Window Into Beliefs, Values, and Past Mathematics Experiences of Preservice Teachers. *Academic Exchange Quarterly,* 5(2), 101-107.

Sliva, J., & Roddick, C. (2002). *Investigating Preservice Elementary Teachers' Attitudes and Beliefs Toward Mathematics.* Paper presented at the 24th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education, Athens, Georgia.

Thompson, A. (1992). Teachers' Beliefs and Conceptions: A Synthesis of Research. In D. Grouws (Ed.), *Handbook of Research on Mathematics Teaching and Learning* (pp. 127-146). New York: Macmillan Publishing Company.

Whitehead, K., Rossetto, M., & Lewis, F. (2005). *Prospective Teachers' Dispositions Towards Teaching and Middle Schooling.* Paper presented at the Australian Curriculum Studies Association Biennial Conference, Sunshine Coast.

Wilson, S., & Thornton, S. (2005). 'I am really not alone in this anxiety': Bibliotherapy and pre-service primary teachers' self-image as mathematicians. In P. Clarkson, A. Downton, D. Gronn, M. Horne, A. McDonough, R. Pierce, & A. Roche (Eds.), *Building Connections: Theory, Research and Practice.* (Proceedings of the 28th Annual Conference of the Mathematics Education Research Group of Australasia, Melbourne, pp. 791-798). Sydney: MERGA.

Wolodko, B., Willson, K., & Johnson, R. E. (2003). Metaphors as a vehicle for exploring preservice teachers' perceptions of mathematics. *Teaching Children Mathematics*, 10(4), 224-229.

Early Notions of Functions in a Technology-Rich Teaching and Learning Environment (TRTLE)

Jill Brown
Australian Catholic University
<jill.brown@acu.edu.au>

This paper focuses on notions of function Year 9 students hold as they begin to study functions. As these notions may be fragile, the questions, tasks, and ways of interacting orchestrated by the teacher to elicit depth of understanding, or allow observation of changing notions, are of interest. Extended tasks where students were required to make choices about solution paths provided opportunities for students to develop and consolidate their concept images. Discussion between small groups provided the best evidence of developing and stable conceptions held by students in contrast to written scripts where the strength of these understandings was not evident.

The Function Concept and Student Understanding

The National Council of Teachers of Mathematics technology principle (2000) states, “technology is essential in teaching and learning mathematics; it influences the mathematics that is taught and enhances students’ learning” (p. 24) and suggests that all students should have access to technology that can allow higher order mathematical thinking to occur. The role of the teacher is vital in this as it is the teacher who “must make prudent decisions about when and how to use technology and should ensure that the technology is enhancing students' mathematical thinking” (p. 24). Opportunities in *Technology-Rich Teaching and Learning Environments* (TRTLE’s) have opened the door for easy access to the multiple representations of functions. The study of “multiple representations of functions is important in secondary school mathematics curricula, yet many leave high school lacking an understanding of the connections among these representations” (Knuth, 2000, p. 500).

Functions have been the focus of much research in recent years (e.g., Yerushalmy & Shternberg, 2001). The complex nature of functions has resulted in many student difficulties being identified (Knuth, 2000). Janvier (1996, p. 233) argues, “the notion of function conceals a wide range of concepts (so much so that one should more correctly speak of the notions (plural) of functions).” This view is not unique, Dreyfus (1990) suggested that due to its many layers of complexity and related sub-concepts “it may well be one of the most difficult concepts to master and teach of all school mathematics” (p. 122). Tall (1996) describes one purpose of functions to be “to represent *how things change*” (p. 289). He also notes that in practice the functions students experience are “first linear, then quadratic” (p. 298). Many schemas and frameworks have been developed to describe and analyse understanding of functions (e.g., Tall, 1996; Vinner & Dreyfus, 1989). Some incorporate the idea of representations (e.g., Moschkovich, Schoenfeld, & Arcavi, 1993) and the impact of technology use (e.g., Confrey & Smith, 1994). Understanding of functions has been the object of study both for students (Sfard, 1992) and teachers (Chinnappan & Thomas, 2003).

Sfard developed a schema allowing “different facets of the same thing” (1992, p. 60) to be applied to algebraic thinking. When the focus of mathematical thought is the function concept, Sfard’s schema describes the dual nature of the function as needing to be understood both as an *object*, that is, structurally, and as a *process*, that is, operationally. Sfard argues that the operational view invariably precedes the structural view. At the structural level the view of a function is of two sets having some relationship or correspondence between them. At the process level a function is some method of determining one value given another. Moschkovich, Schoenfeld, and Arcavi (1993) refine the object-process schema by introducing representations into the schema. They suggest that "from a *process perspective*, a function is perceived of as linking x and y values … From the *object perspective*, a function or relation and any of its representations are thought of as entities" (p. 71). Both perspectives are essential to a full understanding of functions according to many writers (e.g., Moschkovich, Schoenfeld, & Arcavi, 1993; Sfard, 1992). Vinner and Dreyfus (1989) use the constructs of *concept definition* and *concept image* to distinguish between the formal definition held by a learner about a concept and the broader set of images a learner holds about a concept. Vinner and Dreyfus (p. 356) define concept image as "the set of all mental pictures associated in the student's mind with the concept name, together with all the properties characterising them".

Linear Functions and the Notion of Gradient

Leinhardt and Steele (2005) investigated what understandings Grade 5 students can develop about linear functions and the role of classroom discourse in this. The students in this classroom discovered many important ideas including, "that the function rule [2X + 1] is a line" (p. 139), "some students discovered that the graph itself could be used to check … [and] to predict values" (p. 139). Their research suggests that students developed "intuitive ideas of slope and parallel slope" (p. 155) and "recognise[d] patterns in the connections between pairs of pairs [all] quite subtle notions for fifth graders, yet these students generate[d], them spontaneously and they do so publicly" (p. 155). Another study in the primary years, using carefully structured situations, also found that 8 to10 year olds can develop important ideas related to functions (Schliemann & Carraher, 2002). Grade 3 students were able to consider functional relationships; make generalisations, including using mapping notation ($n \to n + 3$) and n "to represent any value" (p. 255); and make connections between situations and the algebraic, numerical and graphical representations of these. With respect to gradient, Schliemann and Carraher report that third-graders "can start to understand how straight lines in a graph represent the same ratio" (p. 263).

Unlike much research in this area, the students who are the focus in this study are in Year 9, just beginning their study of functions. Students' conceptions of gradient within the study of linear and non-linear functions is the major focus of this paper. In this situation, where knowledge is often, understandably, fragile, the research questions of interest are: "What notions related to the function concept do students have?" and "How do we know what they know?"

Methods

In this paper one Year 9 TRTLE is being considered. Students in this class had their own laptops and TI83/83Plus graphing calculators. A qualitative approach was chosen to provide a comprehensive picture of what was occurring within the TRTLE. A case study using an instrumental approach (Stake, 1995, p. 3) was used. The purpose was to study the case to "understand the phenomena or relationships within it" (p. 171) in order to establish what understandings of function students demonstrated in their early study of function in a

TRTLE. Manifestations of the phenomena being studied, namely student understanding of functions, and the conditions that enabled, promoted, or impeded this were identified. Data sources for this paper were teacher interviews, observational notes, audio and video recording of 3 lessons on a teacher designed task building function notions, and a sequence of 14 lessons on linear function, and documentary evidence (student work including assessment scripts, handouts of teacher presentations, and student task sheets).

Teacher Orchestration: Developing Students' Understanding of Function

Student understanding of function was developed through two main arenas in this TRTLE. First, the teacher, Peter (all names are pseudonyms), developed and implemented a 14 50-minute-lesson unit of work focussed on linear functions. Secondly, the students were involved in a series of extended teacher designed tasks, building function ideas, over the course of the year. In both of these arenas, the teacher integrated technology into his teaching program. Two of the extended tasks, *Cunning Running* and *Shot on Goal*, were implemented prior to the linear functions unit. Both tasks were made accessible to Year 9 students by the use of technology. Students were able to recognise structure across repeated by-hand calculations that were then duplicated using LIST formulae to replicate and extend each set of calculations. Subsequent concatenation of these formulae and their transformation to an algebraic function enabled students to work with functions numerically (LISTS), graphically (plot and function graph) and algebraically (symbolic LIST formula and algebraic function) as shown in Figure 1. Both tasks enabled students to develop their function concept image, and specifically to consider what information was offered by each of the various representations.

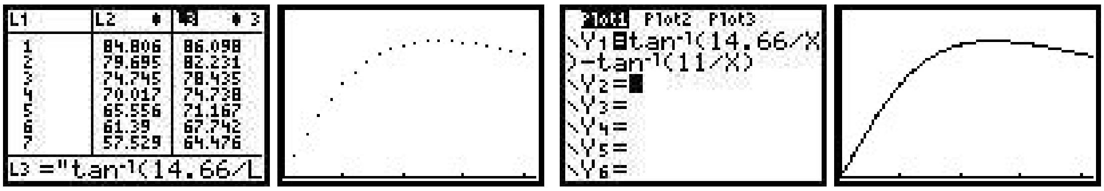


Figure 1. Multiple representations of the *Shot On Goal* function.

Early Notions of Gradient and Optimum Values of Functions

During the task *Shot on Goal*, students were asked to give the positions of the two shot spot distances between which the maximum angle for a shot on a hockey goal occurred. This was in the context of having completed a table of values of angles subtended by the goal mouth at various shot distances from the goal line. Responses to this task allowed insight into students' beginning notion that both discrete functions and continuous functions exist. Ben, for example, considering the function from an object perspective, showed evidence of a developing concept image of a continuous function related to the notion of the maximum value of a function. Ben was searching for where the maximum shot on goal angle occurred for a particular run line.

Ben: Okay so. 9.18, 9.19. So it is between 19 and 20 metres. [Reading from his table of values]
Ken: Why did you say that?
Ben: Because that is 9.19 [at 19m shot spot] and that is 9.18 [at 20m shot spot].
Ken: But then it could be between …
Ben: Could be between 18 and 20.
Ken: It **is** between 18 and 20.

Ben: 18 and 20.
Ken: 18 has a big jump though. [(18, 9.11), (19, 9.19), (20, 9.18)]
Ben: See 18 is 9.11, 20 is 9.18. So between, [pause] the biggest opening is between 18 and … [pause]. The maximum. [pause]
Ken: Ohh, the best angle you can get.
Ben: The maximum angle for the shot on goal [pause] is between 18 and 20 metres, between 18 and 20 metres, two spots—18 and 20.
Ken: 19 and 20.
Ben: It could be there is a higher point between 18 and 19. And it could start dropping again.

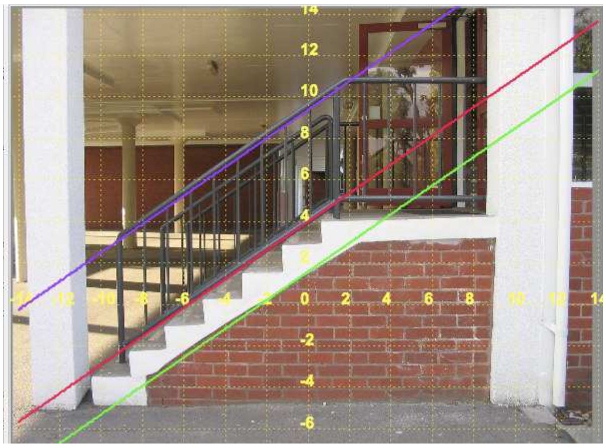
It appears that it was Ken's questioning that resulted in Ben originally expanding his notion of the maximum from being between the two largest calculated angles to a larger set of possibilities. Although Ken initiated the expansion of Ben's concept image, his questioning and Ben's subsequent explanations did not appear to have the same impact for Ken himself. From observations of these and other students attempting several optimising tasks a framework of developing images of the optimal value of a function has been developed (Figure 2). These images are hierarchical however, it is likely that the image held by students is initially fragile and hence they may move between images prior to moving to a stable concept image. In this TRTLE, students were identified at each stage in the framework.

Image A: Students simply see a set of numerical values where the optimal value is obvious – a range of values for the optimum is meaningless if the data are all different as they are discrete values.
Image B: Students have expanded their concept image here to include the graphical representation in addition to the numerical. These students have a mental image of the plot of the function, but still see only discrete values and the highest/lowest data point is the absolute optimum.
Image C: Extending the image of B, students show evidence of considering the situation as represented by a function of continuous values and visualising a "curve" passing through their mental image of the plot of points, however, this image has the curve reaching a maximum/minimum that is one of the data points.
Image D: Students show evidence of considering the optimal point can be at a point other than their discrete values however they consider this to be possible on only one side of the optimal discrete value, e.g., between the two highest/lowest values.
Image E: Students show evidence of recognising that the maximum value of the function could be on either side of the maximum/minimum or at this discrete value.

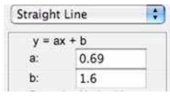
Figure 2. A Framework of images associated with the optimal values.

Linear Functions Unit

Peter changed the way he taught and what he allowed his students opportunities to learn, because within a TRTLE in the linear functions unit, Peter was able to begin the unit in a non-traditional way through the use of the computer application, *GridPic* (Visser, 2004). The use of this software, allowed the students to focus on functions in a new way (Figure 3). Peter (2004) explains "we started with the visual, which *GridPic* allows, [then] with the numerical which they then tried to algebracise to that pattern". He wanted his students to make more sense of slope than simply recalling and applying "rise over run". He felt that his approach using photographs of stairs in *GridPic* developed a deeper understanding. Peter continually emphasised slope as the ratio of the "change in the y values" and "the change in the x values". Peter's belief that previous teaching had not resulted in what he considered to be understanding, was demonstrated when one student commented, "We learnt it last year as run over rise." Peter responded with a laugh, "It was rise over run, so you didn't learn it at all!" Research findings by Walter and Gerson (2007) support Peter's belief that the notion of gradient as a process, rise over run, does not lead to a full understanding conceptually enabling application and explanation of the concept in a variety of contexts. Observation of Peter's classroom suggests that he was striving to develop in his students, higher order reasoning and understanding that would enable such explanation.



Clicking on a point shows the point on the graph screen and adds the coordinates of the point to an adjacent table.



Selecting a straight line graph and entering values for a and b results in the graph of y = ax + b being displayed. Any picture can be used as the background, or the picture can be hidden with a more standard graphical display presented.

Graph 1 : y = 0.69x + 4.0
Graph 2 : y = 0.69x + 9.4
✓ Graph 3 : y = 0.69x + 1.6

Figure 3. Using GridPic: Ned's three linear functions.

The first two lessons in the sequence saw students using *GridPic* for various purposes including to consider functions in a non-traditional way. Tasks included identifying points and trying to match a line to the stair rails (Figure 3). Conceptions related to function that were raised in the first lesson are shown in Figure 4. All these were orchestrated by the teacher except one (indicated with [S]) which was raised by Ned who was the student operating the computer with the display projected for the class to see.

Conceptions of Gradient

The three lessons forming the conclusion of the 14 lesson sequence were spent on a linear functions task. Students were presented in pairs with a task involving either the cost of installing safe drinking water wells or the cost of running small village health clinics. There were five different versions of the task. Introductory information and details of the elements of Part 1 and 2 of one version of the task are given in Figure 5.

The task was implemented as two consecutive lessons totalling 100 minutes on the Wednesday and a single period on the Friday in the last week of term 2. Twenty-four students were present. Two students were present for lessons 1 and 2 but not the third and a final student was present only for lesson 3. Scripts for 19 students were collected with a script for Ken, being recreated from his audio recording. One pair of students was video recorded with a further two pairs audio recorded. Of particular interest here is students' understanding of gradient as elicited by this task.

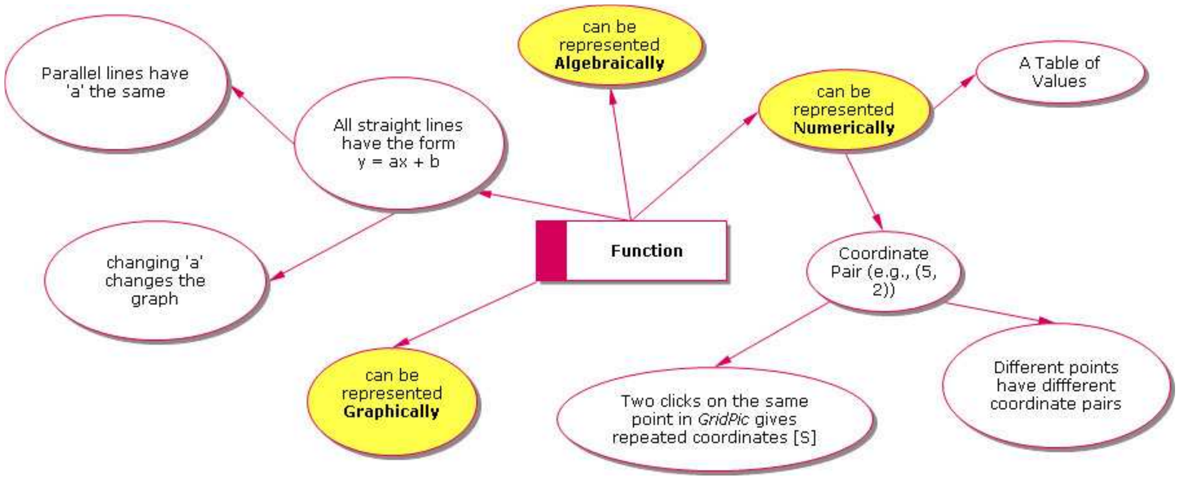


Figure 4. *Webs of meaning: Conceptions related to functions raised during lesson 1.*

A village Health Clinic in Mali

The weekly cost of running a small village clinic at Lake Haogoundou in Mali is a function of a constant weekly value and varies as the number of patients (n) attended. The cost is \$1100 when 50 patients are treated and \$1740 when 90 patients are treated.

Part 1 of the task required students to: Draw a plot showing a linear relationship for an appropriate domain. Identify the relationship decoding from text. Identify the domain, and dependent and independent variables. Using a linear rule C(n), find C given n. Write the linear relationship as an algebraic rule. Find n, given C.

Part 2 of the task [Functions for two other clinics are given: Bamako: COST = 390 + 17.50 × number of patients, Timbuktu: COST = 115 + 19.75 × number of patients] required students to: Find the costs, C_B and C_T, given n = 50, 60, … 200 recording in table of values. Focusing on one specific value of n state which cost is cheapest, C_B or C_T. Calculate $|C_B - C_T|$ for this value of n; Determine the value of n when C_B becomes lower than C_T. Construct a table to support this result. Explain how the table of values supports these ideas. Graph the 2 functions over an appropriate domain. Identify rate for C_T. Identify rate for C_B.

Figure 5. The Linear Functions Task.

Determining the gradient. In Part 1, students were required to determine the equation of the given function, hence this involved the calculation of the gradient. All students demonstrated they were able to calculate the gradient as evidenced by their written response to this part of the task. However, two students, Kit and Rani did not identify the equation of the line that passed exactly through the two points given. Instead, when completing the table of values prior to the determination of an algebraic rule, they used estimates for function values from their graph. However, their responses indicated they knew how to calculate the gradient. Althought from a purely function point of view, one could argue that these students were unable to calculate the gradient correctly, from a modelling perspective, one can equally well argue that the student moves were perfectly valid. Peter, in fact, encouraged them to determine their equation from their graphical representation of the situation.

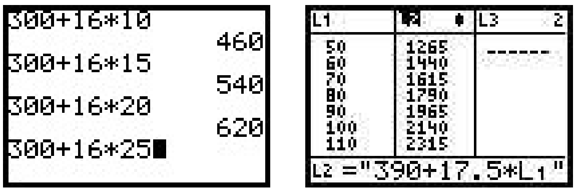
Like most of the students, Ben and Ken, for example, considered the gradient as a ratio to be calculated.

Ken: Well we have two points.
Ben: Yeah, so.

Ken: 1100 minus.
Ben: No, 1340 - 1100. [Calculates (1340 – 1100)/(170 – 90) = 240 ÷ 80]. 240. Divided by 80. So the gradient is three.

Use of the schema of Moschovich, Schoenfeld, and Arcavi (1993) showed that all students considered the gradient as a process, not an object, and most students operated in the numerical representation. For these students a connection was made between the numerical and algebraic representations, as a process was undertaken on numerical objects to determine the value of the gradient or “a” in the function equation $y = ax + b$. Kit and Rani, however, operated across three representations. Having represented the initial numerical information graphically, they were asked to complete a table of values. They completed this by reading values from their graph and subsequently used these values to determine the gradient. They were the only pair to do this in determining the gradient although they and two other students used the graphical representation to determine the y intercept. The majority of the students continued to work in the numerical representation to determine the value of b in $y = ax + b$, using a substitution method.

Opportunities for using the gradient. In Part 2 of the task, students were initially required to complete a table of values for two given functions. This could be completed in a number of ways using the HomeScreen of the graphing calculator to enter each calculation individually as undertaken by Kit and Rani (Figure 6a), or using the LISTs in the graphing calculator as undertaken by Kate and Meg as shown in Figure 6b.



(a) (b)

Figure 6. Completion the table values using (a) the HomeScreen and (b) LIST formula.

A third method, used by Ken and Ben, made use of the gradient employing what is described by Walter and Gerson (2007) as “slope as an additive structure” (p. 227). Ben was quick to see that he could use the gradient to complete the table of values more efficiently. Ben recognised that as the number of patients was given in increments of 10, the corresponding cost values should increase at a rate of 10 times the gradient, that is 140. Having determined the costs for 50, 60, 70, and 80 patients as 855, 995, 1135, and 1275, Ben stopped when he noticed the constant increment.

Ben: Hang on, hang on. Stop for a second, ha, 995 – 855, 140. So I think it is just going up by 140. So 1415 + 140 [Adding 140 to Ken’s last calculation for y(90).]
Ken: What? How did you work that out? It is not going up by 140.
Ben: Yeah, it is.

The students then continued using this additive method for further calculations in the task.

Identifying the gradient. Students were asked to identify the cost for treating each additional patient at the two health clinics being considered. The parallel question for the Water Well versions of the task required students to identify the cost for each metre of well depth drilled. Analysis of student scripts showed that 15 students recorded a correct answer to the questions. Only two students recorded a correct method. Another 11 students recorded no method, so unless other evidence is available it is difficult to infer what understanding these students have. One did not attempt this part of the task. An incorrect solution recorded by Kit and Rani for the first clinic was the result of $y(10) \div 10$, but the question for the second clinic was not answered. Kit originally recorded the cost for one patient, but this was subsequently crossed out. Di and Ann, working together, both recorded the cost for one patient, not appearing to realise that the gradient was required.

It is important to note here that during the final lesson on the task, each pair of students was expected to discuss their work with a second pair who had attempted the same version of the task. This provided students with the opportunity to check and discuss their results. Recordings of these discussions shows significant differences in student understanding of the interpretation of this question. When attempting these questions for the task (Figure 3), the transcript of Kate and Meg shows clear evidence of developing understanding of the concept of gradient. For Kate and Meg, their first function [$C_T = 115 + 19.75n$] was for the cost of treating patients at a clinic in Timbuktu. Initially, the pair was unsure what the question was asking. Meg began by finding the cost for one patient. She appeared to have some sense that they were finding the gradient but this knowledge was fragile.

Meg: What have I got? I think you just, I am going to put in just one [hesitantly].
Kate: Mmm. And then?
Meg: Okay, 19.75 × 1 + 115, [pause] whatever.
Kate: How come you are doing it times 1?
Meg: Umm, because when you find each additional patient after, like from, you go up by one.
Kate: Oh yeah.
Meg: It is hard to explain. Each time it goes up by. Each time it adds on to the 115 [fixed cost].

Later, Meg suggested that she was now confident that they are being asked for more than the cost for one patient.

Meg: Wouldn't it be 20 [approximate difference between C(2) and C(1)]?
Kate: What?
Meg: I thought it asked to do one, but to do each additional one. So what if you times it my two and then take away what you got for one. Like then it is about 20. It will go up by 20 each time you treat somebody.
Kate: Why 20?
Meg: Because you times. If you treated 2 patients you add, yeah you get, well, [how] much it would cost and then you take that away from one [C(1)]. No take 1 away from that. [C(2) – C(1)].
Kate: Wait, what are you saying? You go 19[.75 × 1 + 115] [pause]. Yeah , you do that right?
Meg: Yep, yeah.
Kate: Then what?
Meg: Then there is like one, and then there is like two. And there is like the difference. I think the difference there is like 20. Because if this here is 134.5 [C(1) = 134.75], and this one here is like 15.5, no 14.5. Wouldn't you just take them away? [C(2) = 154.50]
Kate: [incoherent] It is 154.5.
Meg: That is how many if you treat two patients, that is how much it costs.

Kate showed evidence of understanding what Meg was saying, as she then suggested that for a third patient the cost would be an additional $20 compared to the cost for two patients. The pair then proceeded to calculate similar values for the clinic at Bamako. An error in their subtraction, which was identified as they checked their results, led to their recognition that they had found the gradient.

Meg: How come is it 19.75? Oh my god! That is the gradient it goes up by each time. So it is 19.75.
Kate: That is the gradient?
Meg: Yes, the gradient goes up by, each time it goes up by. Oh my god!!

Kate: Yeah, der, because it is ‘ax’. [laughs] Der. [laughs]
Meg: Each additional patient costs 19.75. Because that is what the gradient is. Because each, the gradient is, whenever you go up by one, no whenever you go across [referring to a visual image of a graph]. When you treat one more patient.
Kate: Yeah I know, I get it.
Meg: So it is 19.75?
Kate: For each additional patient. So it is not that? [referring to $134.75 calculated previously].
Meg: No, that is just how much it costs to treat one patient.
Kate: So this one [referring to the similar question focussed on the second clinic] would be the same thing. This one would just be 17.50, I guess.
Meg: Yeah.

For Kate and Meg in particular this sub-task provided the further opportunity to develop understanding of the notion of gradient incorporating this additive structure. Fortunately, their discourse was captured via the audio recording, as the fragility of their knowledge was not evident on their written scripts.

In contrast, for two other students who came together during the final lesson, this sub-task seemed trivial as their notions of gradient were stronger.

Amy: [Reads] “What is the cost for treating each additional patient at Malange?”
Ben: 14.
Amy: Yep.
Ben: And then 12.50.
Amy: Yeah. Gee that is intelligent having to figure that out!
Ben: Yeah I know. [Facetiously] That was the hardest question!
Amy: Yeah.

Scripts for the students discussed here, for this particular sub-task, show little evidence of either the fragile yet emerging understanding of Kate and Meg, or the stable understanding of Ben and Amy. Only through access to their dialogue has this become evident.

Discussion and Conclusions

The students in the TRTLE that was the focus of this study demonstrated both process and object perspectives of various function notions. Both optimisation tasks and the linear functions task provided opportunities for, and evidence of, students thinking about and making connections between functions in each of the numerical, graphical, and algebraic representations. The fragility of these notions was evident at times through the choices of representations selected in solving tasks. Both the schema of Moschkovich, Schoenfeld, and Arcavi (1993) and the concept image construct of Vinner and Dreyfus (1989) proved valuable in exploring students’ developing notions.

The nature of the tasks presented to the students enabled them to make choices, including when to use technology and when to use by-hand methods. Additionally, when choosing to use the available technology, students had to make choices as to methods and representations in which to operate. These opportunities for choice at times allowed students to demonstrate their stable understanding of particular notions related to function. At other times, these same opportunities allowed students, in discussion with others to develop new understandings or to challenge fragile understandings and consolidate these.

The extended tasks where students were working relatively independently of their teacher and placed in the position where they made choices about solution pathways were particularly valuable in providing students with opportunities to develop and consolidate understandings of essential notions related to function. The expanding concept image enabled by engagement with the tasks led to broader, deeper, or more stable conceptions of the function concept. However, evidence of the strength of student understanding was more likely to be detected through attending to classroom discourse, at a private level between two or three members of the TRTLE, rather than through written work as little trace of this was recorded by students on their task scripts.

Acknowledgement. Jill was a doctoral student of the University of Melbourne on RITEMATHS (an ARC funded Linkage Project – LP0453701) when these data were collected.

References

Confrey, J., & Smith, E. (1994). Exponential functions, rates of change, and the multiplicative unit. *Educational Studies in Mathematics, 26*(2-3), 135-164.

Chinnappan, M., & Thomas, M. (2003). Teachers' function schemas and their role in modelling. *Mathematics Education Research Journal, 15*(2), 151-170.

Dreyfus, T. (1990). Advanced mathematical thinking. In P. Nesher, & J. Kilpatrick (Eds.), *Mathematics and cognition, ICMI Study Series* (pp. 113-134). Cambridge, UK: Cambridge University Press.

Janvier, C. (1996). Modeling and the initiation into algebra. In N. Bednarz, C. Kieran & L. Lee (Eds.), *Approaches to algebra: Perspective for research and teaching*, (pp. 225-238). Dordrecht, The Netherlands: Kluwer.

Knuth, E. J. (2000). Student understanding of the Cartesian connection: An exploratory study. *Journal for Research in Mathematics Education, 31*(4), 500-508.

Leinhardt, G., & Steele, M. (2005). Seeing the complexity of standing to the side: Instructional doalogues. *Cognition and Instruction, 23*(1), 87-163.

Moschkovich, J., Schoenfeld, A., & Arcavi. A. (1993). Aspects of understanding: On multiple perspectives and representations of linear relations and connections among them. In T. A. Romberg, E. Fennema, & T. P. Carpenter (Eds.), *Integrating research on the graphical representations of functions* (pp. 69-100). Hillsdale, NJ: Lawrence Erlbaum.

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics*. Reston, VA: Author.

Schielmann, A., & Carraher, D. (2002). The evolution of mathematical reasoning: Everyday versus idealized understandings. *Developmental Review, 22,* 242-266.

Sfard, A. (1992). Operational origins of mathematical objects and the quandary of reification: The case of function. In G. Harel & E. Dubinsky (Eds.), *The concept of function: Aspects of epistemology and pedagogy, MAA Notes 25,* (pp. 59-84). Washington, DC: Mathematical Association of America.

Stake, R. (1995). *The art of case study research.* Thousand Oaks, CA: Sage.

Tall, D. (1996). Functions and calculus. In A. J. Bishop, K. Clements, C. Keitel, J. Kilpatrick, & C. Laborde (Eds.), *International handbook of mathematics education* (pp. 289-325). Dordrecht, The Netherlands: Kluwer.

Vinner, S., & Dreyfus, T. (1989). Images and definitions for the concept of function. *Journal for Research in Mathematics Education, 20*(4), 356-366.

Visser, G. (2004). *GridPic* (Version 1.2.1) [Computer software.] Melbourne: Gerard, Inc.

Walter, J., & Gerson, H. (2007). Teachers' personal agency: Making sense of slope through additive structures. *Educational Studies in Mathematics, 65*(2), 203-233.

Yerushalmy, M., & Shternberg, B. (2001). Charting a visual course to the concept of function. In A. A. Cuoco (Ed.), *The roles of representation in school mathematics. 2001 yearbook* (pp. 251-268). Reston, VA: *National Council of Teachers of Mathematics*.

Communicating Students' Understanding of Undergraduate Mathematics using Concept Maps

Karoline Afamasaga-Fuata'i
University of New England
<kafamasa@pobox.une.edu.au>

The concept map data from a study of Samoan university students constructing *topic concept map*s and *vee diagrams of problems* throughout a semester is presented. Students found that, initially, concept mapping their topic was difficult. However with independent research and multiple critiques, their understanding of the conceptual structure of the topics deepened, becoming integrated and differentiated as evident from the concepts selected, valid propositions and structural complexity of the maps. Students also improved their skills in negotiating meaning, challenging and counter-challenging each others' explanations. Findings imply concept maps can facilitate the effective communication of students' understanding within a social setting.

Introduction

Working and communicating mathematically is being encouraged as part of everyday mathematical learning in schools. Research shows students' perceptions of mathematics learning reflect the way they have been taught mathematics (Thompson, 1984; Knuth & Peressini, 2001; Schell, 2001). In addition, pedagogical decisions teachers make about teaching and assessment are influenced by their mathematical beliefs (Ernest, 1999; Pfannkuch, 2001). Typically, an authoritative perspective views mathematics as a body of knowledge with classroom practices, simply a transmission of information. In contrast, cognitive and social perspectives view mathematics learning and understanding "as the result of interacting and synthesizing one's thoughts with those of others" (Schell, 2001, p. 2), suggesting mathematics knowledge is a social construction that is validated over time, by a community of mathematicians. Hence making sense is both an individual and consensual social process (Ball, 1993). Classroom practices should equip students with the appropriate language and skills to enable the construction of the mathematics that is taught, and critical analysis and justification of the constructions in terms of the structure of mathematics (Richards, 1991). Lesh (2000) argues that, "mathematics is not simply about doing what you are told" (p. 193) while Balacheff (1990, p. 2) posited that "students need to learn mathematics as social knowledge; they are not free to choose the meanings ... these meanings must be coherent with those socially recognized".

Existing problems with mathematics learning in Samoa are perceived as related to students' perceptions of mathematics, ability to communicate mathematically, and critical problem solving. Firstly, the narrow view most undergraduate students have, reflects their school mathematics experiences, found to be mostly rote learning, a problem consistently raised by national examiners. Even the top 10% of Year 13 (equivalent to Year 12 in Australia) students consistently struggle with applications of basic principles to solve inequations/equations and/or graph functions (Afamasaga-Fuata'i, 2001, 2002, 2005a,). Secondly, students justify methods in terms of sequential steps instead of the conceptual structure of mathematics. Thirdly, students may be proficient in solving familiar problems, however, the lack of critical analysis and application becomes evident when they are given novel problems. Such approaches are symptomatic of authoritative classroom practices in which students typically do not question, challenge or influence the teaching of

Proceedings of the 30th annual conference of the Mathematics Education Research Group of Australasia
J. Watson & K. Beswick (Eds), © MERGA Inc. 2007

mathematics (Knuth & Peressini, 2001). The examination-driven teaching of secondary mathematics in Samoa naturally inculcates a narrow view of mathematics (Afamasaga-Fuata'i, 2005a; 2002), As a result, problem solving skills students acquire over the many years of secondary schooling may not necessarily be situated "within a wider understanding of overall concepts" and would probably not be "long-lasting" (Barton, 2001). Against this general background, this paper reports a study, conducted over a semester, to investigate some second year university students' developing understanding of selected topics, as illustrated by individually constructed hierarchical concept maps (cmaps). Before the data are presented, the underlying theoretical framework and methodology are discussed.

Theoretical Framework and Relevant Studies

The difference, between an authoritative perspective of mathematics learning and Ausubel's cognitive theory of meaningful learning, socio-linguistic and social constructivistic perspectives, is the extent to which classroom discourse and social interactions are supported (Wood, 1999). That is, students learn mathematics in meaningful ways, by developing their understanding through the construction of their own patterns of meanings and through participation in social interactions and critiques (Novak & Cañas, 2006; Novak, 2002). In contrast, rote learning tends to accumulate isolated propositions rather than developing integrated, interconnected hierarchical frameworks of concepts (Novak & Cañas, 2006; Ausubel, 2000; Novak, 2002). Guiding the study were Ausubel's principles of assimilation and integration of new and old knowledge into existing knowledge structures through a degree of synthesis (i.e., *integrative reconciliation*) or reorganization of existing knowledge under more inclusive and broadly explanatory principles (i.e., *progressive differentiation*). Both the meaningful learning and social constructivist approaches support the metacognitive development of students' understanding and the active construction of mathematical thought whilst publicly presenting, for example, cmaps and vee diagrams (schematic diagrams), within a social setting. A cmap is a graph consisting of *nodes,* which correspond to important concepts in a domain and arranged hierarchically; *connecting lines* indicate a relationship between the connected concepts (nodes); and *linking word*s describe the interconnections (explanation). A *proposition* is the statement formed by reading the triad(s) "node $\xrightarrow{\text{linking words}}$ node" (Novak & Cañas, 2006). For example, the triad "Functions $\xrightarrow{\text{may be described using}}$ equations" forms the proposition, "*Functions* may be described using *equations*".

Numerous studies investigated the use of cmaps and/or vee diagrams (cmaps/vdiagrams) as assessment tools of students' conceptual understanding over time in the sciences (Novak & Canas, 2006; Brown, 2000; Mintzes, Wandersee, & Novak, 2000), and mathematics (Afamasaga-Fuata'i, 2005b; Schmittau, 2004; Swarthout, 2001); as communication tools (Freeman & Jessup, 2004); and as analytical tools to unpack teachers'/participants' perceptions (Pittman, 2002; Wilcox & Lanier, 1999). Research in secondary (Afamasaga-Fuata'i, 2002) and university mathematics (Afamsaga-Fuata'i, 2004) found students' conceptual understanding of mapped topics was further enhanced after a semester of concept mapping. Research with preservice teachers showed cmaps were useful pedagogical planning tools (Afamasaga-Fuata'i, 2006; Brahier, 2005). Workshops with science and mathematics specialists and teachers found maps/diagrams have potential as teaching, learning, and assessment tools (Afamasaga-Fuata'i, 2002;

1999). The research question for this paper is: “How can hierarchical concept maps illustrate improvements in students’ understanding of mathematics topics?”

Methodology

The study required students to undertake conceptual analyses of topics (identifying relevant major concepts, principles, formal definitions, rules, theorems, and formulas) and illustrate the theoretical results on cmaps. The methodology was an exploratory teaching experiment to investigate students’ developing understanding of particular topics (Steffe & D'Ambrosio, 1996), involving meeting twice a week for 50 minutes each time over 14 weeks with a cohort of students enrolled in a research mathematics course. Cmaps/vdiagrams were introduced as means of learning mathematics more meaningfully and solving problems more effectively. The content was from students’ recent mathematics courses, namely, *limits and continuity*, *indeterminate forms*, *numerical methods*, *differentiation*, *integration*, *motion, multiple integrals*, *infinite series*, *normal distributions,* and *complex analysis*. The epistemological principles, namely, building upon students’ prior knowledge, negotiation of meanings, consensus, and provision of time-in-class for student reflections, guided classroom practices. Hence, the study included a familiarization phase, which introduced the new socio-cultural classroom practices (socio-mathematical norms) of students presenting and justifying their work publicly, addressing critical comments, and then later on critiquing peers’ presented work. Time was set aside between critiques to revise maps/diagrams. The cyclic process was: presenting (to peers or researcher) → critiquing → revising → presenting underpinned the study. Of the 13 students, 3 chose topics outside of mathematics (computer programming, cell biology, and organic chemistry). This paper reports the data from the mathematics cmaps only.

Concept Map Analysis

Although the literature documents a variety of assessment/scoring techniques (Novak & Gowin, 1984; Ruiz-Primo, 2004; Liyanage & Thomas, 2002), a modified version of the Novak scheme was adopted, which used counts of a criterion. The three criteria were the *structural* (complexity of the hierarchical structure of concepts), *contents* (nature of the contents or entries in the concept nodes), and *propositions criteria* (valid propositions).

The structural criteria were in terms of integrative *cross-links* between concept hierarchies, progressive differentiation evidenced by nodes with *multiple branching* (more than one outgoing link) (which create *main branches* and *sub-branches*), and *average number of hierarchical levels per sub-branch*. The contents criteria indicate students’ perceptions of mathematical *concepts* in terms of suitable labels and illustrative *examples*. *Inappropriate* entries include those describing procedural steps (more appropriate on vee diagrams), redundant entries (indicating the need for a re-organization of concepts), and linking words as concept labels (linking-word-type). The definitional-phrase invalid node, although conceptual was too lengthy, its presence signals the need for further analysis to identify “*concepts*” as distinct from “linking words”. The propositions criteria define valid *propositions* as those formed by valid triads (i.e., “valid node $\xrightarrow{\text{valid linking words}}$ valid node”).

Concept Map Data

The data collected consisted of students’ progressive cmaps (4 versions) and progressive vee diagrams of 3 problems (at least 2 versions per problem), and final reports. Only the data from cmaps are presented here. The three criteria were used to assess students' first and final cmaps, to identify any changes. Individual results are presented first before a discussion of general themes. The cmap data for Students 1 to 5 are in Table 1 and those for Students 6 to 10 are in Table 2.

Student 1: Pene – Indeterminate Forms. Despite encountering "indeterminate forms" in first year mathematics, Pene struggled to begin a cmap. As a result of critiques, revisions and independent research, Pene's final cmap became structurally more integrated (increased cross-links from 3 to 10), more differentiated (increased multiple-branching nodes from 8 to 10 and increased average hierarchical levels per sub-branch from 6 to 8), and more compact (decreased sub-branches from 17 to 14) with main branches remaining unchanged (Table 1). However, the percentage of valid nodes (from 77% to 67%) and valid propositions (from 52% to 44%) decreased due to increased definitional-phrase invalid nodes (from 8% to 30%). An example of a definitional phrase is "$g(x) \neq 0$ *for any x in (a, b)*". Despite this, the final cmap was conceptually richer in its choice of concept labels with a structurally parsimonious, network of conceptual interconnections.

Table 1
Concept Map Data for Students 1 to 5

Student	1 Pene		2 Loke		3 Fia		4 Vae		5 Heku	
Criteria	First Cmap	Final Cmap	First Cmap	Final Cmap	First Cmap	Final Cmap	First Cmap	Final Cmap	First Cmap	Final Cmap
Contents										
Valid Nodes										
- *Concepts*	35 (67)	30 (65)	17 (44)	32 (56)	73 (99)	83 (83)	40 (59)	66 (99)	44 (86)	50 (74)
- *Examples*	5 (10)	1 (2)	19 (49)	16 (28)	0 (0)	6 (6)	10 (15)	0 (0)	0 (0)	0 (0)
Invalid Nodes										
-*Definitional*	4 (8)	14 (30)	1 (3)	8 (14)	1 (1)	8 (8)	12 (18)	1 (1)	1 (2)	15 (22)
-*Inappropriate*	8 (15)	1 (2)	2 (5)	1 (2)	0 (0)	3 (3)	6 (9)	0 (0)	6 (12)	3 (4)
Total Nodes	52	46	39	57	74	100	68	67	51	68
Propositions										
Valid Propositions	27 (52)	26 (44)	25 (69)	29 (49)	77 (96)	106 (88)	32 (51)	85 (97)	35 (66)	54 (67)
Invalid Propositions	25 (48)	33 (56)	11 (31)	30 (51)	3 (4)	14 (12)	31 (49)	3 (3)	18 (34)	27 (33)
Total Propositions	52	59	36	59	80	120	63	88	53	81
Structural										
Cross-links	3	10	0	6	9	10	4	17	6	22
Sub-branches	17	14	9	19	26	33	22	19	9	32
Average H/Levels per Sub-branch	6	8	6	8	10	9	7	8	8	7
Main Branches	6	6	5	7	5	8	4	5	6	9
M/Branching Nodes	8	10	5	8	18	19	9	18	9	19

Key H/Levels Hierarchical Levels M/Branching Multiple Branching Count (% of total number)

Student 2: Loke – Differentiation. Loke's first cmap had relatively more illustrative examples (49%) than conceptual entries (44%). As a result of critiques, revisions and independent research, the final cmap was relatively more conceptual (increased valid concept nodes from 44% to 56% and a reduction in examples from 49% to 28%), structurally more expanded (addition of 2 more main branches), more integrated (addition of 6 new cross-links) and more differentiated (increased multiple-branching nodes from 5 to 8 and increased sub-branches from 9 to 19). However, the reduction of valid propositions (from 69% to 49%) was due mainly to increased definitional-phrase invalid nodes (from 3% to 14%). An example of an incorrect proposition is "*Differentiation* also have a *non-differentiable function*". Overall, the final cmap was more differentiated, more integrated and more conceptual than the first cmap.

Student 3: Fia – Numerical Methods. Fia's first cmap had a high percentage of valid propositions (96%) reflecting her careful organization of propositions. As a result of critiques, revisions and further research, the final cmap showed increased number of valid concept nodes (from 73 to 83) and valid propositions (from 77 to 106) but proportionally reduced (valid nodes from 99% to 89% and valid propositions from 96% to 88%) due to increased definitional-phrase and inappropriate nodes (from 1% to 11%). Structurally, the final cmap expanded (increased main branches from 5 to 8), becoming more integrated (increased cross-links from 9 to 10) and more differentiated (increased multiple-branching nodes from 18 to 19 and increased sub-branches from 26 to 33) with more compact sub-branches (reduced average hierarchical levels from 10 to 9).

Student 4: Vae – Limits and Continuity. Vae's first cmap showed inclusion of complete formal definitions as concept labels, which the first peer critique highlighted as problematic. As a result of revisions, and critiques, Vae's cmap progressively evolved into a more conceptual one (increased valid nodes from 74% to 99%) with substantially increased valid propositions (from 51% to 97%), structurally expanded (main branches increased from 4 to 5), more integrated (cross-links increased from 4 to 17), more differentiated (increased multiple branching from 9 to 18 and increased average hierarchical levels per sub-branch from 7 to 8), and more compact (reduced sub-branches from 22 to 19). Evidently, continuous revisions enhanced the hierarchical interconnections such that formal definitions were analysed substantively, with concepts appropriately linked and described to illustrate the conceptual structure of the topic.

Student 5: Heku – Motion. Heku's final cmap became more conceptual with increased number of valid concept nodes (from 44 to 50 but proportionally reduced from 86% to 74%) and increased valid propositions (from 66% to 67%). Structurally, the final cmap was more expanded (increased main branches from 6 to 9), more integrated (increased cross-links from 6 to 22), more differentiated (increased multiple branching nodes from 9 to 19 and increased sub-branches from 9 to 32), but relatively more compact within sub-branches (reduced average hierarchical levels from 8 to 7). Increased invalid nodes (from 14% to 26%) resulted mainly from increased definitional phrases (from 2% to 22%).

Student 6: Santo – Complex Analysis. With repeated cycles of presentations → critiques → revisions, Santo's final cmap (Table 2) still had the same number of main branches, average hierarchical levels per sub-branch, and cross-links, a reduction of valid nodes (from 93% to 90%) while valid propositions increased (from 74% to 79%), and becoming structurally more differentiated (increased multiple branching nodes from 24 to 34) and more compact (reduced sub-branches from 68 to 66).

Table 2
Concept Map Data for Students 6 to 10

Student	6 Santo	6 Santo	7 Fili	7 Fili	8 Pasi	8 Pasi	9 Toa	9 Toa	10 Salo	10 Salo
Criteria	First Cmap	Final Cmap	First Cmap	Final Cmap	First Cmap	Final Cmap	First Cmap	Final Cmap	First Cmap	Final Cmap
Contents										
Valid Nodes										
- *Concepts*	165 (87)	159 (84)	32 (67)	30 (73)	41 (36)	52 (87)	34 (76)	63 (71)	100 (57)	79 (72)
- *Examples*	11 (6)	12 (6)	0 (0)	1 (2)	20 (18)	0 (0)	0 (0)	7 (8)	34 (19)	17 (16)
Invalid Nodes										
-Definitional	1 (1)	2 (1)	3 (6)	3 (7)	3 (3)	7 (12)	6 (13)	15 (17)	8 (5)	6 (6)
-Inappropriate	12 (6)	16 (8)	13 (27)	7 (17)	50 (44)	1 (2)	5 (11)	4 (4)	33 (19)	6 (7)
Total Nodes	189	189	48	41	114	60	45	82	141	92
Propositions										
Valid Propositions	148 (74)	166 (79)	15 (32)	26 (62)	48 (40)	39 (67)	28 (56)	88 (81)	110 (60)	82 (73)
Invalid Propositions	51 (26)	45 (21)	32 (68)	16 (38)	71 (60)	19 (33)	22 (44)	20 (19)	73 (40)	30 (27)
Total Propositions	183	112	47	42	119	58	50	108	183	112
Structural										
Cross-links	8	8	1	4	11	12	13	21	5	6
Sub-branches	68	66	16	13	19	16	10	24	44	35
Average H/Levels per Sub-branch	6	6	6	6	12	9	11	9	9	7
Main Branches	19	19	4	6	12	6	3	10	10	10
M/Branching Nodes	24	34	4	9	13	11	9	17	20	16

Key H/Levels Hierarchical Levels M/Branching Multiple Branching Count (% of total number)

Student 7: Fili – Multiple Integrals. Fili's first cmap illustrated sequential derivations of double and triple integrals, with critical comments targeting invalid nodes. As a result of critiques, revisions, and further independent research, Fili's final cmap became more parsimonious (reduced sub-branches from 16 to 13 and unchanged average hierarchical levels per sub-branch), more integrated (increased cross-links from 1 to 4), more differentiated (increased multiple-branching nodes from 4 to 9), more conceptual (increased valid nodes from 67% to 75%) and valid propositions almost doubled (from 32% to 62%).

Student 8: Pasi – Integration. As a consequence of the cyclic process of presenting → critiquing → revising → presenting, Pasi's final cmap evolved into a substantially more conceptual one (increased valid nodes from 54% to 87%) with increased valid propositions (from 40% to 67%). For example, a new branch illustrated the numerical limit view of integrals from successive approximations of area under a curve and linking it to the limit of the Riemann sum as a definition for the definite integral. The absence of illustrative examples was noticeable. Structurally, the cmap was more compact (reduced multiple-branching nodes (from 13 to 11), reduced sub-branches (from 19 to 16), reduced main branches (from 12 to 6), and reduced average hierarchical levels per sub-branch (from 12 to 9). Overall, the final map was predominantly more conceptual with more valid propositions and a more parsimonious, compact final structure.

Student 9: Toa – Normal Distributions (ND). Toa felt challenged to construct a cmap that included ND, Poisson distributions (PD) and binomial distributions (BD). He wrote: "(it was) hard to think of a concept to start the cmap and then link the others right down to the end when it introduces (BD, PD and ND)." The first peer critique commented the cmap had "too many useful concepts … missing", and the "concepts used were paragraphs". In subsequent revisions, he "tried to break down those paragraphs into one or two concept names" and "re-organized concept hierarchies", eventually resulting in a final cmap that was more conceptual (increased valid nodes from 76% to 79%) with increased valid propositions (from 56% to 81%). Structurally, the final cmap became more expanded (increased main branches from 3 to 10), more integrated (increased cross-links from 13 to 21), more differentiated (substantial increases with multiple branching nodes from 9 to 17 and sub-branches from 10 to 24) and more compact within sub-branches (reduced average hierarchical levels from 11 to 9). Shown in Figure 1 is part of Toa's final map (example of a good cmap) showing examples of integrative crosslinks between two branches (proposition "*Normal Distribution* can be approximately used for *Binomial Distribution → Normal Distribution*"), multiple branching nodes (*bell-shaped curve* and *parameters*) and integrative reconciliation of a number of nodes merging into a single node (nodes x, $n - x$, p, n, $q = 1 - p$, with merging links to *Probability Function*).

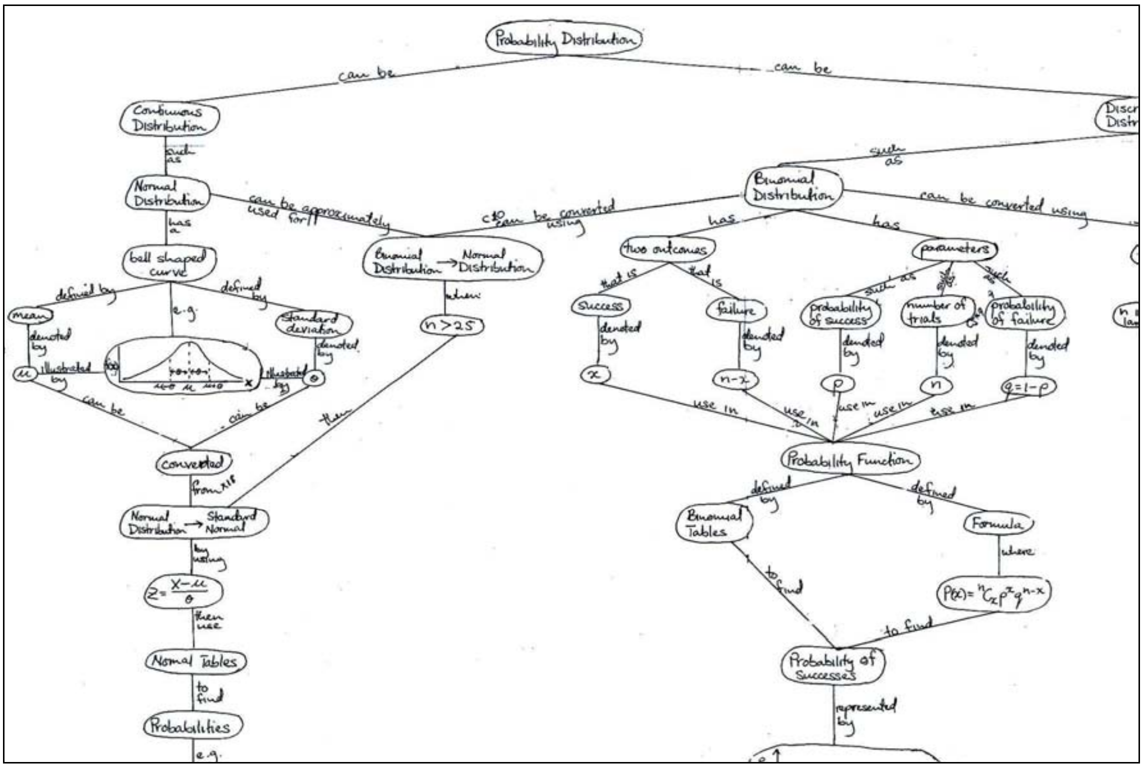


Figure 1. Partial final concept map – Toa.

Student 10: Salo – Infinite Series. The first peer critique targeted the high number of inappropriate nodes (33) with subsequent critiques focussing on the need to improve linking words and appropriate placement of progressively-differentiated concepts. Salo's final cmap became more conceptual (increased valid nodes from 76% to 88%) with increased valid propositions (from 60% to 73%). Structurally, the final cmap was more integrated (increased crosslinks from 5 to 6) and more compact with less differentiation (decreased multiple branching nodes from 20 to 16 and decreased sub-branches from 44 to 35, with a lower average hierarchical levels per sub-branch from 9 to 7) whilst main branches remain unchanged. Overall, the final map was more conceptual with a more enriched network of interconnections and structurally more integrated and more compact than the first cmap.

Discussion

Findings suggested that students' progressive cmaps became integrated and differentiated as students continually strove to illustrate valid nodes and meaningful propositions, in response to concerns raised in social critiques and in anticipation of future critiques. Hence the re-definition of socio-mathematical norms appeared to affect the nature of students' cmaps substantively, particularly as students had to justify their displayed connections, negotiate meanings with their peers, and reach a consensus to revise or not. For example, half the students showed increases in valid nodes, propositions, and structural complexity by the final cmap. There was a marked shift from simply providing formulas, procedural steps, excessive illustrative examples, and entire paragraphs, to seeking out more integrated and differentiated conceptual interconnections, which reflected the impact of the social interactions on an individual's evolving understanding. Also, students necessarily had to reflect more deeply, as individuals, about the conceptual structure of topics than they normally did. Because of the need to communicate their understanding competently in a social setting, over time and with increased mapping proficiency, students became more parsimonious in their selection of concepts and more astute in describing the nature of the relationships between connecting concepts more correctly to minimize critical comments. From students' perspectives, they realized that mathematics has a conceptual structure, the socially validated body of knowledge, which underpins its formal definitions and formulas. By searching for missing relevant concepts to make the cmaps more robust and comprehensive, students eventually realized that an in-depth understanding of topics required much more than re-stating a definition or formula. Concept mapping required the identification of main concepts, an integrated understanding of connections between relevant concepts, visually organising this understanding as a meaningful hierarchy of interconnecting nodes with valid linking words that form valid propositions as socially warranted by a community of mathematicians.

Over the semester, students eventually appreciated the utility of cmaps as a means of depicting networks of conceptual interconnections within topics and of highlighting connections between concepts, definitions and formulas. However, attaining this greater conceptual understanding of mathematics was hard work and required much reflection, social negotiation, and individual research on their part. The findings suggested that with more time and practice students can become proficient and adept at constructing cmaps whilst simultaneously deepening and expanding their theoretical knowledge of the structure of mathematics. Challenges faced by the students included the importance of getting quality feedback from their peers, sustaining students' motivation to seek more meaningful connections by revising inappropriate nodes and incorrect linking words and reorganising concept hierarchies, and developing their self-confidence in presenting mathematical justifications and counter-arguments during social critiques. The progressive quality of students' cmaps over the semester confirmed that students' ways of learning mathematics are very much influenced by the expectations and beliefs of the teacher, the prevailing socio-mathematical norms of the classroom setting, and the socially-validated structure of mathematics. Findings also extended the literature on the impact of social negotiations of meanings, interactions and critiques on the development of students' conceptual understanding of topics, which in this study, was greatly facilitated with the visual mapping of students' progressive conceptions on hierarchical cmaps over time. Finally, using the metacognitive tools promoted a higher level of self-reflection and lateral thinking that generally motivated students to analyse their perceptions of mathematics knowledge critically and specifically encourage deeper, conceptual understandings of topics.

Implications

Findings from the study imply that the concurrent use of concept mapping and social critiques as part of the culture and practices within mathematics classrooms has the potential to promote the development of mathematical thinking, reasoning, and effective communication, which are most desirable skills to succeed in mathematics learning. Doing so as early as the primary level would be an area worthy of further investigation.

References

Afamasaga-Fuata'i, K. (1999). Teaching mathematics and science using the strategies of concept mapping and vee mapping. *Problems, Research, & Issues in Science, Mathematics, Computing & Statistics*, *2*(1), 1-53.

Afamasaga-Fuata'i, K. (2001). New challenges to mathematics education in Samoa. *Measina A Samoa 2000* (Vol. 1, pp. 90-97. Apia: Institute of Samoan Studies, National University of Samoa.

Afamasaga-Fuata'i, K. (2002). A Samoan perspective on Pacific mathematics education. In B. Barton, K. C. Irwin, M. Pfannkuch, & M. O. J. Thomas (Eds.), *Mathematics Education in the South Pacific* (Proceedings of the 25th annual conference of the Mathematics Education Research Group of Australasia, Volume 1, pp. 1-13). Sydney: MERGA.

Afamasaga-Fuata'i, K. (2004). Concept maps and vee diagrams as tools for learning new mathematics topics. In A. J. Canãs, J. D. Novak, & F. M. Gonázales (Eds.), *Concept Maps: Theory, Methodology, Technology.* (Proceedings of the First International Conference on Concept Mapping, Volume 1, pp. 13-20). Navarra, Spain: Dirección de Publicaciones de la Universidad Pública de Navarra.

Afamasaga-Fuata'i, K. (2005a). Mathematics education in Samoa: From past and current situations to future directions. *Journal of Samoan Studies*, *1*, 125-140.

Afamasaga-Fuata'i, K. (2005b). Students' conceptual understanding and critical thinking? A case for concept maps and vee diagrams in mathematics problem solving. In M. Coupland, J, Anderson & T. Spencer (Eds.), *Making Mathematics Vital* (Proceedings of the Twentieth Biennial Conference of the Australian Association of Mathematics Teachers, Vol. 1, pp. 43-52). Adelaide: AAMT.

Afamasaga-Fuata'i, K. (2006). Innovatively developing a teaching sequence using concept maps. In A. Canas & J. Novak (Eds.), *Concept Maps: Theory, Methodology, Technology*. (Proceedings of the Second International Conference on Concept Mapping, Vol. 1, pp. 272-279). San Jose, Costa Rica: Universidad de Costa Rica.

Ausubel, D. P. (2000). *The acquisition and retention of knowledge: A cognitive view*. Dordrecht: Kluwer Academic Publishers.

Ball, D. (1993). With an eye on the mathematical horizon: Dilemmas of teaching elementary school mathematics. *The Elementary School Journal*, *93*(4), 373-397.

Ballacheff, N. (1990). Beyond a psychological approach: the psychology of mathematics education. *For the Learning of Mathematics*, *10*(3), 2-8.

Barton, B. (2001). How healthy is mathematics? *Mathematics Education Research Journal, 13*(3), 163-164.

Brahier, D. J. (2005). *Teaching secondary and middle school mathematics* (2nd ed.). New York: Pearson Education, Inc.

Brown, D. S. (2000). *The effect of individual and group concept mapping on students' conceptual understanding of photosynthesis and cellular respiration in three different levels of biology classes.* Dissertation Abstracts International AADAA-I9970734, University of Missouri, Kansas City.

Ernest, P. (1999). Forms of knowledge in mathematics and mathematics education: Philosophical and rhetorical perspectives. *Educational Studies in Mathematics*, *38*, 67-83.

Freeman, L. A., & Jessup, L. M. (2004). The power and benefits of concept mapping: Measuring use, usefulness, ease of use, and satisfaction. *International Journal of Science Education*, *26*(2), 151-169.

Knuth, E., & Peressini, D. (2001). A theoretical framework for examining discourse in mathematics classrooms. *Focus on Learning Problems in Mathematics, 23*(2 & 3), 5-22.

Lesh, R. (2000). Beyond constructivism: Identifying mathematical abilities that are most needed for success beyond school in an age of information. *Mathematics Education Research Journal, 12*(3), 177-195.

Liyanage, S., & Thomas, M. (2002). Characterising secondary school mathematics lessons using teachers' pedagogical concept maps. In B. Barton, K. C. Irwin, M. Pfannkuch, & M. O. J. Thomas (Eds.), *Mathematics Education in the South Pacific* (Proceedings of the 25th annual conference of the Mathematics Education Research Group of Australasia, pp. 425-432). Sydney: MERGA.

Mintzes, J. J., Wandersee, J. H., & Novak, J. D. (Eds.). (2000). *Assessing science understanding: a human constructivist view.* San Diego, CA: Academic.

Novak, J. D. (2002). Meaningful learning: the essential factor for conceptual change in limited or appropriate propositional hierarchies leading to empowerment of learners. *Science Education*, *86*(4), 548-571.

Novak, J. D., & Cañas, A. J. (2006). *The theory underlying concept maps and how to construct them*. Technical Report IHMC Cmap Tools 2006-01, Florida Institute for Human and Machine Cognition, 2006, available at: http://cmap.ihmc.us/publications/ResearchPapers/TheoryUnderlyingConceptMaps.pdf.

Novak, J. D., & Gowin, D. B. (1984). *Learning How to Learn*. Cambridge: University Press.

Pfannkuch, M. (2001). Assessment of school mathematics: Teachers' perceptions and practices. *Mathematics Education Research Journal*, *13*(3), 185-203.

Pittman, D. B. (2002). *A study examining the role of teacher beliefs and how these beliefs affect the teaching of mathematics*. Dissertation Abstracts International AADAA-I3041385, George Mason University.

Richards, J. (1991). Mathematical discussions. In E. von Glaserfeld, (Ed.), *Radical constructivism in mathematics education* (pp. 13-51). London: Kluwer Academic Publishers.

Ruiz-Primo, M. (2004). *Examining concept maps as an assessment tool.* In A. J. Canãs, J. D. Novak & Gonázales (Eds.), *Concept Maps: Theory, Methodology, Technology* (Proceedings of the First International Conference on Concept Mapping, pp. 555-562). Navarra, Spain: Dirección de Publicaciones de la Universidad Pública de Navarra.

Schell, V. (2001). Introduction: language issues in the learning of mathematics. *Focus on Learning Problems in Mathematics, 23*(2 & 3), 1-4.

Schmittau, J. (2004). Uses of concept mapping in teacher education in mathematics. In A. J. Canãs, J. D. Novak & Gonázales (Eds.), *Concept Maps: Theory, Methodology, Technology.* Proceedings of the First International Conference on Concept Mapping September 14-17, 2004 (pp. 571-578). Navarra, Spain: Dirección de Publicaciones de la Universidad Pública de Navarra.

Steffe, L. P., & D'Ambrosio. (1996). Using teaching experiments to enhance understanding of students' mathematics. In d. F.. Treagust, R. Duit, & B. F. Fraser. (Eds.), *Improving Teaching and Learning in Science and Mathematics* (pp. 65-76). New York: Teachers College Press, Columbia University.

Swarthout, M. B. (2001). *The impact of the instructional use of concept maps on the mathematical achievement, confidence levels, beliefs, and attitudes of preservice elementary teachers*. Dissertation Abstracts International, AADAA-I3039531, The Ohio State University, Ohio.

Thompson, A. G. (1984). The relationship of teachers' conceptions of mathematics and mathematics teaching to instructional practice. *Educational Studies in Mathematics, 15*, 105-127.

Wilcox, S. K. & Lanier, P. E. (1999). Using cases to integrate assessment and instruction. *Mathematics Teaching in the Middle School*, *4*(4), 232-241.

Wood, T. (1999). Creating a context for argument in mathematics class. *Journal for Research in Mathematics Education*, *30*(2), 171-191.

Primary Student Teachers' Diagnosed Mathematical Competence in Semester One of their Studies

Karoline Afamasaga-Fuata'i
University of New England
<kafamasa@une.edu.au>

Paul Meyer
National University of Samoa
<p.meyer@nus.edu.ws>

Naomi Falo
National University of Samoa
<naomi.falo@yahoo.com>

Data collected from a diagnostics mathematics test taken by some primary student teachers are reported. Student responses were analysed using the Dichotomous Rasch Measurement Model. Error analyses enabled the identification of main misconceptions. Findings showed students performed relatively well with basic computations and visually presented data but struggled with word problems. The more complex and abstract the language used, the more difficult it became, implying that the critical skills of interpreting mathematical concepts, representations, and language and problem solving require explicit remediation. Implications for primary teacher education are provided.

Professional Teaching Standards (NCTM, 2005; AAMT, 2006) prescribe requirements such as a deep understanding not only of the teaching and learning processes but also the specific discipline content. Shulman's (1986) teacher knowledge taxonomy included *subject-matter content knowledge*, *pedagogical content knowledge* and *curriculum knowledge*. Although curriculum knowledge is knowledge of curriculum programs and instructional materials (Chick, 2002), Shulman (1986) defines subject-matter knowledge as knowledge of both the substantive structure and syntactic structure. Transforming subject-matter knowledge and curriculum knowledge into pedagogical content knowledge conceptualises "the link between knowing something for oneself and being able to enable others to know it" (Huckstep, Rowland, & Thwaites, 2003). Ma's (1999) study illustrated the need for primary teachers to have *profound understanding of fundamental mathematics* in order to promote and extend student learning. Ball and Bass (2000) argued teachers should be mathematically competent in order to effectively address the diversity of student needs. Research (Shulman, 1986; Ma, 1999; Ball, & Bass, 2000; Huckstep et al., 2003) also showed teachers' content knowledge of the curriculum generally influences their selection of activities and mediation of meaning in the classroom. This paper focuses on the identification of the mathematical competence of a cohort of foundation and primary student teachers in their first semester. Mathematical competence is defined as the ability to solve a set of items, in a written test, based on the Samoan Ministry of Education, Sports and Culture's (MESC) *Primary and Early Secondary Mathematics (PESM) Curricula* (MESC, 2003). Each item is designed to contribute meaningfully to a measure of mathematical competence. Ideally, student teachers should be capable of solving these items by critically (a) interpreting mathematical concepts, multiple data representations, and language describing quantitative relationships, (b) transforming interpretations arithmetically and/or algebraically, and (c) synthesising relevant knowledge and procedures to generate plausible solutions. The presence of mathematical competence is assessed by the *quality of student responses* and *nature of errors*. Therefore, the focus questions for this paper are: (1) How reliable was the test as a tool to measure students' mathematical competence? (2) What are primary student teachers' main mathematical misconceptions?

Methodology and Analysis

The mathematics diagnostic test (MDT1, Appendix A) consisted of thirty items, compiled (Mays, 2005) primarily from the TIMSS 1999-R mathematics paper (Mullis, Martin, Gonzalez, Gregory, Garden, O'Connor, Chrostowski, & Smith, 2000) (code T in the first position) as these have reliability and validity data, and eight items from the misconception literature (code M in the first position). These include five mental computation items (code MMCT) on products of single digit numbers and decimals, percentage of two-digit integers, four-digit substraction, and adding unit fractions (McIntosh & Dole, 2000; Callingham & Watson, 2004) and items on ordering fractions (MFNS08), the student-professor problem (MALG14), and proportional reasoning (MGE029) (Thompson & Saldanha, 2003). The 38 items sampled the content areas of *MESC's PESM Curricula* – fractions and number sense (FNS), measurement (MSR), algebra (ALG), geometry (GEO) and data presentation, analysis (DPA) and probability (PRB) – and five cognitive domains: *knowing, using routine procedures, investigating and problem solving, and mathematical communication* (Mullis et al., 2000). To provide access to students' errors, all 38 items were left open-ended. MDT1 was used at an Australian regional university with different cohorts of primary student teachers (Mays, 2005). A total of 140 Samoan primary student teachers took MDT1. Responses were categorised *Correct*, *Incorrect* or *Blank* and analysed using the Dichotomous Rasch Measurement Model and QUEST software (Adams & Khoo, 1996). Error analysis counted error types by item and identified up to 3 most common errors. The Rasch Model examines only one theoretical construct at a time on a hierarchical "more than/less than" logit scale (unidimensionality). Rasch parameters, item difficulty and person ability, are estimated from the natural logarithm of the pass-versus-fail proportion (*calibration of difficulties and abilities*) whereas estimation of fit is measured by mean square (mean squared differences between observed and expected values) and *t*, infit and outfit values (*estimation of fit to the model*). Fit of the data to the model (infit *t* values (-2, 2)) and reliability of the test (around 1) are examined.

Results

Review and Reliability of the Mathematics Diagnostic Test

The Rasch Model theoretically sets the mean of item estimates at 0 before item and person estimates are calibrated. Infit *t* values showed all items (except TGEO17) fit the model. A 3.70 infit *t* value indicated erratic behaviour. An item analysis from QUEST showed a non-monotonic increase in mean abilities for the 3 response categories, suggesting TGEO17 (difficulty -0.26 logits) might be measuring something different. Item TMSR27 had a zero score, meaning it was too difficult and was not discriminating among students. Thus both TGE017 and TMSR27 should be revised in future testing. Using the (-2, 2) infit *t*-criteria on cases confirmed they all fit the model. Candidate 119 had a zero score, which meant the case was not contributing to the calibrations. Finally, to improve the data's fit to the model, items TGE017 and TMSR27 and Candidate 119 were excluded from the second analysis of 139 cases and 36 items (see Table 1). The person ability mean of -.95 logits suggested the test was hard. A standard deviation of 1.15 indicated the cases were more clumped around its mean whereas the items were more spread out. An item fit map showed all items fit the model hence establishing that the 36 items worked together consistently to define a unidimensional scale. The reliability indices for items (0.97) and cases (0.84) were both high (Bond & Fox, 2001) indicating the test produced a reliable measure of student teachers' mathematical competence of MESC's PESM curriculum.

Table 1
Second Analysis - Summary of Item and Person Estimates

Estimates	Mean	SD	SD (adj)	Reliability	Infit Mean Square		Outfit Mean Square		Infit *t*		Outfit *t*	
					Mean	SD	Mean	SD	Mean	SD	Mean	SD
Item	0.00	1.85	1.82	0.97	0.99	0.10	1.05	0.55	0.00	0.83	0.11	0.90
Case	-0.95	1.25	1.15	0.84	1.00	0.25	1.05	0.95	0.00	1.05	0.17	0.83

The item-person map (Figure 1) corroborates the high reliability indices with its hierarchical distribution of items (represented by item codes) on the right of the vertical line from the most difficult to the easiest, and distribution of cases (represented by 'X') on the left, with both distributions sharing a common logit scale (on the extreme left). The two distributions are not aligned, corroborating that the test was hard for this cohort; further evident from the presence of more difficult items than cases above 2 logits. The model predicts people have a 50% chance of successfully solving items with estimates within their ability band (ability ± standard error), better chances of succeeding with items below the band and less than average probability with items located above the ability band. Items in Figure 1 are spread horizontally along their QUEST-generated logit locations into 6 content areas to facilitate discussions.

Cognitive Developmental Hierarchy of Items

Figure 1 displays both an overall and content-specific cognitive developmental scale of mathematical competence. At the top-end are the most difficult items (>3 logits, TFNS31 and MALG14), which are complex, non-routine word problems on investigation, multi-step problem solving, and algebraically representing multiplicative relationships. At the lower end are the easiest items (<-3 logits) involving basic computation (TALG28 and MMCT01). Above average items but below the most difficult items, involve increasingly less complex, multi-step word problems on likely outcomes, rate, ratio and quantitative descriptions; application of students' fraction understanding and knowledge; interpretation of complex diagrams, and mathematical communication. Below average items but above the easiest items, involve routine procedures (computing/modelling equivalent fractions); mentally computing percentage; algebraically transforming descriptions; mental computation; pattern extension (numerical and geometric); solving simple geometric word problems and linear equation; substitution; and graph interpretation. In summary, there seems to be distinct stages of cognitive development from *basic computations* with whole numbers at the lower end, through to interpretation of *visual data representations* and *explicitly stated quantitative relationships* around the middle (0 logits), and increasingly *implicit and abstract quantitative relationships* towards the top-end. Success rates and some common errors are presented next from the most difficult items and then by content area.

Common Errors and Misconceptions

Most difficult items. Item TFNS31 (4.32 logits, 0.7% success) showed 51 different error types with 26% of the students multiplying the given quantities, 22% responding "*71, 6.5, 500, 3.25, 32.5, 0.1, 0.05, or 1.2*" and 15% "*500/6.5*" with a 28.8% baulking rate. Item MALG14 (3.23 logits, 1.4% success) showed 55 different error types with 14% of the students responding "*16, 16:1* or *1:16*", 9% wrote "*y=16, n=16 or 16n*", and 6% gave "*16S=1P, x+y=16, 16/P, 16S/P, 16/x=n or A=16/n*" with a high 41.1% baulking rate. These errors suggested conceptual and computational difficulties.

```
Item Estimates (Thresholds) all on all (N = 139 L = 36 Probability Level= .50)
Logits              |  Fractions        Algebra     Probability    Geometry      Measurement   Data Presentation
                    |& Number Sense                                                                & Analysis
5.0                 |
                    |
                    |
                    |
                    |TFNS31
4.0                 |
                    |
                    |
                    |
                    |
                    |                 MALG14                                                   Most Difficult Items
3.0                 |.................................................................................................
                    |
                    |
                    |                              TPRB10
                  X |TFNS09           TALG18
2.0                 |TFNS24
                    |MFNS08
                 XX |
                  X |
                    |
                 XX |                 TALG38                                                      TDPA16
1.0               X |                 TALG33       TPRB13                      TMSR15
               XXXX |                                                          TMSR34
                  X |
              XXXXX |TFNS30 TFNS35
             XXXXXX |TFNS06 TFNS26                 TPRB25
 .0        XXXXXXXX |TFNS32.......................................................................Item Mean = 0.00
             XXXXXX |
          XXXXXXXXX |TFNS22 MMCT02
            XXXXXXX |                 TALG11
           XXXXXXXX |MMCT05
        XXXXXXXXXXX |...............................................................Ability Mean = -0.93
-1.0        XXXXXXX |
          XXXXXXXXX |MMCT04        TALG07 TALG37
          XXXXXXXXX |                                           TGEO20
           XXXXXXXX |                 TALG21
                XXX |                 TALG19                    MGEO29
-2.0          XXXXX |
                XXX |MMCT03                                     TGEO23                         TDPA12
               XXXX |                 TALG36
              XXXXX |
              XXXXX |
                  X |.................................................................................Easiest Items
-3.0          XXXXX |                 TALG28
                  X |
                    |
                    |MMCT01
                  X |
-4.0                |
                  X |
                    |
                    |
                    |
-5.0                |
          Each X represents    1 student
```

Figure 1. MDT 1 Item-Person Map.

Fractions and number sense. The hierarchical difficulty order showed the most difficult to be a multiplicative relationship word problem (average weight) followed by a cluster of items on speed and unit conversion, operation with fractions and ordering fractions. Above the item mean is a cluster of items on speed, ratio, ordering decimals, fraction area-model, and fraction of an amount. Below item mean are items on equivalent fractions and mental computations. The latter, in decreasing difficulty, included computing percentage, multiplying decimals, adding unit fractions, 4-digit subtraction and multiplying 1-digit integers.

For item TFNS09 (2.30 logits, 5% success), the three most common errors (from 64 different error types) were "*3x8 =24m/s*" from 24% of the students; 3% wrote "*3 km=8* *min*" and 1.5% responded "*3000/s*" with a 25% baulking rate. Item TFNS24 (2.11 logits, 5.7% success) showed 50 different error types. The three most common errors were "*38*" (15%), "$\frac{1}{3}+\frac{1}{4}=\frac{7}{12}$," "$\frac{1}{3}+\frac{1}{4}=\frac{2}{7}$," "$\frac{1}{2}+\frac{1}{4}+\frac{1}{24}$" or "$\frac{1}{3}\text{x}\frac{1}{4}=\frac{1}{12}$" (14%), and "*8*" (6%) with a 21.3% baulking rate. Item MFNS08 (1.98 logits, 7.9% success), highest error rate of the test (87.9%), showed 16 different error types. The three most common errors were arranging numerators/denominators in ascending order as "$\frac{2}{3},\frac{3}{5},\frac{5}{6},\frac{7}{10}$" (50%) or descending order "$\frac{7}{10},\frac{5}{6},\frac{3}{5},\frac{2}{3}$" (19%) and "$\frac{7}{10},\frac{3}{5},\frac{5}{6},\frac{2}{3}$" (4%) with a 4.3% baulking rate. Errors suggested conceptual and computational difficulties.

Item TFNS30 (0.38 logits, 17.9% success) showed 38 different error types. The three most common errors were "*330-4.5*" (8%), "*330x4.5*" (8%), and "*330/4.5*" (3%) with a 35.7% baulking rate. Item TFNS35 (0.47 logits, 18.6% success) showed 40 different error types where the three most common errors were "*2:3:6*" or "*200:300:600*" (12%), "*2:3*" (4%) and "*200+300+600=1100*" (4%) with a high baulking rate of 41.1%. Item TFNS06 (0.36 logits, 25.7% success) had 39 different error types. The three most common errors were "*0.5, 0.25, 0.037, 0.125, 0.625*" in increasing (21%) or decreasing decimal places "*0.625, 0.125, 0.037, 0.25, 0.5*" (9%), and "*0.625, 0.5, 0.25, 0.125, 0.037*" (6%) with a 2.9% baulking rate. For item TFNS26 (0.26 logits, 25.7% success), the three most common errors (from 11 different error types) were "*3 squares*" (27%), "*8 squares*" (10%), and "*6 squares*" (6%) with a 14.3% baulking rate. For TFNS32 (0.13 logits, 27.1% success), the three most common errors (from 35 different error types) were "*$150*" (15%), "*$182*" (13%), and "*$234.20*" (3%) with a 11.4% baulking rate. Item TFNS22 (-0.19 logits, 33% success) showed 44 different error types. The three most common errors were an incorrect third equivalent fraction (10%), two incorrect fractions (7%) and "$\frac{3}{4},\frac{4}{5},\frac{5}{6}$" (3%) with a 12.1% baulking rate. Errors demonstrated fraction and place value misconceptions and computational difficulties.

Mental computations. Item MMCT02 (-0.20 logits, 35.7% success), one of two items everyone attempted, showed 28 different error types. The three most common errors were "*30% of 50 = 15%*" (40%), "$\frac{30}{50}x100=60\%$" (3%), and "*30*" (1.4%). Item MMCT05 (-0.66 logits, 44.3% success) showed 12 different error types. The three most common errors were "*0.3x0.3=0.9*" (41%), "*0.3x0.3=9*" (2.2%) and "*0.3x0.3=0.03*" (2.2%) with a 0.7% baulking rate. Item MMCT04 (53.6% success) showed 30 different error types. The three most common errors were "$\frac{1}{2}+\frac{1}{3}=\frac{2}{5}$" (9.4%) indicating fraction misconceptions, "$\frac{1}{2}+\frac{1}{3}=\frac{1}{5}$" (6%) suggesting mis-remembered procedures, and "$1\frac{1}{2}+1\frac{1}{3}=2\frac{5}{6}$" (4%) reflecting poor listening skills with a 0.7% baulking rate. Errors from item MMCT03 (-2.13 logits, 71.4% success) showed 23 different error types. The three most common errors were "*5113*" (4%), "*5003*" (3%), and "*4113*" (3%) with a baulking rate of 1.1%. Item MMCT01 (-0.63 logits, 89.3% success), one of two items everyone attempted, showed 8 different error types. The two most common errors were "*48*" (1.4%) and "*8x7*" (1.4%) which reflected poor knowledge of multiplication facts.

Algebra and problem solving. The algebra item hierarchy also reflected the cognitively more demanding non-routine word problems (multiplicative and additive relationships) at the top-end with simple word problems around the middle and routine procedures towards the lower-end. Item TALG18's (2.27 logits, 5.7% success) three most common errors (from 38 different error types) were "*24m*" (18%), "*15m*" or "*9m*" (16%) and "*12m*" (7%) with a 17.1% baulking rate. Item TALG38 (1.26 logits, 9.3% success) showed 39 different error types. The three most common errors were "$\frac{1275x51}{50}$" (7%), "*1275x51*" (6%) and "*275+51=1320*" (2%) with a high 41.4% baulking rate. Item TALG33 (1.09 logits, 14.3% success) showed 44 different error types. The three most common errors were "*57 females, 29 males*" (16%), "*86-14=72*" (14%), and "*86+14=100*" (9%) with a 15% baulking rate. Error responses from TALG11 (-0.54 logits, 38.6% success) showed 34 different error types. The three most common errors were "*12*" (12.2%), "*3*" (3%), and "*1/3x48 =16*" (2.2%) and a 12.9% baulking rate. Item TALG07 (-1.17 logits, 52.9% success) showed 50 different error types. The three most common errors were "*5*" (4%), "*5x7+6=41*" (2%), and "*n(7+6)=41*" (2%) with a 4.3% baulking rate. Item TALG37 (-1.17 logits, 49.3% success) showed 25 different error types. The three most common errors were "*21 blocks*" (4.3%), "*5 blocks*" (3%), and "*13 blocks*" (1.4%) with a 13.6% baulking rate. Item TALG21 (-1.56 logits, 57.9% success) showed 36 different types, and the three most common errors were "*x=2*" (3%), "*x=6*" (2%), and "$x=\frac{42}{18}$" (1.4%) with a 9.3% baulking rate. Item TALG19 (-1.68 logits, 62.1% success) showed 31 different error types with three most common errors being "$\frac{18}{15}$" (4.3%), "*15*" (2.2%), and "*3*" (2.2%) with a 6.4% baulking rate. Item TALG36 (-2.31 logits, 68.6% success) showed 18 different error types. Three most common errors were "*9, 12*" (6.4%), "*10, 12*" (2.2%), and "*10, 20*" (2.2%) with a 9.3% baulking rate. Item TALG28 (-3.17 logits, 83.6% success) showed 9 different error types. Three most common errors were "*3n*" (2.2%), "n^2" (1.4%), and "*1x1x1*" (1.4%) with a 3.6% baulking rate. Errors suggested conceptual and computational difficulties.

Probability. The item hierarchical order reflected the decreasing level of cognitive difficulty from likely outcomes and expected number to application. Item TPRB10 (2.54 logits, 3.6% success) showed 36 different error types. The three most common errors were "*head*" or "*tail*" (13%), "$\frac{5}{10}$" or "$\frac{1}{5}$" (12%) and "*5/2 or 2.5*" (10%) with a 31.4% baulking rate. Errors with TPRB13 (1.09 logits, 11.4% success) showed 35 different error types. Three most common errors were "$\frac{3000}{5}=600$" (10%), "*100-5=95*" (6%) and "*5x100=500*" (5%) with a 30% baulking rate. Item TPRB25 (0.22 logits, 23.6% success) showed 35 different error types. The three most common errors were "$1/11$" (11%), "$3/11$" (10%), and "$1/3$" (3%) with a 23.6% baulking rate. Errors implied conceptual and computational difficulties.

Geometry. The three items displayed a hierarchy of decreasing cognitive difficulty from calculating a missing angle and similar triangles to identification of a 45° angle. Item TGEO20 (-1.44 logits, 54.3% success), showed 24 different error types. The three most common errors were "*115+115+70=290; 360-290=70*" (6%), "*115-70=45*" (3%), and "*180-115=65*" (3%) with a 12.1% baulking rate. Item MGEO29 (-1.75 logits, 62.1% success) showed 17 different error types. The three most common errors were "*12–6*" (9%), "*10-6=4*" (7%), and "$\frac{1}{2}x5x6$" (6%) with a 7.1% baulking rate. Item TGEO23 (-2.20 logits, 72.1% success) showed 10 different error types. The three most common errors were "*R*" (>90°) (5%), "*Q*" (90°) (5%), and "*P*" (90°) (3%) with a 2.1% baulking rate, indicating forgotten basic geometric facts.

Measurement. The two items were basically the same difficulty level on interpreting data from nested geometric shapes. Item TMSR15 (0.93 logits, 17.1% success) showed 27 different types. The three most common errors were "*144*" (15%), "*64*" (14%), and "*96*" (11%) with a 6.4% baulking rate. Item TMSR34 (0.87 logits, 17.1%) showed 30 different types. The three most common errors were "*16*" (19%), "*12*" (16%), and "*15*" (2%) with a 13.6% baulking rate. Errors indicated conceptual and computational difficulties.

Data presentation and analysis. The hierarchical order of difficulty reflected the level of cognitive processing required to determine a pictograph scale and reading histogram data. Of the 35 error types counted for item TDPA16 (1.19 logits, 13.6% success), the three most common errors were "*51*" (18%), "*8*" (18%), and "*Orange: 6 and Lime:7 houses*" (11%) with a 9.3% baulking rate. Item TDPA12 (-2.06 logits, 70% success) showed 14 different error types with the three most common errors being "*5 pupils*" (17%), "*14 pupils*" (3%), and "*8 pupils*" (1.4%) with a 1.4% baulking rate. Errors indicated conceptual and computational difficulties.

In summary, for the fractions and number sense items, the highest error percentage for a single error type was ordering fractions using only the numerators/denominators (50%) followed by the product of 1-digit decimals as a 1-digit decimal (41%), the percentage of a number as a percentage (40%), and area-model of an equivalent fraction using only the numerator (27%). The next two highest error percentages were words problems where students simply multiplied given quantities for average weight (26%) and average speed (24%). For the probability items, the most common errors (≥10%) indicated misconceptions about likely outcomes, expected number and favourable outcomes. The most common misconceptions (≥5%) for the geometry items were about similarity and basic geometric facts. For the measurement items, the most common errors (15 and 19%) were conceptual difficulties extracting relevant information from diagrams while it was incorrect interpretation of the language of the problem and visual data (17-18%) with the data presentation and analysis items. Finally, if mastery of the mathematics content is set at 80% success rate, then mastery level was not achieved for the majority (94% or 34/36) of the items. Overall, two-thirds of the items were quite difficult as evident from the number of above-item-mean items and less-than-50% success rates. Also high baulking rates (41.1%) were noted for three items requiring critical interpretation of multiplicative descriptions and critical organization and synthesis of information (i.e., critical problem solving). Error analyses provided additional, empirical evidence of the nature and extent of students' content-specific misconceptions and computational difficulties.

Discussion

Rasch statistics established that the diagnostic test was a reliable test to produce a unidimensional, cognitive developmental scale for students' mathematical competence. The item-person map and success rates showed students found non-routine word problems with abstract, multiplicative descriptions the most difficult and basic computations the easiest. This general pattern was also reflected within each content-area. Error analyses provided further insights to the most common errors for each item. From the item-person map and error analyses, it appeared that, in addition to weak content knowledge, students generally demonstrated difficulties in three crucial ways, firstly, critically interpreting the meanings of mathematical concepts in word problems (*average weight*, *average speed*, *likely outcomes*, *ratio* and *perimeter*); mathematical representations (*pictographs*, *bar graphs* and *complex diagrams*); and mathematical language (*twice as long*, *14 more females than males*, *16 students to one professor,* and *more than 5 minutes*). Secondly, student teachers demonstrated difficulties critically transforming their interpretations *arithmetically* to obtain numerical values (geometric and numeric pattern extensions, relational description, and operations with fractions); and *algebraically* to communicate general rules (tabular pattern extension and student:professor). Thirdly, students demonstrated difficulties critically managing, selecting and organizing relevant information (i.e. problem solving skills) to generate plausible solutions. Computational errors were also evident after selecting an appropriate procedure (calculating the interior angle and operations with fractions). Finally, findings from this study of Samoan primary student teachers contribute to the literature on preservice teachers' mathematics content knowledge (Shulman, 1986; Ma, 1999; Mays, 2005; Ball, 2000, Chick, 2002) and further confirm findings reported by others on misconceptions with mental computations (Callingham & Watson, 2004) and word problems involving fractions, and multiplicative reasoning with ratios and proportions (Thompson & Saldanha, 2003).

Conclusions and Implications

Findings from this paper show student teachers' content knowledge of the primary and early secondary mathematics curriculum appears to be lacking in conceptual depth in some content areas. Students' main misconceptions may be a mutual interaction of weak: (1) content knowledge of the curriculum, (2) critical interpretation of mathematical concepts, multiple representations, and language of the problem, and (3) critical problem solving skills (CPSS). CPSS permeate and underpin (1) and (2). Student teachers exhibit poorly developed fraction and number sense such as in ordering fractions and decimals, modeling equivalent fractions, and operating with fractions and applying fractions in ratio and proportion (Thompson & Saldanha, 2003). Findings also suggest students have underdeveloped conceptual understanding of probability, weak knowledge of basic geometric properties (similar triangles, quadrilaterals, rectangles and angle types), and weak algebraic skills. Although they mastered mental multiplication of 1-digit whole numbers and simplifying basic algebraic expressions, solving word problems was difficult. As descriptions of quantitative relationships become increasingly abstract, implicit and multiplicative, students struggle to access the mathematics embodied in problem statements and visual representations whereas they cope better with simple word problems and basic computation. Student errors demonstrate poor critical problem solving skills to interpret and analyse given information effectively, represent, and synthesise relevant knowledge and appropriate procedures to generate correct responses. Since pedagogical content knowledge is dependent on subject-matter knowledge and curriculum knowledge, student teachers need to know the mathematics first as learners before they can teach others to know (Huckstep, Rowland, & Thwaites, 2003). Aspiring to become effective teachers of primary mathematics means being proficient problem solvers who are competent at mastery level with the content of the primary and early secondary mathematics curricula. This implies that explicit remediation of student teachers' identified misconceptions needs to form part of their teacher education courses to specifically enhance their content knowledge, and critical skills in interpreting mathematical concepts, multiple representations, and language used in problems.

References

Adams, R. J., & Khoo, S. T. (1996). *QUEST*: *Interactive Item Analysis.* Melbourne: Australian Council for Education and Research.

Australian Association of Mathematics Teachers (AAMT). (2006). *Professional Teaching Standards.* AAMT.

Ball, D. L., & Bass, H. (2000). Interweaving content and pedagogy in teaching and learning to teach: Knowing and using mathematics. In J. Boaler (Ed.), *Multiple perspectives on mathematics teaching and learning* (pp. 83-104). Westport, CT: Ablex Publishing.

Bond, T. & Fox, C. (2001). *Applying the Rasch Model Fundamental Measurement in the Human Sciences.* Lawrence Erlbaum Associates.

Callingham, R., & Watson, J. (2004). A developmental scale of mental computation with part-whole numbers. *Mathematics Education Research Journal, 16* (2), 69-86.

Chick, H. (2002). Evaluating pre-service teachers' understanding of middle school mathematics. In B. Barton, K. C. Irwin, M. Pfannkuch, & M. O. J. Thomas (Eds.), *Mathematics Education in the South Pacific* (Proceedings of the 25th annual conference of the Mathematics Education Research Group of Australasia, Auckland, pp. 179-186). Sydney: MERGA.

Huckstep, P., Rowland, T., & Thwaites, A. (2003). Primary teachers' mathematics content knowledge: what does it look like in the classroom? Retrieved on February 16, 2007 from http://www.leeds.ac.uk/educol/documents/00002534.htm.

Ma, L. (1999). *Knowing and teaching elementary mathematics.* Mahwah, NJ: Lawrence Erlbaum.

Mays, H. (2005). Mathematical knowledge of some entrants to a pre-service education course. In M. Coupland, J. Anderson, & T. Spencer (Eds.), *Making Mathematics Vital.* Proceedings of the Twentieth Biennial Conference of the Australian Association of Mathematics Teachers (AAMT) (pp. 43-52). Sydney: University of Technology.

McIntosh, A., & Dole, S. (2000). Mental computation, number sense and general mathematics ability: Are they linked? In J. Bana & A. Chapman (Eds.), *Mathematics education beyond 2000.* (Proceedings of the 23rd annual conference of the Mathematics Education Research Group of Australasia Incorporated, pp. 401-408). Perth: MERGA.

Ministry of Education, Sports and Culture (MESC). (2003). *Primary and Secondary Mathematics Curriculum.* Apia, Samoa: MESC.

Mullis, I., Martin, M., Gonzalez, E., Gregory, K., Garden, R., O'Connor, K., Chrostowski, S., & Smith, T. (2000). *TIMSS 1999 International Mathematics Report.* Retrieved on February 16, 2007 from http://timss.bc.edu/timss1999/math_achievement_report.html.

National Council of Teachers of Mathematics (NCTM). (2005). *2000 Principles and Standards.* Retrieved on February 16, 2007 from http://my.nctm.org/standards/document/index.htm.

Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher, 15* (2 February), 4-14.

Thompson, P. W., & Salanda, L. A. (2003). Fractions and multiplicative reasoning. In J. Kilpatrick, W. G. Martin, & D. Schifter (Eds.), *A Research Companion to Principles and Standards for Schools* (pp. 95-113). Reston, VA: NCTM.

Appendix A

Text Descriptions of MDT1 items.

Item	Text Descriptions	Item	Text Descriptions
MMCT01	8 x 7 = ?	TGEO20	Missing angle of a quadrilateral given 70°, 115° and 115°.
MMCT02	What is 30% of 50?		
MMCT03	8006 – 2993 = ?	TALG21	Find the value of x if 12x – 10 = 6x + 32.
MMCT04	$\frac{1}{2}+\frac{1}{3}=?$		
MMCT05	0.3 x 0.3 = ?	TFNS22	Write three fractions equivalent to $\frac{2}{3}$.
TFNS06	Write in ascending order 0.625, 0.25, 0.037, 0.5, 0.125.	TGEO23	In the diagram, which angle has a measure closest to 45°?
TALG07	An unknown number *n* is multiplied by 7 and then 6 is added to the result. The final answer is 41. Write this as a mathematical expression.	TFNS24	Penny had a bag of marbles. She gave one third of them to Rebecca She then gave a quarter of the remaining marbles to Jack. If Penny ended up with 24 marbles, how many did she start with?
MFNS08	Write in ascending order $\frac{5}{6},\frac{2}{3},\frac{7}{10},\frac{3}{5}$.		
TFNS09	An athlete ran 3 kilometres in exactly 8 minutes. What was her average speed in metres/sec?	TPRB25	Eleven chips are labelled 2, 3, 5, 6, 8, 10, 11, 12, 14, 18 and 20 respectively. The eleven chips are placed in a bag

TPRB10	If a fair coin is tossed, the probability that it will land heads up is 1/2. A fair coin is tossed 4 times and it lands heads up each time. What is likely to happen when the coin is tossed a fifth time?
TALG11	If 4 times a number is 48, what is one third of the number?
TDPA12	The graph shows the time taken to travel to school by a group of students. How many pupils travel for more than 10 minutes to reach school?
TPRB13	A sample of 100 light bulbs is chosen at random from a complete batch containing 3000 voters. When the sample is tested, it was found to contain 5 faulty light bulbs. How many faulty bulbs would you expect to find in the complete batch?
MALG14	At a particular university, there is an average of 16 students to every professor. Write this as a mathematical equation.
TMSR15	A rectangular garden bed adjoins a building as shown in the diagram.The garden bed has a path on 3 sides. What is the area of the path?
TDPA16	Two streets in a town have 30 houses (Orange St.) and 21 houses (Lime St.) respectively. This is represented in the pictogram. How many houses are represented by the symbol?
TGEO17	Which two of the four triangles are similar?
TALG18	An elevator starts at the first floor of a building. It travels up to the fifth floor, then down to the third floor and back up to the fourth floor. If the floors are 3 metres apart, how far did the elevator travel?
TALG19	If x = 3, what is the value of $\frac{5x+3}{4x-3}$?
	and one is drawn out at random. What is the probability that the number on the chip is a multiple of 3?
TFNS26	Shade 3/8 in the given (6 x 4) grid.
TMSR27	The length of the rectangle is twice as long as it is wide. What is the ratio of the width to the perimeter?
TALG28	Write in simplest form $n \times n \times n$.
MGEO29	In the diagram (of similar triangles), what is the length of the interval BD?
TFNS30	Sound travels at approx. 330 m/sec. A lighting strike was followed 4.5 seconds later by a clap of thunder. How far away did the lightning strike?
TFNS31	A pile of salt contains 500 individual crystals and has a weight of 6.5kg. What is the average weight of a salt crystal?
TFNS32	Laura had $240 but spent five eighths of it. How much money did she have left?
TALG33	A club has 86 members with 14 more female members than male members. How many males and females are members of the club?
TMSR34	What is the area of shaded rectangle?
TFNS35	A fertilizer mix contains 200g of nitrate, 300g of phosphate and 600 g of potash. What is the ratio of the weight of the nitrate to the total weight of the fertilizer?
TALG36	Extension of a geometric pattern.
TALG37	Extension to 2 terms of a numeric tabular pattern based on ALG36.
TALG38	If we produced a figure with 50 rows, we would require 1275 blocks. Explain how to calculate the number of blocks required to construct a figure with 51 rows.

Number in the Item Code corresponds to the Item Number in MDT1.

An Online Survey to Assess Student Anxiety and Attitude Response to Six Different Mathematical Problems

Vincent Anderson
Launceston College and University of Tasmania
<Vincent.anderson@bigpond.com>

Survey results for anxiety responses and attitude responses to six particular mathematics problems are presented for 43 students from grades 4, 5, and 6. These data are analysed for a relationship between mathematics anxiety and attitude to mathematics. An online survey method is used and is found to be a valuable tool for use in a primary school setting. The six mathematics problems vary in type between traditional levelled tasks in the form of basic mathematical operations and rich tasks. Basic operations are varied amongst three levels of difficulty and rich tasks are varied amongst three degrees of complexity of context. A weak relationship is found between mathematics anxiety and attitude to the six mathematics problems presented to students. Some differences are observed between boys and girls for responses to rich tasks. Also, differences in both attitude and anxiety responses are found due to a variation of problem difficulty for traditional basic operations. Further research is suggested that promises to inform the pedagogies of practicing teachers.

Anxiety response to mathematics is a significant concern to educators in terms of the perception that high anxiety will relate to avoidance of mathematics. An internet search quickly shows the broad interest of many in this subject. This paper presents survey responses of a small sample of 43 upper primary school students. The context of the survey is an online survey environment, where students are asked to consider six particular mathematical problems. After each of the six problems, the students are asked to respond to six questions for anxiety response, based on the survey instrument used by Uusimaki and Kidman (2004), followed by a question of their familiarity with the question and then six questions for an attitude response, based on the survey instrument used by Ma and Kishor (1997).

In this study, anxiety response to mathematics is taken to mean an involuntary emotional response to mathematical problems and mathematical language. Attitude to mathematics is taken to refer to a qualitatively different phenomenon, where the nature of the response is considered and couched in thoughtful, cognitive language. Some explanations of attitude to mathematics treat the concept as synonymous with mathematics anxiety, such as a teacher of mathematical economics, Dr Alpha C. Chiang (Huenneke, 2005):

> Unfortunately, studying mathematics is, for many, something akin to taking bitter-tasting medicine, necessary and inescapable, but extremely tortuous. Such an attitude, referred to as math anxiety, has its roots, I believe, largely in the inauspicious manner in which mathematics is presented to students.

Unlike the position taken by Chiang, the present study interprets attitude response to mathematics as a cognitive response. This differentiation between attitude response as cognitive and anxiety as an emotional response is informed by Goleman (1996), Schlöglmann (2001), Hannula (2005), and Ritchhart (2001). This differentiation is adopted by Kabiri and Kiamanesh (2004). Particularly Goleman (1996) describes the anxiety phenomenon of involuntary emotional responses that operate too quickly for cognitive processing to filter them.

Proceedings of the 30th annual conference of the Mathematics Education Research Group of Australasia
J. Watson & K. Beswick (Eds), © MERGA Inc. 2007

The phenomenon of attitude to mathematics is considered in this study as similar to the concepts of Intellectual Character (IC) and Thinking Disposition, as outlined by Ritchhart (2001). Ritchhart refers to the limitations of traditional concepts of intelligence, and proposes an alternative perspective, where the importance is placed on IC, rather than the traditional, fixed measure of IQ. Ritchhart summarises the various interpretations of Thinking Dispositions from the literature and relates these to the concept of IC. Ritchhart differentiates IC from IQ by claiming that ability is only part of performance, where IC is a demonstration of a will or inclination to use these abilities, and a sensitivity to know when particular abilities are appropriate. Thinking Dispositions proposed by Ritchhart are: openmindedness, a curious nature, metacognition, truth seeking, strategic planning, and a sceptical nature.

Measurement of anxiety response for this study is done through an online survey instrument only. Hanula (2005) identifies the difficulty of measuring affective responses, such as anxiety, to mathematics accurately outside of a psychology laboratory. Hanula favours observations of students to assess emotional response when researching in real life situations. The use of such techniques in a classroom environment was not feasible for this survey. The design of the survey is intended to allow the students to respond immediately to the anxiety survey questions after seeing each mathematics problem, augmented by the use of smiley faces and the immediacy of an online survey.

Unlike the research of Ma and Kishor (1997), as well as Uusimaki and Kidman (2004), this study considers student responses to six particular mathematics problems, rather than mathematics in general. Ma and Kishor suggest that their focus, and that of many researchers, on attitude responses to general mathematics could be too broad to show strong relationships that are meaningful and applicable to pedagogy development. They suggest the use of particular problems or fields of mathematics.

> Perhaps the best solution, before more advanced attitude measures are developed, is to measure specific attitudes toward certain mathematical areas or activities (e.g., arithmetic, problem solving) rather than generalized attitude toward mathematics as a whole. (p. 40)

Due to the use of particular mathematics problems, it is possible to survey the students' familiarity with each problem type. The role of familiarity with the problem type can then be considered in relation to students' anxiety and attitude responses. The use of particular mathematics problems also allows the effect of problem difficulty and problem type to be considered. The problems selected for this survey were conceived as three basic operations, ranging in difficulty from easy to difficult, followed by three rich tasks ranging from a familiar and simple context to an unfamiliar and complex context.

The purpose of this paper is not to elaborate on the difference between rich tasks and traditional basic operations. Ritchhart (2001) argues the merits and attributes of rich tasks. Anderson (2005) reviews literature on this topic.

Research Questions

The research questions posed for this study are:

- What is the typical anxiety versus attitude profile of upper primary school students in response to the three basic operations and three rich tasks selected?
- What effect does the difficulty of the selected basic operations have on student responses?

- What effect does the complexity of the selected rich tasks have on student responses?
- What trends are apparent between anxiety response and attitude response to each mathematics problem?
- Are there any implications for teaching practice in the results of this research?
- Does the method of online surveys support efficient and effective research in the environment of a primary school?

Methodology

The school that participated in this research is a primary school of approximately 300 students in a central suburb of a city with a population of about 250,000 people. The socio-economic background of the student population is modestly wealthy and very homogeneous. Nineteen girls and 24 boys in grades 4, 5, and 6 completed the survey.

An online survey is used to measure students' anxiety response, and then attitude response to six particular mathematics problems. Access to the survey is controlled with a password and a page that asks the students if they agree to participate in the survey. The design of the survey initially requests grade level and gender of the student. The structure of the remaining survey shows a sequence of mathematical problems to the student and asks them to *consider* solving each problem, but *not* to solve them. The online survey format reinforces this request by not allowing any area for an answer to be presented by the student and by stating repeatedly that students do not need to solve the problems. Immediately after the presentation of each problem, an anxiety survey instrument of six questions is presented to the students. Smiley face symbols are used to highlight the emotional nature of the response, as shown in Figure 1.

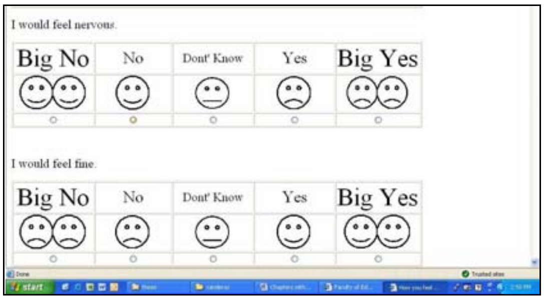


Figure 1. Screen capture of the online survey showing the appearance of radio buttons and formatting.

The anxiety survey instrument is based on that used by Uusimaki and Kidman (2004) and consists of graded responses as shown in Figure 1 to the following six statements:

- I would feel comfortable.
- I would feel nervous.
- I would feel fine.
- I would feel worried.
- I would feel confident.
- I would feel frustrated.

Uusimaki and Kidman (2004) also used an online survey, although with a different format, and they surveyed preservice teachers. The construction of this survey is modified to be more communicative to grade 4, 5, and 6 students through a selection of language, font size, and smiley faces.

To differentiate the attitude section of the survey, the students are specifically asked to respond to what they think about the problem they have seen, marking a deliberate shift to less emotive language. Smiley faces are not used for this reason. Immediately before the attitude survey statements, a familiarity question is posed as a statement, *I have seen problems like this before*, requesting a graded response.

The attitude survey statements that follow are based on those used by Ma and Kishor (1997) and request a graded response, but without use of smiley faces, to the following six statements. As with the survey instrument used for anxiety response, polarity of the questions are alternately reversed in an attempt to neutralize erroneous or random responses.

- I am not good at maths like this.
- I like this kind of maths problem.
- I would not try to answer this maths problem if I didn't have to.
- I think answers to problems like this might be useful in my life.
- I think this problem would not be easy to answer.
- I think maths like this is important in the world.

The six mathematical problems selected are shown respectively in Figure 2 to Figure 5. Problems 1 to 3 are basic operations (Figure 2), or traditional levelled tasks, intended to range from easy to difficult for the sample group. Questions 4 to 6, shown in Figures 3 to 5, are conceived as rich tasks. They demonstrate a broader use of language to describe context and present texts that the students might use in authentic contexts. The screen format in the actual survey for the fifth problem in Figure 4 is much larger than shown here.

Problem 1: $17 - 5 = ?$

Problem 2: $553 + 365 = ?$

Problem 3: $43 \times 17 = ?$

Figure 2. Three basic operation problems used in survey.

If 6 of your friends are coming to your house to share a pie, what shape would you make it so that it would be easy to cut into equal serves? How would you cut it? If one person does not turn up, how could you cut the cake into 5 equal portions?

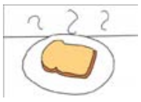
Figure 3. Fourth example problem, sharing a pie between 5 or 6 friends.

You have \$6.30, and your best friend has \$6.90. What items could you buy from the following advertisement?

Do you think that the food in this ad is healthy? How could you measure this?

Figure 4. Fifth example problem, shopping with a friend.

You have 45 friends coming around to your house to eat some French Toast with you. The cook says they have run out of eggs and that you need to run to the store and get enough for everyone. It's up to you to work out how many dozen eggs to get for the recipe shown here.



French Toast

Details.

2 eggs
1/2 cup milk
Pinch of cinnamon
3 to 4 bread slices (preferably stale challah or sourdough)
1 tbsp. butter
Maple syrup, powdered sugar, or orange/raspberry juice concentrate (optional)

What you do.

Crack the eggs into a bowl, add milk, put in pinch of cinnamon.
Whisk until well blended.
Pour the mixture into a pie pan, dip both sides of each slice of bread in the mixture until well soaked
Melt butter in a skillet over medium heat.
Cook the bread for five minutes or until brown underneath, both sides.
Transfer the bread to a clean plate and add the topping of your choice. You've made French toast!
Recipe and illustrations by Mollie Katzen, author of Pretend Soup

Do you think this food will be healthy for everyone who would be coming to share food? What are some reasons that a person might not be able to eat French Toast? How could you include them?

Figure 5. Sixth example problem, needing eggs for a recipe.

The use of an online survey allows the data to be transferred via email for collection and collation. The security of personal data is addressed by de-identifying all of the data, meaning that without the key of the survey, the data cannot be interpreted by third parties. Names are not requested for this reason. The survey is conducted within the classroom in small groups and interaction between the students is not discouraged.

The anxiety and attitude data are analysed using descriptive statistics, including box-and-whisker plots, to compare responses to the six questions for boys and girls. Familiarity data are considered in graphical form as positive, neutral, or negative, again for boys and girls. The relationship of anxiety and attitude responses is summarised in graphical form and the degree of association reported in *r*-squared values.

Results

Anxiety responses are shown in Figure 6 where the discrete data have been normalised between extremes of -1 to +1. The overlap of responses as problems varied was significant. As the sample size was only 43 students, apparent trends should be interpreted reservedly. The most significant feature is the apparent increase in anxiety response to the three basic operations as there level of difficulty increases for both boys and girls. Variation in anxiety between the three rich tasks is less apparent for either boys or girls although there is a suggestion of a slight increase as the problems become more complex.

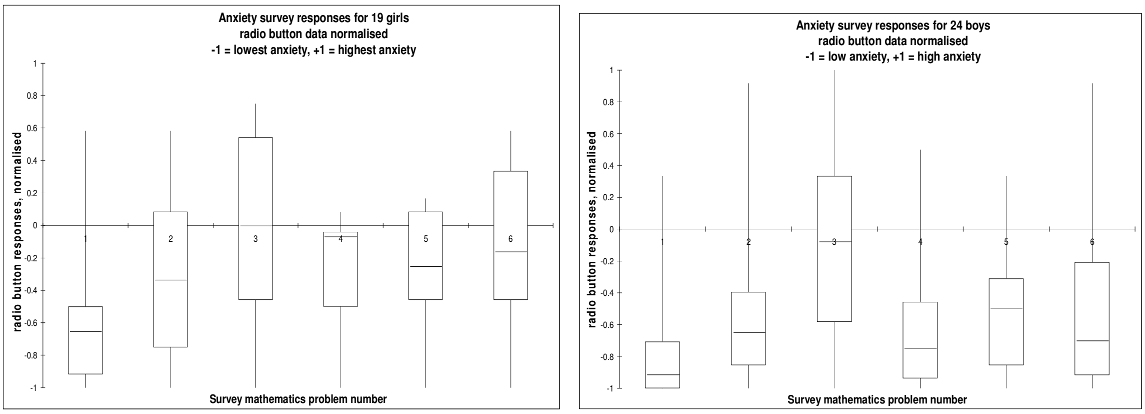


Figure 6. Anxiety survey data spread, 19 girls (left) and 24 boys (right).

The familiarity responses to the six problems are shown in Figure 7, for boys and girls. The responses are grouped as positive, neutral, or negative. Figure 7 shows that the girls familiarity steadily decreases as the problems progress and shows that boys responses are less regular. They show that boys show a potential increase in familiarity with more difficult problems, although a larger sample would be needed to support this interpretation. For five of the six problems, however, the girls express a higher level of familiarity.

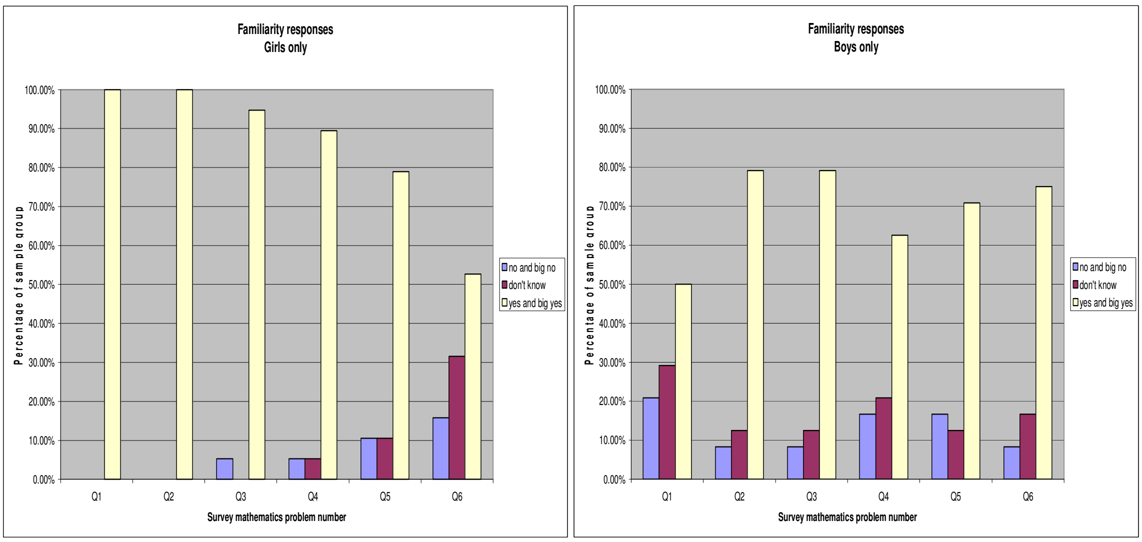


Figure 7. Familiarity responses, 19 girls and 24 boys.

Attitude responses are shown in Figure 8 for boys and girls. The most significant feature is the apparent decrease in attitude response to the three basic operations as the level of difficulty increases for both boys and girls. Variation in attitude between the three rich tasks is not apparent for either boys or girls.

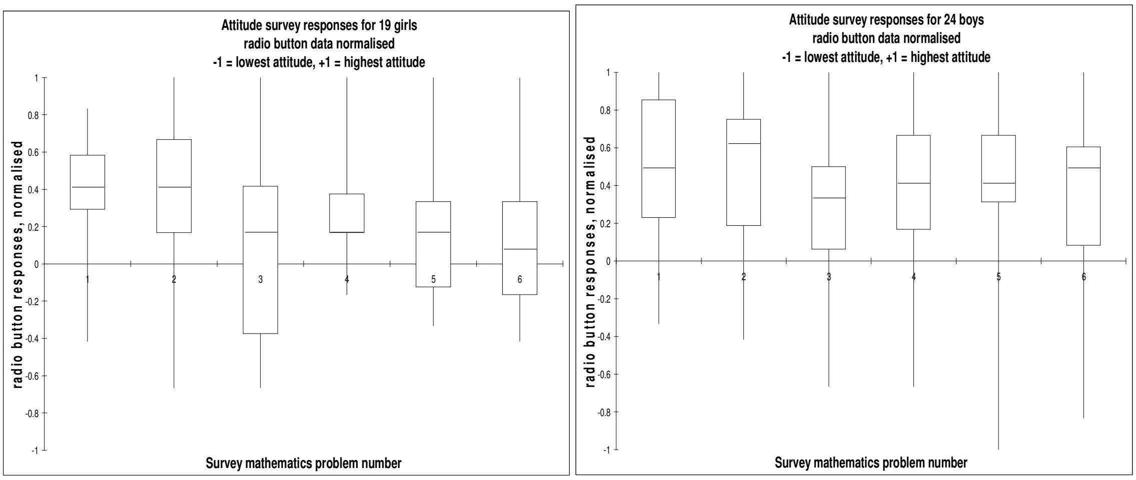


Figure 8. Attitude survey data spread, 19 girls and 24 boys.

An alternative representation of the anxiety and attitude data is shown in Figure 9 with average and standard deviation used instead of the quartiles represented by box-and-whisker plots. Figure 9 supports the apparent sensitivity of anxiety and attitude with relation to difficulty of basic operations for both boys and girls. A general trend of higher anxiety relating to lower attitude response is indicated by responses to the basic operations. Figure 9 shows a potential increase in anxiety for the most complex rich task for girls only. There does not appear to be any other discernable sensitivity of attitude and anxiety with relation to the three rich tasks selected for this survey, particularly for boys.

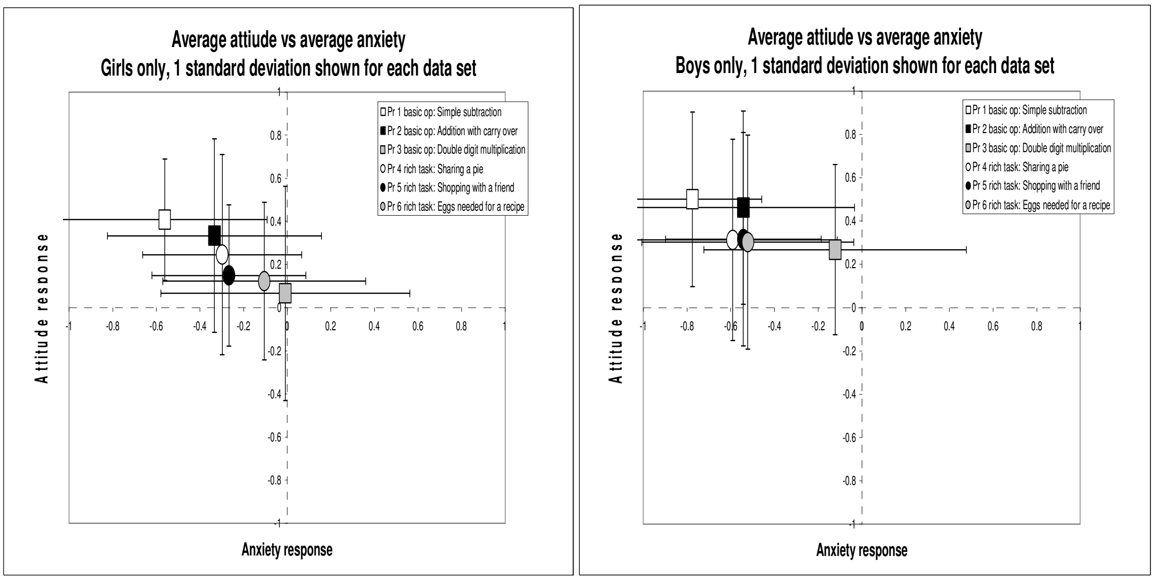


Figure 9. Average attitude versus anxiety for different mathematics problems, girls and boys.

The general trend of a reduction in attitude with an increase in anxiety response is also indicated by Figure 10 where attitude response and anxiety response to all six problems are shown together. Each data point represents one student's response to a particular problem.

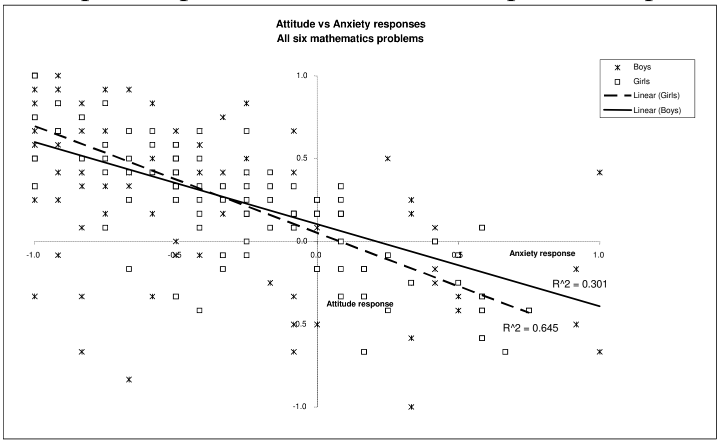


Figure 10. Attitude vs Anxiety responses for all six mathematics problems.

Although not shown in individual graphs here, the squared Pearson's correlation coefficient values for each of the six problems, showing the strength of the association for boys and girls is shown below in Table 1. The sign of all correlation coefficients is negative. For all problems except the first, the association of anxiety and attitude is stronger for girls than for boys.

Table 1
Summary of Pearson's r^2 Values for Variation of Data from Trendlines for Attitude Versus Anxiety

	Boys	*Girls*
Problem 1	0.339	0.336
Problem 2	0.390	0.678
Problem 3	0.302	0.756
Problem 4	0.451	0.672
Problem 5	0.236	0.507
Problem 6	0.231	0.704

Discussion

The role of anxiety response is assumed to be a powerful driver of decision making for students in discontinuing with mathematics or avoiding mathematics and further entrenching an innumerate self perception for those afflicted. Although this survey asks for responses to six particular problems whereas other research asks for responses to mathematics in general, if Figure 10 is considered as indicative of students' responses to mathematics in general, the results compare favourably with other research. Bowd and Brady (2003) cite Hembree (1990), from a meta-analysis of 151 pre-service teachers:

> Hembree also noted that preservice arithmetic teachers were especially prone to mathematics anxiety and that positive attitudes toward mathematics consistently related to lower mathematics anxiety.

Kabiri and Kiamanesh (2004) found a similar relationship with an r^2 coefficient of 0.4 from a survey of 366 Iranian eighth graders.

Answers to Research Questions

What is the typical anxiety versus attitude profile of upper primary school students in response to the three basic operations and three rich tasks selected? Students typically show a reduction in attitude to a mathematics problem that they also show an increase in anxiety response towards. Student responses range almost the full scale of attitude and anxiety with most responses lower than neutral anxiety and higher than neutral attitude. There are very few responses that show a high attitude associated with a high anxiety.

What effect does the difficulty of the selected basic operations have on student responses? The range of basic operations selected caused a surprisingly large difference in anxiety response and an apparent difference in attitude response.

What effect does the complexity of the selected rich tasks have on student responses? The range of rich tasks selected was not associated with significant variation of attitude or anxiety responses.

What trends are apparent between anxiety response and attitude response to each mathematics problem? A weak and negative correlation is found for all problems, where an increase in anxiety response is associated with a decrease in attitude. Squared Pearson's correlation coefficients are shown in Table 1.

Are there any implications for teaching practice in the results of this research? Teachers are invited to interpret these data in terms of their own practice. One feature worthy of note is the small number of responses showing high attitude and high anxiety response. This would indicate that anxiety is not an effective motivator for some students' performances. Knowledge of the potentially stronger association of anxiety and attitude for girls may assist teachers in planning classroom activities and the attitudes they themselves exhibit. Knowledge of increased anxiety with increasing difficulty of basic operations would not be a surprise to teachers but again may assist in planning support for students who struggle.

Does the method of online surveys support efficient and effective research in the environment of a primary school? The online survey method was observed to be engaging to the students. The value of receiving data in an electronic format made translation and analysis of the data easier. This ease of handling data meant that as a researcher in the classroom the researcher could focus on managing the flow of students and answering their technical questions. The format of the online survey was also well received by the three classroom teachers who generously allowed their classes to participate in the survey. There were a handful of students who opted not to participate in the survey of their own accord, whereas two parents responded with withdrawal of consent. Of three classes, the take up rate was very encouraging for the use of web based surveys in the future.

Limitations of the Study and Suggestions for Further Research

The sample size of this study is small. A larger sample group would allow stronger conclusions and invite more detailed analysis of data. A sample that also included a wider range of socioeconomic variation would also allow conclusions to be applicable more broadly to teaching practice.

Anxiety is understood to be an involuntary emotional response, best measured by involuntary responses such as perspiration, body language or twitching, as suggested by Hanulla (2005). Calibration of the anxiety survey instrument in relation to involuntary responses would establish the degree of validity of measuring anxiety response with a survey instrument only.

No deliberate attempt was made explicitly to link the particular skills required to solve the relative rich task and basic operation in this survey design although some similarity was intended. Future survey designs should align the skills required between relative problems of different kinds to allow stronger conclusions when comparing types of problems that teachers might set for students.

This study was not designed to investigate any potential causal relationship between attitude response and anxiety response. A longitudinal study to establish cause and effect between anxiety and attitude would be valuable in terms of informing teaching strategies that might address cause and not effect.

Acknowledgement. This study was conducted as part of the Bachelor of Teaching Honours Program at the University of Tasmania under the supervision of Jane Watson.

References

Anderson, V. (2005). *Investigation of the relationship between anxiety response and attitudinal response to six mathematics problems: upper primary students*. Unpublished Honours Thesis. Hobart: University of Tasmania.

Bowd, A. D., & Brady, P. H. (2003). Gender differences in mathematics anxiety among preservice teachers and perceptions of their elementary and secondary school experience with mathematics. *Alberta Journal of Educational Research, 49*(1). Available at http://proquest.umi.com.ezproxy.utas.edu.au/pqdweb?index=2&did=579671371&SrchMode=3&sid=1&Fmt=3&VInst=PROD&VType=PQD&RQT=309&VName=PQD&TS=1176534708&clientId=20931&aid=1

Goleman, D. (1996). *Emotional Intelligence: Why it can matter more than IQ*. London: Bloomsbury Publishing Plc.

Hannula, M. (2005). *Understanding affect towards mathematics in practice.* Abstract submitted to the Fourth Nordic Conference on Mathematics Education, Relating Practice and Research in Mathematics Education, Trondheim, Norway. Retrieved on 15/7/05 from http://norma05.hist.no/Abstracts/Hannula.pdf

Hembree, R. (1990). The nature, effects, and relief of mathematics anxiety. *Journal for Research in Mathematics Education, 21*, 33-46.

Huenneke, S. (2005). *Does Econ Answer Man have a way for us mortals to do better in economics?* Retrieved 16/9/2005 from: http://warp6.cs.misu.nodak.edu/econ/econanswerman/econans1.html.

Kabiri, M., & Kiamanesh, R. (2004). *The role of self-efficacy, anxiety, attitudes and previous math achievement in students' math performance*. Unpublished paper. Tehran: Institute for Educational Research, Teacher Training University, Tehran, Iran.

Ma, X., & Kishor, N. (1997). Assessing the relationship between attitude toward mathematics and achievement in mathematics: A meta-analysis. *Journal for Research in Mathematics Education, 28*, 26-27.

Ritchhart, R. (2001). From IQ to IC: A dispositional view of intelligence. *Roeper Review, 23*, 143-150.

Schlöglmann, W. (2001). Affect and cognition: Two poles of a learning process. *Proceedings of the 3rd Nordic Conference on Mathematics Education, Kristianstad, Sweden*. Retrieved 9/9/05 from http://www.education.monash.edu.au/centres/sciencemte/docs/vamp/schloglmann2001.pdf

Uusimaki, L., & Kidman, G. (2004, November*). Reducing maths-anxiety: Results from an online anxiety survey.* Paper presented at the Australian Association for Education Research Annual Conference, Melbourne. Available at http://www.aare.edu.au/04pap/kid04997.pdf

Collective Argumentation and Modelling Mathematics Practices Outside the Classroom

Raymond Brown
Griffith University
<ray.brown@griffith.edu.au>

Trevor Redmond
AB Paterson College
<trd@abpat.qld.edu.au>

An important aspect of bringing about change in the mathematics classroom is gauging the efficacy of the change in bringing about learning that has application outside the school classroom. The research reported in this paper is situated within an on-going study where over 20 teachers of mathematics in the middle years of schooling are using the practices of Collective Argumentation to bring about change in their classrooms. This paper reports on one aspect of the study that sought to explore whether students who use Collective Argumentation on a regular basis in their classrooms view mathematics as providing a forum where personal understandings can be expressed, re-considered, shared and co-authored when they go about knowing and doing mathematics in a novel context – an inter-school mathematics modelling challenge. The results of the exploration are discussed and situated within the context of the findings of the on-going study.

The Mathematics Teacher's Lament

"I spent 3 weeks teaching this and the students do not have a clue what I am talking about."

In the day-to-day operation of a school, comments such as the one above are common. When it comes to mathematics, there is evidence to support the idea that many students are disinterested and unwilling to engage in the teaching and learning process (Boaler & Greeno, 2000). Often students question what is being taught in class because they do not see the relevance of what they are doing (Pajares & Graham, 1999). Students question, "Why do we have to know this?"; "Where am I ever going to use this?" Yet it is interesting that many mathematics teachers persist with teaching and learning practices that perpetuate the view that mathematics understandings are transmitted rather than constructed (Schoenfeld, 2004).

In terms of teaching for understanding, Perkins (1992) has identified shortfalls in education. One shortfall that he identifies refers to "inert knowledge" (p. 22), that is, knowledge that is only able to be articulated by the student if the right stimulus is provided by the teacher. A question is asked and that triggers a response that allows the student to give the correct answer. At this level of knowledge acquisition, the teacher may incorrectly assume that a student has developed understanding of a concept only to find later that the student is unable to apply the knowledge to a novel situation.

Another shortfall that Perkins (1992) has identified is "ritual knowledge" (p. 25). In displaying this type of knowledge, students have learnt to play the school game. They are able to use the language of mathematics and they are able to use the correct mathematical procedures to manipulate mathematical expressions such as equations, but they have difficulty modelling the mathematics, for example, building equations, when they are embedded in a novel context.

If, as Perkins (1992) argues "learning is a consequence of good thinking" (p. 8), suitable problem solving routines need to be built in the classroom that allow students to develop understanding through the use of good thinking skills. Students need to be encouraged to use thinking routines (Richart, 2002, p. 89), that may become part of their

Proceedings of the 30th annual conference of the Mathematics Education Research Group of Australasia
J. Watson & K. Beswick (Eds), © MERGA Inc. 2007

repertoires of practice when coming to know and do mathematics. The routine needs to be simple, explicit, and provide students with a mechanism to engage with the task, construct meaning, build a solution, and communicate that solution to others (Richart, 2002, p. 90). One such routine that has been taken up by teachers to privilege student understanding in the mathematics classroom is Collective Argumentation (Brown & Renshaw, 2000).

Collective Argumentation

Collective Argumentation involves the teacher and students in ways of coming to know, do, and value mathematics that reflect the investigative processes and ways of interacting employed by the mathematical community. In simple terms, collective argumentation involves the teacher and students in small group work (two to five students per group) where students are required, initially, to "represent" a problem individually by using pictures, diagrams, drawings, graphs, algorithms, numbers, etc. Students are then required to "compare" their representations with those of other group members. This phase of individual representation and comparison provides the potential for differences in understanding of curriculum content to be exposed and examined. Subsequent talk by the students regarding the appraisal and systematisation of representations is guided by the keywords – "explain", "justify", "agree". Finally, moving from the small group to the classroom collective, the thinking within each group is validated for its consistency and appropriateness as it is presented to the whole class for discussion and validation.

The Teacher's Role in Collective Argumentation

The teacher has an active role throughout each phase of Collective Argumentation. The tasks of the teacher include: (a) allocating management of the problem-solving process to the group; (b) facilitating peer co-operation by reminding students of the norms of participation; (c) participating in the development of conjectures and refutations; (d) modelling particular ways of constructing arguments; (e) facilitating class participation in the discussion of the strengths and weaknesses of a group's co-constructed argument, (f) introducing and modelling appropriate mathematical language; and (g) providing strategies for dealing with the interpersonal issues that may arise when working with others.

This paper explores the effects of Collective Argumentation in making visible students' understandings as they go about knowing and doing mathematics in a novel context – an inter-school mathematics modelling challenge. Specifically, the paper seeks to explore whether a group of students from a Collective Argumentation classroom see mathematics as providing a forum where personal understanding is privileged, that is, as providing a space where personal understandings can be expressed, re-considered, shared and co-authored.

Method

This paper arises from an on-going study of teachers' appropriation of the practices of Collective Argumentation into their everyday teaching of mathematics and/or science. The study is being conducted over a three-year time frame with 20 elementary and middle school teachers of mathematics and/or science from six schools located in South-East Queensland. The study employs a sociocultural design, based on a "design-experiment" (see Schoenfeld, 2006). The "design-experiment" is an extension of Vygotsky's (1987) experimental-developmental method that was designed to capture the determining influence of social and cultural processes on learning and development. From this perspective, the activity of the students, the activity of the teacher/researcher, and the co-constructed activity of the classroom, interrelate at a number of levels to create the “life context” of the mathematics classroom. A “design-experiment”, therefore, requires multiple sources of data to be collected and involves prolonged, systematic inquiry into change through engagement in collaborative cycles of analysis, design, implementation, assessment, and reflection. The authoring of this paper is an artifact of this cyclical process.

Participants. The teacher, whose students are the focus of this paper, had been using the practices of Collective Argumentation to inform his teaching of mathematics to students for one school year. The teacher taught at a P-12 school located in a middle-class suburb of a major city. Thirteen students (seven girls and six boys) from this teacher’s Year 5/6 class and 14 students from other schools had been encouraged by their teachers to participate in the Year 6/7 section of an inter-school mathematics modelling challenge. Three of these students (two girls and one boy) – Helen, Nicole, and Neil – form the focus of this paper. All three students were high-achievers in mathematics.

The Challenge. The challenge was conducted over a 2-day period at the campus of a local university during the last week of November and was attended by 220, Year 4 to Year 11 students from South-East Queensland. Each day of the challenge lasted from 9:00am till 3:00pm and consisted of three sessions broken by morning tea and lunch. During the challenge, students were allocated to a group of four students and invited to work with mathematics educators on authentic mathematical modelling tasks appropriate for their year level. At the completion of the challenge each member of the group, at each year level, judged to have provided the best mathematical model of a solution to a task was awarded a plaque and a calculator valued between $50 and $150.

The Task. The students who form the focus of this paper were allocated to the same group. The group comprised Helen, Nicole, Neil and Aaron (a student from a local state school). Over the two days of the challenge, the students were engaged with the task of designing, building to scale, and mathematically modelling a mini-golf course. Each group was required to design a mini-golf hole on graph paper – complete with blockers, tunnels and other obstacles – and create a theoretical hole-in-one path of the ball such that each angle of incidence equalled the corresponding angle of reflection. Each group was required also to represent their mini-golf hole design on graph paper, provide a spreadsheet showing the segment angles, slopes, and linear equations, and provide a short journal entry of their experience with the challenge. Each group received a poster board, a piece of green felt, wooden blocks, cardboard tubes, and a marble along with graph paper and a criteria checklist. Four computers, connected to the internet were available for the students to use. Clarification of task requirements was provided to each group by a mathematics educator, however no direct teaching of task content was provided.

Data Collection. The targeted group was video- and audio-taped by research assistants at three pre-determined one-hour segments of the mathematics challenge. The first recording occurred in the middle-session of the first day of the challenge, recording sessions two and three occurred in the morning and middle-sessions of the second day. However, the research assistants were present for the entirety of the challenge and video-recorded the targeted group outside pre-determined times when they thought that something of interest to the research was happening. At the conclusion of the challenge, all video- and audio-tapes were transcribed and names were replaced with pseudonyms. Consent was sought and gained from the participants for the transcripts to be used for research purposes.

The sections of transcript provided in this paper were taken from the second pre-determined taping session and from a moment of interest in the challenge when students communicated to other students outside their group. These segments of text were chosen for analysis because they provide instances of students talking about what they learnt and accounts of how they came to know the conceptual elements of the task.

Data Analysis. Bakhtin's (1986) notion of "voice" was used to analyse the transcripts. Bakhtin (1986) formulated a theory of voice that emphasized the active, situated, and functional nature of speech as it is employed by various communities within a particular society. Taking the notion of "utterance" rather than "word meaning" as a basic unit of communication, Bakhtin maintained that in dialogue with others, people align themselves within different speaking positions or voice types as they produce or respond to an utterance or a chain of utterances. Such voice types reflect the social ways of communicating that characterize various group behaviors (eg., professional communities, age groups, and socio-political authorities) that a person has had the opportunity and/or willingness to access. As such, "voice" as used in this paper, encompasses "what" is being said, the "way" in which it is spoken, and the positioning of speakers in relation to the authority framework established within the communication.

Analysis and Discussion

We enter the mathematics modelling challenge when Helen, Nicole, Neil and Aaron (a student from a different school) are preparing a short journal entry of their experience with the challenge. The extract is taken from the second targeted data collection session held in the morning of the second day of the challenge (see Table 1).

Table 1
Maybe our Whole Group Learnt About it

Turn	Speaker	Text
01	Nicole	Aaron learnt, what did you learn?
02	Aaron	I learnt lots.
03	Nicole	Well then tell us.
04	Aaron	I learnt about slope.
05	Neil	Maybe our whole group learnt about it.
06	Helen	I didn't (learn about slope), I had to do it ($y = mx + 3$).
07	Neil	(I learnt) About the equations.
08	Helen	$y = mx + 3$

We enter the script where Nicole is recognising Aaron's "belonging" in the group by asking him what he had learnt from engaging with the mini-golf task. Instead of accepting Aaron's general response (turn 2 - *I learnt lots*) and then moving to record the responses of the other members of the group who were from her school, Nicole encourages Aaron to be reflective and to consider the specifics of what he had learnt (turn 3 – *Well then tell us*). This action reflects a reason Nicole's teacher gave for taking up Collective Argumentation in the classroom, “We want to encourage our students to be reflective and consider how the various concepts (in mathematics) are related”.

Collective Argumentation privileges this level of understanding by requiring students to explain and justify their learning on a regular basis, therefore, making knowledge public. Explaining and justifying involves the gathering and sharing of evidence that satisfies disciplinary constraints associated with coherence and logic. Explaining and justifying allows students to become conscious of others’ ideas and points of view, allowing processes of thought as well as products to become visible.

This privileging of reflecting on understanding continues as the other members of the group comment on their personal understandings relating to “slope”. Here we see students being reflective, considering what they have learnt (turn 7- *About the equations*) and what they did not learn (turn 6 – *I didn’t, I had to do it*). However students saying they have learnt it does not mean they understand it, as Helen reveals in the next sequence of text (see Table 2).

Table 2
I Knew Something that you Didn’t Know

Turn	Speaker	Text
09	Nicole	Helen, what did you learn today?
10	Helen	y = mx + 3
11	Nicole	Didn’t you already know that?
12	Helen	No, how to do it, like I knew what it (slope) was, I just didn’t know how to do it (slope).
13	Nicole	Didn’t you know how to do it (slope)?
14	Helen	You (Nicole) didn’t.
15	Nicole	Yes I did, well I knew how to do it the obvious way, I knew how to do it on a graph, but on quadrant things (quadrants of a full grid).
16	Neil	I knew something that you didn’t know.

Here we see Helen and Nicole linking what they know, considering a different strategy (using y = mx + 3 [turn 10] or graphing a line on a grid [turn 15]) and recognising they are doing the same thing. Through this text, we see Helen and Nicole transferring the mathematical tools they had leant in the classroom to this context and recognising that there are different ways of applying those tools and different levels of knowing about and using mathematical tools.

Collective Argumentation privileges the recognition of multiple representations of a mathematical idea through requiring students to represent a solution or idea about a task individually and to compare their representation with others. When students complete a brief written response to a text, or a solution to a problem, or an evaluation of the effectiveness of an experiment, they are more likely participate in any discussion that follows, ask questions of others, share ideas with others, and to self-monitor their understanding (Gaskins, Satlow, Hyson, Ostertag, & Six, 1994). Comparing representations allows students to see what is the same and what is different about their ideas and interpretations. In the process, it can help students learn by making them view concepts from different perspectives, and can be affirming as students see congruence between ideas and representations (Feltovich, Spiro, Coulson, & Feltovich, 1996).

Through recognising that Nicole is using a graphical approach to completing the task and that Helen is using an algebraic approach, the students pave the way for relating procedural to conceptual understanding, as illustrated in the following sequence (see Table 3).

Table 3
We Used it, but we Didn't Know How

Turn	Speaker	Text
17	Nicole	Neil, what did you learn?
18	Neil	I learnt that, I learnt just that (y = mx + 3).
19	Nicole	What do you mean just that? y = mx + 3?
20	Neil	Just write everybody learnt that (y = mx + 3), because we all did learn that, yeah everybody learnt it.
21	Nicole	I need an eraser.
22	Neil	So you don't have to write just Aaron (learnt y = mx + 3) cause we all learnt it.
23	Nicole	Did anyone else learn anything that's not there (in the journal entry)?
24	Neil	Um maybe we …
25	Nicole	How to use slope or anything?
26	Helen	That (slope) is part of the equation.
27	Neil	Yeah, that's part of the equation. Let's see, what about how to …
28	Aaron	Did you know that equation (y = mx + 3) before we came (to the challenge)?
29	Nicole	We used it (y = mx + 3), but we didn't know how.
30	Helen	That's how to find out 'm'.

Once again (turns 17 & 19) a member of the group, Neil, is asked by Nicole to explicate what he learnt from engaging in the mini-golf task. Neil's admission that he learnt about slope (turn 22 - *So you don't have to write just Aaron cause we all learnt it*) marks a moment in the conversation when this grouping of students from two different schools, have become a group who are willing to take ownership of their learning. In so doing, links are made between "*how to use slope"* (turn 25) and the algebraic equation – y = mx + 3 (turn 26 – *that's part of the equation*) and between the concept of "slope" and its algebraic representation "m" (turn 30).

Collective Argumentation privileges linking conceptual with procedural understanding and linking individual with collective understanding by requiring that the group attain consensus about a response to a task that they can present to the whole class – a response that each member of the group understands. Consensus based on understanding is the end product of a process of considering and critiquing. Students negotiating a common understanding of a representation or idea take learning from the co-operative to the collaborative plane of learning.

This willingness to collaborate in the sharing of understandings continues in the next sequence of text, which was recorded in a moment of interest when the group extended its boundaries to include members from other groups undertaking the challenge. We enter the script where Helen has just explained to her group how to find the slope "m" in the equation y = mx + 3 (see Table 4). During the explanation, students from other groups gather around to listen. The students included Gail (another student from Nicole, Helen, and Neil's school).

Table 4
Let me Explain How to do "m"

Turn	Speaker	Text
31	Nicole	Yeah, I got that, I got it (y = mx +3).
32	Neil	Yeah I do (understand).
33	Nicole	Because I didn't really get it (y = mx + 3) before, but I understand now.
34	Gail	So you (Helen) just explained how to do 'm'?
14	Neil	Let me explain how to do 'm'.
15	Gail	He (points to a boy in her group) needs to figure out also how to do 'b' (the Y Intercept).
16	Helen	mx + 3, equals 'b' equals Y intercept.
17	Gail	He (a boy in her group) doesn't get (understand) it.
18	Helen	Whatever Y intercept is, is 'b'.
19	Gail	He (a boy in her group) doesn't understand.
20	Helen	Y intercept is when the Y, where the point Y is. Well then you (Neil) can explain it then.

The interaction of students in the above text is interesting for students engaged in an inter-school mathematics challenge. The challenge relating to the mini-golf course can be won by one group only. Helen, in demonstrating her understanding of how to find the slope of a line between two points provides an explanation that is attended to by students not in her group. Not only does Helen share her understanding with Nicole (turn 31) and Neil (turn 32), but also she receives a request from Gail (a member of another group) to explain again how to find the slope, as a boy in Gail's group does not understand. Neil requests permission to provide the explanation (turn 14). However, Helen simply revoices the main point of her explanation (turn 16). Upon receiving a signal from Gail that this revoicing is insufficient (turn 19), Helen gives Neil permission to explain. Neil goes on to present an explanation to the gathered audience that results in a number of students from different groups working together to build a model that they can use to make predictions.

Collective Argumentation privileges a view of mathematics as being about engagement in communal practice by requiring each group to present their agreed approach to the class for discussion and validation. Such presentations of group work permit students to engage with the conceptual content of a lesson at their level, to employ their own prior experiences, preconceptions, and language, and to distribute the nature of their knowing across a group rather than in a fashion that focuses on any one individual.

Conclusion

This paper set out to explore the effects of Collective Argumentation in making visible students' understandings as they go about knowing and doing mathematics in a novel context – an inter-school mathematics modelling challenge. The nature of the learning displayed by Helen, Nicole, and Neil as they engaged with the mathematics challenge of designing a hole-in-one mini-golf course, suggests that these students view doing the mathematics as providing a forum where personal understandings can be expressed, re-considered, shared, and co-authored – an unusual stance for students engaged in what might be viewed as a mathematics competition.

The nature of collaboration constructed by Helen, Nicole, and Neil displayed many of the characteristics of Collective Argumentation – a way of teaching and learning mathematics frequently employed by their classroom teacher. The above analysis of student-student interaction suggests that within this group's way of doing the mathematics challenge, understanding emerged around shared practice; that is, a collaborative space emerged where a voice of inquiry was enacted that privileged: (a) the relating of conceptual understanding to procedural understanding (e.g., determining slope and the Y intercept to build the equation, $y = mx + b$); (b) group ownership of learning over individual performance (e.g., designating new learning about slope to the whole group rather than just to an individual within the group); and (c) mathematising, that is, knowing not only the mathematics, but also how and when to use the mathematics, over "ritual" knowing (e.g., as suggested by Nicole's statement – *We used it ($y = mx + 3$) but we didn't know how*).

Within this collaborative space, on-going processes for adding meaning to the mini-golf task such as representing, comparing, and explaining were used by the students in a fashion that allowed their individual representations, ideas, and points of view to become products of the moment, able to be used by others to progress understanding. Students' interactions, as portrayed in the above transcripts, imply that within the collaborative space constructed by the students within the constraints of the mathematics modelling challenge, students not only co-constructed knowledge, but also developed an awareness of the "self" as operating with tools of mathematics (e.g., $y = mx + b$), of the self operating as a mathematician.

This paper has provided some evidence that students who experience Collective Argumentation on a regular basis in their classrooms do see mathematics as providing a forum where personal understandings can be expressed, re-considered, shared, and co-authored when they go about knowing and doing mathematics in a novel context. The interactions between Helen, Nicole, and Neil occurred within a real novel context centred around real mathematics challenges. Rather than displaying individual personalities engaged in competitive intellectual practice, Helen, Nicole, and Neil were drawn into a culture of inquiry that displayed distinct co-operative and collaborative relationships. However, this culture of inquiry did not happen by chance but is, we argue, a result of regular participation in the collaborative partnerships and relationships of Collective Argumentation.

In terms of the larger study in which these students and their teacher are situated, over 80% of the 20+ teachers who commenced doing Collective Argumentation in their classrooms in 2006, have carried these practices over into 2007. The major reasons provided by teachers who ceased participating in the study related to change of school, year level, or status within the school system. The teachers who have continued with Collective Argumentation in 2007 report an increased desire by their students to learn mathematics in the middle school years when doing Collective Argumentation and a corresponding decline in student behaviours that disrupt teaching-learning relationships. Teachers also report a growing need for professional development in the content domains of mathematics as they move away from using textbooks and structured mathematics lessons towards using the practices of Collective Argumentation to scaffold the teaching-learning relationship. In 2007, these teachers will be joined by eight more teachers from their respective schools who, after seeing these teachers successfully negotiate two rounds of reporting to parents, have expressed a desire for their students to use Collective Argumentation to come to know and do mathematics.

Acknowledgements. The research on which this paper is based was funded by the Australian Government through the Australian Research Council Grant DP0667073.

References

Bakhtin, M. M. (1986). *Speech Genres and Other Late Essays.* (C. Emerson & M. Holquist, Eds., V. W. McGee, Trans.). Austin: University of Texas Press.

Boaler, J., & Greeno, J. G. (2000). Identity, agency and knowing in mathematics worlds. In J. Boaler (Ed.), *Multiple perspectives on mathematics teaching and learning* (pp. 171-200). Westport, CT: Ablex.

Brown, R.A.J., & Renshaw, P.D. (2000). Collective argumentation: A sociocultural approach to reframing classroom teaching and learning. In H. Cowie and G. van der Aalsvoort (Eds.), *Social interaction in learning and instruction: The meaning of discourse for the construction of knowledge* (pp. 52-66). Amsterdam: Pergamon Press.

Feltovich, P.J., Spiro, R.J., Coulson, R.L., & Feltovich, J. (1996). Collaboration within and among minds: Mastering complexity, individually and in groups. In T. Koschmann (Ed.), *CSCL: Theory and practice of an emerging paradigm* (pp. 25-44). Mahwah, N.J.: Lawrence Erlbaum Associates.

Gaskins, I.W., Satlow, E., Hyson, D., Ostertag, J., & Six, L. (1994). Classroom talk about text: Learning in science class. *Journal of Reading, 37*(7), 558-565.

Pajares, F., & Graham, L. (1999). Self-Efficacy, Motivation constructs, and mathematics: Performance of entering middle school students. *Contemporary Educational Psychology, 24,* 124–139.

Perkins, D.N. (1992). *Smart schools: Better thinking and learning for every child.* New York: The Free Press.

Richart, R. (2002). *Intellectual character: What it is, why it matters, and how to get it*. San Francisco: Jossey-Bass.

Schoenfeld, A. H. (2004). The math wars. *Educational Policy, 18*(1), 253-286.

Schoenfeld, A. H. (2006). Design experiments. In P. B. Elmore, G. Camilli, & J. Green (Eds.), *Handbook of complementary methods in education research* (pp. 193-206). Washington, DC & Mahwah, NJ: American Educational Research Association and Lawrence Erlbaum Associates.

Vygotsky, L (1987). The development of thinking and speech. In R.W. Rieber & A S Carton (Eds.), *The collected works of L.S. Vygotsky, Volume1: Problems of general psychology* (pp. 191-220). New York: Plenum.

Visual Perturbances in Digital Pedagogical Media

Nigel Calder
University of Waikato
<n.calder@waikato.ac.nz>

Several studies have investigated how the formation of informal conjectures, and the dialogue they evoke, might influence young children's learning trajectories, and enhance their mathematical thinking. In a digital environment, the visual output and its distinctive qualities can lead to interpretation and response of a particular nature. In this paper the notion of *visual perturbance* is explored, and situated within the data obtained, when ten-year-old children engaged in number investigations in a spreadsheet environment.

When learners engage in mathematical investigation, they interpret the task, their responses to it, and the output of their deliberations through the lens of their fore-conceptions; their emerging mathematical discourse in that perceived area. Social and cultural experiences always condition our situation (Gallagher, 1992), and thus the perspective from which our interpretations are made. Learners enter such engagement with fore-conceptions of the mathematics, and the pedagogical medium through which it is encountered. Their understandings are filtered by means of a variety of cultural forms (Cole, 1996), with particular pedagogical media seen as cultural forms that model different ways of knowing (Povey, 1997). The engagement with the task likewise alters the learner's conceptualisation, which then allows the learner to re-engage with the task from a fresh perspective. This cyclical process of interpretation, engagement, reflection, and re-interpretation continues until some resolution occurs.

This echoes of Borba and Villareal's notion of humans-with-media (2005), where they see understanding emerging from an iterative process of re-engagements of collectives of learners, media and environmental aspects, with the mathematical phenomena. Some models of human behaviour likewise incorporate mind, mediating tools and tasks with societal and community influences, for example, activity theory (Engestrom, 1999). Other researchers emphasise the eminence of mental schemes, which develop in social interaction (e.g., Keiren & Drijvers, 2006). In essence the mathematical task, the pedagogical medium, the fore-conceptions of the learners, and the dialogue evoked are inextricably linked. It is from their relationship with the learner that understanding emerges. This understanding is their interpretation of the situation through those various filters.

When learners investigate in a digital environment, some input, borne of the students' engagement with, or reflection on the task, is entered. The subsequent output is produced visually, almost instantaneously (Calder, 2004) and can initiate dialogue and reflection, perhaps internally for the student working individually. This will lead to a repositioning of their perspective, even if only slight, and they re-engage with the task. They either reconcile their interpretation of the task with their present understanding (i.e., find a solution) or they engage in an iterative process, oscillating between the task and their emerging understanding. This allows for a type of learning trajectory that can occur in various media (Gallagher, 1992), but is evident in many learning situations that involve a digital pedagogical medium (Borba & Villareal, 2005).

There are, however, opportunities or constraints associated with the process. This paper is concerned with one aspect that might be perceived as a constraint, *visual perturbances*, but which can offer opportunities for enhanced mathematical understanding. When the students' fore-conceptions suggest an output that is different to that produced, a tension arises. There is a gap between the expected and the actual visual output. It is this visual perturbation that can either evoke, or alternatively scaffold, further reflection that might lead to the reshaping of the learners' perspectives: their emerging understanding. It shifts their conceptual position from the space they occupied prior to that engagement. The learner's reaction, if it emerges as a conceptual tension, is what I am defining as a *visual perturbance*. It is the tension for the learners between what their fore-conceptions indicated and the actual visual output the pedagogical medium produced.

As learners re-engage with the task, informal mathematical conjectures often have their speculative beginnings (Calder, Brown, Hanley, & Darby, 2006). Other researchers have noted that the development of mathematical conjecture and reasoning can be derived from intuitive beginnings (Bergqvist, 2005; Dreyfus, 1999; Jones, 2000). This intuitive, emerging mathematical reasoning can be of a visual nature. In both algebraic and geometric contexts learners have used visual reasoning to underpin the approach taken to conjecturing and generalisation (Calder, 2004; Hershkowitz, 1998). Meanwhile, Lin (2005) claims that generating and refuting conjectures is an effective learning strategy, whereas argumentation can be used constructively for the emergence of new mathematical conceptualisation (Yackel, 2002). Visual perturbances, and the dialogue they evoke, can generate informal conjectures and mathematical reasoning as the learners negotiate their interpretation of the unexpected situation. Research into students' learning in a computer algebra system environment (CAS), likewise revealed that probably the most valuable learning occurred when the CAS techniques provided a conflict with the students' expectations (Keiran & Drijvers, 2006). If the visual perturbance induced by investigating in a digital medium meant the learner framed their informal conjectures in a particular way, it is reasonable to assume that their understanding will likewise emerge from a different perspective.

Method

This paper reports on an aspect emerging from the data of an ongoing study into how spreadsheets, as a pedagogical medium, might influence learning trajectories and filter understanding in problem solving processes. This part of the study involved a group of ten-year-old students, attending five primary schools, drawn from a wide range of socio-economic backgrounds. There were four students from each school, who had been identified as being mathematically talented through a combination of problem solving assessments and teacher reference: eleven boys and nine girls. Their discussions were audio recorded and transcribed, each group was interviewed after they had completed their investigation, and their onscreen output was printed out. For this paper, the transcripts and printouts, together with informal observation and discussion formed the data that were analysed. The data were coded for NVIVO analysis, and then analysed for emerging patterns.

Results and Discussion

The data in this study illustrated the notion of visual perturbance. We examine some of the episodes in the data that illustrate different types of visual perturbance and ways in which they influenced the students' interpretation and learning trajectories. It is interesting to note that they do not necessarily emerge discretely, but that an episode can illustrate several types of visual perturbance in an interrelated manner.

Episode 1

This relates to an activity set in a scenario that allowed the children to explore different ways that they could get a pocket money allowance. This particular dialogue and output relates to investigating one possible option: receiving one cent the first week, and then doubling each week, that is, two cents the second week and so on. The children initially began to enter the counting number sequence into the spreadsheet.

1
2
3

Mike, using his current understandings in number operation, immediately had a conflict between what he saw, and what his more global perspective was telling him it should be. This created the visual perturbance, one that prompted re-engagement of an exploratory nature.

> Mike: Hey, there's a bit of a twist, look, third week he gets 4 cents. We'll have to change it.

His mathematical fore-conceptions and understanding of the situation allowed him to predict with confidence the outcome of 4 cents for the third week, yet the screen displays 3. Hence he recognised the tension and articulated the need to reconcile this. This facilitated the process by which the output is produced. It also suggested a process of re-negotiation of what the task was about: their interpretation of the task rather than the engagement in its investigation. His partner Jay started to enter input into cell A2.

> Mike: No, no, no we'll have to be in C (column C of the spreadsheet).

This was another visual perturbance, but of a different nature. It seemed to be primarily due to his present understandings of the structure and processes of the spreadsheet environment, rather than his mathematical fore-conceptions. Thus, they were addressing a technical or formatting aspect associated with their investigation. Mike was also perhaps looking to show in some way the relationship between the counting sequence, in this case illustrating the number of weeks, and the amount of money received each week. The pedagogical medium through which he engaged the mathematical phenomena was beginning to structure his approach to the task and his thinking. It was this informal indication of a relationship, and the possibility of a pattern to the amount of money received, that was the beginning of the mathematical thinking, however.

Jay entered 1 into cell C1 to represent the cent for the first week. He began to enter a formula into C2, which he simultaneously verbalised:

> Jay: = C1 + 1 + 0

The output in C column was now:

1
2

Mike suggested the next entry:

Mike: = C2 + 2

The output was now:
1
2
4

Jay: Goes up by two. We have to double each week.

He pondered on the input to the next cell (Cell C4).

Jay: = C3 +

He considered which number to add to C3 to continue the doubling pattern. Mike meantime, addressed the same output, but his fore-conceptions were different, so his thinking was too. His interpretation of the question, the spreadsheet, and his mathematical understanding of the processes involved also influenced his thinking.

Mike: According to this it doubles each week.
Jay: How do you make it double?
Mike: Times by two, and star is times.

Mike took over the keyboard and entered =C3*2 into cell C4 then filled down in the cells below.

Jay: Look at the amount of cash you get on double though.
Mike: That's the biggest one.
Jay: See that huge amount of cash.

The spreadsheet has enabled them to process the large amounts of data quickly with the particular medium shaping their investigation in a distinct, structured manner. Their surprise with the difference between what they expected from option 2, and the size of the actual output is illustrative of a visual perturbance. Throughout the process, the visual perturbances, the difference between what their existing understanding suggested and the actual output, influenced their decisions, and hence their learning trajectory. Their mathematical reflection was a function of their interaction with the task filtered by the pedagogical medium through which it was encountered, and their prevailing mathematical discourse. As their perspective was also repositioned through each interaction, the spreadsheet environment has also influenced this aspect.

Episode 2

The next scenario illustrated a different type of visual perturbance. Tension evoked from the variance between the expected and actual output was evident, but in this situation the visual perturbance arose when the actual output was beyond the scope of the children's current conceptualisation. This involved the scientific form of very large numbers. The students sought teacher intervention, for reconciliation of their mathematical fore-conceptions with the output.

This episode related to a traditional Grand Vizier problem with the doubling of grains of rice for each consecutive square of a chessboard, and investigating how long this might feed the world for. This investigation was initiated after the children had already had some experience of using the spreadsheet. They were less tentative regarding the operational aspects of using them, for example, they were more comfortable generating formulas, and had an expectation of what output they might get based on some accumulated experience.

Ana: It goes 1,2,4,8,16 … , so its doubling
Lucy: =A1 times 2.
Ana : Is that fill down.
Lucy: Go down to 64.
Ana: Right go to fill, then down.

They made an initial interpretation of the problem, and immediately saw a way the spreadsheet would help them explore the problem. However, there was some unexpected output in a visual form they could not recognise.

Lucy: What the …
Ana: Eh…
Lucy: What you…
Ana: 9.22337 E+18.

The unexpected outcome produced a significant perturbation as they attempted to reconcile it with their existing understanding. This was a visual perturbance that was associated with an idea or area they had no previous conceptual cognition of, that is, scientific notation. They quickly decided it was beyond their conceptual scope and sought the teacher's input. The teacher gave some explanation about scientific form related to place value. They made sense of this within their current conceptualisation.

Lucy: So that would be the decimal space up 18 numbers.

They wrote it out on paper to get a picture of it within their current frame:
9223370000000000000
They re-engaged with the activity from their repositioned perspective.

Lucy: We have to add it all up.
Ana : Wow it's big.
Lucy: = A1+A2+A3 …
Ana : Takes a long time, because its 64.

Lucy was using a simple adding notation with the spreadsheet, to sum the column of spreadsheet cells A1, A2, A3 etc. Ana realised, and articulated, that there were 64 cells from A1 to A64, so it would take a long time to enter them individually. They acknowledged the scope of this particular task, and intuitively felt the medium offered possibilities for a more efficient approach. They reflected on prior knowledge and earlier experiences, and negotiated a way to undertake their decided trajectory more easily.

Lucy: Sum.
Ana : = sum (A1:A64).
Lucy: 1.84467E19.

Ana: How long will that feed?
Lucy: 1.84467E19 divided by 2000.

The sum of the values in cells A1 to A64 was 1.84467 x 10^{19} that is, 18446700000000000000. There was no reaction to the scientific form of the output at all this time, and they were almost seamlessly moving into the next phase of their investigation with the newly reconciled concept. Their prevailing discourse in this area had been repositioned through the reconciliation of their fore-conceptions with the unexpected output. This reconciliation and subsequent repositioning was initiated by the visual perturbance they encountered as a result of investigating in this particular pedagogical medium.

Episode 3

The next two scenarios related to an activity investigating the pattern formed by the 101 times table.

The two students had entered the counting numbers into column A and were exploring the pattern formed when multiplying by 101 in column B:

1
2
3 etc.

Awhi: =A2 * 101. Enter.
Ben: 202.

Contemplating the output produced from their unique conceptual perspective, they postulated an informal, rudimentary conjecture through prediction.

Awhi: Now let us try this again with three. Ok, what number do you think that will equal? 302?
Ben: No, 3003. They copy the formula down to produce the output below.
1 101
2 202
3 303 etc.

Ben: (continues) 303.

The actual output was different to the output they expected. This created a visual perturbance, which in this case was easily reconciled with their present understanding. The visual perturbance had caused a reshaping of their prediction that allowed them to reposition their conceptualisation. It also initiated the beginnings of a conjecture or informal generalisation.

Awhi: If you go by 3, it goes 3 times 100 and zero and 3 times 1; 303.

They then explored a range of two and three digit numbers, before extending the investigation beyond the constraints of the task.

Awhi: Oh try 1919.
Ben: I just have to move that little number there, 1919.

The following output was produced:
193819
Interestingly, they seemed to disregard this output and form a prediction based on their fore-conceptions.

Awhi: Now make that 1818, and see if its 1818 (the output).
Ben: Oh look, eighteen 3, 6, eighteen.

There was a visual perturbance, which made them re-engage in the activity, reflect on the output, and attempt to reconcile it with their current perspective. It caused them to reshape their emerging conjecture.

Awhi: Before it was 193619-write that number down somewhere (183618) and then we'll try 1919 again.
Ben: Yeh see nineteen, 3, 8, nineteen. Oh that's an eight.
Awhi: What's the pattern for two digits? It puts the number down first then doubles the number. This is four digits. It puts the number down first then doubles, and then repeats the number.

The visual perturbance made them reflect on their original conjecture and reposition their perspective on the initial, intuitive generalisation. It stimulated their mathematical thinking, as they reconciled the difference between what they expected and the actual output, and rationalised it as a new generalisation. This new generalisation was couched in visual terms.

Episode 4

The next episode was part of the same investigation, but with a different pair of children, as they began to explore what happens to decimals. Ant predicted that if they multiplied 1.4 by 101, they would get 14.14.

Bev: I get it, cos if you go 14 you'll get fourteen, fourteen.
Ant: We'll just make sure.

They entered 1.4, expecting to get 14.14 as the output.

Bev: 141.4, it should be 1, 4 (after the decimal point, that is 14.14).

This created a visual perturbance. They began to rationalise this gap between the expected output (14.14) and the actual output (141.4). This visual perturbance caused a reshaping of their conjecture or informal generalisation. In doing so they drew on their current understandings of decimals and multiplication, but also had to amend that position to reconcile the visual perturbance the pedagogical medium has evoked. Again they used a visual lens to do so.

Ant: We're doing decimals so its 141.4.
Bev: So it puts down the decimal (point) with the first number then it puts the 1 on, then it puts in the point single number whatever.
Ant: It takes away the decimal to make the number a teen. Fourteen.
Bev: 141.
Ant: Yeah. It takes away the decimal (14 – my insertions) and then it adds a one to the end (141), and

then it puts the decimal in with the four (141.4).

Bev recognised that this as more of a visual description of this particular case rather than a generalisation. There was still a tension with her existing understanding.

Bev: No it doesn't, not always, maybe. It might depend which number it is.
Ant: Try 21 or 2.1. See what that does.

According to Ant's conjecture from earlier they would be expecting to take away the decimal point (21), add a one to the end (211), and then re-insert the decimal point and the one (211.1). However the output was 212.1, which created another visual perturbance to be reconciled.

Bev: No it doesn't.
Ant: Two, where's point? One two point one.
Bev: Oh yeah, so its like, the first number equals…

They tried to formulate a more generalised conjecture. Bev proffered a definition that they negotiated the meaning of, then situated within their emerging conjecture.

Ant: Takes away the decimal and puts that number down then puts the first number behind the second number. Aw, how are we going to write this?
Bev: It doubles the first numbers.
Ant: Takes away the decimal, doubles the first number, then puts the decimal back in.
Bev: How does it get here?

They then entered 2.4 and made predictions regarding the output in light of their newer conjecture.

Ant: Twenty-four, twenty-four with the decimal in here.
Bev: It will be doubled; twenty-four, twenty-four but the last number has a point in it, a decimal.

Their predictions were confirmed, and they negotiated the final form of their generalisation. They were still generalising in visual rather than procedural terms, and Bev suggested a name for their theory, double number decimals, that they both had a shared sense of understanding of. This mutual comprehension had emerged through the process: the investigative trajectory they have negotiated their way through. The investigative trajectory was directly influenced by the pedagogical medium through which they engaged the mathematical activity. More specifically, the questions evoked, the path they took, and the conjectures they formed and tested were fashioned by visual perturbances: the tension arising in their prevailing discourse by the difference between the expected and actual output. The process should not necessarily stop just there, however. An intervention, perhaps in the form of a teacher's scaffolding question, might initiate the investigation of why this visual pattern occurs.

Conclusions

Each of the above episodes illustrated how the learning trajectory, was influenced by the learners' encounter with some unexpected visual output as they engaged in tasks in this particular domain, through the pedagogical medium of the spreadsheet. The perturbation, and the dialogue that ensued as the learners reconciled their existing perspective with this unexpected output, seemed to create opportunities for the re-positioning of their existing understanding, as they negotiated possible solutions to the situations.

The engagement with the task, and with the medium, often evoked dialogue. This was an inherent part of the negotiation of understanding. When the students' fore-conceptions suggested an output that differed to that produced, a tension arose. This output, in visual form, initiated the learners' reactions, reflections and subsequent re-engagement with the task. The learners posed and tested informal conjectures, and negotiated a common interpretation through dialogue. This facilitated mathematical thinking, and they developed new understanding.

The data in this study illustrated the notion of a visual perturbance. Within this notion there seemed to be several manifestations or variations.

1. When the visual perturbance led to a change in prediction. It caused an unsettling and repositioning of the prevailing discourse, but the re-engagement was of an exploratory nature.
2. When the visual perturbance caused a reshaping of the conjecture or generalisation. This was similar to that above, but the re-engagement was more reflective and global in nature as compared to a specific example. This was more often accompanied by a significant amount of dialogue and negotiation of meaning.
3. When the visual perturbance made them re-negotiate their sense making of the task itself. This was not a distinct process from the investigative trajectory, but interwoven, with each influencing the other.
4. When the visual perturbance was associated with an idea or area they had no previous conceptual cognition of. The tension this evoked often led them towards seeking further intervention, frequently in the form of teacher led scaffolding.
5. When the visual perturbance led them to further investigate and reconcile their understanding of a technical or formatting aspect associated with their exploration. This was often also symbiotically linked to the conceptual exploration, but sometimes in unexpected ways. For instance, the rethinking of their approach to formatting an actual formula due to a visual perturbance was a structural aspect, but they were simultaneously re-engaging with a mathematical process while negotiating their understanding of the format, for example, in this case some form of algebraic thinking.

These episodes illustrated that the particular pedagogical medium of the spreadsheet, at times induced a particular approach to mathematical investigation. This occurred through the tension that arose from the learners' engagements with the task, when the actual output differed from that which their fore-conceptions led them to expect. This output being in visual form, led to the term visual perturbances, and it appeared this was a particular characteristic of the learning trajectory when using spreadsheets. It may be that this is a generic characteristic of learning trajectories in digital media. Certainly the literature suggested that with CAS software, unexpected outcomes that arose while engaging with algebraic tasks through that medium, influenced the learning trajectories and provided rich opportunities for learning (Kieren & Drijvers, 2006). It appears to be an area that would benefit from further investigation.

References

Bergqvist, T. (2005). How students verify conjectures: Teachers' expectations. *Journal of Mathematics Teacher Education*, *8*, 171-191.

Borba, M. C., & Villarreal, M. E. (2005). *Humans-with-media and the re-organisation of mathematical thinking: Information and communication technologies, modelling, experimentation and visualisation.* New York: Springer.

Calder, N. S. (2004). Spreadsheets with 8 year olds: Easy to visualise? *Teachers and Curriculum, 1*, 125-143.

Calder, N. S., Brown, T., Hanley, U., & Darby, S. (2006). Forming conjectures within a spreadsheet environment. *Mathematics Education Research Journal, 18*(3), 100-116.

Cole, M. (1996). *Cultural psychology: a once and future discipline.* Cambridge, MA: Harvard.

Dreyfus, T. (1999). Why Johnny can't prove. *Educational Studies in Mathematics, 38*, 85-109.

Engestrom, Y. (1999). Activity theory and individual and social transformation. In Y. Engestrom, R. Miettinen, & R.-L. Punamaki (Eds.), *Perspectives on Activity Theory* (pp. 19-38). Cambridge: Cambridge University Press.

Gallagher, S. (1992). *Hermeneutics and education.* New York: State University of New York Press.

Hershkowitz, R. (1998). About reasoning in geometry. In C. Mammana & V. Villani (Eds.), *Perspectives on the Teaching of Geometry for the 21st Century* (pp. 29-37). Dortrecht: Kluwer.

Jones, K. (2000). Providing a foundation for deductive reasoning: Students' interpretations when using dynamic geometry software and their evolving mathematical explanations. *Educational Studies in Mathematics, 44*(1-2), 55-85.

Keiren, C., & Drijvers, P. (2006). The co-emergence of machine techniques, paper-and-pencil techniques, and theoretical reflection: A study of CAS use in secondary school algebra. *International Journal of Computers for Mathematical Learning, 11*, 205-263.

Lin, F. (2005). Modelling students' learning in argumentation and mathematics proof. In H. Chick & J.L. Vincent (Eds.), *Proceedings of the 29th Conference of the International Group of the Psychology of Mathematics Education* (Vol. 1, pp. 3-18). Melbourne: PME.

Povey, H. (1997). Beginning mathematics teachers' ways of knowing: The link with working for emancipatory change. *Curriculum Studies, 5*(3), 329-343.

Yackel, E. (2002). What we can learn from analyzing the teacher's role in collective argumentation. *The Journal of Mathematical Behaviour, 21*(4), 423-440.

Professional Experience in Learning to Teach Secondary Mathematics: Incorporating Pre-service Teachers into a Community of Practice

Michael Cavanagh
Macquarie University
<michael.cavanagh@mq.edu.au>

Anne Prescott
University of Technology, Sydney
<anne.prescott@uts.edu.au>

Wenger (1998) and Lave and Wenger (1991) developed a social theory of cognition in which learning takes place as a result of one's *legitimate peripheral participation* in a *community of practice*. In this paper, we apply Lave and Wenger's theory in learning to teach secondary mathematics. We report on clinical interview data concerning the practicum experiences of eight students enrolled in the Graduate Diploma in Education programs at two universities. Factors which influence the pre-service teachers' classroom practice include the pedagogy of the supervising teacher, the academic ability of pupils, and concerns about classroom management.

One of the most significant rites of passage in learning to teach secondary mathematics is the period of school-based professional experience known as the practicum. The practicum is completed under the supervision of a more experienced teacher who is charged with the task of assisting the pre-service teacher develop confidence and expertise in the art of teaching. The practicum is also designed, notionally at least, as an opportunity for novice teachers to experience first-hand the convergence of the theory discussed in their university methods course with the daily practice of the classroom. However, the practicum is far more idiosyncratic in nature than that, and the degree to which these goals are achieved rests almost exclusively with the individual cooperating teacher, who is more likely to see his or her role as one of inculcating the pre-service teacher into the traditional norms of the status quo (Jaworski & Gellert, 2003).

This paper reports on the most recent stage of a project in which we have followed a group of Graduate Diploma of Education (Grad Dip) students in two universities and investigated the pre-service secondary teachers' beliefs about mathematics and mathematics teaching as they progress through their university studies (see Prescott & Cavanagh, 2006 for a report of our earlier work). Here we focus on the pre-service teachers' experience of their practicum and how they intend to teach as they begin their first year of employment in a school.

Learning as Participation in a Community of Practice

Much of what an individual learns about teaching from his or her practicum experience is gained through interactions with others in various communities and so the contexts of these communities are crucial in determining the nature and extent of what is learned (Cooney, Shealy, & Arvold, 1998). In recent years, the work of Lave and Wenger (1991) has proved helpful to researchers in understanding how pre-service teachers come to know and learn about the practice of teaching. For Lave and Wenger, learning is a social activity that is derived from active engagement in the world in a *community of practice*. Such communities are characterised by mutual engagement in valued enterprises that are defined by the participants through a shared repertoire and that hold the community together. Thus the community of practice is by no means a homogeneous grouping since it includes both veterans, who are fully absorbed into the culture of the community, and novices, who are just beginning to gain greater participation in the community and become more knowledgeable about its shared history.

There are four key components in Lave and Wenger's social and situated view of learning. They are: *meaning*, which is a way of discussing how we experience the world as relevant and meaningful; *practice,* a way of talking about the shared social, cultural and historical perspectives that sustain mutual engagement; *community,* which is the unit of organisation in which the joint enterprise is recognised and defined; and *identity,* which describes the role of learning in changing who we are and how we define ourselves. In particular, Wenger (1998) characterises three modes of belonging and sources of identity formation: *engagement* or mutual participation in joint tasks; *imagination,* a willingness to explore and try new things, and then reflect on how these relate with other practices; and *alignment,* which is concerned with the convergence of a common focus, cause, or interest.

Lave and Wenger (1991) describe the position of neophytes within the community of practice as *legitimate peripheral participation*, by which they explain both the developing identity of participants in the community of practice and the very formation of these communities in the world. The newcomers' legitimate peripheral participation provides them with more than a vantage point from which to observe the inner activity of the community, it also necessarily involves a place from which to move to greater levels of participation in the culture of the community. Learning is, therefore, not so much concerned with replicating the performance of others or acquiring knowledge transmitted during instruction, but rather occurs through becoming part of the community and having access to a wider range of ongoing activity in its practice.

An important aspect of legitimate peripheral participation involves learning the language of the community and how to talk to other members, and so Lave and Wenger (1991) distinguish between *talking within* and *talking about* the practice of a community. Talking within the practice of a community is a sign of full participation in its shared repertoire and is essential to the task of negotiating new meanings, transforming identity, and developing greater levels of participation. Talking about the practice of a community from outside of it is usually associated with the formal learning of beginners, since the effect of this talk is not full membership of the practice because it necessarily occurs on the edges of the community. The nature of the learners' discourse can therefore serve as a useful distinction between theory (talking about) and practice (talking within).

Participation in the Community of Practice of Mathematics Teacher Education

A growing number of researchers have employed Lave and Wenger's (1991) theory of communities of practice to describe the experience of learning to teach mathematics (see, for example, Adler, 1998; Goos & Bennison, 2006; Smith, 2006). In particular, the notion of legitimate peripheral participation in a community of practice can provide a rich conceptual framework for understanding pre-service teachers' knowledge acquisition during their practicum because the nature of such participation emphasises both the personal and social nature of learning. In this sense, learning to teach is concerned not so much with developing new skills, but rather with the individual pre-service teacher's socialisation into the ways of thinking and operating of the practicum school, the supervising teacher and the other members of the teaching staff, and how each individual is influenced by membership of these communities.

The role of legitimate peripheral participation also highlights the importance of on-going activity in the actual practice of teaching as the primary means by which a person learns to become a teacher. In Lave and Wenger's (1991) view, becoming a full participant in the community of secondary mathematics teaching involves engaging with the everyday discourse of practising teachers and actively building relationships in that community by doing things together with practising teachers. Access to and use of the tools and artefacts in the community are crucial if pre-service teachers are to legitimise their peripheral participation and make visible the meaning of the shared repertoire of mathematics teaching, thus enabling the development of more complete and richer forms of participation (Graven, 2004).

Participation in a community of practice is not unidirectional. It involves a good deal of give and take on the part of its members because engagement in a community shapes the experience of individuals who, in turn, help to negotiate new forms of community by virtue of the diversity of their interactions within it. In other words, the community of practice of mathematics teaching inevitably grows according to the endeavours of its members, both in terms of what they know and how they act within the community. Pre-service teachers also make an important contribution to their practicum experience by virtue of their personal history and previous experience of schooling, which act as a prism through which they view the practicum classroom. However, their lengthy "apprenticeship of observation" (Lortie, 1975, p. 61) as pupils can also make it more difficult for pre-service teachers to imagine alternative approaches to teaching from those which they received in their own education. The likelihood is that the lessons the pre-service teachers observe during their practicum placement are not radically different from those they experienced when they were in high school and this produces a "familiarity pitfall" (Feiman-Nemser & Buchmann, 1985, p. 56) that is difficult to overcome.

The physical and social settings in which pre-service teachers undertake the activity of learning to teach are an integral part of the learning that takes place within them (Putnam & Borko, 2000). The learning environment is of particular importance when the reform approach to mathematics teaching taken in the university methods course is not matched by a similarly progressive stance in the practicum school and there is growing evidence that the pre-service teachers' interactions with the supervising teacher and the classroom climate of the practicum are powerful influences on pre-service teachers' own practice (Shane, 2002). So, even though pre-service teachers are regularly exposed to progressive pedagogical approaches at university, they nevertheless often shift to more traditional teaching practices as they move into the practicum and begin their teaching career.

Most pre-service secondary mathematics teachers excelled at the subject when they were in high school. They are likely to have been placed in the top mathematics classes and to have responded positively to the traditional teaching that they received, achieving good marks on written tests and examinations. Their initial identity formation as mathematics teachers was shaped by these experiences and Zeichner and Tabachnick (1981) suggest that their traditional views remain latent during the pre-service teachers' university studies only to reappear when they enter the classroom. The prospective teachers are sustained in the culture of teaching they first observed as pupils, a process of identity formation that is reinforced during the practicum (Frykholm, 1999). However, they are still capable of talking *about* reform-oriented mathematics teaching or writing university essays that espouse the benefits of student-centred learning.

During the practicum, opportunities to re-imagine other forms of teaching mathematics are limited, largely because the pre-service teachers tend to focus almost exclusively on the technical aspects of teaching, especially classroom management and organisation. They plan lessons that are often tightly structured and predominantly teacher-centred because they believe that such an approach is more likely to discourage student misbehaviour. During lessons, they are more concerned with monitoring their own actions than attending to students, and often fail to notice whether any significant student learning is taking place. Thus the chance of alignment between the community of the university methods course and that of the practicum school is severely restricted.

This paper focuses on a small group of pre-service secondary mathematics teachers and seeks answers to the following research questions.

1. Which factors influence the pre-service teachers' classroom practice during their practicum experience?
2. Based on their practicum experience, what pedagogical approaches do the pre-service teachers intend to use in their first year of teaching?

Method

The Grad Dip programs at Macquarie University and the University of Technology, Sydney [UTS], are both one-year, full-time equivalent, professional qualifications for secondary teaching. They are comprised of units in education, curriculum, methodology, and a supervised professional experience practicum of 10 weeks duration. The Grad Dip is available to graduates with academic qualifications in mathematics or a related area of study and most students are mature-aged and have decided to train as mathematics teachers after some previous work experience. At Macquarie, the practicum is completed in a single school under the direction of one teacher, sometimes in small blocks of one or two weeks, but predominantly on one teaching day per week over the course of an entire school year. At UTS, students undertake the practicum in two five-week blocks in separate schools and so have a separate supervising teacher in each school.

All applicants for the Grad Dip at Macquarie and UTS were invited to participate in the research project. A random sample of 16 pre-service teachers (eight from each institution) was subsequently taken from those applicants who accepted a place in the Grad Dip at their chosen institution and returned a signed consent form. The students were interviewed immediately prior to commencing the Grad Dip (February), approximately half way through the program after they had completed at least twenty days of the practicum [June], and at its conclusion (November). Eight participants were involved in the middle and final interview rounds, which are reported in this paper.

The pre-service teachers were interviewed individually for approximately 20 minutes on each occasion. The interviews were semi-structured and designed to investigate how the pre-service teachers interpreted their practicum experiences. We were particularly interested in the factors that the participants identified as playing a major influence on their teaching practices. We also wanted to hear about the style of mathematics lessons that the participants observed during the practicum and the extent to which the pedagogy of their supervising teachers differed from the reform approach taken at the university. In the final interview, we also asked the participants to look ahead to their first year of teaching and discuss how they intended to reconcile these apparent differences. All of the interviews were recorded and transcribed for later analysis of recurring themes.

Results

The Practicum Experience

The participants in our study recognised multiple influences on their teaching, both while they completed their practicum and when they reflected on the experience after it was completed. The pre-service teachers often recalled how the classroom practices of their supervising teachers fitted well with memories of their own time as high school mathematics students. The student-teachers reported that most of the mathematics lessons observed during their practicum followed a familiar pattern: reviewing the work from the previous day, some teacher exposition of new material, worked examples on the board, and individual seat work for pupils to practise new skills and procedures. The most common description reported by the pre-service teachers was one of "chalk and talk" lessons where pupils completed many textbook exercises, working predominantly on their own. The student-teachers' own high school experiences bore close resemblance to their practicum observations, a fact which served to reinforce this style of teaching as an acceptable and workable model of pedagogy.

The expectations of university lecturers also had some influence on the pre-service teachers' pedagogy during their practicum, but these were often dismissed as unworkable in the "real world" of the classroom. For instance, some pre-service teachers believed that the reform teaching approaches encouraged by university staff were more useful for high-achieving students than the predominantly low-ability classes they were usually required to teach.

In contrast to the university lecturers, the practicum supervising teachers were far more influential in shaping the participants' teaching styles. Typically, the participants in our study characterised their supervising teachers as "traditional" and claimed that it was difficult to experiment with working mathematically tasks in the classroom because the supervising teacher was dismissive of such an approach. This was most apparent when the student-teacher devised a lesson plan focusing on group work or activity-based learning but the supervising teacher insisted that the plan be changed to a more teacher-centred method of delivery. The pre-service teachers often reported that their mentor teachers complained that the reform approaches encouraged at the university did not allow for the completion of a sufficient number of practice exercises during lessons.

The pressure to conform to the supervising teacher's style was also seen as a factor in determining the kind of final practicum report that each student-teacher would receive. Even though the determination of the student's grade for the practicum rested ultimately with the university, the report of the supervising teacher was a high-stakes document in the minds of the pre-service teachers because they used it in job interviews as evidence of their teaching capabilities. The pre-service teachers concluded that the best way to guarantee a good report was to follow closely the supervising teacher's advice, which usually meant teaching in a traditional way.

Classroom management was an important consideration for most student-teachers and although many commented that the textbook-based lessons of their cooperating teachers were not very effective in terms of student learning, the pre-service teachers felt that such lessons were easier to teach because “you don’t have to prepare as much” and you can have “more control over the class”. The student-teachers wanted to keep a tight rein over their classes until they had established themselves in the role of the teacher and sensed that students respected their authority. They did not feel comfortable in allowing students too much latitude through the use of investigations or open-ended tasks and tended to “write things up on the board and get them [the students] to copy into their books” because they regarded this approach as more likely to lead to compliance from students. The pre-service teachers wanted to concentrate on developing their basic teaching skills and thought this would be easier if classroom management concerns were minimised, and the best way to guarantee this was to use teacher-centred strategies.

Often the student-teachers linked the style of teaching they employed to the academic ability of the class. As one commented, “If you’re teaching a really good class that you can trust to do stuff, then it’s different”. Another noted that “with really weak students … you just [say], ‘this is a result you need to learn’”. However, the supervising teacher was often reluctant to allow group activities with brighter classes because of the perceived need to cover as much content as possible in preparation for examinations and to ensure what the supervisor regarded as the best preparation for the senior years.

Some of the participants in our study did begin to reflect more critically on the teaching they had received as pupils themselves and on the supervising teacher’s lessons they attended during the practicum. One student-teacher stated that the traditional approach “never really fitted with the way I learned” and that it was “a bit of a lazy way to teach”. Another compared his own learning in university methods classes and workshops with observations of pupils during the practicum and concluded that a student-centred approach was a more effective pedagogy. But these student-teachers also reported that they found it difficult to depart too far from the style of the supervising teacher because the pupils reacted against any change from the traditional classroom routines to which they had become accustomed. As one student teacher remarked, “It was their [the supervising teacher’s] school and their classroom, their students”.

The student-teachers were naturally inexperienced and lacked some basic skills in promoting class discussion through questioning and motivating students, so their first, tentative steps in using alternative teaching strategies were usually not very successful and often resulted in minimal student participation or learning. One student-teacher commented, “I said [to the class], ‘Alright, go and start discussing things for yourself’, but they just talked and carried on”. She then concluded that “student-centred [teaching] is a harder way to teach”. Another pre-service teacher recognised that one more likely source of these difficulties was that the pupils, too, lacked experience in this type of classroom interaction.

> The kids are not used to learning that way [group activities] and they don’t really know what to do … They have not yet learned to learn that way, I believe.

Since the student-teachers’ initial attempts at reform approaches fell so short of their expectations, they were reluctant to try them again, particularly when they perceived that the supervising teacher, who would later write their final practicum report, was also unimpressed by these lessons.

Looking Ahead

As part of the interviews, we asked the student-teachers to look ahead to their first year of employment in schools and discuss how they intended to teach, and the factors that they imagined might influence their classroom practice at that time. All of the participants in our study expressed the desire to "eventually" conduct lessons that conformed to the reform practices they had been exposed to at university. However, they expected to find themselves in mathematics faculties much like those they experienced during their practicum: ones where traditional teaching approaches were the norm.

A common theme among the pre-service teachers was that when they started teaching in the subsequent year, they did not believe they would have much support from other mathematics staff members because most of their colleagues would not be accustomed to a reform style of teaching and therefore could not offer practical advice on how to implement it in the classroom. As one pre-service teacher noted, "it means you don't have as many people to ask for help" and, as a result, there would not be the resources and ideas available that could be shared with a new teacher who intended to adopt a student-centred approach.

The student-teachers' practicum experiences convinced them that the workload of a new teacher would be very demanding, especially in terms of lesson preparation. They felt that the additional requirement of imagining activities and organising materials for more creative lessons that were designed for a student-centred approach would be excessive. Therefore, it would be necessary for the beginning teachers to "resort" to a style of teaching they believed to be ineffective in order to survive the early years of teaching while they gathered resources for themselves. As one student-teacher stated "it won't be practical for me to be spending hours doing research for an hour lesson".

Another student-teacher in our sample was concerned that that his colleagues would be unimpressed if he attempted to use activities and investigations with students because they would not regard this as an acceptable form of teaching, especially if there was a lot of noise and commotion coming from his room. He felt that the other staff members would

> see my classroom as messy, as noisy, as not good teaching because for them good teaching is a completely quiet classroom … with their heads down doing their exercises.

To avoid any perceived conflict with other teachers, this student-teacher concluded that he would be a "textbook teacher" (i.e., teach predominantly from the textbook) for a while and then gradually introduce other activities for his students when he thought he could maintain better control over the class. Others noted that students, too, had certain expectations about the kind of lessons they would receive when they arrived for class and that they "expect a certain style of teaching in mathematics", which typically meant a traditional approach. Thus, the pre-service teachers thought that it might be difficult to overcome their students' demand for instrumental rules and procedures and teach for relational understanding using an investigative or discovery style.

As a result of these factors, most of the student-teachers planned to use a mix of approaches as they started their first year of teaching; some believed that they would introduce group work very gradually, whereas others wanted to start relatively early so that they could begin to train their students according to their reform pedagogy. As one participant remarked:

> As the year goes on I think is when you give kids more and more responsibility for themselves … but not let them go too far until you know that when you say "Ok class, now sit back and listen to me", you know they're going to listen to you.

All of the student-teachers in our study commented favourably on the fact that they would no longer have to contend with the difficulties associated with teaching classes that essentially belonged to another teacher. There would not be the conflict and confusion of classes being taught in more traditional methods for most of their lessons by the supervising teacher, and then occasionally using more student-centred approaches by the pre-service teachers, when they were permitted to do so. In a sense, the student-teachers recognised that, from now on, they would be master or mistress of their own destiny.

Discussion

One of the consistent themes to emerge from our interviews is the fact that pre-service teachers struggle with a number of competing (and perhaps conflicting) demands in their professional preparation. This is especially so during the practicum where student-teachers are in a period of significant identity transformation as they begin to participate in the community of practice of secondary mathematics teaching. The process is made more difficult because although the student-teachers have some responsibility for the classes they teach, the ultimate authority still rests with the supervisor. And although the pre-service teachers have some freedom to develop their individual teaching persona, they often feel constrained by the style of their supervisor. Moreover, even though student-teachers can plan lessons according to their own ideas, they must nonetheless present them to the supervising teacher for final approval.

So there is an unavoidable tension between one’s past experiences as a *student* and the brief intermediate period as a *student-teacher*, when one is beginning to engage in the work of a *teacher*, and is still not fully regarded as a member of the teaching community. The high school mathematics lessons when the participants in our study observed the work of their own teachers were formative encounters and clearly influential, both in imagining a life as a teacher and in deciding to embark on a teaching career. To some extent at least, the student-teachers have to overcome the limitations of these experiences in order to develop new ways of imagining themselves as teachers. Like many intending secondary mathematics teachers, they enjoyed the subject at school and responded favourably to the traditional forms of teaching in their own education. Moreover, they tend to believe that their own students will react just as positively to a similar direct instruction model and so they find it difficult to imagine a need to teach in any other way (Ball, 1988).

Notions of what constitutes “good teaching” are thus formed early on and can prove difficult to shake, particularly because they are often based on the personalities of individual teachers rather than on pedagogical principles (Lortie, 1975). Such initial observations are necessarily from the students’ perspective, so the meanings that are attached to them lack any real appreciation for the subtleties of the craft of teaching, which might explain why the pre-service teachers in our study interpreted their practicum experience in fairly simplistic or idealistic terms that conceived teaching primarily as technical competence, particularly in terms of classroom management, rather than as a process of on-going decision-making focused on student learning.

Our interview data suggest a clear division between the social constructivist approaches discussed at university and the more traditional practices of many supervising teachers. Ebby (2000) notes that although practicum classrooms do not necessarily need to be models of constructivist pedagogy, they must provide a place in which student-teachers can at least imagine possibilities beyond traditional norms and experiment with new ways of teaching. However, our research indicates that not only do pre-service teachers have very limited opportunities to observe reform teaching during their practicum, but also they are also unlikely to receive much encouragement to try it for themselves. Pre-service teachers' identity formation is therefore compromised by the disjointed nature of their university and school-based programs, and the tasks of engagement, imagination, and alignment (Wenger, 1998) become more complex and problematic. As a result, student-teachers sometimes struggle to engage meaningfully in what appear to be two separate communities of practice that are, in many respects, at odds with each other.

Conclusion and Further Research

It is commonly quite difficult to place student-teachers in schools for their practicum and the shortage of those who are willing to act as supervisors often means that there is only a rudimentary screening of supervising-teacher applicants. The comments from participants in our study indicate that supervising teachers appear to see their role predominantly as one of giving advice about the practical concerns of classroom routines and organisation rather than in developing the student-teachers' reflective pedadgogy. We plan to investigate the supervising teachers' perceptions of their responsibilities more fully and test this assertion in a follow-up study.

The trainee-teachers we spoke with often used the language of reform teaching but there are doubts about whether they really understood what they were discussing, since, as Lave and Wenger (1991) point out, it is difficult to talk *within* a community and imagine teaching in a particular style if you have never done so in practice. Indeed, like Zeichner and Tabachnick (1981), we sometimes had the distinct impression that the participants were telling us what they thought we wanted to hear rather than what they really believed. It therefore remains to be seen whether the participants latently hold traditional views that will eventually re-emerge when they are on their own, or if these pre-service teachers really do begin to implement the reform teaching approaches they have indicated that they want to try in their first year of teaching. We will investigate the classroom practices of these student-teachers in our future research.

References

Adler, J. (1998). Lights and limits: Recontextualising Lave and Wenger to theorise knowledge of teaching and of learning school mathematics. In A. Watson (Ed.), *Situated cognition and the learning of mathematics* (pp. 161-177). Oxford: Centre for Mathematics Education Research.

Ball, D. (1988). Unlearning to teach mathematics. *For the Learning of Mathematics, 8*(1), 40-48.

Cooney, T. J., Shealy, B. E., & Arvold, B. (1998). Conceptualizing belief structures of preservice secondary mathematics teachers. *Journal for Research in Mathematics Education, 29*(3), 306-333.

Ebby, C. B. (2000). Learning to teach mathematics differently: The interaction between coursework and fieldwork for preservice teachers. *Journal of Mathematics Teacher Education, 3*, 69-97.

Feiman-Nemser, S., & Buchmann, M. (1985). Pitfalls of experience in teacher preparation. *Teachers College Record, 87*(1), 53-65.

Frykholm, J. A. (1999). The impact of reform: Challenges for mathematics teacher preparation. *Journal of Mathematics Teacher Education, 2*, 79-105.

Goos, M., & Bennison, A. (2006). An online community of practice for pre-service and beginning teachers of secondary mathematics. In J. Novotná, H. Moraová, M. Krátká, & N. Stehlíková (Eds.), *Mathematics in the centre* (Proceedings of the 30th annual conference of the International Group for the Psychology of Mathematics Education, Vol. 3, pp. 209-216). Prague, Czech Republic: PME.

Graven, M. (2004). Investigating mathematics teacher learning within an in-service community of practice: The centrality of confidence. *Educational Studies in Mathematics, 57,* 177-211.

Jaworski, B., & Gellert, U. (2003). Educating new mathematics teachers: Integrating theory and practice, and the roles of practising teachers. In A. J. Bishop, M. A. Clements, C. Keitel, J. Kilpatrick, & F. K. S. Leung (Eds.), *Second international handbook of mathematics education* (pp. 829-863). Dordrecht: Kluwer Academic Publishers.

Lave, J., & Wenger, E. (1991). *Situated learning: Legitimate peripheral participation.* Cambridge: Cambridge University Press.

Lortie, D. C. (1975). *Schoolteacher: A sociological study*. Chicago: University of Chicago Press.

Prescott, A., & Cavanagh, M. (2006). *An investigation of pre-service secondary mathematics teachers' beliefs as they begin their teacher training*. In P. Grootenboer, R. Zevenbergen & M. Chinnappan (Eds.), *Identities, Cultures and Learning Spaces* (Proceedings of the 29th annual conference of the Mathematics Education Research Group of Australasia, Vol. 2, pp. 424-431). Sydney: MERGA.

Putnam, R. T., & Borko, H. (2000). What do new views of knowledge and thinking have to say about research on teacher learning? *Educational Researcher, 29*(1), 4-15.

Shane, R. (2002). Context and content: What are pre-service teachers learning about teaching mathematics? In S. Goodchild & L. English (Eds.), *Researching Mathematics Classrooms: A Critical Examination of Methodology* (pp. 119-154). Westport, Connecticut: Praeger.

Smith, T. (2006). Becoming a teacher of mathematics: Wenger's social theory of learning perspective. In P. Grootenboer, R. Zevenbergen, & M. Chinnappan (Eds.), *Identities, cultures and learning spaces* (Proceedings of the 29th annual conference of the Mathematics Education Research Group of Australasia, Vol. 2, pp. 619-622). Sydney: MERGA.

Wenger, E. (1998). *Communities of Practice: Learning, Meaning and Identity.* Cambridge: Cambridge University Press.

Zeichner, K. M., & Tabachnick, B. R. (1981). Are the effects of university teacher education 'washed out' by school experience? *Journal of Teacher Education, 32*(3), 7-11.

Young Children's Accounts of their Mathematical Thinking

Jill Cheeseman
Monash University
<Jill.Cheeseman@Education.monash.edu.au>

Barbara Clarke
Monash University
<Barbara.Clarke@Education.monash.edu.au>

As part of a larger study exploring teacher behaviours that challenge children to probe their mathematical understandings, children were interviewed about their mathematical thinking and asked to reflect on their learning. Fifty-three interviews were conducted in four schools with 5- to 7-year-old children. The subjects were involved in close conversation with their teachers during the mathematics lesson. Video-stimulated recall was used with a conversational interview to prompt children's recollections and reflections. Findings indicate that young children in the first years of schooling are able to recall events in their mathematics lessons to reconstruct their thinking and reflect on their mathematical learning.

Background

The theory of social constructivism underpins this research. Cobb, Wood, Yackel, and McNeal (1992) and Sfard, Nescher, Streefland, Cobb, and Mason (1998) argued that the construction of knowledge occurs within a social and cultural context where discourse is a vital component in establishing an effective learning context. The focus of this research is the meaning constructed between the teachers and children in classrooms.

There has been a long history of interviewing young children to describe their mathematical thinking (e.g., Donaldson, 1978; Gelman & Gallistel, 1978; Hughes, 1986; Irwin,1996). These interviews often involved children performing mathematical tasks to demonstrate their thinking or development. Task-based interviews have also been used to assess and plot the growth of the mathematical thinking of children over time (e.g., Clarke & Cheeseman, 2000). However there appears to be little research that reports young children's reflections on their thinking in post-lesson interviews.

Franke and Carey (1997) conducted interviews to research first-grade children's views about what it means to do mathematics in problem solving classrooms. They found that young children were in fact able to reflect on classroom events.

McDonough (2002) reported procedures that prompted 8- to 9-year-old children to articulate their beliefs about mathematics. Children found it a difficult to talk abstractly about learning, however, they "held beliefs about mathematics, learning and helping factors and could articulate beliefs when prompted" (p. 270). Although acknowledging the scarcity of research in the area, McDonough expressed little surprise that children even younger than those in her study could describe their mathematical thinking and learning after lesson of the day (McDonough, 2007, private communication).

Method

To capture some of the complexities of classrooms settings and to collect rich data, the approach termed *complementary accounts methodology* was used for this study (Clarke, 2001). Although the methodology used for the large study differed from that of Clarke, similar fundamental techniques were used. These include videotaping the whole mathematics lesson, audio taping participants' reconstructions of classroom events, and an analysis of the multiple data sets.

Proceedings of the 30th annual conference of the Mathematics Education Research Group of Australasia
J. Watson & K. Beswick (Eds), © MERGA Inc. 2007

In total, 53 children were interviewed on the day their mathematics lesson was conducted. The children were aged 5 to 7 years from four classes, each in a different school. The four schools were different from each other in geographic, socio-economic and cultural background and the only common characteristic was that each of the teachers was female. The selection of students was based on classroom observation notes of the researcher and where possible, the recommendation of the teacher. In some cases it was not possible to have a conversation with the teacher before children's interviews began.

The interviews were audio taped for transcription and analysis. A video of the lesson was used as a stimulus to recall sections of the lesson directly involving each child. Children were asked to recount events where they were in conversation with the teacher, to say what they were thinking at the time, and to reflect on what they had learned in the mathematics lesson. The interview was conversational in style. Although there was an interview script, it was adapted in order to elicit responses from each child. The scripted questions were:

1. I am interested in the times when teachers talk to kids in maths lessons—you know when they are really just talking to one child. I noticed that your teacher had a talk with you / stopped to work with you / asked you about your work in that maths lesson. Can you remember that? Can you tell me what happened?
2. I think that we got that on video. Would you like to see it?
3. What were you thinking about? (Maybe just watch it at first.)
4. Can you say what was happening?
5. What did you learn in maths today? Was there anything else?

These questions are modelled on those used by Clarke (2001, pp. 13-32). The original research was with secondary students, and so the language used in the questions has been simplified for young children. In fact it was not known whether children as young as 5 years old could give an account of classroom events where they were challenged to think mathematically. Hence the research question: to what extent can young children give a subsequent account of a classroom mathematical event from their perspective?

Video-Stimulus Recall

There appears to be scant literature describing the use of stimulated recall using video with young children. There are reports of Year 8 children, using video-stimulated interviews to reconstruct the learner's perspective (e.g., Williams, 2003) and reports of teachers video-stimulated recall of the events in their classrooms (e.g., Ainley & Luntley, 2005) but there seems to be no use of this methodology in mathematics education with young children.

Because little was known about how young children would respond to video-stimulated interviews, some piloting occurred. In the pilot stage, young children responded to the video of the mathematics lesson in a very different way from that of their teachers. When teachers were shown excerpts of the lesson they were able to jump into the moment and to talk about what was going on and even reconstruct their thinking at the time. Young children though, would watch the video as a passive observer and if asked at the end of the event to talk about what was happening there, they would give a look as if to say "What do you mean? You just watched what was happening!" They seemed to feel that the video required no explanation or interpretation. After a while it became clear that the best way to prompt recall was to play a little of the beginning of an incident of interest to set the scene for the child then to pause the video and to ask, “Do you remember that bit, what was happening there?”

If a child had no recollection of the event, the entire video episode involving them in conversation with the teacher was played and used as a stimulus to help them describe their thinking or reflect on their learning. In general, the video was used as a starting point and it was paused as soon as the child had remembered the event.

Children of 5 to 7 years old are often asked to talk about a piece of work in class, especially when reporting back to the class at the end of the lesson. So, during piloting each child was interviewed with their work sample as well as the video. However having the work in their hands tended to focus their reflection on the output of the lesson and the details of what was on the paper rather than what they were thinking so the technique of having work samples available to the child was discontinued. If a child asked for the work sample to help them to explain it was provided to them.

Data Coding and Analysis

Interviews were digitally recorded. Seventeen interviews were transcribed in full. An analysis of the transcripts resulted in the data being considered in terms of the children’s recall of an incident or task, description of events, explanation of their thinking, and description of their learning. Categories of response emerged as nodes in the data (see Table 1). Descriptors of response were listed in increasing levels of sophistication, with 0 being the least and 3, 4, or 5 as the most sophisticated responses. The category “missing” was used where the question was not asked. This happened because a feature of semi-structured interviews is that the interviewer tries to follow the child’s previous response.

The remaining 36 interviews were coded directly from the audio files. In general, the highest level of the particular category was coded when evidenced anywhere in the interview. Codes were then entered into a statistical analysis program (SPSS) to produce descriptive statistics.

Reliability of Coding

To improve internal reliability, interviews were re-coded. This was done to examine whether there was consistency between researchers and whether similar conclusions could be reached about children’s behaviour (Goldin, 2000, p. 531). An independent person coded a 20% sample of the audio data. This person was skilled at listening to young children describe their ideas as she came from a primary teaching background and mathematics education research. All points of difference were discussed and an agreed understanding of the data was reached. The following matters were raised:

- transcripts would have helped the coder;
- the broad categories that emerged from the data seemed appropriate;
- some descriptors required clarification to better define distinctions in levels of response;
- examples would help the coder/listener/reader;
- the distinction between evidence of description of thinking and correct thinking was reiterated; and
- evidence of a higher level of code was taken as the default.

Based on the combined critical analysis, further interviews were transcribed in full (17 in total) and category descriptions were refined. The entire data set was coded again applying the new protocols without any reference to the previous coding. The results of this second coding form the data reported here.

Table 1
Categories of Response to Aspects of the Interview

Aspects of interview	Categories of response
Recall of the incident/task	no recall
	could talk about the event only after of the entire video excerpt was replayed
	recall with the video paused just before the event of interest or with the video playing in the background with no audible sound
	recall spontaneously with little or no assistance of the video extract
Description of events	no description of interaction with teacher
	describe actions
	describe outcomes only, e.g., a work sample, "I stuck the cats onto the paper."
	describe the event from their perspective
	describe their reasoning and/or justify their thinking
Explanation of their thinking	no explanation
	"account for" the videotape e.g., make up a "story" of the event
	explicit description of thinking
	explain/reconstruct thinking, reasoning, justifying, evaluating thinking
Description of their learning	unable to specify learning
	learned nothing
	learned a behaviour not mathematics e.g., "to share"
	remembered factual information e.g., number facts
	learned how to do something e.g., "to count by 6s"
	described learning at a conceptual level, expressed as a mathematical principle or an insight, e.g., "I can count by 1s, 2s, 3s, 4s, 5s, 6s, 7s, 10s, and 100s and 1000s …once I can count by ten I can count by all the rest. Like 10, 20, 30, 40, 50, and it always has a zero on the end."

Results and Discussion

Recall of Events

Using videotape of events involving each child in the mathematics lesson of the day to stimulate the recall and an account of the episode from the view of the child was largely successful. This is evident from Table 2, which summarises the categories of responses of children's recall of events, where only 2% of children were unable to recall the events of the lesson. Some children needed to watch the entire replay of the videotape where they were in conversation with the teacher in order to talk about it (23%). Many children, having watched the video of the lesson leading up to the event, could recount their version of what had unfolded after the videotape was paused (30%). In addition almost half of those interviewed could recall a conversation with the teacher before the video was replayed.

Table 2
Categories of Response of Children's Recall of an Event

Category of response	Frequency as a percent (*n* = 53)
No recall	2 (1)
Recall with video replay of the event	23 (12)
Recall with video paused or with no audible sound	30 (16)
Recall spontaneously	45 (24)

Description of Event

An analysis of the children's descriptions of events revealed an interesting three-way split of responses (see Table 3). Some children described only what they did (23%). The following example illustrates this category of response. James could be seen on the video interlocking blocks but saying nothing:

> Interviewer: So what was happening here?
>
> James: My brain was counting and I wasn't. [James, J2.3:25]

Other children offered a description from their point of view (36%). For example, Ali explained his counting of five groups of five teddies saying, "It goes 10, 20, 30, 40, 50. You have to count the ears" [Ali, G1, 7:30]. It is hardly surprising that 36% of children who could remember the event described it from their point of view. In fact what was interesting was that such a large proportion described the event with some reconstruction of their reasoning at the time (28%). This was perhaps the most interesting group of responses. For example, Jessica was explaining how to weigh a dog, Joey, who would not stand on bathroom scales:

> Interviewer: Can you tell me about your good idea for maths today please?
>
> Jessica: I thought of holding Joey on the scales. I would know how much Joey weighed. So I hopped on the scales with him and I holded him. And then we took away 19 [from 28] because I was 19 and he was 9 and so that was 9 kilograms and that's what he weighed [Jessica, J3, 0:35].

Table 3
Children's Descriptions of Events

Category of response	Frequency as a percent (n = 53)
No description of interaction with teacher	4 (2)
Describe actions	23 (12)
Describe outcomes only, e.g., a work sample	8 (4)
Describe the event from their perspective	36 (19)
Describe their reasoning and/or justify their thinking	28 (15)
Missing	2 (1)

Explaining Thinking

Table 4 shows the number of children who could explicitly describe their mathematical thinking was high (85%).

Expecting children to be able to communicate their thinking has been an element of mathematics curriculum definition for years (Australian Education Council, 1991; Board of Studies, 2000). Certainly based on classroom observational data from the classrooms of the children interviewed here it is a clear expectation of their teachers that they explain their reasoning. The teachers frequently ask; "How did you work that out?", "What do you think?", "Why are you doing that?", and "How do you know?"

It should be said that these children had been learning mathematics in the classrooms of "highly effective" teachers of mathematics (McDonough & Clarke, 2003) for 8 months. Perhaps this would account for their readiness to describe their mathematical thinking. Whether children in other classrooms can explain their thinking with this frequency is a question that might be explored by further research.

An example of the type of response that shows a child reconstructing and evaluating his thinking is when Tom offered a thinking strategy for his classmates who could not count by four. His idea was to use a count by two.

> Interviewer: Now Mrs A says that's a really complicated way to work it out I can't really hear what you were saying. She was looking at a page that had 8 legs and 4 things on each leg. How were you trying to work that one out?
>
> Tom: Oh a different way. You know, when there's 8 legs and I was thinking if people didn't know how to count by 4, I was splitting 4 in half to make two on each side. Then I did 2 X 8 equals 16 then I have to count by 2s up to 32 what it equals. I have to count by 2s 16 times [Tom, G1, 1:00].

A few children could not explain their thinking and another few gave an explanation of their thinking as if telling a story. In examining the knowledge that experienced mathematics teachers access to operate effectively, Ainley and Luntley (2005, p. 78) made a distinction that may be pertinent here. Teachers were shown episodes of videotapes of their classrooms and in these interviews some teachers gave an "account for" rather than an "account of" their actions. The children who made up a story to suit the occasion may be doing the same thing or perhaps there is a different mechanism at work. No definitive statements could be made based on the evidence collected here. All that can be said is that 3 (6%) children made up a fiction to match the video.

Table 4
Children's Explanation of Their Thinking

Category of response	Frequency as a percent (n = 53)
None	6 (3)
"Account for" or gave an invented story	6 (3)
Explicit description of thinking	43 (23)
Reconstructs thinking, justifies, reasons, evaluates	42 (22)
Missing	4 (2)

Specify Learning

Only 15% of children did not know what they learned in the mathematics lesson (see Table 5). The category of "nothing" proved unreliable because it became clear that young children translated "What did you learn today?" into "What new things did you learn today?" and these two questions are quite different. Therefore this category is not discussed. Some children talked about behavioural learning, for example, "to share." Or they referred to non-mathematical things, for example the learning context, "talking about tools and building" [Michael, Jk2]. Totalling the first 3 categories of Table 5 shows that 30% of the children did not specify mathematical learning.

The three categories of most interest were those that made distinctions between learning factual information (15%), learning how to do something (23%), and learning at a conceptual level (21%).

About one third of the children who remembered facts talked in terms of numbers. For example, Annie who had been talking about measuring with a piece of string when asked what she learned said, "I learned that 9 + 11= 20." Although it is not possible to be certain from these data, it raises a question as to what these young children think constitutes mathematics learning. Is learning mathematics equated to remembering numbers? Lindenskov (1993) found that students' learning can be influenced by their everyday knowledge of what mathematics is. She was also struck by "the students" perceptions of details, even small ones, both in the teaching and in her/his own learning" (1993, p. 153). Certainly the children interviewed for this research described their learning in detail. For example, Tom talked about his learning in the following exchange.

Tom: I think I might have leant some new times tables.

Interviewer: Oh so you sort of had to figure some out?

Tom: Yes.

Interviewer: In which times table?

Tom: I think some were in the, I think some were like 9 X 6. I didn't know that but then I knew it because I just counted by 6 nine times [G1: 6:36].

Some children learned how to do something, for example Jordan, who "learned how to count by nines." Another substantial proportion of the children (21%) reflected on their learning at a conceptual level. For example, Tahani reflected on a lesson where the teacher intended to introduce multiplicative thinking, saying she learned "about groups, to make groups and to count them altogether and I learned to count by 6s." Another example was

Lucas who said he learned "how long things were and how short they were … by counting the blocks."

Table 5
Children's Learning

Category of response	Frequency as a percent (n = 53)
Unable to specify learning	15 (8)
Nothing "new"	9 (5)
Learned behaviour/ not mathematics	6 (3)
Remembered factual information	15 (8)
Learned how to do something	23 (12)
Specified a conceptual level of understanding	21 (11)
Missing	11 (6)

Conclusion and Implications

It can be concluded that young children could give an account of mathematical events from their perspective. Children could recall at least part of their conversations with the teacher during the day's mathematics lesson. These interactions appear to have some lasting effects. If, as we assert, interactions that challenge children to think about their mathematical understandings are a critical factor in their learning, then knowing that many young children spontaneously remember these conversations and can reconstruct their thinking is an important finding.

The sophistication of their descriptions of events in the classroom was fairly evenly split between recounts of actions, descriptions of the event from the child's perspective, and a description that involved some recount of their reasoning. It was impressive to find that such a large proportion of five- to seven-year-old children (42%) could reconstruct their thinking and justify it.

It is assumed that the experiences offered to children in mathematics classrooms contribute to their learning. These data indicate that 59% of children could talk about their learning as a result of the lesson – some at a factual level, some at a procedural level, and some at a conceptual level. Further research might investigate factors that influence different levels of understanding reported by young children.

It is also important for researchers to know that video-stimulated recall can be successfully used with 5- to 7-year-old children.

References

Ainley, J., & Luntley, M. (2005). The role of attention in classroom practice: Developing a methodology. In P. Clarkson, A. Downton, D. Gronn, M. Horne A. McDonough, R. Pierce & A. Roche (Eds.), *Building connections: Research, theory and practice* (pp. 73-80). Melbourne: Mathematics Education Research Group of Australasia.

Australian Education Council. (1991). *A national statement on mathematics for Australian schools*. Carlton: VIC: Curriculum Corporation.

Board of Studies (2000). *Curriculum and Standards Framework Mathematics*. Carlton: Victoria, Author.

Clarke, D., & Cheeseman, J. (2000). Some insights from the first year of the Early Numeracy Research Project. In *ACER Numeracy Conference*. Melbourne: Australian Council for Educational Research.

Clarke, D. J. (2001). Negotiating meanings - An introduction. In D. J. Clarke (Ed.), *Perspectives on practiceand meaning in mathematics and science classrooms* (pp. 1-11). Dordrecht: Kluwer Academic Publishers.

Donaldson, M. (1978). *Children's minds*. London: Fontana.

Cobb, P., Wood, T., Yackel, E., & McNeal, B. (1992). Characteristics of classroom mathematics traditions: An interaction analysis. *American Education Reseach Journal, 29*(3), 573-604.

Franke, M. L., & Carey, D. A. (1997). Young children's perceptions of mathematics in problem solving environments. *Journal for Research in Mathematics Education, 28*(1), 8-25.

Gelman, R., & Gallistel, C. R. (1978). *The child's understanding of number.* Cambridge, MA and London: Harvard University Press.

Goldin, G. (2000). A scientific perspective on structured, task-based interviews in mathematics education. In A. Kelly & R. Lesh (Eds.), *Handbook of research design* (pp. 517-545). Mahwah, NJ: Lawrence Erlbaum, Associates.

Hughes, M. (1986). *Children and number*. Oxford: Basil Blackwell Ltd.

Irwin, K. (1996). Children’s understanding of the principles of compensation and covariation in part-whole relationships. *Journal for Educational Research in Mathematics Education, 27,* 25-40.

Lindenskov, L. (1993). Exploring the students own mathematics curriculum. In J. Malone & P. Taylor (Eds.), *Constructivist interpretations of teaching and learning mathematics* (Proceedings of Topic Group 10 at the 7th International Congress of Mathematics Education, ICME-7, pp. 149-156). Quebec, Canada: ICME.

McDonough, A. (2002). *Naive yet knowing: Young learners portray beliefs about mathematics and learning.* Unpublished PhD thesis. Melbourne: Australian Catholic University.

McDonough A., & Clarke D. M. (2003). Describing the practice of effective teachers of mathematics in the early years. In B. J. N. A. Pateman & J. Z. Dougherty (Eds.), *Proceedings of the 27th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 261-268). Honolulu, HI: College of Education, University of Hawaii.

Sfard, A., Nescher, P., Streefland, L., Cobb, P., & Mason, J. (1998). Learning mathematics through conversation: Is it as good as they say? *For the Learning of Mathematics, 18*(1), pp. 41-51.

Williams, G. (2003). *Social and affective factors that promote and inhibit insightful and creative mathematical thinking.* Paper presented at the conference of the Learners’ Perspective Study international research team, University of Melbourne, December 1-3, 2003.

Mathematical Reform: What Does the Journey Entail for Teachers?

Linda Cheeseman
The University of Auckland
<l.cheeseman@auckland.ac.nz>

This paper presents a case study of the journey a teacher/facilitator took to increase her mathematical content knowledge in order to implement reform-oriented teaching approaches in her mathematics classroom, and subsequently supported other teachers to do the same. In the past decade mathematic education reform has been introduced to teachers in curriculum documents and related in-service professional development programmes promoting an inquiry-based approach to the teaching and learning of mathematics to increase student achievement. Recent research findings suggest that the complex mathematical knowledge embedded in these reforms makes it difficult for many teachers to accommodate the reforms in their entirety. This was indeed the case for the teacher in this study.

Introduction

The case study reported here is part of a wider study that investigated the professional development perspectives of eight teachers and one teacher/facilitator who had participated in the long-term, school-based mathematics programme, the New Zealand Numeracy Development Projects (NDP). The teachers in this wider study found the complex nature of the reforms, for example coming to terms with understanding multiple strategies and moving away from procedural-based algorithms, led to significant shifts in their pedagogical content knowledge. Even so they struggled to accommodate the full extent of these reform teaching approaches without ongoing support (Cheeseman, 2006). Similar struggles were identified by teachers in the nation-wide government evaluations of the NDP (Young-Loveridge, 2004). In this case study the teacher/facilitator talks about her lengthy professional development journey and the types of content knowledge she gained along the way.

Background

Research over the past two decades has identified the teacher and the teaching methodology as the crucial factor for students' ability to learn mathematical concepts with understanding (Skemp, 1986; Wilson & Ball, 1996). Skemp (1986) argued that teachers as poor communicators of mathematics accounted for many students' negative attitude and anxiety towards mathematics and their resulting underachievement in the subject. Studies have confirmed that many teachers lack the content knowledge required to deliver effective teaching programmes in mathematics (e.g. Carpenter, Fennema, Fuson, Heibert, Human, Murray, Oliver, & Wearne, 1999; Hill & Ball, 2004; Shulman, 1986). The results of studies such as these led to mathematical reforms focusing on teacher knowledge and the way that knowledge is delivered so that students became fully engaged in mathematical thinking. The reform-oriented teaching approaches promoted were inquiry-based, "a process in which students reorganise their conceptual activity to resolve situations that they find problematic" rather than procedural "a process of internalising carefully packaged knowledge" (Cobb, Wood, Yackel, Nicholls, Wheatley, Trigatti, & Perlwitz, 1991, p. 5).

These reforms required most teachers to make a major shift in pedagogy from teaching approaches that focused on a procedural approach (standard algorithms and rules) to a conceptual approach focusing on student thinking and reasoning (Stein & Strutchens, 2000; Anderson & Bobis, 2005). To make this pedagogical shift necessitated teachers extending their knowledge of mathematics to include what Shulman (1986) described as pedagogical content knowledge (PCK). Shulman (1986) sees PCK as going beyond knowledge of subject matter per se to the subject matter of teaching that includes knowledge of: how to teach mathematics, mathematics curriculum and resources, and importantly how students learn mathematics. In regard to the latter, Hill and Ball (2004) elaborated that teachers' knowledge of how students learn results from the "interplay between teachers' knowledge of students, their learning, and strategies for improving that learning". This includes the teachers' ability to understand and assess the problem solving strategies used by their students and when a new strategy is used to "determine whether such strategies would be generalizable to other problems" (p. 332). Embedded within PCK is the development of teachers' awareness of sociocultural norms whereby students feel confident to share their mathematical thinking in a non-threatening learning environment (Fraivillig, Murphy, & Fuson, 1999; Yackel & Cobb, 1996).

In New Zealand the Ministry of Education undertook a series of initiatives to provide professional development programmes to assist teachers to accommodate the reform-orientated approaches in mathematics. A small initiative began in the late 1990s with the introduction of a tertiary course (Helping Children Succeed in Maths) at the Auckland College of Education. Those teachers attending the course were introduced to the theory of relational understanding based on Skemp's (1986) research, the notion of students using their own strategies to solve number problems, and the developmental stages of children's mathematical thinking. Nation wide long-term, school-based initiatives followed, commencing with the New South Wales programme Count Me In Too [CMIT] (Wright, 1998), which was later replaced by a New Zealand numeracy project focusing on the junior school, the Early Numeracy Programme (ENP). The professional development programme was then extended to teachers of older students (8 – 10 year olds) with the introduction of the Advanced Numeracy Project (ANP). Both ENP and ANP were designed to up-skill teachers in their ability to teach numeracy by providing The Number Framework, a breakdown of the development stages of students' mathematical knowledge and thinking, and a strategy-teaching model. The Number Framework was the NDP's key tool in developing teachers' knowledge of number concepts and the processes by which these number concepts are best developed. It was intended that teachers' awareness of student mathematical mental strategies be increased and their pedagogy changed (guided by the teaching model) in order to improve student achievement (Thomas & Ward, 2002).

Teacher change as a result of accommodating the reform-orientated approaches was extensive and difficult to achieve for many teachers (Cheeseman, 2006; Young-Loveridge, 2004). This parallels the reports in recent international studies (Anderson & Bobis, 2005; Cady, Meier, & Lubinski, 2006; Stigler & Heibert, 1997). Stigler and Heibert (1997) observed the challenges American teachers encountered while attempting to make changes to their deep-seated beliefs when faced with reforms and as a result only changed some practices. Anderson and Bobis (2005) investigated Australian teacher responses to the reform-oriented approaches recommended by the NSW curriculum and found that overall teachers' agreed with the reforms but many had difficulty fully embracing them. Similarly, a longitudinal study undertaken by Cady, Meier, and Lubinski (2006) observed the development of pre-service teachers to experienced teachers and found variance in the teachers' abilities to implement reform practices in their classrooms as novice teachers. This paper examines the complexities of one teacher's journey in her attempt to incorporate and consolidate the mathematical reforms promoted by the numeracy initiatives in New Zealand into her teaching practice.

Methodology

The study used an interpretive approach to investigate the perceptions of a teacher/facilitator who participated in Ministry of Education numeracy professional development initiatives over an extended period of four years. The mode of enquiry was in the form of a 45-minute to 60-minute face-to-face interview and shorter follow-up telephone interviews. The semi-structured, open-ended interview questions were formulated as a guide for the researcher to follow. The main intention of the researcher was to listen actively so that the interview was shaped by the participant's voice (Denzin & Lincoln, 2003).

This case study was part of a wider interpretive study that explored the experiences of eight teachers who were involved in the NDP professional development programme.

Mathematical Journey

In her first year of professional development Jayne was teaching in the junior area of the school (5 – 7 year olds). She participated in the tertiary mathematics paper, "Helping Children Succeed in Maths", which introduced her to the idea of student strategies for counting, and theory about conceptual or relational thinking.

> The first year I took the paper "Helping Children Succeed in Maths" at ACE [Auckland College of Education]. I was the only teacher attending from my school which was a shame because after each session I would come to school and talk enthusiastically about all these new mathematical ideas. I suspect that most people taught like I've always done and have those same values, and the ideas I was now advocating were quite radical so my colleagues were not keen to listen. The idea that the children could think of their own strategies to solve addition problems seemed alien and their reaction was "but that means there would be more than one answer!"

Jayne took ownership of the new pedagogical ideas and practised the implementation of them with her class. The following year her junior syndicate participated in CMIT. CMIT was a long-term professional development programme (a duration of three school terms) that further consolidated the new pedagogical content knowledge she had gained from the tertiary course. An example of this knowledge consolidation was her increasing familiarity with the number framework outlining the stages of student mathematical thinking.

> Before starting the PD, I remember asking a Year three child to add 8 + 3 and she went 12345678, 123 and then counted up to 11. I thought what is she doing? Why is she doing that? Now I know that [counting all - one to one] is a developmental progression … and now you have to go from this step to teach them [children] to go further.

Her enthusiasm and success in accommodating to and implementing the PCK led to her being asked to become a part-time facilitator for ENP, which allowed her to continue to teach mathematics in her year 4 class as well as introduce other teachers of junior classes in other schools to the reform-oriented teaching practices.

> When I became a facilitator in terms of maths [knowledge] it deepened what I knew rather than changed it so much because by then I'd already changed the way I taught and the way I thought about maths. As a facilitator I became aware of the importance of the numeracy framework in focussing teachers' attention on stages of children's thinking so that you can see which stage each child is at and where to push them to next. … I used to look at my Year four children [8 yr olds] and they would all be using their fingers to count on for a problem like 8 + 5, but now we are teaching them to count smarter by rearranging the groups of numbers [making tens or using doubles strategies]. I think that knowing there is a next step was a big change in my thinking about mathematics and how children learn to add and subtract.

The following year Jayne was asked to become a full-time facilitator, this time working with teachers in both the ENP and ANP professional development courses. She talked about the increase in content knowledge in relation to strategies used by junior school students to those used by students in the senior school (8-9 year olds). She initially struggled to understand some of the more complex strategies, in particular those that involved multiplicative thinking.

> This year the development has been huge because [as a facilitator of ANP] it's been multiplication and division … it is understanding the actual strategies e.g. for a problem like 5 x 18 – can take a long time to actually understand that you are just halving that group and rearranging it [10 x 9]. Initially you just have to see it and do it [using materials]. That knowledge is then extended, to fractions and decimals, which I knew very little about and which is so abysmal in NZ anyway. Both my knowledge and strategies have increased and that would be common with most of the facilitators and a lot of the professional development had been based around that for facilitators.

Jayne's struggle to understand the more complex mathematical strategies parallels the findings of the research that reported on how teachers had coped with the increasingly sophisticated part-whole strategies introduced as part of the ANP professional development. She, as for most teachers in the research project, felt challenged but like the other teachers increased her own mathematical knowledge (Irwin, 2003; Young-Loveridge, 2004). It is crucial that teachers understand the strategies their students are using and provide guidance to extend their students' thinking. Jayne discussed the aspect of teaching strategy as another significant aspect of her increasing PCK and accommodation of the range of strategies to be taught.

> At the ANP level I had to learn firstly, what is the range of strategies children might use and secondly, how do we teach them. I can empathise with the teachers' feeling of "information overload" when learning about strategy because I often felt this too when attending the ANP facilitator professional development days. Sometimes at those PD days I would think if they say another thing I'm going to burst because I don't want to hear any more. It is a lot to take in and it's not just taking it in, it is processing it and then telling and showing that to teachers.

The nature of the NDP required a major shift in pedagogy from teaching approaches that focused on a transmission approach to a teacher facilitation approach focusing on student thinking and reasoning (Stein & Strutchens, 2001). Jayne became more conscious of the importance of listening attentively to students explain their thinking at the ANP level where the strategies were more complex. She was aware both as a teacher and a facilitator of the necessity to elicit, support, and extend students mathematical thinking (Fraivillig et al., 1999) and to model this for teachers.

> Listening to children's thinking – was a huge shift. No longer just wanting answers – asking how did you get that answer or a range of answers. Yes, I accept all the children's solutions without value judging but some ways of getting the answers are more efficient than other ways so that's the way we want to guide then depending on what the numbers are in the problem. The most efficient strategies will vary depending on the nature of the problem. For example, using an algorithm to solve a

> problem with large four to five digit numbers is fine but for a problem like 1003 – 998 you can solve it in your head. However, children do now always see that there is a quicker way to solve it and need guidance to move away from the standard algorithm to see the easier method.

Jayne found that the shift away from teaching algorithms was highly significant in her growth of PCK. This significant aspect was highlighted in the literature where it was noted that the NDP strategy-teaching model required most teachers to change their ways of thinking and learning about mathematics. This entailed a shift away from teaching rules, procedures, and algorithms to guiding students to use multiple strategies to solve a problem (Young-Loveridge, 2004). Very early on in her professional journey Jayne could see the tension caused by teaching algorithms whereby the children were just following a process and not seeing the wholeness of the numbers or looking at them contextually.

> As teachers we would get to that step where the children could count on and then we would teach the algorithm which is not so much to do with mathematical thinking but is more to do with following rules. For example, with a problem like 605 – 308, the children would not see 605 as a whole number but would concentrate on the 5 and 8 ones, each little bit of it and not have a sense of the wholeness – number sense!

Jayne had changed her practice to teach multiple strategies and delay the teaching of algorithms to students until they had a deep conceptual knowledge about operating on number. She then had to convince other teachers to move away from the standard procedures.

> As a facilitator of ANP my biggest challenge was convincing teachers to move away from teaching algorithms [standard procedures to solve all four operations]. And trying to get through that students won't be penalised because they will know so much more about number and they will have a much richer base [strategies] in their heads. And they won't understand an algorithm anyway if you teach it too early. We've been teaching procedures for years and that's fine for the basic facts, counting and number identification, and reading fractions and decimals, but operating on them involves other mathematical thinking.

The journey Jayne took enabled her to increase her mathematical content knowledge and implement reform oriented teaching approaches in her mathematics teaching practice.

Conclusion

Her dual role beginning as a self-motivated teacher and becoming a facilitator gave Jayne multiple opportunities to take advantage of the professional development associated with NDP. Her personal journey took four years and involved the challenges of working with colleagues as well as with students. It seemed for her 4 years laid a solid foundation that may be an optimum result for the numeracy teaching development programmes to be consolidated effectively. The case study outlining Jayne's accommodation of the reform-oriented teaching approaches demonstrates the difficulties faced by teachers embarking on this mathematical self-improvement journey. Recent research findings state "that teachers who are more successful than others at developing effective reform-based practices appear to be self-sustaining, generative learners (Anthony & Walshaw, 2007). This would indeed appear to be the case for Jayne.

References

Anderson, J., & Bobis, J. (2005). Reform-oriented teaching practices: A survey of primary school teachers. In H. L. Chick & J. L. Vincent (Eds.), *Proceedings of the 29th annual conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 65-72). Melbourne: PME.

Anthony. G., & Walshaw, M. (2007). *Effective pedagogy in mathe,matics / pangarau: Best evidence systhesis iteration {BES}*. Wellington: Ministry of Education.

Cady, J., Meier, S. L., & Lubinski, C. A. (2006). The mathematical tale of two teachers: A longitudinal study relating mathematics instructional practices to level of intellectual development. *Mathematics Education Research Journal, 18*(1), 3-26.

Carpenter, T., Fennema, E., Fuson, K., Hiebert, J., Human, P., Murray, H., Olivier, A., & Wearne, D. (1999). Learning basic number concepts and skills as problem solving. In E. Fennema, & T. Romberg (Eds.), *Mathematics classrooms that promote understanding,* (pp. 45-61) Mahwah: Lawrence Erlbaum.

Cheeseman, L. (2006). The numeracy journey: How long does it take to get on board? In P. Grootenboer, R. Zevenbergen, & M. Chinnappan (Eds.), *Identities, cultures and learning spaces* (Proceedings of the 29th annual conference of the Mathematics Education Research Group of Australasia, pp. 131-138), Sydney: MERGA.

Cobb, P., Wood, T., Yackel, E., Nicholls, J., Wheatley, G., Trigatti, B., & Perlwitz, M. (1991). Assessment of a problem-centered second-grade mathematics project. *Journal for Research in Mathematics Education. 22*(1), 3-29.

Denzin, N., & Lincoln, Y. (2003). *The landscape of qualitative research: Theories and issues* (2nd ed.).Thousand Oaks, CA: Sage.

Fraivillig, J., Murphy, L., & Fuson, K. (1999). Advancing children's thinking in everyday mathematics classrooms. *Journal for Research in Mathematics Education, 30*(2), 148-170.

Hill, H., & Ball, D. (2004). Learning mathematics for teaching: Results from California's mathematical professional development institutes. *Journal for Research in Mathematics Education, 35*(5), 350-351.

Irwin, K. (2003). *An evaluation of the numeracy project for years 7-10, 2002.* Wellington: Ministry of Education.

Shulman, L. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher, 15*(2), 4-14.

Skemp, R. (1986). *The psychology of learning mathematics* (2nd ed.). London: Penguin Books.

Stein, M., & Strutchens, M. (2001). Mathematical argumentation: Putting umph into classroom discussions. *Mathematics Teaching in the Middle School, 7*(2), 110-113.

Stigler, J., & Heibert, J. (1997). Understanding and improving classroom mathematics instruction: An overview of the TIMSS video study. *Phi Delta Kappan, 79*(1), 14-21.

Thomas, G., & Ward, J. (2002). *An evaluation of the early numeracy project 2001.* Wellington: Ministry of Education.

Wilson, S., & Ball, D. (1996). Helping teachers meet the standards: New challenges for teacher educators. *The Elementary School Journal, 97*(2), 121-138.

Wright, R. J. (1998). An overview of a research-based framework for assessing and teaching early number. In C. Kanes, M. Goos, & E. Warren (Eds.), *Teaching mathematics in new times (*Proceedings of the 21st Annual Conference of the Mathematics Education Research Group of Australasia , Vol. 2, pp. 701-708). Gold Coast, QLD: MERGA.

Yackel, E., & Cobb, P. (1996). Sociomathematical norms, argumentation and autonomy. *Journal for Research in Mathematics Education, 27*, 458-477.

Young-Loveridge, J. (2004). *Patterns of Performance and progress on the numeracy projects 2001-2003.* Wellington: Ministry of Education.

Year Six Fraction Understanding: A Part of the Whole Story

Doug M. Clarke
Australian Catholic University
<d.clarke@patrick.acu.edu.au>

Anne Roche
Australian Catholic University
<a.roche@patrick.acu.edu.au>

Annie Mitchell
Australian Catholic University
a.mitchell@patrick.acu.edu.au

A range of assessment tasks was developed for use in one-to-one interviews in December 2005 with 323 Grade 6 students in Victoria. In this paper, we summarise briefly the research literature on fractions, describe the process of development of assessment tasks, share data on student achievement on these tasks, and suggest implications for curriculum and classroom practice. Particular emphasis in the discussion is given to students' judgements and strategies in comparing fractions. A particular feature of this report is that one-to-one interview assessment data were collected from a larger number of students than is typically the case in these kinds of studies. Recommendations arising from these data include the importance of teachers understanding and presenting a wider range of sub-constructs of fractions to students in both teaching and assessment than is currently the case, using a greater variety of models, and taking available opportunities to use the interview tasks with their own students.

Theoretical Background

Fractions are widely agreed to form an important part of middle years mathematics curriculum (Lamon, 1999; Litwiller & Bright, 2002), underpinning the development of proportional reasoning, and important for later topics in mathematics, including algebra and probability. However, it is clear that it is a topic which many teachers find difficult to understand and teach (Post, Cramer, Behr, Lesh, & Harel, 1993), and many students find difficult to learn (Behr, Lesh, Post, & Silver, 1983; Kieren, 1976; Streefland, 1991). Among the factors that make rational numbers in general, and fractions in particular difficult to understand are their many representations and interpretations (Kilpatrick, Swafford, & Findell, 2001).

There is considerable evidence that the difficulties with fractions are greatly reduced if instructional practices involve providing students with the opportunity to build concepts as they are engaged in mathematical activities that promote understanding (Bulgar, Schorr, & Maher, 2002; Olive, 2001).

In the Early Numeracy Research Project (Clarke, et al., 2002), a task-based, interactive, one-to-one assessment interview was developed, for use with students in the early years of schooling. This interview was used with over 11 000 students, aged 4 to 8, in 70 Victorian schools at the beginning and end of the school year, thus providing high quality data on what students knew and could do in these early grades, across the mathematical domains of Number, Measurement, and Geometry. There was equal emphasis in the teachers' record of interview on *answers* and the *strategies* that led to these answers.

The use of a student assessment interview, embedded within an extensive and appropriate inservice or preservice program, can be a powerful tool for teacher professional learning, enhancing teachers' knowledge of how mathematics learning develops and knowledge of individual mathematical understanding, as well as content knowledge and pedagogical content knowledge (Clarke, Mitchell, & Roche, 2005; Schorr, 2001).

The success of the interview and comments from middle years' teachers prompted the authors to consider extending the use of the assessment interview to the middle years of schooling (Grades 5 to 8). As a first, major step in this process, it was decided to focus the interview on the important mathematical topics of fractions and decimals. This paper reports the process and findings from this work, with particular emphasis on fractions.

Fractions: Constructs and Models

Much of the confusion in teaching and learning fractions appears to arise from the many different interpretations (constructs) and representations (models). Also, generalisations that have occurred during instruction on whole numbers have been misapplied to fractions (Streefland, 1991). Finally, there appears to be a void between student conceptual and procedural understanding of fractions and being able to link intuitive knowledge (or familiar contexts) with symbols (or formal classroom instruction) (Hasemann, 1981; Mack, 2002). The dilemma for both teachers and students is how to make all the appropriate connections so that a mature, holistic, and flexible understanding of fractions and the wider domain of rational numbers can be obtained.

Kieren (1976) was able to identify several different interpretations (or constructs) of *rational numbers* and these are often summarised as part-whole, measure, quotient (division), operator, ratio, and decimals. For the purpose of this review these interpretations are explained in the context of fractions.

The *part-whole* interpretation depends on the ability to partition either a continuous quantity (including area, length, and volume models) or a set of discrete objects into equal sized subparts or sets. The part-whole construct is the most common interpretation of fractions and likely to be the first interpretation that students meet at school. Lamon (2001) suggested that "mathematically and psychologically, the part-whole interpretation of fraction is not sufficient as a foundation for the system of rational numbers" (p. 150).

A fraction can represent a *measure* of a quantity relative to one unit of that quantity. Lamon (1999) explained that the measure interpretation is different from the other constructs in that the number of equal parts in a unit can vary depending on how many times you partition. This successive partitioning allows you to "measure" with precision. We speak of these measurements as "points" and the number line provides a model to demonstrate this.

A fraction (a/b) may also represent the operation of *division* or the result of a division such that 3÷5 = 3/5. The division interpretation may be understood through partitioning and equal sharing. These two activities have been the focus of much research (Empson, 2003).

A fraction can be used as an *operator* to shrink and stretch a number such as 3/4 x 12 = 9 and 5/4 x 8 = 10. The misconception that multiplication always makes bigger and division always makes smaller is common (Bell, Fischbein, & Greer, 1984). It could also be suggested that student lack of experience with using fractions as operators may also contribute to this misconception.

Fractions can be used as a method of comparing the sizes of two sets or two measurements such as "the number of girls in the class is 3/5 the number of boys", i.e., a *ratio*. Post et al. (1993) claim "ratio, measure and operator constructs are not given nearly enough emphasis in the school curriculum" (p. 328).

Although these constructs can be considered separately they have some unifying elements or "big ideas". Carpenter, Fennema, and Romberg (1993) identified three unifying elements to these interpretations and they are: identification of the unit, partitioning, and the notion of quantity.

Method

Focusing on the rational number constructs of part-whole, measure, division, and operator, and the "big ideas" of the unit, using discrete and continuous models, partitioning, and the relative size of fractions, a range of around 50 assessment tasks was established, drawing upon tasks that had been reported in the literature, and supplemented with tasks that the research team developed. These tasks were piloted with around 30 students in Grades 4 to 9, refined, and piloted again (Mitchell & Clarke, 2004).

Using a selection of the set of tasks, 323 Grade 6 students were interviewed at the end of the school year. The schools and students were chosen to be broadly representative of Victorian students, on variables such as school size, location, proportion of students from non-English speaking backgrounds, and socio-economic status. A team of ten interviewers, all experienced primary teachers, with at least 4 years' experience in one-to-one assessment interviews of this kind, participated in a day's training on the use of the interview tasks, including viewing sample interviews on video.

The tasks were administered individually over a 30- to 40-minute period in the students' own schools, with interviews following a strict script for consistency, and using a standard record sheet to record students' answers, methods and any written calculations or sketches. Each actual response to a question was given a code by the authors, and a trained team of coders took the data from the record sheets, coded each response, and entered it into SPSS. Key findings are provided in the following section.

Results

In this section, data from the 323 Grade 6 students are provided on eight of the tasks, organised around relevant sub-constructs of fractions (Kieren, 1976). In each case, the task is outlined, the mathematical idea it was designed to address is stated, the percentage student success rate is given, and common strategies and solutions, including misconceptions, are outlined.

Part-whole

Three tasks focused on part-whole thinking.

1. *Fraction Pie task* (adapted from Cramer, Behr, Post, & Lesh, 1997). Students were shown the pie model (Figure 1), and asked:
a) What fraction of the circle is part B?
b) What fraction of the circle is part D?

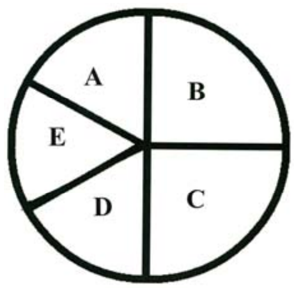


Figure 1. Fraction Pie task.

Part (a) was relatively straightforward, with 83.0% of students answering 1/4. Of the total group, 3.3% offered a correct equivalent fraction, decimal, or percentage answer, whereas 5.6% and 1.9% answered "1/5" and "1/2", respectively. Part (b) was more difficult, with only 42.7% giving a correct answer, with 13.6% answering 1/5 (presumably based on "five parts"). The same percentage answered 1/3, probably focusing only on the left-hand side.

2. *Dots Array task*. Students were shown the array in Figure 2, and asked, "what fraction of the dots is black?" They were then asked to state "another name for that fraction"; 76.9% gave a correct answer, with the three most common answers being 2/3 (35.6%), 12/18 (30.7%), and 4/6 (8.7%). The most common error was 3/4. Only 53.5% of students were able to offer another correct name for the fraction, with 4/6 being the most common response (17.0%). These data indicate that students generally showed a flexible approach to *unitising* (Lamon, 1999).

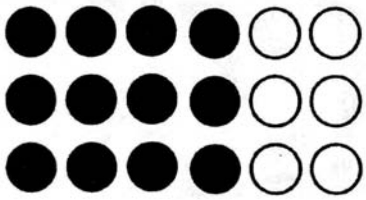


Figure 2. Dots Array task.

3. *Draw me a whole task (part a).* In assessing students' capacity to move from the part to the whole, acknowledged by Lamon (1999) and others as an important skill, students were shown a rectangle (shaded grey in Figure 3), and asked, "if this is two-thirds of a shape, please draw the whole shape," while explaining their thinking. 64.1% were able to do so successfully, with 28.5% of them dividing the original shape into two equal parts first, and 35.6% showing no visible divisions.

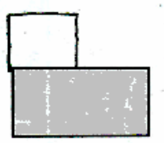


Figure 3. A student's correct solution for Draw me a whole (part a).

Draw me a whole task (part b). Students were presented with a different rectangle (shaded part in Figure 4), told that it was "four thirds," and asked to show the whole. In this case, 40.5% drew a correct shape, with just under half of these breaking the original rectangle into four parts, indicating three of these as the whole.

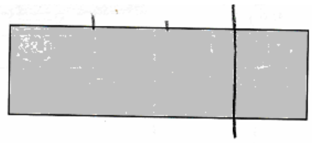


Figure 4. A student's correct solution for Draw me a whole (part b).

Fraction as an Operator

4. *Simple operators.* Students were posed four questions, with no visual prompt, which required students to work out the answer in their heads. They were as follows: "… one-half of six?" (97.2% success); "… one-fifth of ten?" (73.4%); "… two-thirds of nine? (69.7%); and "… one third of a half?" (17.6%). The data on the last item, by

far the most difficult of this set, are interesting in light of the relative difficulty with the related pie task.

Fractions as Measure

5. *Number line (parts a, b & c).* Students were asked to “please draw a number line and put two thirds on it”. If students did not choose to indicate where 0 and 1 should be in their drawing, they were asked by the interviewer, “where does zero go? … where does 1 go?” Only 51.1% of students were successful in correctly locating 2/3 on the number line. A common error was placing 2/3 after 1 (see Figure 5), or two-thirds along some line, e.g., at 4 on a number line from 0 to 6, or two-thirds of the way from 0 to 100 (see Figure 6).

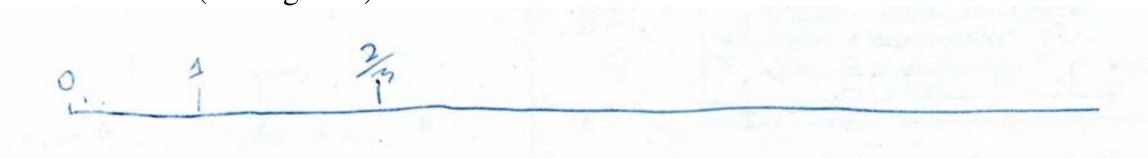


Figure 5. A student’s incorrect solution for placing 2/3 on a number line (part a).

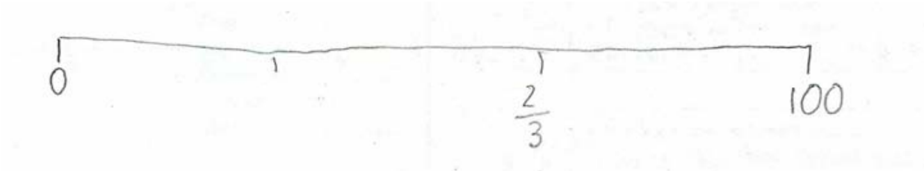


Figure 6. Another student’s incorrect solution for placing 2/3 on a number line (part a).

Given a number line as shown in Figure 7, students were then asked to mark, in turn, six thirds (part b) and eleven sixths (part c) (Baturo & Cooper, 1999). Only 32.8% and 25.4% were successful, respectively. Many placed 6/3 on 6 or 3. Several students located 11/6 well to the right of 6.

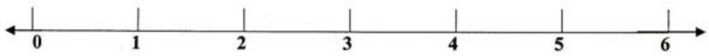


Figure 7. Number line task (part b & c).

6. *Construct a Sum.* In a task designed to get at students’ understanding of the “size” of fractions, we used the Construct a Sum task (Behr, Wachsmuth, & Post, 1985). The student is directed to place number cards in the boxes to make fractions so that when you add them the answer is as close to one as possible, but not equal to one. The number cards included 1, 3, 4, 5, 6, and 7 (Figure 8). Each card could be used only once. The capacity for students to move cards around as they consider possibilities is a strong feature of this task. Only 25.4% of students produced a solution within 0.1 of 1, the most common response being 1/5 + 3/4 (5.3% of the total group). 24.5% of students chose fractions at least 0.5 away from 1, and most of these included an improper fraction. The answer closest to one (1/7 + 5/6) was chosen by only four students.

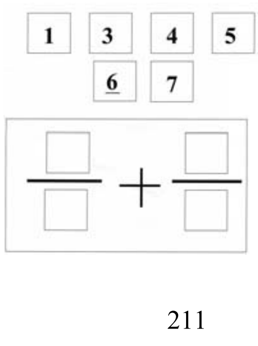


Figure 8. Construct a Sum task.

7. *Fraction pairs task.* Another task that we used to assess the important notion of fraction as a quantity is the *fraction pairs task*. Eight fraction pairs were shown to students, one pair at a time (see Figure 9). Each pair, typed on a card, was placed in front of the student, and the student was asked to point to the larger fraction of the pair and explain their reasoning. No opportunity was given for the students to write or draw anything. Our interest was in mental strategies.

a)	3/8	7/8	e)	2/4	4/2
b)	1/2	5/8	f)	3/7	5/8
c)	4/7	4/5	g)	5/6	7/8
d)	2/4	4/8	h)	3/4	7/9

Figure 9. The eight fraction pairs used in the study.

The intention was that, based on previous piloting, the tasks were presented in order of increasing difficulty. This proved not always to be the case.

For each task, the interviewer circled the student's chosen fraction on the interview record sheet, and recorded the student's reasons, choosing from a list of common explanations. For example, the choices given for the pair 3/4 and 7/9 were:

- Residual with equivalent (2/8 > 2/9)
- Residual thinking (1/4 > 2/9) with proof
- Converts to decimals
- Common denominator
- Higher or larger numbers
- Other …………………………

If the method offered by the student did not correspond to any of the listed strategies, the interviewer noted the method used under "Other", making every effort to record all the words used by the student in the explanation.

Data analysis involved determining the percentage of students who gave the correct answer, and then for both correct and incorrect choices, the percentage of students who used each particular strategy. The list of strategies was expanded during data analysis to incorporate any strategies which were common, from the "Other" category.

Table 1 shows the percentage of students who selected the appropriate fraction from the pair (or indicated both were equal in the case of 2/4 and 4/8) *and* gave a reason for their choice that was judged to be reasonable. The fraction pairs are presented in decreasing order of success.

The most straightforward pair (3/8, 7/8) and the most difficult pair (3/4, 7/9) were easily predicted in advance. Having said that, the percentage success on the easiest pair (77.1%), with *success* being defined as a correct choice coupled with an appropriate explanation**,** was not high. Given that students were interviewed at the end of their Grade 6 year, after probably some years of introductory work on fractions, nearly one-quarter of students do not seem to have a basic, part-whole understanding of fractions.

The vast majority (94.8% of successful students) noted that the denominator was the same (and hence the size of the parts), and therefore compared the numerators. However, 5.2% benchmarked to 1/2 and 1. Also 38.5% of all incorrect solutions (for which 3/8 was chosen as the larger) gave an explanation to the effect that "smaller numbers mean bigger fractions".

Table 1
The Percentage of Grade 6 Students Choosing Appropriately from Fraction Pairs With Appropriate Explanation (n = 323)

Fraction pair		*% correct*
3/8	7/8	77.1%
2/4	4/8	64.4%
1/2	5/8	59.4%
2/4	4/2	50.5%
4/7	4/5	37.2%
3/7	5/8	20.4%
5/6	7/8	14.9%
3/4	7/9	10.8%

The most difficult pair (3/4 & 7/9) proved to be very difficult for various groups of primary and junior secondary *teachers* with whom we have worked in professional development settings. Many teachers have been unable to offer an explanation beyond the use of common denominators, and so the 10.8% success rate for students is probably not surprising. In fact, 54.3% of successful students used common denominators and a total of 40% used some form of *residual strategy* (either 2/8 > 2/9 or 1/4 > 2/9 with some other explanation), whereas 5.7% (two students) converted the fractions to decimals in their heads.

The relative difficulty of the pair (4/7, 4/5) was a surprise to us, with only a 37.2% success rate, indicating that it was more difficult than (1/2, 5/8) and (2/4, 4/2). We did note however that 60.0% of all successful students provided an explanation similar to "there are four pieces in each, but as sevenths are smaller than fifths, so 4/5 will be larger", indicating the most common correct response was a strategy involving number sense rather than procedure. It was of some concern that 20.0% felt the need to convert to common denominators; 9.1% of successful students used benchmarking and 10.8% used residual thinking. This was a task in which *gap thinking* (Pearn & Stephens, 2004) was common, with 21.4% of students who chose 4/5 as larger providing inappropriate gap thinking reasoning (focusing on the difference between 4 and 7 and between 4 and 5). For all students who chose 4/7 as larger, 73.5% of reasons were to do with "larger numbers".

Benchmarking and residual strategies are a couple of the strategies that appear to be used by students displaying a more conceptual understanding of the size of fractions, yet they are not in widespread use by students or teachers in our schools. These strategies would have been most appropriate for the pairs (3/7, 5/8) and (5/6, 7/8) respectively, but the success rates were 20.4% and 14.9%. Of the successful students, 28.8% and 45.8% of students chose to use common denominators for these pairs respectively, thereby choosing a procedure rather than a strategy based more clearly on number sense. Also, 21.2% of all students used gap thinking for (3/7, 5/8) and 29.4% of all students claimed 5/6 and 7/8 were the same, often using gap thinking as their justification.

Student understanding of simple equivalences, appears to contribute to the relative success rate for the pairs (1/2 and 5/8, and 2/4 and 4/8) as most could identify ½ and 4/8 as the same, however, it must be said that 59.4% and 64.4% respectively are still lower than we predicted for students at the end of Grade 6.

The lack of emphasis on improper fractions in primary grades may account for the difficulty in explaining the relative size of 2/4 and 4/2 (42.7%). Also, the language some students use to label fractions may hinder their understanding. For example, some students were noted to read these as "two out of four" and "four out of two", which is not helpful when considering their respective size. "Two-quarters" and "four-halves", on the other hand, may help to create an image about the size of the parts that is more likely to lead to a correct solution.

Fractions as Division

8. *Pizza task.* Children were shown a picture (Figure 10), and told, "three pizzas were shared equally between five girls. … How much does each girl get?" Students were invited to use pen-and-paper if they appeared to require it.

Although 30.3% of Grade 6 students responded with a correct answer, it was apparent that most either drew a picture or mentally divided the pizzas to calculate the equal share. A concerning result was that 11.8% of students were unable to make a start. Greater exposure to division problems and explicit discussion connecting division with their fractional answers, for example, $3 \div 5 = 3/5$ may help lead students to the generalisation that $a \div b = a/b$.

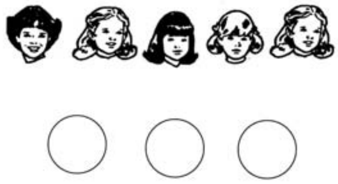


Figure 10. Pizza task.

Discussion

Despite the strong recommendations from researchers that school mathematics should provide students experiences with all key sub-constructs of fractions and the many useful models that illustrate these sub-constructs (Lamon, 1999; Post et al., 1993), it is clear that a large, representative group of Victorian Grade 6 students do not generally have a confident understanding of these and their use.

Generally, performance on *part-whole* tasks was reasonable, although when the object of consideration was not in a standard form and not broken into equal parts (e.g., the Fraction Pie task), less than half of the students could give a correct fraction name to the part. The teaching implications here are clear. Students need more opportunities to solve problems where not all parts are of the same area and shape. On the other hand, the dots array task showed that students handled this discrete situation well, *unitising* appropriately, and usually had access to fractions that were equivalent to a given fraction.

Although simple fraction as an *operator* tasks were straightforward for most students, it seems that only around one-sixth of students being able to find one-third of a half indicates that students may need more encouragement to form mental pictures when doing such calculations. The second part of the Fraction Pie task was closely related, and it is interesting that of the 138 students who solved the pie task correctly, only 47 could give an answer to "one-third of a half". On the other hand, of the 59 students who were successful with the mental task, 47 could solve the related pie task. Once again, the importance of visual images in solving such problems is clear.

The experience of the authors is that Australian students spend relatively little time working with number lines in comparison to countries such as The Netherlands. Given that only around half of the students could draw an appropriate number line that showed 2/3, it is clear that fraction as a *measure* requires greater emphasis in curriculum documents and professional development programs, as many students are clearly not viewing fractions as numbers in their own right. In light of these data, the performance on locating six thirds and eleven sixths was relatively high. The Construct a Sum and fraction comparison tasks revealed similar difficulties with understanding the size of fractions, particularly improper fractions, and a lack of use of *benchmarks* in student thinking. Emphasising these aspects instead of fraction algorithms may be wise.

From our experience, few Australian primary school and middle school teachers and even fewer students at these levels are aware of the notion of *fraction as division*. Most students who concluded that 3 pizzas shared between 5 people would result in 3/5 of a pizza each, either drew a picture or mentally divided the pizzas to calculate the equal share. A very small percentage knew the relationship automatically. This supports the data of Thomas (2002) that 47% of 14 year-olds thought 6÷7 and 6/7 were not equivalent.

In summary, our data indicate clearly that Victorian students (and probably their teachers through appropriate professional development) need greater exposure to the sub-constructs of fractions and the related models, as noted by Post et al. (1993) and other scholars. We would also encourage teachers to use some of the tasks we have discussed in one-to-one interviews with their students, as our experience is that the use of the interview provides teachers with considerable insights into student understanding and common misconceptions, and forms a basis for discussing the "big ideas" of mathematics and curriculum implications of what they have observed.

References

Baturo, A. R., & Cooper, T. C. (1999). Fractions, reunitisation and the number line representation. In O. Zaslavsky (Ed.), *Proceedings of the 23rd annual conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 81-88). Haifa, Israel: PME.

Behr, M. J., Lesh, R., Post, T. R., & Silver, E. A. (1983). Rational number concepts. In M. Landau (Ed.), *Acquisition of mathematics concepts and processes* (pp. 91-126). Hillsdale, NJ: Erlbaum.

Behr, M., Wachsmuth, I., & Post, T. (1985). Construct a sum: A measure of children's understanding of fraction size. *Journal for Research in Mathematics Education, 16*(2), 120-131.

Bell, A., Fischbein, E., & Greer, B. (1984). Choice of operation in verbal arithmetic problems: The effects of number size, problem structure and context. *Educational Studies in Mathematics 15*, 129-147.

Bulgar, S., Schorr, R. Y., & Maher, C. A. (2002). Teachers' questions and their role in helping students build an understanding of division of fractions. In A. D. Cockburn & E. Nardi (Eds.), *Proceedings of the 26th annual conference of the Interntional. Group for the Psychology of Mathematics Education* (Vol. 2, pp. 161-168). Norwich, UK: PME.

Carpenter, T., Fennema, E., & Romberg, T. (1993). Towards a unified discipline of scientific enquiry. In T. Carpenter, E. Fennema, & T. Romberg (Eds.), *Rational numbers: An integration of research* (pp. 1-11). Hillsdale, NJ: Lawrence Erlbaum.

Clarke, D. M., Cheeseman, J., Gervasoni, A., Gronn, D., Horne, M., McDonough, A., Montgomery, P. & Roche, A., Sullivan, P., Clarke, B. A., & Rowley, G. (2002). *Early numeracy research project final report.* Melbourne: Mathematics Teaching and Learning Centre, Australian Catholic University.

Clarke, D. M., Mitchell, A., & Roche, A. (2005). Student one-to-one assessment interviews in mathematics: A powerful tool for teachers. In J. Mousley, L. Bragg, & C. Campbell (Eds.), *Mathematics: Celebrating achievement* (Proceedings of the 100th Annual Conference of the Mathematical Association of Victoria, pp. 66-80). Melbourne: MAV.

Cramer, K. A., Behr, M., Post, T., & Lesh, R. (1997). *Rational Number Project: Fraction lessons for the middle grades—Level 1.* Dubuque, IA: Kendal Hunt Publishing.

Empson, S. (2003). Low performing students and teaching fractions for understanding: An interactional analysis. *Journal for Research in Mathematics Education, 34*(4), 305-343.

Hasemann, K. (1981). On difficulties with fractions. *Educational Studies in Mathematics, 12,* 71-87.

Kieren, T. (1976). On the mathematical, cognitive and instructional foundations of the rational numbers. In R. A. Lesh (Ed.), *Number and measurement: Papers from a research workshop* (pp. 101-144). Athens, GA: ERIC/SMEAC.

Kilpatrick, J., Swafford, J., & Findell, B. (Eds.). (2001). *Adding it up: Helping children learn mathematics.* Washington, DC: National Academy Press.

Lamon, S. (1999). *Teaching fractions and ratios for understanding: Essential content knowledge and instructional strategies for teachers.* Mahwah, NJ: Lawrence Erlbaum.

Lamon, S. (2001). Presenting and representing: From fractions to rational numbers. In A. Cuoco & F. Curcio (Eds.), *The roles of representation in school mathematics, 2001 Yearbook,* (pp.146-165). Reston, VA: National Council of Teachers of Mathematics.

Litwiller, B, & Bright, G. (2002). *Making sense of fractions, ratios, and proportions, 2002 Yearbook,* Reston, VA: National Council of Teachers of Mathematics.

Mack, N. (2002). Making connections to understand fractions. In D. Chambers (Ed.), *Putting research into practice in the elementary grades: Readings from journals of the National Council of Teachers of Mathematics* (pp. 137-140). Reston, VA: NCTM.

Mitchell, A., & Clarke, D. M. (2004). When is three quarters not three quarters? Listening for conceptual understanding in children's explanations in a fractions interview. In I. Putt, R. Farragher, & M. McLean (Eds.), *Mathematics education for the third millenium: Towards 2010* (Proceedings of the 27th annual conference of the Mathematics Education Research Group of Australasia, pp. 367-373). Sydney: MERGA.

Olive, J. (2001). Connecting partitioning and iterating: A path to improper fractions. In M van den Heuvel-Panhuizen (Ed.), *Proceedings of the. 25th annual conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 1-8). Utrecht, The Netherlands: PME.

Pearn, C., & Stephens, M. (2004). Why you have to probe to discover what year 8 students really think about fractions. In I. Putt, R. Faragher & M. McLean (Eds.), *Mathematics education for the third millenium: Towards 2010,* (Proceedings of the 27th annual conference of the Mathematics Education Research Group of Australasia, Vol. 2, pp. 430-437). Sydney: MERGA.

Post, T., Cramer, K., Behr, M., Lesh, R., & Harel, G. (1993). Curriculum implications of research on the learning, teaching and assessing of rational number concepts. In T. Carpenter, E. Fennema, & T. Romberg (Eds.), *Rational numbers: An integration of research* (pp. 327-361). Hillsdale, NJ: Lawrence Erlbaum Associates.

Schorr, R. Y. (2001). A study of the use of clinical interviewing techniques with prospective teachers. In M van den Heuvel-Panhuizen (Ed.), *Proceedings of the. 25th annual conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 153-160). Utrecht, The Netherlands: PME.

Streefland, L. (1991). *Fractions in realistic mathematics education: A paradigm of developmental research.* Dordrecht: Kluwer Academic Publications.

Thomas, M. (2002). Versatile learning of mathematics. In D. Tall & M. Thomas (Eds.), *Intelligence, learning and understanding in mathematics* (pp. 179-204). Flaxton, Queensland: Post Pressed.

Teaching as Listening: Another Aspect of Teachers' Content Knowledge in the Numeracy Classroom

Ngaire Davies
Massey University
<n.m.davies@massey.ac.nz>

Karen Walker
Massey University
<k.walker@massey.ac.nz>

Recent mathematics education reform calls for pedagogical practice that is responsive to students' personal articulations of mathematics ideas. In such initiatives, listening to students is fundamental to advancing students' thinking. Our study explored the relationship between teachers' orientation towards listening and teachers' content knowledge. We investigated how four teachers listened to and made sense of students' ideas, and the influence of content knowledge on their capacity to listen. The study revealed that the depth of teachers' content knowledge – both subject matter knowledge and pedagogical content knowledge – mediated their enactment of effective listening practices.

Content knowledge plays a key role in teacher effectiveness (Ball & Bass, 2000). What teachers do in classrooms is very much influenced by what they know about mathematics (Hill, Rowan, & Ball, 2005; Shulman, 1986). The effective teacher has a sound grasp of mathematical ideas (Askew, Brown, Rhodes, Johnston, & Wiliam, 1997), and from that understanding is able to choose appropriate ways to represent subject matter, to ask questions, to plan activities, and to facilitate discussions. Importantly, a high level of content knowledge provides teachers with the cognitive resources that enable them to move students' thinking forward. Teachers do this by negotiating their understanding of subject matter with their knowledge of the learning of the students in the classrooms (Sherin, 2002). We were interested to see if teachers' subject matter knowledge and knowledge of pedagogical content also influenced the ways in which teachers listened to students.

Careful listening to what students have to say has been shown to be an important aspect of practice (Carpenter & Fennema, 1992; Crespo, 2000; Davis, 1997). Unlike traditional classrooms, in which there is little opportunity for students to engage in extended dialogue about mathematics (Tanner, Jones, Kennewell, & Beauchamp, 2005), teachers in classrooms implementing new initiatives hold the view that talking about mathematics is an essential feature of a quality mathematical experience. Muir (2006) suggests that "encouragement of purposeful discussion" allows teachers to "probe and challenge children's thinking and reasoning" (p. 369). Purposeful mathematical discussion, however, demands focused listening. Effective teachers who listen carefully to students' responses to questions are able to draw out students' understandings (Yackel, Cobb, & Wood, 1990). Franke and Kazemi (2001) have shown from their research that, not only is listening important but also it is fundamental to advancing students' thinking. It is, according to Sherin (2002), one of the key focus areas for initiating more effective mathematics teaching. Indeed, for teachers in research undertaken by Carpenter and Fennema (1992), "listening to their students was the crucial factor" (p. 463) that contributed to more effective mathematics instruction.

Teachers listen to their students through their own mathematical, personal, and social resources (Wallach & Even, 2005). Teachers who do not listen or do not understand their students' thinking, tend to minimise or dismiss it, by imposing their own understandings (Cobb, 1988). Ball (1997) has argued that a teacher's understanding of the subject matter, along with the commitment to the students in the classroom, will significantly influence what the teacher hears. Hill, Rowan, and Ball (2005) go so far as saying that knowledgeable teachers are able to hear their students' methods better because they have a clearer understanding of the structures and connections of mathematics. Teachers with sound content knowledge are able to access the conceptual understandings that students are articulating. They are able to make informed decisions about how those understandings might have arisen and where they might be heading (Shulman & Shulman, 2004). Such teachers listen by drawing on their content knowledge in order to create "more powerful forms of classroom teaching" (Doerr & Lesh, 2002, p. 130).

Conceptual Framework

In our exploration of the relationship between listening and content knowledge we have found the work on communities, developed by social practice theorists, extremely helpful. Lave and Wenger (1991), amongst others, propose that people develop knowledge when they are engaged in immediate, concrete, specific, and meaning-rich activities. What their proposals are able to explain is how learning occurs in the context of shared events and interests. We draw on these ideas about communities of practice to explain an aspect of classroom teaching. We plan to show that the way in which students' understandings of mathematics are advanced within the classroom community is very much influenced by what the teacher hears. In turn, what the teacher hears is informed by his/her content knowledge.

Our analytic strategy is guided by the categories set out by Davis (1997). Davis' three categories of teachers' orientation to listening have proved to be an effective means of understanding classroom phenomena. The conceptual categories are namely, evaluative, interpretive, and hermeneutic. Davis suggests that not all forms of listening are conducive and respectful of students' thinking. For example, teachers with an *evaluative* orientation tend to listen to students' ideas in order to diagnose and correct their mathematical understandings. A correct answer is already in place in the teacher's mind (Crespo, 2000). Typically, if the expected response is not given by the students then often the gaps would be filled by the teacher's response. Thus the teacher strives for unambiguous explanations and to maintain a well-structured lesson that does not deviate.

Teachers with an *interpretative* orientation listen to a student's ideas with the primary purpose of assessing. In relation to the evaluative orientation, there is an increased opportunity for interaction, both between teacher and student, and among students. However, the teacher is accessing rather than assessing the student's understanding (Crespo, 2000). There is an awareness of active participation. However, what is learned is manageable for the teacher within a set of precise steps in order to achieve particular pre-specified understanding. Teachers with a *hermeneutic* orientation continually and interactively listen to a student's ideas. They tend to adopt a more flexible approach to the ever-changing circumstances within the learning process by engaging with them in the "messy process of negotiation of meaning and understanding" (Crespo, 2000, p. 156).

Our interest in these categories of teachers' orientations to listening was to explore the influence of teachers' content knowledge on each. We wanted to investigate the teachers' approach to listening within the classroom as an enactment of their content knowledge.

Attending to teachers' orientations to listening will help us understand effective practice (Davis, 1997).

Description of the Study

We report on the second year of our study on "Teacher Knowledge". The study is one of four research "nests" situated within a larger project, *Numeracy Practices and Change*. The New Zealand Numeracy Development Project (Ministry of Education, 2001) acknowledges that:

> teachers' understanding of subject matter and of pedagogy are critical factors in mathematics teaching. The effective teacher has a thorough understanding of the subject matter to be taught, comprehends how students are likely to learn, and knows difficulties and misunderstandings they are likely to encounter. (p. 2)

The Numeracy Development Project provided the context for our focus on teacher knowledge. In our first year of study we reported on teachers "learning to notice" critical mathematical instances during classroom interactions (Davies & Walker, 2005). The focus of our second year was to move from the supportive community of learners to a closer investigation of the teachers in their classrooms. In order to characterise how content knowledge is enacted we further investigated teachers on the process of noticing significant mathematical moments. This paper reports on one central aspect of teachers noticing, namely, their orientation towards listening. The questions guiding the exploration were as follows:

- Was there evidence of different orientations to listening?
- Did the orientation to listening affect the lesson pathway?
- Was there a link between teachers' content knowledge development and their listening?

To address those questions we used a design research experiment working collaboratively with four teachers from two primary schools. The nature and design experiment methodology allowed us to investigate further classroom incidences of "listening" and the teachers reflection on these incidences. We report here on two teachers, whom we name Mike and Joe, from the same school, both of whom focussed on the topic of fractions for their year 5/6 classes. Initially the teachers were released from their classrooms for a brainstorming/planning session with the researchers. This session focussed on possible teaching points, key fraction understanding, as well as problems and equipment. Within this session teachers' own content knowledge was discussed.

Extensive use was made of video footage recorded by the researcher in each class on five occasions over a two-week period. Following each recording, the teacher was released to view video footage and participate in reflective discussion with the researcher. Teachers were asked to stop the video at significant mathematical moments and discuss. Initially this discussion was to focus on simple questions, such as: What did you notice? What does this mean? However, it soon became apparent that teachers were not noticing key mathematical moments themselves when watching the video footage. As researchers we drew out significant mathematical incidences and refocused our attention on instances of teachers' "listening".

Audiotapes were used to record each reflective session with the teacher, during the time that the video footage of their teaching was reviewed. The video footage of teaching and the resulting audiotapes of discussion, together with researcher's field notes, formed the dataset. At the end of the research period individual interviews recorded on audiotape were transcribed and collated. Relevant excerpts from these were analysed for anecdotes of listening and comments regarding mathematical incidences. Video footage was replayed to transcribe exact instances of listening and resulting teaching and learning pathways.

Results and Discussion

The discussion is developed around two teaching episodes that are drawn from Joe and Mike. Each is intended to highlight moments from a teaching episode characterised by a particular orientation to listening. The purpose of the selection and discussion is to identify, describe and contrast some classroom episodes during which significant mathematical incidences occurred. These classroom episodes illustrate how "listening orientations" can be useful analytic tools for interpreting classroom phenomena and as a starting place for transforming mathematics teaching practice.

Through the extensive use of video in the first year of research teachers "noticed" aspects of their teaching. After viewing his teaching, Joe commented on the types of questions he asked. Joe clarified his need for "better questions".

> Because in a lot of ways, my questioning was directly leading the student to the right answer, I was in some ways influencing their answer and it wasn't giving them a chance to think about the answer and get the right answer, rather than giving it to them. … Asking better questions and more open-ended questions. So why did you think that?

He was aware of the need to follow the student's response and, if needed, to make significant changes in the direction of the lesson. Making changes involves both in-depth subject knowledge and pedagogical content knowledge. In the second year of research Joe again spoke about using questioning to "get inside" the children's heads. This pointed to an effort, on his part, to use a more interpretive approach to listening. In the following problem that Joe gave his class, we track how this happened.

> Amy earns $24 a week. She saves 1/3. She keeps ¼ for clothes, ¼ for hobbies and movies and 1/6 for junk food. How much does she spend on each?

Joe moved around working with small groups of children.

Joe:	Saves 1/3, how much is that?
Child:	8 dollars
Joe:	Great, how did you work that out?
Child:	3 times 8 is 24
Joe:	And you took that away? Good girl.
Joe:	She keeps ¼ of this for clothes. So she keeps ¼ of 16 for clothes
Child:	4 dollars
Joe:	Good, we've got 4, we take 4 away from 16
Child:	12
Joe:	Put the 12 dollars down, once again we've got a ¼ for hobbies … [continues subtracting from total] …
Joe:	We have 9 dollars and she spends 1/6 on junk food …[pauses… rechecks question] … I'll check I have the question right. Wait there.
Comments to researcher:	That's quite hard aye?

Later the class came together to share their solutions and Joe selected Jordan to share his solution.

Joe:	Some people have done it quite differently than how I did it, which is fine. Tell us what you did Jordan.
Jordan:	Well 1/3 of 24 is 8 cause 8 times 3 is 24
Joe:	Yes, did everyone else get that?
Jordan:	Then ¼ for clothes is 6, and so ¼ for hobbies is the same 6, and 1/6 for junk food is 4. [Looks to teacher for support, teacher nods to carry on]
Jordan:	So then 6 plus 6 plus 4 is 16.

Jordan sits down and Joe looked slightly puzzled; not actually convinced of Jordan's thinking.

Joe:	Who got the same answer as Jordan? Who had a different answer? I know I did. But we were doing the same stuff though. As we went through each step I took that money away from the total. Just goes to show that the way you interpret the question can affect your answer.

Joe seemed not to consider the solution suggested by Jordan. He was not attending to the answer given in a way that would help develop understanding. Davis (1994) warns that the listener must be "vigilant to the fallibility of interpretation" (p. 279). Initially Joe worked out the problem and when it proved difficult to solve he thought perhaps he had written it incorrectly. Jordan managed to solve the problem as presented; however, his thinking did not match Joe's. In the discussion that followed the video Joe described his thinking.

Joe:	I was comfortable with that. The group that came to the board had a 1/3 of 24 and then ¼ of 24. They were not using takeaway and decreasing amounts.
Researcher:	Why do you say takeaway?
Joe:	Because that is what she spent, its obvious, she is spending the money so you take it away. [long pause]
Joe:	Now I look back on it, they answered the way it was meant to be. The question wasn't well written though was it? When you think of money you take some away for savings and then you deal with what you have left!

Joe was listening through his own mathematical, personal, and social resources (Wallach & Even, 2005). His subject matter knowledge influenced what he heard as he was unable to access the conceptual understanding that Jordan was articulating (Hill et al., 2005). As an evaluative listener, Joe was seeking a particular response; even though the child's response provided a solution to the problem he did not change his thinking. His own content knowledge let him down.

Two days later Joe gave two similar problems to a group and worked alongside the children as they solved them. In the first problem there were 32 children choosing their favourite sport: ¼ rugby, 1/8 tee ball, etc. The problem solution was discussed and solved satisfactorily. The second problem involved 60 vegetables for an "umu": ¼ were taro, ¼ kumara, 1/3 yams, and 1/6 breadfruit. Joe worked with the children individually questioning their workings and requiring explanations. He confirmed Sam's solution of 15, 15, 20 and 10. When sharing his strategy Sam explained he found a quarter through halving the 60 and halving again. Sam then drew 6 dots to represent the 60 vegetables and proceeded to use the dots to simplify finding a third, by circling two dots, and then a sixth by circling one dot. However Joe took over Sam's explanation using his subtraction method to the obvious confusion of Sam. During the reviewing of the video Joe explained,

> Yeah I saw on his paper he had done it all right but he lost me a bit [when explaining on the board], so I wanted to come back to the point where I knew where he was and that's where I went wrong … I was trying to make Sam think how I was thinking and not getting inside his head.

Although similar problems had been fully explored with both the researcher and through the explanations of some children, Joe continued with his misconception. Joe was listening for the response he wanted. Joe needed to take the child to where Joe could understand and move the solution on the expected pathway. In an effort to "get inside the child's head" and to diagnose and correct the child's understanding he used an evaluative orientation to listening seeking a common understanding. Joe was limited by his content knowledge unaware of the fallibility of his own method.

We now move to our second teacher Mike. On the final day videoed, Mike provided a group with a variety of circular fraction pieces to investigate and to make five statements about them. Mike's expectation was that they would come back with equivalent fractions although he did not communicate this to the group.

> Statements from this group shared with the rest of the class included:
> 1/3 and 1/6 makes a ½
> 2 ¼'s makes ½
> 4 of 1/8's makes a ½
> 1/6 and 1/8 makes ¼.

During the review of the teaching episode Mike explained that although his plan was to focus on equivalent fractions due to the "novel student responses" involving addition he decided to follow the children's lead. A noticeable difference was the increased opportunity for interaction within the group. Mike opened up opportunities for representation and revision of ideas (Davis, 1997). He was surprised that the children were capable of working things out for themselves even though it was not quite what he expected. He said that for them, it was valuable learning. Mike's listening orientation moved to a more interpretative stand.

The teaching continued as he decided to open the discussion to the rest of the class. Once again he deviated from his initial plan indicating a more interpretative approach. After further statements about the circular regions he asked if anyone could make any statements about adding fractions. Within this discussion Mike moved from interpretive listening back to evaluative listening.

> George made the statement 1/3 + 1/6 + 1/3 + 1/6 = 1 whole [Mike wrote this on the board].
> Mike asked could anyone make it a shorter equation.
> Teane wrote ½ + ½ = 1
> When asked to explain this Teane said 1/3+1/6=1/2 [Mike did not seek further explanation].
> Bridget wrote 2/6 + 2/12 = 1 and explained it pointing the 2 1/3's were the 2/6 and the 2 1/6 were the 2/12.

At this stage we see Mike constructing with the learners as they construct their mathematics (Davis, 1997). He was accessing the children's understanding in an interactive way. His purpose of accessing rather than assessing the children's thinking demonstrated an interpretative orientation to listening (Crespo, 2000). He continued,

> Mike asked what the "rectangles group" had learnt when discussing adding fractions in their group?
>
> Child: Not allowed to change the denominators [i.e. not adding them together]
> Mike to Teane: You're not even in that group well done [acknowledges Teane's earlier response].
> Mike to Bridget: I am banning you changing the denominator but if you can change the numerator what would it look like now?
>
> Bridget rubbed out the 6 [in 2/6] and changed it to a 3 [looked for reassurance] then changed the 12 [in 2/12] to 6 and quickly sat down. [She now had 2/3 + 2/6 = 1]

Mike: OK who agrees with the equation Bridget has made? [quick show of hands].Can you explain Bridget why it's true?
Bridget: Cause I didn't change the denominators.
Mike: OK gonna stop there guys cause the bell is going to go but it's certainly something to discuss for next time.
George [quickly said]: That it was the same as 2/3 + 1/3 = 1.

Mike's reliance on procedural mathematics understanding influenced his teaching and his listening orientation. Bridget gave the wrong answer so he gave her a way to fix it and then moved on. Mike was comfortable with George's equation as it demonstrated procedural understanding, which is how Mike operates. George was also one of the children he considered to be good at mathematics. Davis (1994) suggests teachers' orientation to listening is enabled by "who" the teachers are listening to and constrained by "what" they are listening for. During discussions whilst viewing the video Mike commented that he asks George to explain further because he expects a correct response. But he did not ask Teane because Mike thought that he would not be able to explain satisfactorily. Mike began to question his view on Teane's ability after watching the video.

Mike demonstrated a shift from a strong evaluative orientation to listening to the beginnings of an interpretive orientation. However his own procedural understanding and his expectations of children's ability greatly influenced what he heard the children say (Crespo, 2000). It also influenced how he reacted to it drawing him back to a more evaluative orientation.

Conclusions

To teach effectively it is crucial that teachers notice the significant mathematical moments and respond appropriately. If teachers are going to provide students with appropriate mathematical challenges and assist the students to gain meaning, they need to be able to access their own content knowledge whilst engaged in the act of teaching. We expected that a novel student idea would prompt the teachers to reflect on and rethink their instruction (Schifter, 1996). The teachers did initiate questioning and probing in order to assess the students' understanding. To do this they needed to listen to the students' ideas and access their own content knowledge complexes to decide how best to proceed. However the teachers, more often than not, return to their planned lesson rather than exploring the students' ideas. Attention was given to the students' responses with little impact on the development of the lesson as the teacher was only seeking a particular response. Davis (1997) calls this evaluative listening.

It was apparent throughout the 2 years of our research that teachers' orientations to listening varied greatly between individuals and also within lessons. The orientation to listening influenced their ability to negotiate the lesson fully. Their listening orientation was dependent upon the level of their own content knowledge (Ball, 1997). We suggest that the teachers did not have the knowledge of the subject to be able to make connections for the children and for themselves. Teachers' orientation to listening was also influenced by their expectations of the children's mathematical ability. These expectations influenced their decisions concerning which child to call upon, whether to require further explanation of their thinking, or whether they just filled in the gaps (Crespo, 2000). Perhaps we also need to examine the teachers' beliefs about mathematics. If teachers believe that learning mathematics involves only the "acquisition of knowledge" then their orientation to listening takes on quite a different relevance (Davis, 1994, p. 279).

Noticeably absent from this study was Davis' (1997) third category of hermeneutic listening. We are left to wonder whether this orientation to listening is accessible to teachers without further support and what kinds of experiences might bring about a transformation in their practices. For teachers to realise the need for change and to transform their practice they need a strong community of support (Davis, 1997). Due to the complexities of teaching it is difficult to isolate "quick remedies" in developing more effective listeners. Time needed for change is a constraint. In our first year we saw changes in the teachers "learning to notice" after ongoing intensive planning/content knowledge workshops within a supportive community of learners. The more condensed time frame in the second year did not allow opportunities for ongoing and sustained change (Doerr & Lesh, 2003).

The teachers' content knowledge became a central organiser for the lessons and a defining feature of effective teaching. The depth of teachers' content knowledge – both subject matter knowledge and pedagogical content knowledge – mediated their enactment of effective listening practices.

Acknowledgement: The authors wish to acknowledge that this project was funded through the New Zealand Ministry of Education's *Teaching and Learning Research Initiative (TLRI).*

References

Askew, M., Brown, M., Rhodes, V., Johnston, D., & Wiliam, D. (1997). *Effective teachers of numeracy.* London: School of Education, Kings College.

Ball, D. (1997). What do students know? Facing challenges of distance, context and desire in trying to hear children. In B. J. Biddle, T.L. Good, & I.F. Goodson (Eds.), *International handbook of teachers and teaching* (pp. 769-818). Dordrecht, The Netherlands: Kluwer.

Ball, D., & Bass, H. (2000). Interweaving content and pedagogy in teaching and learning to teach: Knowing and using mathematics. In J. Boaler (Ed.), *Multiple perspectives on the teaching and learning of mathematics* (pp. 103-104). Westport CT: Ablex.

Carpenter, T., & Fennema, E. (1992). Cognitively guided instruction: Building on the knowledge of students and teachers. In W. Secada (Ed.), *Researching educational reform: The case of school mathematics in the United States* (pp. 457-470). A special issue of *International Journal of Educational Research*.

Cobb, P. (1988). The tensions between theories of learning and instruction in mathematics education. *Education Psychologist, 23*, 87-103.

Crespo, S. (2000). Seeing more than right and wrong answers: Prospective teachers' interpretations of students' mathematical work. *Journal of Mathematics Teacher Education*, *3*, 155-181.

Davies, N., & Walker, K. (2005). Learning to notice: One aspect of teachers' content knowledge in the numeracy classrooms. In P. Clarkson, A. Downton, D. Gronn, M. Horne, A. McDonough, R. Pierce, & A. Roche (Eds.), *Building connections: Theory research and practice* (Proceedings of the 28th annual conference of the Mathematics Education Research Group of Australasia, Melbourne, pp. 273-280). Sydney: MERGA.

Davis, B. (1994). Mathematics teaching: Moving from telling to listening. *Journal of Curriculum and Supervision, 9*(3), 267-283.

Davis, B. (1997). Listening for differences: An evolving conception of mathematics teaching. *Journal for Research in Mathematics Education*, *28*(3), 355-376.

Doerr, H., & Lesh, R. (2002). A modelling perspective on teacher development. In R. Lesh & H. Doerr (Eds.), *Beyond constructivism: A models and modelling perspective on mathematics teaching, learning and problem solving* (pp. 125-39). Mahwah, NJ: Lawrence Erlbaum.

Doerr, H., & Lesh, R. (2003). Designing research on teachers' knowledge development. In L. Bragg, C. Campbell, G. Herbert, & J. Mousley (Eds.), *Mathematics education research: Innovation, networking, opportunity* (Proceedings of the 26th annual conference of the Mathematics Education Research Group of Australasia, Geelong, pp. 262-269). Sydney: MERGA.

Franke M.L., & Kazemi, E. (2001). Teaching as learning within community of practice: Characterising generative growth. In T. Wood, B.S. Nelson, & J. Warfield (Eds.), *Beyond classical pedagogy: Teaching elementary school mathematics* (pp. 47-74). Mahwah, NJ: Lawrence Erlbaum.

Hill, H., Rowan, B., & Ball, D. (2005). Effects of teachers' mathematical knowledge for teaching on student achievement. *American Educational Research Journal, 42*(2), 371-406.

Lave, J., & Wenger, E. (1991). *Situated learning: Legitimate peripheral participation*. New York: Cambridge University Press.

Ministry of Education (2001). *Curriculum Update: The Numeracy Story. No. 45*. Wellington: Learning Media.

Muir, T. (2006). What does effective teaching for numeracy look like? The design of an observation schedule. In P. Grootenboer, R, Zevenbergen, & M. Chinnappan (Eds.), *Identities, cultures, and learning spaces* (Proceedings of the 29th annual conference of the Mathematics Education Research Group of Australasia, Canberra, pp. 368-375). Sydney: MERGA.

Schifter, D. (1996). *What's happening in math class?* New York: Teachers College.

Sherin, M. (2002). When teaching becomes learning. *Cognition and Instruction, 20*(2) 119-150.

Shulman, L.S. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher*, *15*(2), 4-14.

Shulman, L., & Shulman, J. (2004). How and what teachers learn: A shifting perspective. *Journal of Curriculum Studies, 36*(2), 257-271.

Tanner, H., Jones, S., Kennewell, S., & Beauchamp, G. (2005). Interactive whole class teaching and interactive white boards. In P. Clarkson, A. Downton, D. Gronn, M. Horne, A. McDonough, R. Pierce, & A. Roche (Eds.), *Building connections: Theory research and practice* (Proceedings of the 28th annual conference of the Mathematics Education Research Group of Australasia, Melbourne, pp. 273-280). Sydney: MERGA.

Wallach, T., & Even, R. (2005). Hearing students: The complexity of understanding what they are saying, showing and doing. *Journal of Mathematics Teacher Education*, *8*, 393-417.

Yackell, E., Cobb, P., & Wood, T. (1990). The interactive constitution of mathematical meaning in one second grade classroom: An illustrative example. *Journal of Mathematical Behaviour, 11*(4), 458-477.

Essential Differences between High and Low Performers' Thinking about Graphically-Oriented Numeracy Items

Carmel M. Diezmann
Queensland University of Technology
<c.diezmann@qut.edu.au>

Tom J. Lowrie
Charles Sturt University
<tlowrie@csu.edu.au>

Nahum Kozak
Queensland University of Technology
<n.kozak@qut.edu.au>

This study compared the thinking of five high performing and five low performing primary students on a set of graphically-oriented numeracy items. Generally, their thinking differed in four ways. First, high performers drew on existing knowledge and skills, which low performers appeared to lack. Second, high performers used multiple cues to complete tasks, whereas low performers worked from a single cue or overlooked cues. Third, high performers used simple solution procedures correctly; in contrast, low performers used more mentally demanding procedures with limited success. Finally, high performers were more knowledgeable about everyday graphics than low performers.

Introduction

Worldwide there has been a strong and ongoing emphasis on the development of a numerate populace who can use mathematics effectively in everyday life at home, at work and in the community. Traditionally, numeracy has been characterised by arithmetical competence. However, in the digital age, numeracy also involves proficiency with the various graphics that are commonly used in mathematics (Department for Education and Employment, 1998): "numeracy also demands practical understandings of the ways in which information is gathered by counting and measuring, and is presented *in graphs, diagrams, charts and tables* (emphasis added)" (p. 110). Thus, the achievement of a numerate populace requires that all citizens use graphics effectively in mathematical situations. The students who are most at risk of being innumerate are those who struggle with mathematics. Hence, the achievement of the numeracy goal depends on our ability to educate those students who have difficulty with essential mathematics. These students are of two types. First, there are those students who have special needs due to a problem that impacts on their ability to think or to learn. These difficulties include memory problems, processing or perceptual deficits (Diezmann, Thornton, & Watters, 2003). Second, there are those students who do not have specific learning problems but nevertheless are low performers. Notwithstanding the importance of understanding how to educate students with special needs, this paper focuses on ways to support students who are low performers on numeracy items that incorporate graphics. This support will be informed by the performance of students who consistently demonstrate proficiency with these items because such students can provide an insight into the knowledge and skills that are required to be successful. Thus, this study will contribute towards addressing the paucity of literature on high and low performing Australasian students (Diezmann, Lowrie, Bicknell, Farragher, & Putt, 2004).

Background

To provide a background to the thinking and solution strategies of high and low performers on numeracy tasks, we first provide an overview on graphics in mathematics and high and low performers' use of representations in mathematics.

Graphics in Mathematics

In recent decades, there has been enormous growth in the field of information graphics for the management, communication, and analysis of information (Harris, 1996). Although there are many thousands of graphics in use, they can be categorized into six broad categories that Mackinlay (1999) refers to as "graphical languages" (Table 1). These languages are distinguished by the information that is encoded in the graphic and the relationships among the graphical elements. Knowledge of graphics is fundamental to success on many numeracy items. However, although graphics are visual-spatial rather than linguistic or symbolic representations, many primary students have difficulty interpreting graphics, such as number lines (Diezmann & Lowrie, 2006).

Table 1
An Overview of the Six Graphical Languages (adapted from Mackinlay, 1999)

Graphical Languages	Encoding Technique
Axis Languages (e.g., number line)	A single-position encodes information by the placement of a mark on an axis.
Opposed Position Languages (e.g., bar chart)	Information is encoded by a marked set that is positioned between two axes.
Retinal List Languages (e.g., saturation on population graphs)	Retinal properties are used to encode information. These marks are not dependent on position.
Map Languages (e.g., road map)	Information is encoded through the spatial location of the marks.
Connection Languages (e.g., network)	Information is encoded by a set of node objects with a set of link objects.
Miscellaneous Languages (e.g., pie chart)	Information is encoded with a variety of additional graphical techniques (e.g., angle, containment).

High and Low Performers' Use of Representations in Mathematics

Mathematical proficiency is influenced by students' understanding of a variety of representations including graphics. According to von Glasersfeld (1987), the individual plays an important role as the interpreter or decoder of a representation: "A representation does not represent by itself – it needs interpreting and, to be interpreted, it needs an interpreter" (p. 216). Students' proficiency with representations impacts on whether they will be high or low performers. For example, students who are successful on number line items recognise that it is a measurement model and explain their solutions with reference to distance, proximity, or reference points (Diezmann & Lowrie, 2006). In contrast, some students who are unsuccessful on number line items interpret the number line as a counting model and overlook the proportional distances between marks on the line. Students' capability with linguistic representation also distinguishes high performers from low performers. For example, whereas novices (typically low performers) interpret keywords literally and make links to a limited knowledge base, experts (typically high performers) use keywords as cues to an appropriate knowledge schema (Chi, Feltovich, & Glaser, 1981): "Experts perceive more in a problem statement than novices do. They have a great deal of tacit knowledge that can be used to make inferences and derivations from the situation to the problem statement" (p. 149). The differences between high and low performers in their interpretations of various representations extend to reasoning from the representations. An individual's reasoning must take into account the mathematical conventions that are associated with particular representations. Hence, representations are systems of organised data with inbuilt sets of rules of use. For example, reasoning about distance on a map requires attention to the scale of the map. Galotti (1989) proposes that knowledge includes an appreciation of the various rule-based systems in use in mathematics: "Experts, by virtue of their richer knowledge base and extensive experience with problems within a given domain, have a larger and more differentiated set of rules with which to reason" (p. 347). Thus, being mathematically proficient requires an extensive knowledge of various representations including graphics and the associated reasoning that is used with different types of representations.

Research Design and Methods

This study had two purposes. The educational purpose was to gain insights into the differences between high and low performers with a view to identifying specific ways to support the thinking of low performers. The methodological purpose was to establish whether a comparison between high and low performers was a fruitful avenue for gaining insights into students' thinking about graphically-oriented numeracy items, and hence, would be worthwhile implementing with a more extensive data set.

The Participants

Ten participants were identified for this study from 67 Queensland students who participated in a series of annual interviews about graphically-oriented numeracy items. These participants comprised five of the most high performing students (one boy, four girls) over two annual interviews and five of the most low performing students (two boys, three girls) for the same period. These two groups of students are henceforth referred to as "high performers" and "low performers". The students were aged between 10 and 11 years when they commenced in the study. All students attended one of two similar schools in a moderate socio-economic area of a capital city.

The Interviews

The participants were interviewed on a set of 12 items in each of two annual interviews. These tasks were drawn from the 36-item Graphical Languages in Mathematics [GLIM] test which comprises six sets of numeracy items for each of the six graphical languages (see Lowrie and Diezmann, 2005 for a discussion of the test). Examples from this test are presented in the Appendix. The two easiest items from each of the six language groups were presented to the students in the first annual interview and six pairs of items of moderate difficulty were presented in the second annual interview. (The six pairs of the most difficult items will be presented to students in a third annual interview, which has yet to be conducted.)

Data Collection and Analysis

Interview data comprised students' selections on a multiple choice task and the reasons they gave for their responses. The students attempted each pair of tasks independently, and were then asked to explain their solutions. They were probed about any difficulties that they experienced but no scaffolding was provided to avoid the possibility that support on one item might influence understanding on another item. The interviews were video-taped to facilitate analysis. These data were analysed within an inductive theory-building framework with a focus on description and explanation (Krathwohl, 1993). The tactics for generating meaning were noting patterns and themes, imputing plausibility, and building a logical chain of evidence (Miles & Huberman, 1994).

Results and Discussion

Four themes emerged from a comparison of high and low performers' responses to the 24 GLIM items.

Theme 1: The Use of Mathematical Knowledge and Skills

In interpreting items, high performers were more likely to bring existing mathematical knowledge and skills to bear on the task. Low performers were less mathematically proficient, and worked out their solutions in a more laborious fashion that typically involved counting. Though the strategies low performers selected were appropriate, their strategies were more prone to error. Differences in the use of existing knowledge and skills by high and low performers are illustrated by the following example.

On *The Pie Chart* item, students were asked to determine how many hours were spent on homework based on the information presented (see Appendix). The high performers and low performers used different strategies. The five high performers used a *fractional strategy* successfully. In contrast, four low performers used an *estimate and add strategy* with mixed success and the final low performer misunderstood the question.

The *fractional strategy* required an understanding of quarters as shown in Chloe's (a high performer) response.

> Chloe: About a quarter of it (the time) was Mathematics and that was two hours so there was four quarters … two times four is eight.

By identifying the Mathematics portion of the pie chart as a quarter, Chloe reduced the question to a simple multiplication calculation, which she easily accomplished mentally. That is, two hours of Mathematics multiplied by four (for a quarter of the pie chart) is eight hours of homework in total. Thus, as typical of the other high performers, Chloe's success was due to her ability to use existing knowledge and skills to achieve the correct answer. None of the low performers recognised the opportunity to use a simple fractional strategy or mentioned that Mathematics was a represented by a quarter of the pie chart.

The *estimate and add strategy* was used by four of the five low performers. Two were successful and two were unsuccessful. Although this strategy had the potential to be successful, it required students to estimate the number of hours in each segment of the pie chart accurately and to sum these values to determine the total hours shown on the chart.

An inherent pitfall in applying this strategy was to accurately estimate the value of each portion of the chart, as shown in Bree's (a low performer) explanation.

Bree: Whenever I count, I get to nine … Mathematics is two hours ... each half of Science (is) two, Reading and History … an hour each, and I counted that (Art) as an hour. That's why I (got nine).

Bree used only whole number values when estimating sections of the pie chart. She incorrectly identified Science as 4 hours (actually 3½ hours) and Art as 1 hour (actually ½ an hour). Bree added these incorrect estimates for Science and Art to her correct estimates of two hours for Mathematics and one hour each for Reading and History to reach a total of nine hours instead of eight hours. Similarly, Mike (a low performer) also overestimated the value of some sections of the pie chart. However, two other low performers, Nellie and Helen, correctly estimated values and were successful in their use of the estimate and counting strategy.

Thus, a key difference between these high performers and low performers on *The Pie Chart* was the high performers' selection of an effective but simple strategy incorporating their existing mathematical knowledge of fractions and their multiplication skills. Pie charts are Miscellaneous graphics that encode information through the use of angles (Mackinlay, 1999). In the *fractional strategy,* high performers showed their ability to recognise the value of a key portion of the chart as a quarter of the total time and to use this knowledge efficiently in solution. In contrast, in the *estimate and add strategy,* low performers typically estimated the values of all of the portions, sometimes erroneously, and added these times. This approach was more mentally demanding because half hours needed to be recognised and the addition involved multiple addends including fractions.

Theme 2: The Use of Cues

A further difference between high performers and low performers was their use of cues within the task. High performers were aware of and used multiple cues to solve problems, whereas most low performers were not. The importance of using more than one cue is illustrated by students' responses on the following item.

The Scale item required students to find the mass of an apple by referring to a graphic depicting a traditional set of kitchen scales (see Appendix). On the face of the scales there were three cues in the form of values marked in grams: zero at the top, 100 in the middle, and 200 at the bottom. Between the labelled numbers were unlabelled marks that each represented 10 grams. Use of at least two of the number values was needed to appreciate that the vertical scale was arranged in ascending order.

The five high performers and one of the low performers successfully identified that the scale indicator was at the 170 gram mark. Four of the five high performers noted that the unlabelled mark halfway between 100 and 200 was 150, and proceeded to count in tens to 170. Cody was one of these high performers who used this *midpoint strategy* to successfully find the mass of the apple.

Cody: What I did then is like, do 150, and then went 160, 170.

One low performer, Mike, used exactly the same process as four high performers and found the halfway mark and counted on. Recall that low performers were selected as students who were consistently low performers over 24 interview items. As in Mike's case, this did not preclude them from being successful on a few items. Elise, the fifth high performer, was also successful but her *count all tens strategy* was less efficient. She counted on in tens from 100 grams to 170 grams making no reference to the halfway point between 100 and 200 grams.

In contrast to the successful students (five high performers, one low performer), the unsuccessful students (four low performers) did not detect the ascending order of the scale. These unsuccessful students used a single number value as a cue and then attempted to identify the mass of the apple. Nellie's response was typical of other unsuccessful students in that she focused on the "200" value, which was close to the mass indicator, and incorrectly assumed that the scale was in descending order.

Nellie: I put 230 grams because the arrow was near 200 and then I just counted steps up.

Nellie was efficient in counting by tens from the 200 mark to reach 230 grams, but because she did not account for the directionality of the scale, she counted forwards rather than backwards. Thus, the key difference between all high performers and most low performers was the ability to identify the directionality of the scale. Detecting that the scale was ascending required attention to at least two number values, which acted as cues for directionality.

The Scale item used an Axis graphic to encode information by the placement of a mark on some form of number line (Mackinlay, 1999). Although number lines are commonly used in primary texts and tests, they are difficult for some students. On the (US) National Assessment of Educational Progress, many fourth graders' success using a scale was no better than chance accuracy on a multiple choice item (1 out of 4, 25%) (National Center for Education Statistics, 2003). Here, we have identified directionality as problematic but students also have difficulty with Axis graphics because they interpret the number line as a counting model rather than a measurement model (Diezmann & Lowrie, 2006).

Theme 3: The Solution Approach

A further difference between high performers and low performers was their solution approach. When approaching a task, more high performers than low performers were methodical. They typically broke tasks into components and dealt with these components systematically. In contrast, low performers tended to attempt items more holistically. These differing approaches are illustrated in the following example.

In *The Puzzle* item, students were asked to select which of four puzzle pieces would complete the picture of three triangles (see Appendix). The solution piece needed a portion of each triangle to match the partly shown triangles in the picture. Every high performer was successful on this item whereas only two of the five low performers were successful.

Four out of five high performers selected the correct response by using a *component strategy* involving pieces of the puzzle. Rita's response was typical.

Rita: That bit there can fit into this one, that bit can fit into this one, and that can fit into there.

Rita's response suggests that she examined the sections of the triangles and decided which piece would fit into the larger puzzle. All high performers who chose this strategy were successful but only one of two low performers using the same strategy was successful.

The other approach used by students was a *perceptual strategy*. This strategy was used successfully by one high performer and one of three low performers. Jacob (low performer) used this strategy successfully and like his high performing counterpart made his choice based on what "looked" right.

Jacob: They all looked in place.

On this item, there was overlap in strategy use by high performers and low performers. Students' success using these strategies revealed two points of interest. First, some strategies are more likely to lead to success than others. Overall, the success rates were 83.3% for the *component strategy* (5 out of 6 students) and 50% for the *perceptual strategy* (2 out of 4 students). The *component strategy* was selected by 60% of students (40% high performers; 20% low performers) and the *perceptual strategy* by the remaining 40% of students (10% high performers; 30% low performers). Thus, high performers more than low performers selected strategies that were more likely to lead to success. Second, irrespective of which strategy the high performers selected they were more successful than low performers. All high performers who employed the *component strategy* were successful compared with 50% of low performers. Additionally, the one high performer who used the *perceptual strategy* was successful compared to only 33% of low performers. Thus, high and low performers differed in both their selection of a strategy and in its execution.

The Puzzle item used a Retinal list graphic, which encodes information in various ways including shape, size, and orientation (Mackinlay, 1999). The *component strategy* accommodates each of these visual-spatial characteristics when puzzle pieces are tested systematically to check their fit in the large puzzle. In contrast, the *perceptual strategy* relies more on an overall impression of the goodness of fit of a particular piece rather than whether the shape, size, and orientation of the piece is correct for the puzzle.

Theme 4: Knowledge of Everyday Graphics

Everyday graphics add authenticity to numeracy tasks. However, it cannot be assumed that students are familiar with these graphics or can use them effectively as shown in the following example.

In *The Calendar* item, students were asked to find a certain date on the supplied calendar (see Appendix). Unlike the other items discussed in this paper, there was limited difference in the success rates for high (100%) and low performers (80%). However, high and low performers differed in two ways in their use of the calendar.

First, more high performers (80%) than low performers (40%) used an efficient graphically-oriented strategy. Four high performers and two low performers successfully used a *count back by weeks strategy* in which they read off the numbers in the Thursday column, thereby capitalising on the spatial organisation of the calendar. Anna's (high performer) response is typical of these students.

> Anna: One week was 22, and two weeks would have been 15, and three weeks would have been the eighth.

A less efficient strategy – *count back by days strategy* – was used by two low performers. Although this strategy was used successfully, it was inefficient because the students failed to capitalise on the spatial organisation of the calendar when they counted by days instead of by weeks. The final high performer successfully used a *subtraction strategy* to calculate 21 days earlier. No low performers attempted this strategy.

Second, one low performer demonstrated a lack of understanding of the basic structure of a calendar. Helen appropriately chose the *count back by weeks strategy*. She started counting at 29 but the three "weeks" she counted were the Thursday, Friday, and Saturday columns. Helen selected her answer, the third of May, from the top of the Saturday column.

Helen: I worked it out because… it's one week (indicating the Thursday column), I counted the weeks until the 29th May…

Interviewer: So tell me why you think it's the third (of May)?

Helen: I went back from the 29th and I counted three weeks and it ended up there (3 May).

During the solution process, Helen made four errors in calendar use. Her first error was to treat the columns incorrectly as weeks rather than the rows. Her second error was to count forwards rather than backwards starting at the Thursday column and finish at the Saturday column. Her third error was to count the commencement column as the first week before the 29th May. This meant that she only counted two "weeks" before the initial date instead of three "weeks". Recall her concept of the representation of a "week" as a column on the calendar was incorrect. Helen's identification of the commencing location as one week is another example of primary students' lack of understanding of how to interpret the measures on a graphic. Diezmann (2000) reported that many similarly-aged students incorrectly identified the ground height on a diagram of a tree as one metre. Helen's final error was to select the answer from the top of the Saturday column rather than its base. This step violated her own reasoning that the columns were weeks when she moved up the rows. However, this anomaly might have occurred because the only multiple choice answer option in the Saturday column was "3 May", which was at the top of the Saturday column.

The Calendar is a Miscellaneous graphic that uses a variety of graphical techniques to communicate information. The conventions for using a calendar typically include representing the weeks of a month in seven labelled columns – one for each day of the week – and showing blank cells in the first and last weeks of the month before and after the first and last days of the month if necessary. The four high performers and two low performers who used the *count back by weeks strategy* capitalised on the spatial organisation of the calendar in their solution. In contrast, the spatial structure of the calendar was not recognised by the two low performing students who used the *count back by days strategy.* Though they were successful, these students' strategy is inappropriate because it fails to take into account the structure of a calendar. Similar to using the columns on a hundred board to count forward and backward in tens, students should use the columns on a typical calendar to count forward and backward in weeks. Because a calendar is an everyday graphic, both the low performers who used the *count back by days strategy* and Helen, who made multiple errors in calendar use, need to learn how to use a calendar efficiently.

Conclusion and Implications

Educationally, the comparison of these high and low performers' thinking about the use of graphics in mathematics was instructive in three ways. First, low performers need to develop adequate mathematical and graphical knowledge to be successful on numeracy tasks. Hence, teachers should support low performers to identify any related mathematics that could be used in the solution and to check on their interpretation of the graphics. Second, low performers should be encouraged to draw on implicit information embedded in the graphic to generate further information – which high performers seem to do intuitively. Thus, low performers need to capitalise on the multiple cues within a graphic and reason from this visual-spatial information. Visual reasoning differs substantively from sequential reasoning (Barwise & Etchmendy, 1991). Hence, explicit instruction may be required, such as teaching students how to interpret and reason from a family tree. Third, because some strategies are more likely to lead to success than other strategies, it would be helpful in discussions with students to compare the range of strategies used in terms of the efficiency of strategies and the likely errors using particular strategies. Overall, the comparison of these high and low performers indicated that to become more successful on graphically-oriented numeracy tasks, it is essential that low performers develop and use mathematical and graphical knowledge, generate information from graphics, build repertoires of strategies, and select and use these strategies judiciously.

Methodologically, the comparison of high and low performers' thinking has been fruitful because it provides a means to explore how different approaches to thinking contribute to success. Thus, conceptually high performer-low performer comparison acts as a thought-revealing tool for researchers in a similar way to model-eliciting tasks acting as a thought-revealing tool for teachers and students (see Lesh, Hoover, Hole, Kelly, & Post, 2000 for a discussion of thought-revealing activities).

Acknowledgements. This research was funded by the Australian Research Council (#DP0453366). Special thanks to Lindy Sugars, Tracy Logan, and the other research assistants who contributed to this project.

References

Barwise, J., & Etchemendy, J. (1991). Visual information and valid reasoning. In W. Zimmerman & S. Cunningham (Eds.), *Visualization in teaching and learning mathematics* (pp. 9-24). Washington, DC: Mathematical Association of America.

Chi, M. T. H., Feltovich, P. J., & Glaser, R. (1981). Categorization and representation of physics problems by experts and novices. *Cognitive Science, 5,* 121-152.

Department for Education and Employment. (1998). *The implementation of the national numeracy strategy. The final report of the numeracy taskforce.* London: DfEE.

Diezmann, C. M. (2000). Making sense with diagrams: Students' difficulties with feature-similar problems. In J. Bana & A, Chapman (Eds.), *Mathematics education beyond 2000* (Proceedings of the 23rd annual conference of the Mathematics Education Research Group of Australasia, pp. 228-234). Sydney: MERGA.

Diezmann, C. M., & Lowrie, T. (2006). Primary students' knowledge of and errors on number lines. In P. Grootenboer, R. Zevenbergen, & M. Chinappan (Eds.), *Identities, cultures and learning spaces* (Proceedings of the 29th Annual Conference of the Mathematics Education Research Group of Australasia, pp. 171-178). Sydney: MERGA.

Diezmann, C., Lowrie, T., Bicknell, B., Farragher, R., & Putt, I. (2004). Catering for exceptional students in mathematics. In B. Perry, G. Anthony, & C. Diezmann (Eds.), *Research in Mathematics Education in Australasia: 2000-2003* (pp. 175-195). Brisbane: PostPressed.

Diezmann C, Thornton C., & Watters J. (2003). Addressing the needs of exceptional students through problem solving. In F. Lester & R. Charles (Eds.), *Teaching mathematics through problem solving.2003 Yearbook* (pp. 169-182). Reston, VA: National Council of Teachers of Mathematics.

Educational Testing Centre (2002). *Primary school maths competition: Year 4.* Sydney: University of New South Wales.

Galotti, K. M. (1989). Approaches to studying formal and everyday reasoning. *Psychological Bulletin, 105*(3), 331-351.

Harris, R. L. (1996). *Information graphics: A comprehensive illustrated reference.* Atlanta, GA: Management Graphics.

Krathwohl, D. R. (1993). *Methods of educational research.: An integrated approach.* White Plains, NY: Longman.

Lesh, R. Hoover, M., Hole, B., Kelly, A., & Post, T. (2000). Principles for developing thought revealing activities for students and teachers. In A. Kelly & R. Lesh (Eds.), *Handbook of research design in mathematics and science education* (pp. 591-646). Mahwah, NJ: Lawrence Erlbaum.

Lowrie, R., & Diezmann, C. (2005). Fourth-grade students' performance on graphical languages in mathematics In H. L. Chick & J. L. Vincent (Eds). *Proceedings of the 30th annual conference of the International Group for the Psychology of Mathematics Education* (Vol 3, pp. 265-272). Melbourne: PME.

Mackinlay, J. (1999) Automating the design of graphical presentations of relational information, In S. K. Card, J. D. Mackinlay, & B. Schneiderman (Eds.), *Readings in information visualization: Using vision to think* (pp. 66-81). San Francisco, CA: Morgan Kaufmann.

Miles, M. B., & Huberman, A. M. (1994). *Qualitative data analysis* (2nd ed.). Thousand Oaks, CA: Sage Publications.

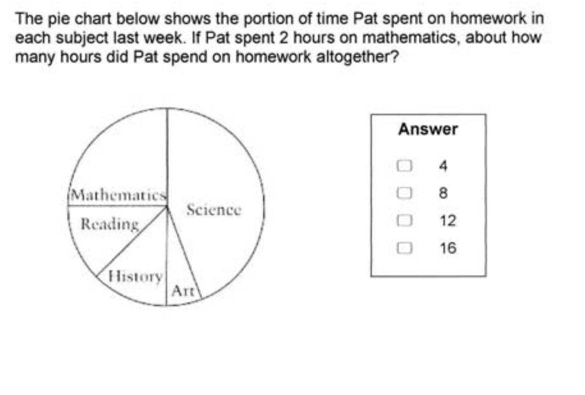
National Center for Educational Statistics. (2003). *NAEP questions:Year 4, Question. 3.* Retrieved 10 January 2003 from http://nces.ed.gov/nationsreportcard/itmrls/search.asp

Queensland School Curriculum Council. (2001). *2001 Queensland Year 5 test: Aspects of Numeracy.* Melbourne: Australian Council of Educational Research.

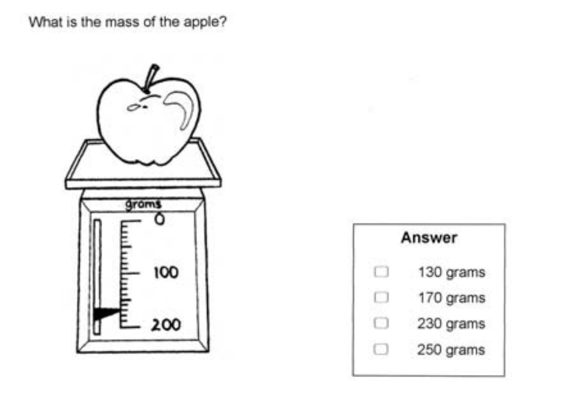
Queensland School Curriculum Council. (2002). *2002 Queensland Year 3 test: Aspects of Numeracy.* Melbourne: Australian Council of Educational Research.

von Glasersfeld, E. (1987). Preliminaries to any theory of representation. In C. Janvier (Ed.), *Problems of representation in the teaching and learning of mathematics* (pp. 215-225). Hillsdale, NJ: Lawrence Erlbaum.

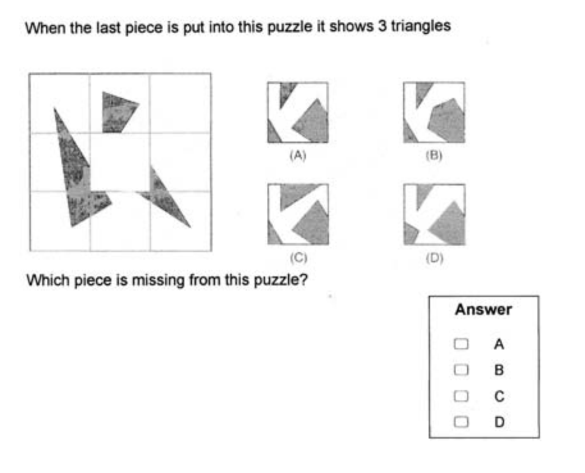
Appendix



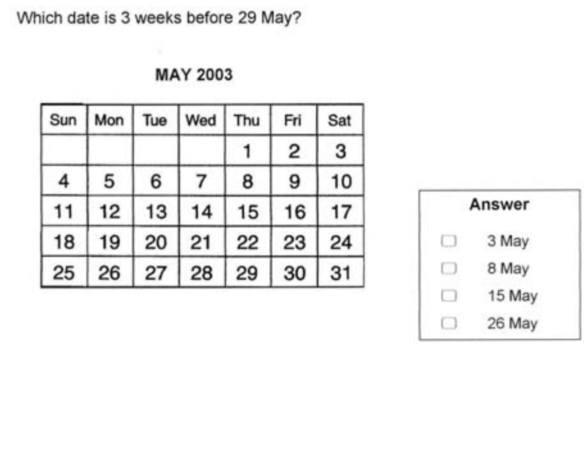
The Pie Chart (National Centre for Educational Statistics, 2003, Year 4, Q. 3)



The Scale (Queensland School Curriculum Council, 2001, p. 31).



The Puzzle (Educational Testing Centre, 2002, p. 8).



The Calendar (Queensland School Curriculum Council, 2002, p. 9)

High School Students' Use of Patterns and Generalisations

Jaguthsing Dindyal
Nanyang Technological University
<jaguthsing.dindyal@nie.edu.sg>

This paper reports how high school students from two different schools used patterns and generalisations while working on some selected problems. The results show that the initial identification of a pattern was crucial in determining the type of symbolic generalisation, which for successful students' seemed to proceed through four sequential stages.

Generalisation is an important aspect in mathematics that permeates all branches of the subject and is a feature highlighted in the teaching of the subject at practically all levels. For example, in Arithmetic a child may generalise that multiplication of a whole number by 5 gives a product ending in 0 or 5. As a statement that is true for all members of some set of elements, theorems in geometry can be considered as generalisations (Mason, 1996). On the other hand, in algebra, we commonly use variables, which Schoenfeld and Arcavi (1988) described as general tools in the service of generalisation. So, what do we mean by generalisation?

Several attempts have been made to explain the term generalisation. Kaput (1999) claimed that generalisation involves deliberately extending the range of reasoning or communication beyond the case or cases considered by explicitly identifying and exposing commonality across the case or the cases. He added that this resulted in lifting the reasoning or communication to a level where the focus is no longer on the cases or situations themselves but rather on the patterns, procedures, structures, and relations across and among them, which in turn become new, higher-level objects of reasoning or communication. This hierarchical aspect is similar to what Sfard (1991) proposed in her theory of reification, in which processes at one level become the new objects at another level. The idea of creating new objects for subsequent actions was also used by Davidov (1972/1990) who described generalisation as "inseparably linked to the operation of abstracting" (p. 13). The link between generalisation and abstraction was also highlighted by Dreyfus (1991). However, Dreyfus used the term generalisation as the recognition of some general characteristics in a set of mental objects and claimed that generalisation involves the expansion of an individual's knowledge structure. Regarding cognitive activities involved in generalising, Harel and Tall (1991) made a distinction between three types of generalisations, (a) expansive generalisation – one that extends the students' existing structure without requiring changes in current ideas; (b) reconstructive generalisation – one that requires the reconstruction of the existing cognitive structure; and (c) disjunctive generalisation – one which adjoins the new particular as an extra case or generates a new structure distinct from the others.

On the other hand, Radford (1996) claimed that a goal in generalising geometric-numeric patterns is to obtain a new result. This new result depends on the observer's conceptualisation of the mathematical objects and the relations involved. Radford added that accordingly, generalisation is not a concept but rather a procedure and that a generalisation procedure g arrives at a conclusion α, starting from a sequence of "observed facts", $a_1, a_2, \ldots, a_n$, which can be written as: $a_1, a_2, \ldots, a_n \rightarrow \alpha$ (α is derived from $a_1, a_2, \ldots, a_n$). The most significant aspect of the generalisation is its logical nature that makes possible the conclusion α.

It should be noted that inductive reasoning, which is commonly used in generalising from patterns, does not necessarily lead to valid conclusions. If there are flaws in the logic then certainly the generalisation would not be valid, and so generalisation as a didactic strategy cannot avoid the question of validity. Burton (1984) claimed that to become robust a generalisation must be tested until it is convincing so that it moves from being personal to public. Burton also mentioned that both inductive learning and deductive learning involve generalising activities. Her view is that inductive learning involves specialising, conjecturing, and generalising in that order, which is the reverse order for deductive learning.

Although generalisation may seem to be omnipresent in school mathematics, there are pedagogical issues that cannot be ignored. In her research, Lee (1996) found that generalising activities led to three types of conceptual obstacles. First, there were obstacles at the perceptual level, which concerned with seeing the actual pattern. Second, there were obstacles at the verbalising level, which involved expressing the pattern clearly. Third, there were obstacles at the symbolisation level, for example using a variable n in a general expression. Thus, generalisation in school mathematics is a very important aspect that needs to be carefully investigated. Accordingly, this study focused on how secondary students used patterns to help them generalise and what were some of their related conceptual difficulties?

Methodology

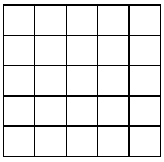
The study reported in this paper is part of a larger study investigating students' use of algebraic thinking in geometrical contexts. The study took place in two large Midwestern high schools in the United States. One geometry class was selected from each of the two high schools: School X and School Y. There were 21 students in the class from School X and 18 students in the class from School Y. The two classes were observed for three months and twelve lessons from each class were videotaped. Three students were selected from each of the two classes based on the results of an algebra test, which was developed in conjunction with the classroom teachers of these two classes and three other experts in the field. Andy, Betty, and Melanie were the focus students from School X whereas Pete, Kristina, and Abby were from School Y (all names are pseudonyms). Andy and Kristina were more able students whereas Betty and Melanie were weaker students from the sample. Each of the six students was interviewed four times for about 40 minutes each time. The interviews were audiotape recorded and then transcribed. The students were asked to solve some problems involving certain aspects of patterns and generalisations. The problems were asked sequentially, in the different interview sessions, as given in the Results section below. The questions were read out to the students and additionally a written version was provided to them. Seven problems that involved some aspect of generalisation in a geometrical context were used with the students. The problems were selected based on the topic coverage in the selected classrooms. Problem 1 has been adapted from the one by Swafford and Langrall (2000) and Problem 7 from the one used by Krutetskii (1976).

Results

In this section, the focus students' generalisation approach in the context of the seven problems is discussed sequentially. The results for the students' performance show some interesting features.

Problem 1

> How many small squares are there in the border of this 5×5 square (square drawn on a rectangular grid)? How many are there in a 6×6 square? How many are there in a 10×10 square? How many would there be in a square of side $n \times n$? If there are 76 border squares in square grid, what is the size of the grid?



In this problem the square grid provides a geometrical context for an algebraic generalisation. The three students from school X used different strategies to find the number of border squares for the 5×5 square grid. Andy did it mentally and later explained that he added 3 + 3 + 5 + 5 to get 16. Betty counted the squares one by one and then wrote 5, 3, 5, and 3 along the border of the grid. This showed that her strategy of using 5 + 3 + 5 + 3 was somewhat similar to Andy's. Melanie responded very quickly that the answer was 20, which was incorrect. When asked to check the answer by actual counting, she was puzzled that it was 16. She did not show any strategy for getting the answer other than by counting.

For a 6×6 grid, Andy did not follow his strategy from the previous part. He said the answer was 25 and added that for a 10×10 it was 81. For an $n \times n$ grid he said it was $(n\text{-}1)^2$. This clearly showed that Andy was not using his previous strategy. He did not mention why he chose $(n\text{-}1)^2$, but it seems that he was mislead by the number of border squares in the 5×5 grid as also being $(5\text{-}1)^2$. On prompting, he changed his answer and was able to come up with the correct generalisation of $4(n\text{-}1)$. He was able to use this formula for the inverse problem to find the size of the grid for which the number of border squares was 76. Betty stuck with her strategy and had no problem getting the answer for a 6×6 or 10×10 grid. She was eventually able to write down the answer for an $n \times n$ grid. She wrote $N + N + (N\text{-}2 + N\text{-}2) = 4N\text{-}4$. Betty was not concerned about the use of N instead of n in the expression. She needed some prompts to be able to set up an equation and solve it to get the size of the grid for which the number of border squares was 76. Melanie could not follow through to get the answer for a 6×6 grid. She thought that it might be 16 + 6 = 22. That is, she thought of adding one additional row of 6 squares to the previous answer of 16 for a 5×5 grid. She could not get to a 7×7 or 10×10 grid. She said she could not do it without a diagram. After several prompts, she was able to finally generalise to $4n\text{-}4$ for an $n \times n$ grid. However, for the inverse problem, she could not get the size of the grid for which the number of border squares was 76.

From school Y, Pete started this problem by actually counting the number of squares in the 5×5 grid. Since no diagram was given for a 6×6 grid, he knew that he had to be more systematic. His revised strategy was to add 5 + 5 + 6 for the 5×5 grid, thinking of the 6 as 3 + 3. He used the same strategy for a 6×6, 10×10, and also for the general case $n \times n$. For this latter case, he wrote $2n + (n\text{-}2)\times 2$, which he simplified to $4n\text{-}4$. For getting the size of the grid for 76 border squares, he wrote 76 + 4 = 80, then he wrote 80/4 = 20, to say that the size of the grid was 20×20. Kristina and Abby were able to get all of the answers and they had very similar strategies for getting the generalised value of $4n\text{-}4$ for the $n \times n$ grid. However, for finding the size of grid with 76 border squares, Kristina just substituted 20 for n to get the answer. This suggested a more trial and error strategy, whereas Abby actually set up an equation and solved for n.

Problem 2

> What is the sum of the interior angles in a triangle? From any vertex, we can divide a quadrilateral into two triangles. What is the sum of the interior angles in a quadrilateral? What is it for a pentagon, hexagon, and a decagon? What would it be for a polygon with n sides?

In this problem, the students had to know the angle sum of a triangle and the names of the polygons up to ten sides. The algebraic skills included the identification of a pattern and subsequently writing down the generalisation from the pattern. All of the focus students except Melanie were able to find the sum of the interior angles in a quadrilateral, pentagon, and hexagon by actually drawing such a figure and then counting the number of triangles they could get by drawing diagonals from one vertex. They had no problem generalising to a polygon with ten sides, even though a diagram was not used. Eventually all of them, except Melanie, were able to get the result that for a polygon with n sides the sum of the interior angles is $(n\text{-}2)\times 180°$. Melanie needed some help with the pentagon and hexagon before writing down the angle sum. For a decagon she did not do any calculation but used an additive strategy and counted on from a hexagon, which implied that she had noted a pattern in her responses, but was not quite able to formalise it. To get the result $n\text{-}2) \times 180$ for a polygon of n sides, a table of values for number of sides and the corresponding angle sum was drawn for her. It was only when this scaffolding was done that she was able to follow the pattern and come up with the generalisation. It seemed that the organisation of the results in a tabular form was important for Melanie in triggering the identification of the pattern.

Problem 3

> What is the relationship between an interior and an exterior angle of a triangle? How many pairs of interior and exterior angles do you have in a triangle? What is the sum of all of the interior and exterior angles of a triangle? What is the sum of the exterior angles of a triangle? What is the sum of all of the exterior angles in a quadrilateral? A pentagon? A hexagon? A polygon with *n* sides?

The main geometrical concepts in this problem are that of interior and exterior angles. The students had to understand that the sum of an interior and the corresponding exterior angles in a polygon is 180°; and that if they knew the sum of all of the interior angles in a polygon then the sum of the exterior angles could be found by subtracting the sum of the interior angles from the sum of all of the interior plus exterior pairs. The students should then have been able to generalise from this result.

Andy was able to follow the argument and he got the sum of the exterior angles of a triangle, a quadrilateral, and a pentagon easily. He was even able to do it for a decagon and although he guessed that the answer had to be 360° for any polygon, he actually did the calculation for a polygon with n sides to confirm his guess. Betty was able to do it for a triangle, quadrilateral, and for a decagon as well. Although she guessed that the result should be 360° for any polygon, she was not actually able to do the calculations to justify the result for a polygon with n sides. Melanie, on the other hand, had some difficulties following the argument even for a triangle. After some prompting, she was able to do it for a quadrilateral and a pentagon but not for a decagon. However, she guessed that the sum of the exterior angles might always be 360° for any polygon. She was not able to do the actual calculation to justify the answer.

All of the three students from school Y were able to follow the arguments and were able to get the exterior angle sum for a triangle, quadrilateral, pentagon, and the decagon easily. They guessed early that the sum of the exterior angles in any polygon would be 360°. They all were able to do the calculation for a polygon of n sides to show that the sum of the exterior angles did not depend on the number of sides of the polygon and that it was always 360°. While checking for understanding, it was noted that the students had difficulty in applying their knowledge about the sum of the exterior angles to find the number of sides of a regular polygon if the size of one exterior angle was known. Although the students knew what a regular polygon was, none of them was able to solve such a problem.

Problem 4

> The equation of a line is $y = 3x + 5$. Write down the equation of another line having the same slope as the given line. What would be the general form of the equation of a line having the same slope as the given line?

This problem refers to the equation of a line in the slope-intercept form. The students were expected to know that in coordinate geometry, the equations of lines having the same slope varied only in the value of the intercepts. All of the six focus students were able to identify the slope of the line as 3. More specifically, Andy, Betty, and Abby wrote 3/1 for the slope. This seemed to be a common practice for writing down the slope of a line from its equation in the slope-intercept form. However, for the general form of a line having the same slope, different answers were obtained. Andy wrote $y = 3x +$ *anything*, then wrote $y = 3x + z$, where z is a number. Betty wrote $y = 3x +$ *number on y-intercept*, and Melanie wrote $y = 3x +$ *anything*. From school Y, Pete wrote $y = 3x + n$, where n is a number. Kristina wrote $y = 3x +$ *something*, whereas Abby was not able to come up with a general form for such a line. In this context where the symbol for the parameter was not provided, students found it difficult to generalise using their own symbols.

Problem 5

> All points on a circle are equidistant from its center. If P(x, y) is a point on a circle having center at the origin and radius 5, what relation can you write connecting x and y? What would be the relation if the radius was 10? What would it be if the radius was r?

To solve this problem, the students were provided with a diagram and the formula to find the distance when the coordinates of two points were given. The students also needed some algebraic skills in the manipulation of the relation that they had to write connecting x and y. All of the focus students were able to write down the relation connecting x and y using the distance formula and even the relation for the general case when the radius was given as r. All of them wrote $5 = \sqrt{\ } [(x\text{-}0)^2 + (y\text{-}0)^2]$, except Melanie who reversed the order in which she used the points in the formula, which was, of course, correct. Melanie wrote $5 = \sqrt{\ } [(0\text{-}x)^2 + (0\text{-}y)^2]$ and then she was not sure how to simplify $5 = \sqrt{\ } [(\text{-}x)^2 + (\text{-}y)^2]$. She even thought that $(-x)^2 \neq x^2$ and $(\text{-}y)^2 \neq y^2$. However, she was later convinced that this could be written as $x^2 + y^2 = 25$. It was interesting to note that four of the focus students Andy, Pete, Kristina, and Abby made the same algebraic mistake when trying to simplify the expression $5 = \sqrt{\ } [(x\text{-}0)^2 + (y\text{-}0)^2]$. They wrote $5 = \sqrt{\ } [(x^2 + y^2]$, but then they went on to write $5 = x + y$ and eventually wrote $x^2 + y^2 = 25$. There seemed to exist some underlying misconceptions about algebraic simplifications.

Problem 6

Two parallel lines are labeled l and m. On line l one point A is marked and on line m three points B, C, and D are marked. How many different triangles can be formed by joining three of the given four points? If the point on line l is kept fixed but one more point is added on line m, how many triangles can be formed in the same way? Can you find out the number of triangles that can be formed under the same conditions if there were 6 points, 10 points, n points on line m?

No. of points on line m	3	4	5	6	10	n
No. of triangles						

A diagram showing the parallel lines l and m and the points A, B, C, and D was given. In this problem the students had to count the number of triangles systematically as the number of points on line m was increased and then they had to come up with some rule for finding the number of triangles in the general case when there were n points on line m.

Andy and the three students from school Y, Pete, Kristina, and Abby, had no difficulty in counting the number of triangles up to $n = 6$. They had a systematic strategy and were then able to extend the result to $n = 10$, without doing any actual calculation, by just following the pattern of numbers they had obtained in the table. However, they could not come up with a general formula for the case when there were n points on line m. Only Andy came up with a recursive formula. He wrote $X + n\text{-}1$ for the number of triangles when there were n points on line m, where X for him represented the previous number of triangles. Both Betty and Melanie were not systematic in their counting of the triangles and so had difficulties in completing the table. Melanie had even more difficulties than Betty. However, once they were able to get the values in the table up to $n = 6$ after some very careful counting and some help, they were both able to identify the pattern and were able to write down the number of triangles for $n = 7$ and $n = 8$ without using a diagram.

Problem 7

Two of the sides of an isosceles triangle have measures 4 inches and 10 inches. What would be its perimeter? Why? A triangle has sides of lengths a, b, and c. What relation(s) can you write connecting a, b, and c?

This problem required knowledge about an isosceles triangle and about the geometrical fact that in any triangle the sum of any two sides is always greater than the third side. The students had to identify this geometrical fact in the first part of the problem and then to generalise it in the second part. The three students from school Y initially thought that there were two answers for the first part namely, 24 and 18. However, they soon realised that 18 was not a possibility and so gave the correct answer as 24. They were able to generalise to any triangle and wrote the relations $a + b > c$, $b + c > a$, $a + c > b$. The only difference in their answers was the inconsistent use of capital and small letters for the length of the sides. Abby used all small letters a, b, and c, but Pete used only capital letters A, B, and C whereas Kristina used a combination of both small and capital letters.

The students from school X had different responses. Andy initially thought the answer was 24 for the first part but then thought that 18 was also possible. It was only after some prompts that he finally realised the impossibility of having 18 as a perimeter. He could not give a general rule for a triangle with sides a, b, and c. However, he did mention that at least one of a or b had to be greater than half of c. This was obviously incorrect, but it seemed that his belief was that "half of a plus half of b" had to be greater than c, rather than "a plus b" was greater than c. Betty was not able to follow the first part of the problem. It was only after some help that she could do so. She wrote $A + B > C$, for the second part and with some further prompts was able to write $c + a > b$ and $b + c > a$. Melanie initially wrote 24 and 18 as an answer for the first part. She thought that both of these answers were possible. After a triangle was drawn for her to illustrate the situation, she understood that 18 was not possible. She knew that a triangle with sides of lengths 2, 3, and 7 units was not possible but she could not generalise this result to a triangle with sides a, b, and c. When the relation $a + b > c$ was written down, she was able to write out $b + c > a$ and $a + c > b$.

Discussion

In Problems 1 to 7 the focus students had to identify a general pattern starting from few specific cases. It was expected that reasoning inductively from a few cases the focus students would be able to generate a general rule or formula. Successful strategies seemed to proceed through the following sequence of stages: a direct modelling stage, the stage of identification of a pattern, the stage of proof testing of the pattern, and the final stage for finding a rule for the general case.

The direct modelling stage involved the focus students actually using strategies such as counting, drawing, or writing down the first few cases systematically. For example, in Problem 1 most of the students counted the number of squares in the 5×5 grid and some of them drew a 6×6 grid and again counted the number of squares before identifying any pattern. In Problem 2, at this stage, the students used the drawing of a quadrilateral, a pentagon, and a hexagon to find the angle sum by drawing inside those figures a certain number of triangles from a given vertex. In Problem 3, the students used the drawing for a quadrilateral, a pentagon, and a hexagon to arrive at a pattern of results for the sum of the exterior angles. In Problem 6, the students counted the number of triangles when there were 3, 4, 5, and 6 points on line m. Thus, in most of the problems the students were doing some direct modelling at this stage.

The second stage was the stage during which the students were actually able to identify some useful pattern. Which pattern one chooses depends on the particular aspect of the pattern that one wishes to observe (Phillips, 1993), and this depended considerably on the students' systematic counting, drawing, or writing/recording from the first stage. For example, in Problem 1 for the 5×5 grid, some students identified the pattern as $5 + 5 + 3 + 3$, and some as $5 + 5 + 6$. Although the two representations do not look very different, they led to slightly different ways of writing the general expression. The generalisation was $n + n + n\text{-}2 + n\text{-}2$ for the first pattern and $n + n + 2\times(n\text{-}2)$ for the second one, which was later simplified to $4n\text{-}4$. Thus a systematic way of counting the number of squares helped the students to generalise. The generalisation was fairly easy when there were sufficient examples to make the pattern quite evident. In Problems 2 and 3, the successful students were able to identify a connection between the number of sides in a polygon and the sum of the interior/exterior angles in the polygon. The systematic way of recording the number of triangles in Problem 6 in a table helped the successful students to identify a pattern in the results. In problems where this was not the case, the students had more difficulties in coming up with a useful pattern. For example, in Problem 7 the students had to come up with a generalisation based on only one initial case. This proved to be hard for the students. Lee (1996) has pointed out that the problem for many students is not the inability to see a pattern but the inability to see an algebraically useful pattern.

In the third stage, the successful focus students tested their conjectures about the patterns by using a particular case beyond the range for them to model directly. For example, in Problem 1 the students were asked to find the number of border squares in a 10×10 grid. They knew that it was not worthwhile to draw a 10×10 grid and then to count the squares one by one. Generally, the students who were able to attain this stage were able to get to the algebraic generalisation later. "Counting on" was a common strategy for some of the focus students to reach a solution for the 10×10 grid, but this was not very helpful as an overall strategy. It was when these students were asked about larger grids such as 100×100 where counting on strategies were not very practical that these students looked for alternative strategies. So, they used their earlier patterns such as $3 + 3 + 5 + 5$ or $5 + 5 + 6$ from the earlier parts to get the answer. In problems 2 and 3, the students were asked to find the sum of the interior/exterior angles in a polygon with ten sides. The students knew that it was not necessary to draw the decagon and had to rely on their previous sequence of results. Similarly, in Problem 6 the students did not put 10 points on line *m* to come up with the number of triangles for this case. They used the patterns they had identified to do so.

In the final stage, the students had to come up with a generalisation. Swafford and Langrall (2000) had claimed that the generalisation of a problem situation might be presented verbally or symbolically. In the problems that were used in this study, the focus students avoided a verbal generalisation and all of them tried to give symbolic generalisations. For the symbolic, this involved constructing an algebraic relation for the pattern they had noticed. Their success in the first three stages of the solution process helped them to come to the right conclusion. The students used the pattern that they had identified earlier to come up with the generalisation. For example, Betty from school X wrote $N + N + (N\text{-}2) + (N\text{-}2)$ which was similar to her $5 + 5 + 3 + 3$ strategy for the 5×5 grid. She overlooked the fact that the grid was $n\times n$ and not $N\times N$, but this minor detail did not seem to bother her. In very much the same way, the students from school Y wrote $2n + 2\times(n\text{-}2)$, following their pattern $5 + 5 + 2\times3$ for the 5×5 grid. In Problem 2, the successful students had no difficulty in coming up with the generalisation $(n\text{-}2)\times180°$ for the interior angle sum of a polygon of n sides. The pattern of results noted from the triangle, quadrilateral, pentagon, and hexagon was essential. By the time they had to find the sum of the interior angles of a 10-sided figure, they already had the pattern for the general case. It was a similar situation in Problem 3, except that the weaker students could only guess that the exterior angle sum would be 360°, but they were not able to justify it. The more successful students were able to show by subtraction of the sum of the interior angles of a polygon of n sides from the sum of all the interior and exterior angle pairs of the polygon that the result came out to be 360°.

Some of the difficulties encountered by the students, such as producing variables on their own, and writing down the relations algebraically, hampered the students' progress. For instance, the students found it very difficult to come up with a symbolic generalisation for Problem 6. Generally, the students were able to fill up the table, but their search was for a linear symbolic relationship. Most of them were able to identify a recursive relationship in the table but only Andy, from school X, gave an explicit recursive formula. His formula was $X + n\text{-}1$, where X stood for the number of triangles from the previous value of n, the number of points on line m. However, he was unable to give an explicit symbolic representation of the non-linear generalisation in Problem 6. Some authors caution that, in their attempt to write symbolic representations, students often focus on inappropriate aspects of a number pattern – particularly the recursive relationship between successive terms in a sequence (MacGregor & Stacey, 1993; Orton & Orton, 1994). Thus, in Problem 6, it might be possible that the students' focus on the recursive relationship was responsible for their inability to produce an explicit generalisation. Even Problem 4 was problematic for some of the students. In Problem 7, the students had difficulties in coming up with the generalisation about the sides of the triangle mainly because a single case illustrated the problem. It seemed that a limited number of initial cases might not be enough for the students to find a pattern and hence a generalisation from the pattern, although Dreyfus (1991) had claimed that sometimes it is better to abstract from a single case.

The three types of conceptual obstacles in generalising activities that Lee (1996) found in her research were also noticed in this study. First, there were obstacles at the perceptual level. For example, Melanie had this obstacle in Problems 1, 2, and 3. She was unable to identify the pattern and this led to her not being able to proceed further on her own. At the perceptual level, the focus students found it easy to identify patterns that showed constant differences between successive terms but not when the pattern was different. The symbolic expressions for the generalisation was obtained easily when constant differences were involved but not in problems where this was not the case as in Problem 6. Second, there were obstacles at the verbalising level. For example, Melanie in Problem 1 was not able to verbalise a useful pattern and this probably led to her incorrect generalisation. Third, there were obstacles at the symbolisation level. For example, in Problem 6 most of the focus students could not come up with a generalisation using appropriate symbols, even when they had identified a pattern. As noted by Lee in her research, the major problem for students was not in seeing a pattern, but in perceiving an algebraically useful pattern. It is important to note that some of the focus students did not verify whether the formula they had generated worked in the simplest of cases. They were generally confident that they had the right symbolic form of the generalisation. Also it is worth noting that when the students were not systematic in their recording of the results then they were unable to identify any patterns and this led them to inappropriate conclusions.

To check for understanding, the students were asked to solve the inverse problems in Problems 1 and 3. In Problem 1 students were asked to find the size of a grid for which the number of border squares was 76. Solving an equation was the most common strategy. Some of the focus students needed prompts to be able to do so. Kristina used a trial and error strategy. In Problem 3, the students were asked to find the number of sides of a regular polygon with a given exterior angle. None of the focus students were able to solve such a problem. Thus, the students seemed to have a loose understanding of the generalisations that they had come up with in the problems.

To conclude, the study shows that the identification of a useful pattern by the students was a significant factor in their successful symbolic generalisation, which seemed to proceed in four sequential stages. However, the students had difficulties with non-linear symbolic generalisations. The students generally avoided verbalising their generalisations. Students with a weaker background in algebra, such as Melanie and Betty, had more difficulties generalising compared to the other students. Even the students with a stronger background in algebra displayed some misconceptions in handling algebraic expressions. In this study, all of the problems had some connections to geometry, which may have added to the students' difficulties. In future studies, a broader range of problems with similar generalising activities may provide a more complete picture.

References

Burton, L. (1984). Mathematical thinking: The struggle for meaning. *Journal for Research in Mathematics Education*, *15*(1), 35-49.

Davidov, V. V. (1972/1990). *Types of generalization in instruction: Logical and psychological problems in the structuring of school curricula* (Soviet studies in mathematics education, vol. 2; J. Kilpatrick, Ed., J. Teller, Trans.). Reston, VA: National Council of Teachers of Mathematics. (Original work published in 1972)

Dreyfus, T. (1991). Advanced mathematical thinking processes. In D. Tall (Ed.), *Advanced mathematical thinking* (pp. 25-41). Boston: Kluwer Academic Publishers.

Harel, G., & Tall, D. (1991). The general, the abstract, and the generic in advanced mathematics. *For the Learning of Mathematics*, *11*(1), 38-42.

Kaput, J. J. (1999). Teaching and learning a new algebra. In E. Fennema & T. Romberg (Eds.), *Mathematics classrooms that promote understanding (pp. 133-155)*. Mahwah, NJ: Lawrence Erlbaum Associates.

Krutetskii, V. A. (1976). *The psychology of mathematical abilities in school children*. Chicago: University of Chicago Press.

Lee, L. (1996). An initiation into algebraic culture through generalization activities. In N. Bednarz, C. Kieran, & L. Lee (Eds.), *Approaches to algebra: Perspectives for research and teaching* (pp. 87-106). Boston: Kluwer Academic Publishers.

MacGregor, M., & Stacey, K. (1993). Cognitive models underlying students' formulation of simple linear equations. *Journal for Research in Mathematics Education*, *24*(3), 217-232.

Mason, J. (1996). Expressing generality and roots of algebra. In N. Bednarz, C. Kieran, & L. Lee (Eds.), *Approaches to algebra: Perspectives for research and teaching* (pp. 65-86). Boston: Kluwer Academic Publishers.

Orton, A., & Orton, J. (1994). Students' perception and use of pattern and generalization. In J. P. da Ponte & J. F. Matos (Eds.), *Proceedings of the 18th annual conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 404-414). Lisbon, Portugal: PME.

Phillips, E (1993). Issues surrounding algebra. In C. Lacampagne, W. Blair, & J. Kaput (Eds.), *The algebra initiative colloquium*,(Vol. 2, pp. 67-79). Washington, DC: US Department of Education.

Radford, L. (1996). Some reflections on teaching algebra through generalization. In N. Bednarz, C. Kieran, & L. Lee (Eds.), *Approaches to algebra: Perspectives for research and teaching* (pp. 107-114). Boston: Kluwer Academic Publishers.

Schoenfeld, A. H., & Arcavi, A. (1988). On the meaning of variable. *Mathematics Teacher, 81,* 420-427.

Sfard, A. (1991). On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the same coin. *Educational Studies in Mathematics, 22*, 1-36.

Swafford, J. O., & Langrall, C. W. (2000). Grade 6 students' preinstructional use of equations to describe and represent problem situations. *Journal for Research in Mathematics Education*, *31*(1), 89-112.

The Teacher, The Tasks: Their Role in Students' Mathematical Literacy

Katherine Doyle
Queensland University of Technology
<km.doyle@qut.edu.au>

This paper reports on part of a larger study and examines the changing nature of mathematics teaching and tasks. Two Year 4 classes were compared after mathematical-modelling tasks were undertaken with and without top-level structuring. The results indicate that mathematical-modelling and top-level structuring tasks can advance mathematical literacy. Where students are guided through information organisation and mathematising through quality teaching, they can make sense of the mathematical world. Also evident was the vital role of the teacher in creating a positive learning environment through facilitating discourse and literacy development in mathematics students. Recommendations for teaching are given. Indications evidenced here warrant further investigation.

The nature of mathematics teaching and classroom activities is changing in an endeavour to meet the needs of today's students. The role of the teacher is changing from that of main instructor, teaching rules and correcting related exercises, to that of facilitator of mathematical activities that promote understanding of mathematics, mathematical thinking and reasoning abilities. In other words, educators today are aiming to provide students with expertise in mathematics, so that students will be equipped to use advanced thinking skills to acquire mathematical knowledge (Kulikowich & DeFranco, 2003). As a result, mathematics, as well as mathematical solutions and representations become powerful tools by which to understand the world. This paper explores two components of mathematics teaching and how they contribute to mathematical literacy and expertise: firstly, the nature of mathematical-modelling combined with top-level structuring (TLS) activities, and secondly, the role of the classroom teacher as a creator of, facilitator of, and participant in the classroom discourse community.

The Evolution of Mathematical Literacy

Mathematics has been described as an empowering tool, by which people can learn to reason and make sense of the world around them (Schoenfeld, 2002). In order to do this, Schoenfeld maintains, students must be active participants in "mathematical sense-making" activities (2002, p. 155). International educational authorities such as, The National Council of Teachers of Mathematics (2000), the United Kingdom National Curriculum (2000), and Queensland Studies Authority (2004) concur that a variety of problem-solving experiences contributes to students' empowerment and ability to function effectively in society.

The key to mathematical empowerment is mathematical literacy because it is the means by which one can actively participate in the process of problem solving, make sense of the problem, and ultimately unlock a solution. Mathematical literacy has been described by Romberg (2001) as having knowledge of the intricacies of mathematical language in order to gather and understand information on concepts and procedures. This information can

Proceedings of the 30th annual conference of the Mathematics Education Research Group of Australasia

then be used efficiently to mathematise various non-routine problems, for example, mathematical modelling problems.

Mathematical modelling is one problem-solving process that aims to provide conditions that facilitate growth in mathematical knowledge. Through participating in a discourse community with peers, students interpret, analyse, reason, seek relationships and patterns between elements, then explain, justify, and predict situations (Lesh & Doerr, 2003a). Through these real-world, open-ended, problem-solving experiences students' develop conceptual systems "to construct, describe or explain mathematically significant systems they encounter" (Lesh & Doerr, 2003a, p. 9). Mathematical-modelling problems involve attaining, managing, and presenting pertinent information through factual and graphic texts, as well as aurally and orally. Therefore, students require a sound level of language literacy in order to construct mathematical literacy. One means of enhancing language literacy is to employ a literacy strategy as a sense-making tool to use in conjunction with mathematical-modelling activities. TLS aids comprehension of oral, textual, or graphic information. It is described by Bartlett, Liyange, Jones, Penridge, and McKay (2001) as a procedure:

> which allows the strategic reader, listener or reviewer to form an opinion on what a writer, speaker or performer considers as essential content and if necessary, then to move on to critical or inferential analysis. Conversely, it allows a strategist as writer, speaker or performer to produce coherent text and to signal what he/she wants to be seen as essential content. (p. 67)

Harel and Sowder (2005) argue that educators must construct meaningful, rational instruction that aims to produce advanced mathematical thinking. They differentiate between mathematical thinking and mathematical understanding, but acknowledge these two as essential modes of knowledge. Meanings gained and given, justifications and assertions constitute understanding. However, generated theories and the expression of reasoning, which is specific to not just one situation, but, "a multitude of situations" (p. 31) portray thinking. These skills equate to those disseminated through mathematical modelling.

Theories of mathematics as described in Kulikowich and DeFranco (2003) provide a framework for the teaching of mathematics. Theorists, such as, Barab, Hay, and Yamagata-Lynch (2001) have argued that situated cognition, that is, "the interaction of individuals and their environments" (p. 149) shapes the setting for the attainment of knowledge, whereas others (Anderson, Reder, & Simon, 1996) have claimed that one processes, stores and organises information in one's head (the information-processing theory). Furthermore, critical theorists like Lambert and Blunk (1998) have focussed on authentic, social activity to provide valuable learning experiences. Small "classroom societies" mirror real-life social practices where professionals gather to discuss/debate and realise solutions to problems. Anderson, Greeno, Reder, and Simon (2000) have acknowledged the importance of both cognitive and social practices perspectives. They have identified the fact that there are simply different foci for learning activities. Learning can occur through both solitary and group activities. Anderson et al. (2000) distinguish two aspects of mathematics teaching: (a) the cognitive perspective where students learn structures, concepts, and procedures individually, and (b) the social, situated learning tasks whereby students can learn the intricacies of mathematical discourse, and how to participate in supportive learning practices.

Theories are many and varied. The theories cited here are only a few examples. Nevertheless, as Kulikowich and DeFranco (2003, p. 149) contend, "no one theory should dictate how to practice … educators should draw from a variety of perspectives in teaching and designing materials for the classroom".

Mathematical-modelling tasks are indicative of activities that draw on a variety of theoretical perspectives. For example, the tasks are individually and environmentally interactive: situated cognition. When coupled with TLS, mathematical modelling's alignment with the information-processing theory is enhanced. Because of their emphasis on storing and organising information, TLS provides a tool which fosters thinking skills as students organise their information in a logical and systematic manner (Bartlett, 2003). Finally, modelling tasks are social, reflecting real-world practices, whereby problem solving takes place via a process of interpreting, discussing, explaining, analysing, justifying, revising, and refining ideas (Lesh & Doerr, 2003a).

Modelling activities are a prime example of a current practice that demonstrates the changing role of the teacher. However, it needs to be emphasised that, to be worthwhile, an activity must be constructed with a true perspective on why the activity will benefit learning (Kulikowich & DeFranco, 2003), and how the activity will benefit learning. This perspective constitutes quality teaching. Furthermore, quality teachers carefully monitor students, and act on cues that indicate when and how activities can be directed to gain most benefit. Significantly, teachers need to be active facilitators of classroom discourse, supporting students' focus on meaningful content as well as their reflections on understandings about the content (Schoenfeld, 2002). In this way, teachers are leaders of functional discourse communities that promote mathematical literacy in students.

Teachers must be proactive, and at the same time modify their views to correspond with student needs (McClain, Cobb, & Bowers, 1998). A mathematics classroom should be a community "of disciplined inquiry" (Schoenfeld, 2002, p. 132). A teacher's role is to create an environment where students are supported to participate actively in "mathematical sense-making" through engaging "collaboratively in reasoned discourse" (Schoenfeld, 2002; p. 151). Therefore, students can become independent thinkers. This is what it means to become mathematically literate.

To demonstrate the issues raised in this literature review, episodes taken from a larger study on mathematical modelling and TLS are presented. These episodes provide a context in which to examine the changing face of mathematics teaching and the changing role of the mathematics teacher.

Design and Method

The section of the study reported here is part of a larger study that was designed to investigate the effects of applying TLS to mathematical-modelling tasks. The study used a design-research approach, also known as a design experiment (Bannan-Ritland, 2003). Data were sourced from video/audio taped evidence, student work samples, and teacher observations and reflections. As well, information was gathered from students' Year 3 Queensland 2004 numeracy and literacy test results. This provided an historical record, which added further credence to the final reporting. Employing multiple-method data collection validated claims and assertions from the research (Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003), which used an interpretational, data analysis approach (Tobin, 2000).

The Participants

The participants consisted of 57 Year 4 students. These students were divided into two classes that formed the study's TLS group (n = 28) and the non-TLS group (n = 29). The students attended a Catholic school on the outskirts of a major Australian capital city. The school is situated in a lower to middle socio-economic area. The school principal and teachers very enthusiastically supported the study. The Year 4 teachers actively participated in monitoring the students' group work, in partnership with the researcher.

The Procedure for the Original Study

Firstly, the TLS group received instruction on and practised the application of TLS over a series of ten lessons. Both the TLS and non-TLS groups participated in a mathematical-modelling task where they had to investigate the best light conditions in which to grow beans. Using a table of data illustrating growth rates of beans growing in shade and sunlight, students analysed and compared data to lead them to a decision. Their decision had to be explained with reasons in a letter to a "farmer". Subsequently, the data collected from the students as they progressed through the modelling process was compared and contrasted.

Secondly, the non-TLS group received instruction on and practised the application of TLS over a series of ten lessons. The two groups then participated in a further modelling task. In this experience, students viewed data on distance, time, and number of attempts made by paper planes in a contest. The students were asked to write a letter to judges to explain the best way to decide on a winner. Data collected from this episode was compared and contrasted, any changes in the non-TLS group's capacity to engage fully in the modelling task was noted, taking into account that this was the second modelling experience for the students.

Mathematical Modelling: Teaching and the Role of the Teacher

When the data from this research were analysed, some unexpected findings became evident. These were particularly of interest because of their potential impact on mathematics teaching and outcomes. Of specific interest, were the effects of the interactions of individual teachers with students as the students participated in their group tasks. As a result, this paper reports on these findings in the light of mathematical modelling and TLS as interactive, sense-making, social components of a discourse community that should positively contribute to mathematical literacy for students. Equally, the findings are discussed with special attention to the teacher role and the effects teacher interaction had in enhancing or diminishing students' mathematical literacy.

Results and Discussion

There are results on the impact of TLS on mathematical modelling reported elsewhere, such as Doyle (2006). Two major assertions can be drawn from the data analysis here. Firstly, with reference to mathematical-modelling tasks coupled with TLS, the analysis indicated that students participated in an active discourse community. Students were able to mathematise as they investigated and analysed data, made connections, explained, and justified their ideas, an indication of students' acquisition of mathematical literacy. Secondly, emerging from these data, was the fact that the teacher plays a vital role as creator, facilitator, and participant in the discourse community. Of major significance were (a) the role of the teacher as listener, and (b) the role of the teacher as questioner. Listening to student discourse proved to be crucial in the teachers' ability to impact positively or negatively on the classroom discourse community. As well, the way in which questions were directed to students influenced the discourse community. When focused, sharp questioning occurred, the result was positive in that the students remained on task and their mathematising was enhanced. When questioning were unclear and prolonged, the students became confused, which detracted from their mathematising.

In the following text example, the students demonstrated that they were comparing and measuring as they participated in the modelling task. They were using the organised language of TLS to explain that they were "comparing" the weights of the beans. The teacher's role here was to focus the students on their mathematising. The students explained their perceptions. The teacher's questioning had a positive impact supporting the students' sense-making of the situation.

Megan: We need to write down Weeks 6, 8 and 10 and rows 1, 2, 3, and, 4 for sunlight and then we'll move on to shade.

Teacher: What is happening here?

Jeff: We're *comparing the weights*.

Teacher: So what is happening when you compare the weights?

Jeff: We're mainly measuring the weights of butter beans after they're in sun and shade.

Teacher: What are you comparing – weeks or rows??

Jeff: *We're comparing like* in row 1, week 1 they have 9 kilos in the sun. They're not growing too well *but in the shade* they are growing heavier.

Teacher: That's row 1 but is that the same for everywhere else?

Jeff: No, not really.

(Students continue to work out and write their results under two headings: The results of the weights of the butter beans in the shade/and in the sunlight.)

Jeff: *We are comparing the results* of the butter bean plants. After 10 weeks we have some results.

Other examples of the teachers' positive role occurred throughout the modelling investigations, such as, the teacher in the following text encouraged mathematical thinking and the need for justification.

Ben: So, sunlight has more kilos *compared to* the shade.

Teacher: And is that true for all of it? You have to make a decision and you have to check that information really carefully.

The teacher's intervention in the next excerpt was necessary to counteract the students' over reliance on their prior knowledge. Where prior knowledge often plays an important part in the construction of new knowledge, in this instance the student was relying on it whilst ignoring the mathematical data and evidence. The teacher had listened to the students as they discussed and was correctly cued to this situation. As a result, the teacher was able to redirect the students' attention to focus on the data to provide mathematical evidence for their explanations.

Kiesha: Well, a plant needs sunlight and shade. If a plant gets too much shade it will die or if a plant gets too much sunlight, it will die.

Teacher: What is this (the table) telling you? This is going to prove your discussion. Why is it saying sunlight is best?

Matt: Its got more kilos than in the shade.

Teacher: Oh! OK, so it has more kilos compared to?

Matt. The shade. So, sunlight has more kilos compared to the shade.

On the other hand, there were examples of instances where teacher intervention detracted from the students'mathematising during the investigation. This was most likely due to the teacher not having listened into the students' discussions accurately. In this next excerpt, the students were reasoning with time, distance, and also realising from the table that "scratching" needed to be taken into account when deciding on a winner.

Isobella: I chose E because it goes for a long distance. It goes for longer seconds and it has no scratches.

Eden: I chose Team E because its scores were higher than the others and it didn't get scratched and it hoes for the longest time." Both girls reasoned with time, distance and took scratches into account.

However, the teacher interrupted and asked to be shown the proof for their claims. Also, a number of questions were asked in a row.

Teacher: How do you know? Where did you work it out? Show me how you worked that out. Show me the numbers.

Students: The numbers?

Teacher: Show me the best number from the others. Make sure you can prove it.

Although the intention to ensure the students had evidence for their claims was good, the timing was unfortunate and only succeeded in misdirecting and confusing the students. It took them away from their sense-making, mathematising, and empowerment. Isobella's final comment ratifies this. She moved from a confident participant to believing that they were all "wrong in their answer". As well as being interrupted, the students were also faced with four directions in a row. This appeared to baffle the students even more. This is demonstrated in the subsequent conversation.

Kristy: I don't get it though.

Eden; What are we meant to do?

Kristy: I don't get what he said.

Isobella: Well, we're not doing it right because he told me to pick the number of each that would be the best one so I circled 13 because it was the biggest out of all of them so that's why I chose E.

Following is an illustration of what teachers can unwittingly cause in the classroom. The students were confidently examining and interpreting the data, analysing, and justifying their ideas. Firstly, the students appeared confident and empowered. This conversation was indicative of the type of discussion that the researcher had witnessed occurring in the classroom.

Eryn: What about Team D?

Kristy: Yeah, *but it has scratches*.

Eden: E doesn't have any scratches.

Kristy: Neither did C and neither did B.

Isobella; I chose E *because* it has 13, the highest number out of ll of them and that's why I chose E.

Kristy: What about C? It goes 9, 11, 11

Isobella: Eden, why did you choose E?

Eden: *Because there were no scratches*. It had the highest number in metres *and because its seconds* were more and so…

Students were examining the data, accounting for variables, looking for patterns, considering length and time, and generally finding mathematical justifications for their explanations. However, despite an urging from the researcher not to interrupt the students as they were actively participating, and were on task, the teacher stopped all groups. This teacher thought it would be better for the students to think individually about their decisions and then share with the rest of their group. The justification for this was that there would be better outcomes for the research because after personal reflection, the students would have more ideas to discuss. As a result, the students' behaviour diminished. When they returned to discussion, they were not on task. They read out their written ideas. It appeared that fellow group members did not listen to these readings.

Conclusion and Implications

Mathematical-modelling tasks coupled with TLS demonstrate how activities can be successful in promoting mathematical literacy. These tasks go beyond traditional views to provide students with opportunities to acquire advanced-thinking skills that are interpretive, organisational, and communicative as students encounter a variety of narrative, graphic, and factual texts (English, 2004; Lesh & Doerr, 2003b; Lesh & Yoon, 2004). The overall study not only clarified this claim, but also opened other windows of opportunity to investigate such issues as the role teachers have in impacting positively or negatively on students' acquisition of mathematical literacy.

The research reported here informs mathematics educators in two specific ways. The first way is to reiterate the view of Harel and Sowder (2005), that mathematical thinking will be best produced if meaningful and rational tasks are constructed. Students must be guided to think mathematically through the activities provided for them, and by the expertise of the teacher. These vital roles of the classroom teacher were demonstrated in the results reported here. Students were given tasks that encouraged mathematical thinking, but the teacher, in certain instances needed to guide the students to mathematical understanding (Harel & Sowder, 2005).

The second way is to impress upon educators the essential role of the teacher to (a) construct quality activities that benefit learning, and (b) act appropriately on indicative cues to benefit learning (Kulikowich & DeFranco, 2003). The episode where the teacher interrupted the whole class was an example of where this teacher could have modified personal views (McClain et al., 1998) to benefit the learning community. This, as well as the other example cited, demonstrates that perhaps we, as educators, all have lessons to learn on how our decisions impact upon our students. Further research investigating teacher impact on students learning could benefit mathematics teaching.

Mathematical modelling with TLS has given a prime example of how tasks can be constructed to reflect a diverse theoretical basis. These interactive tasks are established in situated cognition (Barab et al., 2001). As students are required to store and organise information, the tasks build upon information-processing theory (Anderson et al., 1996). They are social activities (Lambert & Blunk, 1998). They reflect both cognitive and social practices (Anderson et al., 2000).

An environment for mathematical sense-making (Schoenfeld, 2002) must be created. A quality teacher provides the means by which to do so. Mathematical modelling with TLS provides a task by which to do so. Such an environment encourages students to make sense of situations as they participate in a supportive discourse community to advance their mathematical literacy.

References

Anderson, J., Greeno, J. G., Reder, L. M., & Simon, H. A. (2000). Perspectives on learning, thinking, and activity. *Educational Researcher, 29*(4), 11-13.

Anderson, J., Reder, L. M., & Simon, H. A. (1996). Coming to terms: How researchers in learning and literacy talk about knowledge. *Educational Researcher, 24*(4), 5-11.

Bannan-Ritland, B. (2003). The role of design in research: The integrative learning design framework. *Educational Researcher, 32*(1), 21-24.

Barab, S. A., Hay, K. E., & Yamagata-Lynch, L. C. (2001). Constructing networks of action-relevant episodes: An in situ research methodology. *The Journal of the Learning Sciences, 10*, 63-112.

Bartlett, B. J. (2003). Valuing the situation: a referential outcome for top-level structurers. In B.J. Bartlett, F. Bryer & D. Roebuck. (Eds.), *Reimaging Practice: Researching Change* (pp. 16-37). Griffith University, Brisbane: Proceedings of the 1st International Conference on Cognition, Language and Special Education. School of Cognition, Language and Special Education.

Bartlett, B. J., Liyange, I., Jones, S., Penridge, J., and McKay, K. (2001). Science Education through Literacy and Language. In F. Bevan, C. Kanes & D. Roebuck (Eds.) *Knowledge Demands for the New Economies Volume 1*. Brisbane: Academic Australian Press, pp. 64-70

Cobb, P., Confrey, J., diSessa, A., Lehrer, R., & Schauble, L. (2003). Design experiments in educational research. *Educational Researcher, 32*(1), 9-13.

Doyle, K. (2006). Organisational structure for mathematical modelling. In P. Grootenboer, R. Zevenbergen & M. Chinnappan (Eds.), *Identities, Cultures and Learning Spaces:* (Proceedings of the 29th annual conference of the Mathematics Education Research group of Australasia, Vol. 1, pp. 187-195). Sydney: MERGA.

English, L. D. (2004). *Mathematical modelling in the primary school.* (Proceedings of the 27th annual conference of the Mathematics Education Research Group of Australasia, Vol 1, pp. 207-214). Sydney: MERGA.

Harel, G., & Sowder, L. (2005). Advanced mathematical-thinking at any age: Its nature and its development. *Mathematical Thinking and Learning, 7*(1), 27-50.

Kulikowich, J. M., & DeFranco, T. C. (2003). Philosophy's role in characterizing the nature of educational psychology and mathematics. *Educational Psychologist, 38*(3), 147-156.

Lambert, M., & Blunk, M. L. (Eds.). (1998). *Talking mathematics in school: Studies of teaching and learning.* Cambridge: Cambridge University Press.

Lesh, R., & Doerr, H. M. (2003a). Foundations of a models and modelling perspective on mathematics teaching, learning and problem solving. In R. Lesh & H. M. Doerr (Eds.), *Beyond constructivism: Models and modeling perspectives on mathematics problem solving, learning, and teaching* (pp. 3-33). Mahwah, NJ: Lawrence Erlbaum Associates.

Lesh, R., & Doerr, H. M. (2003b). In what ways does a models and modelling perspective move beyond constructivism. In R. Lesh & H. M. Doerr (Eds.), *Beyond constructivism: Models and modelling perspectives on mathematics problem solving, learning, and teaching* (pp. 519-556). Mahwah, NJ: Lawrence Erlbaum Associates.

Lesh, R., & Yoon, C. (2004). What is distinctive in (our view about) models and modeling perspectives on mathematics problem solving, learning and teaching? In H. W. Henn & W. Blum (Eds.). *ICMI Study 14: Applications and Modeling in Mathematics Education* (pp. 151-159): Dortmund, Germany: University of Dortmund.

McClain, K., Cobb, P., & Bowers, J. (1998). A contextual investigation of three-digit addition and subtraction. In L. J. Morrow & M. J. Kenney (Eds.), *The teaching and learning of algorithms in school mathematic., 1998 yearbook*. Reston, VA: National Council of Teachers of Mathematics.

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics*. Reston, VA: National Council of Teachers of Mathematics.

Queensland Studies Authority. (2004). *Mathematics: Years 1 to 10 Syllabus*. Retrieved 24/05/05, from http://www.qsa.qld.edu.aqu/yrsato10/kla/mathematics/index.html

Romberg, T. A. (2001). *Mathematical literacy: What does it mean for school mathematics?* Retrieved 15/03/07, from http://www.wcer.wisc.edu/ncisla/publications/articles/OctMathWASB.pdf

Schoenfeld, A. H. (2002). A highly interactive discourse structure. *Social Constructivist Teaching, 9*, 131-169.

Tobin, K. (2000). Interpretive research in science education. In A. E. Kelly & R. Lesh (Eds.), *Handbook of research design in mathematics and science education* (pp. 487-512). Mahwah, NJ: Lawrence Erlbaum Associates.

U. K. National Curriculum. (2000) *Mathematics: The National Curriculum for England*. Retrieved 18/03/07, from www.nc.uk.net

Informal Knowledge and Prior Learning: Student Strategies for Identifying and Locating Numbers on Scales

Michael Drake
Victoria University of Wellington
<michael.drake@vuw.ac.nz>

This paper reports on one aspect of a larger study into student understanding of scale. Thirteen students from Years 7 and 8 were interviewed, using a diagnostic assessment designed for the purpose, to identify how they went about locating numbers on, and reading numbers from scales. A range of student strategies were identified, most of which can be classed as informal knowledge. These strategies can be sorted into a progression that relates to the level of number thinking involved.

While learning mathematics in New Zealand, by Years 7 and 8 students are expected to develop the ability to work successfully with scales in a wide variety of contexts, including measurement, algebra, and statistics. Scales themselves, however, are not explicitly identified as something that needs teaching (Ministry of Education, 1992). Scales are also widely met in other curriculum areas (e.g., Ministry of Education, 1993, 1997), where the focus is on using them to facilitate other learning. In all of these documents, it is important to note that learning is expressed as statements of the outcomes that students should be able to achieve, and that in taking this approach they omit the *how*.

Commonly used resources that are designed to assist teachers in the delivery of the mathematics curriculum document follow this lead (e.g., Ministry of Education, 2000a, 2000b; Tipler & Catley, 1998; Wilkinson, 2002a, 2002b). They provide exposure to the sort of activity that students are expected to be able to master. Students get to read kitchen scales, draw graphs to display data they have collected, use number lines showing fractions or decimals, and interpret graphs drawn by others. In many of these activities the focus is not on the scales themselves, but the information transmitted through understanding the scales. This leaves teachers the task of realising what the potential stumbling blocks are and providing scaffolding instruction.

Unfortunately for teachers, this may not be an easy task. Research from a number of fields has shown that there are significant issues that need to be addressed as students learn about scale. In relation to linear measurement: the role of zero, the iteration of the unit, whether to count marks or spaces, and the difference between number and measurement, are all significant (see Nunes & Bryant, 1998; Outhred & McPhail, 2000; Bragg & Outhred, 2000a, 2000b; and Irwin & Ell, 2002). For the measure construct of fractions, some of these issues are also identified, as well as where fractions reside in relation to the whole numbers, the nature of the unit, how the scale is marked, and the meaning of fraction symbols (see Behr, Lesh, Post, & Silver, 1983; Lesh, Post, & Behr, 1987; Bright, Behr, Post, & Wachsmuth, 1988; Baturo & Cooper, 1999). In relation to statistical graphs, treating the horizontal axis of a histogram as a scale, scaling, and working between the gridlines have been identified as issues (see Kerslake, 1981; McGatha, Cobb, & McClain, 1998; Friel, Curcio, & Bright, 2001). Research into algebraic graphing, decimals, integers, and the use of the number line to show addition or subtraction problems, also identify issues, though space limitations preclude further development of these ideas.

Proceedings of the 30th annual conference of the Mathematics Education Research Group of Australasia
J. Watson & K. Beswick (Eds), © MERGA Inc. 2007

Given the inherent problems in learning to use scales, and the lack of direction from curriculum documents and commonly used resources, this study aims to identify what understandings students have actually developed.

Methodology

This report focuses on the student interviews undertaken as part of a wider research project on student understanding of scale, and teaching strategies to improve that understanding. In total 13 students from three classes at an urban Wellington intermediate school were interviewed over 3 days. Although a larger sample had been planned, student absence and other school activities restricted the number of students available. The students were chosen by their teachers to provide a range of abilities, and a mix of gender from both Years 7 and 8.

For the research, a diagnostic assessment was developed. This included number line items as well as similar or parallel items from "familiar" mathematical contexts, as identified by the curriculum and text analyses. Questions addressed issues commonly identified in the research literature and involved whole numbers, multiples of whole numbers, fractions, decimals, and integers. The questions in the diagnostic assessment were then used in the form of a *cognitive interview* (Presser et al., 2004). This provided feedback on the assessment and the questions as well as data on how students went about answering scale related questions. These interviews were audiotaped.

Each question was provided individually in written form to the student. Once a question was answered, the student was asked "how did you work that out?" Responses were clarified and recorded by hand. Visual strategies observed by the interviewer as well as the explained strategy were recorded. Where a verbal response was not clear, the observed strategy was sometimes voiced as a clarifying question. Such an approach provided a richer record than the audiotape alone, as it allowed some access to students' initial strategies that were later rejected. However, it is acknowledged this approach is still prone to identify the method that a student considers they used to answer the question successfully, and can explain, rather than provide a record of all the thought processes attempted by the student. In a few cases students were also at a loss to explain their reasoning, and no visual cues were provided, so no strategy could be deduced.

After the interviews, the audiotapes were transcribed, with the transcript compared to the written notes. From this, the solution methods used for the different questions were identified, and categories of response created. This process necessarily required the coder to interpret the responses and draw inferences about the logic used to create them. Here the form of thinking used by the student provided a tool for classification, as some responses clearly relied on counting, whereas others relied on adding or multiplying.

Results and Discussion

Mental Strategies

In conducting the interviews, it quickly became clear that students had a range of mental strategies that they used when working with scale. As these strategies had been nowhere identified in the document analysis (described above) as forming part of scale-related instruction, an alternative explanation as to their existence needed to be found. Mack (1995) identifies the body of skills and understandings students have developed for themselves while working on real tasks outside the classroom as *informal knowledge*. This knowledge may or may not be correct, and can be context related. In this case, it seems that students may have developed these strategies for themselves while working with scales in classroom situations. Alternatively, they may have resulted from informal instruction while focusing on a learning task that happens to involve scales. In either case, the label informal knowledge seems appropriate as the knowledge is probably gained in an incidental fashion.

Mental Strategies as a Window into Student Thinking

In working with scale, a student's written response was not always an accurate indication of how a student obtained the answer. This was particularly true if the answer was correct. Figure 1 shows a number line on which students were first asked to write the missing numbers in the boxes, then to locate the number 11. In follow-up questioning, Student 5 was asked to identify how he worked out that the second box should have the number 42 in it. He responded that "(i)ts going up in sixes and then there's 12 so you have to put another six in there and then that's another six to make 36, and then another six to 42". Meanwhile Student 2 responded "I just counted in sixes and what I did was, there was one, two and three and so I did three times six is 18 and then for 42 I said seven times six is 42". When locating 11, Student 4 explained their strategy as "probably just before the 12, right here", whereas Student 5 explained that "you've got to get it in an even space". This student was dividing the interval into six equal spaces, then counting along five of them.

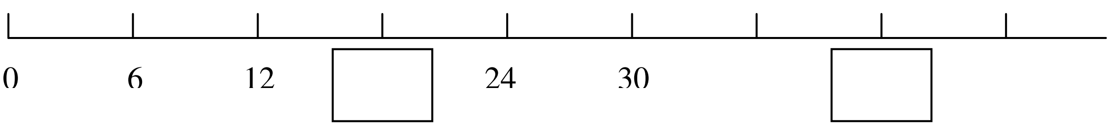


Figure 1. A number line question from the interview.

Both of these pairs of responses illustrate significant features of the identified student strategies. The first is that not all students use the same strategy on the same question, rather the strategy chosen seems to relate to their different understandings of number. For example, in the first quote Student 5 is using a skip count approach, which has links to additive thought. Meanwhile Student 2 is clearly relying on an understanding of multiplication. This provided a way to differentiate student strategies according to a level of sophistication.

The second feature relates to how students located numbers in intervals. In locating 11, Student 4 seems to be using an estimation strategy, whereas Student 2 is using partitioning. A closer look at all of the responses indicated that somewhere in the interview all 13 students used a strategy similar to that of Student 4, finding "a little bit more or a little bit less". For some problems, this strategy was used in conjunction with partitioning strategies. This suggests that finding "a little bit more" is a simple strategy accessible to all. The analysis also identified that "success" with the strategy was varied, as if the size of the "bit" chosen was arbitrary. For example, Student 7 described using both a partitioning strategy (halving) and "a little bit less" when locating 0.4cm on a ruler. His explanation for the placement being "(c)ause like zero point five would be about there [indicates where 0.8 would be], so the one before". Figure 2 below shows his response.

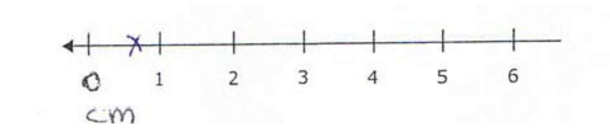


Figure 2. Work from Student 7.

In this question Student 7 did not manage to divide the interval into two equal pieces when halving, so in considering the possibility that students were estimating when using "a little bit more/a little bit less", this perspective was explored. For estimation to be used successfully, it needs to be relational with the "bit" being in proportion to the size of the interval. Given that Student 7 did not halve an interval accurately, it seems to be very bold to suggest that he can work with space proportionally. For this example, better explanations are either that this student is used to working with rulers and "knows" how big a millimetre is, and uses this knowledge, or the piece chosen *was* arbitrary, with a small partition "close to" being taken.

Other questions in the test seemed to access a student's informal knowledge specifically – or rather their assumed understanding of a situation. Figure 3 shows a response from Student 9, who, when asked how she got that answer, did not seem to imagine that a thermometer could have anything other than a unit scale: " 'cause there's ten, that would be twelve". However, on a similar item involving a number line she correctly identified that the scale went up in twos, suggesting her response to the thermometer question was prompted by the context. In other questions, Student 9 showed she had access to a number of different strategies, though not to any that involved the use of multiplication, suggesting she did not have access to multiplicative thinking.

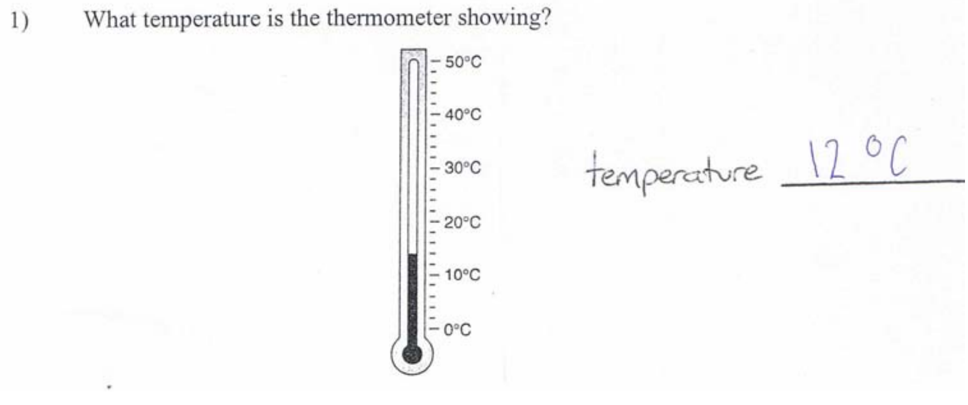


Figure 3. Unit scale thinking.

Error Patterns as a Window to Student Understanding

In explaining her reasoning for her answers to the questions in Figure 4, Student 9 indicated that she was unclear about whether or not she had them correct. For question 3a her logic was " 'cause the one's on zero so it might be like zero point", for 3b "point nought two, or one". To interpret this error pattern, research into measurement understanding seems to offer a better insight into the thinking that Student 9 is applying than fraction based research. For example, Nunes and Bryant (1998) suggest that several problems exist for students when learning to use rulers. One is the issue of counting and measuring, where counting never starts at zero. Another is whether or not to count the gaps or the lines. A third is that "children can conceivably be taught to follow a procedure for reading measurements on a ruler and still have little understanding of the logic of measurement" (p. 86). Student 9 seems to be clearly counting the marks, but uses one as her start point, a counting-measurement confusion. She also seems to be "counting in points", that is counting each mark between the whole numbers as a tenth, regardless of how many there are. Thus this question has opened a valuable window into the understanding that Student 9 has of scales, and suggests several avenues for new teaching.

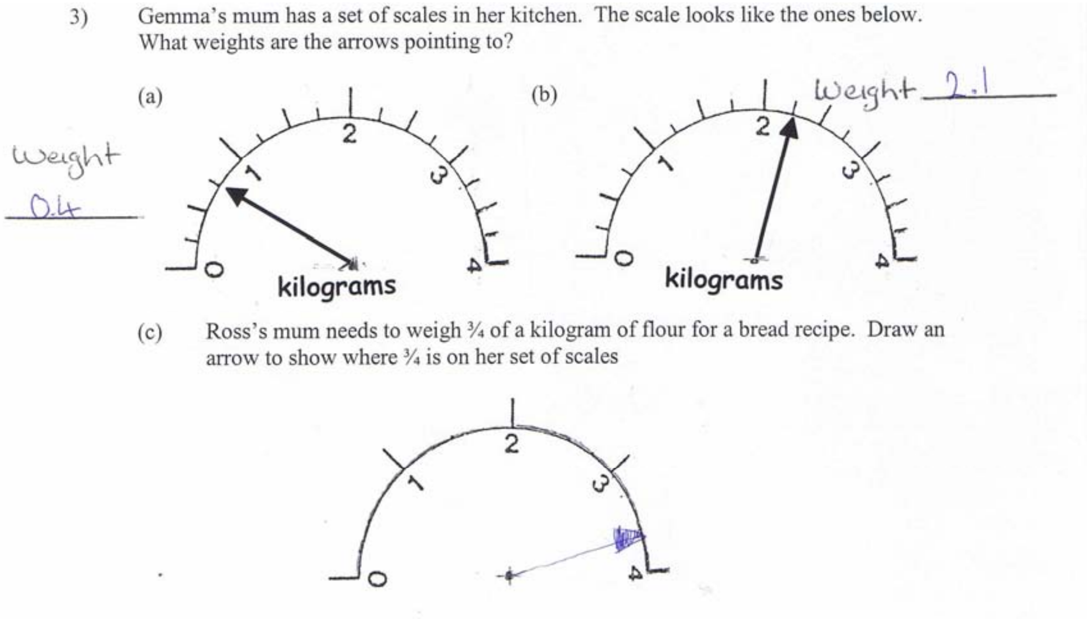


Figure 4. Other examples of thinking from Student 9.

Question 3c adds another perspective to Student 9's understanding. Her logic for placement is " 'cause like, the three then mark quarters, like a little bit away from four". Here fraction knowledge seems to be accessed, though there is confusion as to the meaning of the symbol ¾, apparently confusing ¾ with 3¾. Baturo and Cooper (1999) also found such confusion. One possible explanation is that this could be tied to a developed understanding of fractions like ¾ as "three pizzas, each cut into four slices" in which the three is the number of wholes. This interpretation can be used successfully when sharing (e.g., dividing three pizzas between four people) or when answering questions involving the quotient sub-construct of fractions, but suggests a limited understanding of fraction symbols, and a poor knowledge of the continuity of fractions – that is, where they can be found in relation to the whole numbers.

In the follow-up interview Student 9 indicated that she had met these sorts of problem before, and did not find them difficult. However, her observed strategies indicate significant misconceptions that need to be addressed. How did these arise? Some strategies appear to be the result of specific instruction, and appear to be strategies that have been developed uncritically and have been overgeneralised. (We can almost hear a teacher say to the class learning about the ruler that "each of these little marks between the numbers is a tenth, so its point one, point two, point three…"). Such strategies can be described as *prior learning*. Others appear self-developed and are better described as informal knowledge. For example, ¾ as "three pizzas each cut into four pieces" was not a common approach to the teaching of fraction symbols found in the reviewed texts.

Table 1
Typical Student Strategies for Partitioning Unmarked Intervals

Thinking type.	Strategy name.	Example of strategy.	Useful with…
Counting based.	A little bit more, a little bit less.	Make a mark "a bit" to the left or right.	Locating numbers "just next to" other numbers.
	Halving.	"Eying up" exactly where the middle of an interval is using the point of the pen as a marker.	Can be used repeatedly (to find quarters etc).
Addition based.	*n* equal spaces	Draw in marks for each unit, counting along in ones "to fill in the gap".	Scales involving whole number multiples. Can be successfully transferred to decimals or fractions.
	Mixed methods.	Repeated halving or combining the use of halving and "a little bit more, a little bit less" or "counting in ones".	
Multiplication based.	2, 3, 5 method.	Students know how to accurately partition an interval into 2, 3, and 5 pieces.	
	Mixed methods.	Locating 11 on a scale using multiples of 6 by finding ½ way, and cutting the remaining interval into 3 equal pieces…	Subdividing most intervals into the commonly used number of pieces.

Student 9's answers were typical of the pattern of responses found in the interviews. Students had a range of strategies that they used selectively to answer questions. Overall a finite set of solution strategies was identified and student success with these was influenced not only by the appropriateness of the strategy to the situation, but also by a series of other understandings, for example, whether or not to count the marks or the spaces (and how to do this), whether to start the count at zero or one, and the ability to create intervals of equal size. Table 1 summarises and names the strategies identified as being used by students when answering problems involving partitioning unmarked intervals on scales. In some cases where the type of thought was not obvious, these strategies have been allocated to a stage based on the frequency of their use. For example, halving was used by 12 of the 13 students, though not by one who answered all questions correctly, so has been placed in the counting category. Hart (1981) also talks of one half as an honorary whole number suggesting that students find working with one half easy.

The set of strategies identified allows a "multiplicative" student to partition intervals into the most commonly used number of subdivisions. Strategies used to partition intervals into sevenths, elevenths, thirteenths and the like were not investigated.

Student Responses to Items Involving a Scale where some Marks are not Numbered

The thermometer in Figure 3 and the scales from questions 3a and 3b in Figure 4 are all examples of scales where not all marks are numbered. Students used a different set of mental strategies to those in Table 1 when working with this sort of scale. These are shown in Table 2. As examples of these strategies, when dealing with the fractional question A5 (Figure 5), Student 1 used a "counting in tenths" strategy, referencing the nearest whole number rather than "counting up from the number on the left": "'(C)ause it's zero there [points to zero] and zero point nine, one is after zero point nine … and one point one is after one." Student 5 on the other hand converted the problem to whole numbers, then reconverted to answer the question, a strategy that relies on an understanding of multiplication: "Well, you can't get 4 into 10 so I worked to 100 and stuff." Student 7 meanwhile ignored some of the scaffolding on the problem (the zero at the start of the scale) to turn the problem into one he could understand and solve: "I knew that one before zero is zero and one after one is two."

Table 2
Typical Student Strategies for Numbering a Marked Scale

Thinking type.	Strategy name.	Example of strategy.	Useful with…
Counting based.	Thinking in ones.	Each mark shows one more, as all scales go up in ones…	Unit scales.
	Trial and error.	Students count along in ones and if that doesn't work try twos…	Scales marked in multiples of a number. Can be adapted for decimals.
	Counting in tenths.	If there are marks between the (counting) numbers, count 0.1, 0.2, 0.3, … . Some students also count back in points from the nearest whole number.	Scales marked in tenths.
Addition based.	Skip counting.	A development of the trial and error strategy – using skip counts. For example – "that's a big gap/number to fit on, lets try tens…"	For interpolating and creating a scale. For extrapolating, this just requires a continuation of the scale with the correct "skip".
	Fitting tenths	For example, a scale marked in quarters "that would be 0.3, that o.5 then o.6 0r o.7 then 1"	Scales marked in tenths.
	Bits and "ths".	There are 5 bits (spaces), so each is a fifth.	Fractional and decimal scales.
	Whole number conversion	For example, treating the entire number line as the whole '¼ is 1, ½ is 2, ¾ is 3 and 1 is 4' or reunitising tenths as whole numbers.	Not useful for fractional scales. Decimal version works on scales in tenths.
Multiplication based.	Marks and interval method.	There are 5 marks, the interval is 10, so each mark is 2.	Any non-unit scale. Also useful for decimals.
	Whole number conversion.	A development of the "marks and interval" method. For example, 4 pieces, ¼ of 100 is 25 so ¾ is 75, so its 0.75.	Decimal and fractional scales.
	Treating the fraction as an operator	Treating the entire number line as the unit. For example, 6 is ¾ of 8	Not particularly useful.

With some of these strategies, it is possible that they are simply reconceptualisations of an earlier strategy with a higher level of number understanding. For example, it seems likely that unit counting (thinking in ones) precedes all other strategies, and that "trial and error" relies on the development of the ability to skip count – and the realisation that not all scales go up in ones. "Counting in tenths" likewise appears to be linked to learning that there are numbers *between* the whole numbers. All of this may well be the case, but is likely to need a study of student understanding over Years 1 to 6 to determine a thorough developmental progression.

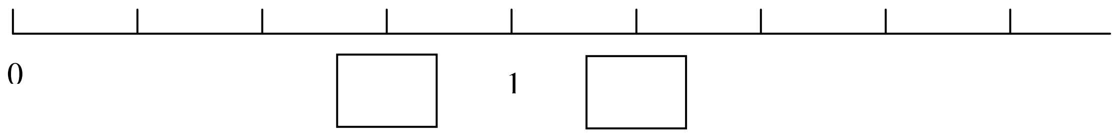


Figure 5. Number line item mathematically similar to questions 3a and b from Figure 4.

Consistency in Student Response Strategies

In designing the diagnostic assessment, one consideration was whether or not students found number lines easier or harder to work with than scales found in "familiar" situations. One measure of this was created by considering the strategies used by a student in the pairs of supposedly similar questions. Students' responses were analysed to see whether or not they had used their strategies for answering the number line question on the contextual question. A "mark" was given if they did. This analysis thus gave a *consistency rating* for a question. Questions with a consistency rating of 13/13 were questions where every student transferred the strategy they used on the number line item to the contextual item. Table 3 shows the results of this analysis.

Table 3
Consistency in Strategy use when Dealing with a Number Line Problem and a Similar Item Presented in a Familiar Context.

Item type	Consistency rating	Percentage
Scales involving multiples of whole numbers	13/13	100
	10/13	77
	10/13	77
Decimal scales	11/13	85
Fractional scales	9/13	69
	3/13	23
Conventions of scales	3/13	23

Most students answered similar questions involving whole numbers and decimals by using the same mental strategy, and gave similar explanations when asked to explain their reasoning. Only in two situations were there significant inconsistencies, in that most of the students changed their mental strategy when answering the "contextual" item. In one case this involved showing understanding of the conventions of a number line, and creating a horizontal axis for a bar graph. Here the issue identified by McGatha et al. (1998) relating to students treating the numbers on the horizontal axis of a bar graph as individual data points or categories can be identified in the students' responses to the question.

The other case involved fractional number lines with marks. The two contextual items involved are shown in Figure 4 (Questions 3a and 3b), whereas Figure 5 shows the similar number line items. Note that although the questions required students to find the similar numbers, the visual cues were different in that the number line item did not go up to four. This may have caused some students to respond differently.

Several patterns were of note when considering student responses to these items. Firstly, in answering the number line question in Figure 5, only three of the 13 students answered correctly and these students were successful with both the number line and the contextual items. Secondly, each of these students identified the missing numbers as 0.75 and 1.25, using a "whole number conversion" approach (see Table 2). Fractional strategies were not found to be used by any of the 13 students for these four fraction questions.

Overall, analysis identified that of the students who answered *any* contextual question incorrectly, in 30 out of 40 instances (75%), the students had changed their response strategies from the equivalent number line question. This suggests that strategy use is unstable in situations where a student is unsure of the mathematics in the situation.

Conclusions and Implications

In the absence of formal guidance from curriculum documents and commonly used resources, these New Zealand students seem to have developed their own understanding of scale. This consists of informal knowledge and prior learning of varying levels of sophistication that students apply to situations in an attempt to make sense of them. In many cases, this understanding was used consistently, in that mathematically similar items utilising a number line and a "familiar" context evoked the same solution strategy. However, this was not always found to be the case. Fraction questions caused students to change their strategy. Also, with the bar graph, most students did not treat the horizontal axis as a scale, instead bringing to the question a particular understanding of the context. Here it can be said that using such a graph as a context for developing an understanding of scale has introduced an element of *contextual pollution*; that is it has introduced context situated knowledge that interferes with the intended learning about another topic. In this particular case the contextual pollution was the common misconception that the horizontal axis of a bar graph is not a scale so, for example, ordinal data recorded on this axis do not need to be placed in order of size. In another situation quoted, a thermometer invoked a unit scale response from a student who could use appropriate mathematics on the similar number line item. The concept of contextual pollution suggests that teachers need to be aware that contexts may not always be helpful and that they need to be alert for signs that students are operating from a different conceptual base to them. In terms of scale, the consistency analysis has suggested that number lines invoked similar strategies from students, so may be a better initial tool for developing students' understanding.

In conclusion, scale is one of the big ideas in mathematics. It underpins significant learning in number, measurement, algebra, and statistics. Scales are met not only in mathematics but also in other curriculum areas. It comes as a surprise that even by Years 7 and 8, not all these students have learned that there are numbers between the whole numbers, and that some students cannot recognise when an interval on a number line (or a weighing scale) has been divided into quarters. This small study has shown that many of these New Zealand students have a lot to learn if they are to become successful users of scale, that an understanding of scale cannot be assumed by teachers, and that more research into this area would be of value. It also suggests that it may be time to reconsider how students are expected to develop their understanding of scale, as current approaches seem to be leaving a great deal to chance.

References

Baturo, A., & Cooper T. (1999). Fractions, reunitisation and the number line representation. In O. Zaslavsky (Ed.), *Proceedings of the 23rd conference of the International Group for the Psychology of Mathematics*

Education volume 2. (pp 81-88) Rehovot: Department of Science Teaching, The Weizmann Institute of Science.

Behr, M., Lesh, R., Post, T., & Silver, E. (1983). Rational number concepts. In R. Lesh & M. Landau, (Eds.), *Acquisition of mathematics concepts and processes* (pp. 92-127). New York: Academic Press.

Bragg, P., & Outhred, L. (2000a). What is taught versus what is learned: the case for linear measurement. *Mathematics education beyond 2000*. (Proceedings of the 23rd annual conference of the Mathematics Education Research Group of Australasia, Freemantle, Western Australia (pp 112-118)). Perth: MERGA.

Bragg, P., & Outhred, L. (2000b). Students' knowledge of length units: do they know more than rules about rulers? In T. Nakahara & M. Koyama (Eds.), *Proceedings of the 24th conference of the International Group for the Psychology of Mathematics Education* (pp. 97-104). Hiroshima: Hiroshima University.

Bright, G., Behr, M., Post, T., & Wachsmuth, I. (1988). Identifying fractions on number lines. *Journal for Research in Mathematics Education*, 19, pp. 215-232.

Friel, S., Curcio, F., & Bright, G. (2001). Making sense of graphs: Critical factors influencing comprehension and instructional implications. *Journal for Research in Mathematics Education, 23,* pp. 124-158.

Hart, K. (1981). *Children's understanding of mathematics: 11-16.* Newcastle upon Tyne: Athenaeum Press Ltd.

Irwin, K., & Ell, F. (2002). Visualising and the Move from Informal to Formal Linear Measurement. In B. Barton, K. Irwin, M. Pfannkuch & M. Thomas (Eds.). *Mathematics Education in the South Pacific* (Proceedings of the 25th annual conference of the Mathematics Education Research Group of Australasia, Auckland, pp. 358-365). Sydney, MERGA.

Kerslake, D. (1981). Graphs. In K. Hart (Ed.), *Children's understanding of mathematics.* pp. 11-16., Newcastle upon Tyne: Athenaeum Press Ltd..

Lesh, R., Post, T., & Behr, M. (1987). Representations and translations among representations in mathematics learning and problem solving. In C. Janvier (Ed.), *Problems of representation in the teaching and learning of mathematics* (pp 33-40) Hillsdale, NJ: Lawrence Erlbaum Associates.

McGatha, M., Cobb, P., & McClain, K. (1998, April). *An analysis of students' statistical understandings.* Paper presented to the annual meeting of the American Educational Research Association, San Diego, California.

Mack, N. (1995). Confounding whole-number and fraction concepts when building on informal knowledge. *Journal for Research in Mathematics Education, 26,* 422-441.

Ministry of Education. (1992). *Mathematics in the New Zealand curriculum*. Wellington: Learning Media Limited.

Ministry of Education. (1993). *Science in the New Zealand curriculum.* Wellington: Learning Media Limited.

Ministry of Education, (1997). *Social Studies in the New Zealand curriculum*. Wellington: Learning Media Limited.

Ministry of Education. (2000a). *Figure It Out measurement level 3*. Wellington: Learning Media Limited.

Ministry of Education. (2000b). *Figure It Out measurement level 3 answers and teachers' notes*. Wellington: Learning Media Limited.

Nunes, T., and Bryant, P. (1998). *Children doing mathematics*. Malden, MA: Blackwell Publishers.

Outhred, L., & McPhail, D. (2000). A framework for teaching early measurement. *Mathematics education beyond 2000*. (Proceedings of the 23rd annual conference of the Mathematics Education Research Group of Australasia, Freemantle, Western Australia (pp 487-494)). Perth: MERGA.

Presser, S., Couper, M., Lessler, J., Martin, E., Martin, J., Rothgeb, J., & Singer, E. (eds.). (2004). *Methods for testing and evaluating survey questionnaires.* Hoboken, NJ: John Wiley & Sons.

Tipler, M., & Catley, K. (1998). *National curriculum mathematics level 3 book 1*. Christchurch: Caxton Educational Ltd.

Wilkinson, C. (2002a). *Pearson primary maths 3b*. Auckland: Pearson Education New Zealand Ltd.

Wilkinson, C. (2002b). *Pearson primary maths book 3b teachers guide*. Auckland: Pearson Education New Zealand Ltd.

Documenting the Knowledge of Low-Attaining Third- and Fourth-Graders: Robyn's and Bel's Sequential Structure and Multidigit Addition and Subtraction

David Ellemor-Collins
Southern Cross University
<david.collins@scu.edu.au>

Robert (Bob) Wright
Southern Cross University
<bob.wright@scu.edu.au>

Gerard Lewis
Catholic Education Office, Melbourne
<GLewis@ceo.melb.catholic.edu.au>

Aspects of students' arithmetic knowledge are described via two case studies of responses to tasks during a videotaped assessment interview. Tasks include reading numerals, locating numbers, saying number word sequences by ones and tens, number word after or before a given number, incrementing and decrementing by ten, addition in the context of dot strips of tens and ones, and addition and subtraction involving bare numbers. On many tasks the students had significant difficulties and responded differently from each other. The paper demonstrates the idiosyncratic nature of arithmetical knowledge, and the significance of context in students' multidigit thinking.

The paper reports on aspects of a current 3-year project that has the goal of developing pedagogical tools for intervention in the number learning of low-attaining third- and fourth-graders (8- to 10-year-olds). These tools include schedules of diagnostic assessment tasks, and a learning framework for profiling students' number knowledge. A particular focus of study has been assessment of student knowledge of multidigit addition and subtraction. Most research on multidigit knowledge is with younger students' initial learning of multidigit arithmetic. For low-attaining older students, who may already have been expected to master 2-digit column algorithms, we wish to establish a profile of their multidigit knowledge. The paper describes two case study profiles.

Literature Review

In the last 15 years, research and curriculum reforms in a range of countries highlight a renewed emphasis on mental computation with multidigit numbers (Beishuizen & Anghileri, 1998; Cooper, Heirdsfield, & Irons, 1995; McIntosh, Reys, & Reys, 1992; Thompson & Smith, 1999). An emphasis on mental strategies may (a) support conceptual understanding of multidigit numbers (Fuson et al., 1997; Heirdsfield, 2005; Hiebert & Wearne, 1996); (b) support development of number sense and important connections to related knowledge (Askew, Brown, Rhodes, Wiliam, & Johnson, 1997; McIntosh et al., 1992; Sowder, 1992); and (c) stimulate the development of numerical reasoning, and flexible, efficient computation (Beishuizen & Anghileri, 1998; Yackel, 2001). Following the principle of beginning instruction with students' informal strategies, researchers now put initial instructional emphasis on strong mental strategies (Beishuizen & Anghileri, 1998; Carpenter, Franke, Jacobs, Fennema, & Empson, 1998).

Place Value and Base-ten Structures

Multidigit knowledge includes knowledge of the numeration system and place value (e.g., Hiebert & Wearne, 1996). However, researchers argue that students may not operate with numbers in symbolic terms, observing that place value tasks become tasks of verbal patterns and symbolic manipulation, without connection to the students' sense of numbers (Cobb & Wheatley, 1988; Treffers, 1991). Thompson and Bramald (2002) make a distinction between quantity value, for example, partitioning 47 into forty and seven, and column value, for example, 47 represents 4 units of ten and 7 units of one. They argue students' mental strategies only depend on quantity value. In this paper we focus on base-ten structures that include aspects of place value knowledge, such as quantity value, which do not involve manipulating written symbols.

Of central interest in students' mental multidigit computation is the developing sophistication of their use of base-ten structures. Researchers have charted learning trajectories from using counting-by-ones strategies, through increasingly powerful uses of units of ten and other base-ten structures. In a synthesis from four research projects, Fuson et al. (1997) proposed a developmental sequence of children's two-digit conceptual structures. The structures incorporate students' relations among written numerals, number words, and quantities: *unitary* (53 as one, two, … fifty-three); *decade and ones* (one, two … fifty; and fifty-one, fifty-two, fifty-three); *sequence-tens and ones* (ten, twenty, … fifty; and fifty-one, fifty-two, fifty-three); *separate-tens and ones* (five tens and three ones); and *integrated-sequence-separate*. A sixth, incorrect conceptual structure was labelled *concatenated single digit* (53 as five and three). Developing the work of Steffe and colleagues, Cobb and Wheatley (1988) distinguished three levels in children's construction of ten as a unit. The levels were evident in children's thinking in additive tasks. Children operating at level 1 manipulate ten units and one units separately, and can not coordinate them. The level 1 construction of ten as an *abstract singleton* is comparable to the *concatenated single digit* structure from Fuson and colleagues. At level 2, children can coordinate counts or collections of tens and of ones, in the context of representations of the quantities, but they cannot "simultaneously construct a numerical whole and the units of ten and one that compose it" (p. 7). Students at level 3 can anticipate, without representations, that a numerical whole consists of tens and ones units, and coordinate operations with these. Significant in these analyses is the consideration of students' thinking in two settings: structured materials and bare numbers. The present study investigates students' use of base-ten structures and units when solving additive tasks in three settings: structured materials, bare numbers, and verbal number words.

Sequence-based Structure and Strategies

When students begin to use base-ten structures in arithmetic, they develop a variety of multidigit addition and subtraction strategies (Beishuizen & Anghileri, 1998; Cooper et al., 1995; Foxman & Beishuizen, 2002; Thompson & Smith, 1999). Sequence-based or jump strategies involve keeping the first number whole and adding (or subtracting) via a series of jumps, for example, 57 + 26 as 57 + 10, 67 + 10, 77 + 3, and 80 + 3. Collections-based or split strategies involve partitioning both numbers into tens and ones, and adding (or subtracting) separately with tens and ones, for example, 50 + 20, 7 + 6, and 70 + 13.

A broad knowledge of number relationships and numeration is important for mental computation (Heirdsfield, 2001). This includes knowledge of sequential structure

(Ellemor-Collins & Wright, in press): jumping by ten off the decade, locating numbers, number word sequences across decades, and making small hops (Fuson et al., 1997; Menne, 2001; Yackel, 2001).

Beishuizen and Anghileri (1998) argued that jump strategies can develop as curtailments of students' informal counting strategies. Beishuizen, Van Putten, and Van Mulken (1997) compared students' use of jump and split strategies and found that jump resulted in fewer errors and enabled making efficient computation choices. In contrast, split strategies led to difficulty in developing independence from concrete materials (Beishuizen, 1993); procedural and conceptual confusion (Klein, Beishuizen, & Treffers, 1998); and slow response times, suggesting a heavier load on working memory (Wolters, Beishuizen, Broers, & Knoppert, 1990). As well, Klein, Beishuizen, and Treffers (1998) found that, among low-attainers, jump strategies were much more successful.

Low-attaining students seem to use jump strategies less frequently and many do not develop knowledge of jumping in tens (Beishuizen, 1993; Foxman & Beishuizen, 2002; Menne, 2001). In Australia in many instances, instruction does not focus on counting by tens off the decade nor on developing sequential structure. Yet, sequence-based strategies can be more successful, and are necessary for integrating sequence-based and collections-based constructions (Fuson et al., 1997). Hence, the focus of this study is on low-attaining students' development of sequential structure and jump strategies.

Low-attaining Students

Students' arithmetic knowledge is componential (Dowker, 2005) and for students of similar ability levels, there can be significant differences in arithmetic knowledge profiles (Gervasoni, 2005). Understanding more about such profiles is one important response to calls for intervention in early number learning (Louden et al, 2000; Department of Education, Training and Youth Affairs, 2000). Further, assessment of students' multidigit knowledge should include a focus on multidigit numerals, number sequence knowledge, ten as a unit, mental computation, in verbal, structured, and bare number settings, and attention to students' strategies, as well as their answers. This paper presents two case studies that (a) describe in detail, low-attaining students' multidigit knowledge; (b) illustrate the idiosyncratic nature of this knowledge; and (c) illustrate the significance of context in students' multidigit thinking.

Method

Study

A screening test of arithmetical knowledge was administered to all third- and fourth-grade students in 17 schools. On the basis of the screening test, 191 students were classified as low-attaining. During their year in the project – 2004 or 2005 – these students were assessed twice, that is, in the second term and in the fourth term. The assessment consisted of an individual interview, videotaped for subsequent analysis. The analysis documents in detail each student's responses and strategies.

Task Groups

The interview used a schedule of task groups. A task group consists of tasks very similar to each other used to document students' knowledge of a specific topic. Some of these tasks are adapted from Cobb and Wheatley (1988) and have been widely used elsewhere (e.g., New South Wales Department of Education and Training, 2003). This paper focuses on eight of 20 task groups in the schedule:

1. *Numerals task group.* These tasks involved identifying and writing numerals. This included numerals with up to 5 digits and 3- and 4-digit numerals with a zero (e.g., 12, 21; 101, 730, 306; 1000, 1006, 3406, 6032, 3010; 10 235).
2. *Locating numbers task group*. Given a piece of paper showing a line with ends labelled as 0 and 100, the task was to mark in turn, 50, 25, 98, and 62.
3. *Number word sequences (NWS) by ones.* These tasks involved (a) saying a forward (FNWS) or backward (BNWS) sequence and included bridging decades, 100s, and 1000; and (b) saying the number before or after a given number.
4. *Number word sequences by tens.* Saying sequences by tens, forward or backward, in the range 1 to 1000, on and off the decade.
5. *Incrementing and decrementing using numerals.* Given a numeral, say the number that is ten more, using: 20, 90, 79, 356, 306, 195, and 999. Similarly, ten less than: 30, 79, 356, 306, 1005; one hundred more than: 50, 306, 973; one hundred less than 108.
6. *Incrementing and decrementing using ten-strips.* A strip with seven dots is placed out, then strips with ten dots are used one by one. The student's task is to state the total number after each successive strip is placed out – 7, 17, 27 etc.
7. *Incrementing using tens and ones.* Strips with the following numbers of dots are progressively uncovered: 4, one 10, two 10s, one 10 and 4, two 10s and 5. The student's task is to state the total number of dots at each successive uncovering. Finally, the 73 dots are covered and the student is asked how many more dots are needed to make 100.
8. *Bare number tasks.* The following are presented in horizontal format for the student to solve without materials or paper for writing: 43 + 21, 37 + 19, 86 – 24, 50 – 27.

Results

The case studies in this paper are based on the first interviews of two students. Of particular interest in the case of Bel are (a) his inability to jump by ten off the decade, in the absence of materials; and (b) his difficulties with addition and subtraction tasks requiring regrouping. Of particular interest in the case of Robyn are (a) her facility with jumping by ten off the decade, and (b) her difficulties with addition and subtraction tasks in bare number settings.

The Case of Bel

Bel was 9 years and 4 months old at the time of his interview, 15 weeks into the third grade (fourth year of school).

Numerals and locating numbers. Bel wrote correctly, all 3- and 4-digit numerals asked (270, 306, 1000, 1005, 2020), and identified all 3-digit numerals (101, 400, 275, 730, 306) and all but one of the 4-digit numerals (1000, 8245, 1006, 3406, 6032, 1300). His error was to identify 3010 as “three hundred and ten”. Bel’s location for 50 on the number line from 0 to 100 was quite accurate. His locations for 25, 62, and 98 were correctly ordered but inaccurate.

Number word sequences. Bel recited four FNWSs and BNWSs in the range 1 to 120. This included two self-corrections. He recited the BNWS from 303 but could not continue beyond 298. As well, he was partially successful with sequences bridging 1000. He recited the sequence from 1010 to 995, but made errors as follows: “1003, 1002, 1001, 999, 998” and “993, 992, 991, 990, 899, 888”. He was successful on nine number word after tasks and ten number word before tasks in the range one to 2000. He made one error only on this kind of task, that is, he stated “seven hundred and sixty-nine” as the number before 170.

Number word sequences by ten and incrementing by ten. Bel recited the sequence of decuples from 10 to 120 forward and backward, and other sequences of decuples up to 1090 but he could not count by tens from 24. As well, he could increment and decrement by 10 on the decade but not off the decade. His errors were to answer “81” as 10 more than 79, “315” as 10 more than 356, “61” as 10 less than 79, and “259” as 10 less than 356. By contrast, he correctly stated 100 more than 306, 100 more than 973, and 100 less than 108. In the context of ten-strips, Bel incremented by 10 off the decade – “27, 37, 47…”, but appeared to count by ones from seven, to figure out 7 dots plus 10 dots.

Incrementing using tens and ones. Bel was partially successful on the task involving strips and incrementing using tens and ones. He incremented 34 by 14, and in doing so, appeared to use a split-jump strategy, that is, 30 + 10, 40 + 4 and 44 + 4, counting by ones to figure out 44 + 4. In attempting to increment 48 by 25, he answered “33” after 43 seconds. When asked to explain, he pointed to each of the two ten-strips in turn, in coordination with saying “58, 68”. He then counted by ones as follows: “69, 30, 31, 32, 33”. He apparently used a jump strategy but could not correctly keep track when counting by ones from 68. Note that (a) Bel used a relatively low-level strategy, that is counting on by ones, to figure out 44 and 4, and 68 and 5. In both cases the items to count were perceptually available. (b) In the context of ten-strips, he incremented 48 by two tens, but (as described earlier), on a verbal task he could not count by tens from 24 and could not state 10 more than 79.

Bare number tasks. Bel used a split strategy to solve each of 43 + 21 and 86 – 24. For 37 + 19 he answered “68”. According to his explanation, he first added 3 and 1. These solutions contrasted with his jump strategy in the context of ten-strips, for incrementing 48 by 25 (as described earlier). For 50 – 27 he answered “28”. According to his explanation, “I took away 2 off that”, indicating the 5 of 50, “then when I got down to 30, I took away 7”.

The Case of Robyn

Robyn was 9 years and 5 months old at the time of her interview, 15 weeks into the fourth-grade (fifth year of school).

Numerals and locating number. Robyn showed fluency with 3-digit numerals, and made three errors with 4-digit numerals. She correctly wrote 270, 306, 1000, 1005, and 4320. When asked to write “one thousand nine-hundred” she wrote “1009”. She correctly identified 101, 400, 275, 730, 306, 1000, 8245, 1006, 3406, 3010; she identified 6032 as “six hundred and thirty-two”, and then corrected herself, and identified 1300 as “thirteen thousand". In the locating numbers task, Robyn placed 50 correctly but, like Bel, her marks to locate 25, 62, and 98 were correctly ordered but inaccurate.

Number word sequences. In the range 1 to 1000, Robyn recited eight number word sequences, and stated the number word before or after for twenty-five given numbers. She made five errors across these tasks, each of which she promptly, spontaneously corrected. Sequences across 1000 and beyond were problematic for Robyn, which we detail further below.

Number word sequences by ten and incrementing by ten. Robyn counted by tens on and off the decade, up to 120. With sequences beyond 120, she had difficulties bridging hundreds saying "170, 180, 190, 800, 810 …", and "177, 187, 197" (pause), "207" pause, "227, 237". Robyn successfully incremented and decremented by ten from on and off the decade in the range to 1000. She was fluent with eight such tasks, but she had significant difficulty with the task of incrementing 195 by ten and her response was indiscernible. Robyn was more successful on these tasks than many of the other low-attaining students. By contrast, Robyn could not increment by one hundred off the hundred: For 100 more than 50 she answered, "five hundred", and for 100 more than 306 she answered, "4006 … 406 … 4006".

Sequences across 1000. Robyn was unsuccessful with tasks that involved bridging 1000, apart from correctly stating the number word before 1000 and after 1000. She stated the forward sequence by ones as, "997, 998, 999, ten hundred, ten thousand (pause), ten hundred and one, ten hundred and two", and the backward sequence by ones as, "1002, 1001, 1000, nine-, 999, 989, 998 (as a correction for 989), 997, 996". For the forward sequence by tens she said, "970, 980, 990, 10 000, 10 010, 10 020", and for the forward sequence by hundreds she said, "800, 900 (six-second pause), 1000, 2000, 300, 3000 (as a correction for 300)". For the task of incrementing 999 by 10, she said "10 009", and for the task of decrementing 1005 by 10, she said "905".

Incrementing using tens and ones. On the task with 48 covered, and two ten-strips and five dots uncovered, Robyn counted subvocally, "48, 58, 68, 69 (pause), 70, 71, 72, 73", that is, she used a jump strategy that involved jumping two tens and counting by ones. Robyn was then asked how many more dots (from 73) would be needed to make 100. She made four attempts to solve this task and all of her attempts were unsuccessful. On the first three attempts her strategy was to count by ones from 73, and keep track of her counts on her fingers, but she seemed to lose track after about ten counts. Her fourth attempt appeared to involve a different strategy. She thought for 30 seconds in conjunction with some finger movements, and then answered "906". Thus Robyn was able to count in tens on the task involving addition with strips but not on the missing addend task.

Bare number tasks. Robyn did not solve successfully the three bare number tasks that were presented to her. For 43 + 21, she answered "604", and for part of her solution she counted by ones using her fingers to keep track. For 37 + 19, she answered "406" and for 86 – 24, she answered "994". On all three problems, Robyn appeared to use a split strategy and to recombine the tens and ones unsuccessfully. She apparently did not assess the appropriateness of her answers.

Discussion

Table 1 sets out descriptions of Bel's and Robyn's responses to numeral identification tasks, sequential structure tasks, and additive tasks. On the sequential structure tasks Bel's and Robyn's responses were significantly different from each other. This suggests that students' learning of topics related to sequential structure such as incrementing by ten or 100 on and off the decade and extending this to beyond 1000 can progress in different ways. Robyn's proficiency with jumping by ten off the decade contrasted significantly with Bel's lack of proficiency. However, Robyn did not use jumping by ten on the bare number tasks. Rather, she used split strategies. As well, on the addition task with ten-strips, Robyn was not more proficient than Bel.

Table 1
Summary Descriptions of Bel's and Robyn's Responses to Assessment Tasks

Task	Bel's response	Robyn's response
Numerals	Successful on all but one 4-digit task	Successful for 3-digit numerals
	Sequential structure tasks	
Locating numbers	Correct order but not accurate	Correct order but not accurate
NWS	Five errors	No errors, four self-corrections
NWS by ten: on decade	Successful to 1000	Successful to 120
NWS by ten: off decade	Unsuccessful	Successful to 120
Increment by ten	Unsuccessful off the decade	Successful to 1000
Increment by 100	Successful to 1000	Unsuccessful
Sequences across 1000	All four correct	Unsuccessful
	Additive tasks	
Ten-strips: 48+25	Jump strategy, could not keep track	Jump strategy
Ten-strips: 73+☐=100	Not assessed	Unsuccessful
Written: 43+21, 86 – 24	Split strategy	Split strategy, unsuccessful
Written: 37+19, 50 – 27	Different strategies, unsuccessful	Split strategy, unsuccessful

Bel's and Robyn's solutions to additive tasks indicate, in different ways, knowledge of the base-ten structure of numbers. On tasks involving ten-strips they used jump strategies and were partially successful. Their coordination of tens and ones suggests a sequence-tens and ones conception (Fuson et al., 1997), and a construction of at least a level 2 unit of ten (Cobb & Wheatley, 1988). Robyn's inability to construct a solution to the subsequent missing addend task suggests she had not yet constructed a level 3 unit of ten. On bare number tasks Bel and Robyn used split-based strategies and were less successful. Bel's different approaches to 37 + 19 and 50 – 27 suggest an integrated-sequence-separate conception. Robyn's responses suggest a concatenated single-digit conception of the written numbers, using only a level 1 unit of ten. Cobb and Wheatley (1988) also observed differences in students' responses to bare number tasks compared with tasks involving ten-strips.

On the additive task of 48 and 25 involving ten-strips, both Bel and Robyn counted by ones to add 68 and 5, and these solutions seemed to require significant effort. Bel counted by ones to add 44 and 4 involving ten-strips, even though elsewhere in the interview he solved 4 + 4 immediately (without counting by ones). Also, in the bare number tasks, Bel made errors adding 7 to 9 for 37 + 19, and subtracting 7 from 30 for 50 – 27. Further, in solving addition and subtraction problems in the range 1 to 20 (not described in the above case studies), both students used counting by ones and had difficulties. Thus Bel and Robyn lacked facility with addition and subtraction in the range 1 to 20 and, when doing addition and subtraction in the range 1 to 100, were not able to apply facts in the range 1 to 20 that they had habituated.

Some researchers have linked low-attainers' difficulties such as those described above, with broader aspects of their thinking. Drawing on Gray and Tall (1994), we observe that Robyn and Bel tended to use procedural thinking, which involves counting by ones and splitting, rather than proceptual thinking which involves for example, using 4 + 4 to work out 44 + 4, and coordinating units. Nevertheless, the students' use of jump strategies on the tens-strips tasks seemed to be more appropriate than their use of split strategies on the bare number tasks. Because of this, we contend that their difficulties can be attributed in part to confronting numbers in settings that do not yet make sense to them (Cobb & Wheatley, 1988). Drawing on analyses of mathematical development (Thomas, Mulligan, & Goldin, 2002), we contend that Robyn's and Bel's weak sense of locating numbers indicate low levels of knowledge of mathematical structure, which is linked with low-attainment.

Conclusions

As shown in the two case studies, the process of documenting a student's current arithmetical knowledge in terms of the eight aspects addressed in this study, highlights the complexities of that knowledge and its idiosyncratic nature (Gervasoni, 2005). Students' knowledge of the sequential structure of multi-digit numbers can be regarded as somewhat distinct from their place value knowledge. This refers to place value knowledge in a collections-based sense (Yackel, 2001). We contend that developing in students a rich knowledge of sequential structure is important and can provide an important basis for the development of mental computation.

The case studies confirm that facility with addition and subtraction involving a 1-digit number is a significant aspect of facility with 2-digit calculation (Heirdsfield, 2001). We contend that low-attainers need to develop their facility with 1-digit numbers in order to develop efficient strategies for multidigit calculations. Also confirmed in the case studies, is that students can learn to read and write numerals well in advance of learning place value in a collections-based sense (Wright, 1998). For this reason, we advocate that assessment frameworks should treat numeral identification (reading numerals) and place value (interpreting numerals) as separate domains of knowledge.

As well, the case studies illustrate that a student's mental strategies and number sense can differ from, on one hand, a context involving base-ten materials to, on the other hand, tasks based on bare numbers. This accords with the finding by Cobb and Wheatley (1988) that "the horizontal sentences and tens tasks were separate contexts for the children. The meanings that they gave to two-digit numerals or number words in the two situations were unrelated" (p.18). Related to this, students' strategies for addition and subtraction in bare number contexts can be relatively unsophisticated. Therefore low-attaining students are likely to need explicit instruction in order to extend their multi-digit number sense from contexts involving materials to contexts involving written arithmetic (Beishuizen & Anghileri, 1998; Heirdsfield, 2005; Treffers & Buys, 2001). Finally, the case studies demonstrate the use of assessment tasks to document students' knowledge and that the assessment should include (a) tasks involving base-ten materials, (b) verbally-based tasks, and (c) bare number tasks.

Acknowledgements. The authors gratefully acknowledge the support for this project from the Australian Research Council under grant LP0348932 and from the Catholic Education Commission of Victoria. As well, the authors acknowledge the contribution of Cath Pearn (partner investigator) to this project. Finally, the authors express their sincere thanks to the teachers, students and schools participating in the project.

References

Askew, M., Brown, M., Rhodes, V., Wiliam, D., & Johnson, D. (1997). *Effective teachers of numeracy: Report of a study carried out for the Teacher Training Agency*. London: King's College, University of London.

Beishuizen, M. (1993). Mental strategies and materials or models for addition and subtraction up to 100 in Dutch second grades. *Journal for Research in Mathematics Education, 24*(4), 294-323.

Beishuizen, M., & Anghileri, J. (1998). Which mental strategies in the early number curriculum? A comparison of British ideas and Dutch views. *British Educational Research Journal, 24*(3), 519-538.

Beishuizen, M., Van Putten, C. M., & Van Mulken, F. (1997). Mental arithmetic and strategy use with indirect number problems up to one hundred. *Learning and Instruction, 7*(1), 87-106.

Carpenter, T. P., Franke, M. L., Jacobs, V. R., Fennema, E., & Empson, S. B. (1998). A longitudinal study of invention and understanding in children's multidigit addition and subtraction. *Journal for Research in Mathematics Education, 29*(1), 3-20.

Cobb, P., & Wheatley, G. (1988). Children's initial understandings of ten. *Focus on Learning Problems in Mathematics, 10*(3), 1-26.

Cooper, T. J., Heirdsfield, A., & Irons, C. J. (1995). Years 2 and 3 children's strategies for mental addition and subtraction. In B. Atweh & S. Flavel (Eds.), *Galtha* (Proceedings of the 18th annual conference of the Mathematics Education Research Group of Australasia, Darwin, pp.195-202). Sydney: MERGA.

Department of Education, Training and Youth Affairs (2000). *Numeracy, a priority for all: Challenges for Australian schools*. Canberra: DETYA.

Dowker, A. (2005). *Individual differences in arithmetic: Implications for psychology, neuroscience and education*. Hove, UK: Psychology Press.

Ellemor-Collins, D., & Wright, R. J. (in press). Assessing pupil knowledge of the sequential structure of number. *Educational and Child Psychology, 24*(2).

Foxman, D., & Beishuizen, M. (2002). Mental calculation methods used by 11-year-olds in different attainment bands: A reanalysis of data from the 1987 APU survey in the UK. *Educational Studies in Mathematics, 51*(1-2), 41-69.

Fuson, K. C., Wearne, D., Hiebert, J., Murray, H., Human, P., Olivier, A., Carpenter, T., & Fennema, E. (1997). Children's conceptual structures for multidigit numbers and methods of multidigit addition and subtraction. *Journal for Research in Mathematics Education, 28*, 130-162.

Gervasoni, A. (2005). The diverse learning needs of young children who were selected for an intervention program. In H. L. Chick & J. L. Vincent (Eds.), *Proceedings of the 29th annual conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp.33-40). Melbourne: PME.

Gray, E., & Tall, D. (1994). Duality, ambiguity, and flexibility: A “proceptual” view of simple arithmetic. *Journal for Research in Mathematics Education, 25*(2), 116-140.

Heirdsfield, A. (2001). Integration, compensation and memory in mental addition and subtraction. In M. Van den Heuvel-Panhuizen (Ed.), *Proceedings of the 25th annual conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 129-136). Utrecht, Netherlands: PME.

Heirdsfield, A. (2005). One teacher’s role in promoting understanding in mental computation. In H. L. Chick & J. L. Vincent (Eds.), *Proceedings of the 29th annual conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 113-120). Melbourne: PME.

Hiebert, J., & Wearne, D. (1996). Instruction, understanding, and skill in multidigit addition and subtraction. *Cognition and Instruction, 14*(3), 251-283.

Klein, A. S., Beishuizen, M., & Treffers, A. (1998). The empty number line in Dutch second grades: Realistic versus gradual program design. *Journal for Research in Mathematics Education, 29*(4), 443-464.

Louden, W., Chan, L., Elkins, J., Greaves, D., et al. (2000). *Mapping the territory: Primary students with learning difficulties*. Canberra: DETYA.

McIntosh, A., Reys, B. J., & Reys, R. E. (1992). A proposed framework for examining basic number sense. *For the Learning of Mathematics, 12*, 2-44.

Menne, J. (2001). Jumping ahead: An innovative teaching program. In J. Anghileri (Ed.), *Principles and practices in arithmetic teaching – innovative approaches for the primary classroom* (pp. 95-106). Buckingham: Open University Press.

New South Wales Department of Education and Training. (2003). *Count me in too: Professional development package*. Sydney, NSW: NSW-DET.

Sowder, J. T. (1992). Making sense of numbers. In G. Leinhardt, R. Putnam, & R. A. Hattrup (Eds.), *Analysis of arithmetic for mathematics teaching* (pp. 1-51). Hillsdale, NJ: Lawrence Erlbaum.

Thomas, N. D., Mulligan, J., & Goldin, G. A. (2002). Children's representation and structural development of the counting sequence 1-100. *Journal of Mathematical Behaviour, 21*, 117-133.

Thompson, I., & Bramald, R. (2002). *An investigation of the relationship between young children's understanding of the concept of place value and their competence at mental addition (Report for the Nuffield Foundation)*. Newcastle upon Tyne: University of Newcastle upon Tyne.

Thompson, I., & Smith, F. (1999). *Mental calculation strategies for the addition and subtraction of 2-digit numbers (Report for the Nuffield Foundation).* Newcastle upon Tyne: University of Newcastle upon Tyne.

Treffers, A. (1991). Didactical background of a mathematics program for primary education. In L. Streefland (Ed.), *Realistic mathematics education in primary school* (pp. 21-56). Utrecht: Freudenthal Institute.

Treffers, A., & Buys, K. (2001). Grade 2 and 3 - Calculation up to 100. In M. van den Heuvel-Panhuizen (Ed.), *Children Learn Mathematics* (pp. 61-88). The Netherlands: Freudenthal Institute, Utrecht University.

Wolters, G., Beishuizen, M., Broers, G., & Knoppert, W. (1990). Mental arithmetic: Effects of calculation procedure and problem difficulty on solution latency. *Journal of Experimental Child Psychology, 49*, 20-30.

Wright, R. J. (1998). Children's beginning knowledge of numerals and its relationship to their knowledge of number words: An exploratory, observational study. In A. Olivier & K. Newstead (Eds.), *Proceedings of the 22nd annual conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp.201-208). Stellenbosh, South Africa: PME.

Yackel, E. (2001). Perspectives on arithmetic from classroom-based research in the United States of America. In J. Anghileri (Ed.), *Principles and practices in arithmetic teaching – Innovative approaches for the primary classroom* (pp. 15-31). Buckingham: Open University Press.

Interdisciplinary Modelling in the Primary Mathematics Curriculum

Lyn English
Queensland University of Technology
<l.english@qut.edu.au>

This paper examines one approach to promoting creative and flexible use of mathematical ideas within an interdisciplinary context in the primary curriculum, namely, through modelling. Three classes of fifth-grade children worked on a modelling problem (Australia's settlement) situated within the curriculum domains of science and studies of society and environment. Reported here are the cycles of development displayed by one group of children as they worked the problem, together with the range of models created across the classes. Children developed mathematisation processes that extended beyond their regular curriculum, including identifying and prioritising key problem elements, exploring relationships among elements, quantifying qualitative data, ranking and aggregating data, and creating and working with weighted scores.

Numerous researchers and employer groups have expressed concerns that schools are not giving adequate attention to the understandings and abilities that are needed for success beyond school. Research suggests that although professionals in mathematics-related fields draw upon their school learning, they do so in a flexible and creative manner, unlike the way in which they experienced mathematics in their school days (Gainsburg, 2006; Hall, 1999; Hamilton, in press; Noss, Hoyles, & Pozzi, 2002; Zawojewski & McCarthy, 2007). Furthermore, this research has indicated that such professionals draw upon interdisciplinary knowledge in solving problems and communicating their findings.

The challenge then is how to promote creative and flexible use of mathematical ideas within an interdisciplinary context where students solve substantive, authentic problems that address multiple core learnings. One approach is through mathematical modelling involving cycles of model construction, evaluation, and revision, which is fundamental to mathematical and scientific understanding and to the professional practice of mathematicians and scientists (Lesh & Zawojewski, 2007; Romberg, Carpenter, & Kwako, 2005). Modelling is not just confined to mathematics and science, however. Other disciplines including economics, information systems, social and environmental science, and the arts have also contributed in large part to the powerful mathematical models we have in place for dealing with a range of complex problems (Lesh & Sriraman, 2005; Sriraman & Dahl, in press). Unfortunately, our mathematics curricula do not capitalize on the contributions of other disciplines. A more interdisciplinary and unifying model-based approach to students' mathematics learning could go some way towards alleviating the well-known "one inch deep and one mile wide" problem in many of our curricula (Sabelli, 2006, p. 7; Sriraman & Dahl, in press; Sriraman & Steinthorsdottir, in press). There is limited research, however, on ways in which we might incorporate other disciplines within the mathematics curriculum.

The study reported here represents one attempt to link children's mathematical learning with their learning in other curriculum areas; in the present instance, the focus is on fifth-grade children's developments in solving a modelling problem situated within the curriculum domains of science and studies of society and environment (SOSE). The problem was created in collaboration with the classroom teachers to tie in with the

Proceedings of the 30th annual conference of the Mathematics Education Research Group of Australasia
J. Watson & K. Beswick (Eds), © MERGA Inc. 2007

children's learning of Australia’s settlement. The problem differed from the children's modelling experiences in the previous year of the study in that it comprised mostly qualitative, rather than quantitative, data (see Appendix). Hence one of the research goals was to explore how the children dealt with data of this nature, for example, whether they quantified and/or transformed the data in some way to solve the problem. Another goal was to document the developments in the children's mathematical thinking and learning as they interacted with the problem and with each other in small-group situations. Given that previous research has highlighted the multiple cycles of interpretation that children display in solving such problems (Doerr & English, 2003; English, 2006), it was anticipated that the children would display a diversity of approaches in solving the problem. Finally of interest, were variations in the models the children created with respect to the mathematical ideas constructed and the mathematisation processes applied.

Mathematical Modelling for the Primary School

Modelling is increasingly recognized as a powerful vehicle not only for promoting students’ understanding of a wide range of key mathematical and scientific concepts, but also for helping them appreciate the potential of mathematics as a critical tool for analyzing important issues in their lives, communities, and society in general (Greer, Verschaffel, & Mukhopadhyay, in press; Romberg et al., 2005). Students’ development of powerful models should be regarded as among the most significant goals of mathematics education (Lesh & Sriraman, 2005). Importantly, modelling needs to be integrated within the primary school curriculum and not reserved for the secondary school years and beyond as it has been traditionally. Recent research has shown that primary school children are indeed capable of developing their own models and sense-making systems for dealing with complex problem situations (e.g., English, 2006; English & Watters, 2005).

The terms, models and modelling, have been used variously in the literature, including in reference to solving word problems, conducting mathematical simulations, creating representations of problem situations (including constructing explanations of natural phenomena), and creating internal, psychological representations while solving a particular problem (e.g., Doerr & Tripp, 1999; English & Halford, 1995; Gravemeijer, 1999; Greer, 1997; Lesh & Doerr, 2003; Romberg et al., 2005). As used in the present study, models are “systems of elements, operations, relationships, and rules that can be used to describe, explain, or predict the behavior of some other familiar system” (Doerr & English, 2003, p. 112). From this perspective, modelling problems are realistically complex situations where the problem solver engages in mathematical thinking beyond the usual school experience and where the products to be generated often include complex artifacts or conceptual tools that are needed for some purpose, or to accomplish some goal (Lesh & Zawojewski, 2007).

Mathematical modelling in the primary school presents children with a future-oriented approach to learning. The mathematics they experience differs from what is taught traditionally in the curriculum for their grade level, because different types of quantities and operations are needed to mathematise realistic situations. The types of quantities needed in these situations include accumulations, probabilities, frequencies, ranks, and vectors, whereas the operations needed include sorting, organizing, selecting, quantifying, weighting, and transforming large data sets (Doerr & English, 2003; English, 2006; Lesh, Zawojewski, & Carmona, 2003). Modelling problems thus offer richer learning experiences than the standard classroom word problems (“concept-then-word problem”, Hamilton, in press). In solving such word problems, children generally engage in a one- or two-step process of mapping problem information onto arithmetic quantities and operations. In most cases, the problem information has already been carefully mathematised for the children. Their goal is to unmask the mathematics by mapping the problem information in such a way as to produce an answer using familiar quantities and basic operations. These traditional word problems restrict problem-solving contexts to those that often artificially house and highlight the relevant concept (Hamilton, in press). They thus preclude children from creating their own mathematical constructs.

In contrast, modelling provides opportunities for children to elicit their own mathematics as they work the problem. That is, the problems require children to make sense of the situation so that they can mathematize it themselves in ways that are meaningful to them. This involves a cyclic process of interpreting the problem information, selecting relevant quantities, identifying operations that may lead to new quantities, and creating meaningful representations (Lesh & Doerr, 2003). Because children's final products embody the factors, relationships, and operations that they considered important in creating their model, powerful insights can be gained into the growth of their mathematical thinking.

As previously noted, mathematical modelling provides an ideal vehicle for interdisciplinary learning as the problems draw on contexts and data from other domains (English, in press). The problem addressed in this paper, *The First Fleet*, complemented the children's study of Australia's settlement and incorporated ideas from science and the SOSE curriculum. Dealing with "experientially real" contexts such as the nature of community living, the ecology of the local creek, and the selection of national swimming teams provides a platform for the growth of children's mathematisation skills, thus enabling them to use mathematics as a "generative resource" in life beyond the classroom (Freudenthal, 1973).

Finally, modelling problems support recent studies of peer-directed group work (e.g., Web, Nemer, & Ing, 2006), which have demonstrated the importance of implementing activities that inherently develop students' discourse in cooperative groups. The problems are designed for small-group collaborative work where children are motivated to challenge one another's thinking, and to explain and justify their ideas and actions.

Design and Methodology

This study adopted a multilevel collaborative design (English, 2003), which employs the structure of the multitiered teaching experiments of Lesh and Kelly (2000). Such a design focuses on the developing knowledge of participants at different levels of learning, including the classroom teachers whose participation is an essential factor. At the first level of collaboration (the focus of this paper), children work in small groups to solve the modelling problems. At the second level, their teachers work collaboratively with the researchers in preparing and implementing the activities. At the third level, the researchers observe, interpret, and document the growth of all participants.

Participants and Procedures

Three classes of fourth-grade children (8-9 years) and their teachers took part in the first year of this 3-year study; the children participated again in the second year, along with their new classroom teachers. The classes represented the entire cohorts of fourth and fifth graders from a private K-12 college situated in a regional Queensland suburb.

At the beginning of each year, the teachers participated in half-day workshops on mathematical modelling and its implementation in the classroom. Meetings during the first term of each year were held to plan the three modelling problems to be implemented in the year, and, in the case of the first year of the study, some preliminary modelling activities (e.g., interpreting and using visual representations; conventionalising representations; explaining and justifying mathematical ideas). Each modelling problem was implemented in four 50-minute sessions per remaining term. Where possible, the four sessions were conducted in the same week so that the children did not lose track of their ideas. Planning and debriefing meetings were held with the teachers prior to and following the implementation of each problem.

The present modelling problem, the First Fleet, was implemented at the beginning of the second year of the study and comprised several components. First, the children were presented with background information on the problem context, namely, the British government's commissioning of 11 ships in May, 1787 to sail to "the land beyond the seas". The children answered a number of "readiness questions" to ensure they had understood this background information. After responding to these questions, the children were presented with the problem itself, together with a table of data listing 13 key environmental elements to be considered in determining the suitability of each of five given sites (see Appendix). The children were also provided with a comprehensive list of the tools and equipment, plants and seeds, and livestock that were on board the First Fleet. The problem text explained that, on his return from Australia to the United Kingdom in 1770, Captain James Cook reported that Botany Bay had lush pastures and well watered and fertile ground suitable for crops and for the grazing of cattle. But when Captain Phillip arrived in Botany Bay in January 1788 he thought it was unsuitable for the new settlement. Captain Phillip headed north in search of a better place for settlement. The children's task was as follows.

> Where to locate the first settlement was a difficult decision to make for Captain Phillip as there were so many factors to consider. If you could turn a time machine back to 1788, how would you advise Captain Phillip? Was Botany Bay a poor choice or not? Early settlements occurred in Sydney Cove Port Jackson, at Rose Hill along the Parramatta River, on Norfolk Island, Port Hacking, and in Botany Bay. Which of these five sites would have been Captain Phillip's best choice? Your job is to create a system or model that could be used to help decide where it was best to anchor their boats and settle. Use the data given in the table and the list of provisions on board to determine which location was best for settlement. Whilst Captain Phillip was the first commander to settle in Australia many more ships were planning to make the journey and settle on the shores of Australia. Your system or model should be able to assist future settlers make informed decisions about where to locate their townships.

The children worked the problem in groups of three to four with no direct teaching from the teachers or researchers. In the final session, the children presented group reports on their models to their peers, who, in turn, asked questions about the models and gave constructive feedback.

Data Collection and Analysis

In each classroom, one group of children was video-taped and audio-taped and another group was audio-taped in each session, with all data subsequently transcribed. All of the children's group reports to the class and their responses to their peers' comments were also video-taped and transcribed. Other data sources included classroom field notes and all of the children's artefacts. All of the data were reviewed several times for evidence of: (a) children's initial interpretation and re-interpretations of the problem components; (b) cycles of mathematical development as the children created their models, including how the children operationalised the given data and ways in which they documented their actions; and (c) diversity in their approaches and model creation. This paper addresses the cycles of mathematical development displayed by one group of children (Mac's group) in working the problem and then summarises the range of models developed across the three classes.

Results

Cycles of Development Displayed by one Group of Children

Cycle 1: Prioritising and assessing elements. Mac's group commenced the problem with Mac stating, "So, to find out, OK, if we're going to find the best place I think the most important thing would be that people need to stay alive." The group then proceeded to make a prioritised list of the elements that would be most needed. There was substantial debate over which elements to select, with fresh water, food (fishing and animals), protective bays, and soil and land being chosen. However, the group did not remain with this selection and switched to a focus on all 13 elements listed in the table of data.

The children began to assess the elements for the first couple of sites by placing a tick if they considered a site featured the element adequately and a cross otherwise. The group then began to aggregrate the number of ticks for each site but subsequently reverted to their initial decision to focus just on the most essential elements ("the best living conditions to keep the people alive"). Still unable to reach a consensus on this issue, the group continued to consider all of the elements for the remaining sites and rated them as "good" and "not so good". The children explained that they were looking for the site that had "the most good things and the least bad things".

Cycle 2: Ranking elements across sites. Next, the group attempted a new method: they switched to ranking each element, from 1 ("best") to 5, across the five sites, questioning the meaning of some of the terminology in doing so. The children also questioned the number of floods listed for each site, querying whether it represented the number of floods per year or over several years. As the children were ranking the first few elements, they examined the additional sheet of equipment etc. on board the First Fleet to determine if a given site could accommodate all of the provisions and whether anything else would be needed for the settlement. The group did not proceed with this particular ranking system, however, beyond the first few elements.

Cycle 3: Proposing conditions for settlement and attempting to operationalise data. The children next turned to making some tentative recommendations for the best sites, with Mac suggesting they create conditions for settlement.

> …like if you had not much food and not as many people you should go to Norfolk Island; if you had a lot of people and a lot of food you should go to Sydney Cove or um Rosehill, Parramatta.

The group then reverted to their initial assessment of the elements for each site, totalling the number of ticks ("good") and crosses ("bad") for each site. In doing so, the children again proposed suggested conditions for settlement.

> And this one with the zero floods (Norfolk Island), if you don't have many people that's a good one cause that's small but because there's no floods it's also a very protected area. Obviously, so maybe you should just make it (Norfolk Island) the best area.

The group devoted considerable time to debating conditions for settlement and then made tentative suggestions as to how to operationalise the "good" and the "bad". One child suggested finding an average of "good" and "bad" for each site but his thinking here was not entirely clear and the group did not take up his suggestion.

> We could find the average, I mean as in like, combine what's bad, we add them together; we can combine how good we think it might be out of 10. Then we um, could divide it by how many good things there is [sic] and we could divide it by how many bad things there is [sic].

Finding themselves bogged down here, the group turned to a new approach.

Cycle 4: Weighting elements and aggregating scores. This new cycle saw the introduction of a weighting system, with the children assigning 2 points to those elements they considered important and 1 point to those elements of lesser importance ("We've valued them into points of 1 and 2 depending on how important they are"). Each site was then awarded the relevant points for each element if the group considered the site displayed the element; if the site did not display the element, the relevant number of points was subtracted. As the group explained,

> The ones (elements) that are more important are worth 2 points and the ones that aren't are 1. So if they (a given site) have it you add 2 or 1, depending on how important it is, or you subtract 2 or 1, if they don't have it.

The children totalled the scores mentally and documented their results as follows (1 refers to Botany Bay, 2 to Port Jackson, and so on):

1 -12 + 10 = -2
2 - 9 + 13 = 4
3 - 5 + 17 = 12
4 - 7 + 15 = 8
5 - 9 + 13 = 4

Cycle 5: Reviewing methods and finalising site selection. The group commenced the third session the next morning by reflecting on the two main methods they had employed to determine the best site, namely, the use of ticks ("good") and crosses ("bad") in assessing elements for each site and trying to operationalise these data, and the weighting of elements and aggregating of scores. Mac commenced by reminding his group of what they had found to date.

> Yesterday we, um, OK, the first thing we did yesterday showed us that the fifth one (Norfolk Island) was the best place, second one (weighting of elements) we did told us … showed us that number three (Rosehill, Parramatta) was the best. So it's a tie between number three and number five. So it's limited down to them, work it out. Hey guys, are you even listening?

After bringing the group back on task, Mac stated, "OK, we're doing a tie-breaker for number three and number five." The group proceeded to revisit their first method, assigning each tick one point and ignoring the crosses. However, on totalling the points, Mac claimed that Rosehill, Parramatta, was the winning site. Bill expressed concern over the site's record of 40 floods and this resulted in subsequent discussion as to whether

Parramatta should be the favoured site. The children finally decided on Norfolk Island because it was flood-free and because it was their choice using their first main method.

Diversity of Models Created Across all Groups

The children's models varied in mathematical sophistication, from limited use of mathematisation processes through to various scoring and ranking systems that included the use of weighted scores as above. Other models across the classes included the following.

Model 1. This was the most common model that was generated across the classes. It entailed taking each site in turn and assessing whether it adequately displayed all or a selection of the elements. Children used ticks, crosses, and highlighting on the given table of data and took a subsequent tally of each site. The site with the highest tally was selected as the place for settlement. As one group explained, "We're just highlighting the best and then we're going to see how many highlighted ones there are (for each site)." Another group explained, "The least bad and the highest good is the best."

Model 2. Here, children selected and prioritized elements to consider for each site ("We chose six things that we thought were important and made a graph"). The children in one group ranked "accessible by sea" as no. 1, "fresh water" as no. 2, "soil quality" as no. 3, "bush tucker" as no. 4, "land available" as no. 5, and "land suitable for livestock" as no. 6. Each site was then assessed in terms of these elements. The site that displayed the most favoured of these elements was chosen (the site that had the "best out of these categories").

Model 3. The third model was an advance on the previous models. Children rated selected elements (accessible by sea, fresh water, soil quality, trees and plants, and local bush tucker) for each site as "very good", "good", "OK", and "bad". The number of times each category appeared for each site was tallied and the site that had the highest tally for the "very good" category was chosen.

Model 4. This model extended model 3. Each of the 13 elements was ranked in turn from 1 to 5 across the five sites (1 = best). The site with the highest number of ranks of one was chosen as the most suitable site.

Model 5. The fifth model extended the previous two models by awarding 3 ticks for "very good", 2 ticks for "good", 1 tick for "average", and a cross for "bad". The site with the highest number of ticks was the chosen site. On totalling the number of ticks, one group claimed the score was "out of 13".

Model 6. This model incorporated a scoring system where each element for each site was assessed and given a score out of 10 or out of 13. The group that used the 10-point system reported to the class as follows.

> Our strategy was using a point score. We did a rating out of 10 for the data headlines, in the importance of, like 10 out of 10. And down the scale we went. We then rated the answers, like accessible by sea, we rated like, accessible by sea, we rated 9 out of 10 for importance. The answer going down the column would only go up to the highest of 9, because it was 9 out of 10. We did this for the whole graph (table), then for the 5 places here we added up the total scores. We ended up with 39 for Botany Bay, 62 for Sydney Cove, Port Jackson, 77 for Rosehill, Parramatta, 66 for Port Hacking and 70 for Norfolk Island. We chose the highest rating; it was Rosehill.

Discussion and Concluding Points

This study represents one approach to introducing interdisciplinary modelling problems into the primary mathematics curriculum. Mathematical modelling has traditionally been confined to the secondary school and beyond, yet this study and other research have shown that such problems contribute effectively to primary school children's learning in several domains. Such problems allow for a diversity of solution approaches and enable children of all achievement levels to participate in, and benefit from, these experiences. In contrast to traditional classroom problem solving, these modelling problems facilitate different trajectories of learning, with children's mathematical understandings developing along multiple pathways. Importantly, children direct their own mathematical learning. That is, they elicit key ideas and processes from the problem as they work towards model construction. In the present case, the children identified and prioritized key problem elements, explored relationships between elements, quantified qualitative data, ranked and aggregated data, and created and worked with weighted scores—before being formally introduced to mathematisation processes of this nature.

Modelling problems engage children in multiple cycles of interpretations and approaches, suggesting that real-world, complex problem solving goes beyond a single mapping from givens to goals. Rather, such problem solving involves multiple cycles of interpretation and re-interpretation where conceptual tools evolve to become increasingly powerful in describing, explaining, and making decisions about the phenomena in question (Doerr & English, 2003). Furthermore, these phenomena can be drawn from a wide range of disciplines.

The interdisciplinary nature of mathematical modelling means that we can create problems that can help unify some of the myriad core ideas within the primary curriculum. For example, problems that incorporate key concepts from science (English, in press) and SOSE can help children appreciate the dynamic nature of environments and how living and non-living components interact, the ways in which living organisms depend on others and the environment for survival, and how the activities of people can change the balance of nature. The First Fleet problem can also lead nicely into a more in-depth study of the interrelationship between ecological systems and economies, and a consideration of ways to promote and attain ecologically sustainable development.

Finally, the inherent requirement that children communicate and share their mathematical ideas and understandings, both within a small-group setting and in a whole-class context, further promotes the development of interdisciplinary learning. The problems engage children in describing, explaining, debating, justifying, predicting, listening critically, and questioning constructively—which are essential to all discipline areas.

References

Doerr, H. M., & English, L. D. (2003). A modeling perspective on students' mathematical reasoning about data. *Journal for Research in Mathematics Education, 34*(2), 110-137.

Doerr, H. M., & Tripp, J. S. (1999). Understanding how students develop mathematical models. *Mathematical Thinking and Learning, 1*(3), 231-254.

English, L. D. (in press). Mathematical modelling: Linking Mathematics, Science, and the Arts in the Elementary Curriculum. In A. Beckmann, C. Michelsen, & B. Sriraman (Eds.), *Proceedings of The Second International Symposium of Mathematics and its Connections to the Arts and Sciences*. University of Southern Denmark.

English, L. D. (2003). Reconciling theory, research, and practice: A models and modeling perspective. *Educational Studies in Mathematics, 54(*2 & 3), 225-248.

English, L. D. (2006). Mathematical modeling in the primary school: Children's construction of a consumer guide. *Educational Studies in Mathematics, 62*(3), 303-323.

English, L. D., & Halford, G. S. (1995). *Mathematics education: Models and processes*. Mahwah, NJ: Lawrence Erlbaum Associates.

English, L. D., & Watters, J. J. (2005). Mathematical modeling in the early school years. *Mathematics Education Research Journal*, 16(3), 58-79.

Freudenthal, H. (1973). *Didactical phenomenology of mathematical structures*, Boston: Kluwer.

Gainsburg, J. (2006). The mathematical modeling of structural engineers. *Mathematical Thinking and Learning, 8*(1), 3-36.

Gravemeijer, K. (1999). How emergent models may foster the construction of formal mathematics. *Mathematical Thinking and Learning, 1*, 155-177.

Greer, B. (1997.) Modeling reality in mathematics classroom: The case of word problems. *Learning and Instruction 7*, 293-307.

Greer, B., Verschaffel, L., & Mukhopadhyay, S. (in press). Modelling for life: Mathematics and children's experience. In W. Blum, W. Henne, & M. Niss (Eds.), *Applications and modelling in mathematics education* (*ICMI Study 14*). Dordrecht: Kluwer.

Hall, R. (1999). *Case studies of math at work: Exploring design-oriented mathematical practices in school and work settings* (NSF Rep. No. RED-9553648), Arlington, VA: National Science Foundation.

Hamilton, E. (in press). What changes are needed in the kind of problem solving situations where mathematical thinking is needed beyond school? In R. Lesh, E. Hamilton, & J. Kaput (Eds.), *Modeling as foundations for the future in mathematics education*. Mahwah, NJ: Lawrence Erlbaum.

Lesh, R., & Doerr, H. M. (Eds.). (2003). *Beyond constructivism: Models and modeling perspectives on mathematic problem solving, learning and teaching*. Mahwah, NJ: Lawrence Erlbaum.

Lesh, R. A., & Kelly, A. E. (2000). Multi-tiered teaching experiments. In R. A. Lesh & A. Kelly (Eds.), Handbook of research design in mathematics and science education (pp. 197-230). Mahwah, NJ: Lawrence Erlbaum.

Lesh, R. A., & Sriraman, B. (2005). John Dewey revisited—pragmatisim and the models-modeling perspective on mathematical learning. In A. Beckmann, C. Michelsen, & B. Sriraman (Eds.). *Proceedings of the 1st international symposium of mathematics and its connections to the arts and sciences* (pp. 7-31). Schwöbisch Gmund, Germany: The University of Education.

Lesh, R., & Zawojewski, J. S. (2007). Problem solving and modeling. In F. Lester (Ed.), *Second handbook of research on mathematics teaching and learning*. Greenwich, CT: Information Age Publishing.

Lesh, R., Zawojewski, J. S., & Carmona, G. (2003). What mathematical abilities are needed for success beyond school in a technology-based age of information? In R. Lesh & H. Doerr (Eds.)., *Beyond constructivism: Models and modeling perspectives on mathematic problem solving, learning and teaching* (pp. 205-222). Mahwah, NJ: Lawrence Erlbaum.

Noss, R., Hoyles, C., & Pozzi, S. (2002). Abstraction in expertise: A study of nurses' conceptions of concentration. *Journal for Research in Mathematics Education, 33*(3), 204-229.

Romberg, T. A., Carpenter, T. P., & Kwako, J. (2005). Standards-based reform and teaching for understanding. In T. A. Romberg, T. P. Carpenter, & F. Dremock (Eds.), *Understanding mathematics and science matters* (pp. 3-28). Mahwah, NJ: Lawrence Erlbaum.

Sabelli, N. H. (2006). Complexity, technology, science, and education. *The Journal of the Learning Sciences, 15*(1), 5-9.

Sriraman, B., & Dhal, B. (in press). On bringing interdisciplinary ideas to gifted education. In L.V. Shavinina (Ed), *The International Handbook of Giftedness*. Springer Science.

Sriraman, B., & Steinthorsdottir (2007, in press). Implications of research on mathematics gifted education for the secondary curriculum. In C. Callahan & J. Plucker (Eds.), *What research says: Encyclopedia on research in gifted education.* Austin, TX: Prufrock Press.

Web, N. M., Nemer, K. M., & Ing, M. (2006). Small-group reflections: Parallels between teacher discourse and student behaviour in peer-directed groups. *The Journal of the Learning Sciences, 15*(1), 63-120.

Zawojewski, J., & McCarthy, L. (2007). Numeracy in practice. *Principal Leadership, 7*(5), 32-38.

Appendix: First Fleet Activity Data Table

First Fleet Activity Data Table

	Accessible by sea	Shark infested waters	Land available for future growth	Able to transport harvested or manufactured items from site	Soil quality	Land suitable for livestock	Trees & plants	Local bush tucker	Fresh water availability	Fishing	Ave temp	Ave monthly rainfall	Records of floods
Botany Bay, NSW	Sea coast over 47km long, open and unprotected	Yes	Yes	Yes by boat & land	Damp, swampy land, may lead to disease, mud flats	Dry	Very large hardwood trees, can't cut down with basic tools	Emu, kangaroo, cassowary, opossum, birds	Small creek to north but low swamp land near it	Yes but unskilled men can only fish from a boat	18°	98mm	3
Sydney Cove, Port Jackson, NSW	Deep water close to shore, sheltered	Yes	Yes	Yes by boat & land	Unfertile, hot, dry even sandy in parts	Rank grass fatal to sheep & hogs, good for cattle & horses	Very large hardwood trees, Red & Yellow Gum, can't cut down with basic tools	Emu, kangaroo, cassowary, opossum, birds, wild ducks	Tank Stream flowing & several springs	Yes but unskilled men can only fish from a boat	18°	98mm	7
Rosehill, Parramatta, NSW	Yes 25km inland up the Parramatta River	No	Yes	By land only	Rich, fertile, produces luxuriant grass	Good for all	Smaller more manageable trunks, hoop & bunya pines – softwood	Plentiful, including eels	On the Parramatta River	Yes but unskilled men can only fish from a boat	18°	98mm	40
Port Hacking, NSW	35km south of Sydney, sheltered port	Yes	Yes	Yes by boat & land	Able to support a variety of natural vegetation	Good for all	Abundant eucalypt trees, ficus, mangroves	Plentiful	On Port Hacking River	Yes but unskilled men can only fish from a boat	18°	133mm	8
Norfolk Island	32km of coastline inaccessible by sea except one small cove, extremely rocky shore and cliffs	Yes	3,455 hectares in total	Only crops not wood due to small cove	Far superior to others, suitable for grain & seed	Good for goats, sheep, cattle & poultry	Yes, pines and flax plant	Green turtles, petrel birds, guinea fowl, flying squirrel, wild ducks, pelican & hooded gull	Exceedingly well watered	Yes but unskilled men can only fish from a boat	19°	110mm	0

Students' Tendency to Conjoin Terms: An Inhibition to their Development of Algebra

Judith Falle
University of New England
<judith.falle@une.edu.au>

When students' responses to a test of introductory algebra items were Rasch modelled, three distinct "ability" clusters occurred. The question then arose as to the mathematical thinking that could characterise each of these groups. Data from the test revealed that the tendency to conjoin terms inappropriately occurred with different frequencies in each of the three groups. Interview data and error analyses provided further insight into the students' thinking that resulted in these types of errors. Implications for classroom practice are considered.

Many students find the demands of shifting their thinking from arithmetic to algebra challenging and, perhaps, frustrating in its strangeness. This is evident from the errors made by students, and the underlying misconceptions held by the students. Many of these misconceptions arise from students' arithmetic experiences that they (with a certain logic) generalise to their new experiences of algebra (MacGregor & Stacey, 1997). These errors seem to persist across the grades, despite increased exposure to algebra. If these errors can be understood as resulting from students' incorrect generalisation from previous (arithmetic) learning rather than as being symptomatic of cognitive immaturity (MacGregor & Stacey, 1994), then they may be addressed, once identified, by appropriate teaching methods (Tirosh, Even, & Robinson, 1998; Hall, n.d.; Tall, 1994).

One type of error made by students beginning algebra is that which arises from students' tendency to conjoin terms inappropriately (i.e., $5x + 3$ is written as $8x$). The tendency can be attributed to various causes, such as: students wanting to "close" or "finish" an algebraic expression (Booth, 1984, 1988; Tirosh et al., 1998; Hall, n.d.); students making false generalisations from an arithmetic context (e.g., 30 + 4 becomes 34, or, 3 + 1/4 becomes 31/4 (Matz, 1982)); or students interpreting brackets in an expression as indicating that the expression inside the brackets is to "be done first" (e.g., when $2(x + 5)$ becomes $10x$) (Linchevski & Herscovics, 1994). The tendency for students to conjoin terms inappropriately appears when they first encounter algebra. If this remains unremarked, and uncorrected, and possibly masked as students deal with more complex algebraic expressions, further development of their algebraic understanding must be inhibited.

The question addressed in this paper is whether students' ability, as measured by their success on a test of algebraic techniques is associated with their tendency to conjoin terms. The discussion draws on data from items in a test given to students as part of a study of their thinking as they carried out simple algebraic techniques. Only the data from students' responses to particular items in the test are discussed in this paper. The items under consideration are those in the test that required students to simplify expressions by collecting like terms or first expanding brackets and collecting like terms, as well as "semi-literal" items that required students to rewrite an algebraic statement[1]. The data discussed

[1] The term "semi-literal" is used to describe items that ask for an algebraic form of a statement, that still uses some numbers. These items are those used, or similar to those used, by Küchemann (in Hart, 1981).

in this paper are a small part of the data collected during the main study, which is described in the methodology.

Methodology

Data Collection

The main study involved participants from three private secondary schools in a regional town (n = 222). The participants were students from Years 8 and 9 when the study began. These students were in the second and third years of secondary school, and so had been studying algebra for two or three years. The study aimed to find associations between language structures used by students to describe their thinking as they carried out various types of algebraic processes and their mathematical ability. The study consisted of two phases. The first phase was a test consisting of forty items based on the beginning algebra techniques outlined in the *Mathematics 7 – 10 Syllabus* (Stage 4, Board of Studies NSW, 2002) and associated textbooks used by the participating schools. Also included, to provide a well-documented basis for comparison, were items from Küchemann's study (1981), or adaptations of those items. The tests were administered in Term 4 of the school year by the class teachers and collected and marked by the researcher. The results were Rasch modelled using QUEST software (Adams & Khoo, 1994).

The second phase of the study consisted of interviews with students from each of the schools. Because of organisational constraints, this phase occurred in the first term of the year following the test. Students were selected for interview on the basis of their test performance so that a range of abilities would be represented at the interviews. The students who were finally interviewed were those for whom the relevant permission and consent had been obtained, and who were available at times suitable to the school, the teachers, and the researcher. These students were representative of the range of abilities as described by the Rasch model.

The interviews were structured using the test items grouped according to syllabus topic areas (Stage 4, Board of Studies NSW, 2002). Students were interviewed individually using a prepared protocol of questions supplemented by further probes or prompts or requests for clarification by the interviewer, depending on the response given to the initial question. The students were presented with each group of items, one group at a time, and asked the initial stimulus question, "What goes on in your head when you see questions like these?" Responses were audio-taped, and transcribed for later analysis.

Results from the interviews were used to complement the test responses. A particular aspect of those responses, namely the conjoining of terms and the language used by students during the interviews, is discussed in this paper.

Data Analysis

Test Items

The test items were marked and the results analysed using Rasch modelling, and later, an analysis of errors. The test responses were coded as either correct (1) or incorrect (0). Test items were marked by the researcher. Only algebraically "complete" answers were marked as correct. Responses where intermediate steps only were written were also counted as "incorrect", as were those instances where students left a blank (baulk).

The Rasch model uses dichotomous data (e.g., correct/incorrect) from a set of items that test a single construct (unidimensional). Item difficulty and participant ability scores are based on a probabilistic scale of successful response to each item by each participant.

Rank order of item difficulty and participant ability are then mapped on the same equal interval scale in *logits* (scale units) (Bond & Fox, 2001). The software used to model the data (QUEST, Adams & Khoo, 1994) enables the reliability of the data, and the extent to which each item fits the construct, to be calculated. These statistics are summarised in Figure 1. Reliability of the *item difficulty estimates* was calculated at 0.99, and of *student ability estimates* at 0.93.

Summary of Item Difficulty Estimates and Fit Statistics					
Item Difficulty Estimates		Item Fit Statistics			
Mean	0.00	**Infit Mean Square**		**Outfit Mean Square**	
SD	1.75	Mean	1.00	Mean	
SD (adjusted)	1.74	SD			
Reliability of estimate	0 .99	**Infit *t***		**Outfit *t***	
		Mean	-0.05	Mean	-0.05
		SD			
0 items with zero scores		0 items with perfect scores			
Summary of Case Ability Estimates and Fit Statistics					
Case Ability Estimates		*Case Fit Statistics*			
Mean	-0.64	**Infit Mean Square**		**Outfit Mean Square**	
SD	1.89	Mean	0.99	Mean	0.99
SD (adjusted)	1.83	SD	0.22	SD	0.77
Reliability of estimate	0.93	**Infit *t***		**Outfit *t***	
		Mean	0.02	Mean	0.17
		SD	0.94	SD	0.73
0 cases with zero scores		0 cases with perfect scores			

Figure 1: Summary statistics for item difficulty and case ability estimates (QUEST, Adams & Khoo, 1994).

The scale of item difficulty and student ability ranged from –5 logits to +5 logits with the mean set at 0. A student with an ability estimate that is the same as the difficulty level of a particular item has a 50% chance of correctly answering that item. Students with an ability estimate greater than the difficulty level of an item have a better than 50% chance of answering that item, in proportion to the linear scale difference.

The software also produces a map of student ability (case estimates) and item difficulty (item estimates). The map, in Figure 2, is a modified version of that produced by the QUEST software. It illustrates a developmental hierarchy of student understanding (ability estimates, designated by an “x” to the left of the vertical line) and concept difficulty (item difficulty estimates, represented by item numbers to the right of the vertical line) within the construct being tested. The construct in this instance is that of algebra.

Distinct clusters of item difficulty and student ability are apparent. There are three main clusters of items (numbers corresponding to items in the test to the right of the centre line in Figure 2). Cluster 1, consisting of 7 items, has a mean difficulty estimate of -2.7 logits; Cluster 2, containing 21 items, has a mean difficulty estimate of -0.32 logits, and Cluster 3, containing 12 items, has a mean difficulty estimate of 2.09 logits. The differences in the means of difficulty estimates for each cluster are significant at the $p<0.05$ level. There are also three distinct clusters of student ability (shaded “x” clusters to the left of the centre line in Figure 2). These clusters are labelled Ability Groups. The mean for Ability Group 1 is -2.34 logits; for Ability Group 2, -0.15 logits; and, for Ability Group 3, 2 logits. These means are significantly different at the $p < 0.05$ level, and closely align with those of the item difficulty means for each of the clusters of items (no significant difference). These data are summarised in Figure 3.

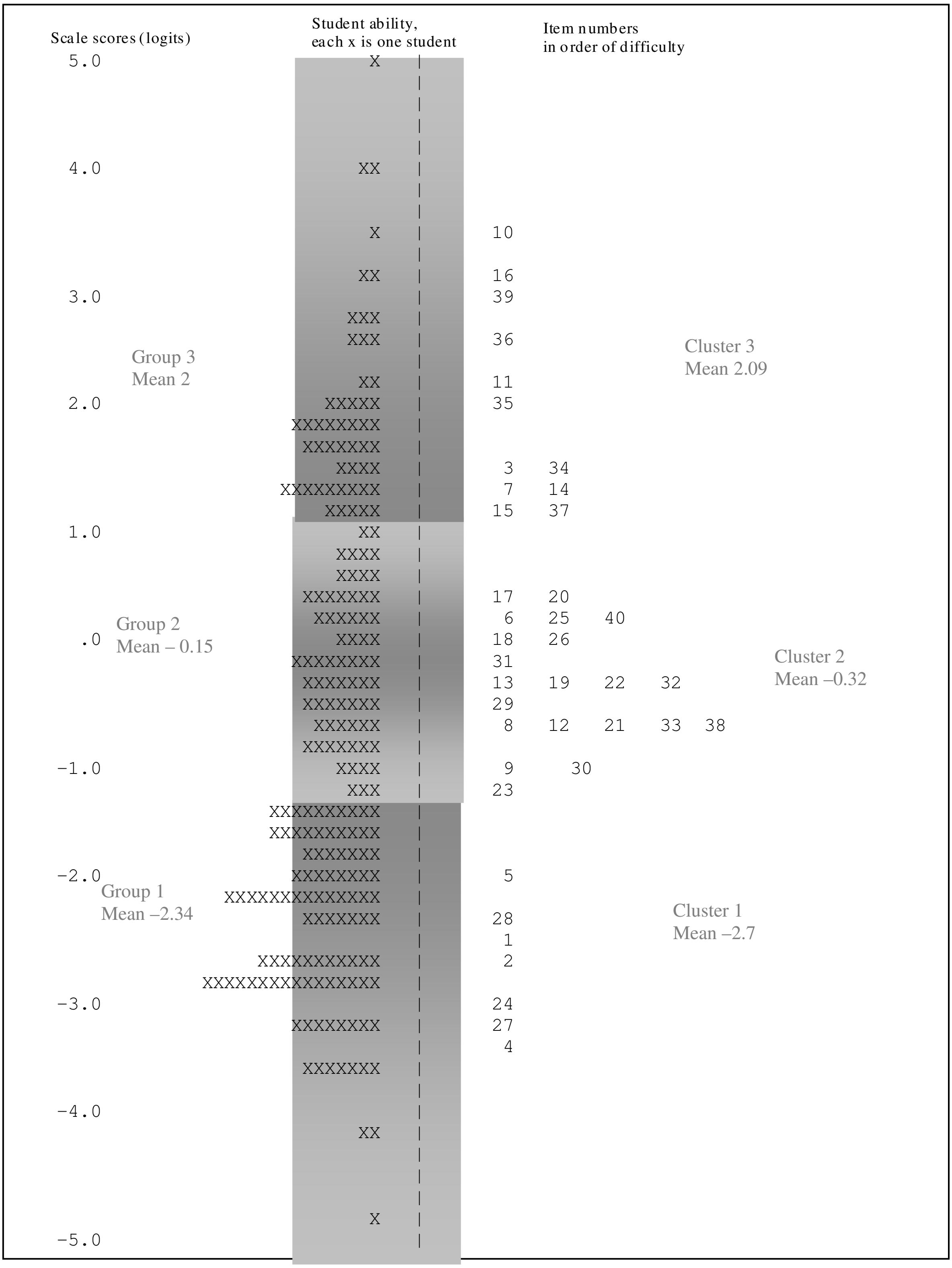


Figure 2: Map of Rasch modelling of algebra test items, showing clusters of items and clusters of student ability estimates (modified from QUEST print out).

Ability Group	**1**	**2**	**3**	**t-test: group ability means**	
Ability Range (Logits)	-1.28 to – 4.89	0.93 to -1.1	4.93 to 1.1	**Group 1-2**	21.75
Number in Group	102	69	52	**Group 2-3**	24.28
Mean ability	-2.34	-0.1472	1.995		
Item Cluster	1	2	3 & 4	**t-test: item difficulty means**	
Difficulty Range (Logits)	-3.27 to –1.98	-1.17 to 0.46	1.18 to 3.56	Cluster 1 - 2	20.49
Mean Difficulty	-2.7	-0.302	2.09	Cluster 2 – 3&4	18.76
t-test: group ability means/item difficulty means	1.8	1.72	0.664		

Figure 3: Summary of student ability group means, item difficulty cluster means and t-test significance at $p<0.05$.

Responses from the Test Scripts

The test responses were also analysed for the types of incorrect responses and the frequency of occurrence of those errors. Blank responses (baulks) were counted separately from other, written, incorrect responses. These data are described only for those responses pertinent to the discussion in this paper. Errors resulting from misreading or misapplication of signs were not considered. Nor were errors resulting from an inability to distribute the multiplier correctly over the brackets and then collect like terms considered. Responses by students are described firstly with respect to the interview sets, and then with respect to the student ability groups.

Responses with respect to the interview sets. The items from the forty-item test that are here discussed were included in interview Sets 1, 3 and 8. These sets are listed in Figure 4, where the particular items are identified, together with their Rasch difficulty estimates.

Set 1: Simplify			**Set 3: Simplify**		
Item No	*Item*	*Difficulty*	*Item No*	*Item*	*Difficulty*
1	3m + 8 + 2m - 5	- 2.53	7	(a – b) + b	1.38
2	5p – p + 1	- 2.6	11	8p – 2(p + 5)	2.28
5	2ab + 3b + ab	-1.98	18	2(x + 4) + 3(x – 1)	0.13
6	5a – 2b + 3a + 3b	0.33	19	2(x + 5) - 8	- 0.27

Set 8: Read aloud and tell me how the following could be rewritten?		
Item No	*Item*	*Difficulty*
20	Multiply x + 5 by 4	0.46
21	Add 4 on to n +5	-0.58
22	Add 3 on to 4n	-0.34
25	Take n away from 3n + 1	0.2
26	If p + q = 5, then p + q + r =?	0.07

Figure 4: Items where students conjoined terms arranged in the sets used in interviews, with Rasch difficulty estimates.

For items in Set 1, the number of baulks was very low – from one only for Item 2 [$5p - p + 1$], to nine for Item 6 [$5a - 2b + 3a + 3b$]. For Set 3 the number of baulks was greater, on average, 36 per item. In both sets 1 and 3, the number of Year 8 students who gave no response, was almost the same as the number of Year 9 students who also baulked. For Set 8 baulk numbers varied from 46 on Item 26 [If $p + q = 5$, then $p + q + r =$?] to more than 20 for Items 21 [Add 4 on to $x + 5$], 22 [Add 3 on to $4n$], and 25 [Take n away from $3n + 1$]. Baulk numbers were higher for items requiring some multiplicative reasoning that also involved the use of brackets, such as Item 20 [Multiply $x + 5$ by 4], or for those requiring logical, but arithmetic, deduction, such as Item 26. In this set, more Year 8 students gave no response than Year 9 students. (e.g., there were 40 baulks for Item 20, 30 of which were Year 8 students, 10 Year 9.)

Some of the most common errors in Set 1 were those in which students conjoined terms inappropriately. For Item 1, 17 responses (out of the 50 errors) were given as $8m$; in the case of Item 2, 38 of the 49 errors involved responses such a 6, $5p$ or $6p$. Item 5 elicited a greater variety of errors than other items in the set; there were 65 incorrect responses, and 33 different responses. The most common error however, involved the conjoining of terms, although there were many different representations. The conjoining of terms was not a common erroneous response to Item 6, and only students in Ability Group 1 gave such responses.

In Set 3, the most common errors were not those that involved the conjoining of terms in Items 7 and 11. However, the conjoining of terms as responses to Items 18 and 19 was common. Item 18 elicited a considerable variety of errors (60 different versions out of 103 incorrect responses), many of which involved conjoined terms either within the brackets, or as a final answer. Item 19 elicited 86 errors, with 17 of those being the response $15x$. Other individual answers also involved the conjoining of terms.

In Set 8, the conjoining of terms was a common error, particularly for students in Ability Groups 1 and 2.

Responses to test items with respect to ability groups. The patterns arising from the error analysis are reflected in the patterns of student responses when considered by the ability groupings of the Rasch model (see Figure 3 and Figure 2). These data are summarised in Figure 5. All errors that are considered the result of terms being inappropriately conjoined are included in the raw numbers. The Rasch difficulty estimates, in logits, are those calculated using QUEST Software (Adams & Khoo, 1994).

Item	Rasch difficulty	Group 1		Group 2		Group 3	
		Numbers	%	Numbers	%	Numbers	%
1	-2.53	*24*	24	*1*	1	*0*	0
2	-2.6	*29*	28	*2*	3	*0*	0
5	-1.98	*24*	24	*4*	6	*3*	6
6	0.33	*19*	19	*0*	0	*0*	0
7	1.38	*14*	14	*5*	7	*0*	0
11	2.28	*31*	30	*14*	20	*2*	4
18	0.13	*34*	33	*4*	6	*0*	0
19	-0.27	*34*	33	*5*	7	*0*	0
20	0.46	*42*	41	*20*	29	*3*	6
21	-0.58	*47*	46	*15*	22	*0*	0
22	-0.34	*69*	68	*20*	29	*1*	2
25	0.2	*57*	56	*20*	29	*4*	8
26	0.07	*23*	23	*5*	7	*1*	2
Students in each group		n = 102		n = 69		n = 52	

Figure 5: Numbers of students who incorrectly conjoined terms in response to items, by ability group and item number [The items are arranged in groups as presented in interviews (see Figure 4). The "groups" are ability groups (see Figure 3).]

Given items in Set 1, students in the Ability Group 1 (mean ability: -2.34 logits) tended to conjoin terms in this set of some of the least difficult items (mean difficulty estimate: -1.7 logits), which required terms to be added and subtracted. No student in the Ability Group 3 (mean ability estimate: 2 logits) did so; with the exception of Item 5 [$2ab+3b+ab$] Ability Group 2 did so (mean ability estimate: -0.15 logits). Item 5 also elicited the greatest number of errors and the greatest variety of incorrect responses that indicated misconceptions and confusions other than that of the appropriateness of conjoining terms.

When required to expand brackets, as in Set 2 (mean difficulty estimate: 0.88 logits), students in Ability Group 2 also tended to conjoin terms, particularly with Item 11 [$8p–2(p+5)$], but not to the extent evident for those in Group 1. Few students in Ability Group 3 did so. The greatest number of conjoining errors occurred with Item 11, although these were of such a varied nature that no particular response could be counted as occurring with great frequency.

The third set of items discussed here were those in Set 8 (mean difficulty estimate: -0.38 logits) of the interview. These items were adapted, or used unchanged, from those in the study by Küchemann (1981). It was in response to these items that the greatest number of conjoining errors occurred in each of the three groups. The absolute numbers remained small in the case of students in Ability Group 3, but greatly increased in the other two ability groups.

Analysis of Interviews: Items in Sets 1, 3, and 8 (Figure 4)

Examination of the transcripts of students in each of the ability groups revealed differences in the verbal responses to the main interview question when the students were directed to the groups of items in Sets 1, 3, and 8 by the instruction to describe their thinking as they dealt with the items in the sets. These responses are described set by set.

Set 1 (Items 1, 2, 5, and 6). Students in each group typically replied: “It’s like terms”, “You put the same/like terms together”; “You add like terms”, etc. Students in Ability Group 1 (mean ability estimate: –2.34 logits) used informal strategies or language such as “Circle the like terms”, “ I use the ones with letters first”, “You put the letters/numbers together”. Only rarely did a student in this group use terms such as “add or “subtract” to describe what they did with the terms. None verbally offered the finished answer to any item. Students in Ability Group 2 (mean ability estimate: -0.14 logits) and those towards the lower end of Ability Group 3 (mean ability estimate: 2) tended to use a mix of both formal language and informal language. Students in Ability Group 2 tended to describe just the sequence of steps involved, although some gave the completed response. Students at the top end of Ability Group 3 (ability estimate >2 logits) tended to use language of a high modality only, describing the steps in the simplification using mathematical terms for the operations, and completing the item.

Set 3 (Items 7, 11, 18 and 19). When presented with expressions containing brackets to be expanded, students, regardless of ability level, responded, “You do them first”. Of the 32 students interviewed, three only directly stated that brackets indicated some form of grouping. All three students had ability estimates greater than 0.75 logits. Most students also described the process of expanding brackets as “getting rid of the brackets”, an informally phrased instruction which implied that the brackets were “unnecessary”, or “you times the outside by the inside”. Most students described the steps in multiplying out the brackets, but did not verbally describe the end result. Only one student (ability 3.8 logits) described what would be done in general, and gave examples, with justifications of the procedural steps. Two out of the seven students interviewed from Ability Group 1 explicitly conjoined terms as they explained their thinking, as did one student in Ability Group 2. Another student in this group seemed unsure of the difference between $5x$ and $x + 5$.

Set 8 (Items 20, 21, 21, 25 and 26). When students were asked to express orally how expressions such as those in Set 8 could be rewritten, many simply repeated the expression, reading it from left to right. This did lead to a "correct" version, although little or no mathematical change occurred, particularly with items such as "Add 4 to n + 5", where many students responded with " Four plus *n* plus five". Only students in Ability Groups 2 and 3 completed the items verbally. Some supplied the answer only without describing the steps in their thinking. Students in Ability Group 1 tended to read aloud the items only, from left to right, and make no mathematical changes. Those in Group 2 tended to make some changes and also were uncertain as how to express, for example, the answer to Item 26 as "five plus *r*" or "5*r*".

Discussion of Results

In many cases, when explaining how they dealt with examples such as those in Set 1 and Set 3, the students spoke about "putting together like terms". However, students in Ability Group 1 tended to "put terms together" by conjoining all the terms. Having identified and isolated "like terms" in Items 1, 2, 5, and 6 circling them, or by rearranging the expression, or simply acting sequentially on each, students in Ability Group 1 "put them together" in a different way to those students in Groups 2 and 3. Students in these two Ability Groups did not tend to conjoin terms in these items. Students in Ability Group 2 tended to do so when dealing with items in Set 3 [those with brackets, Items 18, 19, and 7 and Item 11] and particularly those in Set 8 [Items 21, 22, 25, 26, and Item 20].

Item 11 also prompted some students in Group 2 and Group 3 to conjoin terms. This may be because they failed to take account of the fact that the item indicated a difference between $8p$ and $2(p + 5)$ rather than a multiplicative relationship between the terms, and so multiplied throughout – a case of a stimulus causing an automatic response: when there are brackets in an expression the procedure is to "multiply what is inside by what is outside". This procedural thinking also caused students to have problems with Item 7 [$(a – b) + b$]. Some students simply multiplied $(a – b)$ by b, because the b was outside the brackets. This procedure resulted in the errors such as $ab – b^2$, or ab^2.

The conjoining of terms by students in Ability Group 2 became much more frequent when they were required to answer Items 20, 21, 22, 25, and 26, the "semi-literal" items. These items required students to translate from words to mathematical symbols on their test scripts, showing an awareness of appropriate mathematical syntax and possible ambiguity in the written statement. Students in Group 3 did not tend to make this type of error. In the case of students in Groups 1 and 2, there was a marked increase in the numbers of conjoined-term errors as they responded to these items, compared with that for items in Sets 1 and 3 (Figure 5).

One possible explanation for this is that items in Sets 1 and 3 were typical textbook examples and students could respond to them by carrying out a well-rehearsed procedure, where they had been trained not to "put together" all the terms. Faced with an unfamiliar context, students with little understanding of the mathematical relationships conveyed by arithmetic operators in an algebraic context provided a closed response. The tendency to conjoin terms may help to explain why the group of "semi-literal" items had a higher average degree of difficulty than the group of addition and subtraction items, but which was lower than that for the items with brackets (Set 3) and why the successful response rate for students in Ability Group 2 dropped.

These data suggest that students in the middle and lower ability groups, according to the model of the test responses, have a limited procedural understanding of the algebra presented to them. They have learnt a particular procedure that can be applied to particular examples that have a surface similarity. Tall (1994) suggested that the role of the visual structure of an expression is important in learning algebra, but cautions that the image cannot provide the entire concept. Where students, “search their memories for something previously learnt”, as one student explained in the interview, they are often seeking an image that matches the appearance of the expression in front of them. The image need not encapsulate mathematical meaning, but acts as a visual cue to prompt a series of mathematical manipulative steps whereby the student changes the form of an expression. No meaning need be attached to the steps, or to the expression itself. Responses to Item 11, Item 7 and Item 19, other than those where terms were conjoined, indicate many students see expressions such as these with brackets and react in one way regardless of the structure and the meaning of the expression. This is also evident when students described their procedures in visual terms such as “circling” the like terms, or when they explained their thinking by simply pointing to parts of the expression when being interviewed.

Questions such as those in Set 8 (Kuchemann, 1981) probe the conceptual understanding of the various forms of algebraic expressions without the visual clues provided by more usual examples encountered by students. Such questions are rare in texts and often only appear in the introductory (Year 7, NSW) phases of algebra teaching.

Conclusions: Implications for Teaching

Rasch modelling of algebra test responses resulted in three clusters of student ability estimates. One of the characteristics of the students in these groups is the diminishing tendency for students to conjoin terms as their ability to deal with conceptually more difficult items develops. In other words, in order for students to be able to deal successfully with more complex algebra they need to learn when it is appropriate to conjoin terms (as in algebraic multiplication) and when not. If the tendency to conjoin terms results from students’ understanding arithmetic as much of the literature suggests, then teachers need to become aware of this persistent difficulty and use appropriate teaching strategies, such as those suggested by MacGregor and Stacey (1996) and Tirosh et al. (1998). In particular, students need to encounter arithmetic expressions in different equivalent and unclosed (“unfinished”) forms (Linchevski & Herscovics, 1994).

The data discussed in this paper suggest that students of lower “ability” tend to conjoin terms more often than other students. However, a great number of reasonably successful students have a limited procedural understanding of algebraic techniques. Provided that they have only to deal with standard or familiar examples, they can do so. When challenged by examples requiring an understanding of ways in which mathematical meaning and mathematical structure are connected, they expose their reliance on visual cues (or oversimplified schemata) that prompt the exercise of a particular procedure. In order to provide students with a more comprehensive schema, students need to encounter a variety of forms of expression and to experience being able to write them in several ways without the meaning being altered. Perhaps the use of the instruction “to simplify” is too limiting. Asking students to rewrite expressions in many ways and discussing the mathematical usefulness of their responses may help students to attend to the structure and meaning of expressions and so develop their conceptual understanding.

The data from interviews also suggest that the use of informal language in the classroom may serve to obscure the mathematical ideas. Statements such as "Get rid of brackets", "Do the brackets first" or "Put the like terms together" may not always be correctly interpreted by students, and may contribute to their tendency to conjoin terms because these statements do not convey an exact mathematical message.

The students in Ability Group 3 did not tend to display any marked tendency to conjoin terms in any of the sets of items presented to them. This implies that they have a conceptual understanding of these types of algebraic expression. However, their descriptions of their thinking, although high in modality when they described procedures, lacked depth of explanation or justification. Thus, it could be inferred that their understanding remains largely tacit and, hence, can be articulated only with difficulty. It might also have been that the situation of having to explain their thinking was unfamiliar to the students. This would suggest that class discussion of the various ways in which expressions can be written is a necessary part of developing deeper mathematical understanding. Just as students need to develop a rich vocabulary in their everyday language, so too they also need to experience, and use, a variety of mathematical language and symbols in order to explore and express mathematical meanings. Without this, their algebraic development must be inhibited.

References

Adams, J. & Khoo, S. T. (1994). *Quest: the interactive test analysis system* [computer software]. Canberra: ACER.

Bond, T., & Fox, C. (2001). *Applying the Rasch Model: Fundamental measurement in the human sciences*. Mahwah NJ: Lawrence Erlbaum.

Booth, L. (1984). *Algebra: Children's strategies and errors*. Windsor: NFER-Nelson.

Booth, L. (1988). Children's difficulties in beginning algebra. In A.F. Coxford (Ed.) *The Ideas of Algebra, K-12*, 1988 Year Book of the National Council of Teachers of Mathematics. Reston: NCTM.

Board of Studies NSW (2002). *Mathematics 7 – 10 Syllabus*. Sydney: BOSNSW.

Hall, R. (no date). *An analysis of thought processes during simplification of an algebraic expression*. Retrieved 19 July 2004 from: http://www.ex.ac.uk/~Pernest/pome15/r_hall_expressions.pdf.

Küchemann, D. (1981). Algebra. In K. Hart (Ed.) *Children's understanding of mathematics: 11-16* (pp. 102-119). London: John Murray.

Linchevski, L., & Herscovics, N. (1994). Cognitive obstacles in pre-algebra. In J. P. da Ponte & J. F. Matos (Eds), *Proceedings of the 18th annual conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 176-184). Lisbon, Portugal: PME.

MacGregor, M., & Stacey, K. (1994). Metalinguistic awareness and algebra learning. In J. P. da Ponte & & J. F. Matos (Eds), *Proceedings of the 18th annual conference of the International Group for the Psychology of Mathematics Education* (Vol 3, pp. 200- 207) Lisbon, Portugal: PME.

MacGregor, M. & Stacey, K. (1996). Origins of students' interpretations of algebraic notation. In L.Puig & A. Gutierrez (Eds), *Proceedings of the 20th annual conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 297- 304). Valencia, Spain: PME.

MacGregor, M., & Stacey, K. (1997). Students' understanding of algebraic notation: 11-15. *Educational Studies in Mathematics, 33*, 1-19.

Matz, M. (1982). Towards a process model for high school algebra errors. In D. Sleeman & J. Brown (Eds), *Intelligent tutoring systems*, (pp. 25 –50). London: Academic Press.

Tall, D. (1994). A versatile theory of visualisation and symbolism in mathematics. Plenary Presentation at the *Commission Internationale pour l"Etude et l'Amélioration de l'Ensignement des Mathématiques*, Toulouse, France, July 1994. Retrieved 22 January 2007 from the World Wide Web: http://www.warwick.ac.uk/staff/David.Tall/pdfs/dot1996b -plenary-cieaem-1.pdf.

Tirosh, D., Even, R., & Robinson, N., (1998). Simplifying algebraic expressions: Teacher awareness and teaching approaches. *Educational Studies in Mathematics, 35*, 51-64.

Towards "Breaking the Cycle of Tradition" in Primary Mathematics

Sandra Frid
SiMERR WA, Curtin University of Technology
<S.Frid@curtin.edu.au>

Len Sparrow
SiMERR WA, Curtin University of Technology
<L.Sparrow@curtin.edu.au>

The purpose of this study was to explore the mathematics teaching practices of graduates of a pre-service primary education program designed to develop teachers' capacities to implement non-traditional mathematics curricula. As a complementary component of a large survey study of graduate teachers, eight graduates were interviewed to examine their mathematics teaching practices and influences upon their practices. The teachers were implementing personally developed, constructivist-oriented curricula, while also acting as curriculum leaders. They indicated awareness of how aspects of their pre-service education provided them with the knowledge, skills, and confidence to enact their beliefs about effective mathematics teaching.

A problem facing pre-service mathematics teacher education is the challenge of preparing teachers to "break the cycle of tradition" of mathematics teaching and learning practices that centre on memorisation of facts, and practice of pre-set meaningless procedures, which promote a view of mathematics as lacking creativity, imagination, or critical thought. Research over recent decades indicates that "teachers continue to teach much like their forbears did" (Hiebert, 2003, p. 11), with an emphasis on teaching procedures rather than conceptual understandings. An alternative, non-traditional perspective for mathematics, often referred to as "constructivist", is one in which classrooms are envisioned as places rich in: discourse about important mathematical ideas, the development of mathematical meanings and understandings, and exploration of problems grounded in meaningful contexts (Clements & Battista, 1990; Sparrow & Frid, 2002).

Curriculum renewal and change efforts in mathematics in Australia and elsewhere (e.g., Australian Education Council, 1994; National Council of Teachers of Mathematics (NCTM), 2000) set ambitious goals for schools, teachers, and students by entailing a re-conceptualisation of the nature of mathematics and effective mathematics teaching and learning (Hiebert, 2003; Sparrow & Frid, 2002). To move forward in mathematics education therefore requires substantial learning by teachers and pre-service teachers with regard to their mathematics content knowledge, and their capacities and confidence to plan for and implement "non-traditional" mathematics teaching practices. Thus, there is an ongoing need for research into how to support teachers to develop as professionals who have capacities to break the cycle of tradition.

Background to this Study

The larger research program from which this study arose was designed to tackle the problem of breaking the cycle of tradition in a holistic, ongoing way beginning in pre-service education. Three components of mathematics education – content knowledge, mathematics pedagogical competence, and mathematics professional confidence – formed a foundation for a longitudinal action research cycle of curriculum implementation and evaluation in mathematics pre-service teacher education that was implemented over five

Proceedings of the 30th annual conference of the Mathematics Education Research Group of Australasia
J. Watson & K. Beswick (Eds), © MERGA Inc. 2007

years (and is still in progress). The curriculum initiatives and innovations, along with evaluations of their impact upon pre-service primary and early childhood teachers, are documented in earlier papers (e.g., Frid & Sparrow, 2003, 2004, 2005). However, although there has been evidence of substantial professional learning by the pre-service teachers that indicates they have the content knowledge, pedagogical competence, and professional confidence to begin to break the cycle of tradition upon graduation, the research did not examine the impact of this professional learning subsequent to graduation. In fact, there is little in the research literature regarding the impact of pre-service education subsequent to graduation.

Breaking the cycle of tradition will not occur unless graduate teachers are able to put into practice the non-traditional mathematics curriculum and pedagogical beliefs, ideas, and skills they developed in their pre-service programs. Thus, to begin to address the problem of breaking the cycle of tradition more comprehensively, a graduate survey and small-scale interview study were conducted to examine the questions:

- What are the mathematics teaching practices of graduates from a pre-service program designed to support teachers to break the cycle of tradition in mathematics education?
- What influences these practices?

This paper reports on the findings from the exploratory graduate interview study, while the survey findings are reported elsewhere (Frid, McCrory, Sparrow, & Trinidad, 2007). The significance of this research, as already indicated, is in its potential to inform mathematics educators of mechanisms and outcomes related to the development of beginning teachers as professionals who have the capacities to implement innovative non-traditional mathematics teaching and learning practices.

Theoretical Framework

Within the overall action research program, *teacher professional development* was viewed as a "process of growth in which a teacher gradually acquires confidence, gains new perspectives, increases knowledge, discovers new methods, and takes on new roles" (Jaworski, 1993, pp. 10-11). The curriculum development and implementation of the research program was built upon two main aspects of the literature related to teacher professional development, adult learning theory and professional empowerment, which are summarised below. The framework subsequently developed for the 4-year pre-service primary mathematics education program was named the *Three C's Mathematics Education Framework.* It also is outlined here, to indicate how the 4-year program was designed through analysis and synthesis of the relevant research literature.

Adult Learning Theory

Designing appropriate support for pre-service teachers' learning as mathematics educators requires consideration of how adults learn. Adult learning theory, as proposed by Knowles (1984), emphasises that adults are self-directed learners whose need to learn arises from the interests and challenges of their everyday lives. Further, since adults bring a broad range of experiences, beliefs, values, and ways of functioning to any learning situation, teaching processes that emphasise reflection, self-direction, articulation, scaffolding, and collaboration need to be explicitly recognised and attended to when planning curricula for adults. Learning must be embedded in "contexts that reflect the way knowledge will be useful in real life" (Collins, 1988, p. 2), and key features of related learning environments must include: coaching and *scaffolding* that provides skills, strategies, and cognitive links; *collaboration* to support personal as well as social construction of knowledge; *reflection* to enable meaningful and purposeful learning; *articulation* to consolidate knowledge and foster communication skills; and *integration* of learning and assessment tasks (Herrington & Oliver, 1995).

Teacher Professional Empowerment

Mechanisms for growth and change must ask teachers to act as their own change agents, while gently challenging ideas and fostering critical reflection upon ideas and experiences. Thus, "coming to know" as a professional is based upon ownership of ideas and related teaching practices, a form of professional empowerment. From an empowerment perspective professional development is an educative process in which teachers make meaningful and thoughtful choices about their practices rather than having change imposed externally (Robinson, 1989). What is key is that teachers act as their own change agents for immediate and long-term goals (Richardson, 1994).

The Three C's Mathematics Education Framework

The literature concerning adult learning theory and teacher empowerment guided development of the *Three C's Mathematics Education Framework* (Table 1).

Table 1
Overview of the Three C's Mathematics Framework

Year	**Mathematics *Content*** (content rich learning activities and exploration of curriculum documents)	**Pedagogical *Competence*** (examination of learning theories, teaching resources, technologies, and the literature)	**Professional *Confidence*** (reflection, articulation of ideas, and authentic application of learning)
1st-Year	• focus on the Space strand; overview of other strands • Maths Basic Skills Test	• social constructivist perspectives on learning and related practical implications for teaching mathematics	• develop and implement single and short sequences of mathematics lessons for children
3rd-Year	• Number & Working Mathematically • number sense and mental computation • numeracy	• examination of children's mathematical thinking and meaning-making	• plan for and assess children's learning (implementation with small numbers of children) • incorporate a wide array of resources and technologies into learning activities
4th-Year	• Measurement, Chance & Data, & Working Mathematically	• further examination of broad range of factors that impact on maths learning, including open-ended tasks, inquiry models of learning, games, textbooks, assessment practices, and catering for diversity	• articulate a philosophy of mathematics teaching • develop a mathematics professional teaching portfolio • participate in authentic professional interviews • prepare/implement program for a 10-week school practicum

Method

Purpose of the Interview Study

One of the purposes of the interview study was to explore beyond the quantitative and descriptive data of the larger graduate survey, through the gathering of more elaborated, explanatory data concerning teaching practices. Since the survey design and descriptive findings are reported elsewhere (Frid, McCrory, Sparrow, Trinidad, & Treagust, 2007), this paper aims to go beyond the description of practices to consider possible reasons for their nature. This focus allows for scope in the consideration of specific examples from teaching or other professional experiences, and possible links between graduate teachers' current practices and previous pre-service learning.

Research Sample

The interview research sample consisted of eight graduates selected from over 20 who volunteered when they returned their written survey in the mail. This sample was purposeful in that it was chosen to include graduates from all four years of the graduate survey (2002-2005) and graduates teaching in a range of locations (Table 2). It is acknowledged that this sample is not fully representative of the population of over 300 graduates from 2002-2005, and that their views and practices cannot be generalised to the larger group. However, since the interview component of the study was intended to identify avenues for further research into links between pre-service education and subsequent teaching practices, the diversity of teaching experiences represented by the graduates was considered sufficient as an initial exploration.

Table 2
Teachers Interviewed, Graduation Year, and School Employment History

Teacher (pseudonyms)	**Graduation year**	**School employment history**
Amanda	2002	Metropolitan school
Elaine	2002	Rural and remote schools
Lisa	2003	Metropolitan school
Nicola	2003	Rural school
Nancy	2003	Remote school
Alice	2004	Rural school, then metropolitan school
Yvonne	2004	Rural school
Wendy	2005	Metropolitan school

Data Collection and Analysis

Interviews were semi-structured in nature, with interviewees' initial responses examined further through requests for explanations and specific examples. The interview questions focused on the teachers' experiences related to: how prepared they were in mathematics education for the reality of their first job; factors that helped or limited their mathematics teaching; how they have used their mathematics teaching portfolio; and in what ways they were making an impact on mathematics learning in their classroom or school. These four foci were intentionally broad and contextual in nature, rather than asking an interviewee specifically to outline her teaching practices and related influences. In this way the interview data complemented in a holistic way the survey data that had been obtained from specific, directed questions. The contextual nature of each of the four foci provided opportunity for data to be obtained concurrently for both research questions (practices and influences).

The interviews were conducted in the July 2006 school term break, by telephone or at the university campus. They were conducted by an independent research assistant who was a qualified teacher, did not know the teachers, and had not been involved in their pre-service education program. Interviews lasted 30-45 minutes; they were audio recorded and later transcribed. Data analysis initially involved summarising across all eight teachers the responses for each of the four foci, and then proceeded inductively through a grounded approach (Powney & Watts, 1987). Initial emergence of key themes related to practices and influences upon practices were derived from the summaries and then examined further via re-visiting the transcripts for supporting as well as contrary evidence from the specific examples given by the teachers.

Findings

This section is structured around the two research foci (practices and influences), with the emergent themes each summarised briefly and explicated with examples from the interview data.

Classroom Teaching and Related Professional Practices

Three key aspects of classroom teaching practices emerged: (i) non-traditional teaching; (ii) "fun" mathematics; and (iii) classroom-specific curriculum development. An additional factor emerged as a key aspect of the teachers' broader professional practices related to mathematics: (iv) acting as a curriculum leader.

Non-traditional teaching. All the teachers spoke of teaching in what could be considered a constructivist perspective because it involved students in developing meanings and understandings through active engagement in learning activities (Clements & Battista, 1990). In this regard they also frequently mentioned using "hands-on" materials as a regular and essential feature of supporting students' mathematical thinking and meaning-making. For example, Elaine stated:

> … engaging the children in maths and really getting them to do stuff and working it out in their brains. … Getting the basic concepts across to them [indigenous students at a small school] was a challenge. So to have hands-on, talking about fractions and things, I'd get a cake and we'd cut it in half, … and give them the knife and cut it into quarters, and we'd sort of work our way down and they really got to visualise what it was to have a whole and then a half and then a quarter, and that sort of thing because fractions is a really tricky thing to get across to kids who really don't know much about numbers. (Elaine)

Other aspects of constructivist-oriented rather than more traditional teaching were evident in the teachers' references to how they used open-ended tasks, calculators, or other technology, while also avoiding prescribed textbook or worksheet exercises.

> I do try to think of more open ended activities because I've got such a range of kids. So then I can help the ones that are having problems and give more, and give extra to the ones who can do it all with their hands tied behind their backs. (Wendy)

> I did calculators [in my portfolio] and I try to use those with the kids. … We do lots of fun things and all those sorts of calculator games and stuff like that. (Lisa)

> And so I was really determined to use the influence Len Sparrow had on me. … I didn't use the books in the classroom because they're all those old textbook, workbook things. (Nicola)

"Fun" mathematics. Most of the teachers mentioned attempting to make mathematics experiences "fun", so that students would develop positive attitudes towards mathematics and be motivated to do mathematics. What they meant by "fun" was in fact more than enjoyment. It was learning oriented, involving motivation and enthusiasm, challenge and persistence, success, and a sense that mathematics can be relevant and useful.

> Well I know I'm making a difference because they are meeting the criteria of the outcomes. But the thing, the biggest thing I think is that they actually are enjoying it and are asking to do more. They like the challenge of mental maths and things like that, and "Can we do more?" and "When are we going to do that?" It's the enthusiasm for learning that's been the main thing, and the fact they enjoy maths is great. (Nancy)

> … a lot of the time the kids can be, "Oh, I can't do maths. I just can't do it". And therefore they don't try. But if you do it in an interesting context and in a way that encourages them to think about what they're doing it makes them realise that they can do it and it's not such a big scary thing at all. From the children I have taught I can see their change in attitude. … they can get through it if they are empowered to get through it. (Elaine)

Classroom-specific curriculum development. The teachers spoke of developing their mathematics curriculum locally and flexibly, in the context of their classroom and their students' learning needs. Some had taught in schools in which "you had to follow the textbook", yet even then they made efforts to "be creative" by incorporating hands-on activities and having students use their "brains a bit" (Elaine). In this regard they expressed strong beliefs that a mathematics curriculum cannot be based largely on prescribed textbook or worksheet activities if it is to support effective mathematics learning for the diversity of students in a classroom. Inherent in these beliefs are non-traditional views of mathematics learning and teaching; specifically, the same exercises at the same time are not appropriate for catering for students' developmental and achievement levels. Thus, many of the teachers indicated they preferred to use their professional knowledge and knowledge of their students to make mathematics curriculum decisions.

> We did try to program together for the first term and it just didn't work. It felt like I was banging my head against a brick wall, because her kids do worksheets, lots and lots of worksheets, and they're just five [years old]. (Wendy)

> You can pick and choose the parts that suit you and the different … like using the hands-on stuff, like using calculators. … We make our own lessons up because we said you can't have a textbook in Years 1 and 2. It's a guideline. … there's still room for extending the kids ... if they can do what's in the book you can still go over and above it if you feel they need to, or go back and re-teach a few things if they've missed something. (Lisa)

Acting as a mathematics curriculum leader. There was evidence that some of the teachers, even though they were "novice" teachers, were taking on mathematics leadership roles in their schools. In some cases these roles arose from personal initiatives to do new things in a school related to enhancing mathematics learning, indicating a degree of confidence and professional knowledge on the part of the teachers. Other forms of leadership involved encouraging and supporting other teachers to try new things, by sharing ideas, expertise, or resources. Yet another form of leadership that was mentioned by one teacher was that of acting as a role model, simply by doing different things that later proved to be effective in supporting students' mathematics learning.

> They gave me the opportunity to do the role [maths specialist], which I thought was quite strange because I was very frightened of maths. I thought, "Why me?" … So I go in and I actually give teachers ideas on how they can use the technology with their maths. We've got all these interactive whiteboards, so I train teachers on using the interactive whiteboards in their maths. (Amanda)

> Every time I come back from a conference I report at the following teachers' meeting on what I've learned and show them some stuff. … Last year one of the teachers was particularly receptive to the calculator program I brought back for him … so I had a win there. (Nancy)

> I started, in the newsletter I have a maths corner where I put a maths strategy in for the parents to help their kids. And a maths competition. (Nicola)

> That's actually been really amazing, the difference. ... They have done so much better … from someone who doesn't use the [text]books. … The other year 6/7 teachers, when it's maths they opened up to a certain page in the book and they all did that in the book. Now I never did that, and I was worried about whether they [the students] would be okay with everything. But from the results from different maths tests that they have to do for year 8, it's really shown me I've improved their maths. … I've had some teachers who have said to me, "I've never thought of doing it that way". (Nicola)

Influences on Practices

Two factors emerged as key influences upon mathematics teaching practices: (i) university learning; and (ii) school support or restrictions.

University learning. Since mathematics teaching portfolios (university learning), were specifically asked about in the interviews, their prominence was at least partially a product of the data collection instrument. However, of relevance here is what other aspects of university learning emerged as relevant, and which aspects of mathematics portfolios had an ongoing influence.

With regard to portfolios, specific teaching ideas such as the use of calculators, other technology, games, or mental computation were cited as useful in subsequent teaching. To a lesser extent there was mention of underlying principles for teaching particular mathematics concepts. What received the most mention, however, was the mathematics teaching philosophy developed in the portfolio.

> I've definitely used my maths portfolio, because I looked at maths through technology. So the whole thing was based on how technology can be integrated into our maths. (Amanda)

> I have used my general mathematics philosophy which sort of guides my maths teaching in that I still have the same values I did when I did the portfolio, and I still want to achieve the same things with my children. (Alice)

> The main thing is my philosophy, my beliefs. … I don't think I'll ever stop believing kids need to have fun in their maths, and they need to think and do and play around with stuff, and talk about it. Those are my core beliefs and I don't think they'll change. They might adapt slightly. (Lisa)

Beyond the learning attained at university from development of a mathematics teaching portfolio, what emerged as highly influential were the mathematics education lecturers and how they served as role models.

> My first year out I had year 6/7's and I was determined that if I didn't use the stuff I'd learnt from uni in my first year I never would. And so I was really determined to use the influence Len Sparrow had on me. (Nicola)

> I still think back and think, "What did I do in maths class? How can I teach this concept to my kids?" And I was chatting to some other Curtin graduates at the Beginning Teachers' Seminar and they were saying that they too have Len and Sandra moments. "Oh, what did Len do, what did Sandra do for that to help?" (Wendy)

School support or restrictions. A key feature of this theme was that personal beliefs and values related to mathematics teaching and learning, along with their resonance or incongruence with the beliefs of others at a school, could lead to dissatisfaction with teaching.

> The standard at that school was that you had a textbook and you had to follow the textbook, so I really didn't have a whole lot of room to be creative with those kids. ... I felt restricted because at uni everything was so exciting and energetic and so hands-on. (Elaine)

However, at the same time, some of the teachers noted specifically how their convictions to follow their beliefs, regardless of restrictions or the practices of other teachers in the school, were a guiding source for daring to be different and enacting non-traditional teaching practices.

> Things that have limited it? Simply old ways of thinking. You know you get really good teachers you can collaborate with, that have other experience, but you get other teachers that say, "No calculators in this classroom", or … "My kids aren't using counters for things like that". It's my classroom and if I want them to use counters, well they're going to use counters basically. And that's what my maths beliefs are and it's going to work. And you know what? Sometimes you have to say, "Stuff it". ... You just have to take a bit of a risk sometimes. (Lisa)

Conclusions and Implications

The findings from this study indicate that it is possible to "break the cycle of tradition" in primary mathematics education. More specifically, it is possible to prepare pre-service primary teachers who, subsequent to graduation, have the content knowledge, pedagogical competence, and professional confidence to put into practice non-traditional mathematics curricula. They can develop classroom-specific mathematics curricula that cater for diverse learning needs, use constructivist-oriented teaching strategies, and foster a view of mathematics as a challenging, relevant, enjoyable, and achievable endeavour. Further, they can act as change agents through a variety of forms of curriculum leadership, including serving as a specialist or coordinator, being a role model, fostering collaboration and sharing of ideas, or initiating new ideas and activities at a school.

However, the small scale nature of this study necessitates that these conclusions be made with some qualifications, because the findings cannot be generalised to all graduates. They cannot in fact be claimed for all eight of the teacher interviewees. For seven of the eight teachers the evidence was convincing with regard to the conclusions. The eighth teacher, Yvonne (2004 graduate), was somewhat different from the others in that she spoke of struggling with her mathematics teaching and not knowing what to do with the diversity of achievement levels in her classroom, and she could say very little about what she had learned in her pre-service program or her mathematics teaching portfolio.

The findings do, nonetheless, show what is possible and what is promising. It is in this context that the following discussion of practical implications examines aspects of the teachers' pre-service experiences and current practices that appear to be prominent in their capacities to begin to break the cycle of tradition: (i) development of a mathematics teaching philosophy; (ii) breadth and depth in mathematics pedagogical knowledge; and (iii) professional confidence.

Development of a Mathematics Teaching Philosophy

The fact that most of the teachers, even up to 4 years later, could outline how their mathematics teaching philosophy impacted upon their practices implies the development of a philosophy as a requirement of their pre-service program supported their later teaching endeavours. They spoke of their beliefs and values, but more importantly, of how these guided their practices. This latter point must be noted explicitly in that the development of a mathematics teaching philosophy entails more than outlining beliefs about mathematics teaching. It necessitates translating beliefs into practice, that is, articulating how classroom environments, learning and assessment activities, and teaching strategies can be constructed to attain the goals of one's beliefs. A philosophy is more complex than an outline of beliefs, and thus, this research goes beyond prior research related to the nature and role of beliefs in mathematics teaching. Much previous research has neglected the practical components of an examination of beliefs, by not addressing how to put beliefs into practice in practical ways in the context of actual classroom teaching. A mathematics teaching philosophy and related teaching portfolio require this articulation and application, and hence a practical implication of this research study is that the development of a mathematics philosophy and portfolio can support beginning to break the cycle of tradition.

Breadth and Depth in Pedagogical Knowledge

The teachers showed breadth in their pedagogical knowledge in that they displayed awareness of a wide range of mathematics resources, teaching strategies, and learning activities that can motivate and support meaningful mathematics learning. They showed depth in their pedagogical knowledge in that they could articulate *why* they used particular methods in relation to *how* they facilitate mathematics learning. That is, the teachers displayed understandings of the research on how children learn mathematics, and importantly, how to apply those learning theories to the development of mathematics curricula. The implications here are that teachers who have understandings of mathematics pedagogy, *along with* capacities to translate those understandings into classroom learning experiences, will begin to be able to break the cycle of tradition. What is not as clear here, in comparison the role of the teachers' philosophies, is the degree to which the teachers' pre-service program had direct impact upon their later breadth and depth in pedagogical knowledge. It is however reasonable to note that a key aspect of the pre-service teachers' development of a mathematics teaching portfolio was that they had to justify the content their portfolios. Specifically, they had to use a framework of "what-why-how" (Frid & Sparrow, 2003, 2004) to prepare portfolio items, and then to justify them within authentic interviews with school principals and other educators.

Professional Confidence

Several of the teachers were acting in leadership roles, and some clearly were "daring to be different", even in the face of restrictions and adversity. It takes professional confidence to take the risks needed to enact teaching practices that differ to those of colleagues in a school. The fact that these actions were being taken by "novice" teachers needs further examination. In this study there was evidence that the teachers' professional confidence arose from awareness of their beliefs, values, and philosophy, along with convictions to act in congruence with them. The additional factor in evidence was that they had well-developed pedagogical knowledge of how to translate their beliefs and philosophy into practice. Their professional confidence was not independent of their teaching philosophy and pedagogical competence; they were not separate. Thus, a practical implication here is that pedagogical competence along with related professional confidence can lead to teachers to begin to break the cycle of tradition.

In conclusion, a final statement of what is promising in addressing the problem of breaking the cycle of tradition is that this study implies: it is possible to prepare pre-service teachers to be thinking-acting-leading mathematics teachers – teachers who think critically about their professional practices while also serving as educational leaders who take action and implement changes to enhance mathematics teaching and learning.

Acknowledgement. The authors wish to acknowledge the National Centre of Science, ICT and Mathematics Education for Rural and Regional Australia (SiMERR) for support of this research study.

References

Australian Education Council (1994). *Mathematics–a curriculum profile for Australian schools*. Melbourne: Curriculum Corporation.

Clements, D., & Battista, M. (1990). Constructivist learning and teaching. *Arithmetic Teacher, 38*(1), 34-35.

Collins, A. (1988). *Cognitive apprenticeship and instructional technology.* (Technical report no. 6899). Cambridge, MA: BBN Labs Inc.

Frid, S., & Sparrow, L. (2003). Using mathematics teaching portfolios to empower pre-service primary teachers. In L. Bragg, C. Campbell, G. Herbert, & J. Mousley (Eds.), *MERINO. Mathematics Education Research: Innovation, Networking, Opportunity* (Proceedings of the 26th annual conference of the Mathematics Education Research Group of Australasia, (Vol 1, pp. 356-363). Sydney: MERGA.

Frid, S., & Sparrow, L. (2004). Using reflection as pre-service primary teachers develop a mathematics teaching portfolio. In I. Putt, R. Faragher, & M. McLean (Eds.), *Mathematics education for the third millennium: Towards 2010* (Proceedings of the 27th annual conference of the Mathematics Education Research Group of Australasia, (Vol 1, pp. 239-246). Sydney: MERGA.

Frid, S., & Sparrow, L. (2005). *Selection criteria. Australian Awards for University Teaching 2005.* Unpublished document, Curtin University of Technology.

Frid, S., McCrory, K., Sparrow, L., & Trinidad, S. (2007). *Graduate survey in mathematics, science and ICT education*. Manuscript in preparation.

Herrington, J., & Oliver, R. (1995). Critical characteristics of situated learning: Implications for the instructional design of multimedia. In J. Pearce & A. Ellis (Eds.), *Learning with technology* (pp. 253-262). Melbourne: University of Melbourne.

Hiebert, J. (2003). What research says about the NCTM Standards. In J. Kilpatrick, W. G. Martin, & D. Schifter (Eds.), *A research companion to Principles and Standards for School Mathematics* (pp. 5-23). Reston, VA: National Council of Teachers of Mathematics.

Jaworski, B. (1993). The professional development of teachers–the potential of critical reflection. *British Journal of Inservice Education, 19*(2), 37-42.

Knowles, M. (1984). *Andragogy in action.* San Francisco, CA: Jossey-Boss.

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics.* Reston, VA: Author.

Powney, J., & Watts, M. (1987). *Interviewing in educational research.* London: Routledge.

Richardson, V. (1994). Conducting research on practice. *Educational Research 23*(5), 5-10.

Robinson, I. (1989). The empowerment paradigm for the professional development of teachers of mathematics. In N. Ellerton & M. Clements (Eds.), *School mathematics: The challenge of change* (pp. 269-283). Geelong, VIC: Deakin University.

Sparrow, L., & Frid, S. (2002). Supporting beginning primary mathematics teachers through a 'fellow worker' professional development model. In B. Barton, K. C. Irwin, M. Pfannkuch, & M. Thomas (Eds.), *Mathematics Education in the South Pacific* (Proceedings of the 25th annual conference of the Mathematics Education Research Group of Australasia, (Vol 1, pp. 71-80). Sydney: MERGA.

Exploring the Number Knowledge of Children to Inform the Development of a Professional Learning Plan for Teachers in the Ballarat Diocese as a Means of Building Community Capacity

Ann Gervasoni
Australian Catholic University
<Ann.Gervasoni@acu.edu.au>

Teresa Hadden
Catholic Education Office Ballarat
<THadden@ceo.balrt.catholic.edu.au>

Kathie Turkenburg
Catholic Education Office Ballarat
<KTurkenburg@ceo.balrt.catholic.edu.au>

This paper explores the number learning in 2006 of over 7000 children in the Ballarat Diocese for the purpose of identifying any issues that may inform the development of a Diocesan professional learning plan. The data for each grade level were examined to find if there were any apparent learning, teaching, or curriculum issues. The study found that there was a spread of knowledge within each grade level, and that there were groups of students who may be vulnerable. In particular, it was found that notable numbers of students beginning Grade 6 were not yet able to read, write, order, and interpret four-digit numbers nor use reasoning-based strategies for calculations in addition and subtraction, and multiplication and division. These findings need to inform the professional learning plan.

In 2001, the Ballarat Diocese Catholic Education Office implemented a 5-year Diocesan Literacy and Numeracy Plan with the aim of building the capacity of communities (Howard, Perry, & Butcher, 2006) to improve learning for all students. Indeed, school systems throughout Australia and New Zealand have had a similar focus during the past decade. This emphasis on improving learning has been driven in Australia by the 1997 national literacy and numeracy goal that asserts that "every child leaving primary school should be numerate and able to read, write and spell at an appropriate level" (Department of Education Science and Training, 2001, p. 1). However, it is the sub-goal that "every child commencing school from 1998 will achieve a minimum acceptable literacy and numeracy standard within four years" (Department of Education Science and Training, 2001, p. 1) that focused the attention of school systems in Australia on literacy and numeracy learning in the early years of schooling. This prompted several large research projects (e.g., Gould, 2000; Clarke et al., 2002) that identified strategies for improving mathematics learning and teaching (Bobis et al., 2005).

A common feature of these research projects and also of the *Numeracy Development Project* in New Zealand (Higgins, Parsons, & Hyland, 2003) was the use of clinical interviews so that teachers could identify the current knowledge of each student and plan and customise learning opportunities accordingly. Data obtained and aggregated for a class or school were used to identify particular issues associated with enabling effective teaching, learning, and curriculum development (Clarke et al., 2002) and formed the basis of professional learning for teachers. A similar approach was adopted in the Ballarat Diocese.

This paper examines aggregated data describing number knowledge of over 7000 children attending school in the Ballarat Diocese of Western Victoria for the purpose of identifying issues associated with effective teaching, learning, and curriculum development. It is anticipated that the findings will have implications for the identification

Proceedings of the 30th annual conference of the Mathematics Education Research Group of Australasia
J. Watson & K. Beswick (Eds),

of curriculum and professional learning needs. Of particular interest ultimately is how we may improve the capacity of communities to provide more effective learning opportunities for all.

Using Frameworks and Interviews to Identify Children's Number Knowledge

Clinical interviews are now widely used by teachers in Australia and New Zealand as a means of assessing children's mathematical knowledge. This is due to the experience of three large scale projects that informed assessment and curriculum policy formation in Victoria, NSW, and New Zealand: *Count Me In Too* (Gould, 2000) in NSW, the Victorian *Early Numeracy Research Project* (Clarke et al., 2002) and the *Numeracy Development Project* (Higgins, Parsons, & Hyland, 2003) in New Zealand.

A common feature of each of these projects was the use of a one-to-one assessment interview and an associated research-based framework to describe progressions in mathematics learning (Bobis et al., 2005). Teachers participating in each project indicated that the benefits of the assessment interview, though time-consuming and expensive, were considerable in terms of creating an understanding of what children know and can do, and for subsequently informing planning. Indeed, an important feature of clinical interviews is that they enable the teacher to observe children as they solve problems to determine the strategies they used and any misconceptions (Gervasoni & Sullivan, 2007). They also enable teachers to probe children's mathematical understanding through thoughtful questioning (Wright, Martland, & Stafford, 2000). The insights gained through this type of assessment inform teachers about the particular instructional needs of each student more powerfully than scores from traditional pencil and paper tests, the disadvantages of which are well established (Clements & Ellerton, 1995). Bobis et al. (2005) concluded that one-to-one assessment interviews and associated frameworks assisted to move the focus of professional development in mathematics from the notion of children carefully reproducing taught procedures to an emphasis on children's thinking. This is an important outcome at a time when it is broadly accepted that the traditional focus on taught procedures for calculating can negatively impact on children's number sense (Clarke, Clarke, & Horne, 2006) and may impede children's development of powerful mental reasoning strategies for calculating (Narode, Board, & Davenport, 1993). It is important to consider, therefore, when examining the data presented in this paper, whether students in the Ballarat Diocese use reasoning-based strategies for calculating or not. The evidence may highlight issues to consider when formulating the new Diocesan Professional Learning Plan and identify whether teachers may benefit from opportunities to explore methods that lead to children's development of number sense and reasoning-based strategies for calculating.

The Early Numeracy Interview and Framework of Growth Points

The *Early Years Interview* (Department of Education Employment and Training, 2001), developed as part of the *Early Numeracy Research Project* (ENRP) (Clarke et al., 2002), is one example of a clinical interview and a research-based framework of growth points that describe key stages in the learning of various aspects of mathematics. This interview and the associated growth points were used in the Ballarat Diocese to gather data explored in this paper, so an understanding of them is important. The principles underlying the construction of the growth points were that they would:

1. describe the development of mathematical knowledge and understanding in the first three years of school, through highlighting important ideas in early mathematics understanding in a form and language that was useful for teachers;
2. reflect the findings of relevant international and local research in mathematics (e.g., Steffe, von Glasersfeld, Richards, & Cobb, 1983; Steffe, Cobb, & von Glasersfeld, 1988; Fuson, 1992; Boulton-Lewis, 1996; Mulligan & Mitchelmore, 1996; Mulligan, 1998; Wright, Martland, & Stafford, 2000; Gould, 2000);
3. reflect, where possible, the structure of mathematics;
4. allow the mathematical knowledge of individuals and groups to be described; and
5. enable a consideration of students who may benefit from additional assistance.

The growth points formed a framework for describing children's development in Counting, Place value, Addition and Subtraction, Multiplication and Division, Length, Mass and Time, Properties of Shape, and Visualisation and Orientation. The processes for validating the growth points, the interview items and the comparative achievement of students in project and reference schools are described in full in Clarke et al. (2002).

To illustrate the nature of the growth points, the following are the points for Addition and Subtraction. These emphasise the strategies children use to solve problems.

1. Counts all to find the total of two collections.
2. Counts on from one number to find the total of two collections.
3. Given subtraction situations, chooses appropriately from strategies including count back, count down to & count up from.
4. Uses basic strategies for solving addition and subtraction problems (doubles, commutativity, adding 10, tens facts, other known facts).
5. Uses derived strategies for solving addition and subtraction problems (near doubles, adding 9, build to next ten, fact families, intuitive strategies).
6. Extending and applying. Given a range of tasks (including multi-digit numbers), can use basic, derived and intuitive strategies as appropriate.

Each growth point represents substantial expansion in knowledge, or key "stepping stones" along paths to mathematical understanding (Clarke, 2001). It is not claimed that every student passes all growth points along the way, nor should the growth points be regarded as discrete. However, the order of the growth points provides a guide to the possible trajectory (Cobb & McClain, 1999) of children's learning. In a similar way to that described by Owens and Gould (1999) in the *Count Me In Too* project: "the order is more or less the order in which strategies are likely to emerge and be used by children" (p. 4).

In summary, the framework of growth points can help teachers to understand a possible trajectory for describing children's learning, identify where any child is currently positioned, identify any children who may be vulnerable in a given domain, identify the zone of proximal development for each child in each domain so as to customise planning and instruction, and identify the diversity of mathematical knowledge in a class. Professional learning programs for teachers who use such frameworks may need to build teachers' capacities to use this information to more effectively teach each child.

The interview takes between 30-40 minutes per student and is conducted by the regular classroom teacher. The full text involves around 60 tasks, although no child is presented with all of these. Given success with a task, the interviewer continues with the next tasks in the given mathematical domain (e.g., Place Value) for as long as the child is successful.

The *Early Numeracy Interview* provided teachers participating in the ENRP with insights about children's mathematical knowledge that they reported might otherwise not have been forthcoming (Clarke, 2001). Further, the project found that teachers were able to use this information to plan instruction that would provide students with the best possible opportunities to extend their mathematical understanding. This is important to consider when developing a professional learning plan for the Ballarat Diocese.

Focus on Place Value Knowledge and Reasoning-Based Strategies

A factor in providing effective mathematics learning opportunities for children is the teacher being able to anticipate the difficulties that some children may encounter in order to assist them. Many studies have provided insight about such difficulties. Important to consider in regard to the data presented in this paper are issues associated with children's understanding of Place Value ideas and use of reasoning strategies for calculating.

One important finding is that children who have not constructed grouping and place value concepts often have difficulty working with multi-digit numbers (Baroody, 2004). This is an important idea to explore when examining the data presented in this paper. Also, being able to interpret numerals to order them from smallest to largest is another Place Value challenge for some children. Griffin, Case, and Siegler (1994) observed that this involves integrating the ability to (1) generate number tags for collections, and (2) make numerical judgments of quantity based on the construction of a mental number line (Griffin & Case, 1997; Griffin et al., 1994). This becomes more complex as children encounter two-digit numbers.

Other studies have found that successful problem solving with two-digit numbers depends on children's ability to construct a concept of ten that is both a collection of ones and a single unit of ten that can be counted, decomposed, traded, and exchanged for units of different value (e.g., Cobb & Wheatley, 1988; Fuson et al., 1997; Ross, 1989; Steffe et al., 1988; Young-Loveridge, 2000). Cobb and Wheatley (1988) found that some children develop a concept of ten that is a single unit that cannot be decomposed, and proposed that this type of concept is constructed when children learn by rote to recognise the number of tens and ones in a numeral, but do not recognise that the face value of a numeral represents the cardinal value of a group.

The counting and reasoning strategies children use to solve addition and subtraction problems have also been the focus of many studies (e.g., Clarke et al., 2002; Fuson, 1992; Griffin et al., 1994; Steffe et al., 1988). Counting strategies identified include count-all (including perceptual counting and counting by representing), count-on (from largest and smallest addend), count-back-all, count-down-to, and count-down-from. Reasoning strategies include doubles, near doubles, adding ten, adding nine, commutativity, combinations for ten, part-whole strategies, and retrieving answers from memory (e.g., Clarke, 2001; Fuson, 1992; Griffin et al., 1994; Steffe et al., 1988). Once children have developed a range of strategies, it becomes important to choose wisely among these strategies to fit the characteristics of a strategy to the demands of a task (Griffin et al., 1994). However, not all children choose wisely or have each strategy available.

In order to think multiplicatively, children need to shift from viewing groups as being composed of single items, to viewing the group itself as a countable unit (Clarke et al., 2002; Mulligan, 1998). This is difficult for some. Sullivan, Clarke, Cheeseman, and Mulligan (2001) found that constructing knowledge for abstracting multiplication and division problem solutions provides a significant barrier for many children, and Clarke et al. (2006) found that 16% of children at the end of Grade 4 did not use reasoning strategies in multiplication. These difficulties provide a lens for examining the data presented later.

Improving Mathematics Learning in the Ballarat Diocese

In 2001, the CEO Ballarat implemented the Ballarat Diocese Numeracy Strategy (2001-2005) to improve mathematics learning for primary school students within the Diocese. The strategy was informed by the findings of the ENRP (Clarke et al., 2002) and in a similar way to the ENRP, adopted the Hill and Crévola *Key Design Elements* (Hill & Crévola, 1999) as a means of building the capacity of school communities to provide more effective learning opportunities for all students. These were beliefs and understandings, leadership and coordination, standards and targets, monitoring and assessment, classroom teaching programs, professional learning teams, school and class organisation, intervention and special assistance, and home, school, and community partnerships.

From 2002, schools began to use the *Early Numeracy Interview* to assess all students' number knowledge. All schools were using this interview for all children by 2006. Teachers were encouraged to analyse the data to determine any school-based issues and to identify and assist those students who were at risk of poor learning outcomes. To facilitate this, teachers were invited to train as specialist intervention teachers, so that they could introduce the *Extending Mathematical Understanding* (EMU) intervention program (Gervasoni, 2004) in Grade 1, and provide specialist advice for teachers and parents..

From 2004 onwards, all schools developed a numeracy action plan that addressed each of the nine Key Design Elements. Schools were also funded to enable the appointment of a Numeracy Co-ordinator to guide the implementation and evaluation of the school plan. From 2002, the Diocese provided a professional learning program for all teachers (P-6) and Numeracy Co-ordinators. This included a mix of regionally-based whole-day programs, school cluster workshops, and school-based professional learning team meetings.

The Diocese is now evaluating the effectiveness of the Strategy and considering key issues to focus on to inform a new professional learning program for teachers to build community capacity further to provide effective mathematics learning for all.

Analysing Children's Number Knowledge in the Ballarat Diocese

The data presented in this paper were collected in 2006 from over 7000 children from all 52 Catholic Primary Schools within the Ballarat Diocese. This enabled a rich picture of these children's number knowledge to be formed. The practice in this region is for teachers to assess each student in the first week of school using the *Early Years Interview* for the purpose of gaining insight about each child's current mathematical knowledge. The interview was developed during the ENRP (Clarke et al., 2002). Its development and the associated framework of growth points are reported in detail elsewhere (e.g., Bobis et al., 2005; and Clarke, 2001). However, it is important to note that the growth points describe major learning along a hypothesised learning trajectory (e.g., Cobb & McClain, 1999) and formed the basis for the development of interview assessment items.

Children's responses to assessment items were analysed by the teacher to determine the growth points children reached. To increase the validity and reliability of the data, each teacher followed a detailed interview script, recorded children's answers and strategies on a detailed record sheet, and used clearly defined rules for assigning growth points. Children's growth points were entered into an excel spreadsheet and each school's data were aggregated to form the data set reported on here. The region's Numeracy Advisors and each school's Numeracy Co-ordinator managed this process.

Issues Arising from Examining Children's Number Knowledge

The purpose of the examination of data collected in 2006 within the Ballarat Diocese is to identify any important issues related to learning, teaching and curriculum that need to be addressed to improve learning opportunities for children and that might inform the Diocesan Mathematics Professional Learning Plan. This paper will focus on issues related to the Place Value, Addition and Subtraction, and Multiplication and Division domains.

The percentage of children in each grade reaching each Place Value growth point (GP) is shown in Figure 1. Of particular interest is children's knowledge of multi-digit numbers.

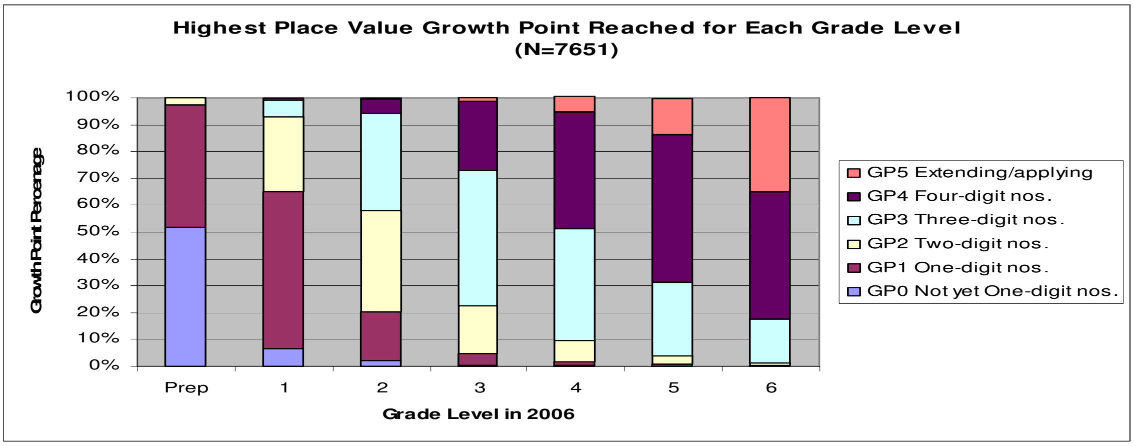


Figure 1. Percentage of children in each grade reaching each growth point at the beginning of 2006

An issue highlighted in Figure 1 is the spread of growth points at each level. This finding has been noted elsewhere (e.g., Gervasoni & Sullivan, in press; Bobis et al., 2005) but highlights the complexity of the teaching process and the importance of teachers identifying each child's current knowledge and knowing ways to customise learning opportunities that meet each child's needs. This has important curriculum and instruction implications for any plan to strategically improve learning outcomes for students.

Another interesting point is that almost half the children beginning Prep, the first year of school in Victoria, can already read, write, order, and interpret one-digit numbers. These children already need opportunities to explore two and three digit numbers, an issue that needs to be addressed in curriculum development and planning. The remaining students require the more traditional Prep experiences that firstly emphasise exploring and constructing knowledge about one-digit numbers. However, right from the beginning of schooling, the data highlight differences in children's knowledge to which the community needs to respond to optimise learning. It is also important to acknowledge that some teachers may not have been able to identify the extent of some children's knowledge because this is sometimes culturally specific, and may not be obvious to the teacher (Gervasoni, 2003). This issue may be another focus for professional development.

Figure 1 also shows that nearly half the Grade 2s and three-quarters of the Grade 3s were already able to interpret three-digit numbers and needed opportunities to explore and construct understandings about four-digit numbers and greater. School communities need to consider how this can be best achieved.

Another feature of the data is the number of students in Grades 4 to 6 within the Diocese who have not yet reached GP 4 and GP 5 (52%, 32%, and 18% respectively). Further examination of these students assessment responses shows that many were able to read and write four-digit numbers, but were not able to either order four-digit numbers and/or answer the questions, "What is 10 more than 2791?" and "What is 100 less than 3027?" As highlighted by Baroody (2004), these tasks require children to appreciate the *quantity* associated with number names and numerals and either to use their mental number line (Griffin & Case, 1997) to find 10 more or 100 less, or to use a reasoning-based strategy that draws upon their number sense. Difficulty with this type of task typifies the children who experience difficulty in Place Value. Certainly, a curriculum emphasis on understanding these numbers as quantities and numbers with positions on the number line is important. A Diocesan professional learning plan may need to address this issue.

A further implication of this finding is that some children in Grades 4 to 6 may be required to solve problems requiring calculations with four-digit numbers and greater (a prominent feature of the curriculum at this level), without an understanding of these numbers as quantities and their position on the number line. It seems fair to assume that many of these children may be reliant on learning procedures for performing calculations without constructing the conceptual underpinnings, and perhaps before they have developed reasoning based strategies for calculating. To explore this conjecture, we first examined the highest growth point reached by students in the Addition and Subtraction Strategies domain (see Figure 2).

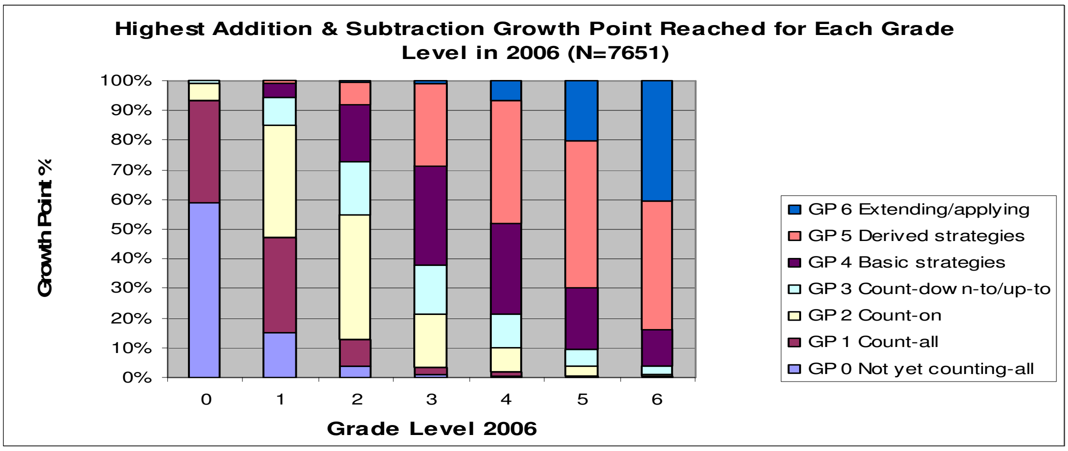


Figure 2. Percentage of children in each grade reaching each growth point at the beginning of 2006 in addition and subtraction (N=7651).

The data show that 51% of children beginning Grades 4 and 30% of children beginning Grade 5 were not yet using derived strategies (GP 5). This is consistent with the findings of a longitudinal study of 323 children who participated in the ENRP (Clarke et al., 2006). Their study found that when children reached Grade 4 and 5, respectively 53% and 37% had not reached GP 5. However, note that in the longitudinal study, data refer to assessment at the end of Grades 3 and 4, so comparisons are indicative only. Figure 2 also highlights that 16% of Grade 6s were not yet using derived strategies. This suggests that these children may rely on rote procedures for performing calculations.

To explore this issue further, we determined the number of Grade 6 students who had not yet reached GP 4 in Place Value, nor used reasoning-based strategies in Addition and Subtraction (GP 5) and Multiplication and Division (GP 4).

Figure 3 shows the number of children who had not reached these growth points and the combinations of domains for which this was the case (N=1195, n=371). It is important to note that 69% of children beginning Grade 6 had meet these minimum targets. Conversely, 31% were vulnerable in at least one of these domains, and these children are the focus of Figure 3. In summary, Figure 3 shows that of the 31% of Grade 6s who were vulnerable in at least one of these domains, 18% were vulnerable in all three domains, and nearly half (45%) were vulnerable in at least 2 domains.

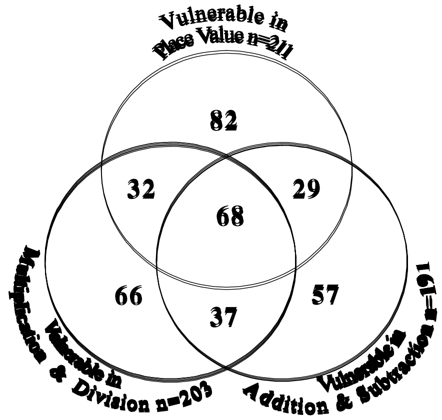


Figure 3. The number and combinations of domains for which Grade 6 children had not yet reached targets in Place value, addition and subtraction, and multiplication and division, (N=1195, *n*=371).

In relation to the question about whether children who had not yet reached GP 4 in Place Value used reasoning-based strategies in Addition and Subtraction and Multiplication and Division contexts, Figure 3 shows that of the 211 Grade 6 children who had not yet reached GP 4 in Place Value, 61% had also not yet reached the growth points associated with using derived strategies in Addition and Subtraction and reasoning strategies in Multiplication contexts. A focus for increasing the capacity of communities to provide effective learning opportunities for these students will include professional learning opportunities that enable Grades 4 to 6 teachers to identify and develop instructional approaches to identify and assist these students. This may also include intervention-style programs aimed at accelerating children's number learning in these aspects.

Conclusion

Examination of the current number knowledge of over 7000 children in the Ballarat Diocese highlights some important issues to consider for developing a professional learning plan to improve mathematics learning outcomes for students. Key issues are the need for communities to provide more effective learning opportunities to assist children interpret four-digit numbers, and reasoning based strategies in Addition and Subtraction and Multiplication and Division. However, it is acknowledged that in formulating a professional learning plan for teachers throughout the Ballarat Diocese, it will be important to explore the views of those living and working in the various communities, and to identify the characteristics of communities that already make a difference.

Discussions with School Numeracy Co-ordinators within the Diocese suggest that although considerable change has occurred in the curriculum and teaching approaches of those involved in the early years of schooling (P-2), and for many teachers working in the later years, some Grades 3 to 6 teachers continue to adopt a more traditional approach to number learning that is based on the rote learning of calculation procedures and number facts. Another point raised was the need for ongoing monitoring and assessment of children's knowledge. Numeracy Co-ordinators suggested that whereas all teachers use the one-to-one assessment interview and framework of growth points at the beginning of the year to inform their curriculum planning, some teachers do not continue to use the framework to monitor children's knowledge and differentiate curriculum and instruction throughout the year. This is another possible focus for the Diocesan professional learning plan.

Overall, it seems that building the capacity of communities to provide more effective learning environments for Grades 3 to 6 children will be an important factor in addressing the learning, teaching, and curriculum issues highlighted by the examination of children's number knowledge, and will be an essential focus for a new professional learning plan.

Acknowledgement. The authors thankfully acknowledge all teachers and numeracy co-ordinators in the Ballarat Diocese who contributed to the data and ideas considered in this paper.

References

Baroody, A. (2004). The developmental bases for early childhood number and operations standards. In D. H. Clements & J. Sarama (Eds.), *Engaging young children in mathematics: Standards for early childhood mathematics education* (pp. 173-219). Mahwah, NJ: Lawrence Erlbaum Associates.

Bobis, J., Clarke, B., Clarke, D., Thomas, G., Wright, R., Young-Loveridge, J. & Gould, P. (2005). Supporting teachers in the development of young children's mathematical thinking: Three large scale cases. *Mathematics Education Research Journal*, *16*(3), 27–57.

Boulton-Lewis, G. (1996). Representations of place value knowledge and implications for teaching addition and subtraction. In J. Mulligan & M. Mitchelmore (Eds.), *Children's number learning: A research monograph of MERGA/AAMT* (pp. 75-88). Adelaide: Australian Association of Mathematics Teachers.

Clarke, D. (2001). Understanding, assessing and developing young children's mathematical thinking: Research as powerful tool for professional growth. In J. Bobis, B. Perry, & M. Mitchelmore (Eds.), *Numeracy and beyond (Proceedings of the 24th Annual Conference of the Mathematics Education Research Group of Australasia,* (Vol. 1, pp. 9-26). Sydney: MERGA.

Clarke, B., Clarke, D., & Horne, M. (2006). A longitudinal study of mental computation strategies. In Novotná, J., Moraová, H., Krátká, M, & Stehlíková, N. (Eds), *Proceedings of the 30th annual conference of the International Group for Psychology of Mathematics Education* (Vol. 2, pp. 329-336). Prague: PME.

Clarke, D., Cheeseman, J., Gervasoni, A., Gronn, D., Horne, M., McDonough, A., Montgomery, P., Roche, A., Sullivan, P., Clarke, B., & Rowley, G. (2002). *ENRP Final Report*. Melbourne: Australian Catholic University.

Clements, M. A., & Ellerton, N. (1995). Assessing the effectiveness of pencil-and-paper tests for school mathematics. In B. Atweh & S. Flavel (Eds.), *Galtha: (Proceedings of the 18th annual conference of the Mathematics Education Research Group of Australasia, pp. 184–188).* Darwin: *MERGA*.

Cobb, P., & McClain, K. (1999). *Supporting teachers' learning in social and institutional context*. In Fou-Lai Lin (Ed.), Proceedings of the 1999 International Conference on Mathematics Teacher Education (pp.18-28). Taipei: National Taiwan Normal University.

Cobb, P., & Wheatley, G. (1988). Children's initial understanding of ten. *Focus on Learning Problems in Mathematics, 10*(3), 1-28.

Department of Education Science and Training. (2001). *National Goals*. Retrieved 5 Aug, 2003, from http://www.dest.gov.au/literacy&numeracy/goals_plan.htm

Fuson, K. (1992). Research on whole number addition and subtraction. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 243-275). New York: Macmillan.

Fuson, K., Wearne, D., Hiebert, J., Murray, H., Human, P., Olivier, A., Carpenter, T., & Fennema, E. (1997). Children's conceptual structures for multidigit numbers and methods of multidigit addition and subtraction. *Journal for Research in Mathematics Education, 28*(2), 130-162.

Gervasoni, A. (2003). Identifying and assisting children who are impeded in learning mathematics. *Australian Primary Mathematics Classroom. 8*(4), 4-9.

Gervasoni, A. (2004). Exploring the diversity of grade 1 and grade 2 children who are vulnerable in learning mathematics. In M. Hoines & M. Fugelstad (Eds.), *Proceedings of the 28th annual conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 376-383). Bergen, Norway: PME.

Gervasoni, A., & Sullivan, P. (in press). Assessing and teaching children who have difficulty learning arithmetic. *Educational and Child Psychology*.

Gould, P. (2000). Count Me In Too: Creating a choir in the swamp. In *Improving numerary learning: What does the research tell us?* (Proceedings of the ACER Research Conference 2000, pp. 23–26). Melbourne: Australian Council for Educational Research.

Griffin, S., & Case, R. (1997). Re-thinking the primary school math curriculum: An approach based on cognitive science. *Issues in Education, 3*(1), 1-49.

Griffin, S., Case, R., & Siegler, R. (1994). Rightstart: Providing the central conceptual prerequisites for first formal learning of arithmetic to students at risk for school failure. In K. McGilly (Ed.), *Classroom lessons: Cognitive theory and classroom practice* (pp. 25-49). Cambridge, MA: MIT Press/Bradford.

Higgins, J., Parsons, R., & Hyland, M. (2003). The Numeracy Development Project: Policy to practice. In J. Livingstone (Ed.), *New Zealand annual review of education* (pp 157–174). Wellington: Victoria University of Wellington.

Hill, P. W., & Crévola, C. A. (1999). The role of standards in educational reform for the 21st century. In D. D. Marsh (Ed.), *Preparing our schools for the 21st century* (Association of Supervision and Curriculum Development Yearbook, pp. 117-142). Alexandria, VA: ASCD.

Howard, P., Perry, B, & Butcher, J. (2006). *Shared ownership and community capacity building*. Paper presented at the Annual Conference of the Australian Association for Research in Education, Adelaide. (http://www.aare.edu.au/06pap/how06148.pdf)

Mulligan, J. (1998). A research-based framework for assessing early multiplication and division. In C. Kanes, M. Goos & E. Warren (Eds.), *Teaching mathematics in new times* (Proceedings of the 21st annual conference of the Mathematics Education Research Group of Australasia, Vol. 2, pp. 404-411). Gold Coast: MERGA.

Mulligan, J., & Mitchelmore, M. (1996). Children's representations of multiplication and division word problems. In J. Mulligan & M. Mitchelmore (Eds.), *Children's number learning: A research monograph of MERGA/AAMT* (pp. 163-184). Adelaide: Australian Association of Mathematics Teachers.

Narode, R., Board, J., & Davenport, L. (1993). Algorithms supplant understanding: Case studies of primary students' strategies for double-digit addition and subtraction. In J. R. Becker & B. J. Preece (Eds.), *Proceedings of the 15th annual meeting of the North American chapter of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 254–260). San Jose: Center for Mathematics and Computer Science Education, San Jose State University.

Owens, K., & Gould, P. (1999). Framework for elementary school space mathematics. Unpublished Discussion Paper.

Ross, S. (1989). Parts, wholes and place value: A developmental view. *Arithmetic Teacher, 36*(6), 47-51.

Sullivan, P., Clarke, D. M., Cheeseman, J., & Mulligan, J. (2001). Moving beyond physical models in learning multiplicative reasoning. In M van den Heuvel-Panhuizen (Ed.), *Proceedings of the 25th annual conference of the International Group for the Psychology of Mathematics* Education (Vol. 4, pp. 233-240). Utrecht: PME.

Steffe, L., Cobb, P., & von Glasersfeld, E. (1988). *Construction of arithmetical meanings and strategies.* New York: Springer-Verlag.

Steffe, L., von Glasersfeld, E., Richards, J., & Cobb, P. (1983). *Children's counting types: Philosophy, theory, and application.* New York: Praeger.

Wright, R., Martland, J., & Stafford, A. (2000). *Early Numeracy: Assessment for teaching and intervention*. London: Paul Chapman Publishing.

Young-Loveridge, J. (2000). How children's understanding of the number system varies as a function of the ethnicity and socio-economic status. In J. Bana & A. Chapman (Eds.), *Mathematics education beyond 2000 (Proceedings of the 23rd annual conference of the Mathematics Education Research Group of Australasia,* pp. 672-679). Sydney: MERGA.

Technology-Enriched Teaching of Secondary Mathematics: Factors Influencing Innovative Practice

Merrilyn Goos
The University of Queensland
<m.goos@uq.edu.au>

Anne Bennison
The University of Queensland
<a.bennison@uq.edu.au>

This paper reports on the initial phase of a research study that is investigating how and why secondary school mathematics teachers use digital technologies to help their students learn. Case studies of a beginning teacher and an experienced teacher, both of whom are regarded as effective users of technology, aim to identify critical factors that support or hinder innovative teaching and learning. The findings are analysed with the aid of Valsiner's (1997) zone theory to study interactions between teachers' knowledge and beliefs, their professional contexts, and their formal and informal professional development experiences.

For some time, education researchers have been interested in exploring the potential for digital technologies to transform mathematics learning and teaching. It is now widely accepted that effective use of technologies such as mathematical software, spreadsheets, graphics and CAS calculators, and data logging equipment offers students new opportunities for fast, accurate computation, collection, and analysis of real or simulated data, and investigation of links among numerical, symbolic, and graphical representations of mathematical concepts (see Forster, Flynn, Frid, & Sparrow, 2004; Goos & Cretchley, 2004). Support for technology use in secondary school mathematics is also found in most Australian state and territory curriculum documents.

A significant body of research has examined the effects of technology use on students' mathematical achievement and attitudes and their understanding of mathematical concepts, but less attention has been given to how teachers use technology in the classroom and how this use is related to their knowledge, beliefs, and professional contexts. Internationally there is research evidence that simply improving teachers' access to technology has not, in general, led to increased use or to movement towards more learner-centred teaching practices (Burrill, Allison, Breaux, Kastberg, Leathem, & Sanchez, 2003; Cuban, Kirkpatrick, & Peck, 2001; Wallace, 2004). Windschitl and Sahl (2002) identified two factors that appear to be crucial to the ways in which teachers adopt (or resist) technology. First, teachers' use of technology is mediated by their beliefs about learners, about what counts as good teaching in their institutional culture, and about the role of technology in learning. Secondly, school structures – especially those related to the organisation of time and resources – often make it difficult for teachers to take up technology-related innovations. These are some of the issues that we are investigating in a 3-year study of technology-enriched teaching in secondary school mathematics. The overarching aim of the study is to generate models of successful innovation in integrating technology into secondary school mathematics teaching. This paper presents findings from the first year of the study, focusing on factors influencing teachers' use of technology.

Theorising Technology-Enriched Mathematics Teaching

The present study builds on a research program informed by sociocultural theories of learning involving teachers and students in secondary school mathematics classrooms (see Galbraith & Goos, 2003; Goos, 2005). Sociocultural theories view learning as the product of interactions with other people and with material and representational tools offered by the learning environment. Because it acknowledges the complex, dynamic, and contextualised nature of learning in social situations, this perspective can offer rich insights into conditions affecting innovative use of technology in school mathematics. The theoretical framework for the study is based on an adaptation of Valsiner's (1997) zone theory to apply to interactions between teachers, students, technology, and the teaching-learning environment.

The zone framework extends Vygotsky's concept of the Zone of Proximal Development (ZPD) to incorporate the social setting and the goals and actions of participants. Valsiner (1997) describes two additional zones: the Zone of Free Movement (ZFM) and Zone of Promoted Action (ZPA). The ZFM represents constraints that structure the ways in which an individual accesses and interacts with elements of the environment. The ZPA comprises activities, objects, or areas in the environment in respect of which the individual's actions are promoted. For learning to be possible the ZPA must be consistent with the individual's possibilities for development (ZPD) and must promote actions that are feasible within a given ZFM. When we consider teachers' professional learning involving technology, the ZPD represents teachers' knowledge and beliefs about mathematics, mathematics teaching and learning, and the role of technology in mathematics education. The ZFM can be interpreted as constraints within the school environment, such as students (their behaviour, motivation, perceived abilities), access to resources and teaching materials, curriculum and assessment requirements, and organisational structures and cultures, whereas the ZPA represents formal and informal opportunities to learn, for example, from pre-service teacher education, professional development, and colleagues at school.

Previous research on technology use by mathematics teachers has identified a range of factors influencing uptake and implementation. These include: skill and previous experience in using technology; time and opportunities to learn; access to hardware and software; availability of appropriate teaching materials; technical support; organisational culture; knowledge of how to integrate technology into mathematics teaching; and beliefs about mathematics and how it is learned (Fine & Fleener, 1994; Manoucherhri, 1999; Simonsen & Dick, 1997). In terms of the theoretical framework outlined above, these different types of knowledge and experience represent elements of a teacher's ZPD, ZFM and ZPA, as shown in Table 1. However, in simply listing these factors, previous research has not necessarily considered possible relationships between the teacher's setting, actions, and beliefs, and how these might influence the extent to which teachers adopt innovative practices involving technology. In the present study, zone theory provides a framework for analysing these dynamic relationships.

Table 1
Factors Affecting Teachers' use of Technology

Valsiner's Zones	Elements of the Zones
Zone of Proximal Development	Mathematical knowledge Pedagogical content knowledge Skill/experience in working with technology General pedagogical beliefs
Zone of Free Movement	Students (perceived abilities, motivation, behaviour) Access to hardware, software, teaching materials Technical support Curriculum & assessment requirements Organisational structures & cultures
Zone of Promoted Action	Pre-service teacher education Professional development Informal interaction with teaching colleagues

Research Design and Methods

Participants in the first phase of the study are four secondary mathematics teachers who are acknowledged by their peers as effective and innovative users of technology. They include two beginning teachers who experienced a technology-rich pre-service program and two experienced teachers who have developed their technology-related expertise solely through professional development experiences or self-directed learning. The beginning teacher participants were recruited from a pool of recent Bachelor of Education (Secondary) graduates from The University of Queensland, whereas the experienced teacher participants were identified via professional networks, including mathematics teacher associations and contacts with schools participating in other university-based research projects. The teachers were selected to represent contrasting combinations of factors known to influence technology integration (as summarised in Table 1).

In the first year of the study the focus was on carrying out highly contextualised investigations of how and under what conditions the participating teachers integrate technology into their practice. There were three main sources of data. First, a semi-structured scoping interview invited the teachers to talk about their knowledge, beliefs, contexts, and professional learning experiences in relation to technology. A diagrammatic representation of the zone theory of teacher learning outlined in the previous section of the paper was used to structure the interviews. Each zone was represented by a circle, with its elements listed as shown in Table 1, and this information was printed on separate overhead transparencies for the three zones. As the zones themselves are abstractions, teachers "filled in" the details that were relevant to their own professional histories and contexts. They were also asked to superimpose the three transparencies to show the degree of overlap between the circles that matched their own circumstances and hence the relationships between their personal zones of influence. The abstract theoretical language for naming the Zones of Proximal Development, Free Movement, and Promoted Action was not used in these interviews. Instead the zones were labelled as Teacher Knowledge and Beliefs, Professional Contexts, and Sources of Assistance respectively.

Additional information about the teachers' general pedagogical beliefs was obtained via a Mathematical Beliefs Questionnaire (described in more detail in Goos & Bennison, 2002). The questionnaire consisted of 40 statements to which teachers responded using a Likert-type scale based on scores from 1 (Strongly Disagree) to 5 (Strongly Agree). The third source of data was lesson cycles comprising observation and video recording of at least three consecutive lessons in which technology was used to teach specific subject matter, together with teacher interviews at the beginning, middle, and end of each cycle. These interviews sought information about teachers' plans and rationales for the lessons and their reflections on the factors that influenced their teaching goals and methods.

The next section draws on the sources of data outlined above to present the contrasting profiles of two participating teachers, Susie (a beginning teacher) and Brian (an experienced teacher).

Teacher Profiles

Susie: A Beginning Teacher

Susie graduated from the university pre-service program at the end of 2003. She was in her third year of teaching at a co-educational independent school with an enrolment of around 600 students in Years 8 to 12. The student population is fairly homogeneous with respect to cultural and socio-economic background, with most students coming from white, Anglo-Australian, middle class families.

Sally's responses to the Mathematical Beliefs Questionnaire suggest that her beliefs were non-rule-based and student-centred (Tharp, Fitzsimons, & Ayers, 1997). For example, she expressed strong agreement with statements such as "In mathematics there are often several different ways to interpret something", and she disagreed that "Solving a mathematics problem usually involves finding a rule or formula that applies". Her beliefs about mathematics teaching and learning, as revealed through the questionnaire, were strongly supportive of cooperative group work, class discussions, and use of calculators, manipulatives and real life examples. Teachers who hold such inquiry-based views about mathematics are more likely to use calculators as a means of developing students' conceptual understanding than simply as tools for checking calculations or graphs done by hand (Simmt, 1997).

Susie's own experience of learning mathematics at school was very structured and content-based, but this is different from the approaches she tries to implement as a mathematics teacher. When interviewed she explained that in her classroom "we spend more time on discussing things as opposed to just teaching and practising it", and that for students "experiencing it is a whole lot more effective than being told it is so". Aged in her mid-20s, Susie feels she was born into the computer age and this contributes to her comfort with using technology in her teaching. Although her first real experience with graphics calculators was in her university pre-service course, she indicated that "the amount I learned about it [graphics calculators] during that year would be about 2% of what I know now".

Our observations of Susie's Year 10 mathematics class provide evidence of how she enacted her pedagogical beliefs. In one lesson cycle we observed, Susie introduced quadratic functions via a graphical approach involving real life situations and followed this with algebraic methods to assist in developing students' understanding. Lessons typically engaged students in one or two extended problems rather than a large number of practice exercises. For example, students worked on a task that asked them to investigate projectile motion as a practical application of quadratic functions. They viewed a computer simulation in which the Sesame Street character Gonzo was shot from a cannon towards a bucket of water some distance away (http://www.funny-games.biz/flying-gonzo.html; see Figure 1). The simulation allowed students to vary the angle of projection and the cannon "voltage" (i.e., muzzle velocity) and observe the effects on the distance Gonzo travelled as they "aimed" him at the bucket of water. They were to use their graphics calculators to tabulate and plot data that would allow them to find a mathematical model for the relationship between this distance and the muzzle velocity. Algebraic methods were then to be used to determine the best cannon settings for Gonzo to hit a target at a given distance.

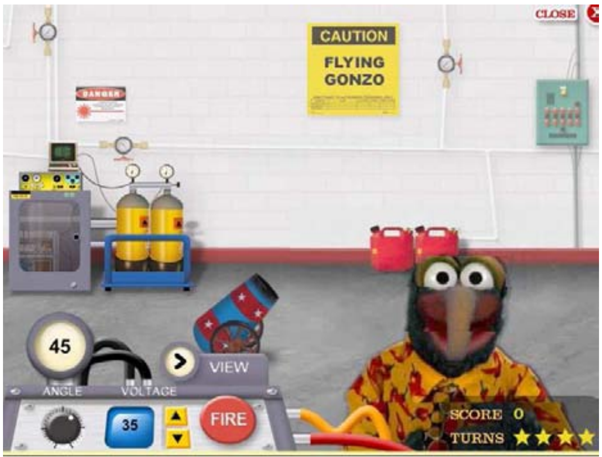


Figure 1. Flying Gonzo simulation.

The questionnaire, interview, and observation data "fill in" Susie's Zone of Proximal Development with knowledge and beliefs about using technology to help students develop mathematical understanding by investigating real life situations and linking different representations of concepts. Elements of her Zone of Free Movement, or professional context, are also supportive of technology integration. Until recently the school's mathematics department was led by a teacher well known for his expertise with technology, and his influence created a culture of technology innovation backed up by substantial resources. Students in Years 9-12 have their own graphics calculators (obtained through the school's hire scheme), there are additional class sets of CAS calculators for senior classes, and data logging equipment compatible with the calculators is freely available. Computer software is also used for mathematics teaching; however, as is common in many secondary schools, computer laboratories have to be booked well in advance. Susie prefers to use graphics calculators so that students can access technology in class whenever they need it. The data projector installed in her classroom also makes it easy for her to display the calculator screen for viewing by the whole class.

Susie spoke enthusiastically of the support she had received from the school's administration and her colleagues since joining the staff: "Anything I think of that I would really like to do [in using technology] is really strongly supported". Nevertheless, as coordinator of the school's junior secondary mathematics programs she has noticed that some of the recently appointed teaching staff are neutral and passive in their attitudes towards technology. Although they are willing to use technology in their teaching if shown how to, they rarely ask questions or engage in discussions about improving existing tasks and technology-based teaching practices. These attitudes do not seem to be related to the number of years they have been teaching or to their previous experience in using technology.

The evidence outlined above suggests that there is a good fit between Susie's Zone of Proximal Development and her Zone of Free Movement, in that her professional environment affords teaching actions consistent with her pedagogical knowledge and beliefs about technology. Susie uses this ZPD/ZFM relationship as a filter for evaluating formal professional development experiences and deciding what to take from these experiences and use in her classroom. For example, she had first seen the Gonzo simulation at a mathematics teachers' conference and realised this was an application of quadratic functions she could exploit with her Year 10 class. Susie had attended many conferences and workshops in the 3 years since beginning her teaching career, but found that most of them were not helpful "for where I am". She explained: "because we use it [technology] so much already, to introduce something else we'd have to have a really strong basis for changing what's already here". Although Susie's exposure to technology in her mathematics pre-service course may have oriented her towards using technology in her teaching, the most useful professional learning experiences have involved working collaboratively with her mathematics teaching colleagues at school. The only real obstacle she faces is lack of time to develop more teaching resources and to become familiar with all of the technologies available to her. For Susie, the most helpful Zone of Promoted Action (sources of assistance) lies largely within her own school, and is thus almost indistinguishable from her Zone of Free Movement (professional context).

Brian: An Experienced Teacher

Brian has been teaching mathematics in government high schools for more than 20 years. For much of this time he was Head of the Mathematics Department in an outer suburban school serving a socio-economically disadvantaged community. In the late 1990s he recognised that the traditional classroom settings and teaching approaches the students were experiencing did not help them learn mathematics. He pioneered a change in philosophy that led to the adoption of a social constructivist pedagogy in all mathematics classes at the school. This new philosophy, expressed through problem solving situations and the use of technology, concrete materials and real life contexts, produced significant improvement in mathematics learning outcomes across all year levels. At the start of 2006 Brian moved to a new position as Head of Department in a different school, also situated in a low socio-economic area. Here he faces many challenges in introducing the mathematics staff to his teaching philosophy and obtaining sufficient technology resources to put his philosophy into practice.

Brian's espoused beliefs, as indicated in his responses to the Mathematics Beliefs Questionnaire, are consistent with the constructivist principles that guide his practice. For example, he expressed disagreement with statements such as "Doing lots of problems is the best way for students to learn mathematics", and he strongly agreed that "The role of the mathematics teacher is to provide students with activities that encourage them to wonder about and explore mathematics". When interviewed, he often emphasised that his reason for learning to use technology stemmed from his changed beliefs about how students learn mathematics.

> When my philosophy changed, it became a question of – what can I put in front of my kids to allow them to access the concepts? So then it didn't really matter what it was, the outcome that I was after

> was them accessing the concept. So it became obvious over time that technology was a way that many students do access concepts that they couldn't, wouldn't normally access.

Some of the lessons we observed dealt with solving trigonometric equations. Brian's method for teaching this topic exemplified his general philosophy in that he initially used a graphical approach to help students develop understanding of the central concepts so they might then see the need for analytical methods involving algebra. He justified this by saying:

> The options are to give them heaps of algebra and watch them fail or try to get them to understand the concepts. If they're confident about what they're doing then I find the algebra's not such a task for them because there's a lot more meaning or reasoning behind it.

A vignette from a Year 11 lesson illustrates this approach. Brian used graphing technologies and probing questions to help students develop a general method for solving trigonometric equations, starting with a straightforward example, $2\sin x + \sqrt{3} = 0$ for $0 \le x \le 2\pi$. He emphasised the critical importance of attending to the domain, as this tells us how many solutions there are. Using his laptop computer and portable data projector, Brian launched the *Autograph* program and displayed the graph of $y = 2\sin x + \sqrt{3}$ shown in Figure 2.

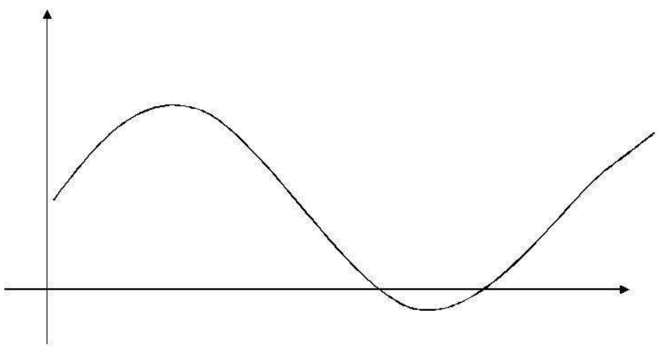


Figure 2. Graph of $y = 2\sin x + \sqrt{3}$.

The students also drew the graph using their graphics calculators, and observed that there are two roots. Brian then announced that they needed to "go into the algebra world", and through careful questioning he led the class through the algebraic process of "unwrapping" the equation. Upon reaching the conclusion that $\sin x = -\frac{\sqrt{3}}{2}$, the students were reminded that they needed instantly to recognise the exact trigonometric ratios for certain angles, in this case 60° or $\frac{\pi}{3}$ radians. Brian explained that "the negative sign tells us a story too", and he guided the students through sketching the unit circle and locating the relevant angles in the third and fourth quadrants as $\frac{4\pi}{3}$ and $\frac{5\pi}{3}$ respectively. The students then used the graphics calculator TRACE function to give meaning to the solutions by entering them as *x*-values and observing that the corresponding *y*-values were zero in both cases: in other words, they had found the points where the curve cut the *x*-axis.

Brian's knowledge and beliefs – his Zone of Proximal Development – were the driving force that led him to integrate technology into his inquiry-based approach to teaching mathematics. When graphics calculators became available in the mid-1990s he attended several professional development workshops presented by teachers in other schools who had already developed some expertise in this area. More recently he won a state government scholarship to travel overseas and participate in conferences that introduced him to other types of technology resources. Apart from these instances Brian has rarely sought out formal professional development, preferring instead to "sit down and just work through it myself". His Zone of Promoted Action, representing sources of assistance for his own learning about technology, is thus highly selective and focused on finding coherence with his personal knowledge and beliefs.

In the 17 years that Brian spent at his previous school he was able to fashion a Zone of Free Movement, or professional context, that gave him the human and physical resources he needed to teach innovatively with technology. However, when he arrived at his current school at the start of 2006 he found little in the way of mathematics teaching resources – "there was a lot of stuff here but it was just in cupboards and broken and not used, and not coherent, not in some coherent program". Mathematics students in this school were not accustomed to technology, even though the use of computers or graphics calculators is mandated by the senior secondary mathematics syllabuses. At the start of the year there were no class sets of graphics calculators and only a few students could afford to buy their own. Because of timetabling and room allocation issues it was also difficult for mathematics classes to gain access to the school's computer laboratories. Exacerbating the problems of limited access to technology resources was an organisational culture that Brian diplomatically described as "old fashioned". Almost none of the mathematics teachers appeared interested in learning to use technology, and it appeared that an atmosphere of lethargy had pervaded the mathematics department for many years. Students demonstrated a similarly passive approach to learning mathematics, expecting that the teacher would "put the rule up and example up and set them up and away they go". Brian responded to these challenges in several ways. First, he lobbied the newly appointed Principal for funds to buy software for the computer laboratories and a data projector for installation in his mathematics classroom. Secondly, he took advantage of the loan schemes operated by graphics calculator companies to borrow some class sets of calculators. He also used his influence as Head of Department to secure a limited number of timetable slots for senior mathematics classes to use the computer laboratories. Brian knows that the Principal is strongly supportive of his teaching philosophy and his plans for expanding the range of technology resources in the school.

Brian evaluates the adequacy of his present Zone of Free Movement, or professional context, by looking through the inquiry-based, technology-rich lens created by the relationship between his ZPD (knowledge and beliefs) and ZPA (previous professional learning). By the end of this first year at his new school, Brian identified his priorities for re-shaping the ZFM as continuing to advocate for the purchase of more technology resources and helping his staff become comfortable and confident in using these resources. He acknowledges that the main obstacles are lack of funds and a teaching culture that resists change.

Conclusion

The research reported in this paper is beginning to examine relationships between factors known to influence the ways in which teachers use technology to enrich secondary school mathematics learning. Our findings so far are consistent with results of other studies of educational uses of technology in highlighting the significance of teachers' beliefs, their institutional cultures, and the organisation of time and resources in their schools (e.g., Windschitl & Sahl, 2002). Although access to technology is an important enabling factor, the profiles of Susie and Brian demonstrate that teachers in well resourced schools do not necessarily embrace technology (compared with Cuban, Kirkpatrick, & Peck, 2001), whereas teachers in poorly resourced schools can be very inventive in exploiting available resources to improve students' understanding of mathematical concepts.

The opportunities that teachers provide for technology-enriched student learning are affected by ways in which they interpret and analyse problems of practice. How do teachers justify and enact decisions about using technology in their classrooms? How do they negotiate potential contradictions between their own knowledge and beliefs about the role of technology in mathematics education and the knowledge and beliefs of their colleagues? How do they interpret aspects of their teaching environments that support or inhibit their use of technology? These questions, when framed within a sociocultural perspective, allow us to investigate systematically conditions affecting teachers' use of technology in mathematics classrooms through the application of Valsiner's (1997) zone theory – where the Zone of Proximal Development represents the possibilities for teacher learning shaped by their knowledge and beliefs, the Zone of Free Movement environmental constraints, and the Zone of Promoted Action the nature of specific activities that promote new pedagogical skills and understanding.

Both Susie and Brian held productive beliefs about mathematics and the role of technology in mathematics learning (ZPDs); however, they differed in the degree of fit between their respective ZPDs and ZFMs. For Susie, the Zone of Free Movement offered by her school was most important in allowing her to explore technology-enriched teaching approaches consistent with her knowledge and beliefs. It may be that the extent of overlap between the ZFM and the ZPD is critical in supporting beginning teachers in further developing the innovative practices they typically encounter in pre-service programs. On the other hand, Brian, as an experienced teacher and Head of Department, relied on his knowledge and beliefs about learning with technology to envision the kind of professional environment, or ZFM, he wanted to create in his new school. For him, the ZPD/ZFM mismatch was a powerful incentive to pursue his goal of technology-enriched mathematics teaching and learning. These tentative proposals will be tested as we continue to work with Susie and Brian, and the other participating teachers, throughout the remainder of the research study.

Acknowledgement. This project is funded by ARC Discovery Grant No. DP0664415.

References

Burrill, G., Allison, J., Breaux, G., Kastberg, S., Leatham, K., & Sanchez, W. (2003). *Handheld graphing technology in secondary mathematics: Research findings and implications for classroom practice.* East Lansing, MI: Michigan State University.

Cuban, L., Kirkpatrick, H., & Peck, C. (2001). High access and low use of technologies in high school classrooms: Explaining an apparent paradox. *American Educational Research Journal*, *38*, 813-834.

Fine, A. E., & Fleener, M. J. (1994). Calculators as instructional tools: Perceptions of three preservice teachers. *Journal of Computers in Mathematics and Science Teaching*, *13*(1), 83-100.

Forster, P., Flynn, P., Frid, S., & Sparrow, L. (2004). Calculators and computer algebra systems. In B. Perry, G. Anthony, & C. Diezmann (Eds.), *Research in mathematics education in Australasia 2000-2003* (pp. 313-336). Flaxton, QLD: Post Pressed.

Galbraith, P., & Goos, M. (2003). From description to analysis in technology aided teaching and learning: A contribution from zone theory. In L. Bragg, C. Campbell, G. Herbert, & J. Mousley (Eds.), *Mathematics education research: Innovation, networking, opportunity* (Proceedings of the 26th annual conference of the Mathematics Education Research Group of Australasia, Geelong, Vic, pp. 364-371). Sydney: MERGA.

Goos, M., & Bennison, A. (2002). Building learning communities to support beginning teachers' use of technology. *Proceedings of the annual conference of the Australian Association for Research in Education, Brisbane*. Retrieved 21 March 2007 from http://www.aare.edu.au/02pap/goo02058.htm

Goos, M. (2005). A sociocultural analysis of the development of pre-service and beginning teachers' pedagogical identities as users of technology. *Journal of Mathematics Teacher Education*, *8*(1), 35-59.

Goos, M., & Cretchley, P. (2004). Teaching and learning mathematics with computers, the internet, and multimedia. In B. Perry, G. Anthony, & C. Diezmann (Eds.), *Research in mathematics education in Australasia 2000-2003* (pp. 151-174). Flaxton, QLD: Post Pressed.

Manoucherhri, A. (1999). Computers and school mathematics reform: Implications for mathematics teacher education. *Journal of Computers in Mathematics and Science Teaching*, *18*(1), 31-48.

Simmt, E. (1997). Graphing calculators in high school mathematics. *Journal of Computers in Mathematics and Science Teaching*, *16*(2/3), 269-289.

Simonsen, L. M., & Dick, T. P. (1997). Teachers' perceptions of the impact of graphing calculators in the mathematics classroom. *Journal of Computers in Mathematics and Science Teaching*, *16*(2/3), 239-268.

Tharp, M. L., Fitzsimmons, J. A., & Ayers, R. L. B. (1997). Negotiating a technological shift: Teacher perception of the implementation of graphing calculators. *Journal of Computers in Mathematics and Science Teaching*, *16*, 551-575.

Valsiner, J. (1997). Culture and the development of children's action: A theory of human development (2nd ed.). New York: John Wiley & Sons.

Wallace, R. (2004). A framework for understanding teaching with the Internet. *American Educational Research Journal*, *41*, 447-488.

Windschitl, M., & Sahl, K. (2002). Tracing teachers' use of technology in a laptop computer school: The interplay of teacher beliefs, social dynamics, and institutional culture. *American Educational Research Journal*, *39*, 165-205.

Supporting an Investigative Approach to Teaching Secondary School Mathematics: A Professional Development Model

Merrilyn Goos
The University of Queensland
<m.goos@uq.edu.au>

Shelley Dole
The University of Queensland
<s.dole@uq.edu.au>

Katie Makar
The University of Queensland
<k.makar@uq.edu.au>

This paper describes a project that supported a group of secondary mathematics teachers in implementing the new Queensland *Mathematics Years 1-10 Syllabus*. The purpose of this paper is to evaluate the effectiveness of the professional development model that was used to assist teachers move towards an investigative approach to "working mathematically". The model integrates a zone-theoretical approach to understanding teacher learning into a framework for designing professional development of mathematics teachers. The effectiveness of the model is evaluated via case studies of teachers' professional learning throughout the project and examination of the impacts on their teaching and assessment practices.

Background

Like all other key learning area syllabuses in Queensland, the recently published *Mathematics Years 1-10 Syllabus* (Queensland Studies Authority, 2004) has an outcomes focus that gives it a different structure from syllabuses previously developed in this state. Instead of specifying what should be learned in particular years or grades of school, the mathematics syllabus is organised around (a) overall learning outcomes that collectively describe attributes of lifelong learners, (b) key learning area outcomes that describe how students think, reason, and work mathematically, and (c) core learning outcomes that describe what students should know and do with what they know in the strands of Number, Patterns and Algebra, Measurement, Chance and Data, and Space. The challenge for teachers implementing the new syllabus lies not only in using the new structure for curriculum planning, but also in designing learning experiences and assessment tasks that take an *investigative approach* to "working mathematically".

An investigative approach to the teaching and learning of mathematics aligns with curriculum reform movements in mathematics education (e.g., National Council of Teachers of Mathematics (NCTM), 2000; Australian Education Council, 1991). Contrasting a traditional rule-based, skill mastery approach to teaching of mathematics, reformist goals include promoting students' communication skills and problem solving capacities, and enabling students to experience the actual processes through which mathematics develops (e.g., conjecture, generalisation, proof, refutation) (Australian Education Council, 1991). These goals resonate with the key learning area outcomes of the Queensland *Mathematics Years 1-10 Syllabus*, which emphasise reasoning, problem solving, communication, and investigation. The importance of an investigative approach to teaching of mathematics has been highlighted in recent classroom based research. For example, the TIMSS Video Study (Hollingsworth, Lokan, & McCrea, 2003) revealed that in Australian classrooms there was little emphasis on developing deep understanding of mathematical concepts or the connections between them. Stacey (2003) described this cluster of features as constituting a syndrome of shallow teaching, where students experience a diet of excessive repetition and problems of low complexity, with very few opportunities for mathematical reasoning. Similar findings were reported by the Queensland School Reform Longitudinal Study (Lingard et al., 2001), a large scale research project involving observations of nearly 1000 lessons across all secondary year levels and subject areas. Mathematics lessons were often found to offer low levels of intellectual quality and connectedness, suggesting that students were given few opportunities to develop higher order thinking and deep understanding, and to appreciate connections between mathematics and the real world. The QSRLS also found that teachers often set assessment tasks that were low in intellectual demand and unconnected to the world outside school.

Implementation of the *Queensland Mathematics Years 1-10 Syllabus* asks teachers to expand their pedagogical and assessment repertoires to include more investigative approaches to "working mathematically"; yet research has revealed how difficult it is for teachers to change their practices to enact curriculum reform (Remillard & Bryans, 2004). This paper reports on a research and development project that supported secondary school teachers in planning and implementing mathematical investigations, consistent with the intent of the new Queensland syllabus. The purpose of this paper is to evaluate the effectiveness of the professional development model that was used to support an investigative approach to mathematics teaching and assessment.

Designing the Professional Development Model

Previous research by Goos (2005a, 2005b) investigated how teachers learn from experience in complex environments, using a theoretical model that re-interprets and extends Vygotsky's concept of the Zone of Proximal Development (ZPD) to incorporate the social setting (Zone of Free Movement, ZFM) and the goals and actions of participants (Zone of Promoted Action, ZPA). In this model, the ZPD represents *teacher knowledge and beliefs*, and includes teachers' disciplinary knowledge, pedagogical content knowledge, and beliefs about their discipline and how it is best taught and learned. The ZFM represents constraints within the *professional context*. These may include teacher perceptions of student background, ability and motivation, curriculum and assessment requirements, access to resources, organisational structures and cultures, and parental and community attitudes to curriculum and pedagogical change. The ZPA represents the *sources of assistance* available to teachers that define which teaching actions are specifically promoted. This assistance is typically provided by colleagues and mentors in a school or by formal professional development activities. To understand teacher learning, it is necessary to investigate relationships between these three zones of influence.

Much is known about designing effective professional development to bring about changes in the way that mathematics is taught in schools (Mewborn, 2003). Change is a long term, evolutionary process that can be supported by giving teachers opportunities to engage with mathematical concepts and focus on their own students' thinking as they struggle to understand these concepts. Professional development is most effective when it occurs in school-based contexts so teachers can try out and validate ideas in their own classrooms. Teachers also need time and opportunities to discuss pedagogical and curricular issues with supportive colleagues as they attempt to implement new practices.

Loucks-Horsley, Love, Stiles, Mundry, and Hewson (2003) created a framework for designing professional development that incorporates the research findings outlined above, and captures the decision making processes that are ideally involved in planning and implementing programs (shaded boxes and "bubbles" in Figure 1).

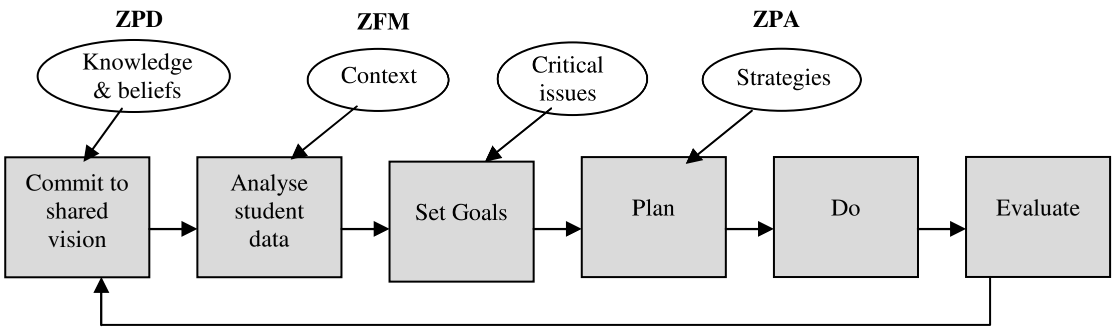


Figure 1. Design framework for professional development.

The planning sequence begins with teachers making a commitment to enhance teaching and learning, thus acknowledging that a tension exists between the current reality and the vision of mathematics teaching offered by new curriculum documents. In practice, it is not always feasible to delay the start of the professional development program until a whole school or group of teachers has established a shared commitment; instead the process of developing this commitment and vision can continue throughout the program and is iterative with other phases of the design. Teacher knowledge and beliefs are an important input into this phase (cf the ZPD discussed above). Analysis of student learning data sharpens the focus on setting targets for improvement and establishing goals for teacher learning and development. It is important here to study the context in order to know who the students are and what teachers know and believe, to identify significant features of the learning environment, and to understand the school's organisational structures and cultures, the local curriculum context, and the views of parents and the community members (cf the ZFM discussed above). The framework suggests anticipating critical issues at the goal setting phase because each of these issues can influence the effectiveness of the program at some point. Planning for professional development can then draw on a wide range of strategies to achieve desired goals (cf the ZPA discussed above).

For this project, we took the Loucks-Horsley et al. (2003) design framework for professional development and considered other literature on effective professional development to plan an overall strategy that is best described as action research. The five key elements of action research (as identified by Loucks-Horsley et al., 2003) guided implementation of the professional development model. The key elements emphasise the need for teachers to own the research project in order for real change to be actualised. They include: (1) teachers devising their own research questions; (2) teachers engaging in the action research cycle; (3) teachers linking with external support mechanisms; (4) teachers working collaboratively; and (5) teachers sharing and disseminating their project with peers. Table 1 summarises how our approach in this project attended to these five key elements.

We envisioned the project as a series of iterative cycles, with teachers coming together to discuss their school-based plans and meet other project teachers, then return to their schools to implement investigative units of work in mathematics, then meet together with project teachers to share their experiences and plan further units of work, as well as an on-site visit to all teachers' classrooms to gain insight into their classroom and school context.

Table 1
Relationship Between Action Research Elements and Project Design

Elements of action research	Project design
Teachers contribute to or formulate their own questions, and collect the data to answer these questions.	Teachers would be encouraged to identify goals relevant to their learning needs and professional context.
Teachers use an action research cycle (set goals, plan, implement, evaluate).	The action research cycle integrates our zone-theoretical model of teacher learning with the Loucks-Horsley et al. (2003) framework for designing professional development.
Teachers are linked with sources of knowledge and stimulation from outside their schools.	The research team would act as a resource for teachers, providing literature on mathematics teaching and assessment as well as exemplary tasks, and advice on collection and analysis of student data.
Teachers work collaboratively.	A pair of teachers would be invited to volunteer from each school so each participant would have continuous collegial support. Pairs would be brought together for professional development meetings with the researchers.
Learning from research is documented and shared.	Teachers would present their work at conferences organised by Education Queensland and attended by key personnel involved in supporting syllabus implementation.

Sources of data to analyse the effectiveness of the project would need to include information about the contexts of the teachers. An audio-taped whole-group interview, where teachers described their teaching situation as well as their personal mathematics teaching history, was planned for the first meeting with the teachers. They would also be asked to complete a Mathematical Beliefs Questionnaire (Frid, 2000) comprising 40 Likert style items about the nature of mathematics, mathematics teaching and mathematics learning. Other planned data sources included the teachers' units of work and student work samples, as well video-taped footage of their classrooms.

Implementing the Professional Development Model

Schools in the region in which the study took place were invited to participate in the project. Schools were specifically requested to nominate pairs of teachers so that they could support each other throughout the project. It was also requested that teachers voluntarily come to this project. The four pairs of teachers who volunteered to participate in the project were from four schools in or near a regional Queensland city. Two schools were in this regional city (Cunningham and Churchill State High Schools), one was in a small rural town nearby (Sugartown State High School), whereas the fourth school was located in a coastal resort approximately 125 km from the regional centre (Seaside State High School). We made three visits to this city to work with the whole group of teachers, in the period from October 2005 to February 2006, each time for two consecutive days. The venue for these meetings was a well equipped computer laboratory in one of the participating schools.

On our first visit to work with the teachers (October 2005) we gathered information about their *knowledge and beliefs* and their *professional contexts* via the Mathematical

Beliefs Questionnaire and the structured audio-recorded whole group interview. This enabled us to map their respective Zones of Proximal Development and Free Movement. The first meeting was also aimed at helping the teachers to identify their personal goals for the project, to demonstrate ways to collect data on their students' beliefs and attitudes towards mathematics, and to provide time and support for them to begin planning units of work to implement in their classrooms. We also wanted to engage these teachers as learners in mathematical investigations on these meeting days. As a *source of assistance* that was deliberately promoting new teaching approaches (Zone of Promoted Action), the professional development model recognized the importance of providing teachers with authentic, practice-based learning opportunities that included examples of mathematical investigations, opportunities to experience these investigations as learners themselves before planning their own investigations and trying them out with their students, and opportunities to share their ideas and experiences with colleagues, including the challenges encountered and their insights into the process. On each of the meeting days, the teachers were provided with some ideas for investigative approaches in particular topics in mathematics. Teachers were also encouraged to share their own ideas of investigative units that they used with their own classes.

In the second visit (November 2005), we facilitated a debriefing discussion of successes and problems each teacher had experienced in implementing their new investigative units. We also modelled the development of assessment criteria for mathematical investigations, and assisted teachers with planning units of work for the start of the 2006 school year. By the third visit (February 2006), sufficient familiarity and trust had been established for us to visit the pairs of teachers in their schools to observe and discuss implementation of the investigative units. The first author visited Sugartown State High School, the second author Churchill and Cunningham, and the third author Seaside. We observed and videotaped at least one lesson during each school visit, and the researcher and teachers then discussed the lesson while watching the video together. This discussion was audio-recorded for later review and analysis. On the second day of this visit we conducted a whole group discussion to evaluate the project and identify implications for extending similar professional development opportunities to teachers in other schools.

Data collected during the project, which comprised interview records, completed questionnaires, videotapes and field notes of lessons, student work samples, and teacher planning documents, were analysed by interpreting the particular circumstances under which elements of the professional development model (Figure 1) were "filled in" with specific people, actions, places, and meanings.

Effectiveness of the Professional Development Model

We evaluated the effectiveness of the professional development model by examining how the teachers negotiated opportunities and hindrances in pursuing investigative approaches to mathematics teaching and assessment. Relevant information is summarised in Figure 2.

From Figure 2, it can be seen that the four pairs of teachers came from quite contrasting school contexts, with three pairs of teachers being rated as having student-centred beliefs and one pair having teacher-centred beliefs. Three of the schools had quite structured and traditional approaches to teaching of mathematics, with one school having a very flexible approach.

Element of PD Model	Schools and Teachers			
	Sugartown Skye & Chris	Seaside Val & Shanti	Cunningham Peter & Ron	Churchill Tony & Ralph
Knowledge & beliefs (ZPD)	Qualified in maths & maths ed. Student-centred beliefs	Mixed quals in maths & maths ed. Student-centred beliefs	Qualified in maths & maths ed. Student-centred beliefs	Mixed quals in maths & maths ed. Teacher-centred beliefs
Professional context (ZFM)	Low achieving Ss Poorly resourced Classes streamed HOD supportive of change Low SES Little parental support for school	Test & textbook dominated practices Poorly resourced Classes streamed via frequent tests Organisational culture resistant to reform approaches	High achieving Ss take Project Maths extension Other subjects use traditional methods Well resourced Flexible timetable Ts plan & teach together	School has strong academic reputation Lecture approach + streamed classes Other Ts resistant to change Tony as HOD of Middle Years seeks curriculum reform
Goals	Engaging learners in meaningful mathematics	Making assessment more authentic and practical	Making Project Mathematics mainstream	Integrating maths with other KLAs

Figure 2. Teacher characteristics, contexts, goals, and longer term impact.

Beyond the information presented in Figure 2, the following summary describes the context and beliefs of the four pairs of teachers when they first started in the project.

Skye and Chris: student-centred beliefs, teaching in a school that has little parental support and low achieving students. However, at this school, a new Year 8 class of students who demonstrated low mathematics outcomes had been created, and the project felt supported at their school to try new approaches with this class.

Val and Shanti: student-centred beliefs, teaching at a school where traditional approaches to mathematics teaching were expected. These two teachers felt that there was a better way to teach mathematics than what was expected at their school. They felt comfortable trying new ways of teaching in their own classrooms, but felt their fellow mathematics teachers disapproved of such approaches.

Peter and Ron: student-centred beliefs, were already implementing a mathematics program which took an investigative approach with extension classes. The goal for these two teachers was to try to make their investigative maths classes integrate into the mainstream classes.

Tony and Ralph: teacher-centred beliefs, high academic student outcomes. Although these two teachers came from the same school, they had quite different reasons for volunteering for the project. One was a science teacher who had just been made middle years coordinator. This teacher wanted to support fellow teachers in the middle years to take a more integrated approach to teaching. The second teacher was teaching a low achieving Year 9 mathematics class and he was hoping to develop new approaches for engaging these learners.

Information in Figure 2 highlights the diversity of professional contexts featuring potentially helpful and unhelpful influences on the teachers participating in the project, however, it does not show how these influences interacted to either support or hinder teacher learning. Therefore, an example of such interactions is provided in the abbreviated case study that follows.

Case Study of Teacher Learning: Skye and Chris

Skye wanted to take a more investigative approach to teaching her new Year 8 Practical Mathematics class of students who were not achieving success in regular mathematics classrooms. Chris was mainly teaching senior classes and wanted help in planning new programs and devising new forms of assessment. Questionnaire responses revealed that they both held similar beliefs about the nature of mathematics, and mathematics teaching and learning. For example, they agreed that there are many ways of interpreting and solving a problem, and that it is important to encourage students to build their own mathematical ideas. However, other responses showed they were uncertain about the benefits of more traditional approaches such as memorisation and practice. This suggests that Skye and Chris were interested in moving towards more student-centred, investigative teaching practices, but that they needed to try out these practices with their own classes to find out whether this would lead to improved learning.

Skye and Chris stated that the most frustrating obstacle in their professional context was the students themselves, and their apparent lack of interest in learning. This was evident in the students' disruptive and uncooperative behaviour, and their frequently stated belief that they were "dumb" and simply could not do mathematics. The experience of teaching unmotivated students led these teachers to formulate a goal of *engaging learners*, or, as Skye explained, "for them to learn maths without being terrified of it". Both saw investigations as a way of presenting mathematics differently that would allow them to make mathematics more interesting for students by engaging them in purposeful tasks with real world relevance.

With full support of their Head of Department, Skye and Chris decided to team-teach the Practical Mathematics class and their teaching timetable was altered to enable this to occur. Skye and Chris's first unit of work asked students to investigate whether it is more economical to buy groceries in Sugartown or drive to the larger regional centre nearby. After reflecting on the mixed outcomes of their first unit, they then planned a unit that they hoped would more closely connected to students' lives. Their "School Rage" investigation asked students to create a Top 20 song list for the school radio station, based on a survey of students attending the school. To make the task more realistic, a letter from the "radio station manager" (one of the mathematics teachers) was given to the students asking for assistance in designing a new radio program similar to the Rage Top 20. The group submitting the best quality report would have their Top 20 songs played on the radio station during a designated lunchtime. Thus the task had an authentic purpose and a real audience comprising the entire school community. Core learning outcomes embedded in this task related to designing and carrying out data collections, using data record templates, organising data and creating suitable displays, making comparisons about data, and working with whole numbers, fractions and percentages.

Classroom observations confirmed the teachers' judgment that students were deeply engaged in the investigation. Overheard comments suggested that the students welcomed this new approach. In their own evaluation of the units, Skye and Chris not only identified the benefits for the students (engagement, confidence, alternative opportunities to demonstrate their learning) but also the challenges the new approach presented to them. They were now spending more class time responding to unanticipated ways students tackled investigations, often by asking questions to scaffold students' thinking, such as "What does it mean if you include the same person twice in your survey?" and "What if this person votes for two different songs?" Skye pointed out that she welcomed such unexpected responses as she regarded it as a sign of growth of sophistication in students' thinking.

Skye and Chris identified several reasons why they had been successful in implementing an investigative approach. They often emphasised the importance of taking into account the students' prior experience and interests, and the local context of the school and community. Access to sample investigations was critical, as was access to human resources in the form of a supportive school administration team, a network of like minded mathematics teachers across the schools participating in the project, and their teaching partner. Planning and teaching as a team, rather than individuals, was a significant benefit for both teachers because they recognised that this reduced their workload, expanded their repertoire of teaching strategies, and provided opportunities for mutual observation and feedback. Skye and Chris's professional learning experience is summarised by the relationships between their knowledge and beliefs (ZPD), professional context (ZFM), and sources of assistance (ZPA), shown in Figure 3. Although they experienced hindrances within their professional context, productive tensions between aspects of the context and their pedagogical beliefs led them to formulate and pursue the goal of engaging learners.

Discussion

Our evaluation of the professional development model was guided by the zone-theoretical model of teacher learning outlined earlier in the paper. For each of the four case studies of pairs of teachers, we were able to identify a different configuration of *teacher knowledge and beliefs* (Zone of Proximal Development), *professional contexts* (Zone of Free Movement), and *sources of assistance* (Zone of Promoted Action), and how these factors came together to shape opportunities for teacher learning. A sample case study illustrated one such configuration. Although there were some differences in the teachers' espoused beliefs about mathematics and how it is best learned and taught, all of them came to the project looking for inspiration and ideas about taking a more investigative approach to their classroom practice, and some were already experimenting with investigative approaches to mathematics teaching. Nevertheless the teachers commented that it was unlikely significant change would have occurred without the impetus provided by this project, because the opportunity to participate validated the changes in teaching and assessment practices that they wanted to achieve. The credibility and authority they gained from participation were vital for helping them deal with relatively inflexible organisational structures and resistance from more traditionally minded mathematics teachers in their schools. Several of the teachers also commented that working with university researchers had enhanced their status as professionals in the eyes of their colleagues. Although these teachers worked in diverse professional contexts that offered both opportunities for, and hindrances to, innovation, all were able to draw on their knowledge and beliefs and the sources of assistance available to them to plan and implement teaching approaches consistent with the intent of the new syllabus.

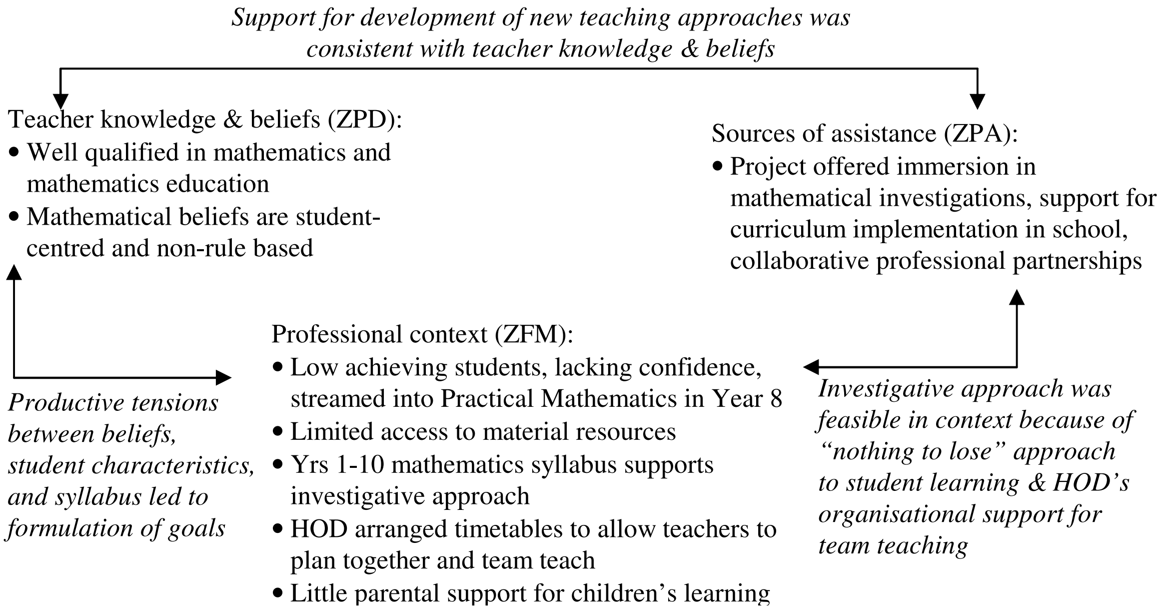


Figure 3. Relationships between professional learning factors for Skye and Chris.

With regard to the professional development model, three clusters of features seemed to contribute to the overall outcomes of the project. The first cluster centres on *professional development processes* involving formulation of realistic goals, provision of long term experiences rather than one-off workshops, and opportunities for teachers to teach and assess student learning during the units implemented. A second cluster of features acknowledges the *resources* required, such as curriculum materials that align with the syllabus, time for planning and reflection with colleagues, and administrative support and commitment. The third set of features focuses on *roles and relationships*, such as the voluntary nature of teachers' participation, acknowledgement of the equal but different contributions made by teachers and researchers, and the importance of broadening participants' perspectives beyond the scope of classroom or school.

Conclusion

Loucks-Horsley et al. (2003) identify several critical issues that must be taken into account when planning a professional development program, preferably in the goal setting phase (see Figure 1). Although we were conscious of these issues throughout the project, their influence is best analysed by looking to the future and asking how might we improve on the conduct of this project in the light of our experiences and what are the implications for extending similar professional development opportunities to secondary mathematics teachers in other schools. One critical issue concerns the need for *building a professional culture* characterised by a strong vision of learning and collegial interactions between teachers. A second issue involves *developing leadership* in teachers who have the capacity to improve the quality of teaching and learning in their schools. Often the most powerful leadership exercised by teachers is simply in modelling new practices for colleagues to demonstrate that they actually work with students. *Building capacity for sustainability* is necessary to ensure that any changes achieved within the life of a professional development project are sustained after it ends. Similarly, *scaling up* is a vital concern for education systems as teachers and school districts implement new teaching and learning approaches. Finally, *gaining public support* for mathematics education is necessary for building consensus around curriculum and pedagogical reform, thus leading to a more informed public understanding of effective methods for teaching mathematics and of the role of mathematics in preparing young people for productive work, leisure, and citizenship.

Acknowledgements. This project was funded by the Queensland Department of Education and the Arts. We gratefully acknowledge the contributions of participating teachers and schools.

References

Australian Education Council. (1991). *A national statement on mathematics for Australian schools.* Melbourne: Curriculum Council.

Frid, S. (2000). Constructivism and reflective practice in practice: Challenges and dilemmas of a mathematics educator. *Mathematics Teacher Education and Development*, *2*, 17-33.

Goos, M. (2005a). A sociocultural analysis of learning to teach. In H. Chick & J. Vincent (Eds.), *Proceedings of the 29th conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 49-56). Melbourne: PME.

Goos, M. (2005b). A sociocultural analysis of the development of pre-service and beginning teachers' pedagogical identities as users of technology. *Journal of Mathematics Teacher Education*, *8*(1), 35-59.

Hollingsworth, H., Lokan, J., & McCrae, B. (2003). *Teaching mathematics in Australia: Results from the TIMSS 1999 video study*. Melbourne: Australian Council for Educational Research.

Lingard, R., Ladwig, J., Mills, M., Bahr, M., Chant, D., Warry, M., Hayes, D., Gore, J., & Luke, A. (2001). *The Queensland School Reform Longitudinal Study*. Brisbane: Education Queensland.

Loucks-Horsley, S., Love, N., Stiles, K., Mundry, S., & Hewson, P. (2003). *Designing professional development for teachers of science and mathematics* (2nd ed.). Thousand Oaks, CA: Corwin Press.

Mewborn, D. (2003). Teaching, teachers' knowledge, and their professional development. In J. Kilpatrick, W. G. Martin, & D. Schifter (Eds.), *A research companion to principles and standards for school mathematics* (pp. 45-52). Reston VA: NCTM.

National Council of Teachers of Mathematics (2000). *Principles and Standards for School Mathematics*. Reston, VA: NCTM.

Queensland Studies Authority (2004). *Mathematics Years 1-10 syllabus*. Brisbane: Office of the Queensland Studies Authority.

Remillard, J., & Bryans, M. (2004). Teachers' orientations toward mathematics curriculum materials: Implications for teacher learning. *Journal for Research in Mathematics Education*, *35*, 352-388.

Stacey, K. (2003). The need to increase attention to mathematical reasoning. In H. Hollingsworth, J. Lokan, & B. McCrae, *Teaching mathematics in Australia: Results from the TIMSS 1999 Video Study* (pp. 119-122). Melbourne: Australian Council for Educational Research.

Identity and Mathematics: Towards a Theory of Agency in Coming to Learn Mathematics

Peter Grootenboer
Charles Sturt University
<pgrootenboer@csu.edu.au>

Robyn Zevenbergen
Griffith University
<r.zevenbergen@griffith.edu.au>

In writing this paper we draw considerably on the work of Jo Boaler and Leone Burton. Boaler's studies of Railside have been particularly poignant in alerting the mathematics education community to a number of key features of successful classrooms, and how such features can turn around the successes for students who traditionally perform poorly in school mathematics. This is supplemented by the more recent work of Leone Burton who worked extensively with research mathematicians in order to understand their communities and ways of working. Collectively these two seminal works provide valuable insights into potential ways to move the field of school mathematics forward. In times where there is international recognition of the plight of school mathematics, there is a need for new teaching practices that overcome the hiatus of contemporary school mathematics.

For a long time now we have known that there have been serious problems with mathematics participation and engagement. The desperate situation facing mathematics has been highlighted recently in Australia by two significant reviews into the mathematical sciences:

1. Statistics at Australian Universities (Statistical Society of Australia, 2005)
2. Mathematics and Statistics: Critical Skills for Australia's Future (Australian Academy of Science, 2006).

Although these reviews were conducted in Australia, a similar story has emerged around the world and it is now approaching a crisis situation. In theses reviews, particular attention has rightly been given to school mathematics and the problems of non-engagement with an increasing number of students in higher level courses of mathematical study. That said, we have known for a long time that mathematics has been unpopular and disliked through the many descriptive studies that have been undertaken since the 1970's, and yet the problems appear to grow unabated and little progress has been made to arrest the decline. At this critical point we want to suggest that it is time to move on from studies that repetitively show that mathematics is suffering from a "poor image" and a "lack of friends", and to try and look forward by offering some positive directions to arrest the decline. To advance this agenda we need more than good ideas that seemed to have worked in a particular context; we need to begin developing a theoretical, robust framework that will address these concerns in a coherent and holistic fashion. In this paper we have drawn on the seminal works of Burton and Boaler to consider mathematical learning from both the discipline knowledge and the mathematical activity perspectives. After reviewing Burton's findings from her study with research mathematicians we briefly highlight some relevant points from Boaler's study of Railside. After presenting an example from teacher education we finish by employing the metaphor of a "dance of agency" (Pickering, 1995) to discuss mathematics learning, particularly in the light of the current crisis.

Proceedings of the 30th annual conference of the Mathematics Education Research Group of Australasia

The Practice of Mathematicians

The two recent reviews of mathematical sciences in Australia mentioned earlier both made significant comment and recommendations for school mathematics education. Interestingly, the authors of these reports were mathematical scientists and there appeared to be little input from mathematics educators and mathematics teachers. Although this is problematic, it does perhaps highlight the gap that seems exist between mathematicians and statisticians, and teachers and educators. This is unhealthy and if the current decline in participation and interest in mathematics is to be arrested then these groups need to engage in dialogue and mutual projects. To this end, the work of Burton (1999, 2001, 2002) is helpful because her research explored the practices of research mathematicians and their implications for the learning of mathematics.

In 1997, Burton studied the practices of 70 research mathematicians in Great Britain and one of the key features she identified was the collaborative nature of their practice. The benefits for collaborating included practical (e.g., sharing the work), quality (e.g., greater range of ideas on problems), educational (e.g., learning from one another) and emotional (e.g., feeling less isolated) reasons. Clearly working together with other mathematicians was seen as important, but there appeared to be a distinction between the public perception of mathematics as a lonely enterprise and the reality of mathematicians' practice where collaboration is highly valued.

Perhaps another anomaly from public perception was Burton's finding that mathematicians have emotional, aesthetic, and personal responses to mathematics.

> … although knowing when you know is extremely important, you have to live with uncertainty. You gain pleasure and satisfaction from the feelings that are associated with knowing. These feelings are exceptionally important since, often despite being unsure about the best path to take to reach your objective, because of your feelings you remain *convinced* that a path is there. … This is particularly poignant in the light of the picture painted of mathematics as being emotion-free … (Burton, 1999, p. 134)

The mathematicians in her study highlighted the power of the "aha!" moment and the joy of mathematical discovery, revealing the clear link between mathematics and those who produce it. Allied to their emotional responses to their mathematical practice were aesthetic reactions. They described mathematics in terms such as "wonder", "beauty", and "delight" and these personal responses provided motivation for continued engagement and fuelled a passion for the discipline of mathematics. Davis and Hersh (1998, p. 169) lamented that "blindness to the aesthetic element in mathematics is widespread and can account for the feeling that mathematics is dry as dust, as exciting as a telephone book …".

Another feature of research mathematicians practice was the importance of intuition or insight. Although the mathematicians were less than clear in describing what intuition and/or insight were, they were unambiguous in highlighting their importance in their mathematical practice. The suggestion was that intuition can be developed through the application of knowledge and experience in mathematical discovery and reflection upon such investigations.

Burton highlighted other features of the practice of mathematicians including the desire to seek and see rich connections between the various branches of mathematics and between mathematics and other disciplines, but her other main agenda was to highlight the pedagogical implications of her findings. Throughout her reports Burton highlights the distinction that is evident between the work and learning practices of research mathematicians, and the learning experiences of mathematics students at almost all other levels from preschool to undergraduate degrees. This led her to assert that "we have a responsibility to make the learning of mathematics more akin to ho" in the absence of any student's need to know" (Burton, 2001, p. 598). Even at a very general level, this would require mathematical pedagogy to be characterised by collaboration and group work with attention paid to the emotional, aesthetic, and intuitive dimensions of the discipline. This encompasses the "doing" of mathematics that has been under-emphasised in education as it has focussed on the "knowing" of mathematics. Indeed, perhaps an issue with the educational recommendations in the Australian review of mathematical sciences was the emphasis on mathematical content knowledge that can be taught largely through a transmission model. On this point Boaler (2003) commented:

> There is a widespread public perception that good teachers simply need to know a lot. But teaching is not a knowledge base, it is an action, and teacher knowledge is only useful to the extent that it interacts productively with all the different variables in teaching. Knowledge of subject, curriculum, or even teaching methods, need to combine with teachers' own thoughts and ideas as they too engage in something of a conceptual dance. (p. 12)

In her seminal work in England, Boaler (1997) explored the mathematical practices of teachers and students in two different sorts of mathematics classrooms. In one group of classes, the mathematical pedagogy was "traditional" and the students learned standard algorithms through worked examples and textbook exercises. The other classrooms were characterised by open-ended projects, group work and discussion. Not surprisingly, she found that the students by and large learned a form of mathematics that was consistent with the mathematical epistemology and pedagogy of their classroom experiences. However, in general the students in the "non-traditional" classes performed better in a range of assessment tasks and overall they developed more positive attitudes towards the subject and a stronger sense of their own mathematical identity. Although the detail is light here, it seemed in short that the experiences of the students in the non-traditional classrooms were akin to the mathematical practices of research mathematicians outlined above.

The Dance of Agency

The claims of Burton and the classroom evidence of Boaler (2003) together seem to make a strong case for considering the learning of mathematics to be like "working as a mathematician". Conceptually, this requires engaging in what Pickering (1995) calls a "dance of agency". In studying the practices of research scientist and mathematicians he noted that they choreographed a complex routine where at times they drew on their own agency as scientists or mathematicians, and yet at other times they would concede authority to the agency of their discipline and associated community of practice. This is like the interplay between the activity of mathematics and the content knowledge of mathematics that was highlighted earlier and rather than seeing the practice or knowledge-base being supreme, it reveals dialectic interdependence where the mathematician (at any level) requires both to meaningfully and to successfully engage in the mathematical enterprise. Likewise, teachers too need to engage in a dance of agency where they appraise and decide when to encourage and support the students' own agency as mathematicians and when to defer to the authority of the disciple (e.g., the requirement to follow a standard procedure or form of presentation). It worth noting that mathematicians do defer to the agency of the discipline in their practice and it is this authority that is credible in a mathematics classroom. However, in traditional mathematics classrooms the authority usually resides with the textbook and the teacher, both of which are temporary aspects of students' mathematical development and they do not endure as the discipline itself does.

Boaler's use of the dance of agency in her recent work (Boaler, 2003) illustrates the importance of learning having a robust and empowering identity in relation to mathematics. Knowing how and when to draw on mathematical ideas to solve problems is a critical part of the dance of agency. Boaler used examples of learners who could not solve tasks but drew on a range of skills, knowledge, and collective wisdom in order to solve such problems. This process is akin to that identified in Burton's work with research mathematicians. The practices offered by Boaler and Burton may offer a way forward and out of the quagmire of contemporary school mathematics that is being identified by many both inside and outside of education.

In the remainder of this paper, we draw on an example taken from a professional development that one of us undertook with a group of primary school teachers. We argue that the level of the learners is not the feature of the analysis as we contend this example can be used across all sectors of learning – primary, secondary, and preservice/inservice education. Rather, the analysis focuses on the ways of working that are the significant aspects of the example. These provide an illustration of how learners, in this case teachers, can draw on previous knowledge to work collectively to achieve a common goal. Collectively the goal is attained but not without considerable input from the learners. The input varies in form and timing, and helps to illustrate the powerful learning made possible when working in ways similar to mathematicians but also having a sense of agency that allows for the legitimate use of learners' understandings that enable the building of deeper understandings. However, as Boaler's work has highlighted, such success is dependent on the learners' sense of identity with mathematics and their sense of agency through which they can "dance" between the known and the unknown in order to build deeper understandings. It is for this reason we have used this example. After describing and illustrating the mathematical practices of these teachers, we draw on their example to discuss the features of mathematical classrooms that promote the development of robust mathematical identities through an authentic "dance of agency". We use this illustrative example to show how the mathematical identity of learners may be constituted through particular practices of mathematics.

The data provided in the following example are drawn from field notes from the professional development activity. The quotes and drawings are those written by the observer and are representative of the discussion made by the participants as no formal recording tools (tape recorders) were used. The data were triangulated with participants so that they are an accurate summation of the interactions in the workshop.

Sum of the Interior Angles of an Octagon: A Working Example

A group of primary school teachers have been working on problems as part of a professional development activity. A standard geometry task is provided where they have to work out the sum of the interior angles of an octagon. There is some discussion as to what an octagon was, and how many sides it had. Once this is clarified, the teachers work in small groups.

I have no idea on how to work this out.

Well if you look at it you can divide it into triangles. See, there are 8 triangles. Each triangle has got 180^o so to work out what the angles are on the bottom of the triangle, you have to work out how many degrees are in the top angle there [draws an arrow to the centre, see Figure 1].

Ah, so that is 360^o divided by 8.

Huh?

Well you know that there are 360^o in a circle [draws a circle around the centre] and you can see there are 8 triangles making up that circle.

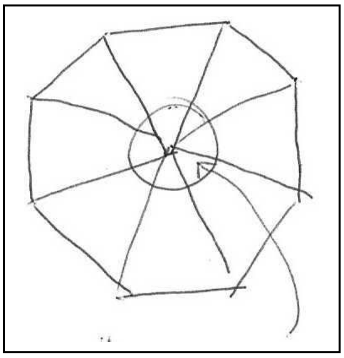


Figure 1. Octagon and eight triangles.

So, 360 ÷ 8 is [some talk on how to work this out, two teachers use pencil and paper for the division] … 45.

OK now what we have to do is work out how big the other angles are. They are the same size so you take 45 from 180 and then divide by 2.

Why?

Well there are two angles [points to the two angles at the bottom of one triangle] and we need to see how big one is.

The discussion continues so that the group identify the size of one of the interior angles of the constructed triangles as being 67.5

There is some discussion that it cannot be right as the leader would not have given them an angle with a half in it. Calculations are checked and the answer is seen to the correct. Some then suggests that they have to multiply it by 8 so it would not be a "half number" any more. Someone else in the group comments that it can not be right as the number they have calculated is less than 90^o which would make for a less than "straight angle" [assumed to mean a "right angle"]. There is some discussion and movement of the shape and then agreement that they have done something wrong.

I know what it is… that is only half of the angle. See look, we have worked out half of the angle, the other part is in the triangle next door.

You're right, so the size of one angle is really double what we found so that makes it 135. And that is bigger than 90 so we must be right now.

Ok, then we multiply by 8 and find out what the total size is.

Someone in the group then multiplies 135 by 8 using a pencil-and-paper method to come to an answer of 1080.

Once the group has finished, the leader then asks them to find out what it might be for a hexagon and some other shapes. The group goes through a similar process, this time drawing the hexagon, finding the magnitude of the central angle and then the size of each interior base angle. This is then doubled and multiplied by 6. At this point, a woman who has not contributed to much of the discussion interrupts and poses the following:

> You know what we are doing… making more work for ourselves. Look at this. You divided the 120 by 2 and got the size of the angle inside the triangle and then you doubled it. We halved and then doubled so we have just done the same thing twice.

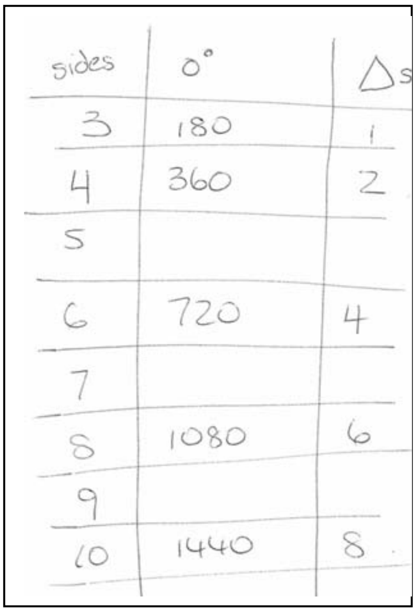
The teachers then go on to do two more shapes of their own choosing. The leader then poses the problem to see if they can make a prediction for any shape and how would they do it. The response is that this means they need to make a formula for the problem.

Group one made a table for their results. (Figure 2.) Aside from the triangle which they knew had 180, they had only made shapes with even numbers of sides so that it looked like:

> Hey, look at that you can see a pattern there. Each time we go up by 2 sides, it gets bigger by 360. That is a square so if we only increased by one side it would be getter bigger by 180^{o} – that is a triangle.

However, this group was unable to move beyond this observation to make a more generalisable statement.

Group two used a similar method and when it came to the discussion at the end of the session where groups shared their findings, this group explained that they found that the pattern was "increasing by 180^{o} each time a side was added to a shape" but you could not go below 1 triangle as this was the lowest point. One teacher explained the generalisation as follows:



sides	o°	△s
3	180	1
4	360	2
5		
6	720	4
7		
8	1080	6
9		
10	1440	8

Figure 2. Table of sides, angles, and number of triangles.

> We found that what the pattern is that each shape is the number of sides takeaway 2 and then you multiply by 180^{o}. So if you use a hexagon as the example, you can see that it has 6 sides but if you takeaway 2, you have 4 and then if you multiply it by 180 you get the sum of the interior angles. We thought you could say it like (number of sides minus 2) and then multiply by 180 so that is (n-2) x 180. We checked it out with the others and it worked. So if you use the triangle. It has 3 sides, so that is 3-1 and then times 180 so that is 180 and that is right.

Coming to Understand "Working as a Mathematician"

In drawing on Burton's and Boaler's work, we propose that there are three elements to developing a sense of working as a mathematician (see Figure 3). There are the cognitive aspects of knowing mathematics and thinking like a mathematician. Burton draws considerably on the cognitive features of working mathematically. Both Boaler and Burton recognise the importance of the social context within which learning occurs. Railside's community has been strongly influenced by Complex Instruction (Cohen & Lotan, 1997; Cohen, Lotan, Scarloss, & Arellano, 1999) in terms of organising the learning environment. Burton draws more closely on the communities of practice literature (Wenger, 1998) to theorise her position and where she sees that "knowledge and the knower are mutually constituted within these dialogic communities" (1999, p. 132). Collectively the two positions provide a more comprehensive picture of the potential for classroom practice. Finally, the focus of both authors, and this paper, is that of mathematics.

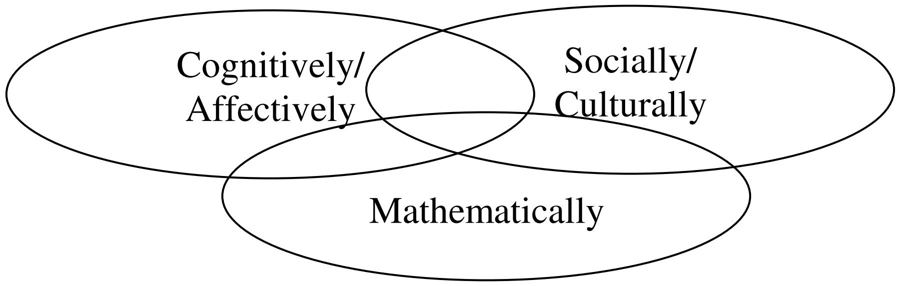


Figure 3. Aspects of working as a mathematician.

What can be seen in this example are a number of features about working as a mathematician. We take from the example used to illustrate aspects of these three constructs related to the notion of working as a mathematician and the importance of agency in this process.

Socially

For us, we define the context within which learning and working is occurring as the social dimension. This includes the ways in which the learning environment is organised along with the social and cultural dispositions that learners bring to that environment. From this example, we can see a number of features that enable learners to work as mathematicians.

Group work. Being part of a group and working as a collective enabled the teachers to share their knowledge, which is often tacit and not well understood. Drawing on this example, the teachers did not know the formula and so relied on bringing their collective wisdom enabled them to fill in gaps in each other's knowledge.

Collaborative talk. The interactions between the participants were focused on the task and enabled them to talk through observations. Having some participants working on the task and other observing enabled the observers to gain insights into the actions. In this case, one of the teachers was able to "see" that her colleagues were halving and then doubling. Being able to provide this input in a non-threatening way to colleagues enabled the group to move forward.

Ethos. The environment established in this session was non-threatening and supportive so that learners could actively engage in the active at levels that met their current needs and understandings. This ethos has been documented in Boaler's studies (Boaler, 2002a, 2002b) as being one that enables learners to participate without threat and hence open up opportunities for participation and learning.

Agency. Participants were able to draw on their own understandings to the situation and use these to develop richer understandings that are strongly mathematical. Being able to draw on existing knowledge to solve the problem in non-traditional ways, enabled the task to be completed but also to allow the participants to gain a strong sense of achievement.

Task. The design of the task may be seen as quite traditional but the leader deviated from those practices often found in classrooms where rote procedures are applied to a range of questions and little opportunity is provided to develop richer understandings.

Extrapolating the task to find the generalisation enabled the teachers to develop ways of thinking mathematically and to construct their own formula/generalisation.

Working as a mathematician. This aspect of the learning environment is very different from the traditional classroom where the format is often as a "consumer" or user of mathematics so that mathematics is the end product rather than the product.

Mathematically

This aspect of working as a mathematician draws on features that can be considered as part of the mathematical content knowledge or the pedagogical content knowledge identified by Shulman (1986). These features are often distinctly mathematical and are what can be seen to differentiate mathematics from other curriculum areas. Unlike traditional classrooms where there is feature of rote-and-drill learning, textbook-based exercises and strong teacher direction, mathematicians employ practices that are quite different from school mathematics practices. Some of these are identified in this example.

Identifying patterns. Creating the table enabled the participants to observe a pattern. For some participants, they were only able to describe the pattern but not the generalisation.

Constructing generalisations. Part of working as a mathematician is about making the generalisable statement. In this case, the development of a formula for the interior angles of a 2-dimensional geometric shape was part of the task. Unlike traditional mathematics classrooms where the generalisation (i.e. the rule) is often the starting point and learners are encouraged to practice on examples, this learning enabled the participants to generate their own generalisation.

Using a simple example to test the hypothesis. Once a potential generalisation had been developed, the participants applied this to a simple example (the triangle) to check its validity. In this case, it worked so it appeared to the participants that the generalisation was valid. They also applied the generalisation to the examples that they had worked out (and recorded in the table) to check that the generalisation was valid in other examples.

Identifying Limits. As noted by one group, the limit in this activity was that the shape had to have more than three or more sides if the generalisation were to work.

Cognitively

Drawing from Burton's work are aspects of cognition and other features of the internal features of working as a mathematician. We have identified particular features of cognition and dispositions that are part of the learners' ways of approaching the tasks.

Thinking styles. Drawing on a range of thinking styles identified by Burton (2001) – visual, analytic and conceptual – we can see how most of the learners used a composite of these styles. From the example used, we can see that the learners engaged using a range of thinking styles which include verbalisation, drawing illustrations, and the use of tables to arrive at insights about the problem, the mathematics, and ways to solve the problem.

Insight/Intuition. Burton's (2001) mathematicians referred to the "light being switched on", which enabled them to see what works and what does not work without being overtly aware of how they gained such insights.

Making connections. What can be seen from this example is that various elements of mathematics have been linked together to form a coherent whole. Burton argues that it is akin to fitting the pieces of the jigsaw together (Burton, 2001). What can be seen in this example is how the teachers have drawn on various aspects of mathematical knowledge, in particular their knowledge of triangles, to pool this knowledge in order to come up with a deeper appreciation of mathematical understanding.

Identity and the Dance of Agency

What becomes possible to see through this example is that the learning situation draws considerably on those aspects of working as a mathematician as identified by Burton's work and on the aspects of classrooms and teaching identified by Boaler's work. Boaler's work has been particularly powerful in illustrating the importance of agency and identity. When we consider the activity identified in this paper, we recognise that the three features – social, mathematical, and cognitive – are critical variables in the provision of quality learning opportunities. If we are to emerge from the current demise in mathematics education as identified at the start of this paper, then reforms are needed to enable change from the current, traditional practices to ones that are more empowering for learners. This requires not only a shift in pedagogy and curriculum but also in the dispositions of learners.- As noted by Zevenbergen (2005) many of the current practices in school mathematics create particular mathematical habitus which are far from empowering for learners and indeed encourage disengagement with the discipline. This example and our analysis of that practice highlight some of the features that foster the characteristics of working as a mathematician that have been identified through the combined work of Burton and Boaler. However, in this final section, we want draw more constructively on Boaler's notion of dance of agency. For her, this construct is critical as it enables the learners to draw on their mathematical understandings, to build on what they know, to construct deeper understandings. This is one of the fundamental premises of much mathematical learning but which is not that possible in many of mainstream classrooms due to the pedagogies being implemented. As shown in the Queensland School Longitudinal Reform Study (Education Queensland, 2001), the teaching of mathematics in schools is one of the most poorly taught areas of school curriculum and dominated by shallow teaching approaches with little scope for students to engage substantially with ideas and deep learning. The example here provides some insights into the ways in which a commonly used activity can be adjusted to allow for depth of learning. However, as Boaler's work highlights, learners must feel some sense of agency to be confident to draw on other forms of knowing in order to solve problems.

We contend that traditional classrooms would have fostered learning activities around the application of a formula for calculating the sum of interior angles. In this example, the participants could not remember this formula (and it was not provided) so they need to rely on their existing knowledge, the collective wisdom of the group and a sense that they could solve the problem. This sense of agency – where they could rely not only on their own knowledge in a legitimate sense, but also on the collective knowledge across the group – enabled them to gain a sense of learning and achievement through the completion of the task. We contend that such practice is far more enabling and develops a strong sense of agency and identity with mathematics.

References

Australian Academy of Science. (2006). *Mathematics and statistics: Critical skills for Australia's future. The national strategic review of mathematical sciences research in Australia*. Canberra, ACT: Author.

Boaler, J. (1997). *Experiencing school mathematics: Teaching styles, sex and setting*. Buckingham: Open University Press.

Boaler, J. (2002a). Exploring the nature of mathematical activity: Using theory, research and `working hypotheses' to broaden conceptions of mathematics knowing. *Educational Studies in Mathematics,* 51(1-2), 3-21.

Boaler, J. (2002b). Learning from teaching: Exploring the relationship between reform curriculum and equity. *Journal for Research in Mathematics Education, 33*(4), 239 - 258.

Boaler, J. (2003). Studying and capturing the case of the dance of agency. In N. Pateman, B. Dougherty & J. Zillox (Eds.), *Proceedings of the 27th annual conference of the International Group of Psychology of Mathematics Education* (Vol. 1, pp. 3-16). Hawaii: PME.

Burton, L. (1999). The practice of mathematicians: What do they tell us about coming to know mathematics. *Educational Studies in Mathematics, 37,* 121-143.

Burton, L. (2001). Research mathematicians as learners – and what can mathematics education learn from them. *British Educational Research Journal, 27*(5), 589-599.

Burton, L. (2002). Recognising commonalities and reconciling differences in mathematics education. *Educational Studies in Mathematics, 50*, 157-175.

Cohen, E., & Lotan, R. (Eds.). (1997). *Working for equity in heterogeneous classrooms: sociological theory in practice*. New York: Teachers College Press.

Cohen, E., Lotan, R. A., Scarloss, B. A., & Arellano, A. R. (1999). Complex instruction: Equity in cooperative learning classrooms. *Theory into Practice, 38*(2), 80-86.

Davis, P. J., & Hersh, R. (1998). *The mathematical experience*. New York: Mariner

Education Queensland. (2001). *The Queensland school longitudinal reform study.* Brisbane: GoPrint.

Pickering, A. (1995). *The mangle of practice: Time, agency and science*. Chicago: The University of Chicago Press.

Shulman, L. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher,* 15(2), 4-14.

Statistical Society of Australia. (2005). *Statistics at Australian universities*: A SSAI-sponsored review. Braddon, ACT: Author

Wenger, E. (1998). *Communities of practice: Learning, meaning and identity*. New York: Cambridge University Press.

Zevenbergen, R. (2005). The construction of a mathematical habitus: Implications of ability grouping in the middle years. *Journal of Curriculum Studies*, 37(5), 607 -619.

Categorisation of Mental Computation Strategies to Support Teaching and to Encourage Classroom Dialogue

Judy Hartnett
Brisbane Catholic Education
<jhartnett@bne.catholic.edu.au>

Mental strategies are a desired focus for computational instruction in schools and have been the focus of many syllabus documents and research papers. Teachers though, have been slow to adopt such changes in their classroom planning. A possible block to adoption of this approach is their lack of knowledge about possible computation strategies and a lack of a clear organisation of a school program for this end. This paper discusses a framework for the categorisation of mental computation strategies that can support teachers to make the pedagogical shift to use of mental strategies by providing a framework for the development of school and classroom programs and provide a common language for teachers and students to discuss strategies in use.

Mental computation has been the focus of a major shift in mathematics education in many parts of the world. Recent curriculum documents in Australia and overseas the United States *Principles and Standards for School Mathematics* (National Council of Teachers of Mathematics, 2000), the new United Kingdom *Primary Framework for Literacy and Mathematics* (DfES, 2007), the Dutch *Specimen of a National Program for Primary Mathematics* (Treffers & DeMoor, 1990), and the Australian *National Statement on Mathematics for Australian Schools* (Australian Education Council, 1991) have indicated that mathematics education needs to change emphasis to match the developments in the world today.

Syllabus documents in all states of Australia advise teachers to take an approach focusing more on mental computation as part of a range of strategies and less on traditional written algorithms. For example, the Level 2 Addition and Subtraction outcome in the Queensland Studies Authority Years 1-10 Mathematics syllabus (2004) states: “Students identify and solve addition and subtraction problems involving whole numbers, selecting from a range of computation methods, strategies and known number facts” (p. 19). The benefits of a focus on mental computation have been widely reported and include the need for school mathematics to be useful and to reflect computational techniques used in everyday life (Australian Education Council, 1991; Clarke, 2003; Irons, 2000; Willis, 1990; Zevenbergen, 2000).

Mental computation strategies are different from written algorithms in that they require more than the application of a remembered procedure. The key difference is the need for some application of a deeper knowledge of how numbers work. Callingham (2005) discussed research in mental computation as focussing on “identifying and describing students’ strategies for addressing particular kinds of calculations, often within a framework of number sense” (p. 193). Number sense has been defined as having a “general understanding of number and operations along with an ability and inclination to use this understanding in flexible ways” (McIntosh, Reys, Reys, Bana, & Farrell, 1997, p. 3). Using mental computation strategies flexibly requires sound number sense and by using a strategies approach to computation, rather than a focus on procedural algorithms, students have opportunities to work with numbers in flexible ways, which in turn, provide

Proceedings of the 30th annual conference of the Mathematics Education Research Group of Australasia
J. Watson & K. Beswick (Eds), © MERGA Inc. 2007

opportunities for them to improve their number sense. Needing number sense for efficient use of computation strategies, and the development of number sense by using such strategies, are very closely interrelated.

Mental Computation Strategies

There has been discussion in the literature of what constitutes a mental computation strategy. Earlier definitions of mental computation focussed on the lack of written recordings. Trafton (1978) described the use of non standard algorithms for the computation of exact answers without the use of pencil and paper. Sowder (1988) defined mental computation as “the process of carrying out arithmetic calculations without the aid of external devices” (p. 182). Threlfall (2002) described strategies, as “where students can be correct by constructing a sequence of transformations of a number problem to arrive at a solution as opposed to just knowing, simply counting or making a mental representation of a ‘paper and pencil’ method” (p. 30). The Queensland Years 1-10 Mathematics syllabus (Queensland Studies Authority, 2004) provides examples of mental computation strategies in early levels such as, “count on and back, doubles, make to ten” (p. 45) and in later levels “making numbers manageable” (p. 46). Some of these “strategies”, for example, “turnarounds (commutativity)” are not strategies as thought process as discussed above, but are skills more related to having sound number sense. These understandings would be used as part of a strategy (i.e., a sequence of transformations of a number problem) to solve a problem but are difficult to consider as strategies themselves.

Strategy Categorisation

In research literature there have been many attempts to describe lists of possible mental computation strategies. A well documented strategy categorisation by Beishuizen (1985) described two main strategies for mental addition and subtraction. The strategy 1010 referred to splitting numbers into tens and ones and dealing with the parts separately, left to right. N10 referred to a strategy where one number is split into tens and ones and the tens of the second number are added to the first number followed by the ones. Many authors refer to these as the two main strategies for addition and subtraction of numbers to 100 (Cobb, 1995; Cooper, Heirdsfield, & Irons, 1996; Fuson, 1992; Reys, Reys, Nohda, & Emori, 1995; Thompson, 1994). Beishuizen, Van Putten, and Van Mulken (1997) extended this list to include a strategy they referred to as A10, where the second number is split to facilitate a bridge to a multiple of ten and then the remainder is added to the first number. This dealt with problems that required bridging of a ten in either addition or subtraction. A further paper by Klein, Beishuizen, and Treffers (1998) discussed another strategy that they called N10C, where the second number is rounded up to a multiple of ten and this number is added to the first number followed by an adjustment or compensation for the rounding. Yackel (2001) described “collections-based” solutions where both numbers are broken into parts, usually tens and ones (compare to 1010), and “counting or sequence based” solutions starting with one number and dealing with the others progressively, part by part (compare to N10).

Cooper, Heirdsfield, and Irons (1996) developed a strategy schema based on work of Beishuizen (1993) to analyse strategies used in a study of young children’s mental addition and subtraction accuracy and strategy usage. Their schema consisted of four strategy categories: i) Counting, ii) Separation (1010) which they further categorised to be right to left, left to right or cumulative, iii) Aggregation (N10), again categorised further as right to left or left to right, and iv) Wholistic, which described strategies involving adjustment of one number by compensation (N10C) or by levelling where both numbers were adjusted to create a new equivalent question. They also included a separate category for students who reported using a mental image of the pen-and-paper algorithm.

Often lists of strategies have been derived from studies where computation problems were presented to students and the strategies that the students actually exhibited were analysed and categories emerged. For example, Reys, Reys, Nohda, and Emori (1995), used a mental computation test in their study of the performance and strategy use of students in Japan. Prior to administering the test the researchers formulated a detailed categorisation of anticipated strategies. Their categorisation reflected similar major grouping as described above and used letters to identify the major strategies and then variations of these strategies were numbered e.g. A1, A2, B1, etc. The categories labelled A involved grouping of tens and ones separately (compare to 1010), those labelled B had one number held constant (compare to N10), and those labelled C involved rounding of one or both numbers to multiples of ten (compare to N10C).

Wigley (1996) described strategies for addition and subtraction where numbers were split and recombined in different ways using knowledge of place value and complementation, which he described as an ability to generate relationships associated with complements in numbers to ten or hundred. He advocated teaching strategies for multiplication that used doubling and halving, including repeated doubling and halving, and the trial and use of multiplication and subtraction to achieve progressively smaller remainders as a strategy for division.

Teaching Mental Computation Strategies

In the literature two different approaches to the teaching of mental computation strategies are described. One focuses on students inventing or using their own intuitive strategies to solve given computation problems (e.g. Buzeika, 1999; Heirdsfield, 2004, 2006) and others describe where particular strategies were the focus of teaching (e.g., Beishuizen, 1999). In all of these studies and others (Buys, 2001; Beishuizen, 2001) students were encouraged to discuss strategies used.

Threlfall (2002) argued that a teaching approach that is intended to foster choice and flexibility by teaching wholistic strategies needs to be underpinned by a coherent way of thinking about the possible choices, “so that they can be taught in an organised and systematic way. In other words, there has to be a categorisation system that makes sense to the teacher” (p. 32). He was concerned that an incomplete set of strategies may lead to efficient strategies not being available for use because they had not been taught. Mental arithmetic needs to be taught using methods quite different from traditional pencil-and-paper methods. Offering only one method is too rigid. Leaving pupils to find their own methods will deprive many of more advanced strategies (Wigley, 1996).

Many teachers in classrooms today were students themselves in a period when mathematics teaching focussed on rote learning of basic facts and on the development of procedures for “successful” completion of traditional written algorithms. These teachers consciously know of very few if any computation strategies other than the use of vertical algorithms in the mind. Although these teachers can see benefits for including mental computation strategies in their teaching programs their lack of knowledge leads to a lack of confidence and lack of teaching ideas to take the idea forward into their practice. If a comprehensive but easy to understand list of possible strategies were organised based on the research in this area a useful tool to change classroom pedagogy and therefore improvement of student learning outcomes could be achieved.

The Mental Computation Strategy Framework

The author of this paper has attempted to create a categorisation framework for the purpose of informing and providing structure for the *teaching* of computation strategies. The intention of the strategy categorisation was to create a small number of general categories with intuitive labels using simple language that would make sense to teachers and also to students. Then a list of sub-categories would make clearer the variations that could be a focus in each category. In all, five major categories and twenty-one sub-categories were identified. It was also an intention that these categories would be applied across the range of the primary school year levels at least, and across the four operations with whole numbers, common and decimal fractions, negative numbers, as appropriate. This way a school could utilise the framework for a whole school program or approach to the teaching of mental computation strategies. With the labels for the categories kept in simple intuitive language it was intended that these names would be used in the classroom as an aid the discussion of strategies used by students and as part of lessons on particular strategies. It is a coherent way of thinking about the possible mental computation strategies that the researcher is interested in providing to meet an identified need from teachers and schools.

A description of the categories and links to other categorisations in the literature are outlined in Table 1. The intention was not to find a single description for each possible strategy but to provide a framework for teachers to base their development of programs of lessons on and for teachers and students to use as a common language to describe ways of working through computation examples.

Method

The focus class consisted of 27 Year 3 students who were approximately 8 years of age in a suburban school in Brisbane, Queensland. There was a wide range of abilities within this class and the teacher was experienced and had taught this year level for many years. Year 3 was chosen for the study as traditionally addition and subtraction algorithms were introduced in this year of schooling. The teacher was interested in the inclusion of mental strategies into the class number program. She perceived there would be benefits for the class by shifting the focus away from the algorithm to the development of mental computation strategies and she was prepared to put teaching of algorithms aside for the whole year.

The class number program was planned to introduce and focus teach one major strategy category from the framework each school term. “Counting On and Back” was the focus in first term, followed by “Breaking Up numbers” in term 2, “Adjusting and Compensating” (also called change and fix especially when working with the students) in term 3 leaving “Doubling and Halving” for fourth term, which linked to other planned focus work on multiplication and division. The “Use Place Value” category was not a particular focus for

Table 1

Categorisation of Mental Computation Strategies and Links to Literature

	Related categorisations and References
Count On and Back:	
Count on to add	Counting *(Cooper et al., 1996)*
	Count on or back *(McIntosh & Dole, 2005)*
Count back to subtract	Counting *(Cooper et al., 1996)*
Count on to subtract	Aggregation (additive) (*Cooper et al., 1996)*
	A10 *(Beishuizen et al., 1993, 1997)*
Count on to multiply	
Adjust and Compensate: (Change and Fix)	
Adjust one number and compensate	N10C *(Beishuizen et al., 1993, 1997)*
	C1, C2 *(Reys et al., 1995)*
	Wholistic compensation *(Cooper et al., 1996)*
	Over jump method (*Thompson, 1999*)
Adjust two numbers and compensate	C3, D1 *(Reys et al., 1995)*
Adjust two numbers	Wholistic levelling *(Cooper et al., (1996)*
Double and /or Halve:	
Use a double or near double to add or subtract	Doubles / near doubles *(McIntosh & Dole, 2005)*
Double to multiply by 2	
Double, double to multiply by 4	
Double, double, double to multiply by 8	Repeated doubling *(Wigley, 1996)*
Half to divide by 2	
Half, half to divide by 4	
Half, half, half to divide by 8	Repeated halving *(Wigley, 1996)*
Double and halve	
Break Up Numbers:	
Break up two numbers using place value	1010 *(Beishuizen et al., 1993, 1997)*
	A1, A3 *(Reys et al., 1995)*
	Separation *(Cooper et al., (1996)*
	Split method (*Thompson, 1999*)
	Split tens method *(McIntosh & Dole, 2005)*
Break up two numbers using compatible nos.	Split jump method *(Thompson, 1999)*
Break up one number using place value	N10 *(Beishuizen et al., 1993, 1997)*
	B1, B2 *(Reys et al., 1995)*
	Aggregation (Cooper et al., 1996)
	Jump method (*Thompson, 1999*)
	Sequential method *(McIntosh & Dole, 2005)*
Break up one number using compatible nos.	A10 *(Beishuizen, Van Putten, & Van Mulken, 1997)*
Use Place Value:	
Think in multiples of ten	
Focus on relevant places	

any term as it is limited to particular problems and was simply introduced where appropriate.

Throughout all instruction and practise activities students were encouraged to show their thinking using any written methods they felt comfortable with. The classroom climate also encouraged discussion and flexibility of choice of strategy. The students completed practise activities for each strategy but when given open computation problems to solve were free to use any strategy they liked. A range of models to support the learning were used throughout the year which included ten frames, numbered lines, open number lines, and number boards.

The students were given a pre-test, mid year test, and post test in which they were asked to complete the computations and show what they were thinking and how they worked out each question. The items were chosen to present addition or subtraction situations that could be solved using some of the strategies they would be taught throughout the year. The items were presented as single computations presented horizontally without context. The intention was to keep the questions as clear and free of distractions as possible. The students were not interviewed, as previous studies, including one quoted in Threlfall (2002), found that written responses attained when students were asked to "work out each answer mentally and write down how they had done it" (p. 33) took the same form as the protocol responses. An aim of the study was to look for evidence of strategy categories in the written responses of the students across the year.

Results and Discussion

The use of the four main strategy categories from the framework as the basic focus of instruction for each of the four terms of the year made sense to the teacher and the students and was an effective program organiser. The teacher was interviewed and stated that this organisation was easy to follow and gave her confidence to teach the strategies. The teacher saw it as clarifying and observed that the students were generally comfortable with the strategies by the end of each term of learning. The students exhibited a growing repertoire of strategies as the year progressed and showed an early ability to use a variety of strategies, evidenced by growth in the number of strategies used for the pre to post tests (See Table 2). The lack of obvious use of strategies did not mean the students did not use strategies but just that they chose not to or, more likely, lacked confidence or methods to record these.

Table 2

Number of Students who used a Variety of Different Strategies

	Pre test	Post test
0 strategies evident	21	0
1 strategy	5	3
2 strategies	3	6
3 strategies	0	4
4 strategies	0	4
5 strategies	0	4
> 5 strategies	0	7

In the mid year and post tests particularly, evidence of the students' use of the strategies in the working and descriptions of the way they solved the problems showed strategies named specifically using the framework. Figure 1 shows four examples of such responses.

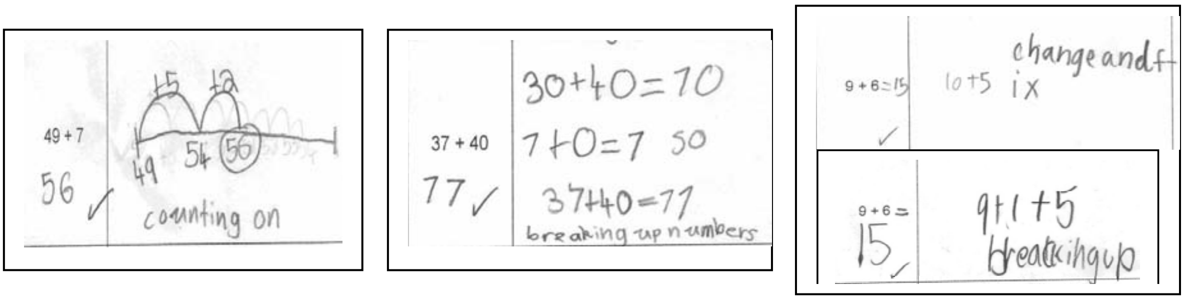


Figure 1: Student work samples showing use of the strategy categorisation framework.

There was also a variation between strategies used by the same students on different instruments. One instrument inadvertently was given to the students by the researcher and again by the class teacher one week apart. There was a large number of students who used a completely different strategy on the same item on each test.

Conclusion

This study was only for one year and was in a year early in primary school. For the framework to be evaluated, a longer period of sustained use for teaching and learning is required. Further monitoring is required on using this framework to plan a whole school program across all year levels, all types of numbers (ie., including decimals, common fractions, etc) and across all operations. The focus school is currently using this framework to do just this with the assistance of the researcher.

References

Australian Education Council (AEC). (1991). *A national statement on mathematics for Australian schools*. Carlton, Vic: Curriculum Corporation.

Beishuizen, M. (1985). Evaluation of the use of structured materials in the teaching of primary mathematics. In B.S. Alloway & G.M. Mills (Eds.), *New Directions in education and training technology. Aspects of educational technology* (Vol 18, pp. 246-258). London: Kogan Page.

Beishuizen, M. (1993). Mental strategies and materials or models for addition and subtraction up to 100 in Dutch second class. *Journal for Research in Mathematics Education, 24*(4), 194-323.

Beishuizen, M. (1999). The empty number line as a new model. In I. Thompson (Ed.), *Issues in teaching numeracy in primary schools* (pp. 157-168). Buckingham: Open University Press.

Beishuizen, M. (2001). Different approaches to mastering mental calculation strategies. In J. Anghileri (Ed.), *Principles and practices in arithmetic teaching* (pp. 119-130). Buckingham: Open University Press.

Beishuizen, M., Van Putten, C.M., & Van Mulken, F. (1997). Mental arithmetic and strategy use with indirect number problems up to one hundred. *Learning and Instruction, 7*, 87-106.

Buys, K. (2001). Progressive mathematisation: Sketch of a learning strand. In J. Anghileri (Ed.), *Principles and practices in arithmetic teaching*. (pp. 15-32). Buckingham: Open University Press.

Buzeika, A. (1999). Invented algorithms: Teachers face the challenge. In J. M. Truran & K. M. Truran (Eds.), *Making the difference (*Proceedings of the Sixteenth Annual Conference of the Mathematics Education Research Group of Australasia pp. 128-134). Sydney: MERGA.

Callingham, R. (2005). A whole-school approach to developing mental computation strategies. In H. L. Chick & J. L. Vincent, (Eds), *Proceedings of the 29th Annual Conference of the International Group for the Psychology of Mathematics Education, (*Vol. 2, pp. 201-208). Melbourne: PME.

Clarke, D. M. (2003, June). *The teaching of algorithms in the primary years: Research, issues, and suggested* *approaches.* Paper presented to the Midsummer World Mathematics Conference, Goterborg, Sweden.

Cobb, P. (1995). Cultural tools and mathematical learning: A case study. *Journal for Research in Mathematics Education*, *26*, 362-385.

Cooper, T. J., Heirdsfield, A., Irons, C. J. (1996). Children's mental strategies for addition and subtraction word problems. In J. Mulligan & M. Mitchelmore (Eds.), *Children's number learning (*pp 147-162). Adelaide: Australian Association of Mathematics Teachers, Inc.

DfES (Department for Education and Studies). (2007). *Primary Framework for literacy and mathematics: Guidance paper: calculation.* http://www.standards.dfes.gov.uk/primaryframeworks/mathematics/ Papers/Calculation/*.* Accessed 21-03-07.

Fuson, K.C. (1992). Research on whole number addition and subtraction. In D.A Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp.243-275). New York: Macmillan.

Heirdsfield, A. M. (2001). *Mental computation: The identification of associated cognitive, metacognative and affective factors.* Unpublished doctoral dissertation. Queensland University of Technology, Brisbane, Australia.

Heirdsfield, A. M. (2006). One teacher's role in promoting understanding in mental computation. In H. L. Chick & J. L. Vincent, (Eds), *Proceedings of the 29th International Group for the Psychology of Mathematics Education,* (Vol 3, pp. 113-120). Melbourne: PME.

Irons, C. (2000). Mental computation: Why we must do more. *Teaching Mathematics,* 25 *(3),* 16–19.

Klein, A.S., Beishuizen, M, & Treffers, A. (1998). The empty number line in Dutch secong grades: Realistic versus gradual program design. *Journal for Research in Mathematics Education*, *29*, 443-464.

McIntosh, A., & Dole, S. (2005). *Mental computation: A strategies approach*. Hobart: Department of Education Tasmania.

McIntosh, A., Reys, B., Reys, R., Bana, J., & Farrell, B. (1997). *Number sense in school mathematics: Student performance in four countries.* Perth: MASTEC, Edith Cowen University.

National Council of Teachers of Mathematics (NCTM). (2000). *Curriculum and evaluation standards for school mathematics*. Reston, VA: NCTM.

Queensland Studies Authority. (2004). *The Queensland years 1 – 10 mathematics syllabus.* Brisbane: QSA.

Reys, R.E., Reys, B.J., Nohda, N., & Emori, H. (1995). Mental computation performance and strategy use of Japanese students in grades 2,4,6, and 8. *Journal of Research in Mathematics Education, 26*(4), 304-326.

Sowder, J. (1988). Mental computation and number comparison: their role in the development of number sense and computational estimation. In J. Hiebert & M. Behr (Eds.), *Number concepts and operations in the middle grades* (pp. 182-197). Hillsdale: Lawrence Erllbaum Associates.

Thompson, I. (1994). Young children's idiosyncratic written algorithms for addition. *Educational Studies in Mathematics, 26*, 323-345.

Thompson, I. (1999).Mental calculation strategies for addition and subtraction. *Mathematics in School.* London: The Mathematical Association.

Threlfall, J. (2002). Flexible mental calculation. *Educational Studies in Mathematics, 50*, 29-47.

Trafton, P. R. (1978). Estimation and mental arithmetic: Important components of computation. In M. Suydam & R. E. Reys (Eds.), *Developing* C*omputational Skills* (pp. 196-213). Reston, VA: National Council of Teachers of Mathematics.

Treffers, A., & DeMoor, E. (1990). *Proeve van een national programma voor het reken-wiskunde-on-derwijs op de basisschool. Deel 2: Basisvaardigheden en cijferen [Specimen of a national programme for primary mathematics teaching. Part 2: Basic mental sills and written algorithms].* Tilburg, Zwijsen.

Wigley, A. (1996). *Last Chant for Times Tables. Times Educational Supplement*. 24 May. Retrieved from www.tes.co.uk\section\story\?story_id=23573&window_type=print on 24 March 2007

Willis, S. (Ed.) (1990). *Being numerate: What counts?* Melbourne: Australian Council for Educational Research.

Yackel, E. (2001). Perspectives on arithmetic from classroom based research in the United States of America. In J. Anghileri (Ed.), *Principles and practices in arithmetic teaching* (pp. 15-32) Buckingham: Open University Press.

Zevenbergen, R. (2000). Using mental computations for developing number sense. *Mathematics education beyond 2000.* (Proceedings of the 23rd annual conference of the Mathematics Education Research Group of Australasia, pp. 672-679). Sydney: MERGA.

Student Experiences of VCE Further Mathematics

Sue Helme
University of Melbourne
<sueh@unimelb.edu.au>

Stephen Lamb
University of Melbourne
<lamb@unimelb.edu.au>

This paper examines student experiences in VCE Further Mathematics. In a survey conducted in 2006, 866 year 12 graduates who had studied Further Mathematics the previous year were asked about their experiences of Further Mathematics classes and their views on the subject and the teacher. The students who did Further Mathematics as their only mathematics subject were less confident about doing well, had a less positive view of the classroom as a learning environment and more negative attitudes towards their mathematics teachers, compared to students who studied both Further Mathematics and Mathematical Methods. The practice of allowing Mathematical Methods students also to study Further Mathematics may contribute to higher results in Further Mathematics for these students, but it may inhibit the capacity for teachers and schools to cater properly to the needs of those for whom the subject was initially designed.

This paper comes from a broader ARC-funded project examining the extent to which young people from different family backgrounds access different "locations" within the Victorian Certificate of Education (VCE) curriculum. It explores the quality of their instructional experiences, their academic outcomes, and the post-school destinations connected with the places they occupied in the curriculum. The broad objective of the project is to make the curriculum more transparent with respect to underlying social patterns and processes.

The formal role of the VCE is to prepare young people for a successful transition to further study and work. In this context, the VCE needs to be both equitable in the range of learning opportunities it provides, and effective in the range of valued destinations to which it leads.

Some subject areas in the VCE are organised to accommodate a broad range of student skills and abilities. Mathematics is designed to do this through provision of a hierarchical set of subjects designed around different skill levels. The mathematics subject Further Mathematics was designed to,

> provide access to worthwhile and challenging mathematical learning in a way which takes into account the needs and aspirations of a wide range of students. It is also designed to promote students' awareness of the importance of mathematics in everyday life in a technological society, and confidence in making effective use of mathematical ideas, techniques and processes. (Victorian Curriculum and Assessment Authority (VCAA), 2005, p. 1)

It is meant to be widely accessible, providing general preparation for employment or further study, in particular where data analysis is important. According to the Victorian Parliamentary Enquiry into the Promotion of Mathematics and Science Education (2006),

> it is suited to students who require some mathematical literacy in their further study or work but not high level applications of pure mathematics or high level conceptual mathematics… it is the easiest of the VCE Unit 3 and 4 mathematics subjects (p. 54).

Further Mathematics has consistently been the most popular Unit 3 and 4 mathematics subject, and is gaining in popularity. According to the Victorian Parliamentary Enquiry (2006) enrolments in Further Mathematics have increased from 37% of the Year 12 cohort in 2000 to 47% in 2004. This is in contrast to enrolments in Mathematical Methods (stable at about 37%) and Specialist Mathematics (13%). The number of students who sat for the Further Mathematics examinations was 21,815 in 2005, a slight increase over 2004 (21,216) (VCAA, 2006).

Participation rates in Further Mathematics are much the same for males and females (see Table 1, which shows participation rates in 2005). This is in contrast to Mathematical Methods, where there is a large gender gap in participation favouring boys, particularly in lower SES bands. The social composition of Further Mathematics is also much more democratic. In contrast to Mathematical Methods, enrolment levels in Further Mathematics are high amongst all groups, but peak in the middle social bands. They are lowest amongst students in the highest quintile of SES. The high overall levels of enrolment in Further Mathematics reflect a range of different orientations to the subject, and contribute to a flattening of the social trend, since students from a wide range of social backgrounds take the subject, either as their only mathematics subject or in conjunction with Mathematical Methods.

Table 1
Participation in VCE Mathematics, by SES and Gender: Year 12 Students, 2005

	Further Mathematics		**Mathematical Methods**		**Specialist Mathematics**	
SES quintile	Males	Females	Males	Females	Males	Females
Lowest	44.4	44.9	34.2	23.5	11.7	6.0
Lower middle	45.2	46.1	35.7	23.9	12.7	5.8
Middle	47.1	46.9	36.5	26.3	12.5	5.7
Upper middle	46.0	45.1	41.6	32.0	16.7	8.7
Highest	40.7	39.0	52.1	44.5	21.0	13.8
Total	44.6	44.5	40.3	29.7	15.1	7.9

Source: Unpublished VCAA data

Lamb and Helme (2007) have reported a pattern in some schools of high rates of enrolment in Further Mathematics associated with high rates of enrolment in Mathematical Methods. In about a fifth of secondary schools in Victoria, 21% or more Further Mathematics students were enrolled in Mathematical Methods. The authors also found that schools in which many Further Mathematics students also studied Mathematical Methods tended to have higher than predicted achievement for Further Mathematics. The strategy of combining Further Mathematics and Mathematical Methods leads to significantly higher achievement levels in Further Mathematics. The results show that the strategy of combining Mathematical Methods and Further Mathematics gives some schools a competitive advantage in VCE scores (and also possibly in terms of ENTER scores). The practice may have benefits for the students in the schools that employ the strategy, however, it may make it more difficult for students in schools where the practice does not occur to achieve the same levels of success.

Further Mathematics is designed for a diverse range of abilities, and particularly for students who do not want to be exposed to the rigorous and challenging intellectual demands of Mathematical Methods or Specialist Mathematics. The growing tendency for students to combine Further Mathematics and Mathematical Methods suggest that Further

Mathematics has been open to use by able and high achieving mathematics students seeking a competitive advantage in the race for VCE results, a situation that may further depress the opportunity for success of students genuinely wanting to continue to learn mathematics at an appropriate level.

This paper examines the impact of these practices on students' classroom experience of mathematics, and investigates a number of questions.

- Do Further Mathematics students who also do Mathematical Methods experience their Further Mathematics classroom in a different way to students who just do Further Mathematics?
- Do Further Mathematics students who also do Mathematical Methods experience their Further Mathematics teacher in a different way to students who just do Further Mathematics?
- Do any differences in student experiences of Further Mathematics classes help explain the performance differences discussed above?

Methodology

The data for this study were derived from a sample of students surveyed as part of a larger study of the VCE curriculum in a group of selected Victorian secondary schools. The aim of the larger study is to look at student experiences in schools that vary in terms of effectiveness, measured on the basis of VCE results. Schools that were selected were those where VCE results (measured as an aggregate as well as across eight key learning areas) were either (a) well above what could be predicted based on SES intake, General Achievement Test (GAT) scores, location, size, resource levels, and sector, (b) about the level that would be expected given those characteristics, and (c) well below expected performance levels based on student intake characteristics. The schools represent a range of SES, GAT achievement, and regional characteristics. For the present paper, 23 of the original schools are represented.

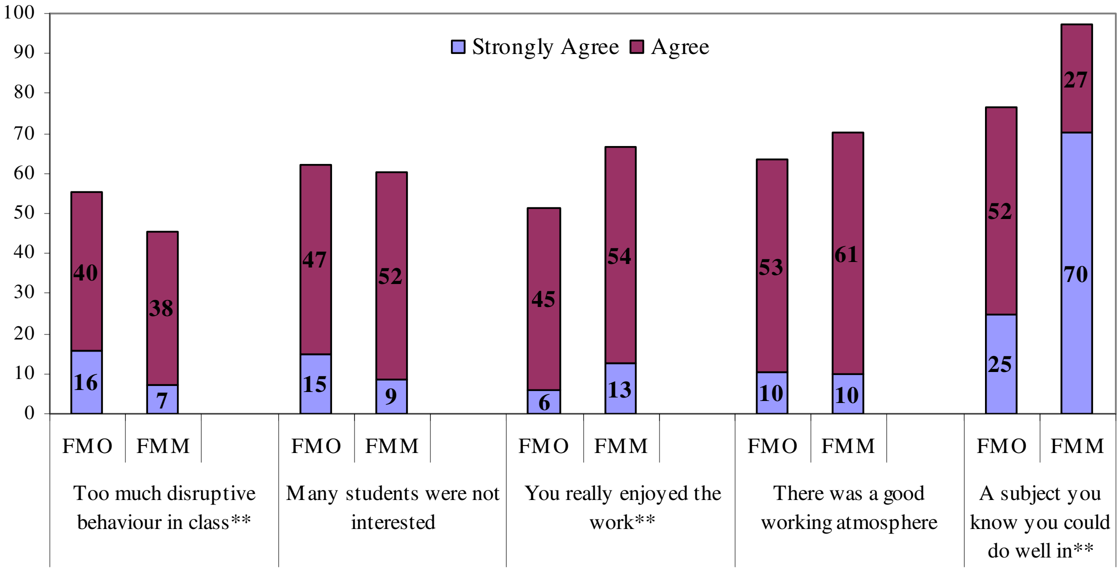
Year 12 VCE graduates from these schools were surveyed in April 2006, the year after they completed VCE. The survey included questions on their experiences of mathematics in VCE. It was done in conjunction with the annual *On Track* data collection. *On Track* is an annual telephone survey of Year 12 completers conducted in March-April in the following year.

Data were obtained from 1368 Year 12 students who confirmed in the survey that they had studied Further Mathematics and/or Mathematical Methods during VCE. A sample of 866 of the respondents indicated that they had studied Further Mathematics and 659 confirmed that they had studied Mathematical Methods, whereas 157 reported that they had enrolled in both subjects. It was possible, on this basis, to distinguish between Further Mathematics only students (FMO) and those who had completed both Further Mathematics and Mathematical Methods (FMM). The samples represented 65.7% and 66.8% respectively of the total enrolments in these subjects across the schools. The response rates compare favourably with the overall response rate for *On Track*, which in 2006 was 66.5% of all Year 12 or equivalent completers (Teese, Nicholas, Polesel, & Mason, 2007).

Two sorts of analyses are presented. The first is a set of descriptive results presenting information on student views on Further Mathematics including on classroom climate, attitudes towards the subject, and enjoyment, and views on their Further Mathematics teacher and his or her qualities and methods. The second is a set of results from a regression analysis using Hierarchical Linear Modelling (HLM) to model both student-level and school-level influences on student experiences of Further Mathematics. Student-level factors included gender, GAT scores, and mathematics subject combination (Further Maths Only or Further Mathematics and Methods). School-level factors included mean SES of the student body at the school, school size (measured as the number of Year 12 enrolments in 2005), and the percentage of Further Mathematics students also studying Mathematical Methods in each school.

Student Views of the Mathematics Classroom

Student responses to a range of items on their experiences of mathematics are shown in Figure 1. It compares the perceptions of students who did Further Mathematics as their only mathematics subject (FMO) with the perceptions of those who also did Mathematical Methods (FMM).



Note: Levels of significance based on Chi-square tests: * p<0.5, ** p<0.01

Figure 1. Student views of mathematics and mathematics classrooms.

Figure 1 reports significant differences in the perceptions of the two groups of students. The most striking aspect of the results is the difference between the two groups in their perceptions of how well they expected to do. Students who combined Further Mathematics with Methods were significantly more likely to report that they knew that they could do well in Further Mathematics (70% strongly agreed, compared with only 25% of FMO students). Indeed, almost all of the FMM students (97%) agreed or strongly agreed that it was a subject they expected to do well in. These findings confirm the strategic value to these students of combining the study of Mathematical Methods and Further Mathematics.

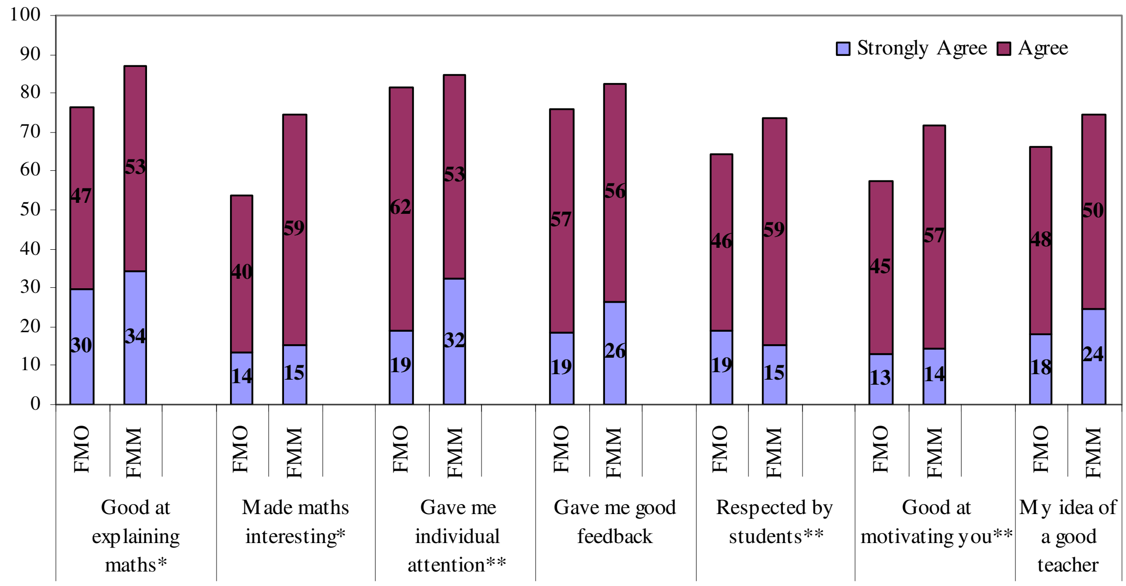
In addition to their perceived advantages over their peers in terms of preparation and confidence the FMM students were significantly more likely to report that they really enjoyed the work.

The other item for which there was a significant difference between the two groups was in relation to student perceptions of classroom behaviour. FMO students were significantly more likely to report that there was too much disruptive behaviour in their classes.

Although statistically significant differences were not evident for the remaining two items, the trend in responses was consistent with the results reported above, that is, FMM students appeared to experience the mathematics classroom in a more positive way than FMO students.

Student Views of their Further Mathematics Teacher

Figure 2 examines student views of their Further Mathematics teacher, in relation to several dimensions of perceived teacher expertise. Similarly to Figure 1, it compares the responses of FMO students with the responses of FMM students.



Note: Levels of significance based on Chi-square tests: * p<0.5, ** p<0.01

Figure 2. Student views of their Further Mathematics Teacher.

There were some strong and significant differences between the two types of students with regard to their perceptions of their mathematics teacher. Students who did both subjects were significantly more likely to report that their maths teacher made the subject interesting. They were also more likely to report that their teacher gave them individual attention when they needed it and was good at motivating them to do their best. Their teacher was also significantly more likely to be reported as good at explaining things clearly, and to be well respected. Results for the remaining two items – “gave you good feedback on your work during the year” and “was your idea of a good teacher” – although not statistically significant using Chi-square, were consistent with the trends for the other items.

Clearly, students who did both subjects had a much more favourable view of their Further Mathematics teacher, compared to those who did Further Mathematics only.

The results indicate that the FMM students experience Further Mathematics differently from the FMO students. They have a more positive experience of their Further Mathematics classroom and perceive their Further Mathematics teacher as responding more to their needs.

A Closer Examination of the Differences

There are a number of student-level and school-level factors that could account for the differences in perceptions, separately from whether or not students were enrolled for both Further Mathematics and Mathematical Methods. At the student level, these include academic aptitude (as measured by GAT), gender, and SES. For example, differences in confidence between FMO students and FMM students may simply be due to FMM students being more academically able, or comprising a higher proportion of male students.

Similarly, school level factors such as size, average socioeconomic status or the proportion of Further Mathematics Students also doing Mathematical Methods may influence student perceptions.

Regression analysis using Hierarchical Linear Modelling was conducted to model both student-level and school-level factors that may influence student experiences and dispositions. The results of the analysis are shown in Table 2.

Student-level Effects

1. GAT. The higher students' GAT scores, the more likely they were to express confidence in their ability to do well in Further Mathematics. Moreover, after controlling for other factors, the higher the GAT the more likely students were to perceive their teacher as good at explaining maths ($p<0.01$).

2. Mathematics Subject Combination. In this analysis, the control group was the FMM students. There were two significant differences between the FMO group and the FMM group, independent of other factors. First, FMO students were significantly less likely to express confidence in their ability to do well ($p<0.001$) and, second, to report that their teacher made mathematics interesting ($p<0.001$), all else equal.

3. Gender. Gender was a significant factor on two items only. Female students were significantly more likely to perceive their Further Mathematics teacher as well respected ($p<0.01$) and as good at motivating them ($p<0.1$). Interestingly, there were no significant differences between male and female students in their confidence in doing well and their enjoyment of the subject.

School-level Effects

1. Socioeconomic status. The mean SES level of a school tends to have a negative relationship with student perceptions, independent of all other factors. That is, the higher the SES of the school, the less that students report enjoying the work. The patterns may reflect a higher propensity for weaker mathematics students in middle class settings to continue in a subject area that they do not enjoy, responding to school policies to include a mathematics subject in Year 12, parental pressure to do mathematics, and/or the desire to keep their options open for further study.

Table 2
Variation among Students and Schools in Perceptions of Further Mathematics and Further Mathematics Teachers

	Student-level			*School-level*			*School effects*	
	GAT	Further Maths Only	Female	SES	School size	% Maths Methods	Without school-level controls	With school-level controls
Views on Further Maths								
It is a subject you knew you could do well in	0.009***	-0.121***	-0.041	-0.000	-0.060	0.008***	13.3	7.1
There was too much disruptive behaviour in your class	-0.005	0.022	-0.014	-0.000	-0.045	-0.006**	17.1	14.7
You really enjoyed the work	-0.003	-0.091	0.040	-0.002***	0.135	0.003	24.2	21.9
There was a good working atmosphere	0.000	-0.005	0.014	0.000	0.119**	0.005***	12.5	8.7
Many of your fellow students were not interested	0.002	-0.012	-0.020	-0.000	-0.152**	-0.009***	19.4	10.7
Views on Further Maths teacher								
Was good at explaining maths	0.007**	-0.014	0.031	-0.001*	0.254**	0.005	34.9	31.7
Made maths interesting	-0.005	-0.069***	0.039	-0.001	0.227**	0.005*	23.5	18.8
Gave you individual attention	0.001	-0.026	-0.010	-0.000	0.078	0.004**	12.5	8.7
Gave you good feedback	-0.000	-0.022	-0.026	-0.000	0.150**	0.003	16.0	12.5
Was well respected by students	0.000	-0.018	0.060**	-0.001*	0.176*	0.003	24.3	22.2
Was good at motivating you	-0.002	-0.029*	0.041*	-0.001	0.218**	0.007**	27.3	22.6
Was your idea of a good teacher	0.004	-0.011	0.025	-0.001	0.182	0.006*	27.8	25.7

Note: * $p<0.1$, ** $p<0.05$, *** $p<0.01$

2. *School size.* The smaller the size of the VCE cohort, the more likely are students to report negative views of Further Mathematics and their Further Mathematics teachers, independent of all other factors. These findings may reflect the differences between smaller and larger schools in the size of their mathematics department, in that larger schools have greater numbers of qualified and experienced teachers from which to draw in staffing their VCE mathematics classes. These results parallel the relationship between school size and achievement in mathematics, whereby the smaller the school, the lower the performance in mathematics (Lamb & Helme, 2007).

3. *Proportion of FMM students.* Independent of all else, the strongest effects on students' views of mathematics and mathematics teachers is the proportion of students in Further Mathematics classes who are also doing Mathematical Methods. As this proportion increases, there is a significant increase in the proportions of Further Mathematics students who view the subject as one they could do well in, a significant decrease in the proportion of students who view classrooms as one in which there is too much disruptive behaviour, a significant increase in the proportion who consider their classroom to have a good working atmosphere, and a significant decrease in the proportion who claim that many students are not interested. Thus in the schools where there are larger numbers of Mathematical Methods students also doing Further Mathematics, students are more confident about doing well, feel they are learning in a good working atmosphere, and sense that other students are well motivated. These findings extend to their views of teachers, who are more likely to be perceived as making mathematics interesting, providing the individual attention they need, motivating them to do well, and conforming to their idea of a good teacher.

4. *School effects.* The data in the last two columns of Table 2 indicate that school level factors can account for much of the variation in students' views of Further Mathematics and Further Mathematics teachers. The second last column presents the amount of variance in the student view that can be explained by between-school differences, before taking account of the school-level factors. The final column presents the amount of between-school variance after controlling for the school-level factors. On certain items, there is a substantial reduction in the amount of school level variation after controlling for school-level factors. For example, between-school differences accounted for about 13.3% of the variation in responses to the item that Further Mathematics is "a subject you knew you could do well in". The school-level factors accounted for almost 50% of the between-school effects, reducing the unexplained variance to 7.1%. The school-level factors identified in this study (SES, size and proportion of FMM students) account for a large proportion of the school effect and can reduce the amount of school-level variance by up to half. This is the case for several items, including students' confidence in doing well, their claims of a good working atmosphere, and their reports of receiving the individual attention they needed.

Conclusions

This paper demonstrates that the students who do Further Mathematics as their only mathematics subject have a different experience of Further Mathematics than do students who combine Further Mathematics and Mathematical Methods. Further Mathematics-only students are less confident about their ability to do well, have a poorer experience of the mathematics classroom, and have more negative views of their mathematics teachers.

Further Mathematics was originally designed to cater to less-skilled mathematics students. The practice of allowing Mathematical Methods students to also study Further Mathematics may contribute to higher results in Further Mathematics for these students, but this may inhibit the capacity for teachers and schools to cater properly to the needs of those for whom the subject was initially designed.

Those in the mathematics education community with an interest in equity need to question the strategies that are being used to provide some students with an unfair advantage both within schools and across the school system, at the expense of the "traditional" Further Mathematics student. The recent decision to allow students to undertake all three mathematics subjects in the VCE without penalty will only exacerbate this problem, further expanding the gap between the "winners" and the "losers".

Acknowledgements. The project on which this work is based was done in collaboration with the Victorian Curriculum and Assessment Authority (VCAA). The views expressed in this paper are those of the authors.

References

Lamb, S., & Helme, S. (2007) *Organising for success in VCE Further Mathematics*. Unpublished manuscript.

Teese, R., Nicholas, T., Polesel, J., & Mason, K. (2007). *The Destinations of School Leavers in Victoria. On Track 2006.* Melbourne: Department of Education.

Victorian Curriculum and Assessment Authority (2005). *Study Summary. Mathematics: Further Mathematics 2006-2009*. Melbourne: Author.

Victorian Curriculum and Assessment Authority (2006). *2006 examiners report: VCE Further Mathematics*. Melbourne: Author.

Victorian Parliament (2006). *Enquiry into the Promotion of Mathematics and Science Education*. Melbourne: Government of Victoria.

Video Evidence: What Gestures Tell us About Students' Understanding of Rate of Change

Sandra Herbert
University of Ballarat
<s.herbert@ballarat.edu.au>

Robyn Pierce
University of Melbourne
<r.pierce@unimelb.edu.au>

This paper reports on insights into students' understanding of the concept of rate of change, provided by examining the gestures made, by 25 Year 10 students, in video-recorded interviews. Detailed analysis, of both the sound and images, illuminates the meaning of rate-related gestures. Findings indicate that students often use the symbols and metaphors of gesture to complement, supplement, or even contradict verbal descriptions. Many students demonstrated, by the combination of their words and gestures, a sound qualitative understanding of constant rate, with a few attempting to quantify rate. The interpretation of gestures may provide teachers with a better understanding of the progress in their students' thinking.

Introduction

Rate of change, with its many everyday applications, is an important concept throughout the mathematics curriculum. However it is fundamental to the understanding of early calculus: without a conceptual understanding of rate of change differentiation becomes an exercise in applying rules and executing routines. This paper reports on data, from a larger study, collected to explore the variation in pre-calculus students' understandings of rate of change. Experience (e.g., Kelly, Singer, Hicks, & Goldin-Meadow, 2002) has shown that analysing students' gestures as well as their utterances will provide greater insight into their thinking. In this paper, five gesture episodes are considered in detail. The aim of the exercise was to identify complementary, supplementary, or conflicting information conveyed by the students' gestures that was not conveyed by the oral text.

The section of the interviews that forms the focus of this paper provides data relating to students' understanding of rate of change in a non-motion context. The scenario was classified as "non-motion" because, for this example, the students were not asked to discuss change in position over time. Detailed analysis, of both the sound and images, of video-recorded interviews with individual students as they explained their reasoning about a computer-based simulation, provides insights into their thinking. Dynamic geometry (Geometers' SketchPad) was used to simulate a blind blocking sunlight coming in a window. This scenario provided a focus for each student's explanations as they grappled with the words needed to describe rate of change in the area of window exposed as the blind is raised, allowing sunlight to enter.

In the following sections, the conceptual framework is described; details of the interviews and the computer-based simulation are provided; and the manner in which the results can be analysed, by attending to gesture, is discussed.

Rate of Change

In this section we draw attention to students' likely school mathematics background related to rate of change; its importance as a pre-calculus concept; and the rationale for choosing to ask the students to discuss a "non-motion" scenario.

Proceedings of the 30th annual conference of the Mathematics Education Research Group of Australasia
J. Watson & K. Beswick (Eds), © MERGA Inc. 2007

According to the curriculum advisory documents (Victorian Curriculum and Assessment Authority (VCAA), 2005) and text books (Bull, Howes, Kimber, Nolan, & Noonan, 2003) the students, in this study, would have studied rate of change in conjunction with ratio, proportion, and percentage, usually, in Year 8. The topic is included in texts for that level. Typical of these is Bull et al. (2003) who describe rate as a measure of how one quantity changes with respect to another. This relationship between two changing quantities may be described qualitatively, such as increasing quickly, or quantitatively with units, such as dollars per year.

Researchers, writing about calculus students' understanding of rate of change, commonly provide more formal or more abstract definitions. For example, Hauger (1997) stresses the importance of the unit change in the independent variable resulting in a change in dependent variable. They consider this to be a very important foundational concept for a sound understanding of derivative.

The traditional approach (Thomas, 1969; 2008) to the introduction of derivative presents students with a formal, abstract definition and rule ($\lim_{h\to 0}(f(x+h)-f(x))/h$), then requires students to manipulate the symbolic representations of functions. Some students become quite competent in this manipulation and can accurately produce the symbolic representation of the derivative, but may not appreciate its meaning and connection to other mathematical concepts studied in earlier years. Indeed, Tall (1991) asserts that although calculus is broken up into small chunks and presented in a sequential, logical (at least to the teacher who can see the whole picture) series of lessons, "students may see the pieces as they are presented in isolation, like separate pieces of a jigsaw puzzle for which no total picture is available" (p. 17). Students may not even be aware that there is a big picture.

It was evident in Pierce and Atkinson's (2003) study in which a number of students, who, when asked to prepare a worksheet for novice calculus students, based on a graphical computer simulation for the tangent to a trigonometric function, ignored rate of change and focused on a rule for differentiating polynomials! Making students aware of the big picture involves linking the new concept to their previous understandings (Hiebert & Carpenter, 1992), which, in the case of differentiation, means being aware of pre-calculus students' understandings about rate of change.

A typical abstract introduction is often then followed later by a motion (change in position over time) example where velocity is aligned with derivative. It appears to be assumed that speed is a well-understood, familiar concept on which to build an understanding of derivative. However, rate may also appear in non-motion contexts, such as the rate of change of area of a circle with change in radius.

This study explores pre-calculus students' thinking about rate of change, in a non-motion context, by analysing video evidence. The next section discusses the reasons for choosing video as a data collection method.

Data Collection Methods

Data may be collected from a variety of sources such as: written tests or students' worksheets; teachers' reports; classroom observation; audio recording of interviews with teachers, individual students, pairs of students or small focus groups; and video. Each method has strengths and limitations. Even though video-recording interviews may be more time-intensive and problematic, for example, some students or their parents are less likely to give consent, the decision was made to use this method because video provides a comprehensive record for later detailed analysis.

Video enables the researcher to formulate interpretations of the gaps in the audio record. Data captured by video may provide a more comprehensive understanding of the learning demonstrated by students (Pea, 2006). Such data may include sound and images containing facial expressions, tone of voice and gestures, together giving insights into emotions and depth of understanding of concepts. Fine-grained analysis discloses insights into students' understanding not otherwise available (Alibali & Goldin-Meadow, 1993).

The next section explores how we may analyse the non-verbal data privileged by the videos.

Gesture

One of the advantages of using video for data collection is that it captures non-verbal communications. The importance of gestures, in conveying information regarding students' understanding of mathematical concepts, has become the focus of much research in recent years. Goldin-Meadow, Kim, and Singer (1999) assert the importance of teachers' gestures in the learning of mathematics in their study of eight teachers, teaching mathematical equivalence to students of age eight to ten years. Noble (2003) reports on the use of gestures in the development of the new mathematical knowledge of connecting graphs of motion with the student's own motion, for one student over three teaching episodes. Sabena (2004), who studied of secondary students understanding of the integral function, reports that gesture was instrumental to the development of this concept. Similarly, Arzarello, Robutti, and Bazzini (2005) suggest "students' cognitive activity is strongly marked by rich language and gesture production" (p. 64), as the 11- and 12-year-olds, in their study, construct "meanings related to the concept of function" (p. 55). They advocate that teachers should encourage the use of language, body-related motion, and gestures and include these in the planning of their lessons. Edwards (2005a), in her study of pre-service teachers, reports that gestures played an important role in their recall of procedures related to fractions. Williams (2005) refers to "gesture as part of an integrated communication system with language and … mathematics" (p. 146). When interactions between students are videoed and the visual images are examined, these images may record instances where one student facilitates the learning of another, and possibly their own learning, by drawing their attention to a particular aspect of a task (Rasmussen, Stephen, & Allen, 2004).

Other researchers, such as Goldin-Meadow (2004) and Arzarello and Robutti (2004), also support the claim that the use of gesture aids an individual's learning of mathematics, perhaps by replacing some of the cognitive load of problem-solving or explanation with gesture (Goldin-Meadow, Nusbaum, Kelly, & Wagner, 2001). This suggests that gesture may not necessarily be used to convey information to another people, but also performs the function of assisting gesturers to clarify their own thoughts.

Of particular interest is that gestures may convey information that differs from the information provided by speech. Gestures may provide additional, complementary information, but may also contradict speech (Alibali & Goldin-Meadow, 1993). The gesture-speech mismatch may afford teachers an opportunity to guide students towards a more correct and complete understanding of a mathematical concept (Alibali, Flevares, & Goldin-Meadow, 1997). Hence, it is important for researchers and teachers to learn more about the hidden meanings of students' gestures (Kelly et al., 2002). However, the interpretation of gesture is often difficult as the gestures may be ambiguous (Williams & Wake, 2004). Interpretation may be facilitated by the classification of gestures. McNeill (1992) defines four gesture categories: beat giving emphasis; deitic or pointing; iconic imitating physical phenomena; and metaphoric, which represent meaning of some kind, but are less easy to interpret. Edwards (2005b) refers to the need for additional categories to enable clearer interpretation of students' gestures. She suggests that the iconic classification may be divided into iconic-physical, for iconic gestures matching physical phenomena, and iconic-symbolic, for iconic gestures referring to "a remembered written inscription for an algorithm or mathematical symbol" (p. 136). Further, she proposes "the nature of mathematics as a discipline may require an even more refined categorization of gestures" (p. 138). Indeed, Arzarello and Robutti (2004) define iconic-representational gestures, as gestures that refer "to a graphical representation of a phenomenon" (p. 307).

The next section describes the methodological considerations of this study.

Method

The seven students whose data are reported in detail this paper were selected from the 25 Year 10 students from five different secondary schools interviewed for the full study. These students were selected because the videos of their interviews demonstrate clear examples of gestures that were commonly used by many of the students in the study. A Geometers' Sketchpad (GSP) file simulating two windows with blinds (Figure 1) was prepared.

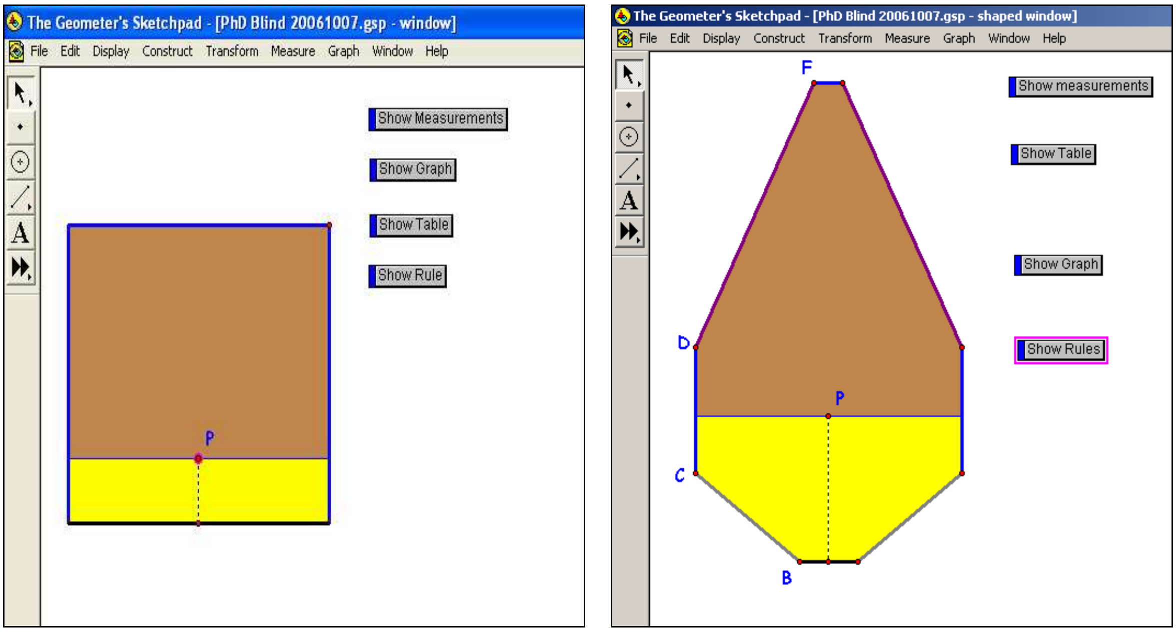


Figure 1. GSP simulation of windows.

The simulation shows two windows, one rectangular, one arched, both with blinds, which could be raised or lowered by dragging. This had the effect of changing the variables: area of sunlight and height of blind above the bottom of the window. Possible constant rate variation associated with the simulation was illustrated using multiple mathematical representations: numeric, graphic, and symbolic.

This simulation and a photograph of an arched window were used as catalysts to explore students' understanding of the constant rate of change. Similarly, the non-rectangular window was used to probe students' understanding of the differences between constant and variable rate. In this way, GSP facilitated exploration of constant and variable rate in multiple representations. These simulations, which were first trialled by a pilot group of students, provided visual material for students to point to, in order to clarify their explanations of their understanding of rate.

Students were videoed as they responded to the interviewer (first author) who prompted them to discuss the rate of change in the area of sunlight exposed as the height of the blind was changed. Students were encouraged to explain their reasoning and think aloud as they were presented with different representational forms of rate of change: the simulation, table of values, graph, and symbolic rule.

The videos were each viewed several times and the students' use of gesture was coded and checked. Coding, as is detailed below, combined McNeill's (1992) deitic and metaphoric categories with the refinements of the iconic classification of iconic-representational (Arzarello & Robutti, 2004) and iconic-physical and iconic-symbolic (Edwards, 2005a). The five episodes described below (pseudonyms used) illustrate the complex use of gesture students called upon to supplement their utterances in order to explain their thinking about rate.

Findings and Discussion

In the twenty-five video-interviews recorded, one student gestured frequently, two students only used deitic gestures to indicate locations on the screen, and the remaining twenty-three students used deitic, iconic, and metaphoric gestures especially when struggling for words to describe their understanding. The simple deitic gestures add to the audio record by clarifying exactly what the students are referring to and emphasising the feature they see as important in their explanation of rate of change.

Straight-sided arch shape for small distances	Two hands for larger distances	Deitic gestures indicating parts of the screen

Figure 2. Rate-related gestures.

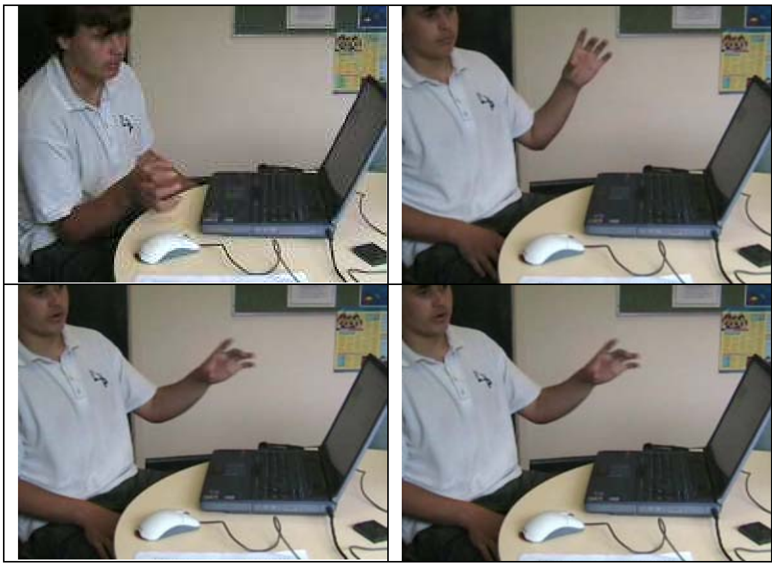
Many students used one hand to form a straight sided arch to represent small distances and two hands held apart for larger distances (Figure 2). These are examples of iconic-physical gestures (Edwards, 2005a). Many students employed what Rasmussen et al. (2004) chose to call a "slope hand gesture" (Figure 6) representing the shape of the linear graph. Rasmussen et al. (2004) found that this was commonly used by students to infer constant rate.

In addition to noting specific static forms of rate-related gestures seen in Figure 2, five gesture episodes were examined in greater detail. "Moving slope gesture" (Figure 3), where a hand was held straight and rigid with the arm pivoted at the elbow, when Annie was describing what the graph would look like if the window were narrower, indicated a change in constant rate in the same manner as Rasmussen et al. (2004) describe.

Figure 3. Moving slope gesture.

The next example demonstrates this student's thinking about the variables involved in this constant rate context.

> Researcher: what does the table tell you about the rate that the height is changing?
> Jason: it goes up three point two meters [pause] every half a meter



It seems that Jason is using the same straight-sided arch shape others used to indicate a small distance, but in a different way.

He seems to demonstrate a unit measurement by making a straight-sided arch shape with his right hand and matching that with three movements of the same-sized, straight-sided arch shape with his left hand.

Figure 4. Speech-gesture mismatch.

In this episode (Figure 4), Jason's gestures and words do not match. He indicates the 0.5 m, from the table, with his right hand as he says "it goes up three point two meters" and makes just three movements with his right hand whilst saying the words "every half a meter". This may suggest that Jason is uncertain about which variable, area or height, is involved in the unit change. These arch gestures could be classified as iconic-symbolic, in this case, as they appear to represent a unit of measurement. However, this episode also suggests that he has some notion that rate involves a change in one variable related to the unit change of another variable. Such a gesture-speech mismatch may provide an opportunity for guidance by a teacher (Alibali & Goldin-Meadow, 1993) to clarify the variables.

Interestingly John, in Figure 5, when he was looking at the simulation of the non-rectangular window and describing the rate in the rectangular section of the window, also uses the same movement of the straight-sided arch gesture as Jason, as he talks about constant rate when referring to the rectangular section of the window.

Changing at a constant pace …the rates will be going up by the same amount

Figure 5. Constant rate gesture.

The next example demonstrates the manner in which gesture can be used to supplement words. There were many instances, in the data, of gesture being used in this way, as shown in Figure 6.

Researcher: what do you think the graph [of the arched window in this photo] might look like? Sue: I don't think it would be a straight line because [pause]	Researcher: What makes you think that? Sue: because this window's not square hmm I dunno because its up the top umm curved there	Sue: [pause] umm a straight line means its always the same

Figure 6. Gesture supplements words.

It appears Sue is grappling with the difference between graphs for constant and variable rate but does not have the words to express her understanding. She is using her hands to indicate what she is thinking. The iconic-representational gesture in the first frame (Figure 6) appears to represent the shape of the graph of constant rate. In the middle of Figure 6, her gesture is iconic-physical as she is showing the physical shape of the top of the window. Finally, in the right frame of Figure 6, the "slope hand gesture" is repeated, indicating her understanding that constant rate will result in a linear graph. Sue has identified the key difference in the two scenarios presented by a rectangular window and an arch window. Her gestures communicate her understanding, demonstrating her awareness that the graph for the curved section of the window would not be the same as the rectangular section. The distinction between the iconic-representational gesture and the iconic-physical gesture indicates that her thinking had not yet progressed to transferring her understanding of the physical situation into a graphic, mathematical representation. Her gestures provided additional information not available in her words. This presents an ideal opportunity for a teacher (Alibali & Goldin-Meadow, 1993) to assist by supplying suitable words to describe her correct thinking and extend her understanding of variable rate to include the graphical representation.

The final example demonstrates a student's thinking about the shape of the graph for variable rate, as she is considering the graph for the rectangular window. The deitic gestures have been used to supplement words rather than just indicating an aspect of interest on the screen.

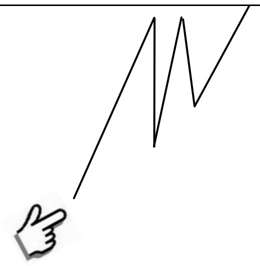
		
Claire: It's going at a average pace. It's equal um even pace equals yeah its even. Researcher: The rate's even?	Claire: yeah like if it was like uneven like it would be like say like pull up this much of it this much light showing	Claire: like it would probably go like that

Figure 7. Gesture for variable rate.

In this episode (Figure 7), Claire also uses the pointed fingers of both hands to indicate a larger distance, similar in meaning to the manner in which other students have used two straight hands. In Figure 7, the diagram illustrates the up and down motion of the index finger of her right hand. It appears that Claire is contrasting "even" rate by trying to demonstrate "uneven" rate with these deitic gestures. She seems to be associating variable rate with this collection of linear segments, varying the slope from positive to negative in an upward trend.

The gesture episodes observed, offer insights into students' understanding of rate of change, which were absent from their words alone. The images in Figure 2 show some commonly used rate-related gestures. In Figure 3, Annie used the "moving slope gesture" to indicate change in constant rate. Jason's gestures, in Figure 4, suggested confusion between the variables, involved in the rate, which was not evident in the written transcript. John (Figure 5) uses the same shaped gesture as Jason, but matches his words to the gesture. In Figure 6, Sue was unable to verbalise why the graph would not be straight for variable rate, but demonstrated by the repeated use of the "slope gesture" that constant rate would result in a linear graph. Claire's statement, in Figure 7, "like it would probably go like that" could not have been interpreted, whereas her deitic gestures suggest she does not fully understand either constant or variable rate. Gestures augmented the verbal descriptions to give greater depth to the researcher's understanding of the meaning of the students' utterances.

Implications

The concept of rate involves an understanding of quantities and their measurement. The episodes, described in this paper, demonstrate examples of gesture related to constant rate; gesture related to variable rate; gesture supplementing utterances; gesture contradicting utterances; and gesture consistent with the classification of other researchers. For the students in this study, gestures provided an intermediary stage. They were able to articulate qualitative, but not completely correct quantitative, descriptions of rate; gesture enabled them to communicate their understanding by using non-standard units (e.g., Figure 4).

Analysis of the video evidence showed that these students had a sound conceptual understanding of constant rate of change but some students had difficulty in verbalising this. The use of gesture enabled many students to communicate ideas related to the less abstract graphic and numeric representations but most students, although able to describe operations with the symbolic representation, could not link this to rate of change. Some students were able to use gesture to supplement their utterances relating to variable rate, but none could describe their thinking with words alone. Indeed, many students demonstrated little conceptual understanding of variable rate in this non-motion context.

Students' gestures provided a rich source of evidence from which to evaluate their understanding of rate of change. Such evidence is not always available in written tests where only the words are valued. Attention to gestures may enable teachers to comprehend better the depth and accuracy of students' understanding of mathematical concepts and allow teachers to target interventions appropriate for individual students. For example, when students' words and gestures match it is likely that they have a clear understanding of the concept. Such students are ready to explore more advanced concepts. When students cannot find words to express themselves, but can demonstrate concepts through gesture, there is an opportunity for the teacher to build on their understanding by targeting vocabulary and symbolic representations. In the case where students use gestures that contradict their utterances, there is an indication that the students do not, as yet, fully understand the concept. Such a mismatch may alert the teacher to the need, both, to further probe the students' understandings, and also to provide suitable tasks to help the students clarify their understandings. Attending to gesture as well as words helps the teacher more accurately chart their students' growth in understanding of rate of change. The examples included in this paper highlight the advantages of including analysis of gesture in the repertoire of both teachers and educational researchers.

Acknowledgements. This study is part of the RITEMATHS project, led by Kaye Stacey, Gloria Stillman, and Robyn Pierce. The researchers thank the Australian Research Council, our six partner schools and Texas Instruments for their financial support of this project, and especially the teachers and students who took part in this study. Special thanks to Gary Asp for sharing his GSP skills.

References

Alibali, M. W., & Goldin-Meadow, S. (1993). Gesture-speech mismatch and mechanisms of learning: What the hands reveal about a child's state of mind. *Cognitive Psychology*, 25(4), 468-523.

Alibali, M. W., Flevares, L. M., & Goldin-Meadow, S. (1997). Assessing knowledge conveyed in gesture: Do teachers have the upper hand? *Journal of Educational Psychology, 89*(1), 183-193.

Arzarello, F., Robutti, O., & Bazzini, L. (2005). Acting is learning: Focus on the construction of mathematical concepts. *Cambridge Journal of Education, 35*(1), 55-67.

Arzarello, F., & Robutti, O. (2004). Approaching functions through motion experiment. *Educational Studies in Mathematics 57*, 305-308.

Bull, I., Howes, B., Kimber, K., Nolan, C., & Noonan, K. (2003). *Maths for Vic 8*. South Melbourne, VIC: Longman.

Edwards, L. (2005a). *Metaphors and gestures in fraction talk.* Paper presented at the CERME 4, Fourth Congress of the European Society for Research in Mathematics Education, Sant Feliu de Guíxols, Spain.

Edwards, L. (2005b). The role of gestures in mathematical discourse: Remembering and problem solving . In H. L. Chick & J. L. Vincent (Eds.), *Proceedings of the 29th annual conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 135-138). Melbourne: PME.

Goldin-Meadow, S., Kim, S., & Singer, M. (1999). What the teacher's hands tell the student's mind about math. *Journal of Educational Psychology, 91*(4), 720-730.

Goldin-Meadow, S., Nusbaum, H., Kelly, S. D., & Wagner, S. (2001). Explaining math: Gesturing lightens the load. *Psychological Science, 12*(6), 516.

Goldin-Meadow, S. (2004). Gesture's role in the learning process. *Theory Into Practice, 43*(4), 314.

Hauger, G. S. (1997). *Growth of knowledge of rate in four precalculus students.* Paper presented at the Annual Meeting of the American Educational Research Association. Chicago, IL: American Educational Research Association.

Hiebert, J., & Carpenter, T. (1992). Learning and teaching with understanding. In D.Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 65 - 97). New York: Macmillan.

Kelly, S. D., Singer, M., Hicks, J., & Goldin-Meadow, S. (2002). A helping hand in assessing children's knowledge: Instructing adults to attend to gesture. *Cognition & Instruction, 20*(1), 1-26.

McNeill, D. (1992). *Hand and mind: What gestures reveal about thought.* Chicago: University of Chicago Press.

Noble, T. (2003). *Gesture and the mathematics of motion.* Cambridge, MA: TERC.

Pea, R. D. (2006). Video-as-data and digital video manipulation techniques for transforming learning sciences research, education and other cultural practices. In J. Weiss, J. Nolan & P. Trifonas (Eds.), *International Handbook of Virtual Learning Environments* (pp. 1321-1393). Dordrecht: Kluwer Academic Publishing.

Pierce, R., & Atkinson, L. (2003). Interactive animation provides a vehicle for exploring students' understandings of derivatives. In L. Bragg, C. Campbell, G. Herbert and J. Mousley (Ed.s), *Mathematics education research : innovation, networking, opportunity : Proceedings of the 26th annual conference of the Mathematics Education Research Group of Australasia, Volume II*, (pp.578-585). Pymble NSW: MERGA.

Rasmussen, C., Stephan, M., & Allen, K. (2004). Classroom mathematical practices and gesturing. *Journal of Mathematical Behavior, 23*(3), 301-323.

Sabena, C. (2004). The role of gestures in conceptualisation: An exploratory study on the integral function. In M. Hoines & A. Fuglestad (Eds.), *Proceedings of the 28th annual conference of the International Group for the Psychology of Mathematics Education. Volume 4,* (pp.145-152). Bergen, Norway: PME.

Tall, D. (Ed.). (1991). *Advanced mathematical thinking.* Dordrecht: Kluwer.

Thomas, G. (1969) Calculus and analytical geometry. Reading, MA: Addison-Wesley.

Thomas, G. (2008) *Calculus*. Massachusetts:Pearson-Addison-Wesley.

Victorian Curriculum and Assessment Authority (VCAA). (2005). *Victorian Essential Learning Standards: Mathematics Level 6*. Retrieved May 31, 2005, from http://vels.vcaa.vic.edu.au/essential/discipline/mathematics/level6.html

Williams, J. S., & Wake, G. D. (2004). Metaphors and cultural models afford communication repairs of breakdowns between mathematical discourses. In M. Hoines & A. Fuglestad (Eds.), *Proceedings of 28th annual conference of the International Group for the Psychology of Mathematics Education. Volume 4,* (pp.409-416). Bergen, Norway: PME.

Williams, J. (2005). Gestures, signs and mathematisation. In Chick, H.L. & Vincent, J.L. (Eds.). *Proceedings of the 29th annual conference of the International Group for the Psychology of Mathematics Education,* (Vol. 1, 146-148). Melbourne: PME.

The Role of Dynamic Interactive Technological Tools in Preschoolers' Mathematical Patterning

Kate Highfield
Macquarie University, Sydney.
<kate.highfield@aces.mq.edu.au>

Joanne Mulligan
Macquarie University, Sydney.
<joanne.mulligan@mq.edu.au>

This paper presents case study data from an exploratory study investigating six preschoolers' patterning skills using three learning modes: concrete materials, screen-based technological tools, and combined modes. Children using *dynamic interactive software* and *virtual manipulatives* to solve pattern-eliciting tasks engaged in more "experimental" representations and created more patterns and transformations than children using concrete materials. However, there were no qualitative differences observed between children's understanding of simple repetition. This research highlights new ways of mathematics learning that can be enhanced through explicit techniques afforded by technology.

The importance of early patterning and pre-algebraic skills have been articulated in several recent research projects (Dougherty & Slovin, 2004; English, 2004; Fox, 2005; Mulligan, Prescott, Papic, & Mitchelmore, 2006; Papic & Mulligan, 2005). These studies have highlighted children's potential to develop simple repetition, growing patterns and functional thinking (Blanton & Kaput, 2004; Warren, 2005). Patterning skills have also been found critical to the development of other mathematical processes, such as analogical reasoning and transformation (Lehrer, Jenkins, & Osana, 1998).

In preschool settings, patterning is readily observed in children's play (Ginsburg, Lin, Ness, & Seo, 2003) however few teachers harness, or mathematise, these moments (Clements & Sarama, 2007; Fox, 2005). Patterning forms an integral part of the school mathematics curriculum and young children are required to engage in simple through to complex patterning (Board of Studies NSW, 2002). Generally these patterning experiences involve the use of concrete materials and representations of patterns through drawing and traditional media. Young children, particularly preschoolers, are rarely given the opportunity to create a range of patterns on-screen, yet they are capable of producing powerful mathematical ideas (Perry, Dockett, Harley, & Hentschke, 2006).

New technologies, such as *virtual manipulatives* and *dynamic interactive software* may allow young children to create mathematical representations that have increased potential mathematically (Clements & Sarama, 2007). For example, the development of simple repetition, and transformation skills such as reflection, rotation and scaling are enhanced through on-screen manipulations. Virtual *Pattern Blocks* and dynamic interactive software can provide representations of concrete manipulatives that allow children to experiment with a broader range of patterns with ease and flexibility.

Background to the Research

A number of researchers have highlighted the importance of linking concrete mathematical experiences with symbolic representations, a transition that may be assisted by using computer-based manipulatives (Clements, 1999; Clements & Sarama, 2007; Kaput, 1992; Moyer, Niezgoda, & Stanley, 2005). Virtual manipulatives are particular forms of mathematical software that can be defined as "interactive, Web-based visual representation of a dynamic object" (Moyer, Bolyard, & Spikell, 2001, p. 373). For

Proceedings of the 30th annual conference of the Mathematics Education Research Group of Australasia
J. Watson & K. Beswick (Eds), © MERGA Inc. 2007

example, Pattern Blocks (see Table 4, following) have considerable mathematical potential because they can be easily transformed and recorded, simulating the manipulations that children make with concrete materials. Other programs utilising dynamic drawing tools, such as *Kidpix* (Brøderbund, 2004) have the added advantage of changing properties of objects.

It appears that dynamic processes afforded by these tools can enable children's spatial visualisation skills and experimentation with size, shape, orientation and simple repetition. Although there is little research on the use of technological tools with preschool-aged children, some key research has been conducted with elementary students (Moyer, Bolyard, & Spikell, 2001). Clements (1999) and Moyer et al. (2001; 2005) highlight benefits of virtual manipulatives for classroom use. For example, virtual Pattern Blocks have colours that can be changed, they can be "snapped" into position, unlike concrete material and they "stay where they're put" (Clements, 1999, p. 51). Although virtual manipulatives may seem advantageous there is little research explicating how young children make connections between concrete and dynamic representations. Reimer and Moyer's work with third graders highlights some possible benefits of virtual manipulatives as a "dynamic visual model" (2005, p. 22) with potential for multiple representations of concepts.

In a study of Kindergarten children's patterning, Moyer et al. (2005) found that children's patterns were more creative, complex and prolific using virtual manipulatives compared with patterns formed with concrete materials. It is not known whether these findings would be supported in studies of preschoolers, who are likely to have less developed computer skills and limited mathematical patterning abilities. There is also scant research on young children's use of dynamic interactive software in early mathematical development. The work of Hong and Trepanier-Street (2004), although not specific to mathematics education, does show that young children's representations employing dynamic interactive software, such as *Kidpix* are more detailed than representations produced off-screen.

This raises a broad research question: In what ways can the use of dynamic interactive software and virtual manipulatives advantage the development of mathematical patterning skills in preschool children? This study focuses on the potential advantages of using such technologies in developing early patterning and transformation skills.

Method

This project took the form of a constructivist teaching experiment, integrating elements of a developmental design approach, using six collective case studies (three dyads) of preschool children, aged between four and five years (Hunting, Davis, & Pearn, 1996). This mixed-method approach allowed for teaching episodes to be constructed and scaffolded systematically, based on the continual reassessment of each child's progress.

Prior to commencing the teaching episodes each child was assessed for numeracy using *I can do maths* (Doig & de Lemos, 2000) and patterning skills using an *Early Patterning Assessment (EPA)*, (Papic & Mulligan, 2005). Three key tasks were administered in the EPA – "imagine and draw a pattern", "make a pattern" with materials and "repeating pattern tasks" (tower tasks). Following the initial assessment children were paired into one of three dyads, balanced for gender. Each dyad then participated in six, 40-minute teaching episodes, conducted by the researcher over a 4-week period at a participating preschool. Each dyad was assigned to one of three learning modalities using:

1. concrete materials (such as blocks, counters, animal pictures, stamps, paint, pencils);
2. a combination of concrete materials, dynamic interactive software (*Kidpix*) and virtual manipulatives (virtual *Pattern Blocks*), and
3. dynamic interactive software (*Kidpix*) and virtual manipulatives (*Pattern Blocks*).

The aim of the teaching episodes was to engage the children in *pattern-eliciting* tasks, based, in part, on recent studies of mathematical modelling (English, 2006) and early patterning (Papic & Mulligan, 2005). Three pattern-eliciting tasks: making "wrapping paper", creating "wall paper borders" and "threading beads" required the construction of simple repetition in different forms, with opportunity for multiple, alternate representations. Where possible the tasks directly related to the children's context, such as creating a new wallpaper border to replace an existing border. These tasks allowed children to play with mathematical patterns but were structured sufficiently to promote mathematical thinking. Tasks were matched across each of the three modalities with concrete materials replicating on-screen resources and tools (and visa versa). Teaching procedures and the order of tasks remained consistent, although it was anticipated that the solution strategies used by each child would differ. The researcher encouraged multiple responses and encouraged children to create and discuss their own representations regardless of the learning mode. Following the six teaching episodes, the children were re-assessed, using the same assessment instruments. Multiple data sources (audio and digital media, work samples, and "researcher as participant observer" records) were compiled throughout the teaching experiment. All data were collated to enable a descriptive analysis for each child, and in turn, each dyad's progress. Children's responses to the tasks in each teaching episode were coded for the type and sophistication of patterning and transformational skills, supported by transcriptions of discourse between dyad and researcher.

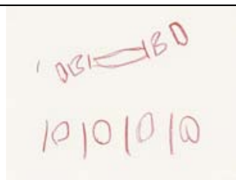
Results

Some initial findings are drawn from pre- and post-assessment data and the analysis of patterning strategies developed throughout the teaching episodes. The discussion provided here focuses primarily on differences between children's patterning and transformational processes afforded by the use of technological tools.

Pre- and Post-Assessment Responses

Pre- and post-assessment data from the EPA indicated that all six children's responses progressed from idiosyncratic to more formalised representations containing a *unit of repeat*. This development appeared independent of the learning mode employed in each dyad. Using the descriptors developed by Papic (Papic & Mulligan, 2005), the children's images of pattern ("imagine and draw a pattern", and "make a pattern with blocks") were initially analysed and coded. Table 1 provides an example of a typical pre- and post-assessment response for the task, "imagine and draw a pattern" using this coding.

Table 1
Sample Pre- and Post-Assessment for "Imagine and Draw a Pattern"

Child ID	Pre-assessment – imagine and draw	Pre-assessment Category and Image Description	Post-assessment – imagine and draw	Post-assessment Category and Image Description
Nicholas		Code 1: Inanimate objects and things. An image of a yellow star. No pattern is evident		Code 3: Informal notations showing evidence of symmetry. Code 4: Icons/ Symbols, A simple AB repetition using shape.

The most important finding that emerged at this stage of the analysis was that no child represented pattern depicting a unit of repeat at the pre-assessment. Although some diagrams showed evidence of symmetry and regularity, the children were seemingly unaware of any pattern features. The pre-assessment representations contrast with the post-assessment data where children depicted pattern as simple repetitions using a unit of repeat. Although it is clear that significant changes were made between assessments, it is not possible to infer whether the children's initial idiosyncratic images of pattern remained or whether they had been reconstructed through new representational processes. Moreover it is feasible that the children learned, through the teaching episodes, to present simple linear repetitions in the way the researcher had scaffolded the learning.

Responses to pre- and post-assessment for the "make a pattern with blocks" task showed similar patterns of response to the first task. Examples of two typical pre- and post-assessment responses are provided in Table 2.

Table 2
Pre- and Post- Assessment Responses for "Make a Pattern with Blocks"

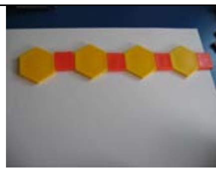
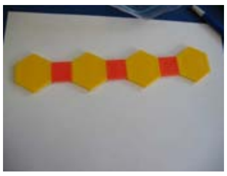
Child ID	Pre-assessment	Pre-assessment Category and Image Description	Post-assessment	Post-assessment Category and Image Description
Joshua		Code 5: Spatial structure. This image of "shapes" shows both tessellation and symmetry, using multiple shapes.		Code 3: Simple single variable repetition This is an example of a complete ABAB pattern.
Isabelle		Code 1: Random arrangement. This is a picture, of "butterflies and flowers", and does not show repetition or regularity.		Code 3: Simple single variable repetition. This is an example of an incomplete ABAB pattern.

Table 2 provides a pre-assessment response by Joshua depicting a pattern with transformation symmetry. Similar pre-assessment responses were produced by two other children. Although some structure is evident in Joshua's work, no child produced a pattern with a unit of repeat central to its design, such as ABCABC. This was in sharp contrast with post-assessment responses, where the children made patterns containing a unit of repeat or an incomplete unit of repeat. Table 2 also provides an example of pre- and post-assessment responses for Isabelle, where an incomplete unit of repeat is shown at post-assessment. However, it was not possible to determine whether these children were aware of using symmetry or a unit of repeat in their designs.

In four repeating patterns tasks (tower tasks), the children used multilink cubes to extend, make and draw simple and complex repetitions, identify hidden elements, break the tower into elements, and record from memory (six modes of response). Table 3 indicates the number correct responses (six responses are possible) for the four tasks at pre- and post-assessment. At pre-assessment all children could continue simple AB repetitions but found most other tasks difficult. Matthew was an exception to this, as he was able to respond correctly to most tasks except for the "breaking into elements" strategy.

Table 3
Children's Performance of Tower Tasks at Pre- and Post-Assessment

		Pre-assessment				Post-assessment			
		AB	ABC	AAB	ABBCCC	AB	ABCABC	AABAAB	ABBCCC
Dyad 1 Materials	Tina	1	0	0	0	4	4	0	0
	Joshua	3	2	0	0	6	5	0	0
Dyad 2 Combined	Nicholas	3	0	0	0	6	4	0	0
	Yvette	2	0	0	0	4	0	0	0
Dyad 3 Technology	Isabelle	4	0	0	0	6	4	0	0
	Matthew	5	5	5	3	5	5	5	4

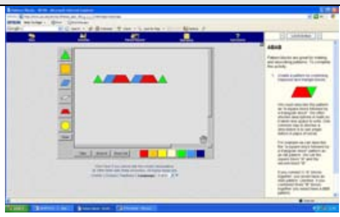
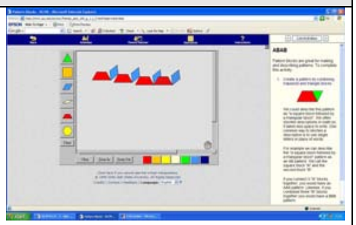
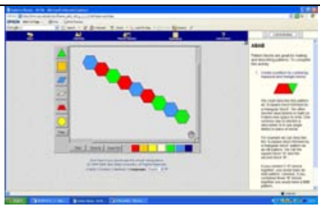
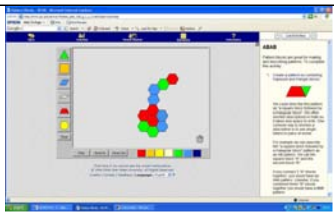
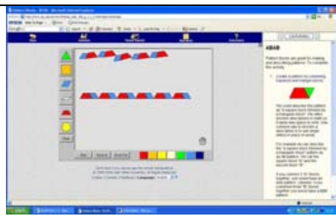
Table 3 indicates that all of the children progressed in their understanding of simple repetition (ABAB) with four of these children also constructing complex repetitions (ABCABC). By the post-assessment all children had progressed significantly in both complexity and awareness of pattern.

The overall progress shown for individuals between pre- and post-assessments across all three EPA tasks was evident but the differences in responses between dyads was too small, or not consistent, to be noteworthy. Further reporting of the individual patterns of response is required to describe individual progress within learning modalities.

Teaching Episodes

Increased representations using technology. Technological tools allowed ease of representation, with children in dyads 2 and 3 consistently engaged in increased experimental patterning. Children working on-screen produced a broader range of patterns, and edited or deleted them before completion. In part, this could be attributed to the "delete tools" that held "novelty value", with the children enjoying "rubbing out" and "chucking" things in the "bin". The figures provided in Table 4 provide examples from each dyad, of children's experimentations from the third teaching episode, where they re-visited a "beading" task, seeking alternate patterns. In this teaching episode, as in all teaching episodes, the more permanent nature of the concrete materials meant that children using traditional representational tools were less likely to experiment with their representations. In contrast, children using technological tools were motivated to experiment with, and produce more patterns. For example, in dyads 2 and 3, Yvette and Isabelle cloned pattern elements following demonstration by the researcher.

Table 4
Three Children's Work Samples for Beading Task "Mum's Beads"

	Teaching Episode 3: Sample 1	Teaching Episode 3: Sample 2	Teaching Episode 3: Sample 3
Dyad 1: Materials		Neither child in this dyad would produce further patterns.	
	Joshua created a complex repetition (ABC) compared with previous pattern.		
Dyad 2: Combined			
	Yvette reconstructed previous pattern (AB) with shapes.	Yvette constructed an ABC repetition created with assistance using 'cloning' technique.	Yvette produced an AB repetition created independently using cloning technique.
Dyad 3: technology			
	Isabelle produced a sequence of hexagons, carefully aligned, using three colours without use of unit of repeat.	Isabelle produced a 'pendant' using a different arrangement of hexagons, without use of unit of repeat or cloning.	Isabelle constructed two AB patterns including a unit of repeat using a cloning procedure.

Transformation skills. Both dyads using technological tools engaged in more explicit transformative actions, such as reflections, translations and rotations, and shearing or scaling of images (see Figures 1 to 3). Although children using concrete materials did engage in transformations such as sliding, rotating and flipping of materials, these actions were not as defined as those actions performed with technology. As well, these children did not engage in shearing or scaling of images, as this was not easily performed off-screen. Children using technology also engaged in rich mathematical discussions about their transformations (see excerpt accompanying Figure 3). Discussion of mathematical actions was not forthcoming from dyad 1. The prevalence of transformative actions on-screen had not been anticipated by the researcher and subsequently this was explored further in response to the children's experimentation. Transformations, such as rotation and translation were identified in representations and discussions of the dyads working on-screen across all teaching episodes. The transformations produced by the children working with traditional materials were not explicit, nor were these discussed spontaneously by the children. Transformative actions for off-screen dyads only occurred in one teaching episode, after the researcher modelled reflections and rotations.

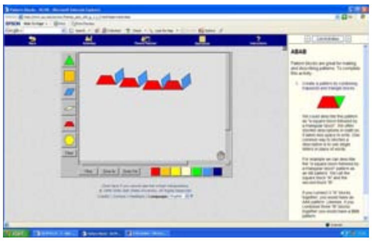


Figure 1. Yvette's repetition, with rotational transformation of parallelograms.

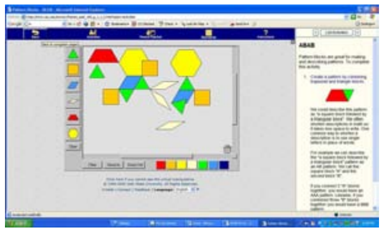
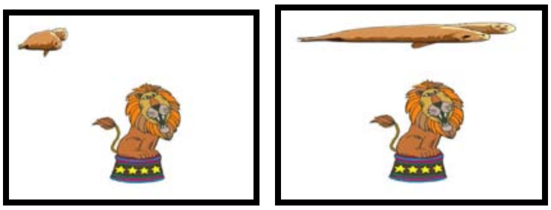


Figure 2. An image created by Isabelle, showing transformations of shapes.



Nicholas: Oh he's really big now. He's really, really big. Wee … Oh … Big … Fat *(as he scaled the lion, enlarging it)*

Yvette: Make him long *(pointing to the seals).*

Nicholas: Flat *(after shearing the seal).*

Yvette: They're both flat *(pointing to the seals).*

Figure 3. Screen shots of shearing and scaling lion and seal icons, with accompanying transcript.

Accuracy of representations afforded by technology. Both dyads using technological tools produced more mathematically accurate representations on-screen. Use of shape icons and stamps ensured that all representations using virtual Pattern Blocks contained geometrically accurate features, compared with those drawn by the children.

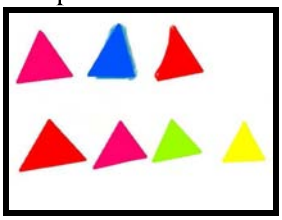


Figure 4. Isabelle's triangles, created using "sticky straight string".

Other tools, such as the "sticky straight string" (Figure 4) available in *Kidpix,* allowed children to present geometric shapes more accurately and with structure. This may not have been permitted with some children's limited fine motor skills.

Discussion

The findings of this study indicate some potential advantages and disadvantages of using technological tools in early patterning. Dyads working on-screen were enabled by the technological tools to pursue alternate learning trajectories. Children restricted to concrete materials still produced patterns using a unit of repeat. However, without the dynamic appeal of on-screen tools they were not motivated to investigate other mathematical processes such as cloning a unit of repeat, or transformations such as shearing and scaling.

Potential Advantages of Screen-Based Tools

The observation that technological tools motivated children to experiment more readily and practice patterning skills is an important insight gained from this study. An increase in on-screen patterns was also described by Moyer, Niezgoda, and Stanley (2005) in their study of Kindergarten children's patterning with virtual manipulatives.

Dynamic interactive software and virtual manipulatives provide tools whereby the children can easily link units and clone or copy units of repeat which can promote mathematical processes such as unitising and multiplicative reasoning. Some of these technological functions have been partially investigated with older children (Clements, 1999; Moyer et al., 2005). In this small-scale study there were few spontaneous examples of cloning units of repeat observed but with teacher guidance and further experience the children may have been able to develop this process independently.

Transformative actions exhibited by the children working on-screen provided a powerful example of the potential of technological tools to enhance geometric concepts and related mathematical processes. The use of technology also exposed children to novel techniques for exploring concepts such as scaling and shearing, fundamental to the development of proportional reasoning.

Representational Detail and Accuracy

The children's on-screen representations elicited more detailed and more mathematically accurate images. Similar results were presented by Moyer, Niezgoda, and Stanley (2005) and Clements (1999), who found that virtual manipulatives offer opportunities for explicit representations that were previously unavailable to young children. Although the children's use of pre-formed, readily available images on-screen allow representations to be more detailed, there is also a risk that exclusive use of these images may limit the development of off-screen representations. It was not possible in this study to ascertain whether a child who exclusively used pre-drawn shapes on-screen had developed the drawing skills to produce these shapes off-screen. On the other hand, it is possible that some drawing tools, such as the "sticky straight string", allowed representations to be scaffolded until the child's fine motor skills were sufficiently developed to enable similar representations off-screen.

Potential Disadvantages of Technological Tools

Despite the advantages, there were two main features of virtual manipulatives and dynamic interactive software that may impede children's patterning skills. The first of these relates to the computer skills that children need to use these tools. In this study, the children initially found the mouse control and the skills needed to manipulate objects on-screen challenging. The importance of modelling and demonstration of processes in early childhood settings is described by Plowman and Stephen as "guided interaction" (2005, p. 152). Without teacher support, scaffolding and practice this impediment could limit learning. Limited mouse control also leads to unexpected actions, such as accidentally spinning shapes with virtual Pattern Blocks.

The second feature that may impede children's learning while using these tools is the distracting nature of some features. This was particularly evident with *Kidpix*, where the tools had the potential to distract children's attention from the learning, and limit dialogue. Again, guided interaction and adequate experience would allow the children to become familiar with these features, thus reducing a novelty effect. Teacher scaffolding of learning also enables children to re-focus attention on mathematical concepts and skills.

Limitations of Study

Pattern-eliciting tasks were designed to encourage repetitions and transformation skills. To some extent the children may have perceived the tasks as somewhat contrived. Thus, the patterns they produced may not have represented their intuitive and emergent patterning concepts that were reflected in the pre-assessment phase. This was further constrained by the limited number of teaching episodes, and the time frame for each episode that may have inhibited further experimentation. The learning may have also been constrained because the children had no access to the materials or a computer in the preschool until the researcher's next visit. Further, it was not possible to ascertain the explicit connections that children made between representations of their patterning and other learning experiences.

Implications and Conclusions

This exploratory study highlights the need for further research investigating the complex representational processes that children engage in when learning mathematical concepts with dynamic technological tools. The preschoolers in this project engaged in mathematical processes usually placed in the K-6 school curriculum. However, it was observed that these children were capable of constructing and representing complex patterns in a variety of ways. It was apparent from discussions with preschool staff that this potential learning had not been harnessed. Staff were intrigued by the dyads' use of technological tools but were apprehensive about continuing such activities because of their lack of pedagogical knowledge and technological skills. Professional development programs in both preschool and formal schooling may assist in promoting the appropriate use of technology in early learning.

This study supports current research advocating that virtual manipulatives and dynamic interactive software have the potential, when used with appropriate teacher support, to be powerful mathematical tools (Moyer et al., 2005). A longitudinal study would provide the opportunity to investigate whether the child's ability to manipulate virtual materials has a significant influence on their conceptual development of patterning and transformation skills. New research might also draw attention the need for integrated, multidisciplinary approaches to investigating the role of technological tools in the early development of mathematical concepts.

References

Blanton, M., & Kaput, J. (2004). Elementary grades students' capacity for functional thinking. In M. J. Høines & A. B. Fuglestad (Eds.), Proceedings of the 28th Conference of the International Group for the Psychology of Mathematics Education (Vol. 2, pp.135-142). Bergen, Norway: PME.

Board of Studies, NSW (2002). *Mathematics K-6 Syllabus.* Sydney: Author.

Brøderbund, (2004). *Kidpix Deluxe 4*, Novato, CA. Riverdeep. Inc.

Clements, D. (1999). 'Concrete' manipulatives, concrete ideas. *Contemporary Issues in Early Childhood*, *1* (1), 45-60

Clements, D. & Sarama, J. (2007). Effects of a preschool mathematics curriculum: summative research on the *Building Blocks* project. *Journal for Research in Mathematics Education*, *38*, 136-163.

Dougherty, B. & Slovin, H. (2004). Generalised diagrams as a tool for young children's problem solving. In M. J. Høines & A. B. Fuglestad (Eds.), Proceedings of the 28th annual conference of the International Group for the Psychology of Mathematics Education (Vol. 2, pp. 295-302). Bergen, Norway: PME.

Doig, B., & de Lemos, M. (2000). *I can do maths*. Melbourne: ACER Press.

English, L. (2004). Promoting the development of young children's mathematical and analogical reasoning. In L. English (Ed.), *Mathematical and analogical reasoning of young learners*. Mahway, NJ: Lawrence Erlbaum Associates.

English, L. (2006). Introducing young people to complex systems through modelling. In P. Grootenboer, R. Zevenbergen & M. Chinnappan (Eds.), *Identities, Cultures and Learning Spaces* (Proceedings of the 29th annual conference of the Mathematics Education Research Group of Australasia, Vol. 2, pp. 408-415). Sydney: MERGA.

Fox, J. (2005). Child-initiated mathematical patterning in the pre-compulsory years. In H.L. Chick & J.L. Vincent (Eds.), Proceedings of the 29th Conference of the International Group for the Psychology of Mathematics Education, (Vol. 2, pp. 313–320). Melbourne: University of Melbourne.

Ginsburg, H., Lin, C., Ness, D. & Seo, K. (2003), Young American and Chinese children's everyday mathematical activity. *Mathematical Thinking and Learning*, *5* (4), 235-258.

Hong, S. & Trepanier-Street, M. (2004). Technology: A tool for knowledge construction in a Reggio Emilia inspired teacher education program. *Early Childhood Education Journal*, *32* (2), 87-94.

Hunting, R. P., Davis, G., & Pearn, C. A. (1996). Engaging whole number knowledge for rational number learning using a computer-based tool. *Journal for Research in Mathematics Education*, *27*, 354-379.

Kaput, J. (1992). Technology and mathematics education. In Grouws, D. (Ed.) *Handbook of research on mathematics teaching and learning* (pp. 515-556). Reston, VA: NCTM.

Lehrer, R., Jenkins, M. & Osana, H. (1998). Longitudinal study of children's reasoning about space and geometry. In Lehrer, R. & Chazan, D. (Eds.) (1998). Designing learning environments for developing understanding of geometry and space, (pp.137-168) London: Lawrence Erlbaum & Associates.

Moyer, P., Bolyard, J., & Spikell, M. (2001). Virtual manipulatives in the K-12 Classroom, In A. Rogerson (Ed) *Proceedings of the International Conference on New Ideas in Mathematics Education* (pp184 - 187) (Palm Cove, Queensland, Australia, August, 2001).

Moyer, P., Niezgoda, D., & Stanley, M. (2005). Young children's use of virtual manipulatives and other forms of mathematical representation. In W. Masalski & P. Elliott (Eds.), *Technology-supported mathematics learning environments* (pp. 17-34). Reston, VA: NCTM.

Mulligan, J.T., Prescott, A., Papic, M. & Mitchelmore, M.C. (2006). Improving early numeracy through a Pattern and Structure Mathematics Awareness Program (PASMAP). In P. Grootenboer, R. Zevenbergen & M. Chinnappan (Eds.), *Identities, cultures and learning spaces* (Proceedings of the 29th annual conference of the Mathematics Education Research Group of Australasia, Vol. 2, pp. 376-383). Sydney: MERGA.

Papic, M., & Mulligan, J. (2005). Preschoolers' mathematical patterning. In P. Clarkson, A, Downton, D. Gronn, A. McDonough, R. Pierce, & A. Roche (Eds.), *Building Connections: Theory, research and practice* (Proceedings of the 28th annual conference of the Mathematics Education Research Group of Australasia, Melbourne, Vol. 2, pp. 609-616). Sydney: MERGA.

Perry, B., Dockett, S., Harley, E. & Hentschke, (2006). Linking powerful mathematical ideas and developmental learning outcomes in early childhood mathematics. In P. Grootenboer, R. Zevenbergen & M. Chinnappan (Eds.), *Identities, cultures and learning spaces* (Proceedings of the 29th annual conference of the Mathematics the Mathematics Education Research Group of Australasia, Canberra, Vol. 2, pp. 408-415). Sydney: MERGA.

Plowman, L. & Stephen, C. (2005) Children, play, and computers in preschool education. *British Journal of Educational Technology* 36, 2, 145 – 157.

Reimer, K, & Moyer, P. (2005). Third graders learn about fractions using virtual manipulatives: A classroom study. *Journal of Computers in Mathematics and Science Teaching, 24*(1), 5-25.

Warren, E. (2005). Patterns supporting the development of early algebraic thinking. In P. Clarkson, A. Downton, D. Gronn, M. Horne, A. McDonough, R. Pierce, & A. Roche (Eds.), *Building connections: Research, theory and practice* (Proceedings of the 28th annual conference of the Mathematics Education Research Group of Australasia, Melbourne, Vol. 2, pp. 759-766). Sydney: MERGA

Students Representing Mathematical Knowledge through Digital Filmmaking

Geoff Hilton
University of Queensland
<s3078952@student.uq.edu.au>

During initial attempts at filmmaking by my Year 7 class my focus was on the technology. However, I observed many positive learner behaviours inherent in the filmmaking process. Could these positive learner behaviours be harnessed, through filmmaking, to improve learner outcomes in mathematics? Two trials were conducted comparing a mathematical mini-documentary making revision strategy with pen-and-paper revision. It was concluded that mini-documentary makers retained information at least as effectively as, if not better than, students who used pen-and-paper strategies. This implies that mathematics educators can be confident of positive effects on students' knowledge retention through student filmmaking in mathematics.

As a teacher of Year 7 students I was attempting to incorporate digital filmmaking technologies into my classroom. Initially my aim was to develop students' familiarity with the technology. The first production attempts were not aligned to mathematics. Production was slow and we encountered many difficulties, which were overcome, and from which, we learnt a great deal. For me, at this stage, the focus had remained on the mechanics of the filmmaking, but my attention was drawn from these processes to the discussions of the students.

Filmmaking required the students to make many decisions about their learning and how it would be portrayed. Discussions occurred about relevance of details, the priorities and sequences of information, the best ways of portraying ideas, and how to link segments. Disagreements among students required them to defend their point of view, rationalise choices, and think deeply about what they had learnt. Critical thinking and higher order thinking were occurring without any prompting from me. I had reached the stage identified by Wiberg (1995-1996) where the focus moved away from the technology to how it could be used to enhance critical thinking. I became interested in the possibility of linking filmmaking to mathematics thereby harnessing the many positive learner behaviours inherent in the filmmaking process for the benefit of the students' mathematical learning. I realised that the making of mini-documentaries about specific mathematical topics would offer my students much more than simple exposure to a new technology. My students could use filmmaking as a vehicle for enhancing their learning through collaboration, investigation, communication, expression, performance, understanding, manipulation of information, and the making of a product. This research was to investigate how these filmmaking technologies could be used as a revision strategy in mathematics and to determine if the strategy resulted in a positive impact on student learning.

Background

Current Information and Communication Technologies Research Agenda

The nature of constant technological change has many authors calling for more and more research to keep pace with how the technologies can be used to enhance student-

Proceedings of the 30th annual conference of the Mathematics Education Research Group of Australasia
J. Watson & K. Beswick (Eds), © MERGA Inc. 2007

learning outcomes. Roblyer and Knezek (2003), for example, discussed the research agenda for education and technology and emphasised the importance of showing why teachers should use particular information and communication technologies (ICT) advancements as a means of finding solutions to educational problems. There is a strong belief that pervades the literature that when used correctly ICT have a positive influence; it seems that it has been very difficult to find the evidence to prove it. Slavin (2004) believed that research, such as this mathematical filmmaking study, comparing alternative approaches with traditional has a valid function to serve. This is in line with Robler and Knezek who promoted research that showed the relative advantage of a technology-based teaching method over another because, before teachers accept a new method they must be convinced of its relative advantage. The practicalities of educational technology research are such that most researchers will continue to work in their local environments, solving problems pertinent to their situations.

Meta-Analyses Involving ICT in the Classroom

A meta-analysis by Waxman, Lin, and Michko (2003) suggested that there were few reviews of research on the effects of classroom use of technologies on student outcomes. They deemed this to be a significant gap in available research as they believed that the use of technologies often changed the teaching practices in a classroom from teacher-centred to more student-centred with subsequent improvement in student outcomes. They believed that the dramatic, present day improvement in quantity and quality of technologies available in classrooms would at least provide the opportunity for improved learning outcomes from teaching and learning with those technologies. Through this research project I investigated these opportunities in the mathematics classroom.

Kozma (2003) completed a meta-analysis of worldwide case studies showing how ICT were incorporated in innovative pedagogical practices. Some similarities became apparent. ICT were able to provide frameworks to support and improve student learning by developing the skills deemed to be of particular importance in the 21st century, such as handling information, problem-solving, communication, and collaboration. Filmmaking in mathematics is an excellent medium for eliciting these skills acknowledging Kozma's emphasis that it is how teachers effectively incorporate a technology that will lead to improved learner outcomes.

Processes Associated with Filmmaking

Primary students can now incorporate words and pictures (still and moving) in representations of their knowledge. Recognising this, Bull and Thompson (2004) called for investigations into academic strategies that utilised this combination of words and pictures into today's instructional objectives.

Strategies do not emerge from the new technology as much as from how it can be applied (Ross, Yerrick, & Molebash, 2003). This observation fits the model of Reeves (1998) who divided ICT in classroom use into learning *from* computers and learning *with* computers. In the context of this research, this analogy can be taken to the broader realm of ICT, in that it aims to discover how students can work *with* film in mathematics to improve learning outcomes.

It was while students were working *with* the video cameras and editing equipment that I noticed a great deal of communication and collaboration. McGrath (2004) believed that learners needed skills such as tracking and communicating, reflecting on ideas and understandings, and designing to make understandings visible to others. These are definitive skills required by young mathematicians and filmmaking seems purpose-built to avail their development.

Other authors have completed studies that hint at further advantages to be gained by filmmaking in mathematics. Yerrick, Ross, and Molebash (2003) noted through their observations of the use of digital video in science, that collaboration and communication were present in the learning process via multiple student voices and ideas. They went on to say that it was possible to improve content understanding using desktop digital video editing when the students were authentically engaged in the production process. This increased authenticity of learning through the use of digital cameras was also highlighted by Sharp, Garofalo, and Thompson (2004). Using a post-test only design Hopson, Simms, and Knezek (2001) found that an enriched technology environment could develop students' higher order thinking skills.

Application of Filmmaking Technologies and Practices to Classrooms

Currently, most research examples of filmmaking in specific subject areas are from tertiary institutions or from high schools. For example, Mills, Kelley, and Jones (2001) showed how digital cameras could be used to capture images in a micro-biology class. The cameras allowed students who previously suffered from the inaccuracies of hand drawing what they saw to now being able to have an accurate image of what they were studying. The advantages of using digital cameras were listed and included rapid collection of images, archiving, class discussion and comprehension, better use of class time, and student empowerment.

For students to produce a film effectively to represent their mathematical knowledge they need to produce a storyboard. This allows the articulation of content, concepts, and sequence prior to the actual filmmaking process. Storyboarding is the planning stage. It requires the makers of the film to visualise what is to be filmed. It is during this stage that a number of educational benefits pertinent to the learning of mathematics have been noted by several authors. Reeder (2005) described storyboarding in the broader sense of preparation for the design of products, not just film, and spoke of its value of communicating intention, sequence, and needs. Storyboarding requires effective communication and requires students' traditional oral and written communication skills before the use of the digital communication begins. In science class applications of digital video and editing, Ross et al. (2003) stated that in preparation for filming, communication of ideas for script, settings, camera angles, examples, and data were required through storyboarding. They added that in an educational context, an accurate storyboard was important as it allowed the teacher to check for accuracy of concepts to be portrayed by the students before the film was started. Storyboarding, they believed, added greatly to the learning process of the students as it was where connections were made between the content of the lesson and the creative aspects of communication of ideas.

A study by Pearson (2005) drew attention to the urgent need to understand the educational implications of classroom digital video and editing, to enable filmmaking to be used with maximum benefit in teaching and learning. One of the implications of Pearson's study was that ways to embed the use of video making across the curriculum needed to be encouraged.

Method

Overview

The project involved two trials, the second replicating the procedures of the first but swapping the tasks of groups, thereby allowing all students to complete a mathematical mini-documentary. The two trials used different mathematics topics.

Mathematics Topics

In trial one the topic was transformation of shapes. As a class we investigated reflection, translation, and rotation of two dimensional and irregular shapes, initially through links to an animated web page and then through practical constructions. Students investigated how to use these transformation functions to create three dimensional shapes from two dimensional, which included instructions such as degrees of turn for a rotation, and movement to new coordinates for a translation. As an extension activity, the works of M.C. Escher were examined and replicated using some of the skills learned. Assessment for this topic focussed on construction using transformation and demonstration of understanding of the mathematical terminology.

The topic for trial two was measures of central tendency. Students had an understanding of average but were introduced to the terms mean, median, and mode. Scenarios were discussed to determine which measure would give the most accurate portrayal of the specified circumstances. Practice was given for the methods of calculation for each measure of central tendency. Assessment focussed on the students' ability to choose the appropriate measure for a particular situation and also their ability to calculate and manipulate the measures.

Participants

Participants were the members of a Year 7 class at a Queensland State Primary School. Students were between 11 and 12 years of age. There were 13 girls and 14 boys in the class. Following school policy the class was assigned students with a balanced range of ability levels. The total number of participants giving their consent to be involved in the project was 27. In the both trials there were 25 students involved and 2 students absent at some point.

Instruments

The instruments used to assess learning were teacher generated class tests, based on information taught in class as per the Queensland mathematics syllabus. These tests were pencil and paper completion items. Scoring of responses was by a right-or-wrong marking scheme. The test given at the end of the week of instruction served as the pre-test. The test without notice, administered two weeks later was the post-test. The raw scores from these tests were used in the data analysis.

Apparatus

Cameras used in these trials were Sony Digital Still Cameras (3.1 mega pixels) that had limited but adequate capacity to take digital video with audio. Sony 128mb memory sticks were used to store data.

A combination of classroom computers and laptops was available with Windows XP. The movie editing software was Movie maker 2. To transfer movies to a central location for review purposes, they were burned to CDs.

Design

These trials were intended to determine the effect of making mathematical mini-documentaries as a revision strategy compared to a more traditional pen-and-paper revision approach. A variation of a pre-test/post-test control group design was used to determine the students' retention of the mathematics topics.

Procedure

For each trial the Year 7 class was instructed in the chosen mathematics topic. At the end of the week in which instruction took place, students completed a teacher-generated test on the topic that served as the pre-test. The following week the students were randomly assigned to one of two groups. These groups undertook a revision lesson on the mathematics topic. Group 1 revised their work by making mini-documentaries and group 2 completed their revision by the more traditional method of pen-and-paper work sheets. Two weeks later, without notice, students were administered the post-test.

These mini-documentary making teams of three or four students were organised just prior to the start of the revision period. Their instructions included a recommended break up of the working time, which was ten minutes for storyboarding and resource collection, twenty minutes filming, and half an hour film editing. The mini-documentaries were to outline the major concepts of relevant mathematics topic. Filming was allowed in and around the classroom. The students' brief was to make a mini-documentary, which could be viewed by other students for future revision purposes or as a teaching and learning tool in future years. Time efficiencies were achieved by encouraging students to not be overambitious with acting or camera shot selection. Groups were asked to do all speaking parts live and not to use voice over recordings during the editing phase as it was not time efficient. Finally, during the editing process students were asked to address the basics first. Only after clips were dragged onto the film timeline and correct sequence achieved were additional tasks of title slides, on screen wording, transitions, and credits to be completed.

The pen-and-paper group was instructed to complete the revision of the maths lesson on the worksheets given to them. On completion, these students could continue with related activities. They were allowed to work by themselves or with others.

The revision period was of one hour's duration. Most of the pen-and-paper revision group completed the set tasks and moved on to related activities.

All mini-documentary teams completed the film within the allocated time. These films were generally less than 90 seconds duration because the time constraints of one hour would not allow for more extensive productions. Also, the editing software had a tendency to freeze with films longer than the proposed duration (Microsoft Corporation, 2003).

Students were given the post-test without notice two weeks after the revision lesson.

Results

A univariate analysis of variance (ANOVA) was conducted to determine the significance of changes in mean (see Table 1) from the pre-test to the post-test for each revision strategy trialled. The analysis of the data was to determine the variation in learner outcomes from the mini-documentary making group to the pen-and-paper revision group. Levine's test showed that the scores had homogenous variance from pre-test to post-test.

In trial one the mini-documentary group's means showed no significant effect from pre-test to post-test $F(1, 24) = 0.300$, $p > 0.05$. As the post-test was administered without notice two weeks after the revision lesson, the results indicate that the mini-documentary makers' retention of the maths concepts, although not improved, were not significantly diminished in the interim.

Table 1
Means of the Two Revision Groups from Pre-test and Post-test in Trial One

	Pre-test		Post-test	
	M	SD	M	SD
Mini-documentary	18.7	2.3	18.1	2.7
Pen-and-paper	19.6	2	17.3	3.3

The pen-and-paper revision group means showed a significant effect from pre-test to post-test: $F(1, 22) = 4.450$, $p < 0.05$. The students in this group had a significantly lower post-test mean than pre-test mean. In the interval between completing the revision worksheets and the post-test these students showed some lack of retention of the mathematics topic of transformations.

In trial two, data were analysed as per trial one (see Table 2). Levine's test showed that the scores of both revision groups had homogenous variance from pre-test to post-test.

Table 2
Means of the Two Revision Groups in the Pre-test and Post-test in Trial Two

	Pre-test		Post-test	
	M	SD	M	SD
Mini-documentary	21.5	3.3	22.2	2.6
Pen-and-paper	20.7	2.7	20.3	2.9

The mini-documentary group means showed no significant effect from pre-test to post-test $F(1, 24) = 0.351$, $p > 0.05$. As for the previous trial the post-test was administered without notice two weeks after the revision lesson and the results indicate that the mini-documentary makers' retention of the mathematics concepts had improved slightly but not significantly. The pen-and-paper revision group means showed no significant effect from pre-test to post-test $F(1, 22) = 0.106$, $p > 0.05$. Contrary to the earlier result, the pen-and-paper revision group's retention of the mathematics concepts was not significantly diminished in the interim.

Discussion

The results of these two trials indicated that the use of mini-documentary making as a revision strategy in mathematics provides a valuable alternative to traditional approaches. Knowledge retention of mini-documentary makers was as good as, if not slightly better than, that of students who completed traditional pen-and-paper revision worksheets. This gives a positive starting point for the incorporation of filmmaking into primary schools as a legitimate form of expression in the teaching and learning process in mathematics. As this technology improves and becomes more available, as teachers and students become more adept at its uses, and as more varied applications are attempted, it seems reasonable to expect that learning outcomes will improve.

A Storyboard Example

As only ten minutes were allocated to storyboarding, the focus had to remain on the concepts to be portrayed and what techniques would best achieve them. As decisions were made, the students recorded them on their storyboards, quickly developing a plan for their films. Students used their mathematics exercise books and graph books to focus their thoughts on what concepts needed to be covered. Decisions were made as to how to portray the concepts through film. Some of these were filming of manipulatives and diagrams, students pretending to be teachers, comedic role play, and coloured chalk examples on the concrete playground.

A Mini-documentary Example

A review of a mini-documentary will give a brief insight into the production processes and the final product. From a teacher's perspective, I was keen to see that the focus of the one-hour period remained on the portrayal of the mathematical concepts. It was important that students did not lose sight of the mathematical goals while engaged in the technological process. The success of the mini-documentary revision strategy as a viable alternative to traditional pen-and-paper methods depended on students maintaining this focus. As the films were to be viewed by peers, and possibly future year seven classes, the students were keen to ensure that information portrayed was correct. They felt that they were producing an authentic product. The following review is an example of the overall process.

This was a film on transformations produced by a team of three girls. The students used a demonstration format where the viewer only sees the hands that manipulate materials. The demonstrations of transformations were effective as this team used a white L shape on a black background that gave visual clarity. Also included was an example of how a three-dimensional shape can be drawn by sliding a two dimensional shape and joining corresponding points. The film concluded with a summary of the concepts using interplay between two of the team members that was to the point and effective. The mathematical concepts were well explained and the film was visually appealing. The team's effort to convey the concepts showed an understanding of the needs of the intended audience. The film was completed within the allocated time.

Observations of the Mini-documentary Making Process

The mini-documentaries produced in these trials varied in approach and technical skill but accurate content was generally present in all. The effectiveness of these films varied owing to the different film techniques and presentation methods used by the teams. Although the filmmaking process as a revision strategy for the film-makers has proved effective in these trials, the reasons for this effectiveness have not been determined.

The process of film-making entailed a number of observable behaviours that may have contributed, in varying degrees, to the effectiveness of this revision strategy, and could be the focus of further research. These observations were of students who:

- Were motivated and engaged, staying on task throughout the activity.
- Collaborated with team members through discussions, decision making, sequencing, role sharing, and task allocation.
- Were active rather than passive in their leaning.
- Used a common mathematical language in their discussions.
- Needed to think about their thinking (metacognition) to portray concepts correctly.
- Showed pride and were creative in scripting, acting, filming, and editing.

Students' comments and interactions during the production process also give an insight into the value of the filmmaking strategy especially from a meta-cognitive perspective.

- A great deal of peer correction occurred when determining the portrayal of the mathematical concepts, sometimes requiring clarification from me if the team had reached an impasse.
- A common challenge expressed by the students was that they understood the concepts but found it difficult to explain them. The filmmaking process forced them to clarify their thoughts.
- A comment from one student was, "We had to talk maths."

Practical Implications

Mathematics Syllabus and Filmmaking

The Queensland mathematics syllabus articulated the contribution of this key learning area to lifelong learning. The following selected phrases from the Queensland Studies Authority (2004) describe these lifelong learner attributes; these attributes can be considered in terms of how filmmaking in mathematics could contribute to their development.

.

- Knowledgeable person with deep understanding
    - Learners' understandings are enhanced through active engagement in mathematical investigations and in communicating their thinking and reasoning in ways that make sense to themselves and others.
- Complex thinker
    - Learners … analyse and synthesise information ….
- Responsive creator

    - [Learners] use a range of representations to communicate mathematical understandings and to transfer knowledge from one situation to another.
- Active investigator
    - [Learners] manipulate concrete materials and make a variety of representations and displays … to assist their mathematical thinking and reasoning.
- Effective communicator
    - [Learners] understand and use the concise language of mathematics, both verbal and symbolic.
    - [Learners] select appropriate mathematical language to convey, logically and clearly, their mathematical understandings, thinking, and reasoning.
- Participant in an independent world
    - Learners cooperate, collaborate, and negotiate in groups to plan, think, reason, and resolve mathematical investigations….
- Reflective and self-directed learner
    - Learners reflect on their learning as they become metacognitively aware and self-regulating. (pp. 2-4)

From analysis of students' mathematical filmmaking processes and products it is clearly apparent that there is alignment with these descriptions of life long learning attributes.

Practical Relevance of Study

This research targeted a revision strategy using filmmaking technologies. It has shown how students can effectively work *with* rather than *from* these technologies.

For teachers to use mini-documentary making confidently in their classrooms within a reasonable time limit, students must be familiar with the filmmaking process and they must be encouraged to use simple techniques. My class had been taught the basic filmmaking process earlier in the year as a part of our Arts program. For the purposes of this research, time parity between revision strategies was very important but, for general classroom use, the strict time limit concerns for making mini-documentaries could be relaxed. This would be especially important if younger grades were involved. The process could be broken into shorter periods; storyboarding, filming, and editing could be completed in a series of short lessons.

The Queensland Studies Authority (2004) through the rationale of the mathematics syllabus has regularly confirmed the link of mathematics with technology (though not always ICT). Filmmaking seems to fit naturally into the Queensland Studies Authority's understandings of, (a) thinking, reasoning, and working mathematically, (b) the attributes of lifelong learning, (c) the cross-curricular priorities of literacy, numeracy, lifeskills, and futures perspective, and (d) understandings about learners and learning.

The findings from this study have shown the effectiveness of a filmmaking revision strategy that teachers may choose to apply across the curriculum and across grades. The next step in my research is for my students to record on film, salient points of a unit of work during, or at the end of each lesson on a given topic. By building a body of recorded information students will then produce a documentary as a culminating activity or investigation. These documentaries could fulfil the same role as a written assignment or "write up" of class work. They could also be used as an assessment piece. The research focus will be to determine not only if student learning outcomes are improved but also why.

The concept of students making films in specific subject areas such as mathematics brings with it enormous educational potential as it requires metacognitive processes, collaboration, and communication as well as technical skills to be successful. A technology that allows students to communicate their mathematical ideas incorporating, spoken word, still and moving images, in everyday mathematics classroom settings has the ability to influence positively the teaching and learning process of mathematics.

References

Bull, G., & Thompson, A. (2004). Establishing a framework for digital images in school curriculum. *Learning & Leading with Technology, 31*(8), 14-17.

Hopson, M. J., Simms, R. L., & Knezek, G. A. (2001). Using technology-enriched environment to improve higher-order thinking skills. *Journal of Research on Technology in Education, 34*(2), 109-120.

Kozma, R. B. (2003). Technology and classroom practices: An international study. *Journal of Research on Technology in Education, 36*(1), 1-14.

McGrath, D. (2004). Strengthening collaborative work: Go beyond the obvious with tools for technology enhanced collaboration. *Learning & Leading with Technology, 31*(5), 30-34.

Microsoft Corporation. (2003). *Movie maker 2 stops responding when you try to save a large movie file* [Electronic Version]. Retrieved October 19, 2004 from http://support.microsoft.com/default.aspx?scid=kb;en-us;614836.

Mills, D. A., Kelley, K., & Jones, M. (2001). Use of a digital camera to document student observations in a microbiology laboratory class. *The American Biology Teacher, 63*(2), 119-124.

Pearson, M. (2005). Splitting clips and telling tales: Students' interactions with digital video. *Education and Information Technologies, 10*(3), 189-205.

Queensland Studies Authority. (2004). *Mathematics year 1 to 10 syllabus.* Brisbane: Office of the Queensland Studies Authority.

Reeder, K. (2005). Using storyboarding techniques to identify design opportunities: When students employ storyboards, they are better able to understand the complexity of a product's use and visualise areas for improvement. *The Technology Teacher, 64*(7), 9-11.

Reeves, T. C. (1998). *The impact of media and technology in schools: A research report prepared for The Bertelsmann Foundation* [Electronic Version]. Retrieved October 3, 2004 from http://www.ed.gov/Technology/TechConf/1999/whitepapers/paper3.html.

Roblyer, M. D., & Knezek, G. A. (2003). New millennium research for educational technology: A call for a national research agenda. *Journal of Research on Technology in Education, 36*(1), 60-70.

Ross, D. L., Yerrick, R., & Molebash, P. (2003). Lights! Camera! Science? Using digital video in elementary science classrooms. *Learning & Leading with Technology, 31*(3), 18-22.

Sharp, B., Garofalo, J., & Thompson, A. (2004). Digital images in the mathematics classroom. *Learning & Leading with Technology, 31*(8), 30-33.

Slavin, R. E. (2004). Education research can and must address "what works" questions. *Educational Researcher, 33*(1), 27-29.

Waxman, H. C., Lin, M.-F., & Michko, G. M. (2003). A meta-analysis of the effectiveness of teaching and learning with technology on student outcomes. [Electronic Version]. *Learning Point Associates*. Retrieved May 5 2005 from www.learninpt.org.

Wiberg, K. (1995-1996). Changing teaching with technology. *Learning & Leading with Technology, 23*(4), 46-48.

Yerrick, R., Ross, D., & Molebash, P. (2003). Promoting equity with digital video: Technology can demonstrate varying scientific strategies, which can be extremely successful in making science accessible to a diverse population of students. *Learning & Leading with Technology, 31*(4), 16-20.

What Does it Mean for an Instructional Task to be *Effective*?

Lynn Hodge
The University of Tennessee
<lhodge4@utk.edu>

Jana Visnovska
Vanderbilt University
<jana.visnovska@vanderbilt.edu>

Qing Zhao
Vanderbilt University
<qing.zhao@vanderbilt.edu>

Paul Cobb
Vanderbilt University
<paul.cobb@vanderbilt.edu>

In this paper we discuss the considerations and challenges in designing instructional tasks that support both students' mathematical engagement and their developing mathematical competence. We draw on Dewey's work and take the perspective that cultivating students' content-related interests should be an instructional goal in their own right rather than solely serving the instrumental purpose of supporting students' conceptual understanding. We reflect on our learning from two classroom design experiments to offer illustrations of issues related to supporting students' interests. We offer these illustrations, not as exemplary cases, but instead, as points of reflection and discussion. In this paper, we focus specifically on instructional tasks by presenting a retrospective analysis on the role of tasks in supporting students' interests and access to important content ideas.

Introduction

Reform recommendations have called attention to the use of real world contexts in mathematics problems (National Council of Teachers of Mathematics, 1989, 2000) and culturally relevant pedagogy has emphasised drawing on students' local and broad communities as a source for engaging problem topics (Ladson-Billings, 1995). In this paper, we focus on instructional tasks and their role in supporting both students' mathematical interests and their developing mathematical competence. In doing so, we develop what it means for an instructional task to be effective from our perspective as mathematics educators. Our discussion centres on two ideas: (a) how a task holds potential for supporting students' development of mathematical interests and (b) how a task holds potential for providing students with access to important mathematical ideas. We believe that instructional tasks are deemed effective according to how well they respond to both of these points. We use the term *task* to refer to problems that are designed and presented to students in mathematics class. We use the term instructional *activity* to refer to how these tasks become realised in the course of discussions and interactions in the mathematics class. In focusing on the design of instructional tasks we emphasise intent and potential. Additionally, we must examine how instructional activities become constituted in a classroom in order to test and refine what we understand about designing effective tasks. Therefore, our focus is on the considerations and challenges in the design of effective instructional tasks while at the same time exploring tensions that might emerge as these tasks become realised in the classroom.

In order to discuss effective tasks in this way, it is important for us to delineate more specifically how we might evaluate tasks in how they provide access to interests and content ideas. To this end, we initially lay the foundation for the analysis to come. We do so by clarifying an orientation toward students' development of content-related interests that draws heavily on the ideas of John Dewey (1913/1975). This orientation has implications for how we think about the specific role of tasks in supporting students' mathematical interests. Secondly, we provide background to two design experiments from which the retrospective analysis draws data. We then share insights from the analysis in order to

- clarify a two-part process of cultivating students' mathematical interests
- examine the potential of task situations in supporting students' development of mathematical interests
- explore the role of tasks in supporting the emergence of particular mathematical topics in whole-class discussions.

These three parts of the analysis relate to each other in that the first describes a way of cultivating students' mathematical interests whereas the second and third parts clarify the role of tasks in supporting this process.

An Orientation on Cultivating Students' Interests

As we have indicated, our purpose in this paper is to examine characteristics of instructional tasks that can contribute to supporting both students' interests and their access to important mathematical ideas. For this reason, we draw on the work of Dewey since his perspective encourages us to think about the resources teachers can draw on to support students' interests within the context of the classroom. In this paper, we focus primarily on tasks, but as will become apparent, classroom discourse and the role of teacher serve as resources in this process as well.

Dewey's ideas have been helpful in that he describes interests as something that individuals can cultivate rather than characteristics that are inherent aspects of people. From his perspective, students' current interests act as leverages from which students' content-related interests could be developed. In this process, current interests could afford opportunities from which content interests, such as mathematics, could be developed. Dewey used the term *cultivation* to indicate that he regarded it a teacher's responsibility to support the development of students' disciplinary interests. He argued that disciplinary interests are an inherent aspect of disciplinary literacy, and as such their development should be an instructional goal in their own right.

Importantly, Dewey's view on interests also highlights the *nature of students' interests*. His focus was on students' interests in particular content ideas that could be cultivated over time in a class, and subsequently a series of courses. His view is in contrast to the more typical emphasis on engaging students to participate in particular activities in the classroom without necessarily noting what students are becoming interested in as they engage in such activities. This orientation on cultivating mathematical interests reflects a developmental perspective that emphasises the deeply cultural nature of students' interests. In this way, Dewey anticipated Vygotsky's argument that interests cannot be adequately accounted for by either biological desires or skill acquisition but are culturally developed (compared with Hedegaard, 1998; Vygotsky, 1987).

From this orientation, cultivating students' mathematical interests becomes a challenge for both instructional design and teaching. As instructional tasks are the most visible means of organizing students' mathematical activity, we examine their potential as a resource in cultivating students' mathematical interests. In doing so, we attempt to discern characteristics of tasks that support students' long-term interests in learning mathematics.

The kinds of tasks we identified as effective are quite different from activities and problems that connect with what can be identified as students' current interests but are weak in providing access to significant mathematical ideas.

The Design Experiments

The classroom design experiments on which we draw focused on supporting students' increasingly sophisticated forms of statistical reasoning. A member of the research team served as the teacher in both experiments, which were conducted in an urban middle school in the United States. Twenty-nine seventh-grade students participated in the first experiment that was conducted over a 12-week period and involved 34 classroom sessions of approximately 40 minutes in length. This experiment was conducted in the students' regular mathematics classroom and focused on the analysis of univariate data. The following school year, a smaller contingent of students from the same class (now eighth graders) participated in a 14-week experiment involving 41 classroom sessions of 40 minutes that focused on the analysis of bivariate data.

Analyses that we have reported elsewhere indicate that the teacher was generally successful in supporting students' development of increasingly sophisticated forms of statistical reasoning (P. Cobb, 1999; P. Cobb, McClain, & Gravemeijer, 2003). The relatively impressive nature of the students' learning encompasses both the sophistication of the data-based arguments that they developed and the depth of their understanding of issues related to the process of generating data such as the representativeness of samples and the control of extraneous variables (P. Cobb & Tzou, 2000). Additional analyses (P. Cobb, Gresalfi, & Hodge, 2007; P. Cobb, Hodge, Visnovska, & Zhao, 2007) reveal that students during the course of the design experiments came to view analyzing data as an activity that was worthy of their engagement. The findings of these prior analyses indicate that the design experiments provide a rich context from which to examine the role of instructional tasks in supporting students' mathematical engagement and their developing competence.

Instructional Tasks in the Design Experiment Class

A basic design principle that guided the development of instructional tasks during both experiments was that they should support students' analyses in involving the investigative spirit of exploratory data analysis from the outset (cf. G. W. Cobb & Moore, 1997). As a consequence, we attempted to develop instructional tasks in which the students analyzed data sets that they viewed as realistic for purposes that they considered legitimate. Most of the instructional tasks involved comparing two data sets in order to make a decision or judgment (e.g., determining whether installing airbags in cars does have an impact on automobile safety). To support the students' engagement further in what might be termed genuine data analysis, they were required from midway through the first experiment to write a report of their analyses for a specific audience that would act on the basis of their reports (e.g., a police chief who wanted to know whether a speed trap had been effective in reducing traffic speed).

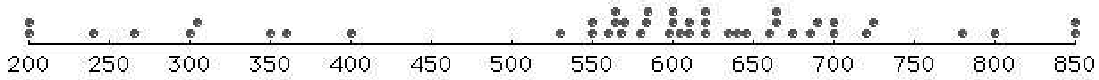
In most of the instructional tasks, the students did not collect data themselves. Instead, the teacher introduced each task by engaging the students in an introductory discussion that was often times lengthy. In the course of these discussions, the class talked through the process by which data might be generated. Specifically, the teacher and students together delineated the particular phenomenon under investigation, clarified its significance, identified relevant aspects of the phenomenon that should be measured, and considered how they might be measured. The teacher then introduced the data as having been generated by this process and the students conducted their analyses individually or in small groups. The final phase of an instructional activity consisted of a whole-class discussions of the students' analyses. The resulting organization of an instructional activity often spanned two or more class sessions.

Data Sources and Method of Analysis

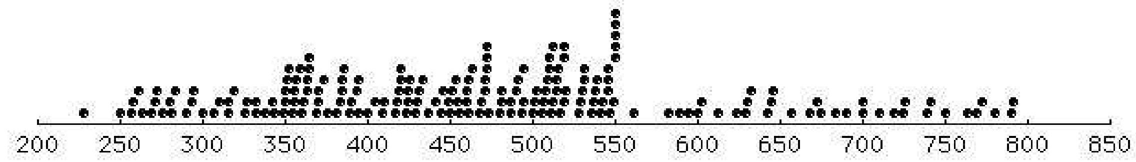
Our analysis of instructional activities draws from data that include video-recordings made with two cameras of classroom sessions, copies of all student work, and two independent sets of field notes of all the classroom sessions. Our central question had to do with discerning which instructional tasks were constituted as worthy of students' engagement and those that were not. Three members of the research team used video-recordings of one productive and one unproductive introductory discussion from the second design experiment as test cases initially in which to develop, test, and refine these criteria. They focused on these introductory discussions because it was during these discussions that the teacher and students negotiated the intent of the activities by talking through the significance of the problem at hand and the relevance of analyzing the situation from a mathematical point of view. This procedure was repeated by reexamining two further productive introductory discussions. As a result, the following criteria were established to determine whether an instructional task was constituted as worthy of students' engagement: (a) at least half of the students contributed to the data generation discussion, (b) the number of turns taken by students in the discussion was equal to or greater than the number of turns taken by the teacher, and (c) the majority of student contributions concerned ways to address the question under investigation by generating and analyzing data (e.g., relevant aspects of the phenomenon that should be measured, how these aspects might be measured, and how data might be generated). These criteria are generally consistent with Engle and Conant's (2002) contention that evidence of engagement can best be seen by considering questions such as: "How are students participating? What proportion of students is participating? And how are students' contributions responsive to those of other students?" (p. 402). Three members of the research team subsequently used these developed criteria to analyze the video-recordings of introductory discussions of all 14 tasks presented in the first experiment independently in order to determine which of these tasks were constituted as worthy of students' engagement. All researchers agreed that eight of the tasks were constituted as worthy of students' engagement whereas six were not. A comparative analysis was conducted to gain insight into the characteristics of the instructional tasks that contributed to the differences documented in students' engagement. We discuss findings from this analysis at a later point in this paper.

A Two-Part Process: Cultivating Pragmatic Interests and Mathematical Interests

Our learning in the design experiments sheds some light on processes that are involved in supporting students' mathematical interests (P. Cobb et al., 2007). One aspect of our learning concerns a two-part process of supporting students' development of disciplinary interests. This two-part process involved first cultivating students' *pragmatic* interests or interests in the problem situation presented in the instructional task. These pragmatic interests we describe relate to an interest in pursuing the specific problem at hand. To illustrate what we mean, one of the instructional activities in which students engaged in the latter part of the seventh-grade design experiment involved analyzing data on the T-cell counts of AIDS patients who had enrolled in a standard treatment program and an experimental treatment program. The datasets presented to students are shown in Figure `1.



Experimental Treatment



Traditional Treatment

Figure 1. AIDS Data.

A pragmatic interest that we encouraged students to develop related to investigating which treatment was more effective rather than solely an interest in the broad topic of AIDS. It seemed from our observations that the issue of AIDS was relevant to few if any of the students' personal daily lives. In other words, they did not know anyone, including family and friends, who had been diagnosed as having AIDS. However, they appeared to have developed a genuine interest in the issue as they engaged in an introductory whole-class discussion that clarified the instructional task and took place prior to the students conducting their own analyses. The teacher typically initiated these introductory discussions by posing a general problem or issue. In the ensuing conversation, the teacher and students clarified why this problem or issue would be significant to them or to a particular audience.

During the AIDS introductory discussion, the teacher and students talked about the general topic of AIDS, the importance of finding an effective treatment, and how data might be collected to help the class decide which of the two AIDS treatments had better results. The initial focus on the students' knowledge of AIDS led to a conversation about both the relevance of finding an effective treatment for AIDS and measures that could indicate to what extent an applied treatment is effective. We conjecture that many students became interested in the instructional activity as they came to see the relevance of developing effective treatments for AIDS within the context of wider society. In this way, students' pragmatic interests were cultivated as they engaged in a discussion that clarified the overall relevance of the task investigation and how data might be used to address this issue. This first phase of cultivating students' pragmatic interests in issues of social relevance was crucial in students coming to see a reason for analyzing the data sets with which they were presented. As we later discuss, our analysis of effectiveness of instructional tasks indicated that the tasks, which did not afford leverage for cultivation of students' pragmatic interests in the problem at hand, were not instructionally effective. As will become apparent, although critical, cultivation of students' pragmatic interests was only one part of cultivating students' interests in mathematics.

As part of their attempts to cultivate students' mathematical (or, specifically statistical) interests, the research team supported students' participation in the emergence of practices consistent with those in which data analysts might genuinely engage. The students' participation in these practices involved identifying relevant patterns in the data, presenting data-based arguments, writing a report to a decision maker summarising their analyses, and judging the adequacy of arguments presented by others. During the whole-class discussion that focused on the students' analyses of the AIDS data, it became apparent that all the students in the class had concluded that the new treatment was more effective than the traditional, standard treatment. However, a lengthy, whole-class discussion ensued that focused on different ways of structuring and organizing the data. It appeared in this discussion, at least on the surface, that students were becoming interested in developing data-based arguments and judging the adequacy of these arguments in the context of this class session in spite of their consensus on which treatment was more effective. We refer to these developing interests, related to practices of *doing* mathematics, as mathematical interests. The following excerpt illustrates the nature of the whole-class data analysis discussion students were afforded. This excerpt focuses on one group's analysis (Figure 2), in which the students proposed an inscription to show the global differences in the way the two sets of data were distributed.

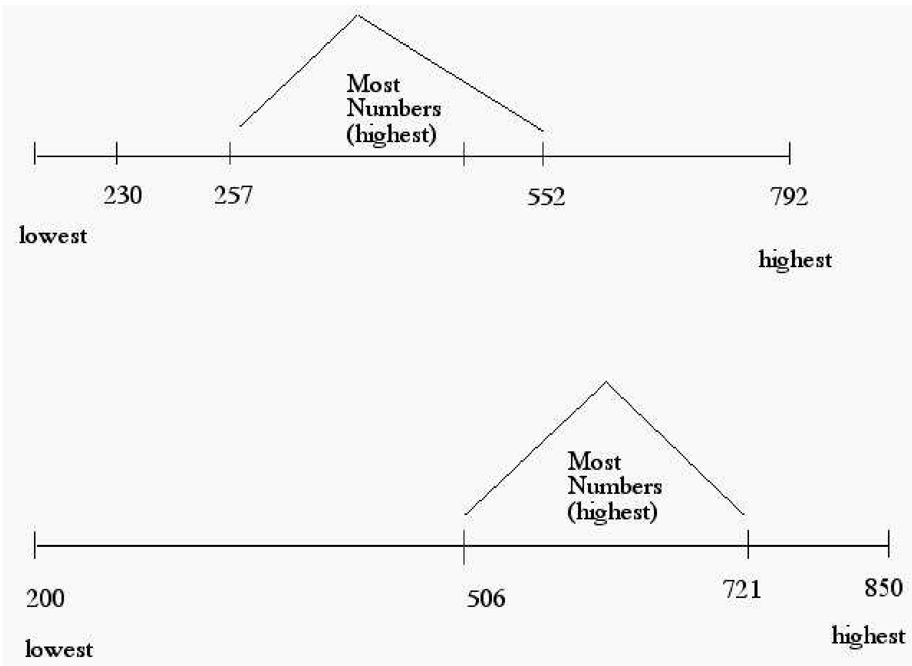


Figure 2. One student group report.

Janet:	I think it's an adequate way of showing the information because you can see where the ranges were and where the majority of the numbers were.
Dan:	What do you mean by majority of the numbers?
Teacher:	Dan doesn't know what you mean by the majority of the numbers.
Janet:	Where the most of the numbers were.
Teacher:	Sue, can you help?
Sue:	What she's talking about, I think what she's saying, like when you say where the majority of the numbers were, where the point is, like you see where it goes up.
Teacher:	I do see where it goes up (indicates the "hill" on Figure 2)
Sue:	Yeah, right in there, that's where the majority of it is.
Teacher:	Okay, Dan.
Dan:	The highest range of the numbers?
Sue:	Yes.
Teacher:	The highest range?
Students:	No.
Teacher:	Valerie.
Valerie:	Out of however many people were tested, that's where most of those people fitted in, in between that range.
Teacher:	You mean this range here (points to lower and upper bounds of one of the "hills")?
Valerie:	Yes.

In this excerpt, students clarified Janet's use of the term "majority" in relation to the datasets. In doing so, majority as related to the notion of relative proportions became an explicit topic of conversation in the classroom. This opportunity to clarify statistical ideas was prompted by both the task situation and the design of the specific data sets to make comparisons of unequal data sets necessary. Furthermore, this excerpt is illustrative of the discussions that constituted the second part of a two part process that sought to cultivate students' interests in learning mathematical ideas. As we reiterate later, the tasks that would not allow for a meaningful mathematical discussions to develop based on students' mathematical contributions make it difficult for teachers to cultivate students' mathematical interests effectively.

Task Situations and Their Potential for Cultivating Students' Pragmatic Interests

Students' development of pragmatic interests was critical in providing a reason to engage in discussions about specific mathematical ideas. We conjectured that "effective" task situations drew from topics that were located within students' zones of proximal development. These situations and topics were located within a space of topics that students were likely to find engaging when supported through discourse and interactions within the classroom. During the design experiments, we found issues that were of a personal or societal relevance to be the most effective in engaging students. This finding is understandable given adolescents' growing interest in their place in society and their sense of power in affecting change on society and their immediate community (Hedegaard, 1998).

During the design experiments, we made a number of modifications to the instructional tasks in light of the instructional agenda, students' mathematical learning, as well as what we learned about ways to cultivate students' interests. In a retrospective analysis on instructional tasks, we found four distinguishing characteristics of effective instructional activities. As an illustration, we draw on the AIDS task that was deemed as a success in engaging students in both pragmatic and mathematical issues. We discuss the four characteristics of task situations that were engaging to students:

- Students have developed some familiarity with or awareness of the phenomenon either in school or out-of-school (e.g., the topic of AIDS, batteries, etc.)
- Students have developed a prior awareness of the specific question to be investigated and initial familiarity with the processes involved (e.g., finding an improved treatment for AIDS patients, AIDS involves your immune system, the physical effects of AIDS on the body).
- Students came to view the specific question to be addressed as significant during the course of a discussion that introduced the instructional activity (e.g., finding a more effective treatment for AIDS would be important to patients and to medical staff).
- Students came to view addressing the question from a mathematical perspective as reasonable during the course of a discussion that introduced the instructional task (e.g., the analyses of AIDS patients' T-cell counts to assess the effectiveness of the two treatments).

It is important to note that we documented examples of ineffective instructional activities in which different ones of the four listed key characteristics were violated. In this sense, we propose that each of the characteristics was *necessary* for cultivating students' interests in the statistics design experiment classroom.

Many would argue that statistics lends itself to real world task situations whereas this is not the case with all mathematical topics or ideas. At this point, we would not make the claim that all effective instructional tasks require a real world scenario; however, we would make a two-fold argument that (a) an introductory discussion that clarifies the intent of the task and its significance (to society or to the students' mathematical learning) is critical in providing all students opportunities to understand the task and to become engaged in it and (b) a real world situation may be useful in engaging students, but the task situation must also be scrutinised in terms of the mathematical ideas that it affords.

Interests, Learning, and the Space of Possible Mathematical Topics

In retrospect, we found it helpful to consider task situations and questions posed in these tasks specifically in light of the space of possible mathematical issues that might emerge in whole-class discussions. This would involve considerations of how students might interpret and reason about the task and what conversations might come about from clarifications and comparisons of these ways of reasoning. It is not surprising that instructional tasks that do not adequately support teachers' efforts in building on students' reasoning towards instructional goals are also generally not effective in supporting students' mathematical learning. Similarly, in order to cultivate students' mathematical interests, it is critical to provide students with access to mathematical ideas that would enable them to solve problems that they come to see as pragmatically important.

In the case of the AIDS activity, the research team purposefully constructed data sets with a significantly different number of data points when we developed the activity so that the contrast between absolute and relative frequency might become explicit. This in turn required a task scenario in which the inequality in the size of the data sets would seem reasonable to the students and which they would view as significant and engaging. The data sets for this activity were therefore designed so that 46 people enrolled in the experimental treatment and 186 people enrolled in the traditional treatment. Additionally, the total number of data points in the larger data set was not a multiple of the total number of points in the smaller data set. These design decisions were made in order to support students' examination of the data in proportional terms. During the whole-class discussion of the students' analyses, a number of significant mathematical issues emerged during the conversation. These include the meaning of the term majority, the distinction between absolute and relative frequency, the usefulness of percents in specifying relative frequencies and the interpretation of graphs in which data sets were partitioned into four groups that contained the same number of data points.

When mathematics becomes a tool for students to solve significant problems they can be supported to see mathematics as relevant and interesting in its own right. We concur with Clarke (2005) that mathematics as it becomes realised in the classroom can be *relevant* in different ways when situated within multiple contexts. In his description of Chinese classrooms, mathematics can be seen to be situated within the broader cultural context in which it is respected and valued as both a pragmatic and intellectual tool (Svan & Clarke, 2007). Additionally, Clarke describes classrooms in South Africa in which mathematical learning is in the service of informing a broader agenda, that of addressing social issues such as substance abuse or AIDS (Sethole, Adler, & Vithal, 2002). In our reflection, we have emphasised the importance of the task situation, the mathematical ideas, and the relationships between the two. When constructing effective mathematical tasks, the multiple ways in which mathematics can become relevant to students should be considered. For our part, we have focused on what can be done *in the classroom* to support students' development of mathematical interests in situations when the students do not necessarily see mathematics as relevant to their lives from the outset.

Discussion

In closing, we refer to two points that we have emphasised in this paper. First, we have argued that when designing instructional tasks, it is important to consider how the task holds potential for cultivating both students' pragmatic and mathematical interests. We have described both of these aspects as closely related and as phases of a process that serves to cultivate students' mathematical interests. We acknowledge that considering both of these aspects at the same time when designing a task is challenging. Similarly, as an instructional activity becomes constituted, addressing both of these aspects in teaching is challenging as well. Tensions can and often do arise between addressing pragmatic interests and content-related interests (Azevedo, 2002). This emphasises the need for analyses that investigate how instructional tasks can serve as resources for teachers as they navigate such tensions and how classroom practices mediate this process.

Second, critics of the use of real world contexts argue that not all students have experiences that support their understanding of such contexts. Some would say that some students are advantaged over others (Lubienski, 2002). Introductory discussions and the ideas of *pragmatic interests as accomplishments* emphasise topics that are located within a zone of proximal development and substantive discussions that support students' access to understanding the task context and its significance. In this way, the *meaningfulness* of a task is seen to be supported and developed through discussions, interactions, and other resources within the social context of the mathematics classroom.

References

Azevedo, F. (2002). *Personal excursions: Investigating the dynamics of student engagement*. Unpublished manuscript, University of California Berkeley.

Clarke, D. J. (2005). *Essential complementarities: Arguing for an integrative approach to research in mathematics classrooms*. Paper presented at the 28th annual conference of the Mathematics Education Research Group of Australasia. Retrieved January 2007, from http://www.merga.net.au/publications/counter.php?pub=pub_conf&id=33.

Cobb, G. W., & Moore, D. S. (1997). Mathematics, statistics, and teaching. *American Mathematical Monthly*, *104*, 801-823.

Cobb, P. (1999). Individual and collective mathematical learning: The case of statistical data analysis. *Mathematical Thinking and Learning*, *1*, 5-44.

Cobb, P., Gresalfi, M. S., & Hodge, L. (2007). *An interpretive scheme for analyzing the identities that students develop in mathematics classrooms.* Manuscript submitted forpublicaion.

Cobb, P., Hodge, L., Visnovska, J., & Zhao, Q. (2007, April). *Cultivating students' mathematical dispositions*. Paper presented at the annual meeting of the American Educational Research Association, Chicago, IL.

Cobb, P., McClain, K., & Gravemeijer, K. (2003). Learning about statistical covariation. *Cognition and Instruction*, *21*, 1-78.

Cobb, P., & Tzou, C. (2000, April). *Learning about data creation*. Paper presented at the annual meeting of the American Educational Research Association, New Orleans, LA.

Dewey, J. (1913/1975). *Interest and effort in education*. Carbondale, IL: Southern Illinois University Press.

Engle, R. A., & Conant, F. C. (2002). Guiding principles for fostering productive disciplinary engagement: Explaining an emergent argument in a community of learners' classroom. *Cognition and Instruction*, *20*(4), 399-483.

Hedegaard, M. (1998). Situating learning and cognition: Theoretical learning and cognition. *Mind, Culture, and Activity*, *5*, 114-126.

Ladson-Billings, G. (1995). Making mathematics meaningful in multicultural contexts. In W. G. Secada, E. Fennema & L. B. Adajion (Eds.), *New directions for equity in mathematics education* (pp. 329-348). New York: Cambridge University Press.

Lubienski, S. T. (2002). Research, reform, and equity in U.S. mathematics education. *Mathematical Thinking and Learning*, *4*, 103-127.

National Council of Teachers of Mathematics. (1989). *Curriculum and evaluation standards for school mathematics*. Reston, VA: Author.

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics*. Reston, VA: Author.

Sethole, G., Adler, J., & Vithal, R. (2002, September). *When AIDS goes to a mathematics class: Is it a hindrance or a source for revising the mathematics classroom culture*. Paper presented at the ICMI Study - Mathematics education in different cultural traditions: A comparative study of East Asia and the West, Hong Kong.

Svan, K., & Clarke, D. J. (2007). *Expanding the relevance paradox*. Manuscript in Preparation

Vygotsky, L. S. (1987). The development of thinking and speech. In R. W. Rieber & A. S. Carton (Eds.), *The collected works of Vygotsky, L.S. (vol.1): Problems of general psychology* (pp. 191-220). New York: Plenum.

A School-Community Model for Enhancing Aboriginal Students' Mathematical Learning

Peter Howard
Australian Catholic University
<p.howard@mary.acu.edu.au>

Bob Perry
Charles Sturt University
<bperry@csu.edu.au>

Strong relationships established between schools and communities can improve the mathematical learning outcomes for Aboriginal students. The 2005-2006 *Building Community Capacity* project sought to identify key aspects of meaningful engagement between schools and communities focusing on the development and implementation of contextualised, relevant and connected mathematics curriculum and appropriate teaching and learning strategies to enhance Aboriginal students' mathematics outcomes. Using case study methodology within two school sites in New South Wales, the paper identifies critical elements of community engagement and provides underlying principles, which other communities might consider in their own community capacity building.

From 1999-2005, the Board of Studies, New South Wales (NSW) in conjunction with the NSW Department of Education and Training, Australian Catholic University, and University of Western Sydney, has worked with schools and community members at two sites: one urban site in western Sydney and one rural site in western NSW in the *Mathematics in Indigenous Contexts* (MIC) project. These two sites were selected because of the significant enrolment of Aboriginal students in the schools. MIC focused on establishing a learning team comprising teachers, Aboriginal educators, and local Aboriginal community people to develop contextual multistage mathematics units that suited the learning needs of Aboriginal students. The mathematics activities reflected each community's knowledge, engaged the students in meaningful learning, created closer school/community links, and brought cross-cultural groups together. An underlying principle of the project was having the school seen as central to the community, with both working together to develop curriculum which enhanced the knowledge and the capacity of the Aboriginal students, community, and school. Building community capacity was a key element of the MIC project. The MIC participants included: Aboriginal educators, Aboriginal parents and community people, primary and secondary teachers, teacher mentors, Aboriginal and non-Aboriginal students, NSW Board of Studies personnel, and university mentors.

MIC was based upon the principle that the mutually beneficial engagement of people and cultures is essential in developing a community's capacity for educating Aboriginal students. According to Matthews, Howard, and Perry (2003), "educating Aboriginal students requires Aboriginal and non-Aboriginal teachers to understand the needs and cultures in which each Aboriginal student lives" (p. 18).

Mathematics and Aboriginal Students' Learning

All education, including mathematics education, needs to be a "place of belonging" for Aboriginal students. Aboriginal students need to feel that schools belong to them as much as to any other child. School success for Aboriginal students is dependent upon "cultural appropriateness, development of requisite skills and adequate levels of participation"

(Elson-Green, 1999, p. 12). To move towards the achievement of potential by Aboriginal students it is important that Aboriginal culture and language are accepted in the classroom and students have a sense of belonging (Matthews et al., 2003). Developing a shared understanding and appreciation of the beliefs of the culturally diverse groups involved in mathematics learning can help lessen cultural conflict in the mathematics classroom and place more focused attention on the learning potential of Aboriginal students. Parents, teachers and students need to come together to develop and implement relevant curriculum and teaching strategies that utilise and value Aboriginal peoples' knowledge and values (Howard, Perry, & Butcher, 2007). Cultural identity is a major issue for Aboriginal people. No matter where an Aboriginal child lives it is likely he/she will identify with aspects of Aboriginal culture. Identity is personal and evolves as individuals grow in the knowledge of their cultural backgrounds and as they respond to varying places and circumstances.

Howard (2001) reported Aboriginal beliefs about mathematics, mathematics teaching, and mathematical learning. The identification and reporting of these mathematical beliefs help inform teachers and Aboriginal communities about required reform in mathematics teaching to enhance Aboriginal students' mathematical learning. Learning mathematics is a process of sociocultural interaction (Sfard & Prusak, 2005). All students, Aboriginal and non-Aboriginal, will meet cultural conflicts in their mathematics classrooms. For Aboriginal students, such cultural conflict may occur through the teaching strategies being used, the lack of relevance of mathematics activities, confusion in the mathematics language being used or the lack of awareness of the social, cultural and historical issues that Aboriginal students bring to the mathematics classroom. Teachers have to become aware of, and appreciate, the cultural diversity and hence the cultural conflicts that can occur amongst teachers, students, parents, and the curriculum content. They need to understand where the school conflicts originate for Aboriginal students in order to implement effective pedagogy. Appropriate curriculum can enhance the mathematics achievement of Aboriginal students through its relevance, appreciation of the complexity of the mathematical language and presentation of practical mathematical learning activities (Howard & Perry, 2005).

Community Capacity: Setting the Scene and Identifying Challenges

The *Building Community Capacity* project seeks to analyse the success of the *Mathematics in Indigenous Contexts* project and encourage its generalisation into other communities and contexts by examining the:

- place of community capacity building within current political, social, and educational contexts;
- nature of community capacity building; and
- challenges such community capacity building provides for teachers and communities.

MIC was funded by the Board of Studies NSW at a time when there were limited, if any, formal channels for Aboriginal communities to have a representative voice in local curriculum development initiatives. Within MIC, priority was given to the voices of Aboriginal people and students as an essential means to enhancing the cultural appropriateness and educational potential of learning goals and strategies for Aboriginal students.

Nature of Community Capacity Building

Community capacity can be described as the bringing together of the community's knowledge, skills, commitment, and resourcefulness to build on community strengths and address community challenges (McGinty, 2002). Community capacity building involves both attending to the foundations of the capacity and taking the capacity beyond where it is at present. Engagement that is respectful of, and sensitive to, the values of these communities and cultures are key to community capacity building.

Challenges of Community Capacity Building

Community capacity building involves school leaders, teachers, students, community leaders and members in a process of mutually beneficial engagement through a discourse of relationships and exploration. Relationships of respect and trust are the gateway to effective engagement. School leaders and teachers are challenged in the first instance to move beyond the educational model of "teacher and taught" to one of mutual respect and engagement with the Aboriginal community as learning partners.

Community capacity building challenges schools and teachers to use learning approaches that are based upon the mutual engagement of the school and the community. One of the criteria for quality teaching and learning (NSW Department of Education and Training, 2003) is that schools must move beyond approaches that assume they alone have responsibility for ensuring learning is related or applied to students' contemporary world and cultural contexts. A second challenge requires educators to move beyond a model of minority children's school achievement that deals only with factors that educators can potentially influence (Okagaki, 2001). Leadership and power lie within and across the school and Aboriginal communities rather than with the school alone. The integrity of leadership lies in the capacity to engage and explore in an alternative and open discourse which will inform approaches to education and learning for Aboriginal students.

A third challenge for schools and teachers lies in their stance with respect to quantifiable measures of student capacities such as student attendance, progression and retention data which are used as benchmarks for public reporting and accountability. Key among these measures in NSW are Basic Skills Tests, Secondary Numeracy Assessment Program, English Language and Literacy Assessment, School Certificate, and Higher School Certificate data. These "evidence-based" measures, which report upon student behaviours, performance, and competencies, inform one's understanding of the learner and learning but do not define or bring closure to a student's capacity. In the case of Aboriginal students' learning, such "informing" requires educators to consider a further register of indicators and evidence that are both informative and culturally inclusive.

A fourth challenge for schools and teachers is to engage with communities in a shared understanding of how home, community, and school can work together in supporting student learning. Alton-Lee (2003) found that for most effective development of student learning outcomes there needs to be an alignment of capacities across student, teachers, and the school community as a whole. This requires teachers to value community contexts and their strengths. Schools and teachers are challenged to engage with the community and the cultural contexts of the students' worlds (Howard, Perry, & Butcher, 2006a) in ways that impact upon school and teacher approaches that are aligned with these contexts. School leaders and teachers develop the cultural and educational alignment of school and community through enhancing their own capacity to think with the cultural perspectives of the students and their communities (Bernstein, 1996).

A fifth challenge underlying a school's and teacher's capacity to enhance Aboriginal students' education lies in teachers developing their own personal and collective efficacy for community engagement. Educators and researchers are challenged to see teacher efficacy as being multi-dimensional including not only their current pedagogical focus on teaching and classroom management (NSW Aboriginal Education Consultative Group Inc./NSW Department of Education and Training, 2004), but also their efficacy to engage with the community (Labone, 2004).

In summary, community capacity building for enhancing the education of Aboriginal students presents schools and teachers with the *five challenges* of developing:

- mutual respect between the Aboriginal community and the school community;
- mutual engagement with the community in developing learning approaches based upon alternative and creative discourses;
- evidence-based discourses to inform one's understanding of learners and learning;
- home-school-community alignment for enhancing student learning; and
- personal and collective efficacy for community engagement.

These five challenges pose a framework for engaging with the school and Aboriginal communities and exploring their community capacity building to enhance the education of Aboriginal students. Within the MIC project the curriculum focus of Aboriginal students' learning of mathematics was the specific vehicle for enhancing community capacity.

Methodology

The *Building Community Capacity* project focused on three NSW Department of Education and Training schools in the two sites – a primary school in an urban community and both a primary and secondary school in the rural site. These schools were chosen based on the collaboration between the Aboriginal community and school in previous *Mathematics in Indigenous Contexts* activities. Each site identified an Aboriginal educator as the key project link between the school and the Aboriginal community.

Qualitative data about building community capacity through meaningful engagement in the *Mathematics in Indigenous Contexts* project were collected by the authors during visits to each site. Semi-structured interviews with Aboriginal community members, Aboriginal Educational Assistants, Aboriginal students, teachers, and school principals were the principal data collection strategies. During 2005, three visits were made to each site. There was a fourth visit to each site in 2006.

The *Building Community Capacity* project focused on investigating attitudes of teachers *(primary and secondary, including school executive)* in respect to parent/community *(Aboriginal)* involvement, issues impacting upon community *(Aboriginal)* involvement, and the possible ramifications on student engagement *(Aboriginal and non-Aboriginal*) in school *(primary and secondary).* The three key research questions were the following.

1. What are the critical interactions between Aboriginal communities, increased community capacity, and positive Aboriginal student engagement with education?
2. What are the critical issues that impact on developing sustainable community capacity projects between schools and Aboriginal people?

3. What activities and processes underpin the development of effective school community capacity projects?

All interviews were audio-taped and transcribed. An initial categorization of the qualitative data was established using a grounded theory approach. Coding was conducted by the authors and identified four constructs linked to the research questions. These constructs formed the *Framework for Successful Community Capacity Building*:

- *Context* – data related to the physical, social, economic, cultural and historical factors in each site;
- *Engagement and Learning* - data related to levels of involvement of Aboriginal students and community with the schools;
- *Sustainability* – data related to factors influencing the continuity of initiatives established during the *Mathematics in Indigenous Contexts* project; and
- *Activities and Processes* – data related to the effective interactions that facilitated school/community engagement.

Each site formed a case study that informed conclusions and recommendations for the building of community capacity through a mathematics curriculum development project.

The Sites

The urban site is situated in Western Sydney. The primary school was established in the mid-1970s. In 2005, approximately 140 of the 450 students at the school were Aboriginal. Most of the people in the community are long-term residents, and many of the children at the school are second generation students.

In 2002/2003, Year 4 teachers volunteered to be involved in the *Mathematics in Indigenous Contexts* project. In collaboration with the Aboriginal Education Assistant (AEA) and the Aboriginal community, mathematics units were developed around a mural theme, use of the local Aboriginal reserve, and group-based activities that focused on building specific mathematical skills such as measuring, numeracy, basic operations, and geometry.

The rural site is a harmonious western New South Wales community of about 3000 people, approximately one-third of whom are Aboriginal. Almost half of the primary school students and one in five of the high school students are Aboriginal. Most of the people in the community are long-term residents.

A key focus of the MIC project in the rural site was building Aboriginal students' specific mathematical skills in measuring, mapping, enlarging, estimating, using compasses, and understanding volume and fractions. The students completed in-class mathematics activities, mapped changes in land use near the school with the help of a local community member, and described directions using compasses. Following these activities, the students visited "The Pines", an area where the Aboriginal community lived from the 1950s to the mid-1970s. The area was well known to the Aboriginal educators at the school and community members. The non-Aboriginal staff knew little of the history of this land. A concept map was generated by the mathematics teachers and Aboriginal community members to identify what type of mathematical knowledge and understanding students could gain from activities utilising the site. The site, as suggested by its name, was covered in pine trees. These became a key resource in the development of mathematical activities such as measuring the heights and circumferences of the trees and estimating their age. The teachers developed a mathematics unit of work about the central theme of the environment of The Pines. Non-routine problems involving orienteering through The Pines highlighted position, angle and direction. Other activities included drawing, naming and categorising various flora. These processes reinforced 2D representation from 3D objects. Plans/maps were drawn of various sections of the site with students generating scales and keys. The project day included a talk from Aboriginal Elders about their life on the site and how the families lived from day to day. The integration of mathematics and history engaged the students in the learning and enabled them to become more aware and appreciate a critical element of the history of their town (Handmer, 2005).

For Harry (Head Teacher, Mathematics), the culminating day of the project evidenced,

> … a sense of achievement in that we had got so far from where we had set off. I know it was a maths unit that we were asked to do, but then we decided ourselves that there was far more importance on the fact that we should acknowledge, appreciate and know about the Aboriginal people out in that area. [It was enlightening] going out there and seeing, the interaction of the children, and seeing the Aboriginal children take an ownership role of their little groups. All the kids learnt something about the identity of the Aboriginal people who lived there. For a lot of the non-Indigenous kids The Pines was an area that you drove past and thought "Oh, so what?" but now it means something. We need to do far more to acknowledge that part of our history.

Harry felt that it was now important for the school community to acknowledge the past: "If you don't know where you came from and have an identity, you flounder for the rest of your life. You're wondering. Yes, you're always wondering."

The *Mathematics in Indigenous Contexts* project enabled part of the Aboriginal history of the rural community to come to the surface. For Harry, the mathematics project was "a good learning experience for both of them *(Aboriginal and non-Aboriginal students).* It was an excellent learning experience for the staff – not only the people involved but the reaction that went through the whole school community."

Results, Analysis, and Discussion

The *Mathematics in Indigenous Contexts* project gave priority to the voices of Aboriginal people as an essential means of enhancing the cultural appropriateness of mathematical teaching and learning for Aboriginal students. It was based upon the rights of Aboriginal people to be engaged as decision makers in local policies regarding the nature and form of mathematics education.

The Aboriginal community members interviewed expressed the view that the *Mathematics in Indigenous Contexts* project enriched the engagement of Aboriginal and non-Aboriginal students in their mathematics learning, acknowledged the relevance of community-based mathematics teaching strategies, and increased the capacity of the community to engage in effective mathematics curriculum reform. Further detail of the data gathered and its analysis is presented here within the previously developed *Framework for Successful Community Capacity Building.*

Context

All three school sites involved in the project were physically welcoming to the Aboriginal community, through significant displays of art and photographs both inside and outside the school buildings and a general feeling of overall calm. There was an obvious sense of pride in the presentation of the schools and this was respected by their communities, staff, and students. There was a sense of self-respect amongst the students and staff of each school. As well, the schools were seen as important centres within the communities.

> There's an exchange of knowledge there when you're getting Aboriginal people that come into schools. OK, they're not very well educated but they know a lot about how Aboriginal people live. And the teachers can see how they relate to the kids and the kids relate to them and you're learning off each other all the time. (Aboriginal community member, rural site)

Staff, students, and community at all of the schools commented that there was really no overt racism. When isolated instances of conflict occurred, those involved were clearly told by school or community that it was just not acceptable in these locations. People from all groups took the responsibility for ensuring harmony.

In all the sites, there are key members of the communities and the school leadership teams who have shown long-term commitment to their roles in developing the strengths of the schools and their engagement with their communities. Of particular note are the roles played by some school executive members and the Aboriginal Education Assistants. The data identify people at both sites who provide role models for other sites in terms of their skills and knowledges and the ways in which they act and interact to build community capacity. Of particular importance in the sites studied were the following people.

Urban site	Aboriginal Education Assistant; Principal; Assistant Principal.
Rural site	Aboriginal Education Assistants (primary and secondary schools); Principal (secondary school); Head Teacher, Mathematics; Assistant Principal (primary school).

The participants in both sites expressed their beliefs that they wanted to go beyond an involvement of the community with the schools through traditional parent/teacher meetings, school barbecues and sports days. They wanted to move towards a purposeful engagement of community in providing appropriate learning opportunities for Aboriginal students. This willingness was evident in a long-term commitment to build relationships between schools and communities and mutual trust and respect among all involved.

Engagement and Learning

By coming together and engaging in community capacity building, all participants are engaged in learning. The teachers were mentored by the Aboriginal educators and community people in developing a different appreciation of the learning ways of their Aboriginal students.

> There is a lot of ignorance of Aboriginal culture. We have to educate them to what we are made of, what we are and where we have come from. We have to open their eyes to see that their way, while it's a good way, it's not the only way to do things. (Aboriginal community member, rural site)

When Aboriginal people and the community are engaged in the school curriculum, with their knowledge and presence valued, they come to feel a greater part of the school. In MIC, such engagement has developed a greater awareness amongst all participants of Aboriginal culture and the importance of education and learning.

> Change is coming. It has been gradual but I think now there's a bigger focus on it whereas before it was ignored. I think getting people into the school to raise the teacher's awareness is helpful. It makes the students feel more a part of the school. It's that awareness that's changed in non-Aboriginal people and leads to other changes. I reckon it's making the kids more aware of their education and the need for education. (Aboriginal community member, rural site)

Sustainability

The effects of many educational initiatives are short-term and unsustained. One of the features of the approach taken in MIC was to endeavour to have the changes last well beyond the intervention period. There was a commitment to an engaged presence of the Aboriginal community within the schools and a clear purpose in the tasks undertaken. Commitment, explaining and timing were seen to be critical elements in facilitating change.

> The people involved in it from the beginning got to be committed and they've got to go out and first be here with their Elders and with the community and not give up on them. So you go back there now and you find another way of doing it, it may work. But you've got to keep at it … it's just explaining yourself more. If they don't understand what they're getting into, well, they're not going to have a go. You've got to catch them at the right time. Things are going on in their lives where it's impossible for them to do things. So if you get them at that right time, you're right. Sometimes you just can't so you just have to keep going back. And you don't try to push it on them, you explain it to them and if they don't understand it, if you haven't explained it properly then you will go back and you'll think about it and go back again … you got to have compassion. (Aboriginal community member, rural site)

The indications from these participants are that they now feel in a position to continue similar initiatives generated from within their own schools and communities.

The coming together of the knowledges of all participants has led to an enhanced understanding of each others' roles within community and a deeper appreciation of the complementarity of these roles. Key features of the sites that have made this possible are:

- an environment of openness and trust;
- mutual respect;
- sincerity in establishing and maintaining relationships;
- a shared commitment to the tasks involved;
- effective leadership from both the school and community;
- willingness to do more than might be seen as one's duty;
- knowledgeable and confident Aboriginal Education Assistants;
- confidence, resilience, efficacy, and initiative of Aboriginal community people;
- expressed recognition and celebration of the value of Indigenous knowledge;
- the presence of key Aboriginal and non-Aboriginal community members with a history of harmonious engagement;
- an appreciation of the risks that need to be taken to engage purposefully and a willingness to take these risks;
- active listening;
- a sharing with other schools and community of what had been achieved;
- managing the subtle prejudicial behaviours that might emerge; and
- tangible products and outcomes from the work undertaken.

When these features are achieved in a project, then there would seem to be an excellent chance for sustainability in building community capacity.

> If our kids are going to thrive, we need our community members, and the only way to get them is to let them know what is going on and let the school know what and who is available out there. I'm like a contact person, liaison person and also make sure that the Aboriginal people that are in the school are comfortable. We want them to come back and do what they are good at doing. (Aboriginal Education Assistant, urban site)

Activities and Processes

What mathematics is done in a project such as MIC is less important than how it is done, providing it does offer opportunities for all participants to engage in meaningful, relevant and interesting tasks. However, there is much evidence that the mathematical excursions to The Pines in the rural site and to the Aboriginal Reserve in the urban site were most worthwhile activities in their own rights. They enhanced student mathematical outcomes in special and particularly relevant ways. As well, they helped the adult participants understand each others' cultural history in ways that would be impossible using traditional classroom-based teaching approaches.

> What we did with these projects was brought it back to relevance, not only for just the Aboriginal kids but for the non-Aboriginal kids too. It would be better for the community if they've got awareness of the history of it, the town they're living in and the people in it. So that must feel better. (Aboriginal community member, rural site)

From the perspective of community capacity building, the actual mathematics learned was a pathway along which people travelled to reach a greater understanding of each other and their communities.

Conclusion

Through these two case studies and the reporting of MIC's impact upon the communities, key features have been identified that other communities could use in enhancing their own community capacity building efforts. Clear progress has been made towards meeting the five challenges for community capacity building that were described earlier in the paper. This framework, as well as the *Framework for Successful Community Capacity Building,* have been useful in analysing the achievements of MIC. The frameworks provide a structure whereby communities can evaluate to what degree they are achieving the key components of a successful capacity building program.

In the past, too much has been left to chance as well-meaning groups of people strived to improve the lot of Aboriginal people without Aboriginal people having a direct engagement in the process. The *Mathematics in Indigenous Contexts* project has provided a strong model for a shift in approach which does ensure that Aboriginal communities play a leading role in the development of their capacity.

References

Alton-Lee, A. (2003). *Quality teaching for diverse students in schooling: Best evidence synthesis*. Wellington, NZ: Ministry of Education.

Bernstein, B. (1996). *Pedagogy, symbolic control and identity: theory, research, critique*. London: Taylor & Francis.

Elson-Green, J. (1999). Equity for Aborigines is top priority: Kemp. *Education Review*, Nov/Dec, 12 - 13.

Handmer, A. (2005). Teach your children well. *Teacher*, December, 22-24.

Howard, P. (2001). *Beliefs about the nature and learning of mathematics in Years 5 and 6: The voices of Aboriginal children, parents, Aboriginal educators and teachers*. Unpublished PhD dissertation. Sydney: PME.

Howard, P., & Perry, B. (2005). Learning mathematics: Perspectives of Australian Aboriginal children and their teachers. In H. L. Chick & J. L. Vincent (Eds.), *Proceedings of the 29th annual conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 153-160). Melbourne: University of Melbourne.

Howard, P., Perry, B., & Butcher, J. (2006a). *Community capacity research project*. Sydney: New South Wales Board of Studies.

Howard, P., Perry, B., & Butcher, J. (2007). *Shared ownership and community capacity building.* In P. L. Jefferies (Ed.) *Engaging Pedagogies*, Adelaide, November 26-30, 2006. [CDROM]

Labone, E. (2004). Teacher efficacy: Maturing the construct through research in alternative paradigms. *Teaching and Teacher Education, 20*(4), 341-359.

Matthews, S., Howard, P., & Perry, B. (2003). Working together to enhance Australian Aboriginal students' mathematics learning. In L. Bragg, C. Campbell, G. Herbert, & J. Mousley (Eds.), *Proceedings of the 26th annual conference of the Mathematics Education Research Group of Australasia* (pp.17-28). MERGA: Sydney.

McGinty, S. (2002). Community capacity building. In P. L. Jefferies (Ed.), *Australian Association of Research in Education 2002 International Education Research Conference Proceedings*, Brisbane. Retrieved May 31, 2007 from http://www.aare.edu.au/02pap/mcg02476.htm

NSW Aboriginal Education Consultative Group Inc./NSW Department of Education and Training. (2004). *The report of the review of Aboriginal education. Yanigurra muya: Ganggurrinmya yaarri guurulaw yirrigin.gurray (Freeing the spirit: Dreaming an equal future).* Sydney: NSW Department of Education and Training.

NSW Department of Education and Training. (2003). *Quality teaching in NSW public schools*. Sydney: NSW Department of Education and Training.

Okagaki, L. (2001). Triarchic model of minority children's school achievement. *Educational Psychologist, 36*(1), 9-20.

Sfard, A., & Prusak, A. (2005). Telling identities: The missing link between culture and learning mathematics. In H. L. Chick & J. L. Vincent (Eds.), *Proceedings of the 29th conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 37-52). Melbourne: PME.

Benchmarking Preservice Teachers' Perceptions of their Mentoring for Developing Mathematics Teaching Practices

Peter Hudson
Queensland University of Technology
<pb.hudson@qut.edu.au>

A literature-based instrument gathered 147 final-year preservice teachers' perceptions of their mentors' practices related to primary mathematics teaching based on five factors for mentoring (i.e., personal attributes, system requirements, pedagogical knowledge, modelling, and feedback). Results indicated acceptable Cronbach alpha scores for each factor: 0.91, 0.77, 0.95, 0.90, and 0.86, respectively. Furthermore, less than 45% of mentors were perceived to provide specific practices associated with mentoring system requirements. This paper discusses possibilities for using the survey instrument including benchmarking mentees' perceptions of their mentoring for developing their mathematics teaching and as a reference point for delivering professional development for mentors.

University-community engagement is a high national priority. Although university-community collaboration has not been a traditional strength of higher education (Holland, 2004, p. 11), there appear to be considerable benefits through university-community engagement. Institutions have found university-community engagement has strengthened and expanded the scholarship and teaching at the academic level (Brukardt, Holland, Percy, & Zimpher, 2004, p. 1), particularly as "Community-based research can be a bridge between the academy and the community" (Heffner, Zandee, & Schwander, 2003, p. 3). These effective partnerships align goals with adequate time to establish partnerships (Kriesky & Cote, 2003). Determining the progress of university-community engagement requires some form of measurement. Many educators have advocated benchmarking as a means for measuring successful practices and as a useful tool for balancing outcomes and processes (Garlick, 2003). Garlick argues that benchmarking must "…begin with an extensive consultation program" (2003, p. 5) and, certainly, university and community consultation needs to be part of the benchmarking process. There are various types of university-community engagement that have the potential for benchmarking practices.

Mentoring is prominent in education systems throughout the world (Hawkey, 1997; Power, Clarke, & Hine, 2002; Starr-Glass, 2005) and mentors (i.e., supervising teachers or cooperating teachers) in professional experience settings (i.e., practicum, field experiences, internships) are well positioned to assist preservice teachers in developing their practices (Crowther & Cannon, 1998). Mentors' responsibilities for developing preservice teachers' practices are increasing as mentoring continues to amplify its profile in education (Sinclair, 1997). Primary teachers in Australia generally work across all key learning areas (KLAs) and hence, in their roles as mentors, are expected to facilitate quality mentoring to preservice teachers across these KLAs. However, primary teachers will not be experts in all KLAs and research shows some areas receive considerably less attention than others (e.g., science (Goodrum, Hackling, & Rennie, 2001) and art (Eisner, 2001)). As the curriculum is so diverse for primary teachers, they may need assistance in their roles as mentors with particular mentoring practices focused on subject-specific areas (Hodge, 1997; Jarvis, McKeon, Coates, & Vause, 2001), which also appears to be the case for mentoring in mathematics education (Jarworski & Watson, 1994; Peterson & Williams, 1998).

Similar to teaching practices, professional development in mentoring practices may enhance the mentors' knowledge and skills. Also similar to teaching practices, mentors operate in their own environment, where they may or may not receive further ideas for developing their practices. Yet, mentoring cannot be left to chance (Ganser, 1996) and needs to be purposeful in order to be more effective with explicit practices (Gaston & Jackson, 1998; Giebelhaus & Bowman, 2002; Jarworski & Watson, 1994; Jonson, 2002). Guidelines for subject-specific mentoring can aid the mentors' development by increasing confidence for raising issues, and providing topics for discussion and observation of specific teaching practices (e.g., see Jarvis et al., 2001). Although there are various models for mentoring (e.g., Colley, 2003; Jarworski & Watson, 1994; Jonson, 2002; Herman & Mandell, 2004), there is little literature on subject-specific mentoring in mathematics education for preservice teachers.

A five-factor model for mentoring has previously been identified, namely, Personal Attributes, System Requirements, Pedagogical Knowledge, Modelling, and Feedback (Hudson & Skamp, 2003), and items associated with each factor have also been identified and justified with the literature (see Hudson, Skamp, & Brooks, 2005). For example, statistical analysis of preservice teachers' responses (n=331) from nine Australian universities on the five-factor model indicated acceptable Cronbach alpha scores for internal reliability on each key factor, namely, Personal Attributes (mean scale score=2.86, SD=1.08), System Requirements (mean scale score=3.44, SD=0.93), Pedagogical Knowledge (mean scale score=3.24, SD=1.01), Modelling (mean scale score=2.91, SD=1.07), and Feedback (mean scale score=2.86, SD=1.11) were 0.93, 0.76, 0.94, 0.95, and 0.92, respectively. The five factors and the development of the Mentoring for Effective Primary Science Teaching (MEPST) instrument are well articulated in the literature (see Hudson et al., 2005) for which this study provides a direct link. To illustrate, providing feedback allows preservice teachers to reflect and improve teaching practices, and this includes practices in specific subject areas such as mathematics. Six attributes and practices, which may be associated with the factor Feedback for developing mentees' primary mathematics teaching, require a mentor to: (1) articulate expectations (Christensen, 1991; Ganser, 2002); (2) review lesson plans (3) observe practice (Jonson, 2002; Portner, 2002); (4) provide oral feedback; (5) provide written feedback (Ganser, 1995, 2002); and, (6) assist the mentee to evaluate teaching practices (Long, 2002; Schon, 1987).

This study explores and describes 147 Australian preservice teachers' perceptions of their mentors' practices in primary mathematics education within the abovementioned five factors linked to a literature-based instrument (Appendix 1). This study aims to determine the transferability of the science mentoring instrument (MEPST) to the development of an instrument based on mentoring preservice teachers in primary mathematics teaching. It also aims to benchmark preservice teachers' perceptions of mentoring practices for developing their primary mathematics teaching.

Data Collection Method and Analysis

The "Mentoring for Effective Mathematics Teaching" (MEMT) survey instrument in this study evolved through a series of preliminary investigations on Mentoring for Effective Primary Science Teaching (MEPST) (Hudson, 2003; Hudson & Skamp, 2003; Hudson, 2004a, b; Hudson et al., 2005), which also identified the link between the literature and the items on the survey instrument. A pilot study was conducted on 29 final-year preservice teachers by administering the MEMT survey instrument at the conclusion of their professional experiences (Hudson & Peard, 2005). Analysis of this pilot test indicated the possibility of a relationship between the MEPST instrument and the MEMT instrument; however further investigation was needed to verify results. For this study, 147 preservice teachers' perceptions of their mentoring were obtained from the five-part Likert scale (i.e., strongly disagree=1, disagree=2, uncertain=3, agree=4, strongly agree=5) MEMT instrument

(Appendix 1). The data provided descriptive statistics for each variable, which also provided an indication of the statistical relationship between variables and within each of the factors. Mean scale scores were derived through a statistical analysis package (SPSS) by analysing specific items associated with each factor. For example, there were six items associated with the factor Feedback, that is, the mentee (preservice teacher) perceived the mentor to: review the mentee's lesson plans before teaching mathematics; observe the mentee teach mathematics before providing feedback; provide oral feedback on the mentee's mathematics teaching; provide written feedback on the mentee's mathematics teaching; discuss evaluation of the mentee's mathematics teaching; and, articulate expectations for improving the mentee's mathematics teaching. Cronbach alpha scores were used as an indication of internal reliability with scores greater than .70 considered acceptable (Hair, Anderson, Tatham, & Black, 1995). The data examined preservice teachers' perceptions of their mentors' mentoring in primary mathematics teaching.

Results and Discussion

These preservice teacher responses (109 female; 38 male) provided descriptors of the participants (mentors and mentees) and data on each of the five factors and associated attributes and practices. Responses were gathered at the conclusion of their final professional experience (i.e., practicum, field experience).

Backgrounds of Participants

Twenty-five percent of these mentees (*n*=147) entered teacher education straight from high school, with 93% completing mathematics units in their final two years of high school (i.e., Years 11 & 12). Seventy-seven percent of mentees had completed two or more mathematics methodology units at university, and 86% had completed three or more block professional experiences (practicums) with 54% completing four professional experiences. There were no professional experiences under three weeks. Ninety percent of mentees taught at least four mathematics lessons during their last practicum with 81% indicating they had taught 6 or more lessons. Most of the classrooms for the mentoring in mathematics were in the city or city suburbs (69%) with 31% in regional cities and in rural towns or isolated areas. Mentees estimated that most mentors (male=22, female=125) were over 40 years of age (55%) with 28% between 30 to 39 years of age, and 16% under 30. Mentees also noted that 86% of mentors modelled one or more mathematics lessons during their mentees' professional experiences, with 59% modelling five or more lessons during that period. Finally, 41% of mentees perceived that mathematics was their mentors' strongest subject in the primary school setting.

Five Factors for Effective Mentoring in Mathematics

Each of the five factors had acceptable Cronbach alpha scores greater than 0.70 (Kline, 1998), that is, Personal Attributes (mean scale score=3.96, *SD*=0.81), System Requirements (mean scale score=3.31, *SD*=0.90), Pedagogical Knowledge (mean scale score=3.58, *SD*=0.94), Modelling (mean scale score=4.01, *SD*=0.78), and Feedback (mean scale score=3.76, *SD*=0.88) were 0.91, 0.77, 0.95, 0.90, and 0.86, respectively (Table 1). Data from items associated with each factor were entered in SPSS13 factor reduction, which extracted one component only for each factor. The associated eignevalues accounted for 59-69% of the variance on each of these scales (Table 1).

Table 1
Confirmatory Factor Analysis for Each of the Five Factors (n=147)

Factor	Eigenvalue*	Percentage of variance	Mean scale score	*SD*	Cronbach alpha
Personal Attributes	4.13	69	3.96	0.81	0.91
System Requirements	2.05	68	3.31	0.90	0.77
Pedagogical Knowledge	7.19	65	3.58	0.94	0.95
Modelling	4.70	59	4.01	0.78	0.90
Feedback	3.64	61	3.76	0.88	0.86

* Extracting only one component with an eigenvalue >1 is considered acceptable (see Hair et al., 1995).

The following provides further insight into specific data on mentees' perceptions of mentors' attributes and practices associated with each factor.

Personal Attributes.

When analysing the mentees' responses on their mentors' "Personal Attributes", a majority of mentors were supportive towards their mentees' primary mathematics teaching (89%) with mentors appearing comfortable in talking about mathematics teaching (86%, Table 2). However, more than a quarter of mentees believed that their mentors had not aided their reflection on mathematics teaching practices (i.e., 73% of mentees agreed or strongly agreed their mentor facilitated this practice), instilled positive attitudes for teaching mathematics (69%), listened attentively to their mentees about mathematics teaching (67%) or instilled confidence for teaching mathematics (64%). Table 2 provides mean item scores (range: 3.67 to 4.35; *SD* range: 0.85 to 1.08) and percentages on mentees' perceptions of their mentors' Personal Attributes.

Table 2
"Personal Attributes" for Mentoring Primary Mathematics Teaching (n=147)

Mentoring Practices	%*	*M*	*SD*
Supportive	89	4.35	0.85
Comfortable in talking	86	4.25	0.88
Assisted in reflecting	73	3.87	1.01
Instilled positive attitudes	69	3.92	0.88
Listened attentively	67	3.67	1.07
Instilled confidence	64	3.75	1.08

* %=Rank-order percentages of mentees who either "agreed" or "strongly agreed" their mentor provided that specific mentoring practice.

System Requirements

Items displayed under the factor "System Requirements" presented a different picture from the previous factor. The percentages of mentees' perceptions of their primary mathematics mentoring practices associated with System Requirements were all below 50%, that is, 44% of mentors discussed the aims of mathematics teaching, 41% of mentors discussed the school's mathematics policies with the mentee, and only 29% outlined mathematics curriculum documents (Table 3). Implementing departmental directives and primary mathematics education reform needs to also occur at the professional experience level, yet the data indicated (mean item scores range: 2.71 to 3.15; *SD* range: 1.14 to 1.24, Table 3) that many preservice teachers may not be provided these mentoring practices on System Requirements for developing their mathematics teaching within the school setting.

Table 3
"System Requirements" for Mentoring Primary Mathematics Teaching

Mentoring Practices	%*	*M*	*SD*
Discussed aims	44	3.15	1.14
Discussed policies	41	3.06	1.18
Outlined curriculum	29	2.71	1.24

%=Rank-order percentages of mentees who either "agreed" or "strongly agreed" their mentor provided that specific mentoring practice.

Pedagogical Knowledge

Mean item scores (3.31 to 3.84; *SD* range: 1.08 to 1.24, Table 4) indicated that the majority of mentees "agreed" or "strongly agreed" their mentor displayed "Pedagogical Knowledge" for primary mathematics teaching. However, in this study, more than 20% of mentors may not have mentored pedagogical knowledge practices (see Table 4 for rank-order percentages). For example, 64% of mentors were perceived to assist in the planning stages before teaching mathematics, 67% discussed timetabling the mentee's mathematics teaching, and 71% assisted with mathematics teaching preparation (Table 4). Furthermore, teaching strategies need to be associated with the assessment of students' prior knowledge, yet nearly half the mentors were perceived not to discuss assessment or questioning techniques for teaching mathematics (52%). Many mentors also appeared not to consider content knowledge and problem-solving strategies for teaching mathematics (57%) and providing viewpoints on teaching mathematics was not considered a high priority (61%, Table 4). This implies that many final-year preservice teachers may not be provided with adequate pedagogical knowledge in the primary school setting to develop successful mathematics teaching practices.

Table 4
"Pedagogical Knowledge" for Mentoring Primary Mathematics Teaching

Mentoring Practices	%	*M*	*SD*
Discussed implementation	77	3.84	1.08
Assisted with classroom management	73	3.77	1.08
Guided preparation	71	3.69	1.14
Assisted with teaching strategies	68	3.73	1.16
Assisted with timetabling	67	3.74	1.16
Assisted in planning	64	3.61	1.04
Provided viewpoints	61	3.51	1.17
Discussed problem solving	57	3.51	1.08
Discussed questioning techniques	57	3.45	1.11
Discussed content knowledge	52	3.31	1.24
Discussed assessment	52	3.50	1.19

* %=Percentage of mentees who either "agreed" or "strongly agreed" their mentor provided that specific mentoring practice.

Modelling

Modelling mathematics teaching provides mentees with visual and aural demonstrations of how to teach and, indeed, mean item scores (3.81 to 4.30; *SD* range: 0.83 to 1.19, Table 5) indicated that the majority of mentors were perceived to model mathematics teaching practices. Even though more than 75% mentees believed their mentors modelled practices for teaching mathematics including modelling a rapport with their primary students (85%), modelling the teaching of primary mathematics (79%), displaying enthusiasm for teaching mathematics (78%), and using language from the mathematics syllabus (78%), more than a quarter of mentees indicated their mentors had not modelled a well-designed lesson or effective mathematics teaching (see Table 5 for rank-order percentages).

Table 5
"Modelling" Primary Mathematics Teaching

Mentoring Practices	%	*M*	*SD*
Modelled rapport with students	85	4.30	0.83
Modelled classroom management	82	4.11	0.97
Demonstrated hands-on	81	4.03	1.04
Modelled mathematics teaching	79	4.14	0.90
Displayed enthusiasm	78	4.02	1.00
Used syllabus language	78	3.97	0.89
Modelled a well-designed lesson	73	3.81	0.99
Modelled effective mathematics teaching	71	3.83	1.19

* %=Percentage of mentees who either "agreed" or "strongly agreed" their mentor provided that specific mentoring practice.

Feedback

Mean item scores (3.31 to 4.18; *SD* range: 0.97 to 1.38, Table 6) indicated that the majority of mentees "agreed" or "strongly agreed" their mentors provided "Feedback" as part of their mentoring practices in primary mathematics teaching. Yet, surprisingly, mentees perceived that 82% of mentors observed their mathematics teaching with only 63% articulating their expectations for the mentees' teaching of mathematics. More surprising is that 4% of mentors provided oral feedback without observation. Fifty-nine percent were perceived to provide written feedback and only 55% of mentors reviewed lesson plans, which is necessary to provide feedback before teaching commences for enhancing instructional outcomes (Table 6).

Table 6
Providing "Feedback" on Primary Mathematics Teaching

Mentoring Practices	%	*M*	*SD*
Provided oral feedback	86	4.18	0.97
Observed teaching for feedback	82	4.08	1.00
Discussed evaluation on teaching	81	3.97	1.08
Articulated expectations	63	3.55	1.16
Provided written feedback	59	3.48	1.38
Reviewed lesson plans	55	3.31	1.25

* %=Percentage of mentees who either "agreed" or "strongly agreed" their mentor provided that specific mentoring practice.

Further Discussion and Conclusions

There appeared to be transferability of the MEPST survey instrument (Hudson et al., 2005) to the MEMT instrument, which was supported by acceptable Cronbach alpha scores and descriptive statistics (Table 1). Even though the Likert scale differentiated the degree of mentoring (e.g., strongly disagree to strongly agree), the quality of these mentoring practices requires further investigation. Also, the mentoring indicated in this study only focused on the mentors' practices and attributes and not on mentees' involvement in the mentoring processes. Nevertheless, 93% of these preservice teachers had completed at least three professional experiences (practicums) and nearly four years of a tertiary education degree in teaching before responding to this survey on their final-year Mentoring for Effective Mathematics Teaching (MEMT, Appendix 1). Mentees' perceptions of mentors not providing the above practices may be interpreted in two ways: the mentor did not provide the particular mentoring practice or the mentoring practice was not apparent enough for the mentee to perceive it. Either way, mentors need to provide such practices that are clearly evident to their mentees. Anecdotal evidence suggests mentors vary their mentoring practices considerably, and as there are national standards for teaching and assessing mathematics (e.g., NCTM, 1991, 1992, 1995), a set of standards for mentoring practices for mathematics appears a logical sequence. The MEMT instrument provided a way to collect data for benchmarking mentees' perceptions of their mentors' practices in primary mathematics teaching occurring in various Queensland schools. Such benchmarks can aid toward developing mentoring programs that enhance mathematics teaching practices.

The inadequate mentoring outlined in this study may be initially addressed through specific mentoring interventions that focus on effective mentoring (i.e., attributes and practices associated with the five factors: Personal Attributes, System Requirements, Pedagogical Knowledge, Modelling, and Feedback). As each item associated with the MEMT instrument is linked to the literature, a mentoring intervention for developing mentees' mathematics teaching can be based around these items. Benchmarking mentees' perceptions can provide starting points for designing well-constructed mentoring programs that provide professional development for mentors to enhance not only their own mentoring practices but possibly their mathematics teaching practices. Further benchmarking may occur using the MEMT instrument with mentoring early-career mathematics teachers. For example, a mentoring intervention based on early-career teachers' perceptions of their mentoring may aid induction processes, particularly in the form of programs for mentors to provide adequate mentoring support for mathematics teaching. Additionally, the MEMT instrument may be used by tertiary institutions or departments of education to benchmark the degree of mentoring in primary mathematics and, as a result of diagnostic analysis, plan and implement mentoring programs that aim to address perceived issues.

References

Brukardt, M., Holland, B., Percy, L., & Zimpher, N. (2004). *Calling the question: Is education ready to commit to community engagement?* Report from Wingspread conference, University of Wisconsin.

Christensen, L. (1991). *Empowerment of preservice teachers through effective mentoring: Course requirements.* Alabama: University of Alabama.

Colley, H. (2003). *Mentoring for social inclusion: A critical approach to nurturing mentor relationships*. London: RoutledgeFalmer.

Crowther, D. T., & Cannon, J. R. (1998, January). *How much is enough? Preparing elementary science teachers through science practicums.* Paper presented at the Annual Meeting of the Association for the Education of Teachers of Science. Minneapolis, MN.

Eisner, E. W. (2001). Should we create new aims for art education? *Art Education 54*(5), 6-10.

Ganser, T. (1995, April). *A road map for designing quality mentoring programs for beginning teachers.* Paper presented at the annual conference of the Wisconsin Association for Middle Level Education, Stevens Point, WI.

Ganser, T. (1996). Preparing mentors of beginning teachers: An overview for staff developers. *Journal of Staff Development, 17*(4), 8-11.

Ganser, T. (2002). How teachers compare the roles of cooperating teacher and mentor. *Educational Forum, 66*(4), 380-385.

Garlick, S. (2003, July). *Benchmarking 'good practice' university-region engagement efficiency.* In Proceedings of the InsideOut conference on higher education, University of Queensland.

Gaston, J. S., & Jackson, J. F. (1998). Mentoring and its implications. Idaha, US. (ERIC Document Reproduction Service No. ED 426 990)

Giebelhaus, C. R., & Bowman, C. L. (2002). Teaching mentors: Is it worth the effort? *Journal of Educational Research, 95*(4), 246-254.

Goodrum, D., Hackling, M., & Rennie, L. (2001). *The status and quality of teaching and learning in Australian schools*. Canberra, Australia: Department of Education, Training and Youth Affairs.

Hair, J. F., Anderson, R. E., Tatham, R. L., & Black, W. C. (1995). *Multivariate data analysis with readings* (4th ed.). New York: Prentice-Hall.

Hawkey, K. (1997). Roles, responsibilities, and relationships in mentoring: A literature review and agenda for research, *Journal of Teacher Education, 48*(5), 325–336.

Heffner, G. G., Zandee, G. L., & Schwander, L. (2003). Listening to community voices: Community-based research, a first step in partnership and outreach. *Journal of Higher Education Outreach and Engagement, 8*(1), 127-139.

Herman, L., & Mandell, A. (2004). *From teaching to mentoring: Principles and practice, dialogue and life in adult education.* London: RoutledgeFalmer.

Hodge, S. R. (1997). Mentoring: Perspectives of physical education graduate students from diverse cultural backgrounds. *Physical Educator, 54*(4), 181-195.

Holland, B. (2004). Analyzing institutional commitment to service: A model of key organizational factors. *Michigan Journal of Community Service Learning,* 4, 30-41.

Hudson, P. (2003). Mentoring first-year preservice teachers. *Action in Teacher Education*: *The Journal of the Association of Teacher Educators, 15*(3), 91-99.

Hudson, P. (2004a). Specific mentoring: A theory and model for developing primary science teaching practices. *European Journal of Teacher Education, 27*(2), 139-146.

Hudson, P. (2004b). Toward identifying pedagogical knowledge for mentoring in primary science teaching. *Journal of Science Education and Technology, 13(2),* 215-225.

Hudson, P., & Peard, R. (2005, November). *Mentoring for effective mathematics teaching.* Paper presented at the Eight International Conference, Johor Bharu, Malaysia.

Hudson, P., & Skamp, K. (2003). Mentoring preservice teachers of primary science. *The Electronic Journal of Science Education, 7*(1). Retrieved 2 August, 2006, from http://unr.edu/homepage/jcannon/ejse/ejse.html

Hudson, P., Skamp, K., & Brooks, L. (2005). Development of an instrument: Mentoring for effective primary science teaching. *Science Education, 89*(4), 657-674.

Jarvis, T., McKeon, F., Coates, D., & Vause J. (2001). Beyond generic mentoring: Helping trainee teachers to teach primary science. *Research in Science and Technological Education, 19*(1), 5-23.

Jarworski, B., & Watson, A. (Eds). (1994). *Mentoring in mathematics teaching.* London: Falmer Press.

Jonson, K. F. (2002). *Being an effective mentor: How to help beginning teachers succeed.* Thousand Oaks, CA: Corwin Press Inc.

Kriesky, J., & Cote, L. S. (2003). Extension/academic service-learning: Benefits and lessons learned. *Journal of Higher Education Outreach and Engagement*, *8*(1), 45-58.

Long, S. (2002). Mentoring: A personal reflection. *New Library World, 103*(3), 94-97.

National Council of Teachers of Mathematics. (NCTM). (1991). *Professional standards for teaching mathematics*. Reston, VA: Author.

National Council of Teachers of Mathematics. (NCTM). (1992). *Curriculum and evaluation standards for school mathematics addenda series, grades K-6 sixth-grade book*. Reston, VA: Author.

National Council of Teachers of Mathematics. (NCTM). (1995). *Assessment standards for school mathematics.* Reston, VA: Author.

Peterson, B. E., & Williams S. R. (1998). Mentoring beginning teachers. *Mathematics Teacher, 91*(8), 730-734.

Portner, H. (2002). *Being mentored: A guide for proteges.* Thosand Oaks, CA: Corwin Press.

Power, A., Clarke, M., & Hine, A. (2002, February). *The internship: A journey of professional learning through reflection*. Paper presented at 'Challenging Futures Conference'. University of New England, Armidale, Australia.

Schon, D. (1987). *Educating the reflective practitioner.* San Francisco, California: Jossey Bass.

Sinclair, C. (1997). Redefining the role of the university lecturer in school-based teacher education. *Asia-Pacific Journal of Teacher Education, 25*(3), 309-324.

Starr-Glass, D. (2005). From teaching to mentoring: Principles and practice, dialogue and life in adult education. *Journal of Transformative Education,* 3, 185-189.

Appendix 1

Mentoring for Effective Mathematics Teaching (MEMT)

The following statements are concerned with your mentoring experiences in mathematics teaching during your last professional experience (practicum/internship). Please indicate the degree to which you agree or disagree with each statement below by circling only one response to the right of each statement.

Key
SD = Strongly Disagree
D = Disagree
U = Uncertain
A = Agree
SA = Strongly Agree

During my final professional school experience (i.e., field experience, internship, practicum) in mathematics teaching my mentor:

1. was supportive of me for teaching mathematics.	SD	D	U	A	SA
2. used mathematics language from the current mathematics syllabus.	SD	D	U	A	SA
3. guided me with mathematics lesson preparation.	SD	D	U	A	SA
4. discussed with me the school policies used for mathematics teaching.	SD	D	U	A	SA
5. modelled mathematics teaching.	SD	D	U	A	SA
6. assisted me with classroom management strategies for mathematics teaching.	SD	D	U	A	SA
7. had a good rapport with the students learning mathematics.	SD	D	U	A	SA
8. assisted me towards implementing mathematics teaching strategies.	SD	D	U	A	SA
9. displayed enthusiasm when teaching mathematics.	SD	D	U	A	SA
10. assisted me with timetabling my mathematics lessons.	SD	D	U	A	SA
11. outlined state mathematics curriculum documents to me.	SD	D	U	A	SA
12. modelled effective classroom management when teaching mathematics.	SD	D	U	A	SA
13. discussed evaluation of my mathematics teaching.	SD	D	U	A	SA
14. developed my strategies for teaching mathematics.	SD	D	U	A	SA
15. was effective in teaching mathematics.	SD	D	U	A	SA
16. provided oral feedback on my mathematics teaching.	SD	D	U	A	SA
17. seemed comfortable in talking with me about mathematics teaching.	SD	D	U	A	SA
18. discussed with me questioning skills for effective mathematics teaching.	SD	D	U	A	SA
19. used hands-on materials for teaching mathematics.	SD	D	U	A	SA
20. provided me with written feedback on my mathematics teaching.	SD	D	U	A	SA
21. discussed with me the knowledge I needed for teaching mathematics.	SD	D	U	A	SA
22. instilled positive attitudes in me towards teaching mathematics.	SD	D	U	A	SA
23. assisted me to reflect on improving my mathematics teaching practices.	SD	D	U	A	SA
24. gave me clear guidance for planning to teach mathematics.	SD	D	U	A	SA
25. discussed with me the aims of mathematics teaching.	SD	D	U	A	SA
26. made me feel more confident as a mathematics teacher.	SD	D	U	A	SA
27. provided strategies for me to solve my mathematics teaching problems.	SD	D	U	A	SA
28. reviewed my mathematics lesson plans before teaching mathematics.	SD	D	U	A	SA
29. had well-designed mathematics activities for the students.	SD	D	U	A	SA
30. gave me new viewpoints on teaching mathematics.	SD	D	U	A	SA
31. listened to me attentively on mathematics teaching matters.	SD	D	U	A	SA
32. showed me how to assess the students' learning of mathematics.	SD	D	U	A	SA
33 clearly articulated what I needed to do to improve my mathematics teaching.	SD	D	U	A	SA
34. observed me teach mathematics before providing feedback?	SD	D	U	A	SA

Relational or Calculational Thinking: Students Solving Open Number Equivalence Problems

Jodie Hunter
Massey University
<jodiehunter@slingshot.co.nz>

Student transition from arithmetic to algebraic reasoning has been acknowledged as an essential but problematical process. Previous research has highlighted the difficulties in shifting students towards relational thinking when solving equivalence problems. This paper reports on an investigation into students' use of relational thinking to solve equivalence problems after they have been in classrooms where specific focus has been on developing flexible, efficient computational strategies. The results reveal that most students used computational strategies to solve the equivalence problems rather than relational strategies. Many of the common errors, students made reflected a lack of understanding of the equal sign.

Introduction

Over the past decade, there has been increased focus, both in national and international research and reform efforts, on the teaching and learning of algebraic reasoning (e.g., Irwin, & Britt, 2005; Knuth, Stephens, McNeil, & Alibabi, 2006; Ministry of Education (MOE), 2006; National Council of Teachers of Mathematics (NCTM), 2000). Such emphasis has arisen from growing acknowledgment of the insufficient algebraic understandings students develop during schooling and the role this has in denying them access to potential educational and employment prospects (Knuth et al., 2006). In response, there have been significant curricular reforms designed to support students making the transition from arithmetic to algebra (Freiman & Lee, 2004; Kaput, 1999). One approach has been to promote students' ability to work flexibly with numbers. By developing the students' computational strategies it is claimed that their "structural thinking can then be exploited to develop their understanding of algebra" (Hannah, 2006, p. 1). This paper explores the strategies that students within the age band of 9 to 13 years old used when solving open number equivalence problems. It also investigates the common errors students made when solving these problems due to their lack of understanding of the equal sign. This study adds to previous research of student understanding of equivalence through the analysis of the common errors the students made.

Concepts of equivalence and understanding of the equal sign are essential to algebraic understanding (Freiman & Lee, 2004; Knuth et al., 2006). The foundations of the transition from arithmetic to algebraic reasoning requires that students are able to abstract key concepts including those associated with equivalence and relations. For students to abstract their structural numerical reasoning across to algebraic reasoning it is necessary they understand the equal sign relationally as an equivalence symbol meaning the "same as" (Knuth et al., 2006; McNeil & Alibabi, 2005). The seminal research of Kieran (1981) illustrated that students often have inadequate understanding of the equality symbol. Recent research continues to show that many primary and middle school students lack deep understanding of the equal sign (Carpenter, Franke, & Levi, 2003; Falkner, Levi, & Carpenter, 1999; Knuth et al., 2006; McNeil & Alibabi, 2005). Students with limited understanding of the equal sign view it as an indication of where to put the answer, or alternatively, equate the symbol with doing something – a left to right action or carrying out an operation (Carpenter et al., 2003; Rivera, 2006; Warren & Cooper, 2005). Inadequate understanding of the equality symbol can lead to difficulties for students solving symbolic expressions and equations (Kieran, 1981; Knuth et al., 2006). Furthermore, a limited understanding of the equality symbol can make the transition to algebra difficult for students (McNeil & Alibabi, 2005).

Errors made by students when solving open number equivalence problems reflect their understanding of the equal sign. Freiman and Lee (2004) demonstrated that open number sentence problems in the form of a + b = d + c involving a blank in the last two positions consistently caused difficulties across grade levels. Carpenter et al. (2003) argue that students' errors in solving open number sentence problems are errors of syntax. Students erroneously interpret the rules for how the equal sign is utilised. For example, when solving 9 + 6 = __ + 5, students may put 15 in the blank space considering that the equal sign is an indication to put an answer. Alternatively other students may put 20 in the blank space. These students overgeneralize the property of addition and assume the sequence of symbols in the number sentence is unimportant.

Understanding of the equality symbol as a sign of relational equivalence is a hallmark of the transition between arithmetic to algebraic thinking (Carpenter et al., 2005). Students with a relational view of the equal sign view it as a symbol of equivalence or quantitative sameness. Relational understanding enables students to solve open number sentence equivalence problems such as 8 + 4 = __ + 5 successfully (Falkner et al., 1999). However, within this group of students who understand the equal sign as a symbol of equivalence further distinctions can be made. These distinctions are between students who use computational forms of thinking or those who use relational forms of thinking to solve open number sentence problems.

Stephens (2006) defines relational thinking as dependent on whether children are "able to see and use possibilities of variation between numbers in a number sentence" (p. 479). Students who are able to use relational thinking to solve open number sentence problems consider the expressions on both sides of the equal sign. They are able to solve the problem by using the relation between both expressions without carrying out a calculation. In contrast, students who use computational thinking view the numbers on each side of the equal sign as representing separate calculations. These students perform a calculation to solve open number sentence problems (Carpenter et al., 2003; Stephens, 2006). Students who successfully use relational thinking to solve equivalence problems are also able to identify the direction in which the missing number will change, in order to maintain equivalence. Direction of variation in equivalence problems involving addition is different from those problems that involve subtraction. Stephens maintains that this can cause further difficulties for students.

Warren and Pierce (2004) propose that the difficulties that students encounter may be due to differentiation in requirements for algebraic reasoning and arithmetical reasoning. Some researchers have suggested that classroom mathematics experiences in the early years of schooling are the basis for many problems. This is particularly when emphasis is placed on computation and students are presented with the equal sign as a signal to carry out a calculation (e.g., Carpenter et al., 2003; Knuth et al., 2006; Warren & Pierce, 2004). Warren (2003) also argues that there may be potential problems associated with current reform shifts that focus on a need for number sense and identification of computational patterns. She maintains that these need to be balanced with explicit abstracting of arithmetic structures.

Advocates of mathematics curriculum reform initiatives have suggested teaching algebra and arithmetic as an integrated strand across the curriculum (e.g., Carpenter et al., 2003; NCTM, 2000; MoE, 2006). This approach focuses on building early algebraic thinking through focusing on students' informal knowledge and numerical reasoning. Teachers who use this approach provide students with learning situations which challenge their notions of equality and encourage them to think about relations. This supports students' transition from computation to relational thinking (Carpenter et al., 2003). Stephens' (2006) comparison of two Australian schools found that students exhibited higher levels of relational thinking to solve open number equivalence problems within a school that had a specialist mathematics teacher who explicitly focused on teaching of relational approaches. However, acquiring understanding of equivalence and developing relational thinking is acknowledged as a complex and difficult task and one which necessitates substantial time and explicit teacher attention (Carpenter et al., 2003; Freiman & Lee, 2004).

Method

This study was exploratory in nature and used a qualitative case study design. The aim of the study was to explore student understanding of the equal sign and equivalence. In particular, the study addresses the following research questions.

- What strategies do students use to solve open number equivalence problems?
- What errors are commonly made by students when solving open number equivalence problems?

Participants

The participants were 361 primary and intermediate school students (37 Year 5 students aged 9-10; 47 Year 6 students aged 10-11; 145 Year 7 students aged 11-12; 132 Year 8 students aged 12-13). The study was conducted at a New Zealand urban primary school. The students came from a predominantly middle socio-economic home environment. They were primarily from a European New Zealand ethnic grouping (67%), with students of Maori ethnic grouping (5%), Pacific Island ethnic grouping (10%), Asian ethic grouping (7%), and Indian ethnic grouping (11%).

The school was in its third year of participating in the New Zealand Numeracy Project and algebra had been taught as a separate strand from the number (arithmetic) strand.

Data Collection

The students were given a pen and paper questionnaire derived from a questionnaire developed by Stephens (2006). This consisted of equivalence balance problems with missing numbers and a question about the equal sign. The questionnaire was completed by each individual in regular class time and adequate time was provided to complete it. The students were advised that the questionnaire was not a test but a way to find out how students would solve the problems.

This study reports on the students' responses to the following sets of open number equivalence problems. All the problems were presented in the form of

$a + b = c + d$ or $a – b = c – d$.

Each set of questions began with the words: “Write a number in each of the boxes to make a true statement. Explain your working.”

Table 1
Sets of Open Number Equivalence Problems

Group A	Group B	Group C
23 + 15 = 26 + __	39 – 15 = 41 – __	746 – 262 + __ = 747
73 + 49 = 72 + __	99 – __ = 90 – 59	746 + __ – 262 = 747
43 + __ = 48 + 76	104 – 45 = __ – 46	
__ + 17 = 15 + 24		

Data Analysis

Data analysis used the scoring rubric devised by Stephens (2006), which categorised students thinking using a five-point scale. This scale categorised students’ responses according to whether they used arithmetical thinking or different levels of relational thinking. Further to this categorisation, responses were grouped into three categories according to students’ stability in using different types of thinking to solve the various sets of problems. These were: stable arithmetic thinkers, the students who used only arithmetic strategies; stable relational thinkers, the students who used only relational strategies; and the unstable relational thinkers, students who used a mixture of relational and arithmetic strategies.

The data set was then re-analysed to identify common error types exhibited across the four year levels. In particular, incorrect responses, which indicated a lack of understanding of the equal sign, were identified and analysed. Common erroneous responses were grouped into categories identified in Freiman and Lee’s (2004) study. These included: direct sum, responses when the blank was in the c or d position and students ignored the number in the c or d position and entered the sum of a and b; complete the sum, responses when the blank was in the a or b position and students filled in the blank to complete the equation to a number in the c or d position; and a sum of all terms category, when students added or subtracted all the numbers in the equation.

Results and Discussion

All students in this study had teachers who had completed the professional development associated with the New Zealand Numeracy Project (MoE, 2004). The New Zealand project aims to develop student facility to work flexibly with numbers through developing their computational strategies. An espoused intention of the project is to use the structural thinking the students construct as a foundation for understanding algebra and developing early algebraic reasoning (Hannah, 2006). Despite this intended focus, the findings of this current study reveal that 46% of all the students only used arithmetic strategies, 28% of all students only used relational strategies, and 26% of students used a mixture of arithmetic and relational strategies.

How Consistent was the Student's Strategy Use Across the Year Levels?

Table 2 illustrates the distribution of students at each year level in each category. Consistently at every year level from Year 5 to Year 8 students predominantly used arithmetic strategies to solve the open number sentence problems.

The data in the table illustrate that the number of students classified as "stable relational" increased across the year levels. The most significant increase was between the Year 5 and Year 6 level. Increases in use of relational thinking between Year 6 and Year 7 students and Year 7 and Year 8 students were relatively small. The number of students classified as "stable arithmetic" decreased from Year 6, to Year 7, and Year 8 with some corresponding rises in the number of students classified as "stable relational" or "non-stable relational".

Table 2
Percentage of Students at Each Year Level in Each Category

	Stable Arithmetic	Stable Relational	Non-stable relational
Year 5	50	19	31
Year 6	53	27	20
Year 7	46	31	23
Year 8	35	34	31

How Were the Problems Solved by Students Using Relational Thinking?

This section outlines the student responses that represent relational thinking given in response to the open number equivalence problems. Responses in this category indicated that the students were able to identify the relation between each side of the equal sign and use this to solve the problem. They were also able to use the correct direction of variation between the uncalculated equations on each side of the equal sign to solve the problem.

Year 5 student: 73 and 72 are 1 apart leaving 73 as the bigger number so I know that I need to make 72's partner 1 bigger than 49.

Year 5 student: 41 is two more than 39 so I have to take away 2 more to make the same answer.

Year 6 student: 48 is 5 more than 43. To make it fair the number in the box has to be 5 bigger than 76.

Year 7 student: If 99 is 9 more than 90, you would need 9 more than 59 to equal it out.

Year 8 student: Subtraction is different to addition. You have to add the 2 on to the first number, you also have to add it on the second to get the same answer 39 + 2 = 41 so you have to add two on to the other number 15 + 2 = 17.

What Were the Common Errors Students Made When Solving the Open Number Equivalence Problems?

A range of student errors were identified. Many of these errors were due to miscalculations as the students attempted to solve the problems using computation. A significant number of errors was also made in group B equivalence problems that involved subtraction. These errors were due to students failing to identify the correct direction of variation between the uncalculated equations. The following examples illustrate that the students have not identified the correct direction of variation in the uncalculated subtraction equations.

> Year 5 student: There is 9 between the two numbers so 99's partner needs to be 9 less than 59.
>
> Year 6 student: 46 is 1 bigger than 45 so I minused the 1 from 104 to get 103.
>
> Year 7: I did 13 because it is two less than 15 so 39 – 15 and 41 – 13 would have to have the same answer.

Examination of the data revealed that when the blank space was in specific positions the student responses indicated a lack of understanding of the equal sign. Predominantly across all year levels the students displayed an error identified by Freiman and Lee (2004) that they termed "complete the sum". This error occurred when the blank was in position A or B of an equation such as A + B = C + D. This error suggests that these students viewed the number on the right of the equal sign as providing the answer. The data in Table 3 shows the percentage of students at each year level demonstrating this error in their response. Responses showing this error remained consistent across students from Year 5 to Year 7 but decreased at Year 8 level.

Table 3
Percentage of Student Responses Which Were Classified as the "Complete the Sum" Error

	Year 5	Year 6	Year 7	Year 8
43 + **5** = 48 + 76	16	19	15	3
43 + **33** = 48 + 76				
7 + 17 = 15 + 24	3	6	7	0
99 – **9** = 90 – 59	14	17	14	3

Freiman and Lee (2004) identified a common student error they termed "direct sum". As illustrated in the data when the blank space was in position C or D the students treated the equivalence problem as a direct sum. In this case they ignored the other number and put the answer to A plus B in the blank space. This error suggests that these students view the equal sign as an indication to write the answer. The data in Table 4 reveal the percent of students making this error decreased slightly over the year levels. However it should be noted that this was the most common error still occurring in the Year 8 students' responses.

Table 4
Percentage of Student Responses Which Were Classified as the "Direct Sum" Error

	Year 5	Year 6	Year 7	Year 8
23 + 15 = 26 + **<u>38</u>**	0	2	7	1
104 – 45 = **<u>59</u>** – 46	3	11	11	5
39 – 15 = 41 – **<u>24</u>**	8	6	5	5

Students also made errors of adding all the numbers in the equivalence problem and putting their sum in the blank space. Freiman and Lee (2004) label this error as "sum of all terms". This error indicates students have over-generalized the property of addition and have ignored the importance of the sequence of symbols in the problem. The data in Table 5 displays the percentage of students at each year level showing this error in their responses. This error was less commonly made by students at the higher year levels.

Table 5
Percentage of Student Responses Which Were Classified as the "Sum of all Terms" Error

	Year 5	Year 6	Year 7	Year 8
73 + 49 = 72 + **<u>194</u>**	8	6	1	1
23 + 15 = 26 + **<u>64</u>**	8	6	3	2
104 – 45 = **<u>13</u>** – 46	3	4	7	3

Conclusion and Implications

This study sought to examine students' use of strategies to solve open number equivalence problems. The results indicate that relatively few students made consistent use of relational strategies across the different sets of equivalence problems. In contrast, many students consistently used arithmetic strategies across all the sets of equivalence problems. However the results also showed that the number of students using only arithmetic strategies decreased across the year levels with more students making some use of relational strategies in combination with arithmetic strategies.

All students within this study had been involved in a mathematics program that focused on strengthening their use of efficient computational strategies for the past three years. However, although emphasis had been placed on developing a flexible range of strategies, many students demonstrated an inability or disinclination to use relational thinking to solve the equivalence problems. These results highlight a need to balance teaching of computational strategies with explicit attention to the fundamental concepts of algebraic reasoning such as relational thinking.

Examination of common student errors when solving the equivalence problems also highlighted some students' lack of understanding of the equal sign. Errors the students made reflected their view of the equal sign as an indication to carry out an operation. Although many of these errors occurred more frequently in the earlier year levels, the frequency of these errors occurring in Year 7 is of some concern, as is the persistence of the "direct sum" error in Year 8. These results support other researchers' contention that greater attention needs to be paid to developing students' understanding of the equal sign through primary and middle school (Carpenter et al., 2003; Falkner et al., 1999; Knuth et al., 2006).

Implications of this study would suggest that an emphasis on increasing numerical reasoning is not adequate to develop deep powerful understandings of essential algebraic concepts. To develop students' algebraic reasoning, explicit attention needs to be given to developing relational forms of thinking. This also requires focus on developing students' notions of the equal sign as representing relational equivalence.

References

Carpenter, T., Franke, M., & Levi, L. (2003). *Thinking mathematically: Integrating arithmetic and algebra in elementary school*. Portsmouth: Heinemann.

Carpenter, T., Levi, L., Berman P., & Pligge, M. (2005). Developing algebraic reasoning in the elementary school. In T. Romberg, T. Carpenter, & F. Dremock (Eds.), *Understanding mathematics and science matters* (pp. 81-98). Mahwah, NJ: Lawrence Erlbaum.

Falkner, K., Levi, L., & Carpenter, T. (1999). Children's understanding of equality: A foundation for algebra. *Teaching Children Mathematics, 6*(4), 232-236.

Freiman, V., & Lee, L. (2004). Tracking primary students' understanding of the equality sign. In M. Hoines, & A. Fuglestad (Eds.), *Proceedings of the 28th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 415-422). Bergen: PME.

Hannah, K. (2006). Foreword. In R. Harvey, J. Higgins, T. Maguire, A. Neill, A. Tagg, & G. Thomas (Eds.), *Evaluations of the 2005 secondary numeracy pilot project and the CAS pilot project* (pp. 1-4). Wellington: Learning Media.

Irwin, K., & Britt, M. (2003). Algebraic thinking in the Numeracy project: Year two of a three-year study. In F. Ell, J. Higgins, K. Irwin, G. Thomas, T. Trinick, & J. Young-Loveridge (Eds.), *Findings from the New Zealand Numeracy development project* (pp. 46-54). Wellington: Learning Media.

Kaput, J. (1999). Teaching and learning a new algebra. In E. Fennema & T. Romberg (Eds.), *Mathematics classrooms that promote understanding* (pp. 133-155). Mahwah, NJ: Lawrence Erlbaum.

Kieran, C. (1981). Concepts associated with the equality symbol. *Educational Studies in Mathematics*, *12*, 317-326.

Knuth, E., Stephens, A., McNeil, N., & Alibabi, M. (2006). Does understanding the equal sign matter? Evidence from solving equations. *Journal for Research in Mathematics Education, 37*(4), 297-312.

McNeil, N., & Alibabi, M. (2005). Knowledge change as a function of mathematics experience: All contexts are not created equal. *Journal of Cognition and Development, 6*(2), 285-306.

Ministry of Education. (2004). *Book 1: The Number Framework*. Wellington: Learning Media.

Ministry of Education. (2006). *The New Zealand curriculum: Draft for consultation 2006*. Wellington: Learning Media.

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics*. Reston, VA: Author.

Rivera, F. (2006). Changing the face of arithmetic: Teaching children algebra. *Teaching Children Mathematics, 12*(6), 306-311.

Stephens, M. (2006). Describing and exploring the power of relational thinking. In P. Grootenboer, R. Zevenbergen, & M. Chinnappan (Eds.), *Identities, cultures and learning spaces* (Proceedings of the 29th annual conference of the Mathematics Education Research Group of Australasia, pp. 479-486). Sydney: MERGA.

Warren, E. (2003). Number combinations and arithmetic structure: Implications for early algebra. In L. Bragg, C. Campbell, G. Herbert, and J. Mousley (Eds.), *Mathematics education research: Innovation, networking, opportunity* (Proceedings of the 26th annual conference of the Mathematics Education Research Group of Australasia, pp. 712-719). Sydney: MERGA.

Warren, E., & Cooper, T. (2005). Young children's ability to use the balance strategy to solve for unknowns. *Mathematics Education Research Journal, 17*(1), 58-72.

Warren, E., & Pierce, R. (2004). Learning and teaching algebra. In B. Perry, G. Anthony, & C. Diezmann (Eds.), *Research in mathematics education in Australasia 2000-2003* (pp. 291-312). Teneriffe, Qld: Post Pressed Publishers.

Scaffolding Small Group Interactions

Roberta Hunter
Massey University
<r.hunter@massey.ac.nz>

In the current reform of mathematics classrooms teachers are required to develop discourse communities in which all students have equitable opportunities to engage in productive discourse. The challenge is for teachers to engage students in the mathematics talk across a range of classroom situations. In this paper I address how a teacher used interactional strategies to scaffold participation of her diverse students in small group interactions. I report on the actions the teacher took to shift the patterns of discourse from a disputational form to one in which the students collectively constructed group explanations and justification.

Over recent years significant changes have occurred in how mathematics classrooms are conceptualised as best able to meet the needs of students in the 21st century. An important hallmark of the changes is a vision of students actively engaged in mathematical discourse within classrooms that resemble learning communities (Manouchehri & St John, 2006). In New Zealand, the responsibility falls on teachers to design "learning environments that foster learning conversations and learning partnerships, and where challenges, support, and feedback are readily available" (Ministry of Education, 2006, p. 24). Similarly, the National Council of Teachers of Mathematics (2000) charges teachers with the responsibility to "establish and nurture an environment conducive to learning mathematics through … the conversations they orchestrate" (p. 18). To achieve such learning communities teachers are required to establish ways in which students can engage in multiple forms of interaction. These include whole class discussions and also small co-operative problem solving groups. But, although the use of small interactive groups is promoted in recent New Zealand policy document (Ministry of Education, 2006a) no guidance for teachers is provided for how these should be established (Irwin & Woodward, 2006). Although there is considerable research available that describes the learning that occurs within small groups and the factors that influence the mathematical learning, there appear to be limited studies that have explored the teacher's role in establishing and maintaining effective small co-operative groups. Therefore, the purpose of this paper is to outline how two teachers created and maintained effective small interactive mathematics groups. The focus of the paper is on the interactional strategies the teacher used and how these resulted in the students engaging socially and cognitively with each others' thinking.

The potential for positive social and cognitive outcomes of working in small groups has been widely recognised (e.g., Blunk, 1998; Mercer & Wegerif, 1999a; Yackel, Cobb, & Wood, 1991). Proponents of collaborative grouping maintain that through providing the individual students with opportunities to articulate their thinking not only do they learn to exchange mathematical ideas – but also they make available their reasoning for examination and critique (Artzt & Yaloz-Femia, 1999; Rojas-Drummond & Zapata, 2004). In addition, through opportunities to explain and justify reasoning, explainers are able to review and reconstruct their mathematical thinking, and extend and build stronger arguments (Whitenack & Yackel, 2002). Other advocates who support teacher use of small groups propose that this structure better meets the needs of the diverse or at-risk students (Baxter, Woodward, Voorhies, & Wong, 2002; Boaler, 2006; Rojas-Drummond & Zapata, 2004; White, 2003). These researchers illustrate that through small group interactions, these students are provided with opportunities to participate in and contribute to productive mathematical discourse without being in the public eye. Within the small supportive groups it is the peers who provide an important forum for the diverse students to develop and extend their mathematical reasoning. In turn, through listening and making sense of their peers' explanations they are able to integrate their reasoning with that of others. Moreover, their peers serve as important models for how they are to recognise and make sense of task demands, make conjectures, and extend their mathematical explanations and justification.

In contrast however, other studies have shown problems that may occur when small group organisation is used by teachers. These relate specifically to the enacted communication patterns and how different members of the group are positioned both socially and academically. For example, Barnes (2005) illustrated how cognitive development of specific individuals was limited by both the communication patterns and social relationships in the small group activity. She reported how specific students within the group were attributed lower status and therefore actively positioned by the others as "outsiders". As a result, their contributions were both interrupted and ignored as irrelevant to progress collective understanding. Likewise, Irwin and Woodward (2006) in a New Zealand Numeracy Project classroom noted the way in which the communication and social relationship patterns limited the collective reasoning of the group. The teacher consistently modelled inquiry discourse patterns when working with the whole class. However, Irwin and Woodward's close examination of groups working independently revealed a predominant use of competitive talk both student to student, and between the boys and girls. Although the teacher had directed them to work cooperatively in these groups she had provided no specific guidance. Similarly, the extensive studies of Mercer and his colleagues (e.g., Mercer & Wegerif, 1999a, 1999b; Rojas-Drummond & Mercer, 2003; Rojas-Drummond & Zapata, 2004) illustrate that without teacher guidance student talk is often of a disputational or cumulative form. In disputational talk the students rather than trying to reach joint agreement work through cyclic assertions and counter-assertions as they struggle for control and status. In the cumulative form a collective view is reached but without evaluative discussion.

Therefore, if students are to engage in productive small group activity teachers need to scaffold specific interactional strategies that support equitable outcomes for all participants. To do this Mercer (2000) promotes the use of a specific programme for teachers to use which he terms "talk lessons". Mercer and his colleagues in a range of studies illustrated how teachers implementing "talk lessons" utilise a number of interactional strategies. These are used to scaffold student participation in mutual inquiry and exploration of the reasoning used by the group members. The teachers use a set of ground rules that emphasise sharing of information, a need for group agreement and responsibility for decisions. But the ground rules also focus on challenge and justification of the collective reasoning. Similarly, Alrø and Skovmose (2002) describe teacher use of an interactional structure they term an "inquiry co-operation model", which aims to engage students in mutual inquiry of open-ended problems. Descriptions of studies that have used this model focus on how the teachers specifically scaffold active listening and identification of varying perspectives of the participants. However, when the reasoning is clarified, it is then subjected to challenge and debate before a collective view is accepted.

Boaler (2006) extends the thinking related to how teachers use interactional strategies to scaffold productive discourse in small groups to include ways teachers have used these with diverse learners. In her research Boaler (2006) examined how teachers used an approach she terms complex instruction. In this approach she outlines the use of heterogeneous grouping and open-ended problems to draw multiple ways to value student contribution. She includes as important group roles for students and responsibility for each others' learning. Within the notion of group responsibility Boaler illustrates the importance of justification and reasoning and the way in which the "teachers carefully prioritised the message that each student had two important responsibilities – both to help someone who asked for help, but also to ask if they needed help" (p. 6). In this model the importance of teacher's high expectations, their affirming effort over ability and their assigning competence is emphasised. Competence is assigned when teachers raise the status of students through public recognition of the intellectual value of their reasoning. Boaler also showed how the diverse students learnt valued learning practices through the teachers explicitly noting which specific actions best supported their learning.

The theoretical framework of this study is derived from a sociocultural perspective. From this perspective mathematical teaching and learning are inherently social and embedded in active participation in communicative reasoning processes (Lerman, 2001). In this environment, students successively gain increased levels of "legitimate peripheral participation" (Lave & Wenger, 1991, p. 53) as they access and participate in productive mathematical discourse.

Research Design

This research reports on one teacher case study from a study that involved four teachers in a one-year collaborative teaching experiment. The study was conducted at a New Zealand urban primary school where students came from predominantly low socio-economic home environments. Students were predominantly of Pacific Nations and New Zealand Maori ethnic groupings with many speaking English as their second language. Based on the results from the New Zealand Numeracy Project Assessment tool (Ministry of Education, 2004) members of the 8-, 9-, and 10-year-old group were achieving at significantly lower numeracy levels than comparable students of similar age grouping in New Zealand schools at the beginning of the study.

Collaborative teaching experiment design (Cobb, 2000) was used in order to direct teacher and researcher attention on the social process of the mathematical discourse, while retaining awareness of the mathematical product of the activity. In recognition of the two central characteristics of teaching experiment design research, the iterative cycles of analysis and an improved process or product, a tentative communication and participation trajectory was used to map the progression of the discourse toward inquiry and to provide focus for the subsequent shifts in participation and communication. For example, after Ava (pseudonym for the teacher) had completed teaching a unit of work that focused on number and before she taught a rational number unit, the types of questions Ava and the students could use and the patterns of interactions anticipated to scaffold a further shift toward inquiry and justification of reasoning were considered and mapped out.

Data collection over one year included three semi-formal teacher interviews, classroom artefacts, field notes, twice-weekly video-captured observations of lessons, diary notes of informal discussions during and after lesson observations, written and recorded teacher reflective statements and teacher recorded reflective analysis of video excerpts. The ongoing data collection and analysis maintained a focus on the developing mathematical discourse. This supported the iterative cycles and revision of the interactional strategies. Data analysis occurred chronologically using a grounded approach in which codes, categories, patterns, and themes were created. Through use of a constant comparative method, which involved interplay between the data and theory, trustworthiness was verified and refuted.

Results and Discussion

At the beginning of the study in line with the New Zealand Numeracy Project (Ministry of Education, 2006), Ava regularly used a small group format in which the students were required to construct explanations of their solution strategies. However, examination of the group interactions in the first lesson observations revealed that the students predominantly used either cumulative or disputational talk (Mercer, 2000). For example, a group of three students are solving a fraction problem.

Hinemaia: What I think is five is a quarter of ten.

Candice: Yeah. No but what about …

Helen: You put five in each paddock and then all the five because you have got two paddocks equal ten plus another five will equal ten and ten plus ten will equal twenty. We need a fraction.

Hinemaia: Oh maybe a ten is a half quarter of twenty. Now we need to think more in our mind.

Helen: Well me and you Hinemaia are thinking. All you are doing is sitting and saying yeah true. You are not doing any maths thinking [to Candice]

Candice: Well I am trying to …

Hinemaia: You have got to think there actually [points to her head].

In this discussion the erroneous reasoning was left unexamined. The third member of the group was positioned by the other two in such a way that she was not able to contribute to the discussion. They consistently interrupted or discounted her explanation or questioning. Then they attributed to her a lower social and academic status because they stated that she had not demonstrated "thinking".

Developing a Shared Perspective in Small Group Interactions

To change the interaction patterns, in the first instance Ava focused on how the students participated together in small group activity. In accord with the trajectory, she placed a focus on their need to engage actively in listening, discussing and making sense of the reasoning used by others. After the students had individual time to think about a solution strategy she directed them:

Ava: You are going to explain how you are going to work it out to your group. They are going to listen. I want you to think about and explain what steps you are doing, each step you are doing, what maths thinking you are using. The others in the group need to listen carefully and stop you and question any time or at any point where they can't track what you are saying.

Ava emphasised their responsibility to develop understanding of the reasoning from the perspective of each member of the group. She discussed the roles of members in the group and placed particular importance on the need for justification and reasoning to develop a collective view. For example, she observed the students as they worked together and then noting that some members of the group were accepting uncritically the explanations from other group members she instructed them:

Ava: Argue your maths. Explore what other people say. Listen carefully bit by bit and make sense of each bit. Don't just agree. Check it all out first. Ask a lot of questions. Make sure you can make sense that you understand. What's another important thing in working in a group?

Alan: Share your ideas. Don't just say I can do it myself that adds on to teamwork.

Ava: That's right. We do need to use each other's thinking … because we are very supportive and that's the only way everyone will learn. So we have to be discussing, talking, questioning, and asking for clarification. Whatever it takes to clarify what you understand in your mind.

Thus, Ava had emphasised that they were required to understand the reasoning from the perspective of others. In addition, she had outlined their need to question and she had reminded them of their responsibility to respond and clarify their reasoning when questioned by other group members.

To further develop group consensus of their reasoning Ava introduced the use of only one pen and one piece of paper in each group. She also required that every member of the small group could explain to her or to a larger sharing group the collective explanations. This was illustrated when Ava instructed a group before they began work:

Ava: Together you need to know what you're … saying and what you are doing. You may need to use your fractions pieces and lots of different ways to make it make sense to all of you in the group … When it comes to the sharing time you need to be able to explain and justify what you are saying in lots of different ways. We are all going to need to be able to see what you are saying, see your reasons behind your explanations. I am going to ask anybody in the group to explain. So you have to make sure that everybody in the group can explain anything you are asked.

The group explored three different solution strategies and then they discuss which one to provide to the larger group.

Rachel: About this one, it's a bit hard to understand because it was so fast.

Tipani: Okay. The truth is this is the most efficient way. That's a good way. That's a good way. But that's the most efficient.

Rachel: Yeah but that one is the most efficient because it's easier to understand. This is more confusing even if it is the fastest. So let's go with the one we know everyone will understand.

In their discussion they illustrated that they recognised that their responsibility to make their reasoning clear extended to a wider audience. They knew that they needed to consider how their explanation would be understood from the perspective of the listeners.

Ava was aware that different students had different status in her class. Although she focused on their need to consider the reasoning used by all the participants in the group she also actively positioned specific students. For example, after she had observed a shy

Pasifika student making an explanation to the small group she began the large group sharing by asking:

Ava: Aporo do you mind if we kick off with you because you were doing some really good talking and explaining to your group and I think this will be a really good opportunity for you to show your maths thinking.

When Aporo began his explanation in a quiet voice Ava requested that the other students listen closely. Then when another student began to prompt him and he hesitated she told the student:

Ava: He knows. He knows. You don't have to prompt him because he knows where his thinking is going.

As this point Aporo became more confident and completed his explanation using a louder voice and making notated recordings to illustrate his reasoning further. Through her actions and her direct focus on the intellectual value of Aporo's reasoning, Ava had shifted Aporo's social status within the group. She had positioned him so that he had a voice and confidence to use it.

Learning Ways to Disagree and Challenge Politely

Engaging in questioning and inquiry involved considerable challenge to how many of these diverse students had experienced mathematics previously. Therefore, in accord with the trajectory Ava introduced the use of open-ended tasks and problems. These supported the notion that there were multiple ways the students in their small groups could develop and support each other in the construction of explanatory reasoning and justification. Ava explicitly directed their attention to the many different roles the individuals in the group could take in developing the collective reasoning. She affirmed those students who preferred to begin by using concrete materials and drawings. She emphasised that these actions were part of the different ways all the members contributed to group activity. She also often stopped groups shortly after they had begun working together and discussed with them the different ways they had selected to approach the problems. She would explore with them where their reasoning had begun and what actions and ideas the different group members were working with. Alternatively, she would join a group and listen closely and then question a group member quietly:

Ava: So how are you going Ruru? How are you going with your thinking?

Ruru: I am trying to explain it to them.

Ava: You are trying to explain it. Are they listening?

Hinemoa: No. He just said he already knows that they have eaten the same.

Ava: That's all right. He has started you thinking. Now you need to listen to him. He needs to explain step by step.

Ruru: I don't know yet.

In response, Ava affirmed the role he had played in beginning the development of a group solution strategy.

Ava: That's fine. You have started the thinking. Now other people in the group may have other ways of thinking and explaining.

Hinemoa: I think he is wrong because if they both ate the same. But I am not sure. He said they both ate the same but there's only five. There's two fifths there and you have to cut it in half but you can't cut it in half if you have only five.

Aroha: Yes you could if you actually had a half, if you halved the piece.

Ava, listening to the students' discussion, realised that they were engaging with the thinking Ruru began. She then advanced their reasoning by suggesting the use of an alternative means to clarify their ideas.

Ava: What about drawing what you mean?

Aroha: You could go like that. So that halved that piece in the middle so it would be equal [Draws a cakes and then uses her hand to show a half of a shape and a half again].

Ava's actions in the group had shown them that she valued the multiple ways the group members contributed to the group discussion. The students were learning what Boaler (2006) terms multidimensionality, which highlights that "when there are many ways to be successful, many more students are successful" (p. 3). These students were learning that every contribution they made in their groups provided a valid basis for open discussion and a way to progress the group reasoning.

Ava recognised the social and academic risks students took when they disagreed or challenged the reasoning of others. Therefore she carefully structured ways in which the students in their small groups could approach disagreement and challenge. She would watch the students working together in their small groups and then she would ask specific members if they agreed or disagreed with the reasoning being used. She also consistently required that they provide justification for the specific stance they took. As the groups worked together she reminded them:

Ava: Please feel free to say if you do not agree with what someone else has said. You can say that as long as you say it in an okay sort of way. If you don't agree then a suggestion could be that you might say I don't actually agree with you. Could you show that to me? Could you perhaps write it in numbers? Could you draw something to show that idea to me? That's fine because sometimes when you go over and you do that again you think…oh maybe that wasn't quite right and that's fine. That's okay.

Ava would also place herself as a participant in small group activity and model behaviour that tuned the students into becoming more aware of other participants responses revealed in their body language. She would actively prompt and probe for agreement or disagreement when she noted a frown on participants' faces or a querying shift in their bodies. Her active prompts to voice agreement and disagreement were appropriated by the students when they worked independently. They would explain a solution strategy step by step, watching the other group members carefully. When they saw a hesitant or querying look on a peer's face the explainer would halt the explanation and respond by asking:

Rachel: Tama you look confused? Do you need to ask some questions?

Tama: Well three times three? Isn't it three plus three plus three not the times way?

As a result the students took ownership of their reasoning and they recognised their collective responsibility to ensure that it was understood by all group members. Justification and reasoning had become key components of the collaborative interactional strategies the groups used.

Learning the Practices of Mathematics

Ava consistently interacted with the students, exploring and discussing with them interactions that supported them learning the practices of mathematics. When she heard a student persistently questioning another group member's reasoning she stopped the group and told them:

> Ava: One thing I will say about you Jo you are never scared to question. It makes other people start to question what their own thinking is.

Her description affirmed that a sound learning practice was to question until sense-making was achieved. At another time she stopped the groups to focus attention on the way in which a student had persistently worked at a problem.

> Ava: Did you see that? Rona has been working this way and that way. She went down one path and then down another and she never gave up. That's how you learn, thinking and rethinking, starting and starting again and that's okay, that's how you learn.

Ava had used Rona as a model to illustrate to the students that both persistence and effort were valued attributes in mathematics.

Ava wanted the students to examine the reasoning used by the members of their small group closely. In the first instance she would halt a group when she heard a students ask a question that clarified or challenged the reasoning. Or she would ask the students to formulate questions they could ask each other when they approached her for support. However, she knew that they required more scaffolds than her directives to them to question and challenge. Therefore, using the trajectory as a guide, at regular intervals during the year she introduced a different set of questions and prompts. She began with a set of questions that the students could use to elicit more information about mathematical explanations. They included such questions as "what", "where", "is that", "can you show us", "explain what you did". When the students were using these ably she introduced a range of questions that challenged and drew justification of the reasoning other group members used. The questions included "but how do you know it works", "why", "how", "convince us", "so what happens if", "are you sure". The final set of questions she introduced, were designed to draw generalisations. They included "so why is it", "does it always work", "does it work for all numbers", "is it always true", "why does that happen", "is there a different way". She actively modelled the use of these questions and prompted the students to use them as she participated in their small groups. She also displayed them on charts on the wall. When she heard a student use a different form of one of the questions she would halt the group and draw their attention to the question and how it was being used. Then she would add it to the wall chart.

Conclusions and Implications

Within the teaching design experiment the communication and participation trajectory was used successively to review and map out the interactional strategies Ava used to scaffold the students in small group interactions. Over the year, Ava implemented a wide range of interactional strategies that focused the students' attention on the development of a collective view. Many of the interactional strategies that Ava emphasised matched those described by Boaler (2006). These included the importance of open-ended problems and tasks that supported a range of ways to contribute to the group processes. However, of key importance in the development of productive group processes and discourse in Ava's classroom was the emphasis she placed on group responsibility to each other. As Boaler (2006) described, central to the group responsibility was the requirement for the students to justify and provide valid reasoning for their solution strategies.

The observations of group processes at the start of the study confirmed what Mercer (2000) and his colleagues describe. The students encountered many difficulties when asked to participate in small groups. As Mercer describes, the students predominantly used unproductive talk and poor social behaviour. Ava employed specific strategies to position her diverse students. She scaffolded them to take a stance and agree or disagree with the reasoning and she also ensured that they were viewed as academically competent. Her actions are similar to those described by other researchers including White (2003) and Boaler (2006).

The findings of this research reveal that the consistent attention Ava directed toward developing different forms of questioning scaffolded the students' skills to examine and analyse the reasoning group members used. Although she did not use specific programmes like those described by Mercer (2000) or Alrø and Skovsmose (2002), her carefully considered scaffolding of student interactions and questioning paralleled their work.

Effecting change in the small group interactions was a lengthy process. It required on-going attention by Ava of the discourse used in the groups. It also required her active participation as a model of the interaction patterns in the group and her highlighting student behaviour to demonstrate valued interaction patterns. Further research is needed to examine other factors that are important in enacting and maintaining diverse learners' use of productive discourse.

References

Alrø, H., & Skovsmose, O. (2002). *Dialogue and learning in mathematics education: Intention, reflection, critique.* Dordrecht, The Netherlands: Kluwer Academic Publishers.

Artzt, A. F., & Yaloz-Femia, S. (1999). Mathematical reasoning during small-group problem solving. In L. V. Stiff & F. R. Curcio (Eds.), *Developing mathematical reasoning K-12: 1999 yearbook* (pp. 115-126). Reston, VA: National Council of Teachers of Mathematics

Barnes, M. (2005). 'Outsiders' in a collaborative learning classroom. In M. Goos, C. Kanes, & R. Brown (Eds.), *Mathematics education and society* (Proceedings of the 4th International Mathematics Education and Society conference, pp. 58-68). Brisbane: Griffith University.

Baxter, J., Woodward, J., Voorhies, & Wong, J. (2002). We talk about it, but do they get it? *Learning Disabilities Research and Practice, 17*(3), 173-185.

Blunk, M. L. (1998). Teacher talk about how to talk in small groups. In M. Lampert, & M. L. Blunk (Eds), *Talking mathematics in school* (pp. 190-213). Cambridge: Cambridge University Press.

Boaler, J. (2006). "Opening our ideas": How a detracked mathematics approach promoted respect, responsibility, and high achievement. *Theory into Practice, 45*(1), 1-11.

Cobb, P. (2000). Conducting teaching experiments in collaboration with teachers. In A. Kelly, & R. Lesh (Eds), *Handbook of Research Design in Mathematics and Science* (pp. 307-333). Mahwah, NJ: Lawrence Erlbaum Associates.

Irwin, K., & Woodward, J. (2006). Advancing Pasifika students' mathematical thinking. In F.Ell, J. Higgins, K. C. Irwin, G. Thomas, T. Trinick, & J. Young-Loveridge (Eds.), *Findings from the New Zealand numeracy development projects* (pp. 80-90). Wellington: Ministry of Education.

Lave, J., & Wenger, E. (1991). *Situated learning: Legitimate peripheral participation.* New York: Cambridge University Press.

Lerman, S. (2001). Cultural discursive psychology: A sociocultural approach to studying the teaching and learning of mathematics. *Educational Studies in Mathematics,* 46, 87-113.

Manouchehri, A., & St John, D. (2006). From classroom discussions to group discourse. *Mathematics Teacher, 99*(8), 544-552.

Mercer, N. (2000). *Words & Minds*. London: Routledge.

Mercer, N., & Wegerif, R. (1999a). Is “exploratory talk” productive talk? In K. Littleton & P. Light (Eds.), *Learning with computers* (pp. 79-101). London: Routledge.

Mercer, N., & Wegerif, R. (1999b). Children's talk and the development of reasoning in the classroom. *British Educational Research Journal, 25*(1), 95-112.

Ministry of Education. (2004), *Book 2: The diagnostic interview.* Wellington: Ministry of Education.

Ministry of Education. (2006), *Book 3: Getting started*. Wellington: Ministry of Education.

Ministry of Education, (2006a). *The New Zealand curriculum draft for consultation 2006*. Wellington: Learning Media.

National Council of Teachers of Mathematics, (2000). *Principles and Standards for School Mathematics.* Reston, VA: Author.

Rojas-Drummond, S., & Mercer, N. (2003). Scaffolding the development of effective collaboration and learning. *International Journal of Educational Research, 39(1-2)*, 99-111.

Rojas-Drummond, S., & Zapata, M. (2004). Exploratory talk, argumentation and reasoning in Mexican primary school children. *Language and Education, 18(6)*, 539-557.

White, D. Y. (2003). Promoting productive mathematical classroom discourse with diverse students. *The Journal of Mathematical Behaviour, 22*(1), 37-53.

Whitenack, J., & Yackel, E. (2002). Making mathematical arguments in primary grades: the importance of explaining and justifying ideas. *Teaching Children Mathematics, 8(*9), 524-528.

Yackel, E., Cobb, P., & Wood, T. (1991). Small-group interactions as a source of learning opportunities in second-grade mathematics. *Journal for Research in Mathematics Education, 22*(5), 390-408.

Numeracy in Action: Students Connecting Mathematical Knowledge to a Range of Contexts

Chris Hurst
Curtin University of Technology
<C.Hurst@curtin.edu.au>

This qualitative multiple case study involved eight Year 6 and 7 students and six classes and investigated their capacity to recognise, apply, and question the use of mathematical ideas embedded in a range of contexts. It also considered the extent to which students' capacity to connect mathematical knowledge to other contexts could motivate them to learn mathematics. In particular, it investigated the effect of the *Mathematical Search* strategy in achieving these ends. It found that student thinking about mathematics and their attitudes towards it could be enhanced by targeting mathematical connections through the use of the *Mathematical Search*.

In recent times, much has been written about numeracy. One common aspect in most definitions of numeracy is the disposition and ability to apply widely one's mathematical knowledge. In her discussions of the notion of statistical literacy, a related concept, Watson (1995; 2004) was concerned not only with the quantity of statistical information that continually bombarded the general population, but also that many people ignored it, misunderstood it, or did not bother to check if its associated claims were valid. In a similar vein, Peter-Koop (2004) found that, when working with worded problems, primary school students often failed to identify the key mathematical ideas involved and tended to randomly apply numbers contained in the text of such problems to arbitrarily chosen mathematical operations. Both of these ideas are encapsulated in Perso's (2006) statement that "since numeracy involves both the mathematics you know and the disposition to use it, teaching must focus on both of these" (p. 25).

The inference for teachers is clear – it is not only necessary to teach the mathematical content but also important to provide students with strategies for recognising and applying mathematics in a range of contexts. Therefore, a main research issue addressed by this study is the investigation of the effectiveness of teaching and learning strategies in helping students to connect their mathematical knowledge to various contexts and situations.

The overall study (Hurst, 2006), on which this paper is based, investigated three ideas:

- The effectiveness of strategies like the *Mathematical Search* in enhancing student ability to recognise mathematics in context.
- The extent to which such strategies enhance student motivation towards mathematics.
- The value that teachers see in using such strategies to enhance student thinking and motivation.

This paper focuses on the first of the above issues, as embodied in the following research question: To what extent does the *Mathematical Search* enhance student capacity to recognise mathematical ideas embedded in a written context, and to display contextual and strategic thinking about mathematical ideas embedded in written contexts?

Theoretical Framework

This study drew upon a wide range of research-based writing in developing the research questions and methodology and there were several key points that emerged. First, at the very heart of the numeracy debate, are the notions of situated cognition and transferability of learning. Boaler (1993) noted that traditional approaches to developing student numeracy were based on the assumption that "mathematics can be learned in school, embedded within any particular learning structures, and then lifted out of school to be applied to any situation in the real world" (p. 12). However, as Kemp and Hogan (2000) pointed out, "evidence suggests that students do not automatically use their mathematical knowledge in other areas" (p. 13). Indeed, if learning were freely transferred from the mathematics classroom to any of a number of outside situations, it is unlikely that the numeracy debate would have begun, or at least, reached the proportions it has.

Second, the idea of teaching "numeracy across the curriculum" emphasises that numeracy is more than mathematical knowledge and that students learn best when "the richness of a context helps them to make sense of mathematical ideas" (Willis, 1998, p. 8). This is closely allied to the previous point as students who tackle mathematics in restricted contexts will be likely to develop limited cognitive structures (Coles & Copeland, 2002). The importance of embedding mathematical learning in a range of contexts was underlined by Morony, Hogan, and Thornton (2004):

> Education must be about enabling people to understand and interact with the world. The skills, habits of mind and dispositions developed through effective attention to numeracy across the curriculum are clearly key components of understanding and interacting with the world. (p. 2)

The above ideas about numeracy are encapsulated in the *Numeracy Framework* developed by Willis and Hogan (Hogan, 2000; Morony et al., 2004; Willis, 1998). The framework incorporates three perspectives on numeracy, a blend of which was required for students to display intelligent mathematical action in context. The three types of knowledge are:

- Mathematical knowledge – the knowledge needed for intelligent mathematical action
- Contextual knowledge – the ability to link mathematics to experiences
- Strategic knowledge – the ability to ask questions about the application of particular mathematical knowledge

A Conceptual Framework – The Model for Teaching Numeracy in Context

The ideas related to numeracy outlined above, particularly the *Numeracy Framework* (Hogan, 2000; Willis, 1998), informed and were incorporated in the *Model for Teaching Numeracy in Context* (Figure 1) that became the conceptual framework for the study. This model was based on the notion that the different modes of thinking in the *Numeracy Framework*, that is mathematical, contextual and strategic thinking, could be developed by using the *Mathematical Search* and associated teaching and learning strategies. The *Mathematical Search* was devised by the researcher and was used on four occasions by the researcher during the course of the study. It was developed with the intent of ascertaining whether or not a specific strategy of that type could enhance the capacity of students to recognise and use mathematical ideas embedded in a variety of contexts. In the study, only written contexts were used. Students had not used the *Mathematical Search* prior to their involvement in the study.

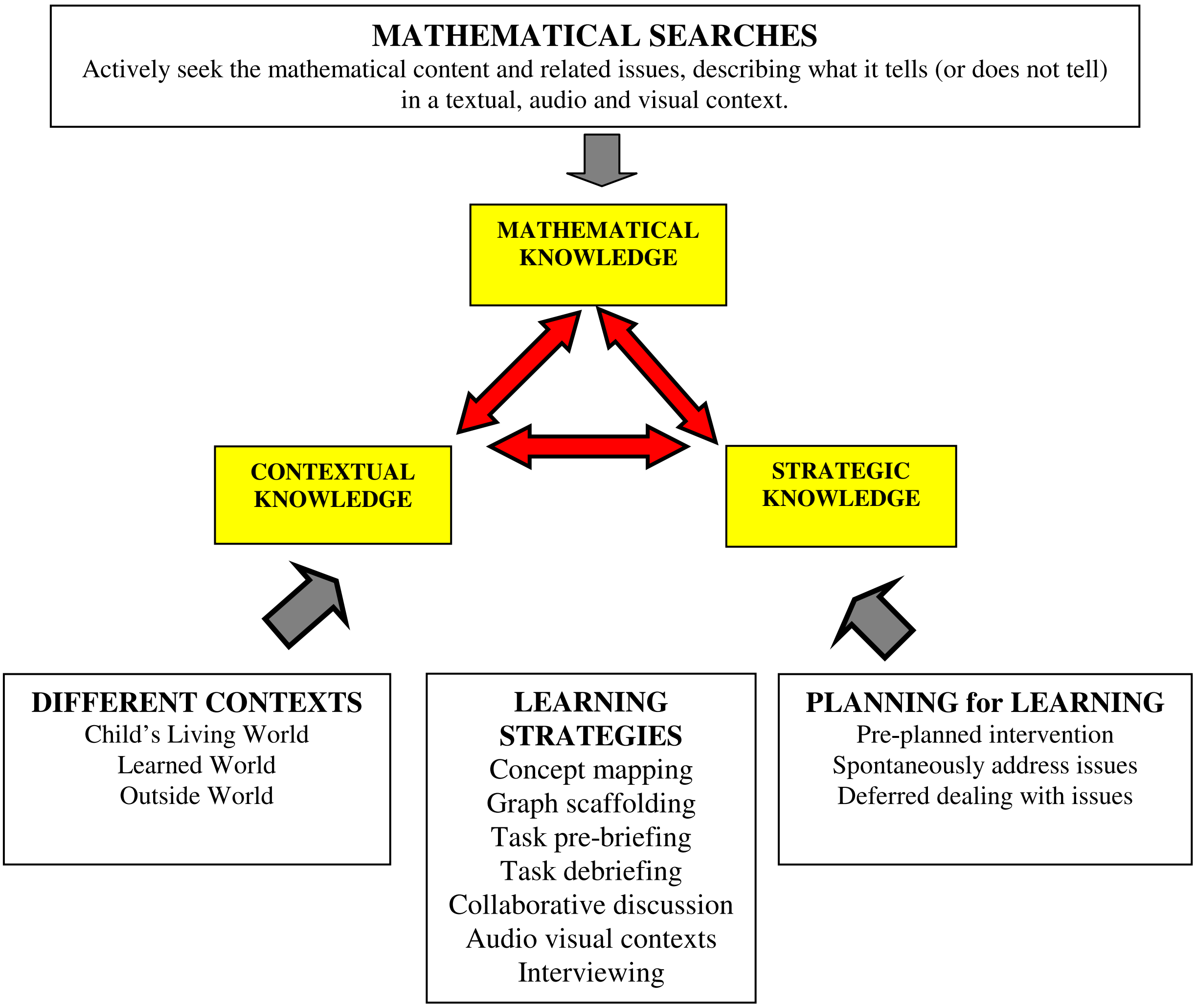


Figure 1. Conceptual framework (Model for Teaching Numeracy in Context).

In a *Mathematical Search,* students were given a body of text to read. These were based on themes and topics that were being taught in classes, such as Indigenous Australians, Gold Rushes, and Environmental Pests. Their task was to describe the mathematical ideas in the text and what the mathematics told about the main ideas in the text, and to use the mathematical ideas to explain some of the patterns, trends and any apparent inconsistencies in the text. The purpose of the *Mathematical Search* was to encourage students actively to seek mathematical concepts and facts embedded in any of a variety of contextual situations. In this study, students were also asked to pose questions about the text using the mathematical ideas described. The *Mathematical Search* was supported by other teaching and learning strategies such as concept mapping, graph scaffolding, debriefing discussions following a *Mathematical Search*, and one-to-one interviewing.

Design and Methodology

In order to generate the rich data required, the study made use of qualitative methods, specifically, a multiple case study approach. This involved a group of eight female Western Australian primary school students, aged 11 or 12 years, in six Years 6 and 7 classes.

Frankel and Wallen (2003) and Yin (2003) noted that evidence from multiple case studies was generally more convincing compared to that from a single case study and could lead to useful and valid generalisations.

Over a period of 6 months, evidence gathered from the multiple case study was supported by evidence from a *General Sample* of students, this consisting of the remaining students in the six classes from which the case study students were drawn. In order to ensure the validity of the data, and increase the possibility of making reasonable generalisations from the results of the study, data triangulation was achieved using multiple sources of evidence, as shown in Table 1.

Table 1
Data Collection Instruments Used During the Study

Instrument	Purpose
Pre-Project Student Survey	Identify students who display a positive attitude towards mathematics, for possible selection in the case study group.
Pre-Project Benchmark Task in Mathematics	Provide a benchmark for later comparison in order to have a basis for assessing changes in student thinking.
Western Australian Literacy and Numeracy Assessment (WALNA) data for each student	Enable selection of students to be confirmed on the basis of a combination of researcher-generated criteria (Student Survey, Pre-Project Interview) and standardised testing.
Pre-Project Teacher Interview	Provide an understanding of the level of experience, commitment to numeracy teaching, teaching style and general philosophy of project teachers. Act as a reference point for later comparisons after implementation of project tasks.
Pre-Project Student Interview	Provide an understanding of current student thinking about mathematics in context, the importance of mathematics and how success in mathematics is judged. Act as a reference point for later comparisons after implementation of project tasks.
Project tasks *Mathematical Searches* (four) and other tasks	Provide students with opportunities to identify, discuss meanings of, and apply mathematical knowledge in a variety of contexts. Generate work samples to serve as indicators of student thinking and progress.
Researcher's Reflective Journal and Anecdotal Notes	Record details of observations made during classroom visits to administer project tasks. These visits occurred at least monthly over a six month period.
Teacher Progress Interviews	Provide anecdotal information about case study students from the perspective of the class teacher.
Post-Project Benchmark Task in Mathematics	Provide a benchmark for comparison with Pre-Project Benchmark Task in order to have a basis for assessing changes in thinking.
Post-Project Teacher Interview	Act as a reference point for comparisons with earlier interview after implementation of project tasks. Ascertain extent of changes to teacher thinking about the value of the project tasks.
Post-Project Student Interview	Act as a reference point for comparisons with earlier interview after implementation of project tasks. Ascertain extent of changes to student thinking about the value of the project tasks, mathematical learning in context, importance of aspects of mathematics, and how mathematical ability is recognised.

The interviews with students were in part "task-based" in that students were given samples of articles, maps, and advertisements, about which they were asked questions to probe the development of their thinking. The benchmark tasks were based on tabular information and students were to identify key ideas that the information showed and also give possible explanations for the variations in that information. Benchmark Task 1 contained a table of information based on school fund raising and Benchmark Task 2 was about a school traffic counting activity. One associated strategy used in the study was task debriefing. This followed each *Mathematical Search* and consisted of whole class discussions in which the possible responses and thinking were modelled by the researcher.

During the initial phase of data analysis, interview transcripts, work samples, and field notes were analysed and some thirteen empirical assertions were developed from the data. An empirical assertion could be described as a contention, statement, declaration or claim that something in particular is likely to occur, based on the contender's observations and experiences (Erickson, 1986). Two of the empirical assertions generated from Research Question 1 that are discussed in this paper are contained in Table 2.

Table 2
Empirical Assertions Generated from Research Question 1

1. Students will display an improved capacity to recognise mathematical ideas in a written context, and to use contextual and strategic thinking when considering mathematical ideas embedded within a written context, having used the *Mathematical Search* strategy on several occasions.
2. Students will display a greater capacity to recognise mathematical ideas embedded in a written context, and to use contextual and strategic thinking when considering mathematical ideas embedded in a particular written context, when they are personally interested in that context.

It was important not to set the boundaries of the research too wide and therefore some potential variables were eliminated from the sample. For example, it was not intended to make wide-ranging comparisons involving gender, different year levels, different types of schools (such as Government, Catholic, or Independent), or other issues such as school policy, socio-economic characteristics of school intake areas, and student ethnicity. Consequently the sample for the study was restricted to Years 6 and 7 female students.

Findings and Discussion

The discussion that follows is based only on Research Question 1 and the two empirical assertions listed in Table 2.

Empirical Assertion 1

> Students will display an improved capacity to recognise mathematical ideas in a written context, and to use contextual and strategic thinking when considering mathematical ideas embedded within a written context, having used the *Mathematical Search* strategy on several occasions.

In attempting to warrant or reject Assertion 1, evidence from the multiple case studies is presented here. To begin with, the responses to Benchmark Tasks 1 and 2 are considered. It is apparent from a comparison of responses by the eight case study students to Benchmark Tasks 1 and 2 that gains were made in terms of the various modes of thinking, that is, mathematical, contextual, and strategic thinking. Mathematical thinking is characterised by the recognition, reiteration, and/or application of specific mathematical information to perform a mathematical operation. For example, a student working with an advertisement showing a price reduction and "new" price for a sale item might use the information to calculate the "normal" price of the item. Contextual thinking may involve the interpretation of data or the posing of questions that require such interpretation. For example, a student working with a similar advertisement to the above might consider a claim made in the advertisement that the product "whitens in fourteen days" and pose the question such as "Does the container last for fourteen days?" Strategic thinking may involve the synthesis of data to produce a new idea or the evaluation of data for consistency and the identification of anomalies. For example, a student working with an advertisement claiming that "Everything is reduced by 15%" might test the claim by comparing original and discount prices to see if the claim was accurate.

The basis on which "gains" are considered to have been made is whether or not a student has displayed modes of thinking that were not displayed earlier in the project. For example, a student displaying mathematical thinking on Benchmark Task 1 is deemed to have made "substantial gains" if, on Benchmark Task 2, he/she displayed contextual thinking, as well as mathematical thinking. A student is considered to have made "reasonable gains" if, for example, emerging contextual thinking on Benchmark Task 1 had developed into established contextual thinking on Benchmark Task 2. Similar criteria described "very substantial gains", "no gains", or "loss".

Five of the eight students made "substantial" or "reasonable" gains and three made "no gain". For each of the eight students, the quality and frequency of responses for Benchmark Task 2 were higher than for Benchmark Task 1. In Benchmark Task 1, students may have displayed emerging contextual thinking without applying mathematical ideas or they may have displayed genuine contextual thinking but only gave one example. For Benchmark Task 2, all students provided multiple responses incorporating mathematical ideas relevant to the context of the task. Responses by the student Tania were typical of those of the other seven students and are shown here in Table 3. It can be seen that Tania gave more responses and more detailed responses to the second task compared to the first. In addition, during the second task, she displayed strategic thinking that was not evident in her responses to the first task.

Table 3

Comparison of Responses by Tania for Benchmark Tasks 1 and 2

Benchmark 1	Benchmark 2
Contextual Thinking – Learner User	Contextual Thinking – Learner User
I think Year Seven raised the most because there are more children in that year.	Brett took 28 minutes and Craig took 12 minutes. Is this because Brett was on a road with a traffic jam or the speed limit was low, or made up some of the answers? Maybe they were at different times of the day or more populated cities.
	Strategic Thinking – Critical User (emerging)
	There must have been at least four emergencies because it shows four emergency vehicles on the chart. But that might not be true because it says at the top that they're all from different schools so they might not be in the same city or did it on a different day. Each time the sedan cars were the most seen. Maybe because they were the cheapest or the most useful?

The development represented in Table 3 was typical of the case study students. Where there was not a "reasonable" or "substantial" gain in modes of thinking, there was at least an increase in the quantity and variety of responses. Similar gains in modes of thinking were noted when case study student responses for the Pre-Project and Post-Project Interviews, and responses to the first and final *Mathematical Search*es, were compared. All case study students displayed both mathematical and contextual thinking during the first interview and all three modes of thinking, mathematical, contextual, and strategic, during the second interview. For six students, this represented a "substantial" gain, for one a "very substantial" gain, and for one, a "reasonable" gain. A summary is contained in Table 4 where "M" represents mathematical thinking, "C" represents contextual thinking, "S" represents strategic thinking, and (em) represents emerging thinking.

Table 4
Comparative Gains for Student Responses to Mathematical Searches 1 and 4, and Interviews 1 and 2, for the Case Study Students

Student	Mathematical Search #1 to #4	Gain	Interview #1 to Interview #2	Gain
Mary	M(em) to M/C	Very Substantial	M/C to M/C/S	Substantial
Sara	M(em) to M	Reasonable	M/C to M/C/S	Substantial
Jenny	M to M/C/S (em)	Very Substantial	M/C to M/C/S	Substantial
Tania	M to M/C	Substantial	M/C to M/C/S	Substantial
Kerryn	M/C to M/C	No gain	M/C(em) to M/C/S	Very Substantial
Louise	M to M/C	Substantial	M/C to M/C/S	Substantial
Lexie	M/C(em) to M/C	Reasonable	M/C to M/C/S(em)	Reasonable
Sonia	M/C to M/C/S (em)	Reasonable	M/C to M/C/S	Substantial

Responses from the Post-Project Interviews support Assertion 1 in that the eight case study students unanimously thought that the *Mathematical Search* helped them to develop their thinking about mathematics. The following responses were made in reply to the interview question "Do you think that doing these tasks [*Mathematical Search*es] helped you to understand mathematics better and if so, how did they help you with your thinking about mathematics?"

> I think they've helped my mind expand and look at things in a different way that I haven't seen them before, to make it easier and different to learn, and I think it's helped a lot. Instead of just looking at a picture or something once, I look at it closely and see if I can find any maths in it. (Jenny, student, Post-Project Interview, November 18, 2005)

> Well, ever since the first task, it really made me think, just looking around at things. It really, really did make me think about everywhere maths is and I talked about it a lot to my parents and they realised a lot too. I know some things I probably wouldn't have noticed as well about maths and I realised that there was heaps of maths everywhere. (Kerryn, student, Post-Project Interview, November 20, 2005)

> Yeah, 'cause it helped me understand maths because I didn't know there was maths in writing. I thought there was just maths in numbers, but there's maths in writing as well. (Lexie, student, Post-Project Interview, November 25, 2005)

The level of gain in student thinking as well as sentiments expressed by students during Post-Project Interviews provide sufficient evidence to establish a warrant for Empirical

Assertion 1. That is, capacity to recognise and use embedded mathematical ideas and to display contextual and strategic thinking is enhanced by using the *Mathematical Search* on several occasions.

Empirical Assertion 2

> Students will display a greater capacity to recognise mathematical ideas embedded in a written context, and to use contextual and strategic thinking when considering mathematical ideas embedded in a particular written context, when they are personally interested in that context.

In attempting to warrant or reject Assertion 2, examples of evidence from teacher and student interviews, and the Researcher's Reflective Journal are presented here. The interview responses from teachers support the assertion that context is an important consideration. The following comment from Karen (teacher) was made in response to a question about the level of reading involved in the *Mathematical Search* tasks. Around the time that her class completed the first *Mathematical Search* task, the context of which was about Indigenous Australians, an indigenous student of a similar age, and known to her students, had died. This gives the following comment considerable weight in terms of the importance of context.

> The reading with the first one [*Mathematical Search* task 1] . . . the level was fine, but I'm not sure if they found the content engaging until this child's death, because then it became more interesting to them because it was their real world. (Karen, teacher, Post-Project Interview, November 25, 2005)

Another teacher, Georgie, made the following comment in response to an interview question about the value of the *Finding the Maths* task. This task was the third *Mathematical Search* where students chose the context and samples to analyse. Typical things chosen by students were "junk mail" catalogues, advertising material, and newspaper articles.

> When they actually found the context, they became active learners and they were putting their skills into practice. I thought that was the most valuable task, but they had to have experienced texts presented to them to begin with but then when they did that [pause] in fact if we gave them that task now, having done two more practices at presenting them with texts, I think the results would be even better. (Georgie, teacher, Post-Project Interview, November 25, 2005)

The following excerpt from the Researcher's Reflective Journal, compiled immediately after a Post-Project Interview with Nick (teacher), provides another example of the importance of considering the context in which mathematics may be embedded.

> The idea of context has arisen again. Today's interview with Nick was very enlightening from several viewpoints; one being that Nick considered that the choice of context for written texts was very important when devising text samples to use with the *Mathematical Search* tasks – he felt that student interest was quite dependent on the information contained in the text. (Researcher's Reflective Journal, December 3, 2005)

Responses from students also supported the assertion that context was an important consideration when considering whether or not students might be able to recognise and apply mathematical ideas contained in that context. The following exchange from a Post-Project Interview provides an example of this view.

> Interviewer: Was there any one of the tasks that was more useful for you than others or more enjoyable for you to do?
> Louise: I really enjoyed the Finding the Maths where you could go out and think where you could find it yourself in the real world, so that's like, real world things you can do.

Interviewer: So because it was real world thing, you thought it was particularly good?
Louise: Yeah, that way you think of things outside the class, things like catalogues and things.
Interviewer: So, if it's something that you're interested in do you tend to think more about it maybe?
Louise: Yep.
(Post-Project Interview, November 30, 2005)

On the basis of the above evidence presented, a warrant for Assertion 2 was established. Teachers and students both indicated that familiarity with, or interest in a particular context enhanced student capacity to recognise and use embedded mathematical ideas. It seems as though the concerns about student numeracy that were illuminated in the review of research literature may have been partly addressed by using the *Mathematical Search.* For instance, the inability of people to recognise embedded mathematical ideas, and to understand and apply them (Peter-Koop, 2004; Watson, 1995, 2004), and the lack of disposition by people to use such mathematical ideas (Perso, 2006) inferred that teachers need to use specific strategies designed to address those problems. On the basis of empirical evidence presented in this study, it appears that the *Mathematical Search* may be such a strategy that could be used successfully.

It is also important to note that the Conceptual Framework for the study, the *Model for Teaching Numeracy in Context*, incorporates a number of other teaching and learning strategies.When used in tandem with the *Mathematical Search*, these strategies, such as task debriefing, concept mapping, graph scaffolding, and interviewing can be effective in enhancing the capacity of students to recognise and apply embedded mathematical ideas. Task debriefing was conducted by the researcher following each *Mathematical Search* task and involved modelling of how to recognise and apply the embedded mathematical ideas. As well, the task debriefing sessions incorporated concept mapping in which typical examples of embedded mathematical ideas were developed around the central theme of the particular *Mathematical Search* context.

Conclusions and Implications

This study has shown, through the warranting of Empirical Assertions 1 and 2, that student thinking and capacity to connect mathematical learning to a range of contexts can be enhanced by using particular dedicated strategies. In other words, the *Mathematical Search* strategy can enhance student performance, subject to some qualifications. These qualifications included regular use of the strategy, application of associated strategies such as task debriefing, and choice of context in which mathematical ideas are embedded. Other aspects such as teacher style and philosophy, and student reading ability had an impact on student performance. Hence, this study has begun to address the important research issue of investigating the effectiveness of teaching and learning strategies in helping students connect their mathematical knowledge to various contexts and situations. The following implications can be made for both teaching and research.

Implications for Teaching Practice

The *Mathematical Search*

- has been shown to be an effective link between classroom mathematics and other learning areas and contexts in which mathematics might be embedded;
- is an effective tool in helping students recognise and connect their own mathematical knowledge;

- helps students develop mathematical, contextual and strategic thinking when working with a variety of contexts;
- could be successfully applied to audio visual and pictorial contexts, as well as written texts; and
- is effective when used in tandem with a range of other strategies, shown in Figure 1 as "Learning Strategies".

Implications for Further Research

Further research could replicate the study or focus on the use of the *Mathematical Search* and associated strategies where other variables could be considered such as

- both male and female students,
- different age groups,
- socio-economic status of students,
- students with varying reading ability,
- use of the *Mathematical Search* in audio-visual contexts, and
- use of the *Mathematical Search* over extended periods of time, perhaps beginning at a younger age.

References

Boaler, J. (1993). The role of contexts in the mathematics classroom: Do they make mathematics more real? *For the Learning of Mathematics, 13*(2), 12-17.

Coles, D., & Copeland, T. (2002). *Numeracy and mathematics across the primary curriculum.* London: Fulton.

Erickson, F. (1986). Qualitative methods in research on teaching. In M. C. Wittrock (Ed.), *Handbook of research on teaching* (3rd Ed.) (pp. 119-161). New York: MacMillan.

Fraenkel, J. R. & Wallen, N. E. (2003). *How to design and evaluate research in education* (5th Ed.). Boston: McGraw Hill.

Hogan, J. (2000). Numeracy – Across the curriculum? *The Australian Mathematics Teacher, 56*(3), 17-20.

Hurst, C. (2006). *Numeracy in action: Students connecting mathematical knowledge.* Unpublished doctoral dissertation. Perth: Curtin University of Technology

Kemp, M., & Hogan, J. (2000). *Planning for an emphasis on numeracy in the curriculum.* Adelaide: Australian Association of Mathematics Teachers. Retrieved September 16, 2003, from http://www.aamt.edu.au

Morony, W., Hogan, J., & Thornton, S. (2004). *ANSN Snapshot: Numeracy across the curriculum*. Lindfield, NSW: Australian National Schools Network.

Peter-Koop, A. (2004). Fermi problems in primary mathematics classrooms: Pupils' interactive modeling processes. In I. Putt, R. Faragher, & M. McLean (Eds.), *Mathematics education for the third millennium: Towards 2010.* (Proceedings of the twenty-seventh annual conference of the Mathematics Education Research Group of Australasia, Vol. 2, pp. 454-461). Sydney: MERGA.

Perso, T. (2006). Issues concerning the teaching and learning of mathematics and numeracy in Australian schools. *The Australian Mathematics Teacher*, *62*(1), 20-27.

Watson, J. M. (1995). Statistical literacy: A link between mathematics and society. In A. Richards (Ed.), *FLAIR: Forging links and integrating resources.* (Proceedings of the fifteenth biennial conference of the Australian Association of Mathematics Teachers, pp. 12-28). Adelaide: AAMT.

Watson, J. M. (2004). Quantitative literacy in the media: An arena for problem solving. *The Australian Mathematics Teacher, 60*(4), 34-40.

Willis, S. (1998). Which numeracy? *Unicorn*, *24*(2), 32-42.

Yin, R. K. (2003). *Case study research – Design and methods*. Thousand Oaks, CA: Sage.

A Story of a Student Fulfilling a Role in the Mathematics Classroom

Naomi Ingram
University of Otago
<ningram@maths.otago.ac.nz>

This paper presents a case study of a secondary school mathematics student in New Zealand. Stories about this student relating to the context of mathematics form his mathematical identities and are told by his parents, his teachers, his peers, himself, and the researcher. The student's negative affective responses to mathematics are explored through these stories. The student was found to have very positive beliefs, values, attitudes, feelings, and emotions about mathematics. He "loves" mathematics because of his enjoyment of mathematics as a discipline, and because he is good at it compared with his classmates. He is perceived to be in the top group of mathematicians in his school, a role endorsed by himself, the school, his teachers, and his peers. During the year however, he becomes less positive about some aspects of mathematics as he struggles to continue to fulfil this role.

Introduction

Exposure to mathematics seems to generate a range of emotions and feelings in secondary school students. These affective responses are often negative and are thought to influence both learning and achievement (Gómez-Chacón, 2000; McLeod, 1992; Reyes, 1984). Students seem to become less resilient to negative emotions and feelings about mathematics as they move through school (McLeod, 1992), and it is important to capture this process of change by conducting research in mathematics classrooms over a period of time to understand its effect on the students' learning of mathematics (Leder & Grootenboer, 2005).

This paper presents a case study of one secondary school student and forms part of a continuing research project. This project investigates a group of students over two years to capture their mathematical identities (who they are in mathematics) and explore the students' negative affective responses to mathematics. The main data for the research are stories told by the teachers, the parents, peers, the researcher, and the students themselves.

In the next section, the theoretical background of the affective domain and the use of stories for investigating learning are outlined. The methodology is then detailed and the student described in terms of his mathematical identities and his negative affective responses to mathematics.

Theoretical frameworks

The Affective Domain

There have been varied definitions of the affective domain in the literature (Leder & Grootenboer, 2005). This research however uses Douglas McLeod's definition. McLeod (1992), a mathematics educator, described the affective domain as a "wide range of beliefs, feelings, and moods that are generally regarded as going beyond the domain of cognition" (p. 576). This domain had three components: beliefs, attitudes, and emotions. Further research enlarged McLeod's model to include values (Goldin, 2002), and an understanding of the relationships of different parts of the domain. Leder and Grootenboer (2005) summarised the different conceptions and relationships in Figure 1. The different elements of the domain lie on a continuum of stability and intensity of responses, and levels of cognitive and affective involvement. This model of the affective domain is useful as a beginning framework to inform this research.

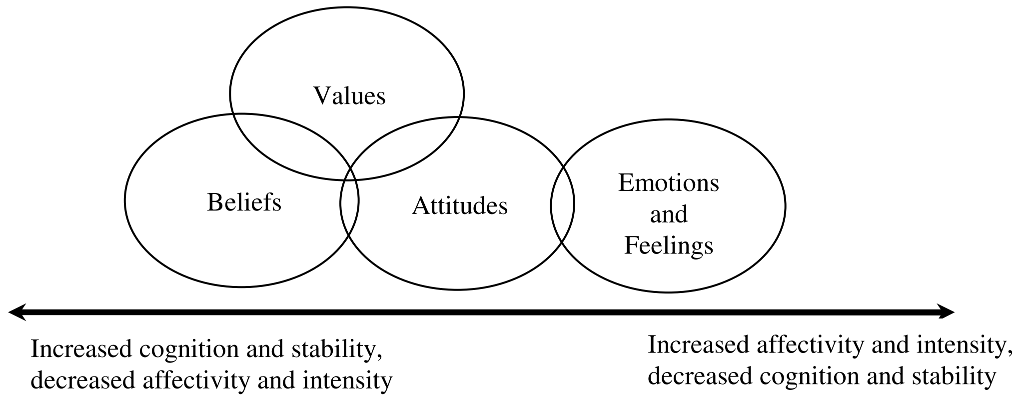


Figure 1. The affective domain (Leder & Grootenboer, 2005).

Rather than study just one of the elements of the affective domain, it is the relationships among the elements that are significant in understanding the effects on learning (Schuck & Grootenboer, 2004). Emotional responses, for example, may result from a perceived conflict with beliefs (McLeod, 1994), and when a person repeatedly experiences an emotion, this may lead to more stable attitudes and beliefs (Zan et al., 2006). For the purposes of this paper therefore, an *affective response* is thought of as a reaction to mathematics that could relate to any part of the affective domain. This reaction could be for example, joy, anxiety, fear, avoidance, frustration, or boredom. Only negative affective responses are considered here, which are defined more operationally in the methodology section of this paper.

Mathematical Identities

As socio-cultural theories have become prominent and there is focus on understanding individuals' actions, there has been renewed interest in the notion of identity (Sfard & Prusak, 2005). Grootenboer et al. (2006) suggest *identity* is a connective construct containing multiple elements such as beliefs, attitudes, emotions, cognitive capacities, and life histories, defining it as "how individuals know and name themselves … and how an individual is recognised and looked upon by others" (Grootenboer et al., 2006, p. 612).

Anna Sfard and Anna Prusak believe identity to be a narrative "constantly created and recreated in interactions between people" (Sfard & Prusak, 2005, p. 15). They see identity as individuals' visions of their own and other's experiences. They "equate identities with stories about persons. No, no mistake here: We did not say that identities were finding their expression in stories – we said they were stories" (Sfard & Prusak, 2005, p. 14). More operationally, they define an identifying story to be:

- *Reifying* - through the use of the words *be, have, can, always, never, usually;*
- *Endorsable* - with the identity-builder (the person the story is about);
- *Significant* - if any change in it is likely to affect the storyteller's feelings about the identified person particularly with regard to membership of a community.

People therefore have a number of stories relating to them. They have multiple identities. Sfard and Prusak (2005) split these multiple identities into *actual identity* (*I am, he is* – stories about the actual state of affairs) and *designated identity* (*I should be* – a state of affairs expected to be the case now or in the future). In this paper, stories told about the student by the participants in the social context of the mathematics classroom are viewed as *mathematical identities.*

Methodology

Cresswell (2003) prescribes for research a methodological framework with three elements: "philosophical assumptions about what constitutes knowledge claims, general procedures of research called strategies of inquiry, and detailed procedures of data collection, analysis, and writing, called methods" (Cresswell, 2003, p. 3). In terms of knowledge claims, my research is guided by social constructivist principles. The data are filtered through my personal and cultural values and experiences that form my own identity and I need to acknowledge that filter. I am an experienced secondary school mathematics teacher with previous research interests in cooperative learning and therefore I am strongly influenced by social dimensions of learning. This research therefore is largely classroom based; it is important that I spend time with the students in their classroom environment to try and understand their processes of engagement and interaction because this is the major arena for developing mathematical identities.

Although theoretical perspectives on affect and identity help to inform my research, I am also informed by a grounded theory approach to the methodology and this approach is my strategy of inquiry. Grounded theory is the derivation of theory from data "systematically gathered and analysed through the research process" (Strauss, & Corbin, 1998, p. 12). This approach is useful for this investigation because it allows the research to be inductive. Decisions I make about each stage of the data collection process are grounded in the data itself and the emerging categories and themes (Strauss, & Corbin, 1998). Using each piece of data to learn more about each student, the class, and the context, I am better able to direct each phase of my data collection and analysis.

This is an instrumental case study in the sense that I am exploring in depth one individual and collecting rich data about that individual over a period of time (Cresswell, 2003). The case itself however, is of secondary interest to the purpose of the research, and the overall project is a analysis of multiple case studies to understand negative affective responses in mathematics (Stake, 2005).

Participants

The participants in the larger study are 30 students aged 14-15 who, in 2006, were in the same mathematics class in a co-educational, medium SES, urban secondary school in New Zealand. The students were chosen to be in the achievement class of their year level because they demonstrated excellence in one or more fields, not necessarily mathematics. The students' mathematical abilities range from average to high.

The participant chosen for this initial case study is Colin. He is of high ability according to standardised testing, and was chosen for this paper, rather ironically, because he demonstrates very few negative affective responses to mathematics. He is, indeed one of the most positive students I have come across. It makes the negative responses he does have significant in their rarity and because of this there is an element of clarity about them.

Data Collection and Analysis

Methodology in affective research needs to be broad enough to capture the complexity of the issue (McLeod, 1994; Zan et al., 2006), and therefore I am using a variety of instruments and techniques in the study. The expanding data set for the larger study consists of audio-taped and video-taped observations, interviews with students and teachers, student and parent auto-biographical questionnaires, an anxiety questionnaire (adapted from Chiu & Henry, 1990), metaphors collected from the students about mathematics (Buerk, 1996), students' drawings of mathematicians, assessments, exercise books, school reports, academic prizes, disciplinary reports, student subject choices (initial and actual), enrolment information, and student journal writing.

Each piece of data that pertains to Colin or his social and physical context is seen as a story and therefore identified according to Sfard and Prusak's (2005) operational definition of identity and represented by ${}_{B}A_{C}$ where A is the identified person, B is the author, and C is the recipient. This creates a structure to differentiate between multiple identities of an individual; for example, a story told about Colin by the teacher to the researcher would be ${}_{\text{Teacher}}\text{Colin}_{\text{Researcher}}$. I then took a subset of these identities and highlighted instances of when Colin displayed or experienced negative affective responses. Operationally, a negative affective response is seen as a negative reaction to mathematics that could be:

- Physiological – a physical reaction, such as going red, or becoming agitated;
- Psychological – feelings such as dislike, boredom, worry, panic, frustration;
- Behavioural – an overt and observable reaction to mathematics endorsed by the student, for example, poor classroom behaviour, avoidance of mathematics.

A microanalysis was performed on these stories to understand how Colin's negative affective responses position themselves within his mathematical identity.

Results

Describing Colin

Colin is a tall, angular boy who has the loose-limbed carelessness of a teenage boy, too big already at 14 for the school desks. Colin is the oldest child of two. His parents describe him as an imaginative, caring, and helpful boy with a good sense of humour, and a strong sense of justice. Colin, his teachers, and his peers endorse this view of his personality. He does well in all his subjects, in particular music, where he is viewed as gifted.

During the observational phase he always had with him the necessary mathematics books and equipment. His teachers describe him as well behaved, with sound work habits, and a positive manner. This was observed, in general, during the course of 2006, and Colin agrees he works hard in mathematics. Colin's squared exercise book shows that he completes the set work neatly on a ruled and dated page. There is little working shown when he is completing exercises from the textbook, the main activity in the class. His working on starter problems however, are written out of the squares in a larger, more fluid style.

Colin loves mathematics, thinks it is fun, and is excited by it. "Maths is a thing for me … I just feel like I have a thing for maths" (${}_{\text{Colin}}\text{Colin}_{\text{Researcher}}$). He concurs he loves it for two reasons; one because of his enjoyment of the field of mathematics itself, and the other because he perceives he is good at it compared with his classmates.

Colin the Mathematician

Colin's own definition of mathematics is that it is "a language we use to evaluate situations and predict what will happen next" ($_{\text{Colin}}$Colin$_{\text{Researcher}}$). His metaphors for mathematics are all scientifically oriented. He believes that mathematics is like

> 1. The universe. It is infinite and all encompassing. 2. An atom because it makes up everything. 3. The entire worldwide ecosystem. It fits together like a giant jigsaw puzzle. 4. The colour white. It is a blending of all the colours of light like all the elements of maths ($_{\text{Colin}}$Colin$_{\text{Researcher}}$).

Importantly, he believes mathematical learning to be of great value and not restricted to the curriculum or institutional structure.

> When stuff is really repetitive it motivates me to actually do my work … perhaps if I can get all of these done I can have some free time at the end of the period and think about music or other maths … I don't see maths as a subject itself, I think of it more as a thing that goes everywhere ($_{\text{Colin}}$Colin$_{\text{Researcher}}$).

Colin does not suffer from significant test anxiety. Colin, early in the research period, expressed only very mild, probably facilitative anxiety, when doing a mathematics test, and being given a mathematics test he was not told about. He also does not worry about getting tests back. Colin enjoys being challenged during mathematical activity, and is patient when he does not immediately understand something, knowing that he will in a few minutes, a few days, or a few years. The more he learns, the better he feels about it.

> There's no problem that I haven't found out the answer to … I have a big book at home full of brain teasers … and you learn how they work eventually and some I just don't get and I come back and I'm like oh I know what that word means now so … or I know the answer to that now. That makes sense ($_{\text{Colin}}$Colin$_{\text{Researcher}}$).

When asked to draw a mathematician, Colin drew a trendy man with dreadlocks and wrote:

> Say hello to Simon. He is *the* mathematician. He has cool sunglasses to prevent UV rays getting into his eyes and going into his brain. He is a normal person. He's really cool. In other words, anyone can be a super mathematician. So instead of drawing a stereotypical nerd, I drew my form teacher … [he] is actually [not] that great a mathematician, but hey ($_{\text{Colin}}$Colin$_{\text{Researcher}}$).

Refreshingly, Colin feels that there is little social stigma attached to being good at mathematics.

> No one really cares about whether you're a nerd or not any more … people are my friends regardless. It's great. I love it. I'm lucky to be born at this time … nerds don't really exist as much any more ($_{\text{Colin}}$Colin$_{\text{Researcher}}$).

Being Good at Mathematics

Colin enjoys mathematics because he perceives he is good at it compared with his classmates. "I always like being better … than other people … I like the feeling of knowing that no-one usually understands that but I kind of do"($_{\text{Colin}}$Colin$_{\text{Researcher}}$). He clearly acknowledges he is one of the top mathematicians in the class and indeed the school.

Colin, with Peter and Angela, are three students that recognise themselves and are recognised by others as being top in the class. "Everyone wants to be in my group when we do maths things … it's like [calling out] Peter, Colin, Angela, come over here" ($_{\text{Colin}}$Colin$_{\text{Researcher}}$). Colin's name was mentioned (unsolicited) by ten students a total of 18 times during interviews as being a part of this group or the top student in the class. Peter and Angela were only mentioned a couple of times, and others in the class at the most once. One student, whose mathematics capability is similar to those in this group, identifies Colin and at the same time distances himself because of how he perceives his own behaviour. “I feel like there are … people like Colin and Angela that just get down to it. I probably don’t feel like [I’m in the top group] because I just slack off when I can”($_{\text{Finlay}}$Colin$_{\text{Researcher}}$). Other students think Colin knows the mathematics automatically “If it’s something hard, it takes a couple of weeks to get through my head … but the really brainy ones like Angela and Colin … it’s just like they know it”($_{\text{Tia}}$Colin$_{\text{Researcher}}$).

Colin is especially competitive with Peter and Angela and they are observably competitive with him. Colin knows all of their results in mathematics for the last few years and thinks about their learning processes. “I do think about it more complicatedly than [Angela] does sometimes. She’s a better learner and more motivated person than I am sometimes because she’s a girl” ($_{\text{Colin}}$Colin$_{\text{Researcher}}$). Other people in the class do not feel competitive, and do not see that label as applying to them, only to people in the “top group”, again identifying Colin as one of them.“That’s not like me. That’s between Peter and Colin and people like them. I don’t compare myself with them. They’re a lot better than me at maths” ($_{\text{Fern}}$Colin$_{\text{Researcher}}$).

The main role Colin has in the class is that of unofficial teacher or tutor, a role he enjoys. “I like it when people ask me things. I could be a teacher when I grow up … and even when someone else might [be able to help] … I feel like they think I’m just the person who knows it really” ($_{\text{Colin}}$Colin$_{\text{Researcher}}$).

The School and the Teachers

In the three years before 2006, Years 7, 8, and 9, Colin received mathematics honours awards at prize-givings (only 2 or 3 are given per year). Colin himself endorsed these rewards as being important to him; he remembers what he and several of his classmates got in all their subjects over several years. He had, until the middle of Year 10 in 2006, received mostly Excellence grades or near 100% in his mathematics assessments. Colin’s school reports reinforce his ability in mathematics to Colin and his family with strong identity statements. “Colin is an excellent student” ($_{\text{Year 7 Mathematics Teacher}}$ Colin $_{\text{Colin/Parents}}$). “Colin is brilliant. He has never really been tested this year in class, however he has stayed focussed and set his sights only on excellence. His exam results were impressive. He is certainly deserving of the [honours award]” ($_{\text{Year 9 Mathematics Teacher}}$ Colin $_{\text{Colin/Parents}}$).

Colin is always included in mathematics competition teams or external mathematics enrichment activities. Other students who have the potential to be in the top group of students but are not recognised as such, perceived themselves to be excluded from mathematics competitions and external projects only requiring two or three people, often because the teacher automatically asked those in the “top group” or had an expectation it is those people who go.

During the observational phase of this research, the teacher frequently named Colin to the class, to encourage others to get help off him, or to highlight his work or assessment results. He spoke openly to the researcher in front of the students about who was good at mathematics in the class. Colin frequently put his hand up to answer questions and was well received by the teacher, sometimes to the exclusion of others in the class. During one observation, the teacher asked a series of verbal questions to check students’ understanding. Except for one other person, who was not asked to contribute, only two people put their hands up for the entire session, one of whom was Colin and the other Angela.

Negative Affective Responses

During 2006, some negative emotions and feelings could be observed in Colin. Early on in the research period, I asked him if he wanted any help with a difficult starter I had seen him struggling with after the class had gone on to the main lesson. Colin seemed immediately flustered and defensive and told me that was not the focus of the day's lesson. He seemed genuinely surprised that I asked. The teacher laughed when I told him about it and said Colin was not used to being asked if he wanted help. When I asked the class to hand in their exercise books that day, Colin did not want to and hid his book, as he did on another occasion when he had perceived he had done little work.

A number of times Colin reacted badly when he got marked assessments back and seemed defensive and secretive about his marks. He explained he got a bit down in class when he did not do well in an assessment (which he defined as achieving Excellence, 100% or performed comparatively to the others in the top group). This was highlighted during the interview when he was asked what his worst mathematics experience was.

> Whenever I only get one wrong. I feel like [I] can't get 100% in a test. I shouldn't be making silly mistakes. I check it like three times. I wish I could like start school again and then get 100% in every test … and then be able to say I got 100% in every test I ever did at school ($_{\text{Colin}}$Colin$_{\text{Researcher}}$).

During the year Colin had a moderate number of absences mostly due to music commitments and a noticeable change of focus. During one term, he had 13 absences out of 34 periods, missing around 40% of the lessons. In Colin's individual interview he said that he found it hard when he had missed out on work by being away and someone told him what to do when he would normally tell him or her what to do. "Then when I get it I am back on top" ($_{\text{Colin}}$Colin$_{\text{Researcher}}$). Different too from the beginning of the year he now said that he felt anxious in mathematics when he did not know something, and he perceived everyone one else knew how to do it.

In the latter part of 2006, Colin talked about his parents. "They … don't want to push me, but they end up pushing me because I've got Excellences all the time and they get a bit worried when I don't get an Excellence, I just get a Merit" ($_{\text{Colin}}$Colin$_{\text{Researcher}}$). Colin's mother was already aware his level of focus had changed. "His interest in maths extension opportunities has decreased in direct relation to the increase in his music interest/social activities" ($_{\text{Mother}}$Colin$_{\text{Researcher}}$). This change in Colin was further highlighted at the end of year exam. He only studied for mathematics for "two minutes" and got mostly Merit rather than Excellence marks. Significantly, because of his placing in the class, he did not get an Honours award for the first time. His behaviour changed in class after the exam results came back and became more casual. He wrote in an end of year questionnaire that Mathematics was his worst subject in the exams.

Discussion and Conclusions

The stories told in this paper capture Colin at the start of 2006 as being very positive about mathematics. He has a mature and well-developed understanding of what mathematics is, and values it highly as a discipline for life-long learning. Colin loves mathematics as a subject but also, perhaps equally, because he feels he is good at it compared with his peers. Grades, marks, place in the class, and prizes, in particular, are all institutional narratives for declaring who Colin *is*, and they all reinforce he is one of the best mathematicians in the school. Furthermore, Colin has received a number of reinforcements from teachers and peers to believe in himself as an excellent mathematician, strongly reifying stories that make up his actual mathematics identity.

For most of the year, Colin had a *role* in class of a top mathematician, a role again endorsed by himself, his teachers, the school, his parents, and his peers. This role can be seen as his designated identity, and therefore there were a number of expectations of Colin; doing extremely well in, or being the "best" in assessments, answering the teachers' questions, helping others, working consistently, being organised, behaving well, and always understanding everything in class.

The Gap Between Colin's Designated and Actual Identities

Until mid 2006, there was no discernable gap between Colin's actual and designated identities. He was fulfilling his role of being a top mathematician. During the year however, because of the role he had been given (designated identity), he felt he needed to continue to fulfil that role or prove himself in the classroom and at assessment time. This became more difficult to do because of a change in focus, absences, and a lack of study and application. He did not do so well in assessments, did not prepare for the exams, did not understand everything in class, and did not get the coveted Honours award for mathematics.

These instances can be seen as critical stories that would make Colin feel as if his whole identity had changed. Sfard and Prusak (2005) concur that assessment results that are not up to expectation have a particular capacity to replace stories that have been part of a student's designated identity. When there is a perceived and persistent gap between actual and designated identities there is likely to be a sense of unhappiness in that person. Colin began to experience negative affective responses to mathematics because of this new gap. He was no longer fulfilling the expectations of his role, or his designated identity. Colin began to show negative affective responses to mathematics for the first time. He became anxious when he did not know immediately how to do something, he worried about what his parents might think about his results, and his behaviour changed in class. There was erosion in his emotions and feelings about mathematics and a concurrent drop in performance.

This is early in the story of the gap between Colin's actual and designated identities, and he, in general, remains very positive about mathematics and he continues to value it highly, but he has lost some of the positive feelings he got from being good at it compared with his classmates. Repeated instances of this could lead to negative change in his stable beliefs and values, particularly when he starts the assessment driven Year 11 NCEA Level One in 2007.

Lessons Learnt from Colin

This is Colin's story (or stories) and the lessons learnt from Colin need to be considered in terms of other students. Angela, for example, continues to have very little gap between her actual and designated identities. She however is different from Colin because she values doing well in the subject more than enjoying and valuing the subject itself. If she is unable to maintain her designated identity, there is likely to be higher consequences than for Colin in terms of increased negative affective responses and related learning outcomes or choices. Other students who are not viewed as top mathematicians (designated identity), but whose results and class work indicate they are excellent mathematicians (actual identity) feel excluded and as a result feel compounding frustration because of lack of acknowledgement. Average mathematicians in the class have the strongest negative affective responses, perhaps compounded by their utter exclusion from the top group of mathematicians, class discussions, and their own reinforced and very much endorsed role as low in the class.

Colin's teachers from the last few years are significant narrators in his story and therefore are strong reinforcers of Colin's role of a top mathematician. Although the teachers' reinforcements may be seen as having high expectations of Colin, they need to be aware that a student's identities are being re-shaped constantly. A teacher therefore can exacerbate expectations that may become unrealistic for a developing, and sometimes mecurial, adolescent. Teachers, as both professionals and mathematics educators, need to understand and take responsibility for not only the effect that this reinforcement has on the individuals they perceive as being the best, but also the effect it has on the learning environment, and the other students in the class.

By capturing Colin's multiple mathematical identities, a context is provided for understanding his affective responses in mathematics. Colin's rare, but increasing, instances of negative affective responses can be seen as a result of a gap between his designated and actual identities. Other students too are affected by a gap, which contributes to their greater level of negative affective responses. By understanding these gaps, and especially the impact that a teacher can have, students with a potential gap can be identified, and the students helped to become more resilient to negative affective responses in mathematics.

References

Buerk, D. (1996). Our open ears can open minds: Listening to women's metaphors for mathematics. *Focus on Learning Problems in Mathematics, 18*, 26-31.

Chiu, L. H., & Henry, L. L. (1990). Development and validation of the mathematics anxiety scale for children. *Measurement and Evaluation in Counseling and Development, 23*(3), 121-127.

Cresswell, J. W. (2003). *Research design: Qualitative, quantitative, and mixed method approaches* (2nd ed.). Thousand Oaks, CA: Sage Publications.

Goldin, G. A. (2002). Affect, meta-affect, and mathematical belief structures. In G. C. Leder, E. Pehkonen, & G. Torner (Eds.), *Beliefs: A hidden variable in mathematics* (pp. 59-72). The Netherlands: Kluwer Academic Publishers.

Gómez-Chacón, I. M. (2000). Affective influences in the knowledge of mathematics. *Educational Studies in Mathematics, 43*(2), 149-168.

Grootenboer, P., Smith, T., & Lowrie, T. (2006). Researching identity in mathematics education: The lay of the land. In P. Grootenboer, R. Zevenbergen, & M. Chinnappan (Eds.), *Identities, cultures and learning spaces. Proceedings of the 29th annual conference of the mathematics education research group of australasia* (Vol. 2, pp. 612-615). Sydney: MERGA.

Leder, G., & Grootenboer, P. J. (2005). Affect and mathematics education. *Mathematics Education Research Journal, 17*(2), 1-8.

McLeod, D. B. (1992). Research on affect in mathematics education: A reconceptualization. In D. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 575-596). New York: Macmillan.

McLeod, D. B. (1994). Research on affect and mathematics learning in the jrme: 1970 to the present. *Journal for Research in Mathematics Education, 25*(6, 25th Anniversary Special Issue), 637-647.

Reyes, L. H. (1984). Affective variables and mathematics education. *The Elementary School Journal, 84*(5, Special Issue: Mathematics Education), 558-581.

Schuck, S., & Grootenboer, P. (2004). Affective issues in mathematics education. In B. Perry, C. Diezmann, & G. Anthony (Eds.), *Review of mathematics education in australasia 2000-2003* (pp. 53-74). Sydney: MERGA.

Sfard, A., & Prusak, A. (2005). Telling identities: In search of an analytic tool for investigating learning as a culturally shaped activity. *Educational Researcher, 34*(4), 14-28.

Stake, R. E. (2005). Qualitative case studies. In N. K. Denzin & Y. Lincoln, S. (Eds.), *The SAGE handbook of qualitative research* (pp. 443-465). Thousand Oaks, California: Sage Publications.

Strauss, A., & Corbin, J. (1998). *Basics of qualitative research: Techniques and procedures for developing grounded theory*. California: Sage Publications.

Zan, R., Brown, L., Evans, J., & Hannula, M. S. (2006). Affect in mathematics education: An introduction. *Educational Studies in Mathematics, 63*(2), 113-121.

Secondary-Tertiary Transition: What Mathematics Skills Can and Should We Expect This Decade?

Nicolas Jourdan
University of Southern Queensland
<jourdan@usq.edu.au>

Patricia Cretchley
University of Southern Queensland
<cretchle@usq.edu.au>

Tim Passmore
University of Southern Queensland
<passmore@usq.edu.au>

We report on the mathematics competencies of 206 Engineering and Science students commencing an algebra and calculus course at an Australian university in the first semester of 2006. To inform course design in the face of growing student diversity, skills were assessed via a pre-test covering six fundamental areas. These data were also compared with the 1997 to 2001 data. The findings revealed reasonable skills with arithmetic, fractions, and index laws but ongoing weaknesses in areas of algebra, functions, and trigonometry. These findings have important implications for planning in Australian universities. Implications for school curricula are also considered.

Introduction: The Australian Context

Secondary-tertiary transition and mathematics under-preparedness for tertiary studies have long been the focus of educational interest in Australia. Much was written on skills, misconceptions, and related issues in the 1990's, especially in the context of the development of support structures in universities in Australia (Taylor, 1999). The challenges of addressing under-preparedness for university mathematics studies continue and have also been reported internationally (Ulovec, 2006).

Examining the Australian context, it is clear that widening tertiary entry policies generally, and the lowering of mathematics pre-requisites in many Engineering and Science programs in particular, have had a dramatic effect on the mathematics skills of students commencing tertiary studies (Wood, 2001; Coutis, Cuthbert, & MacGillivray, 2002). In a recent report, University of Sydney academics Britton, Daners, and Stewart (2006) observed that many students are "not ready for the sophisticated level of mathematics at university". In response, many Australian universities now offer what were mathematics foundation courses as full courses in Science and Engineering programs, to build basic competencies (Carmody, Godfrey, & Wood, 2006). While this flexibility has opened tertiary studies to more students, lower mathematics entry requirements have taken a serious toll on mathematics studies in Australia generally. Not only is it harder to persuade school students to do advanced mathematics subjects in Years 11 and 12, but accommodating school content in Science and Engineering degrees has also reduced the study of higher level tertiary mathematics subjects.

These and other factors have contributed to the general downward spiral in commitment to studies in the mathematical sciences in Australia and elsewhere. Declining numbers of mathematics majors have resulted in Australian universities closing Mathematics Departments. In the recent National Strategic Review of Mathematical Sciences Research in Australia (Australian Academy of Science, 2006), international leaders reported that "Australia's distinguished tradition and capability in mathematics and

Proceedings of the 30th annual conference of the Mathematics Education Research Group of Australasia
J. Watson & K. Beswick (Eds),

statistics is on a truly perilous path". Key findings were that Australian students are abandoning higher-level mathematics in favour of elementary mathematics, that not enough trained mathematics teachers are entering the high school system, and that many university courses such as engineering that should include a strong mathematics and statistics component, no longer do. Key recommendations included encouraging greater numbers of high school students to study intermediate and advanced mathematics, significantly increasing the number of university graduates with appropriate mathematical and statistical training, and ensuring that all mathematics teachers in Australian schools have appropriate training in the disciplines of mathematics and statistics to the highest international standards.

Against this background, declining numbers of tertiary mathematics teachers are endeavouring to support and retain students in their studies, and to provide courses appropriate for their needs. Faced with the challenge of assessing academic readiness quickly and efficiently, to counsel students and steer them into courses appropriate for their needs, there is a need to assess mathematics skills tests alongside other factors. Clear information on current entry-level skills is needed to inform support programs for under-prepared students, and to guide course and curriculum development at tertiary level. Empirical data provide information on the long effect of school studies on both school-leavers and mature-age students.

Skills Tests and Assumptions

Much of the early mathematics skills-testing in secondary-tertiary transition and adult learning was done by specialists in the area of bridging and support (Taylor, 1999; Wood, 2002). However, diagnostic tests of entry-level mathematics competencies are increasingly being used in mainstream first-year university mathematics and statistics courses, to identify, advise, and support students who may be at risk of failing. In recent work, University of Sydney academics Britton, Daners, and Stewart (2006) administered a diagnostic skills test with the objective of better informing students on their suitability for first-year university mathematics studies. The findings were also used in conjunction with school results to gain a better predictor of students' success in university courses.

With similar concerns, Sydney University of Technology academics Carmody, Godfrey, and Wood (2006, p. 24) claimed that one reason for the high mathematics failure rates is the "differing mathematical backgrounds of students who enter university". Their response was to administer a diagnostic skills test in the first week of the semester, and use the results to advise students on doing support studies or doing a foundation course to build skills. The diagnostic test was found to be useful in "alerting those students who were seriously under prepared for mathematics at university".

Queensland University of Technology academics Coutis, Cuthbert, and MacGillivray (2002, p. 97) reported the sharp increase in the diversity of academic preparedness as follows: "a substantial proportion of commencing students taking mathematically based university subjects do not have the prescribed assumed knowledge requirements". Using diagnostic skills tests they identified students with weak mathematical background, and offered a range of support programs which they concluded were effective in bridging the gap between the students' assumed and actual knowledge. Similarly, other reports on the effectiveness of interventions that attempt to address such gaps report positively on students' participation and affective response. However, scanning the literature reveals no sustained objective research into the effects on learning and performance, and in fact, Wood (2002) claimed that short programs are not effective for what are termed “weak” students.

The emphasis in most Australian reports on the use of diagnostic tests has been on skills testing to inform student support and counselling. Certainly, there have been few attempts to compare the mathematics skills of students entering Science and Engineering in Australia now with the skills of those who entered a few years ago. Obvious reasons for this gap in the literature are that changes in student population and curriculum emphases in many university courses make comparisons difficult. However, clearly university programs must respond to these changes, and comparisons are valuable for informing both school and university curricula.

This paper describes the findings of a study that addresses this gap in the literature. We report on the core mathematics skills of students on entry to an Australian tertiary-level mathematics course in 2006, and compare these with the skills of students entering the same course five years earlier. We also consider the implications of the findings.

The Study and the Skills Test

The investigation targeted students entering Algebra & Calculus I at the University of Southern Queensland (USQ). The topics in this course are typical of those traditionally studied by Science and Engineering students on entry to their university studies: single-variable calculus, complex numbers, vectors, and matrices. With declining entry skills however, an increasing number of students now study a foundation mathematics course first, to develop skills that were previously established in school studies.

In the first week of their studies in 2006, Algebra & Calculus I students were encouraged to complete a diagnostic test covering six areas: basic numeracy and arithmetic, fractions and percentages, index laws and scientific notation, algebra, functions and graphs, and trigonometry. An existing test was used, to facilitate comparison with data from past years. Developed and administered by Janet Taylor and others in USQ’s support division some years before, the test comprised 51 questions covering key skills academics had come to expect recent school-leavers to have on entry to Engineering and Science. This team also gathered the 1997-2001 data. Their contribution is noted with thanks. Evolving curricula and use of technology have made some questions on this test dated, but we retained all to capture maximum information and to facilitate comparison with earlier years. The findings of this study have been used to inform the development of a new test for subsequent stages of our work.

Of the 331 students enrolled initially, just over half were studying externally (52.6%). We administered the test electronically, but marked by hand. Submission was voluntary, but the response rate was good, 206 students (62.2%) completing the test. The majority (135) were engineering students, 54 were in science, 11 in education, and the remaining 6 in other faculties.

Analysis and Findings

Appendix A lists most of the questions on the test, and the success rates for each, in 2006 and the years 1997 to 2001. In this earlier period, data were only captured for on-campus Engineering students. Hence two sets of data are provided for 2006: the full group of 206 students, and the 75 on-campus Engineering students, a subgroup. Because of limited space, data for 13 questions are omitted: those on which performance was consistently high, over 80 or 90%, largely basic calculations and percentages.

Skills Data for 2006

The overall 2006 test results were disappointing. Converted to percentages, the mean and standard deviation of marks were calculated to be 62.7% and 20.0%, respectively. Sixty students (29.1%) scored less than 50% overall. Figure 1 shows the overall mark distribution for all 206 students.

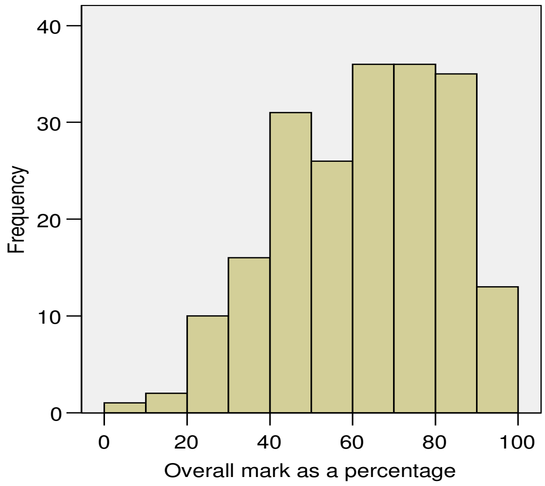


Figure 1: Distribution of test marks in 2006.

Of the six areas tested, questions on basic arithmetic, fractions, and the index laws were generally well answered. However, students' skills in the areas of algebra, functions, and trigonometry were cause for concern. Table 1 shows the percentage of students who scored less than 50% in each of these areas.

Table 1
Percentage of Students Scoring Less Than 50% in Each Area

Arithmetic	Fractions	Index Laws	Algebra	Functions	Trigonometry
1.0	1.0	10.7	48.5	37.4	44.2

See the boxplots in Figure 2 for more information on the spread of marks within each area. Algebra skills were very disappointing:

- 40% could not factorise the quadratic $6x^2 + x - 12$.
- 42% could not solve the quadratic equation $3x^2 + 4x - 8 = 0$.
- 43% could not rearrange the equation $y = (8t + 3)^3 + 4$.
- 44% could not expand $(x + 1)(-2x + 1)(x - 3)$.
- And 59% could not subtract two algebraic fractions.

Given current curriculum emphases, some success rates were expected to be low:

- Only 28.6% could solve a cubic equation.
- Only 21.8% could solve $|3x+3| < 6$.
- Only 15.4% could complete the square in a quadratic expression. Hence questions such as finding the centre and radius of a circle, given its equation, were poorly answered.
- Only 20% knew that $\sin 2\theta = 2\cos\theta\sin\theta$.

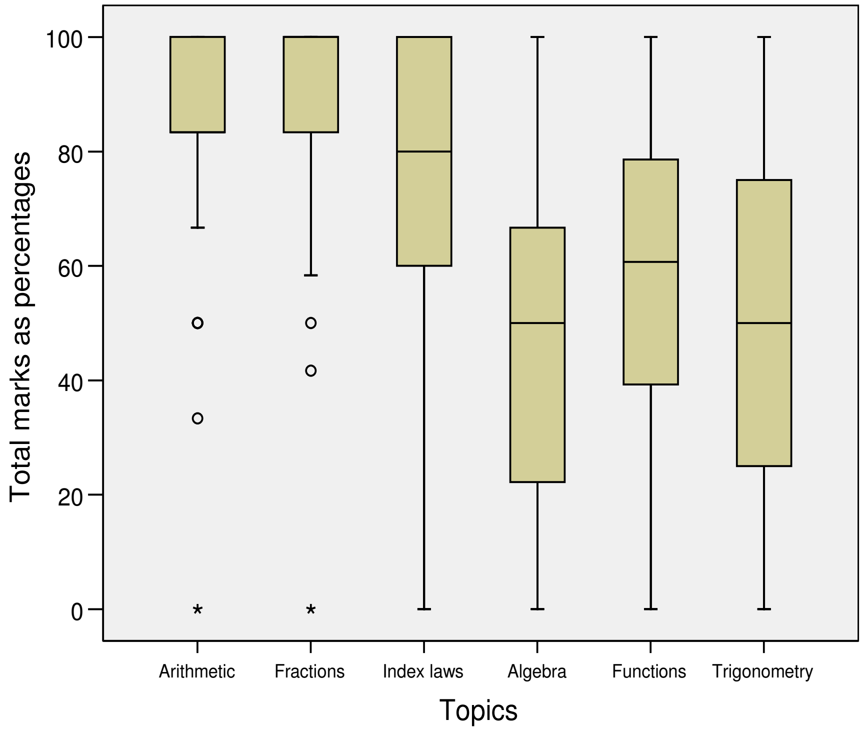


Figure 2: Boxplots showing the distributions of marks in these areas.

Graphing skills were also disappointing:

- 70% could draw the graph of a parabola, given its equation.
- But only 51% were able to sketch the graphs of sine and cosine functions.
- Only 34% could sketch $y = e^x$ and $y = \log_e x$.
- Only a quarter could find the domain and range of $g(x) = \sqrt{x-1}$.
- Less than a third could solve $x^2 - 1 = \sqrt{x}$ graphically.
- Similarly, only a third could sketch $y = \dfrac{1}{x-2}$.

Function notation skills were very limited. Given $f(x) = x^2 + 1$ and $g(x) = \sqrt{x-1}$.

- 64% could calculate $f(-1)$.
- But only 39% could find $f(x+\mathrm{h})$.
- And only 47% could find $f(g(x))$.

Straight line skills were mixed:

- 82% could find the equation given slope and y-intercept.
- But only 54% were able to find the equation of a line given 2 points.
- And 61% could write the equation of a line, given a simple graph.

Trigonometry skills were dismal:

- 68% knew the basic trigonometric identity $\cos^2\theta + \sin^2\theta = 1$.
- But when asked to find all angles between 0 and 2π that satisfy $\sin A = 0.4$, less than a third gave both angles. Using their calculators didn't help much either: only another 15% managed to use a calculator to give one angle correctly.
- Only around 44% could use the cosine rule to find one side of a triangle.

- Similarly, only about 45% could solve a simple word problem involving trigonometry.

Comparison with Previous Years

As noted above, skills data were only gathered for on-campus engineering students in the years 1997 to 2001. Therefore, for fairer comparison with the 2006 data, the skills of the subgroup of 75 on-campus engineering students in the 2006 class were compared with those of the 2000 and 2001 cohorts, comprising 86 and 71 students, respectively.

For these cohorts, no statistically significant differences were found in the six broad skills areas. However, differences were found for particular skills in algebra, functions and graphing, and trigonometry. These include a decline in ability to substitute $x + h$ into a given function $f(x)$, a trend continued in 2006. The success rate for sketching the basic trigonometric functions dropped from above 60% in the 1990's to below 50% in 2006. The ability to multiply out three given linear factors of a cubic polynomial was also disappointing, with success rates well below 50% in three out of the 6 years measured, and only 44% in 2006.

On the positive side, some skills showed improvement, but only one improved significantly to a success rate of over 50%: finding the equation of a straight line given the coordinates of two points. All other improved skills remained at low success rates, with increases generally from 10-20% to 30-40%. These include simplifying a fraction and writing it with no negative powers, determining the centre and radius of a circle, using a graph to find the solution to an equation, and using the cosine rule to find the side of a triangle. These general weaknesses are especially disappointing, given that 61 out of these 75 students had spent at least one semester in Foundation Mathematics, which covers these skills.

Further Analysis of the 2006 Data

T-tests were conducted on the following groups to assess differences in skills associated with the following factors:

- Mode of study (on campus versus external).
- Foundation Mathematics (studied versus not studied).
- Faculty (engineers versus non-engineers).
- Age-group (school-leavers versus older students).

Mode of study revealed the biggest differences, with externals (98 students) performing better in algebra than their on-campus counterparts (108 students) on four out of nine algebra questions (p-values ranging from 0.010 to 0.043). These include factorising a quadratic expression, subtracting two algebraic fractions, solving an inequality containing an absolute value, and completing the square. External students also performed better on two trigonometric questions, namely using the cosine rule ($p = 0.031$), and solving a real world problem ($p = 0.011$).

Foundation studies, faculty and age-group yielded no overall statistical differences in each of the six skills areas. However, differences were found for some specific questions. For example, non-engineers (71 students) performed better than engineers (135 students) on some tasks, including simplifying a fraction containing negative powers ($p = 0.020$), expanding three linear factors ($p = 0.047$), and substituting into a quadratic function ($p = 0.019$).

Students who did not do foundation mathematics (96) performed better than those who did (110 students) on the following tasks: solving a cubic equation, solving a system of linear equations, and recalling the trigonometric identity $\sin 2\theta = 2\cos\theta\sin\theta$. Note, however, that success rates for these three questions were low for both groups. For example, around 40% versus 25% success rate for expanding the cubic equation. Note too that Engineering now recommends that its students do foundation mathematics studies, but it can no longer be assumed that those who do not do foundation studies are those who come better prepared from school.

Data for age-groups were available for only 41 students. The school-leavers (14 students) performed better than the older students (27 students) on a number of tasks. The younger students were better with quadratic functions: describing its graph (p = 0.000), using the graph to predict y-values (p = 0.031), and finding the turning point (p = 0.041). They also performed better with fractions (p =0.003), finding the equation of a line given slope and y-intercept (p = 0.050), and sketching the sine and cosine functions (p = 0.018).

Discussion and Implications

The competencies of 206 students who completed a pre-test on entry to Algebra & Calculus I in 2006 were measured in six areas: basic numeracy and arithmetic, fractions and percentages, index laws and scientific notation, algebra, functions and graphs, and trigonometry. Data are reported for the 2006 cohort, and the 1997 to 2001 cohorts, as measured by the same test.

The 2006 findings revealed reasonable skills on arithmetic, fractions, and index law tasks, many of which could be done with the aid of a calculator. Of concern, however, are findings that reveal ongoing weak skills in areas of algebra, functions, and trigonometry. And these skills such as rearranging a straightforward equation, solving quadratic equations, finding the equation of a straight line, sketching sine and cosine, and finding angles from a sine value are fundamental for studies in calculus, vectors, and linear algebra.

Comparing the 2006 data with those of previous years, no significant differences were found in overall skills in each of the six areas described in this paper. There were differences in some specific skills, many related to functions and graphing, but the few that showed improvement remained at a low level. This was disappointing considering that the majority of the engineering students of 2006 had studied the foundation subject. Furthermore, the 2006 data revealed that students who had done the foundation studies performed significantly worse on two algebraic and one trigonometric task. It seems that these are not students who simply need some time to refresh these skills. More likely it is a warning that many have never engaged deeply enough with these fundamentals to internalise the concepts.

A significant 2006 finding was that the external students showed stronger algebraic skills overall than their on-campus counterparts in four out of nine algebra tasks. This may reflect a range of differences, including study habits. The differences between faculties were less pronounced, non-engineering students performing better than the engineers in just one algebra task and one function task. As expected, school leavers performed better than the older students on a few tasks, especially in the area of function and graphing. Nevertheless their skills levels were disappointing.

These findings have important implications for course and program planning in Australian universities. Algebra & Calculus I used to be the entry-level mathematics course for students in Engineering and Science, but declining levels of mathematical preparedness have resulted in many of these programs now placing students in foundation studies first. Enrolment in Foundation Mathematics at this university alone has risen by close to 6%, to around 900 students, the majority of these studying externally.

It is clear that in many Australian universities, foundation mathematics studies are now an essential part of the degree studies for increasing numbers of students. Should these students pay extra for these studies? Or should universities give credit points to students who enter having done advanced mathematics subjects at school? Either way, current tertiary entry-level skills tests are wish-lists; the reality is different. It is clear that tertiary teachers must radically re-examine the skills they assume their students have on entry to university mathematics courses, and tertiary programs and curricula need restructuring to respond appropriately. And it seems likely that non-foundation courses will need to sustain integrated and effective strategies to develop the core algebra, graphing, and trigonometry skills students need to facilitate even basic studies in calculus, vectors and linear algebra for higher studies in mathematics, sciences, and engineering.

The evolving nature of current tertiary mathematics studies raises questions about the implications for school mathematics curricula and assessment. If universities must respond to widening entry by incorporating current school content in tertiary courses, are school curricula freed from some content and constraints? Can focus be on depth in core skills and content, rather than breadth? We propose that the time is right for secondary-tertiary collaboration on the best path forward for Australian mathematics education at both levels.

References

Australian Academy of Science. (2006). *Mathematics and statistics: Critical skills for Australia's future. Report on the national strategic review of mathematical sciences research in Australia.* Retrieved March 23, 2007, from http://www.review.ms.unimelb.edu.au/Report.html

Britton, S., Daners, D., & Stewart, M. (2006). *A self-assessment test for enrolling students.* Abstract for paper presented at the 3rd International Conference on the Teaching of Mathematics at Tertiary Level, Istanbul, Turkey.

Carmody, G., Godfrey, S., & Wood, L. (2006). Diagnostic tests in a first year mathematics subject. *UniServe Science 2005 Conference Proceedings.* Retrieved March 23, 2007, from http://science.uniserve.edu.au/pubs/procs/2006/carmody.pdf

Coutis, P., Cuthbert, R., & MacGillivray, H. (2002). Bridging the gap between assumed knowledge and reality: A case for supplementary learning support programs in tertiary mathematics. In *Proceedings of the Engineering Mathematics and Applications Conference* (pp. 97-102). City: The Institution of Engineers.

Taylor, J.A. (1999). Undergraduate mathematics and the role of mathematics learning support. In W. Spunde, R. Hubbard, P. Cretchley (Eds.), *Proceedings of the Delta '99 Symposium on Undergraduate Mathematics* (pp. 212-217). Rockhanpton: The Delta '99 Committee.

Ulovec, A. (2006). *Reducing drop-out rates by bridging courses.* Paper presented at the 3rd International Conference on the Teaching of Mathematics at Tertiary Level, Istanbul, Turkey.

Wood, L.N. (2001). The secondary-tertiary interface. In D. Holton (Ed.), *The Teaching and Learning of Mathematics at University Level: An ICMI Study* (pp. 87-98). City, The Netherlands: Kluwer.

Wood, L.N. (2002). Bridging and Support Services for Engineering Students. In M. Pemberton (Ed.), *Proceedings of 5th Biennial Engineering Mathematics and Applications Conference*. Brisbane, Australia: The Institution of Engineers.

Appendix A. *Results of the Mathematics Testing of On Campus Bachelor of Engineering Students 1997-2001, 2006 (Right-most column shows the results for the whole class)*

Question		Percentage correct						
		1997 (n=65)	1998 (n=76)	1999 (n=70)	2000 (n=86)	2001 (n=71)	2006 (n=75)	2006 (n=206)
1 (e) Estimate $56 + 23 \times 9246 \div 125$ by using appropriate rounding		49.2	35.9	39.9	69.8	78.9	60.0	62.6
(f) Evaluate $\{20 - 3[3 + (12/4)^2]\}^3$		52.3	50.0	62.0	57.0	59.2	60.0	67.5
2 (f) Evaluate $\frac{1}{4} \div \frac{5}{6} + \frac{3}{4} - \frac{5}{2} \times \frac{4}{3}$ and express your answer as a fraction		66.2	74.4	73.2	77.9	59.2	72.0	71.8
3 (d) Express $\frac{16(a^2b^4)^{-1/2}}{b^{-3}}$ as a simple fraction involving no negative powers		30.8	20.5	19.7	18.6	21.1	36.0	39.3
4 (a) Factorize $6x^2 + x - 12$		52.3	50.0	25.4	47.7	47.9	54.7	60.2
(b) Expand $(x + 1)(-2x + 1)(x - 3)$		76.9	39.7	39.4	61.6	71.8	44.0	56.3
(c) Write this expression as a single fraction with no common factors $\frac{1}{x-3} - \frac{4}{x-2}$		40.0	33.3	32.4	32.6	35.2	32.0	40.8
(d) Make t the subject of the equation $y = (8t + 3)^3 + 4$		70.8	41.0	53.5	62.8	59.2	56.0	57.3
(e) Solve the quadratic equation for x, $3x^2 + 4x - 8 = 0$		61.5	51.3	33.8	50.0	53.5	58.7	58.3
(f) Solve the cubic equation for x, $x^3 - 4x^2 + x + 6 = 0$		21.5	14.1	8.5	16.3	22.5	25.3	28.6
(g) Solve for x, $\lvert 3x + 3 \rvert < 6$		20.0	9.0	15.5	16.3	11.3	16.0	21.8
(h) By completing the square, find the values of a and b where $x^2 + 3x + 1 = (x + a)^2 - b^2$		15.4	10.3	5.6	12.8	9.9	16.0	22.8
(i) Solve the following set of simultaneous equations $x + y + z = 0$, $x - 3y + 2z = 1$, $2x - y + z = -1$		35.4	35.9	28.2	27.9	32.4	32.0	39.3
5 (a) $f(x) = x^2 + 1$ and $g(x) = \sqrt{x-1}$ are given.	(i) Calculate $f(-1)$	81.5	83.3	73.2	82.6	74.6	62.7	63.6
	(ii) Find $f(x + \mathrm{h})$	61.5	57.7	63.4	64.0	47.9	29.3	38.8
	(iii) Find $f(g(x))$	53.8	52.6	56.3	55.8	57.7	42.7	47.1
	(iv) What are the domain and range of g?	15.4	6.4	19.7	20.9	19.7	25.3	25.2

(b) Write an equation for a straight line with slope of –4 and y-intercept of –3	73.8	85.9	73.2	73.3	78.9	80.0	82.5
(c) Find the equation of the straight line passing through the points (–3,1) and (–1,–2)	69.2	34.6	43.7	45.3	33.8	58.7	53.9
(d) Write an equation for the straight line below. (Sketch not shown here.)	58.5	53.8	52.1	66.3	54.9	54.7	60.7
(e) Sketch the graph of $y = -x/2 + 2$	70.8	53.8	46.5	66.3	60.6	68.0	72.3
(f) (i) Draw the graph of $y = x^2 + 7x + 6$	60.0	50.0	22.5	65.1	56.3	66.7	69.9
(ii) Use the graph drawn in (f) (i) to predict the y value when $x = -2.5$	58.5	44.9	19.7	68.6	67.6	72.0	69.9
(g) What is the turning point of the function drawn in (f)?	53.8	41.0	35.0	41.9	40.8	57.3	52.9
(h) Determine the centre and radius of the circle $x^2 + y^2 - 2x + 3y = 25$	13.8	3.8	5.6	9.3	12.7	30.7	24.3
(i) Sketch the graph of $y = 1/(x-2)$	33.8	41.0	22.5	44.2	40.8	26.7	34.5
(j) Indicate by a labelled sketch how you would graphically approximate the solution to the equation $x^2 - 1 = \sqrt{x}$	13.8	16.7	12.7	18.6	18.3	32.0	30.6
6 (a) Sketch a graph of $y = e^x$ and $y = \log_e x$	36.9	20.5	21.5	33.7	38.0	34.7	34.0
(b) Make x the subject of the equation $y = 3e^x + 2$	44.6	34.6	43.7	43.0	45.0	38.7	42.2
(c) Evaluate using the logarithmic rules (do not use your calculator) $\log_2 4 - \log_2 2 + \log_2 1$	40.0	25.6	35.2	33.7	39.0	44.0	48.1
7 (a) Convert 329° to radians	61.5	66.7	64.8	68.6	76.4	76.0	75.2
(b) Find all the angles between 0 and 2π radians that satisfy the equation $\sin A = 0.4$	15.4	21.8	9.9	19.8	32.4	20.0	32.5
(c) In the triangle below find x. (Diagram not shown here.)	26.2	21.8	18.3	26.7	15.5	40.0	44.2
(d) On the same set of axes sketch and label the graphs of $y = \sin x$ and $y = \cos x$ for $-2\pi \le x \le 2\pi$	63.1	64.1	47.9	64.0	59.2	48.0	51.0
(e) Complete the following statements (i) $? + \cos^2\theta = 1$	75.4	78.2	12.7	73.3	71.8	72.0	68.0
(ii) $1 + ? = \sec^2\theta$	52.3	46.2	11.3	33.7	32.4	52.0	51.9
(iii) $\sin 2\theta =$	12.3	12.8	11.3	11.6	9.9	14.7	20.4
(f) A surveyor attempting to find the height of a vertical cliff makes the following observations: The angle of elevation from the ground to the top of the cliff is 30° at a certain distance away from the bottom of the cliff. But, the angle of elevation is 45° when 20m closer to the cliff. What is the height of the cliff?	38.5	24.4	14.1	29.1	32.4	40.0	45.1

The Power of Writing for all Pre-service Mathematics Teachers

Keith McNaught
University of Notre Dame, Fremantle
<kmcnaught@nd.edu.au>

Jane's decision to write her maths-autobiography came as she witnessed the benefits achieved by other preservice teachers at UNDA undertaking the same task. However, unlike fellow students, Jane did not suffer from Mathematics Anxiety. Jane's autobiographical writing demonstrates the potential uses and benefits for a non-anxious preservice teacher. Her autobiography provides insights for teachers and teacher educators into the everyday experiences of the classroom and students. For teacher educators, it further demonstrates the value of various writing styles as tools for self-growth. Jane's writing contains a number of examples that demonstrate that her childhood experiences and subsequent writing about those times, directly impact on her emerging teaching philosophy and future professional work. Jane's writing also demonstrates the transformative potential of writing a mathematics autobiography for preservice teachers.

Jane's Journey to Writing a Maths-Autobiography

Jane was a high achieving mature-age student. Several of her friends had enrolled in a Directed Individual Study unit, coded ED4810, for the purpose of writing their maths-autobiography. Jane was aware that her peers in that group suffered from varying levels of Mathematics Anxiety, and was interested in the impact the autobiographical writing was having on them. They had informally shared many of the experiences they were writing about, and Jane was interested in the obviously increased confidence levels they were displaying. In one of our conversations, Jane suggested she "join in" and write her maths-autobiography, to tell the story of a transient student. She was aware that she did not suffer from Mathematics Anxiety, and reflected that whilst there were adequate precipitating factors in her own life, there were also numerous protective factors. Jane was interested as she had a strong sense developing that she wanted to teach mathematics very differently to the way she had experienced it for most of her school life. Although her recollections are subjective, from a phenomenological viewpoint what she experiences *is* what she experiences. Jane understood the dangers of solipsism and her writing echoes an awareness of the lack of objectivity that can exist in autobiographical writing.

Stepping into Jane's Story

Jane's autobiography recounts a series of critical events within her school life, including changing schools, moving house, relocating to the country, forming new social sets of friends and interacting with new teachers. It is her relationships with teachers, however, that are at the forefront of her memorable experiences. It is well recognised that teachers play a vital and significant role in student learning. Jane's writing reiterates a recurring theme that student-teacher relationships are central to learning.

> My earliest recollection of mathematics was in Year 1. I was six years old and the youngest of eight children. I was very eager to start school just like my big brothers and sisters. I couldn't wait to be in Mrs. H.'s class. Mrs. H. was the Year 1 teacher at M.P. Primary School in a small country town in South Australia. The very first incident I can recall about mathematics was learning to count using an abacus. I loved the colours of all the balls and enjoyed sliding the balls along the wire. The Year

Proceedings of the 30th annual conference of the Mathematics Education Research Group of Australasia
J. Watson & K. Beswick (Eds), © MERGA Inc. 2007

> 1 classroom had a lot of mathematics equipment on the mathematics shelves. There was a shelf full of abaci in the classroom and every morning we were allowed to play on the mat with them.
>
> On the mathematics shelf were all sorts of amazing things to play with. One of my favourite resources was the till with money in it. Mrs. H. used to set up shopping stalls all around the room, one stall would sell fruit, another would sell groceries and another would sell stationery. I liked it best when I got to be the shopkeeper and had to sell the items on my stall and take the money. I don't recall actually being able to calculate the money, although I was able to imitate what I had experienced when I went to the shop with Mum. I regularly went to the local shop for Mum by myself because the local store was only about 4 doors down from where we lived. In the 1970's most people had a monthly account at the store so I never really got to use money in a real life situation.
>
> I also remember learning to write the numbers up to 10. My favourite number to write was the number two (2), Mrs. H. taught me how to turn the number two into a beautiful swan. She used to use lots of coloured chalk and make all the numbers into pictures, for instance the number eight was a teddy bear. I can't actually recall learning to count or finding it difficult to count. I know I loved school and really liked the mathematics shelf. Mrs. H. really gave me a great start to my school years, especially in mathematics. I can only ever remember doing mostly concrete activities with her and she always made it fun.
>
> The mathematics experience that I can recall is learning to regroup, this was really tricky. We used to have to write the numbers under each other and then draw a line and add them up, my answers were always wrong. I don't remember who my teacher was but I can remember having to write a lot in mathematics. I remember that there was a big emphasis on getting things right, I didn't like it when things were wrong because then I thought I was stupid and couldn't do mathematics. I only stayed at that particular country school until the end of Year 4.

It is instructive to note the differences in Jane's descriptions as she recalls her experiences over time. She provides a detailed description of her feelings and learning activities with Mrs. H., the teacher who made school fun and who really engaged her with mathematics. This contrasts with the sparse description in the final paragraph of the next three years of school. She does not remember the names of her teachers and the relational elements that were detailed in the first grade paragraphs are missing. She does, however, remember several other things that were instilled in her: that you had to get things right, and that when you got things wrong you felt like you were stupid and "couldn't do mathematics". Jane's writing illustrates the human dimension of mathematics learning.

Teachers Make the Difference

Teachers have a major impact on student learning. For example, Turner and Patrick (2004) found that student participation is highly related to teacher practices. Such practices will either be supportive or undermining of the development of student work habits. Jane's autobiography illustrates the relationship between varying learning environments and factors such as resilience, teacher-student relationships, learning programs, and teaching style.

The Impact of Classroom Culture on Learning

The relationship that teachers establish with students is reflective of the culture of the classroom (Dix, 1993). Classroom culture includes the beliefs, attitudes, and values that are then manifest in actions, symbols, icons, and relationships (Good & Brophy, 1994). For example, in a very tidy and well-organised classroom, it could be conjectured that order and organisation things the teacher considers to be important. A classroom with an attractive reading corner, with cushions, beanbags, and books available, would indicate the importance of reading. The presence of open and effective classroom meetings, with a rotating student-chair, would indicate that democratic principles are valued. Classroom culture does not exist in isolation from the broader aspects of the school culture, family culture and social culture (Jones, 1996). A classroom culture might be attempting overtly to be counter-cultural. For example, a school in a socially disadvantaged area with racial tension might focus on developing a respectful, harmonious community "feel" to the classroom. Classroom culture includes the norms and values that teachers establish within the classroom (Stoll & Mortimer, 1995).

Positive Classroom Culture Reduces Mathematics Anxiety Related Factors

A positive and effective classroom culture results from a broad range of operational factors including appropriate classroom management techniques, a sustained focus on learning, cooperative, and respectful attitudes and harmonious relationships (Cavanagh & Dellar, 1997). A practical example of how mathematics teachers can support a positive classroom culture is by ensuring that students' "put down" remarks about other students' mathematical performance, and student behavior that belittles others, is not tolerated. A positive classroom culture encourages "risk taking" and so "at risk" students need to feel safe that they will not be humiliated or criticised for making errors. A classroom culture that fosters tension, anxiety and discord provides fertile ground for breeding anxiety about mathematics.

Good teachers are able to create a learning environment in which students have high and positive expectations about their learning, co-operative behaviour is pronounced, and the culture encourages learning to occur.

Patrick, Turner, Meyer, and Midgley (2003) found that three different types of classroom psychological environments can be established by teachers in the first days of school: supportive, ambiguous, and non-supportive. In supportive environments teachers exhibited behaviours such as expressing enthusiasm for learning, respect for their students, appropriate use of humour and sharing of expectations that all students would and could learn in their classrooms. By contrast, teachers in non-supportive environments tend to use authoritarian control and emphasised extrinsic motivation. Students in supportive classrooms exhibited significantly less work avoidance behaviour and significantly more on-task behaviour than existed in the ambiguous or non-supportive environments (Gallimore & Tharp, 1990). Although these findings were specific to the mathematics learning environment created in the classroom, it is likely that the same applies to other learning areas (Meyer, 1993). Central to the supportive environment is a teacher whose focus is student centred and encourages intrinsic student motivation (Anderman & Midgley, 1998).

The traditional view of "impartially imparting objective knowledge" implies the existence of a passive learner. Jane's writing illustrates that from a young age, children are active constructors in the learning process.

Jane's Emerging Beliefs about Mathematics Learning

> My family then moved to the city of my father's work and I started Year 5 in C.C.C.. This is the first time that I can remember using MAB's. I think in Year 5 the concept of ten finally sunk in, that was

> my "aha" moment. I enjoyed seeing how many ways I could make ten using the MAB's, numbers really started to make sense to me and I loved mathematics. I finally understood the concept of regrouping; using MAB's made it so much easier. I liked trading the ones for tens and tens for hundreds. Increasingly over the year most of the mathematics that we did was copied from the board, but we were allowed to access the MAB's if we needed to. Measurement was a major part of our mathematics in this year, we got to use the big measuring tapes from the sports shed and we went onto the school oval and did lots of measuring activities. This is about the time that the real importance of 10 in mathematics started to make sense to me. I learnt that 10 millimetres makes 1 centimetre and that 100 centimetres made 1 metre. I did not like having to do problem solving that was embedded in number stories such as, Sally travelled 10km to school and 10km home from school, how many kilometres did she travel each day? I struggled with problem solving where I had to read a story and solve the problem. Overall, I enjoyed my one year at that particular school.
>
> L. in the city was my next port of call. This school was much closer to my home and a vacancy became available in Year 6 so my parents decided to move me from C. to L. I settled easily into this new school and my teacher Mrs. R. quickly realised that my reading age was only marginally above my chronological age. During this year I worked extensively on my reading and comprehension skills and gained a lot of ground with my reading skills. My comprehension skills were below average and I had difficulty recalling information. I was becoming increasing frustrated and I was eventually put into a special reading program called the PACE reading program. The PACE program made a big difference to my ability to read and comprehend information from the text.
>
> I liked the positive praise that the students who finished first received from Mrs. R.. I began to rush my work in order to be one of the first finished and receive the praise that the other children were given. Once I had been in the classroom for a few weeks I think Mrs. R. realised what I was doing and she spent a lot of time sending me back to my desk to complete my work to a better standard. I found this very frustrating and on reflection I now understand what a wonderful teacher Mrs. R. was. She always gave a lot of positive praise when you did things correctly and a lot of encouragement when things were not exactly right. Mrs. R. really knew how to get the best work from me and she was my turning point at school. She taught me that mathematics was not about getting it right all the time, it was often about the process that helped you get the answer.

In the opening sentence of the previous paragraph Jane identifies one of the key strategies of an effective teacher as giving frequent encouragement and affirmation to students in the class. This was something that Jane was seeking, as evidenced by her description of her desperate need to finish first so that she too would be praised.

As a preservice teacher Jane has developed a clear understanding of the essence of good quality mathematics teaching when she states that she knows that the process is more important than the right answer. She attributes this insight directly to her teacher. Every time Jane rushed to complete activities, Mrs. R. instructed her to return to her desk to produce a better standard of work. This vignette also provides an insight into how students respond to constructive feedback. It is clear that Jane's relationship and self-esteem were not being adversely affected by Mrs. R.'s insistence on high quality work. She was able to impart to Jane that it was important and yet at the same time, Jane felt affirmed and valued as a child in that classroom.

Affect Attunement

The term "affect attunement" refers to the emotional connectedness between individuals (Stern, 1995). It can be observed in various life-long relationships, such as between parent and child. It can be also be found in relationships between close friends and couples. All children have a basic need for emotional attachment with other people. It is a powerful part of their growing confidence to learn, their willingness to take risks, and their ability to build relationships with significant people who will assist in their learning processes. Some children display a heightened need for emotional attachment to other adults, influenced by factors such as their age, developmental stage, personality factors, previous experience of adult-child relationships, or their experience of teacher-student relationships within their lives (Fennema, 1989; Garden, 1997). Affect attunement can be significantly impacted upon by a wide range of factors either within or external to the classroom. Factors include teacher personality, subject matter, class size, emotional needs of students, behavioural management needs and demands of the class, emotional and psychological problems of a student or students, and the dynamics of the whole school community and its processes (Grootenboer, 2001).

Poulsen and Fouts (2001) found that attuned teaching, in which teachers and students share close relationships, has a considerable positive impact on academic performance in comparison with "traditional" non-attuned teaching relationships. The same research (p. 189) found that improvement occurred within the context of a single lesson and that the effect of attunement was both "immediate and powerful".

Jane writes:

> The classroom was split into three groups for mathematics; I started in the lowest group. Mrs. R. persevered with each of us giving us lots of encouragement and she allowed us to feel comfortable in taking risks in order to learn. I think she allowed us to learn by mistakes, but because of the wonderful way she encouraged us, it never felt like you were wrong. She made the process of mathematics feel like you do when you are doing a jigsaw puzzle; sometimes the pieces don't fit together the first time, but if you try a different piece eventually, through perseverance, you begin to put the puzzle together. At the end of June I took my report card home to my parents with a huge amount of pride because it read "Jane has fast moved up to the middle group which would indicate that she has grasped the basic concepts and is now ready for some extension."
>
> My reading and mathematics continued to improve and at the end of Year 6 the mathematics learning area on my report card read "Definitely Jane's best area. She has come along in 'leaps and bounds' since coming up to the middle group." I contributed this improvement in my mathematics to the fact that my reading and comprehension had improved so much. I found it much easier to complete number sentences or problem solving tasks where mathematics was required.

Exploring the Jigsaw Metaphor

At the end of her autobiographical writing, I asked Jane if the metaphor of the jigsaw was important for her mathematics teaching. She explained that it was very important to her. It described her concept of small parts joining to form a larger picture but, unless you knew what the large picture was and what you were working towards, you could never make the small pieces come together. This was a powerful metaphor that she was able to articulate. She elaborated on the impact this had on her own teaching and the processes that she intended to engage in when she worked with students in her own care. This discussion was transformative for Jane, based on both her later feedback, and my immediate impression of her responses as she spoke. In real terms, she was developing her personal metaphor to describe "connected" teaching and learning.

Jane attributes her successes again to Mrs. R., who identified her reading problems and provided additional literacy support. Jane sees strong, directive teaching as being something that generates significant life-long change. In her experience it has impacted positively upon other learning areas and fostered life long learning.

> I remember liking the fact that mathematics was so easy, considering I found English a real struggle; it was nice to feel like one of the kids who “got it”. However, if it was not for Mrs. R. identifying my reading problems and her encouragement and support, I think I would have stayed in the bottom group in mathematics and I would have slowly hated school on the whole. I continued into Year 7 feeling very confident in mathematics and was really disappointed when I had to again go into a remedial reading class. I think that I felt that I had to prove myself in the area of mathematics and began to rush to try and finish first. Very quickly I started to make mistakes in calculating sums and my work was showing more and more errors. As the work got increasingly more difficult I was beginning to find some new concepts difficult to understand but I still generally enjoyed mathematics. My final report card for year 7 read “Works well. Errors in mechanics due to impatience. Highly satisfactory grasp of work covered.”

Humiliation as Destructive to Learning

In some classrooms when students make mistakes, teachers use humour as a way of dealing with the issue at hand. In some circumstances they may be reinforcing the notion of “put down” albeit in a situation that is funny for students at the time. To be a participant in the humour may be a funny and warm moment. To be the victim of the humour may be a very negative personal experience that can have far reaching and long term impacts on student learning, and on the learning of other students who are vicariously involved in the situation.

The profoundness of this memory is a significant part of Jane’s autobiography. The trauma of the teacher ridiculing her about her spelling has stayed with her into adulthood. It is apparent that one person can quickly erode confidence that has previously been built by another. Positively affective and effective teachers do not ridicule or make fun of students. They create learning environments in which students feel positive about themselves, and where they know that they are protected from ridicule and humiliation. Humiliation is known to be a significant risk factor for Mathematics Anxiety (Burns, 1998). Jane demonstrates the vulnerability of students to be damaged by a teacher reaction or comment, long after the event.

> My teacher in Year 7 was not like Mrs. R. She did not give me much praise and often would belittle us if we did something wrong. I remember when I gave her some written work and I had misspelled a word and she said in a very condescending tone “and I suppose you would put two t’s in writing”.
>
> This has always stuck in my mind because at the time I don’t think I knew if writing had two t’s or one. The sad thing is that those words have stuck with me for 24 years. She took away all my confidence in those few seconds that Mrs. R. had spent a whole year building up. I could feel my stress levels increasing and I can never remember feeling relaxed with this particular teacher. I was always hesitant to hand work into her in case I had made an obvious mistake and she would make fun of me in front of the whole class. The work that was displayed in the classroom was only ever the very best work and therefore mine never quite reached the display board. I always felt as though that particular teacher had no confidence in me, or perhaps she just didn’t like my chatty personality.
>
> My Year 7 teacher took away all my confidence in the area of mathematics, I felt scared to try anything new and often struggled with fear and nerves when it came to test time. Because she made me feel nervous I did not like to take risks in case I got the answer wrong. When she explained a new concept I did not like to ask questions for fear of being ridiculed by her in front of the class. It was not until I reread my reports from Year 7 that I realised that the teacher did think I was quite a good student.
>
> The final report for Year 7 showed that I had achieved above average in all subject areas for effort and ranged from average to above average for achievement.

The Transformative Potential of Autobiographical Writing

This part of Jane's autobiography demonstrates that the writing process was a cathartic process for her. Interestingly, Jane's writing might not be transformative for the reader; the transformative potential of an autobiography does not need to extend to readership, explaining why many autobiographies and journals are never published. The authors of such do not desire publication; it is the need to tell the story, more than the need to have others read, that can be a significant motivator for the writer. When she started to write about this time she went back to her old reports and re-read them. She was struck by the fact that her perception (of how the teacher felt about her) was not accurate. Her perception was challenged, and this had a positive impact on Jane.

> One particular area that I always felt I struggled with was my times tables. The times tables were a major part of the class learning and therefore this was an ongoing problem for me. We were tested regularly and had to get 100% in our test before we could move onto the next lot of tables. This, on reflection, was only taught through rote learning, and at no stage did anyone explain to me that 5 x 7 was the same as 7 x 5. The whole class kept moving at the teacher's pace, and if you did not have an understanding of the topic being taught that week it did not matter, the teacher moved on anyway.
>
> Again my parents moved to the country. This time they had purchased a hotel. L. was a boarding school but there were no vacancies for me in the boarding section so it was back to the country and B. Community School was my new school. I started Year 8 feeling very confident and felt as though I had a good handle on the level of academic achievement expected. This school was very different to my previous school; this was an open plan school and very stark and had boys in it. My previous two schools had been all girl schools. I do not remember seeing any resources or concrete materials for mathematics and I quickly became bored. Everything was presented on a white board and I can remember having great problems understanding "area". I just really struggled with the concepts that were being presented to me, possibly because of how they were presented to me. The teacher style was very much chalk and talk style. I do not remember seeing any sort of teaching aids other than perhaps an overhead occasionally and lots of worksheets. We did have a mathematics book that we worked through from front to back with very little variation from that particular book. I do know that the answers were in the back of the book, so often we would copy the answers into the book and the teacher would mark it and we would move onto the next page.
>
> There was never any group work or group discussion; it was very much students sitting in rows working independently. If you were game enough you might put up your hand and ask for help occasionally but usually only if you were very desperate to get some help. The teacher did not encourage discussion between students and if you did discuss a particular mathematics problem with another student it was considered as cheating and you were normally punished.

Jane's reflection on her Year 8 experiences is sadly an all-too-common picture of lower secondary mathematics for many students. Learning that is teacher-centred and utilizes didactic pedagogy is likely to alienate students and reduce their interest in a learning area (Kohn, 2000).

Jane observes that whenever students worked together it was perceived as cheating and they were punished rather than encouraged to engage in co-operative or collaborative work activities. In discussion, Jane affirmed that a fundamental belief she holds about effective mathematics teaching is to have students to work together, to talk, to interact, and to learn from each other. The constructivist philosophy that has been embedded into her tertiary mathematics learning area lectures, combined with her experiences, is becoming evident in her own beliefs about how she will teach mathematics in the future.

Classroom Teaching-Learning Styles

In the social constructivist classroom the learning environment and teaching practices are student-centred. Mrs. R. adopted a student-centred approach that was reflected in her ability to identify and cater for the various needs of the individual students in the class. The positive impact this has had upon Jane's learning is evident in her writing.

Jane is able to differentiate between the social constructivist environment that she chooses to create as a pre-service teacher, and the non-social "traditional" classroom model where interaction and talk are actively discouraged rather than being seen as a powerful technique for learning and understanding. Jane has also described the positive impact motivation has upon learning and the adverse impact the incorrect use of negativity has as a de-motivator for learning.

The Effects of Teacher Expectation and Affirmation

The *Pygmalion in the Classroom Project* (Rosenthal & Jacobsen, 1968) found that "teacher expectation of student performance" was the most significant variable to impact on student learning. Teachers were given grouped ability students with the groups incorrectly labelled. High ability students were described as low ability students and vice versa. The study revealed that students performed as the teachers had expected them to, despite the lack of correlation between the expectations and their actual abilities. Other research has demonstrated that teachers expect better performance from students about whom they have higher expectations of ability, and lower performance from students whose academic ability they doubt. These expectations are matched by student performance. Students who experience low expectations make fewer efforts to seek teacher attention and gradually withdraw psychologically from the learning environment. In effect, what teachers believe about the educational potential of their students has a pronounced effect on their performance and achievement. Mrs. R. communicated her expectation to Jane that she was able to produce good quality work. Her Year 7 teacher communicated her low expectations of Jane with her comment, "and I suppose you would put two t's in writing".

Writing Leading to Reflection

Jane commented that after writing her mathematics autobiography she began to think about her teaching philosophy. She stated that it was another "aha" moment when she determined that she really wants students in her classes in the future to have a strong sense of developing understanding and grasping the "big picture" rather than being overly focused on "minor tasks being correct". She was able to articulate the importance of process orientated, conceptually based learning as opposed to superficial, topic focused learning.

Jane commented that writing her autobiography had been an interesting and demanding process. She felt that she had learned a lot about herself as a learner, as a person, and as a teacher. Teachers with whom she had positive learning experiences and attuned relationships were the ones she wanted to model herself upon. She intended to reject the practices of those teachers with whom she had non-attuned relationships and negative learning experiences. Having completed three core units of mathematics education, Jane said that she felt confident in tackling mathematics in the classroom and making it a subject which would be a positive learning experience for the students in her future care.

Jane's mathematics autobiography recalls her educational pathway as a passive learner. Yet her attitudes, values and intentions as a preservice teacher are to encourage her students of the future to be active learners. In our post-writing discussions, Jane expressed her thoughts and feelings that it was a composite range of factors that had led to this transformation. For her, these factors included the autobiographical writing process, her own life-long disappointing memories of mathematics learning, the mathematics education courses she had undertaken, her relationships with the lecturers and tutors in those units, and, importantly, her practicum experiences. In her practicum work she was able to see highly effective teachers of mathematics, and felt energized by the students appearing really to enjoy mathematics work, noting the difference from her own feelings about mathematics at school.

Jane did not develop Mathematics Anxiety. There are several potential factors that appear to insulate her from this condition:

- There were a number of positive teacher relationships which developed her confidence as a learner.
- She was aware of making academic progress, of growing in knowledge, thus feeling she could manage new material.
- Transient movement can build resilience – the need to be self-sufficient, manage change, form new social groups – experiences that potentially provide protection from anxiety.
- Jane's transient life-style was "positive" – each move related to changed employment for her parents, not homelessness, family breakdown, financial difficulties or being "forced" to move – which can be more common in transient students.

Conclusion

Although the use of reflective writing in mathematics is most often used as a therapeutic tool, Jane's biography has the potential to be used as a discussion starter with both preservice and practicing teachers. It could be used to explore the deep impact of teacher "throw away" remarks, transient families, resilience to prevent anxiety, informal or unplanned career guidance, the power of writing and transformative readership understandings. A piece of maths-autobiography, or journalling, once de-identified, has numerous potential usages as a tool for readers and practitioners. The use of reflective writing for all students in mathematics education units, as demonstrated by this example, would indicate that it could be a powerful tool for self-awareness which may have considerable impact on future teaching performance.

References

Anderman, L.H., & Midgley, C. (1998). *Motivation and middle school students*. ERIC Clearinghouse on Elementary and Early Childhood Education. (ERIC Document Reproduction Service No. ED 421 281) ERIC digest, Champaign, IL.

Burns, M. (1998). *Math: Facing an American Phobia*. Sausalito, CA: Math Solutions Publications.

Cavanagh, R. F. & Dellar, G. B. (1997). *Towards a model of school culture.* Paper presented at the 1997 annual meeting of the American Educational Research Association.

Dix, T. (1993). Attributing Dispositions to Children: An Interactional Analysis of Attribution in Socialization. *Personality and Social Psychology Bulletin*, *19*(5), 633 – 643.

Fennema, E. (1989). The study of affect and mathematics: A proposed generic model for research. In D.B. McLeod & V.M. Adams (Eds.), *Affect and Mathematical Problem Solving* (pp. 205-219). New York: Springer-Verlag.

Gallimore, R., & Tharp, R. (1990). Teaching mind in society: Teaching, schooling and literature discourse. In L.C. Moll (Ed.), *Vygotsky and education: Instructional implications and applications of sociohistorical psychology* (pp. 175-205). Cambridge: Cambridge University Press.

Garden, R. (1997). *Mathematics and science performance in middle school: Results from New Zealand's participation in the Third International Mathematics and Science Study.* Wellington: Ministry of Education, Research and International Section.

Good, T., and J. Brophy. (1995). *Contemporary Educational Psychology* (5th ed.). New York: Harper Collins.

Grootenboer, P. (2001). How students remember their mathematics teachers. *The Australian Mathematics Teacher,* 57(4), 14-16.

Jones, V. (1996). Classroom Management. In J. Sikula, T. Buttery, & E. Guiton (Eds.), *Handbook of research on teacher education*. (pp. 503-521) New York: Macmillan.

Kohn, A. (2000). Burnt at the high stakes. *Journal of Teacher Education*, *51*(4), 315 – 327.

Meyer, D. (1993). What is scaffolded instruction? Definitions, distinguishing features, and misnomers. In D.J. Leu & C.K. Kinzer (Eds.), *Examining central issues in literacy research, theory, and practice. Forty-second yearbook of the National Reading Conference* (pp. 44-53). Chicago: National Reading Conference.

Patrick, H., Turner, J., Meyer, D., & Midgley, C. (2003). How teachers establish psychological environments during the first days of school: Associations with avoidance in mathematics. *Teachers College Record*, *105*(8), 1521 - 1558.

Poulsen, J., & Fouts, G. (2001). Facilitating academic achievement through affect attunement in the classroom. *The Journal of Educational Research*, *94*, 185-190.

Rosenthal, R, & Jacobson, L., (1968). *Pygmalion in the classroom: Teacher expectations and pupils' intellectual development.* New York: Holt, Reinhart and Winston.

Stern, D. N. (1995). *The motherhood constellation: A unified view of parent-infant psychotherapy.* New York: Basic Books.

Stoll, L., & Mortimer, P. (1995). *Viewpoint no 2: school effectiveness and school improvement.* London: Institute of Education.

Turner, J. & Patrick, H. (2004). Motivational influences on student participation in classroom learning activities. *Teachers College Record*, *106*(9), 1759-1785.

"Connection Levers": Developing Teachers' Expertise with Mathematical Inquiry

Katie Makar
University of Queensland
<k.makar@uq.edu.au>

One of the challenges in research is in understanding processes and systems that enable teachers to build their expertise and commitment to reform-based pedagogies. A qualitative study documented the influence that a set of support mechanisms, or *connection levers*, had in assisting upper primary teachers over the course of a year in developing confidence in teaching mathematics through inquiry.

The research literature in mathematics education has been fairly clear that students benefit from posing and investigating meaningful, open-ended problems (e.g., Diezmann, Watters, & English, 2001; Boaler, 1997). Inquiry is one means to learning that incorporates these ideals. Although inquiry has been embraced in other content areas (e.g., National Research Council, 2000), it continues to be under-utilised in mathematics. One reason for this is likely difficulties that teachers have changing conventional practice (Stigler & Hiebert, 1999; Cuban, 1990). Calls for reform in teaching have been with us for decades (Dewey, 1938/1997; Tyler, 1949/1969; Schwab, 1978; Ball, 2002), but little is known about the *processes* by which teachers alter their practice.

This paper reports on outcomes from the first year of a research project designed to understand better the processes and experiences of teachers learning to develop expertise in teaching mathematics with inquiry. In the first year of the project statistical inquiry (Wild & Pfannkuch, 1999) was used to segue into mathematical inquiry because of its natural connections to context and interpretive epistemology, and its potential as a tool for understanding problems in multiple disciplines. The goal of this paper is to understand how a number of support mechanisms, called *connection levers*, enabled the teachers in the study to develop their expertise, confidence, and commitment to teaching mathematics through inquiry.

Literature

In inquiry, students often engage in epistemological processes of coming-to-know using ill-structured problems, where the initial definition of the problem is ambiguous or has many open constraints (Reitman, 1965). Several obstacles arise in teaching and learning with inquiry because it requires skills unfamiliar in conventional mathematics classrooms. In solving ill-structured problems, the solution phase (where nearly all teaching is focused in schools) requires a relatively small proportion of the cognitive effort compared to the process of structuring and seeing the problem through to completion. The skills required for conducting inquiry have been shown to pose multiple difficulties for learners (Diezmann et al., 2001). In statistical inquiry, for example, there are challenges in designing a measurable question (Confrey & Makar, 2002), collecting and organizing data, and relating findings back to the original question (Hancock, Kaput, & Goldsmith, 1992).

Previous research by the author suggests that *initial* experiences with inquiry pose unique challenges because learners start with a very narrow perspective of the inquiry process. Her research found that these first experiences can result in frustration and poor outcomes and that learners need to undergo *multiple iterations* of inquiry with a number of support mechanisms – time, feedback, support, reflection, and validation – before they can begin to understand the nature of the inquiry process (Makar, 2004; Makar & Confrey, 2007). For example, inquiry often raises more questions than it answers and learners typically believe they have failed in their inquiry if their initial question (often overly simplistic and broad) is left unanswered, even if through the inquiry they have gained a much deeper understanding of the question under investigation.

Inquiry is equally challenging for teachers. It requires the ability to embrace uncertainty, foster student decision-making by balancing support and student independence, recognize opportunities for learning in unexpected outcomes, maintain flexible thinking, hold a deep understanding of disciplinary content, and tolerate periods of noise and disorganization (National Research Council, 2000). These often go against learning trajectories traditionally held in mathematics of neat and orderly classrooms with well-defined learning goals. Because mathematics is not envisioned as a field requiring inquiry, it is unusual for teachers to teach mathematics with this approach. If they do, the difficulties encountered in an initial experience likely dissuade them from continuing. Like learners, Makar (2004) speculated that teachers would need similar elements – time, feedback, support, reflection, validation, and multiple experiences – to develop expertise in teaching mathematics with inquiry in a program of effective professional development.

Research on teachers' learning has provided insight into principles of effective professional development. For example, in a large-scale study of relationships between teachers' professional development and their teaching practices, Cohen and Hill (2001) found that the only professional development approaches that appeared to influence teachers' classroom practices significantly involved a sustained focus on reform curriculum they were to teach, and collaborative analysis of student work. Ball (1996) has argued that professional development must provide teachers with opportunities to learn content in an environment that models effective teaching. And Elmore (2002) contends that professional development must be purposefully connected to student learning of core content, sustained for long periods of time, focus on the curriculum and pedagogy of teachers' classrooms, provide feedback, and develop within a collaborative environment.

Method

The study was developed using a design research framework (Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003), in which the researcher simultaneously studies and tries to improve the study context. The main question was: *How do teachers come to develop expertise, confidence and commitment to teaching mathematics with inquiry in a supported environment?* This paper reports on links between support and the teachers' development.

Four teachers of students in Years 4 and 5 (ages 8-11) at a government school in Queensland volunteered for the study. Teachers participated in four professional learning days during the year, once per term (approximately every 10 weeks). On these days, teachers were engaged as learners on various aspects of statistical inquiry. Time was also set aside for sharing of teaching experiences and planning their inquiry units. Sessions were recorded and portions transcribed for more detailed analysis. Teachers committed to teach an inquiry-based unit in their classrooms each term (see Table 1). They designed the units themselves, sometimes using published materials as a base. Lessons were videotaped to capture the flavour and content of the units, enculturate the researcher into the teachers' classroom practices, provide ongoing support, and gather episodic evidence of teaching and learning issues that arose while teaching the units. Teachers were interviewed at the beginning and end of units to gather data on goals, challenges that arose, unexpected outcomes and opportunities, what they learned and would change next time, and particular aspects that supported and moved forward their emerging expertise and confidence. As part of the process of supporting the teachers to improve and sustain these practices, the researcher continually sought their input into elements that had impact on their practice, working to both improve on their learning and to investigate links between these supports and evidence of the teachers' development.

Table 1
Units Designed by the Teachers each Term

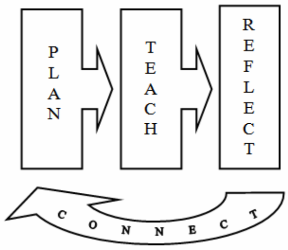
TERM	Year 4 Units (Kaye & Carla)	Year 5 Units (Naomi & Josh)
1	*Can you roll your tongue?* - Exploring hereditary traits	*Are athletes getting faster?* - Investigating winning times at the Commonwealth Games
2	*What's in your lunchbox?* - Investigating healthy lunches	*Kangaroos!* - Modelling and interpreting data from a predator-prey game on the oval (Naomi) *How fast is a blue-tongued lizard?* - Class negotiated investigation (Josh)
3	*Tibia mystery* - Estimating height from a tibia bone found at a archaeological dig	*Who is a "typical" Year 5 student?* Developing a survey and exploring "typical" (Naomi)
4	*How many commercials does a typical Year 4 student watch in a year?* (Kaye) *Comparing students' ages* (Carla)	*Investigating paper airplane designs* (Naomi) *Designing a parachute for an egg* (Josh)

An initial list of support mechanisms relevant to the context was developed using literature (Table 2). This list was used as a framework to code and mark episodes in the transcripts where the teachers discussed these support mechanisms or raised additional possibilities. Special focus was given to supports articulated by the teachers that helped them to connect their learning from one unit to apply to subsequent units and their evolving practice. Based on the episodes retrieved, the list was refined and illustrative examples were drawn from the interview data, focusing on those elements that demonstrated strong links to the development of the teachers' evolving practice (Figure 1). Due to the role these support mechanisms had in helping teachers to apply learning from one teaching experience to subsequent ones, they were called *connection levers*.

Table 2
Initial List of Support Mechanisms

• Developing content knowledge in an environment that models reform-based practices • Reform curriculum	• Collaborative environment • Sustained involvement • Time	• Feedback & Support • Validation • Multiple iterations • Reflection

Connection Levers



CONNECTION LEVERS		
• Inquiry experiences as a learner • Multiple iterations • Validation • Resources	• Sustained support and feedback • Collegiality • Development of deep disciplinary knowledge	• Time & support for reflection • Relevance • Accountability

Figure 1. Connection levers to support teachers' learning to teach with innovative pedagogies.

Results

Inquiry Experiences as a Learner

One of the most compelling experiences for the teachers in learning to teach using inquiry was having the opportunity to work through inquiry problems themselves. The first learning seminar focused on the ambiguity and uncertainty associated with ill-structured problems by having the teachers work together to design an ergonomic chair (adapted from TERC, 1998). They spoke throughout the year about the impact the activity had on them.

Kaye: I thought it was helpful to actually physically throw us into the deep end and say "I want you to investigate chairs". And for a lot of us that's very different to what we've done before and for us, even as a group, it was quite a hard task for us to maintain some sort of focus and to have a direction moving forward. And I think putting us in that situation was good because I think it showed us that some of the things the kids can happen – it gave us a little bit of an insight as to where we might need to help kids move forward.

Carla: You know it made you see sort of phases [of an inquiry process] didn't it? It made you see well, perhaps you need to just brainstorm this part first.

The teachers believed in principle that inquiry was a beneficial approach for learning, but before being immersed in a problem as learners they were unsure what an inquiry-based problem felt like. Experiences with the open-endedness of the initial activity therefore raised a number of issues they had not considered. They worried about teaching students to work collaboratively, managing student diversity in dealing an inquiry, and coming up with good problems. For one pair of teachers, they reflected on their own struggles managing ambiguity and decided they should carefully structure the first unit for their students.

Kaye: We've seen how difficult it is for us, that we'll try to make the introductory process less stressful for them. ...

Carla: Yeah, it might have to be more constrained the first time.

Kaye: Or they would probably need more teacher input or adult input or someone just to sit and focus them. Like you had to come back ... [and] focus on trying to subtly pulling us back to where you want us to go without dominating our investigation but you would hope that we even now, given the same task or a different task after lunch, would be more focused.

After teaching their first unit, Carla and Kaye again mentioned their experiences with the chair problem and how it caused them to decide to scaffold the first unit.

Carla: I was going to do less guidance or less modelling at the beginning but I'm glad I haven't. Otherwise yeah I could foresee that my kids would just go 'oh well, I don't know what I'm supposed to do so, oh well, why bother doing it', those kinds of questions.

Kaye: It's not only children. Let's have a look at four of us up a few, two or three weeks ago when we were given 'Do an investigation on a chair'. How much time did we spend, really without any direction? We were going off in all different planets. But we, as adults, we found it difficult to do, so children will find it difficult to do. I guess even as adults we like structure and we like a scaffold. I guess that's why Carla and I went for a scaffold and we're pleased that we did.

The reaction to the chair activity was quite different for the other pair of teachers. They wanted to give students more control and designed their unit to incorporate this.

Naomi: I think to a large extent this is how it does work in the world. ... It's not as if the boss is standing there saying, 'well this is what the end product has to look like and these are the steps you're going to take', which is what we do in the classroom.

Although the teachers had different responses to their experience with the ambiguity of an open-ended task in designing their first unit, it was clear that it was an important experience for them to think back to during the year.

Multiple Iterations

Regardless of whether they were structured or open-ended, all of the teachers ran into difficulties in their first units.

Naomi: The first one, we were more uncomfortable with it. ... We wanted something that was absolutely, you know, out of this world and we didn't, we didn't plan properly where it was going and whether or not we had the tools to get it to go in the right direction. ... *that* was a *steep* learning curve!

Kaye: It's like all things that we introduce to kids to start, we think the results you get on the first thing you do are probably not going to follow what we want, but probably the more that we do the better they get.

In the second unit, both pairs of teachers designed units that were more balanced between structure and open-endedness. Over the course of the year, they experimented with different phases of the inquiry cycle, sometimes focusing on data collection and other times on interpreting findings or communicating results. At the end of the year, Naomi reflected on how through multiple iterations, both she and her students came away with a robust sense of what statistical inquiry could do.

Naomi: The first unit we looked straight at data collection really, and the interpretation of that data. ... [The second unit] was, yeah, just collecting data and having a look at the data. Then the third unit we extended it a little bit further and we looked at devising our own [survey] forms with which to collect data. And then, interpreting the data to the extent of saying, well, you know, "What was a typical Year 5 student?" But the last one is by far my favourite one because it went right from collecting the data all the way through using that data. And then creating something from that data then using, um, taking more measurements and using that data to see what could be improved and keeping a cycle going. So the children could actually then look at the data and say, "Ok, well, this is what we can realize from it and this is what we need to do next time". It was so much more of a practical use in how we would really use that sort of data in the outside world.

Naomi's statement was indicative of observations by the other teachers as well. In nearly every case, the final unit was the most complex and well-designed. This suggests the iterations were central to the teachers' abilities to build their expertise.

Validation

Having the support of the researcher and the other teachers in the study helped them to build confidence and persistence. Particularly in the beginning of the project, the teachers had concerns about whether they were "doing it right". When things did not go well, they often blamed themselves for not anticipating issues in advance.

Naomi: I said it was the worst day because it was all the stuff I should have anticipated and allowed for so I was blaming myself. You have lessons where something goes wrong and it's outside your control—that's one of those things. But this was well within my control and I didn't account for it.

When I asked the teachers what helped them persist through the units, Josh commented that the validation that their experiences were normal was important to his ability to persist when things did not go as anticipated or unexpected school events disrupted the plans.

Josh: Well, to start with, … you're always there saying, "look, this is a normal classroom".

The students also validated the teachers' efforts through their enthusiasm and learning. Naomi recounted a particularly challenging day for her, but when she reflected on the kind of lifelong skills the students had gained from the unit, she felt validated.

Naomi: There was one day I could have thrown my hands up and said 'I'm not doing this' but I could see that the children were enjoying it. ... [And] the way they're now approaching things and saying "yes, but, what if - ? Could it be that - ?" And that's just wonderful.

Resources

Several times during the study, the researcher asked the teachers what they would suggest to someone attempting to teach mathematical inquiry.

Kaye: I do believe where teachers feel a bit threatened or are doing something new, they work better if they've got a structure to work from. They're more inclined to have a go at it. Like I don't know if we would have gone down the path that we have or had the ideas to go down the path that we have without the resource that we've used.

Naomi and Carla both talked about how they used the resources for inspiration and guidance to generate ideas.

Naomi: The other thing that really helped is that *TeachStat* book [Gideon, 1996] because just flipping through there was a really good place to start to get ideas. Because right from the start, it was well, "Ok, this is a great principle, great in theory. How do I do it? ... What do I *do*? How do I come up with ideas?" So that *TeachStat* book was actually full of some really good ideas. And one of them gave me the idea for "The Typical Year 5 Student" [her third unit]. ... [Otherwise] the ideas are hard to generate sometimes.

Carla: I'm sure that if Kaye and I didn't have that resource we'd be racking our brains trying to think of a good one that's going to try and interest as many people as possible.

Sustained Support and Feedback

Ongoing feedback and technical support were also important for the teachers. In an interview at the end of the year, Kaye recalled a suggestion to consider stacked plots instead of a single graph to allow students to compare, not just describe, their data.

Kaye And actually the support, the throwing in of things that we could do, I appreciated it. A couple of times when you came in, [and suggested] 'this is how you can do this'. ... For me, somebody that, I often learn a lot better and work a lot better when there is input. ... A classic example was stacked line plots, which was something that, you know, I hadn't even registered that stacked line plots made it so easy for the students to interpret the data. And from there that's something that they have been able to do a lot easier, doing it that way rather than putting it [a single graph] on their presentation. Yet in *all* the books I read through, it hadn't *mentioned* stacked line plots! So without your input there, I wouldn't have been able to fly the way I did.

Collegiality

The teachers also expressed how important it was for them to interact together and how this contributed to their ability to develop.

Josh: I think one of the most beneficial things about today, has just been listening to each other.

Naomi: We all had problems, it was ok because we could learn from each other's problems.

Kaye: I think has been one of the major aspects of [Carla] and I just actually working, and bouncing off [ideas] – "oh well this is what we can do, let's try it with this" or "let's use this resource", so that has been professionally very good for us.

Both the professional sharing of teaching in teams and the opportunity to share their experiences with others trying the same innovation was important to their development. It not only helped them continue the momentum, but also enabled them to learn each other.

Development of Deep Disciplinary Knowledge

Another connection lever that the teachers said helped them to sustain and develop expertise in teaching inquiry-based mathematics was their new understanding of statistics. This new learning changed the way that they focused their students' learning.

Carla: Now at the end of the year, I know what it might mean to understand a statistical investigation or working with data, where at the beginning of the year [I only considered] "can they draw that graph?" ... [But now we know] what to look for to say this child understands what working statistically means. [*To the others*] Wouldn't you say?

Naomi: Oh, definitely. I'll be honest, I used to look at chance and data and say, yeah, "if they can draw a graph – good, if they can work out the probability of tossing a head when tossing a coin – that's done. Chance and data's out of the way".

Carla: But now you can say, "Wow, this person can interpret that data and make this assumption".

Time and Support for Reflection

The time to think, to plan, to talk, to try things, and to generate ideas away from school was talked about by the teachers. This was time for them to reflect on what they had learned in a supportive environment with others sharing in the same experience.

> Naomi: Once you're out at the university, or anywhere else that's away from school, you stop thinking about what's going on at school. ... We could just shut out school completely, and just sit and talk and focus completely on maths. And that was really valuable.

Through supported reflection, the teachers drew on their experiences of each unit in planning subsequent units and to stand back, abstract from their experience, and consider how they would apply it to improving their practice.

> KM: What about for you professionally? What do you think that you gained?

> Naomi: Well a couple of things. First of all, I'd never actually thought to use an inquiry approach in mathematics before. We use it in science commonly but not in mathematics. So to see that there was a way that we could incorporate that into the classroom was wonderful. It was, uh, a learning curve for me though because I've realised now there's a lot more planning that I have to do in inquiry maths than I would in a normal maths unit. Simply because I have to try and anticipate now where the unit could go to make sure the children have those underlying skills.

Relevance

The project immersed the teachers in thinking about teaching with inquiry. The way that the professional learning opportunities were directly linked to the teachers' classroom practice and were sustained throughout the year became important support mechanisms for the teachers. Taken together, the inquiry experiences they had during professional development, the opportunity to participate in a community of learning about what they were doing in their classroom, and knowing that others were thinking through the unit with them as they were teaching it, all contributed to their ability to build their expertise. The opportunity to integrate their learning with their teaching was relevant to their classroom work and day-to-day practice. They were excited when they saw that the work they were doing was at the forefront of teaching mathematics and that the inquiry approach they were teaching was being promoted as well by state and local initiatives.

> Kaye: We've had to really look deeply at what an investigation really is and investigations really do form a major part of the new maths syllabus. ... One of the new [mathematics] outcomes ... was about children creating and interpreting and analysing data, which is all what we've been doing the whole year. So I guess this whole thing we've been doing has been excellent for us getting a handle on the sorts of things that we can do.

> Josh: There was a classroom magazine that a friend of mine had the other day and there was a big [article on] inquiry. ... I looked at it and I thought, "Oh! That's what we did!"

Accountability

A big issue for these teachers was juggling the demands on their time. With good teachers, there is intent to try new things, but sometimes the best intentions get buried. Naomi spoke about the fact that she would not have gone beyond the first iteration had I not been there expecting a unit to watch each term.

Naomi: [The accountability] kept me going. Otherwise, ... you go to the conference, you sit there and you write it all down. You say 'this looks wonderful', and you go back and you drop it on your desk. And about six months later when you sort out the pile of things that's built up on your desk. You go, "oh, that looks interesting, I'll put it in a file and I'll try and read that later". And that's kind of it. Whereas this was good. The first one [unit], yep, we did it. We did what we were supposed to do. It was good, I can see some value in it and I can honestly say, that I probably would have then said, "ok, well, I'll try that next year". Maybe! And then probably forgotten. Whereas because there was an expectation to do one every term, by the time you got to the last one, you felt comfortable with it, the unit was great, the kids took it to places that I just, and showed understandings that I didn't think they would be capable of. ... So, I'm completely sold, but it would have taken more than one to do that. ... The accountability, and the fact that you had to rehearse it, effectively, over and over, kind of solidified the skills.

Discussion and Implications

Over the past two decades, there has been a paradigm shift in the teaching and learning of mathematics. In this shift, the ideal for mathematical instruction transforms from an emphasis on skills, facts, and procedures towards greater stress on developing children's mathematical conceptions and proficiency at applying mathematical tools to new situations: in particular, open-ended, complex and everyday problems. In order for teachers to make these shifts in designing innovative learning experiences for their students, they must develop capability with this approach and be able to envision and embrace it. This project examined the process of learning to teach mathematical inquiry in a supported environment. The preliminary results presented here suggest that these connection levers enabled the teachers to reflect on their iterative experiences in teaching mathematical inquiry towards building their emerging expertise. The teachers described how these connection levers supported their ability to persist beyond the challenges encountered during the initial teaching experiences, and continue to sustain them, building their confidence and commitment in the process.

The teachers in this study developed a great deal of expertise in the course of a year, more than was predicted. It must be cautioned, however, that this is partially due to the fact that the teachers in the study already possessed beliefs about learning that were consistent with an inquiry-based environment. Quite possibly progress would be slow unless teachers first commit to an inquiry-based epistemology. Similar work in research on middle schooling suggests that unless teachers' philosophy is consistent with the reform, any apparent change in practice is not sustainable (Pendergast et al., 2005).

Although these findings are tentative and preliminary, many of the connection levers named by the teachers were consistent with research on good professional development (Elmore, 2002; Ball, 1996; Cohen & Hill, 2001). There was no magic in these levers; none are beyond the reach of schools or districts with creative leadership. The challenges the teachers faced and the supports they named were in the context of work in authentic classrooms with diverse student needs. The use of a design experiment further supported the applicability of the research and layers of iterative learning by the researcher, teachers, and students. On one hand, the excerpts from the teachers and the support mechanisms they list point to the complexity of moving teachers from a stage of orientation about teaching mathematical inquiry towards a commitment to teaching with this approach. On the other hand these supports are consistent with moves in education to support more collaborative engagement of teachers throughout their careers in the learning profession.

Postscript

An additional support that has been discussed and will be trialled this year is having the researcher model particular teaching approaches with the teachers' students in their classrooms. This kind of interaction, if equally effective, would further support recommendations for expanding partnerships between schools and universities (Loucks-Horsley, Love, Stiles, Mundry, & Hewson, 2003). The teachers are already being utilised by their schools to begin training their colleagues in this approach. In addition, they are presenting their work at teachers' conferences both locally and nationally.

Acknowledgement. This research was funded by an Early Career Grant from the University of Queensland.

References

Ball, D. (1996). Connecting to mathematics as part of learning to teach. In D. Shifter (Ed.), *What's happening in math class?* (Vol. 2, pp. 36-45). New York: Teachers College Press.

Ball, D. (2002). *Mathematical proficiency for all students.* Santa Monica, CA: RAND.

Boaler, J. (1997). *Experiencing school mathematics*. Buckingham, UK: Open University Press.

Cobb, P., Confrey, J., diSessa, A., Lehrer, R., & Schauble, L. (2003). Design experiments in educational research. *Educational Researcher, 32*(1), 9-13.

Cohen, D. & Hill, H. (2001). *Learning policy: When state education reform works*. New Haven: Yale Press.

Confrey, J. & Makar, K. (2002). *Developing secondary teachers' statistical inquiry through immersion in high-stakes accountability data*. Paper presented at the 24th Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education, Athens, GA.

Cuban, L. (1990). Reforming again, again, and again. *Educational Researcher, 19*(1), 3-13.

Dewey, J. (1938/1997). *Experience and education*. New York: Simon and Schuster.

Diezmann, C., Watters, J., & English, L. (2001). *Difficulties confronting young children undertaking investigations*. Paper presented at the 26th Annual Meeting of the International Group for the Psychology of Mathematics Education. Utrecht, the Netherlands.

Elmore, R. (2002). *Bridging the gap between standards and achievement: The imperative of professional development in education*. Washington, DC: Albert Shanker Institute.

Gideon, J. (Ed.), (1996). *Teach-Stat activities: Statistics investigations for grades 3-6.* Parsippany, NJ: Dale Seymour Publications.

Hancock, C., Kaput, J. J., & Goldsmith, L. T. (1992). Authentic inquiry into data: Critical barriers to classroom implementation. *Educational Psychologist, 27*(3), 337-364.

Loucks-Horsley, S., Love, N., Stiles, K., Mundry, S., & Hewson, P. (2003). *Designing professional development for teachers of science and mathematics*. Thousand Oaks, CA: Corwin Press.

Makar, K. (2004). *Developing statistical inquiry*. Doctoral Dissertation, University of Texas at Austin. Online at http://www.stat.auckland.ac.nz/~iase/publications/dissertations/dissertations.php. (Accessed 29/3/2007).

Makar, K. & Confrey, J. (2007). Moving the context of modeling to the forefront. In W. Blum, P. Galbraith, H-W. Henn, & M. Niss (Eds.), *Modelling and Applications in Mathematics Education* (pp 485-490). New York: Springer.

Pendergast, D., Flanagan, R., Land, R., Bahr, M., Mitchell, J., Weir, K., Noblett, G., Cain, M., Misich, T., Carrington, V., & Smith, J. (2005) *Developing lifelong learners in the middle years of schooling.* Brisbane: University of Queensland.

National Research Council. (2000). *Inquiry and the National Science Education Standards: A guide for teaching and learning*. Washington, DC: National Academy Press.

Reitman, W. (1965). *Cognition and thought: An information-processing approach*. New York: Wiley.

Schwab, J. J. (1978). *Science, curriculum, and liberal education*. Chicago: University of Chicago Press.

Stigler, J., & Hiebert, J. (1999). *The teaching gap*. New York: Free Press.

TERC. (1998). *From paces to feet.* Parsippany, NJ: Dale Seymour Publications.

Tyler, R. (1949/1969). *Basic principles of curriculum and instruction*. University of Chicago Press.

Wild, C., & Pfannkuch, M. (1999). Statistical thinking in empirical enquiry. *International Statistical Review, 67*(3), 223-265.

Acquiring the Mathematics Register in te reo Mäori

Tamsin Meaney
University of Otago
<tamsin.meaney@otago.ac.nz>

Uenuku Fairhall
Te Kura O Te Koutu
<uen_fai@koutu.school.nz>

Tony Trinick
University of Auckland
<t.trinick@auckland.ac.nz>

Acquiring the mathematics register is often assumed to occur when learning mathematics. However, when students learn in a second language and are taught by teachers who are also not teaching in their native language, this may not be a straight-forward process. This paper describes the strategies that teachers in a Mäori immersion school (kura kaupapa Mäori) used to scaffold and model the mathematics register. Although most strategies could be seen in many classrooms, there were some strategies that seemed to be related to the students and teachers using te reo Mäori as the language of instruction.

Kö ta te rangatira kai he körero

As described in the whakatuaaki above, language is the food of chiefs because fluency in it provides access to and control of learning. Khisty and Chevl (2002) summarised the impact of this by stating, "[i]n essence, those with power are literate or in control of a discourse" (p. 167). Analysis of the student data from the Mäori medium numeracy project (*Te Poutama Tau*) found that language proficiency was a significant factor in student achievement in the higher stages of the number framework (Christensen, 2003). In kura kaupapa Mäori, students often have te reo Mäori as a second language, with various degrees of fluency in it. This means that there is a need to understand more about how to support students learning mathematical content at the same time that they are learning te reo Mäori and in particular the mathematics register, te reo tätaitai. This situation is complicated by the newness of this register in te reo Mäori (Christensen, 2003; Meaney, Fairhall, & Trinick, 2006).

During 2005 and 2006, the scaffolding and modelling of students' mathematical language by the teachers in a kura kaupapa Mäori was documented. It involved a partnership between seven teachers of mathematics and three researchers who are the authors of this paper. The kura teaches mathematics to students from Year 0 to Year 13. The teachers in the primary section of the school were also participating in *Te Poutama Tau* and felt that this research would complement that project. The final stage of the research investigated how this knowledge affected the teaching practice of those involved and this enabled an appropriate evaluation of the research for its practical value to be undertaken. Better understanding of how the mathematics register is acquired is likely to be of benefit not just to kura kaupapa teachers and their students but to others considering language issues in other content areas.

This paper provides information on the first part of the project, the strategies that the teachers used to support students learning te reo tätaitai. The role of the teacher has been emphasised in providing the environment in which learning should occur (Anghileri, 2002). This learning includes expectations about the interpretation and production of mathematical language (Khisty & Chevl, 2002). Research by Khisty and Chevl (2002) showed the importance of the teacher's own use of mathematical language when students were learning in a second language. When teachers did not use mathematical language fluently, their students were unable to describe mathematical ideas.

The two main ways that support is provided to students to learn and use the mathematics register are modelling and scaffolding. Modelling is when a teacher uses mathematical language within an appropriate context. For example, if a student provides a response to a mathematical task in everyday language, a teacher might rephrase it in more appropriate mathematical language (Chapman, 1997).

Scaffolding is when a teacher provides part of a response with the student completing the rest. Wood, Bruner, and Ross (1976) originally described the scaffolding by an adult as that which "enables a child or novice to solve a problem, carry out a task or achieve a goal which would be beyond his unassisted efforts" (p. 90). As time goes on, a teacher would expect to reduce the amount of scaffolding and modelling that is provided, thus transferring the responsibility for using the language from the teacher to the student. However, as Williams and Baxter (1996) stated, there is a risk that this transfer of responsibility fails to occur in many classrooms: "Edwards and Mercer pointed out that handover, or the process of gradually shifting control of learning from teacher to student, was missing in the classrooms they observed" (p. 25).

Although the work of Bickmore-Brand and Gawned (1990) would suggest that the effect of modelling and scaffolding of mathematical language has been known for some time, there has been limited research on what are effective modelling and scaffolding strategies. Chapman's (1997) study would be the most comprehensive. From watching a secondary mathematics class for a term, Chapman described how teachers reframed student responses so that they: clearly showed the relationship to the theme of the lesson; focussed on the typical linear, metonymic structure rather than the metaphorical content; and became more certain and less hesitant (what she labelled as high modality). Although Chapman concentrated on the teacher's role within the interactions, researchers such as Rogoff (1988) showed that students themselves have a major influence on the types of scaffolding and modelling that are offered to them.

There is also cross-cultural research on mother-child interactions which suggests that the ways that scaffolding are undertaken are culturally determined (Kermani & Brenner, 1996). Research in reading classrooms for Hawaiian students suggested that reading achievement increased when discourse interaction patterns more closely matched those of a traditional Hawaiian cultural activity, such as talk story (Au, 1980). Therefore, Mäori teachers teaching Mäori children in te reo Mäori may not use scaffolding strategies similar to those identified by Chapman. Nelson-Barber and Estrin (1995) suggested that:

> [u]nfortunately much of the knowledge on culturally influenced notions of good teaching remains unrecorded and unformalized because, as a whole, educators (researchers and practitioners alike) have made little effort to elicit the perspectives and experiences, or study the classrooms, of teachers who are highly effective with non-mainstream students (p. 5).

Methodology

The ethnographic research tradition was used in this research for two reasons. The first is that research in kura kaupapa Mäori needs to be in alignment with Kaupapa Mäori or Mäori-centred research tradition. The second was because the project was about evaluating the effectiveness of different modelling and scaffolding strategies requiring an in-depth consideration of what this meant. Christensen (2003) summarised the five dimensions that contribute to Kaupapa Mäori research. Each of these dimensions is described in the following paragraphs, with an indication of how they were met in this project.

A Mäori World View

There is a need for the unique Mäori world view to be reflected in what is researched, how it is analysed and written up. In considering how te reo Tätaitai is scaffolded, there is a need to be aware of those strategies that are unique to the language and culture of the teachers and the students. If Mäori students are to improve their educational achievement, the role of culture in learning needs to be acknowledged. It cannot be assumed that good teaching for students from diverse backgrounds will always look the same (Alton-Lee, 2005). It is therefore important that effective practices that resonate with cultural practices are documented, and this was one of our aims for this project.

Culturally Safe Research Practices

There is a need for Mäori to feel that they will not be exploited as a consequence of being involved in research. Irwin (1994, cited in Christensen, 2003) suggested "mentoring by kaumätua and research being undertaken by a Mäori researcher as two aspects of culturally safe practices" (p. 14). In our project, two of the principal researchers are respected Mäori mathematics educators. Their involvement has provided a mentoring role for the teachers who were involved in researching their own practices. Regular meetings with teachers meant that the project could evolve to meet the needs of the kura as the teachers' opinions and ideas were incorporated into what was being researched and how this was being done.

Challenges to Existing Power Relationships

It is important that Kaupapa Mäori research results in Mäori development. In order to do this, the way that Mäori have traditionally been portrayed needs to be reconsidered. This will support students' active movement into the wider society as the primary benefactors from the research. By documenting effective strategies and acknowledging their relationship to culture, we anticipate that the impact of this research will not just support students at this kura but be of value to students at other kura.

Accountability and Mediation

There is a need to ensure that control of the research remains with Mäori so that "the research is worthwhile and contributes to Mäori development" (Christensen, 2003, p. 15). This will ensure continued validation of the research so that it reflects a Mäori world view and culturally safe research practices. In our research, we did not have a supervisory group. However, the project was jointly run by the researchers, two of whom were Mäori, with frequent meetings with the teachers who were also researchers of their own practice. As a group research project, there were opportunities for reassessment as it progressed. The project therefore was accountable to the people who were involved in it.

The Researcher is Concerned with Mäori Advancement

The positioning of the researcher is important in Kaupapa Mäori in order for the different issues of doing research, such as the need for Mäori development, ethics, and being systematic, to be considered. This research was a joint activity that valued the different skills and experiences brought to the research project. This ensured that the various demands of the research were dealt with adequately. All of those involved in the project are concerned with Mäori advancement.

Method

Data was primarily collected through videoing each of the seven teachers' mathematics lessons in both 2005 and 2006. The classroom interactions were transcribed and the teachers then watched them with a university researcher. The joint analysis involved identifying the modelling and scaffolding strategies that the teachers used in the classroom. These were arranged around the stages in the Mathematics Register Acquisition (MRA) model (Meaney, 2006). These stages and their strategies are described in the next section.

Findings

Our original research question had been about identifying the *effective* strategies used by teachers to support students in acquiring aspects of the mathematics register. However, it soon became clear from our analysis that a scaffolding or modelling strategy could not be judged as effective in isolation from the whole lesson or in fact from classroom practices in general.

Noticing

The Noticing stage is when the teachers introduce new terms or expressions or add extra meanings to ones that students are already familiar with. The function of this stage is to make students aware of new aspects of the mathematics register, whether these are new layers of meaning for already known terms or previously unheard terms or expressions. The strategies that were identified for this stage were:

- providing opportunity for the new terms to be used appropriately
- using linguistic markers to highlight what was to come
- using intonation to emphasise a correct term after students used an incorrect one
- repeating new terms and expressions several times in appropriate places
- rephrasing the expressions by using other terms
- writing the new term in an equation which is related to what has just been discussed
- giving definitions verbally and through diagrams
- emphasising the relationship between ideas using diagrams or physical materials and words
- modelling a new term/skill (idea) as it is being explained
- after teacher explanation, having students say back the new term
- having students repeat the final answer after the teacher has modelled finding the solution
- relating new terms to already known ones
- using a set of leading questions so that students are channelled into using a particular term
- using fill-in-the-blank sentences
- acknowledging the difficulty of learning some terms (ideas)
- providing a rationale for the need to learn a new term (idea)
- requesting students' attention before introducing a new term
- describing a new term as being important in a subsequent lesson

This stage is characterised by teachers doing almost all of the cognitive work. They engineer the activity so that the new terms are needed. They ensure that the words are used frequently, mostly by themselves but also by the students. Quite often when a new term is being introduced, the teachers repeat it many times, often associating it with activities.

In one of Teacher 6's (T6) lessons on introducing division, she used *whakawehe* (division) 41 times and the students (äkonga) used it 10 times. These repetitions were spaced, giving students time to absorb the vocabulary. Spacing repetition has been noted as important in vocabulary acquisition in second language learning (McNaughton, MacDonald, Barber, Farry, & Woodard, 2006). In the extract, the teacher had the students separate blocks into groups. This allowed her to introduce the term *whakawehe,* which then became the focus of the lesson.

T6: Nā ka ono, waru, tekau, i kaute ahau i ngā ( ) ana ( ). E hia ngä mea paraone? E whä ngā röpü. He rereke, äe. He mahi mämä tënei. Ko te tumanako, he mahi mämä mä koutou. E whä ngā röpü taki, ana. E hia ngā tae ia tae. E whä ngā röpü. Nuku atu i ö koutou pukapuka kia taea e koe te waiho ngā mataono ki mua i a koe, kua pënei koe.

Äkonga: Äe.

T6: Kua whakatakoto koe i o mea pënei [teacher observes students]: Nö reira, titiro mai, he mea kowhai i përä hoki koe.

Äkonga: Käo.

T6: Anä, he aha te pätai mä koutou? I tënei rä. Käore au i te hoatu te whakawehe ki a koutou nërä mä koutou. Kia whakaaro, āe, me whakaaro pea e koutou. Mehemea i ahau e rua ngā röpü takitoru. E hia te katoa o ia takiwha? E hia te katoa o ngā tor- toru?

Äkonga: Ono

T6: Ka tahi, rua, toru, whä, ono, ko tënei te whakarau aha e ono.

Äkonga: Toru, toru

T6: Tuhia te whakawehe möku. E hia te katoa ehara ko te toru [throws pen to child]

Äkonga: ( )

T6: Timata i te aha, ka pai.

It would seem that for a strategy to be an effective, it must contribute to students hearing new vocabulary or grammatical expressions frequently and gaining meaning from them. At this stage, the understanding that students are expected to acquire is usually a definition. However, the teachers giving a rationale also provided another kind of meaning to the new aspect of the register that they were highlighting.

Intake

By this stage, some of the cognitive load has shifted to the students. They now need to give definitions and examples, rather than just being expected to notice and interpret those provided by the teacher. However, the teacher is still very much in control and students' contributions are usually short, thus providing them with little opportunity to provide inappropriate responses.

Teacher check on their students' understanding by asking them for definitions. If the definitions were concise and clear, then the students were at the Output stage. When the teacher or other students had to provide extra clarification, prompts, and/or information, then the students were more likely still to be learning how to use the terms and so would be at the Intake stage. In the extract, the teacher commanded a student to explain what was happening when two lines met on the graph (*tutakitanga* and *rerekë*). The student went up to the whiteboard and was helped in the explanation by suggestions from other students and from the teacher.

T7: Inanahi, i tuhi au ngä rärangi e rua me te pätai ki a koutou. Ah, käre, i te pätai he tono ki a koutou, körerohia mai te tutakitanga o ngä rärangi e rua. Nö reira, Äkonga 1 haere ki te tuhi i ngä rärangi e rua

Äkonga 1: E ai ki töku mea

T7: Oh, koinä täu e kï ai he rerekë

Äkonga: Whä ripeka, oh, mäku e tuhi engari, pätai mäu e whäki mai
Äkonga 1: [stands up and goes towards whiteboard]
T7: Äkonga 2, hoki ki a koutou kei te pai kë mehemea i tino pango te rärangi o waenganui o ngä tua, he uaua te kite i runga i tënä
Äkonga 2: Oh
Äkonga 1: Oh he aha tënä?
Äkonga 2: Oh
Äkonga: Whä kei runga rua ki te taha
Äkonga: I whakaaro au i tuhi au e rima
T7: Koinä te tutakitanga, në?

The strategies that teachers used at the Intake stage of scaffolding students' acquisition of te reo tätatai were:

- choral responses with the students
- having students as a group do choral responses
- giving the first syllable of a term so that students are reminded of the term and then complete it
- asking students for names, definitions, or explanations of terms
- having students model the use of terms/skills (ideas)
- asking students for examples of a term
- using the similarities between concepts (e.g. 7 + 3 and 70 + 30) as an entry into having students reflect on the differences
- having students draw their own diagrams or use materials to show a particular term
- repeating or having students repeat appropriate responses
- elaborating on students' responses in words and with diagrams
- asking further questions to help students reflect on what they were describing and to check on what they know or have done
- having students provide a rationale for what they are learning
- ignoring inappropriate answers and just acknowledging appropriate ones
- querying students' inappropriate responses
- suggesting that students' inappropriate responses are close
- having students work backwards from an inappropriate answer to the question which was asked
- using specific amounts to illustrate a general rule (idea)
- focusing students back onto the main idea being discussed to help solve a problem
- using student-devised terms in giving an explanation
- going over an activity which requires the use of the new language as a whole class before expecting students to do the activity as individuals
- showing students the relationship between what they already know and can do and the new language term or skill
- having students answer a series of closed questions to lead them to using the new term/skill (idea)
- after modelling how a new term or skill is used, having students repeat the action
- recording in writing what had been discussed or done
- students can query obvious errors by the teacher or another student

The function of the Intake stage is for students to form understandings of when and how new aspects of the mathematics register are to be used. Effective strategies, therefore, are ones that support students exploring when and how to use these new aspects of the mathematics register. This support would include providing students with both positive and negative feedback about their experimentation with the new aspects.

Integration

By the Integration stage, students have a good understanding of the new aspects of the mathematics register. They just need to be reminded that they have good skills and knowledge and that they should be making use of them. For example, listening is a skill that students need to become fluent in. In the following discussion, the teacher seemed to be predicting that some students would struggle to follow the logic so she used words and commands to ensure that they paid full attention to the important sections.

T1: Tekau ngä tapa, tekau ngä mata me ngä akitu, tekau mä rua ngä tapa tapirihia kia rua, ä, ka tekau mä rua kë tërä. He oi anö, i mutu i te karaehe i kï mai kë tëtahi; "Whaea kei te hë tëtahi o ngä mahi, me kï ngä kaute, kua hë tëtahi o ngä wähanga." Ko [Äkonga 1] tërä, he aha täu i kite ai?
Äkonga 1: E waru ngä tapa?
T1: E?
Äkonga 1: Waru ngä tapa.
T1: Me whai kë mehemea kei te tika ia. Tahi, rua, toru, whä, rima, ono, whitu, waru, nä reira, käore ko te tekau! Nä reira, kei te tika te maha o ngä mata me ngä akitu?
Äkonga: Äe!
T1: Äta whakaaro koa!
Äkonga: Äe!
T1: Äe, i te mea he aha tëtahi atu huarahi i kite kë?
Äkonga: Tapirihia te rima ki te rima?
T1: Nä reira, kei te körero, i rongo koe, koutou i a ia e kii ana? Körero mai anö koa, tama.
Äkonga: Tapirihia te rima ki te rima*?*
T1: Tapirihia te maha o ngä mata ki te maha o ngä akitu, kua puta kë ko te tekau, nëhä? Te maha o ngä tapa me kï waru inäianei, he aha te huarahi e whai ake? Äe!

This was part of a discussion of how Euler's rule (Vertices + Faces – Edges = 2) worked on a pyramid and how some of what had been discussed on the previous day had been incorrect. The *kë* highlighted for the listeners that they should notice and be surprised by what follows. It, therefore, acted as a scaffolding device for students' listening. They needed to listen so that they could understand the differences between what had been said on both days. This was further emphasised by the teacher with the command "*Äta whakaaro koa!*", which was to understand carefully and occurred a few turns later. Once the student had responded to the initial question, the teacher emphasised that the students needed to listen. She then had the student repeat what he had said. All of these examples suggest that the teacher was confident that the students would understand what was being discussed but, because of its complexity, she needed to remind them to be careful so that they would not miss the information.

The function of this stage is to have students use new aspects of the mathematics register but in a situation where the teacher is able to step in and provide support if necessary. Consequently, the teacher's role has become one of reminding students of what they know and can do. The students are the ones who have the major responsibility for making use of the language that they have gained. If the student seems unable to operate at this level, the teacher is quickly able to supply more support, thus recognising that the student is still at the Intake stage. The strategies at the Integration stage included:

- using commands and linguistic markers to highlight for listeners that they need to pay extra attention to what they are hearing and doing
- encouraging students to make contributions to the teacher and to each other
- reminding students to think about what they already know
- asking a student to repeat a good response
- if a slight correction is needed, the teacher repeat the response correctly
- summarising what a student has said
- if a slight correction is needed, the teacher can model doing the action so that the student self-corrects their own response

- prompting in a general way for more details
- having students write a summary of, or record as a diagram what they have learnt
- facilitating an environment where students will correct each other
- asking students to say whether an answer/term is correct
- repeating the question if the students appear to have responded to a different one
- having students complete appropriate actions as they respond to questions

Effective strategies are ones that allow students to have major control of their use of the mathematics register but enable the teacher to remind students about what they know and can do.

Output

The final stage of the MRA model allows students to show their fluency in using the mathematics register. Its function is for students to be able to show what they know and can do without any support from the teacher. At this stage, there are only the two following strategies:

- providing opportunities for students to use their acquired aspects of the mathematics register between themselves and with the teacher
- providing an environment in which the students can query the language use of the teacher

The teacher's role is simply to provide opportunities for students to make use of the fluency that they have acquired. An effective strategy is, therefore, one that supports this provision. This extract comes from T1's fifth lesson, where a student had to describe the arrangement of five blocks to another student. The second student could not see the arrangement and relied entirely on the first student's description. Many students struggled initially with being able to describe the arrangement of groups of different coloured multi-link blocks. However, it was clear from this student's response that he had full control of the location expressions and knew how to use them to give a clear description in this activity.

Ākonga: E rua ngā mea o te kōwhai ki te taha, kotahi te mea kōwhai, oh, e rua ngā whero ki te taha. Kotahi te mea köwhai o ia huapae.

Combining Strategies

When considered in isolation, some strategies employed by teachers at the various stages of the MRA model could be considered less effective than others. For example, having students repeat an answer, after the teacher has gone through an explanation to reach it, is perhaps not going to highlight for students new aspects of the mathematics register very effectively. However, when this is one strategy of many, all designed to support students to become aware of these new aspects, then it could be seen as having more value. In each of the lessons, if the teachers used strategies from any of the MRA stages, they would always use more than one strategy. Combining a range of strategies, therefore, seems to be part of what makes effective support for students who are operating at the different stages.

Mäori Scaffolding and Modelling Strategies

In considering the modelling and scaffolding strategies for supporting the acquisition of the mathematics register, all of the strategies can be considered culturally appropriate because they were used by these teachers. Many of the strategies used by the teachers in this project would also be seen in English medium classrooms both in New Zealand and in other countries. However, the use of the linguistic resources within te reo Mäori for scaffolding is one strategy that is unique. Words, such as *ara* and *kë*, that warn listeners about the type of material that will follow are not found in English. Given that Mäori immersion education was set up to reverse the decline in Mäori language (Spolsky, 2003), there has been a recognition that "the authenticity of the language is maintained" (Christensen, 2003). Concerns have been raised about the possible implications for te reo Mäori as a consequence of its use for discussing mathematics (Barton, Fairhall, & Trinick, 1998). It, therefore, is interesting to find authentic resources within te reo Mäori that can be of value in the teaching of mathematics.

Another feature, although not unique to kura kaupapa Mäori classrooms but that seemed to be more strongly observed in the video recordings, was the number of student contributions to the interactions. Even at the Noticing stage, which is where teachers have the most responsibility for doing the cognitive work, students have an active role in contributing to the discussions. It was quite clear that the originators of interactions could be students as often as it was the teacher. Video recordings of pairs of senior students show them working together as "teacher" and "learner". The lack of reticence in taking up either role is considered to be an outcome of the valued *tuakana-teina*, older-younger sibling, relationship. Mäori children do not traditionally segregate themselves into age-based peer groups, rather there is the expectation that they will take responsibility for each other, whether younger or older. This can be seen in interactions around the learning of mathematics.

It would seem that strategies that reflect a Mäori world view are those that use the features of te reo Mäori effectively and those that support students to become active participants in interpreting and producing the mathematics register appropriately.

Conclusion

The setting up of kura kaupapa Mäori was done to support the revival of te reo Mäori. Consequently mathematics has been taught through this language to students who are not only learning mathematics but also learning the mathematics register in te reo Mäori. This research has begun an investigation with teachers about how they support students to learn te reo tätaitai.

In this paper we have outlined the strategies that teachers used in the four stages of the MRA model. It was noted that all of the teachers used a variety of strategies when operating at each of the stages, except for the final stage, Output. As this stage was about the students fluently using te reo tätaitai in authentic situations, it was unsurprising that the teacher's role was one of providing appropriate opportunities that would allow students to us the new aspects of the mathematics register.

We also documented strategies that seemed to be related to the language and the culture of the students and their teachers. These strategies are interesting because they encourage the use of what is already present to be incorporated into the mathematics teaching. For teachers in other kura, this information means that they no longer have to rely only on adapting what is considered best practice in English medium classrooms.

Acknowledgements. This research was funded by the New Zealand Ministry of Education through a Teaching and Learning Research Initiative, administrated by the New Zealand Council of Educational Research.

References

Alton-Lee, A. (2005, May). *Quality teaching for diverse learners: How an evidence-based approach can help.* Keynote Address to ACSA (Australian Curriculum Studies Association) Forum, Quality teachers: Quality teaching – Creating a new agenda for action by practitioners, researchers and policy makers. Melbourne.

Anghileri, J. (2002). Scaffolding practices that enhance mathematics learning. In A. Cockburn & E. Nardi (Eds.) *Proceedings of the 26th annual conference of the International Group for the Psychology of Mathematics Education* (Vol 2, pp. 49-56). Norwich, UK: PME.

Au, K. H. (1980). Participation structures in a reading lesson with Hawaiian children: Analysis of a culturally appropriate instructional event. *Anthropology and Education Quarterly, 11*(2), 91-115.

Barton, B., Fairhall, U., & Trinick, T. (1998). Tikanga reo tatai: Issues in the development of a Maori mathematics register. *For the Learning of Mathematics, 18*(1), 1-17.

Bickmore-Brand, J. & Gawned, S. (1990). Scaffolding for improved mathematical understanding. In J. Bickmore-Brand (Ed.), *Language in mathematics* (pp. 43-51). Melbourne: Australian Reading Association.

Chapman, A. (1997). Towards a model of language shifts in mathematics learning. *Mathematics Education Research Journal*, *9*(2), 152-172.

Christensen, I. (2003). *Exploring issues in mathematics education.* Wellington: Ministry of Education.

Kermani, H., & Brenner, M. E. (1996, April). *Maternal scaffolding in the child's Zone of Proximal Development: Cultural perspectives*. Paper presented at the American Educational Research Association, New York City.

Khisty, L. L., & Chevl, K. B. (2002). Pedagogic discourse and equity in mathematics: When teachers' talk matters. *Mathematics Education Research Journal*, *14*(3), 154-168.

McNaughton, S., MacDonald, S., Barber, J., Farry, S., & Woodard, H. (2006). *Ngä Taumatua*. Wellington: Ministry of Education. Retrieved on August 26, 2006 from: http://www.minedu.govt.nz/index.cfm?layout=document&documentid=10606&data

Meaney, T. (2006). Mathematics Register Acquisition. *set* (3), 39-43.

Meaney, T., Fairhall, U., & Trinick, T. (2006, February) *The role of language in ethnomathematics: Does the language make it ethnomathematics?* In the Proceedings of the Third Ethnomathematics Conference, Auckland. Retrieved on August 26, 2006 from: www.math.auckland.ac.nz/~poisard/ICEm3/ICEm3.html

Nelson-Barber, S., & Estrin, E. (1995). *Culturally responsive mathematics and science education for native students.* San Francisco, CA: Regional Educational Laboratory Network.

Rogoff, B. (1988). The joint socialisation of development by young children and adults. In A. Gellatly, D. Rogers & J. A. Sloboda (Eds.), *Cognition and social worlds* (pp. 57-82). Oxford: Oxford University Press.

Spolsky, B. (2003). Reassessing Mäori regeneration. *Language in Society*, *32*, 553-578.

Williams, S. R., & Baxter, J. A. (1996). Dilemmas of discourse-orientated teaching in one middle school mathematics classroom. *The Elementary School Journal, 97*(1), 21-38.

Wood, D., Bruner, J. S., & Ross, G. (1976). The role of tutoring in problem solving. *Journal of Child Psychology and Psychiatry*, *25*, 89-100.

Teaching Ratio and Rates for Abstraction

Mike Mitchelmore
Macquarie University
<mike.mitchelmore@mq.edu.au>

Paul White
ACU National
<paul.white@acu.edu.au>

Heather McMaster
Macquarie University
<heather@workingmaths.net>

A group of practising teachers implemented the Teaching for Abstraction method for the Year 8 topic "Ratio and Rates". The authors first constructed materials for a unit in which students explored familiar ratio and rates contexts, searched for similarities in their mathematical structure, defined the two concepts, and learned to apply these concepts to other contexts. After an introductory workshop, teachers taught the topic in six 1-hour lessons. They experienced considerable difficulties adapting the approach to the abilities and interests of their particular classes, but all students showed evidence of learning. It was concluded that, although Teaching for Abstraction shows promise, there are many factors that need to be taken into account if it is to be implemented in practice.

Teaching for Abstraction is an approach to teaching that takes account of the fact that most elementary mathematical ideas are abstractions from experience (Mitchelmore & White, 2004a). It consists of four steps, in which the teacher helps students to:

- *familiarise* themselves with the structure of a variety of relevant contexts,
- *recognise* the similarities between these different contexts,
- *reify* the similarities to form a general concept, and then
- *apply* the abstract concept to solve problems in related contexts.

The rationale for this approach is the theory of empirical abstraction (Mitchelmore & White, 2004b), where an abstract concept is seen to be "the end-product of ... an activity by which we become aware of similarities ... among our experiences" (Skemp, 1986, p. 21).

Teaching for Abstraction was originally developed from research on the learning of angle concepts in primary school (Mitchelmore & White, 2000). It has been applied successfully to the teaching of angle concepts in Stage 2 (NSW Department of Education and Training, 2003) and has also been trialed with decimals in Year 4 (Mitchelmore, 2002) and percentages in Year 6 (White & Mitchelmore, 2005). The project reported in this paper is one of two studies conducted in 2006 in which we continued to investigate student learning of multiplicative relations through Teaching for Abstraction.

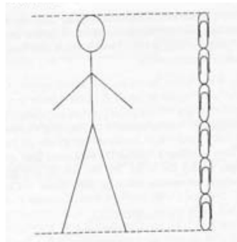
Multiplicative Relations

A cursory look at the school mathematics curriculum shows that multiplicative relations underpin almost all number-related concepts studied in school (e.g., fractions, percentages, ratio, proportion, rates, similarity, trigonometry, and rates of change). Vergnaud (1983) called this set of concepts the multiplicative *conceptual field.* There is a long history of research showing that many children have considerable difficulty understanding these concepts (Behr, Harel, Post, & Lesh, 1992; Carpenter, Fennema, & Romberg, 1993; Harel & Confrey, 1994).

Proceedings of the 30th annual conference of the Mathematics Education Research Group of Australasia
J. Watson & K. Beswick (Eds), © MERGA Inc. 2007

Ratio is a crucial multiplicative concept, and one of the most difficult ideas for students to come to terms with. Although working with ratios of the form *a:b* when *a* is a multiple of *b* causes few problems, cases where *a* is not a multiple of *b* can be highly problematic. Particularly prevalent is the so-called additive error (Misailidou & Williams, 2003) illustrated by the following, taken from a seminal study of a sample of 2257 English students aged 13-15 (Hart, 1982).

> You can see Mr Short's height measured with paperclips. When using matchsticks, Mr. Short's height is 4 match sticks. His friend Mr. Tall's height is 6 match sticks. How many paper clips are needed for Mr. Tall's height?



Mr Short

Only about one third of the students could correctly answer this question, with the majority opting for the answer of 8 (focusing on the additive difference 6 – 4 rather than the multiplicative 6:4).

An additional problem is the confusion that often arises between ratios and fractions. For example, if two boys and three girls sit at a table, the fractions $\frac{2}{5}$ and $\frac{3}{5}$ and the ratio 2:3 (often equated to $\frac{2}{3}$) arise. When a ratio connects two parts of the same whole, students may not adequately differentiate the part-part from the part-whole relationship (Clark, Berenson, & Cavey, 2003).

In New South Wales, the syllabus (NSW Board of Studies, 2002) suggests that rates be taught after ratios in Year 8 in a purely arithmetical context without any reference to slope or linear relations. It seems reasonable to expect the same errors as for ratio. The need to take account of different units may introduce additional errors. However, the fact that the two components clearly relate to different variables may reduce the prevalence of the additive error and eliminate the ratio-fraction confusion.

The Present Study

We hypothesise that students' poor performance on multiplicative tasks is at least in part due to the fact that curriculum materials rarely highlight their multiplicative nature. An emphasis on the underlying structure, including helping students to differentiate multiplicative from additive relations, could help students understand ratio and rates more deeply and enable the formation of stronger links to other multiplicative concepts. We propose Teaching for Abstraction as one way of focussing on this underlying structure.

A Teaching for Abstraction approach to ratios and rates would proceed as follows: Students would firstly explore various familiar situations involving ratios where they can solve simple problems without difficulty. They would then look for structural similarities between these calculations, explore the concepts involved, generalise and practise the procedure, and apply what they have learnt to new situations. This process would then be repeated for rates, emphasising the similarities and differences between rates and ratios.

This paper reports a research project designed to investigate whether it is possible for classroom teachers to implement the Teaching for Abstraction approach to ratio and rates in Year 8. A teaching unit was developed, teachers familiarised themselves with the approach and the content and then taught the unit, and we collected data on teacher and student learning. The study parallels a similar study of teaching percentages in Year 6 that is reported separately (White, Wilson, Faragher, & Mitchelmore, 2007).

Method

Participants

The students and teachers from six Year 8 classes in four schools participated. Two classes were from a selective boys' school, two were from comprehensive girls' schools and two were from comprehensive co-educational schools. Of the comprehensive classes, two contained high ability students, one average ability, and one low ability (as described by their teachers). In each class, five students were selected as a representative "target group".

Teaching Materials

Teachers were supplied with a unit consisting of six lessons, each intended to fit into a 60-minute period, covering the Ratio and Rates section of Outcome NS4.3 in the NSW Mathematics Syllabus (NSW Board of Studies, 2002). The materials included, alongside an orientation to Teaching for Abstraction, a suggested outline for each lesson together with black line masters that could be used for duplicating student worksheets. The six topics were as follows:

1. *Relative and absolute comparisons*
   Students explore a number of situations requiring the comparison between two values, and decide when it is more informative to compare them as they stand (absolutely) or in relation to each other or to other values (relatively).
2. *The concept of ratio*
   Students abstract the concept of ratio by looking for similarities between a variety of different situations where relative comparison is appropriate, and then explore its properties.
3. *Calculating with ratios*
   Students explore various methods of carrying out ratio calculations, including the unitary method, and are introduced to the concept of gradient.
4. *Fractions and ratios*
   A variety of practical situations is used to help students understand the similarities and differences between a ratio and a fraction.
5. *The concept of rate*
   Students explore a number of rate situations, and then explore the similarities and differences between rates and ratios.
6. *Calculating with rates*
   Students extend their skill at ratio calculations to similar calculations with rates, and explore the concept of speed.

Instruments

A short, task-based interview was used to assess students' understanding of the multiplicative structure of ratios and rates. It consisted of four questions focussed around four familiar multiplicative situations. Students were asked to perform various calculations and justify the methods they used. The content of the items is described in the Results section.

A 15-item unit quiz was constructed to assess students' calculation skills at the end of the unit. There were five items on simplification of decontextualised ratios, two on dividing in a given ratio, five on simplification of contextualised ratios, and three rates problems. Students were not asked to explain their answers because it was felt that deep understanding was better assessed through the interviews.

Procedure

The study took place in Term 3, 2006. In a one-day orientation workshop, teachers were introduced to Teaching for Abstraction and the proposed teaching unit.

They then taught the unit over a period of 2-3 weeks, and returned for a second workshop to share their experiences and assess the effectiveness of the unit.

The third author visited schools regularly during the teaching period. On her first visit, she interviewed all target students to assess their initial understanding of ratio and rates. On subsequent visits, she observed two lessons for each class and discussed each lesson with the teachers afterwards. On her final visit, she again interviewed the target students. Teachers also collected work samples from the target students in their class, and administered the unit quiz at the end of the teaching period.

The effectiveness of the teaching unit was assessed on the basis of the following data:

1. *Lesson evaluations* as shown by teachers' comments after each lesson and at the second workshop, the third author's observations, and student work samples;
2. *Student learning* as shown by the change in their understanding between the two interviews and their performance on the unit quiz.

Results

The topics taught in each lesson varied from school to school depending on the length of each period (varying from 40-80 minutes) and the ability level of the students. A further complicating factor was teacher unavailability: Four of the six classes were taught by at least one teacher who had not attended the orientation session. In two classes, the assigned teacher taught less than half the lessons.

The average- and low-ability students were only able to complete the first four lessons of the unit. These two classes also had one lesson in which only half the students were present, and there was no time to repeat the lesson. Students in the other four classes completed all the materials provided.

The results show that the students in the two selective schools performed at about the same level as the high-ability students in the two non-selective schools, so we have often pooled their data in the following.

Lesson Evaluations

Lesson 1 commenced with reports of a survey that students had been asked to administer, in which respondents were asked to indicate whether certain deductions from given data were valid. For example, given that "over the last 20 years in Australia, 10 people have died from crocodile bites and 12 people have died from dog bites", is it valid to deduce that being bitten by a dog has been more dangerous than being bitten by a crocodile? This was followed by discussions of the rationale for deciding The Biggest Loser (a well-known television program) and for assessing animal ages in human equivalents. Finally, the terms *relative* and *absolute* were defined and practiced.

These activities generated much heated discussion. Many students had enjoyed giving the survey to their parents and were amazed at the variety of responses. Most students seemed to understand that the survey data needed to be interpreted relatively, that percentage weight loss was the fairest criterion for The Biggest Loser, and that animal ages should be assessed relative to their average life span. However, many students were hindered by calculation difficulties – graphical displays sometimes helped. The teachers realised the activities were stimulating and felt that all students had understood the difference between relative and absolute comparisons. But they were clearly unused to leading discussions; two teachers found it difficult to curtail digressions peculiar to particular contexts and focus on the essential mathematical content.

In *Lesson 2*, students were asked to do some simple calculations involving "3 for the price of 2" sales, gear wheels, cinema queues on different nights, making playdough, scale drawings, and maps. It was expected students would solve these problems using their contextual knowledge. They were then asked to look for similarities between how they had solved each problem and to derive some generalisations. It was suggested that students use a bar model for making ratio comparisons. The concept of *equivalent ratios* was then introduced and practiced in a number of practical situations (sharing chocolate, making muffins, comparing fertilisers, and balancing voices in a choir).

Although there were a few context-related difficulties (especially with the map item), students seemed to be able to solve the given problems and recognise that they were each dealing with a relative comparison. Teachers said they would not normally have spent so much time on each context, but they seemed to be more familiar with these contexts and showed more skill in highlighting the underlying mathematical structure than in Lesson 1. The computation of equivalent ratios caused different problems for different students. The high-ability students recognised the similarity with equivalent fractions, but could not see how (for example) they could use a recipe for damper if they did not have a measuring cup to measure out the stated quantities. The students at the other end of the spectrum experienced mathematical difficulties (finding equivalent ratios) similar to those they had reportedly experienced with fractions.

Lesson 3 introduced the unitary method for solving proportion problems, and students applied it to some problems from the previous lesson. They then looked at the idea of gradient as a ratio and compared the gradients of some given slopes. The high-ability students enjoyed this lesson, but the other students again had difficulties calculating fractions and often confused the order of the two components of a ratio. The low-ability students attempted to work through all the examples but became confused and did not reach the intended outcomes. At this point, two of the three teachers of that class believed that the Teaching for Abstraction approach was not suitable for their students, so they decided to revert to their previous way of teaching the topic.

Lesson 4 was intended to address a problem, referred to in the introduction, that teachers had identified at the first teachers' meeting: the confusion between a ratio (relating two parts of a whole) and a fraction (relating a part to the whole). The two concepts were computed in a number of practical contexts and their different significance compared. The process of dividing a quantity in a given ratio was then addressed, after which the relation between ratio and percentage was explored.

The high-ability students had little difficulty with this lesson. One teacher supplemented the unit materials by beginning with a "drill and practice" exercise, but students did not make any errors on these calculations. The teacher of the low-ability class, who had reverted to the traditional approach, gave the students drill and practice after stating the rules to be followed. But students had difficulties both with the computations and with knowing which computations to do, and repeatedly questioned the rules they had been given. The teacher of the average-ability class used what seemed to be a more successful approach that certainly engaged the students. She worked only on the example with the smallest numbers. Students worked in pairs, and were required to explain their methods. The teacher then gave several slight variations before generalising and setting students a similar problem for homework. Unfortunately, this was one of the lessons for which only half the class was present.

Lesson 5 explored rates in several contexts (including run rates at cricket), encouraged generalisation by comparing rates and ratios, and addressed the issue of changing units. Gender differences appeared in relation to the cricket calculations and many students experienced difficulties converting units.

Lesson 6 gave more practice in rates, with a special emphasis on the rate-gradient relation in graphical representations (including distance-time graphs). The selective students seem to have covered all these topics previously. Some of the students had the same difficulties calculating with rates as they had experienced with ratios, especially when fractions or decimals were involved.

To summarise: Teachers and students liked having so many practical problems to discuss but were often distracted by contextual peculiarities. Teachers enjoyed "watching students think", and students enjoyed the challenge of making mathematical sense of interesting situations. The higher ability students had little difficulty abstracting the mathematical structure of ratios and rates, but the lower ability students were often hindered by difficulties manipulating fractions and decimals and often got frustrated. All the teachers agreed that they would be more selective of examples and teach the unit better next time.

Student Learning: Interview Results

Thirty students were interviewed before and after the unit had been taught. Figure 1 summarises the results.

Item 1, comparing the performance of basketball players who shot 20 goals from 40 shots or 25 goals from 50 shots, was answered well by all but two students before the unit was taught and by all students afterwards.

Item 2 posed three questions relating to mixing a given cordial. Only one student gave any additive answers (the same student before and after the unit). Among the others, the number of correct answers that were correctly explained increased from an average of 17 to 25.

Item 3 gave the positions of two runners at the start of a 100 m handicap race and 10 seconds into the race. Students were asked to predict the winner and the winning time. Only a few students from the lower ability classes showed any evidence of additive thinking, and the number of students giving correct responses increased from 17 to 22.

Item 4 asked students to suggest how the nutritional information on a food package could be used by people wishing to restrict their fat intake. The number of correct responses increased from 21 to 25. Interestingly, the number of students referring to the need to compare different foods decreased from 10 to 6, whereas the number stating that the information could be used to compare different serving sizes increased from 11 to 19.

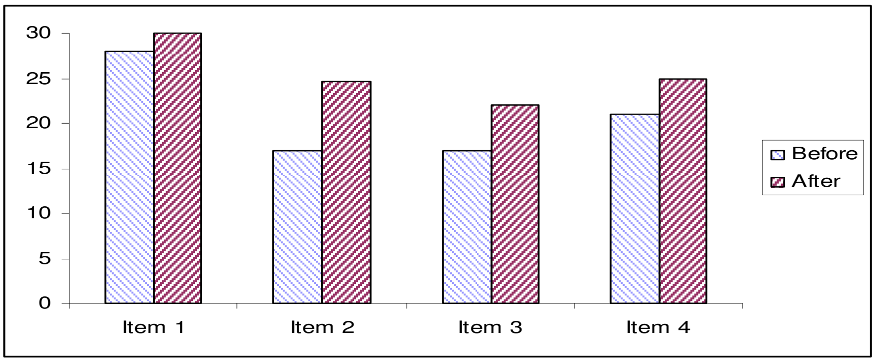


Figure 1. Number of correct interview responses before and after teaching.

To summarise: Most high-ability students had already learned to think multiplicatively before this unit was taught. During the course of the unit, some of the average- and low-ability students started to think multiplicatively and learnt how to perform the correct calculations.

Student Learning: Unit Quiz

Although there were no data available from a pre-test or from comparison classes, the 139 responses obtained to the unit quiz were still informative.

Students in the selective and top-stream classes performed at about the same rate (88% versus 83%), whereas students in the other two classes gave averages of 53% and 39% correct responses, respectively (partly because they did poorly on the rates questions, which they had not studied). The types of errors students made were also different in the three groups. In the high-ability classes, about 50% of the errors were related to units. Among the average-ability students, the most common error (30%) was incorrect multiplication or division. In low-ability students, the most common error (29%) was failing to reduce a ratio to its simplest form.

Discussion

We have learnt a great deal about the implementation of the Teaching for Abstraction method from this study. We discuss our findings under three headings: teaching, learning, and assessment.

Teaching

The teachers were all unfamiliar with the methodology of Teaching for Abstraction. In particular, they were not sure about when to let a discussion ramble, when to cut it off to draw out a mathematical point, and when to supply information or conventional terminology. Some teachers felt it was more difficult to maintain control when so many students wanted to talk at once. As a result, more time was taken than would normally have been available.

The contextualisation of the mathematics appeared to have been beneficial in arousing student interest, especially when teachers could bring in their own experiences (e.g., in raising rabbits). However, the converse also applied when teachers or students were unfamiliar with a context. For example, some teachers were not familiar with "The Greatest Loser" and some students were not interested in cricket, so these examples produced more mystification than enlightenment.

But the major difficulty that teachers experienced lay in adapting the given unit to the prior understanding of the students in their classes. In the higher ability classes, students were generally set to work through all the questions supplied after a minimal introduction, and teaching mainly resulted from discussion surrounding the more difficult questions. Some of these students were clearly frustrated at having to work through problems that did not challenge them. In the lower-ability classes, students could not cover all the material provided because of the calculation difficulties they experienced. Teachers had difficulty selecting exercises that would avoid these difficulties and still allow students to learn the concepts of ratio and rate.

Despite these difficulties, students generally seemed to enjoy the teaching approach and contributed willingly to the discussions. Teachers believed that, as a result, they came to know their students and appreciate their ways of thinking better. However, some students found it difficult to explain their thinking and others preferred working on their own, guided by answers at the back of the textbook.

Learning

There was some evidence of additive thinking in this study, although it never occurred among the high-ability students. Even students in the average- and lower-ability classes made relatively few errors due to additive thinking in the final interview and the quiz – most at least attempted to use multiplication or division. However, this may have been a result of the teaching unit's emphasis on multiplicative relations and may not represent any generalisable learning.

In the classes that had not studied rates, additive methods were more prevalent on the rates question in the quiz. Given a medicine label which says "Use 2 mL for each 5 kg body weight" and asked how much one should use for someone weighing 75 kg, one student proceeded to make a long table starting with 2 – 5, 3 – 6, 4 – 7, and ending with 75 – 78. He finally decided that you should use 72 mL for a 75 kg person. The same student used multiplication and division for all the questions on ratio. Without teaching, he clearly saw no connection between ratios and rates.

The major difficulty for the students in the higher-ability classes was in partitioning a given quantity in a given ratio. It appears that they often omitted the units because they believed ratios did not need units. There were also frequent errors in converting units.

Students in the average- and lower-ability classes had two main difficulties. Firstly, they often confused the ratio of two parts with the fraction for each part of the whole. This difficulty was known beforehand, but apparently Lesson 4 had not adequately addressed this misunderstanding – and student absence in the average-ability class only exacerbated the problem. Secondly, students often could not convert ratios to their simplest form because they were unable to recognise common factors. Converting ratios to unitary form was much easier because students could use their calculators for this. Unfortunately, no attempt was made to show students how to use the fraction mode on their calculators to reduce a ratio to simplest form.

All students, but particularly those in the lower ability classes, found the graphical representation of a ratio by a partitioned bar to be helpful. It would have been even more helpful had it previously been used in the teaching of fractions and percentages. Greater familiarity with the bar model could have enabled more students to relate the representation to the mathematical operations involved.

Assessment

In this study, we had to infer student understanding of ratios and rates mainly from lesson observations and teacher comments. Neither the interview nor the quiz adequately assessed non-routine learning (e.g., the connection with other topics) and what were judged as favourable responses could have been due to the influence of recent teaching. The absence of a pre-test on calculational skills was also a limitation on this study. Closer attention needs to be given to the assessment of multiplicative understanding in future studies.

Conclusions

This study has highlighted the difficulties in implementing Teaching for Abstraction in practice. Teachers were unfamiliar with the approach, and were unable to assimilate it in the one day briefing session. Furthermore, the frequent replacement of teachers during the course of the study meant that several classes were taught by teachers who had not been exposed to the philosophy behind the study unit at all. As a result, teachers were not in a position to do what they normally do as a matter of course: adapt the approach and the materials to the needs of the students in their class.

Encouragingly, all teachers said they would use at least some of the unit materials the following year, with modifications to suit their class. At that time, they would be more familiar with the approach, better able to choose appropriate contexts, and more confident about how to adapt the method to students' ability levels. Future implementation might be more successful if teachers, after an initial introduction to Teaching for Abstraction, were involved in the development of a revised ratio and rates unit. It may also be necessary to plan a general implementation of the model over a longer period of time, and not just for one unit. Professional development is obviously a key issue here.

Despite these difficulties, we still believe that Teaching for Abstraction holds promise. In this study, it appeared that many students were able to abstract the ratio concept from discussion of several contexts, and that more would have been able to do so if the contexts had been more appropriate for them. However it is clear that, in planning the practical implementation of the method, much more attention needs to be given to what needs to occur between the recognition of a concept and the application of that concept to new contexts – that is, the reification stage.

This study shows that pre-existing computational fluency plays an important role in the reification of ratio and rates concepts. Students who cannot recognise simple common factors or cannot perform simple multiplication and division calculations will have difficulties recognising multiplicative structure even in familiar contexts. Consequently, they will not be able to generalise across different contexts and abstract the desired concepts. Drill and practice exercises focussed on multiplication and division skills are unlikely to be helpful, and may only reinforce a feeling of failure. It is also likely to be unhelpful to restrict the examples to simple numbers that provide no challenge, because the multiplicative structure may then completely escape students' attention. More likely to be successful is careful grading in the difficulty of the arithmetical computations involved, more widespread use of graphical models, and the provision of electronic assistance once the underlying structure has been recognised.

The other side of the coin is that many Year 8 students may already have acquired the necessary computational fluency, even in ratio and rates problems. Instead of repeating unstimulating practice, such students would best deepen the reification of ratio and rates concepts by exploring the limitations of ratios and rates in practice as well as the links between them and other multiplicative concepts such as slope and enlargement.

References

Behr, M. J., Harel, G., Post, T., & Lesh, R. (1992). Rational number, ratio, and proportion. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 296-333). New York: Macmillan.

Carpenter, T. P., Fennema, E., & Romberg, T. A. (1993). *Rational numbers: An integration of research*. Hillsdale, NJ: Lawrence Erlbaum.

Clark, M. R., Berenson, S. B., & Cavey, L. O. (2003). A comparison of ratios and fractions and their roles as tools in proportional reasoning. *Journal of Mathematical Behavior, 22*, 297-317.

Harel, G., & Confrey, J. (Eds.). (1994). *The development of multiplicative reasoning in the learning of mathematics*. Albany, NY: SUNY Press.

Hart, K. M. (Ed.). (1982). *Children's understanding of mathematics: 11-16.* London: John Murray.

Misailidou, C., & Williams, J. (2003). Diagnostic assessment of children's proportional reasoning. *Journal of Mathematical Behavior, 22*, 335-368.

Mitchelmore, M. C. (2002). The role of abstraction and generalisation in the development of mathematical knowledge. In D. Edge & Y. B. Har (Eds.), *Mathematics education for a knowledge-based era* (Proceedings of the Second East Asia Regional Conference on Mathematics Education and the Ninth Southeast Asian Conference on Mathematics Education, Vol. 1, pp. 157-167). Singapore: Association of Mathematics Educators.

Mitchelmore, M. C., & White, P. (2000). Development of angle concepts by progressive abstraction and generalisation. *Educational Studies in Mathematics*, *41*, 209-238.

Mitchelmore, M. C., & White, P. (July, 2004a). *Teaching mathematical concepts: Instruction for abstraction.* Invited regular lecture presented at the 10th International Congress on Mathematical Education, Copenhagen, Denmark.

Mitchelmore, M. C., & White, P. (2004b). Abstraction in mathematics and mathematics learning. In M. J. Høines & A. B. Fuglestad (Eds.), *Proceedings of the 28th annual conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 329-336). Bergen, Norway: PME.

NSW Board of Studies (2002). *Mathematics 7-10.* Sydney: Author.

NSW Department of Education and Training. (2003). *Teaching about angles: Stage 2.* Sydney: Author.

Skemp, R. *The psychology of learning mathematics* (2nd ed.). Harmondsworth, UK: Penguin.

Vergnaud, G. (1983). Multiplicative structures. In R. Lesh & M. Landau (Eds.), *Acquisition of mathematical concepts and processes* (pp. 127-174). New York: Academic Press.

White, P., & Mitchelmore, M. C. (2005). Teaching percentage as a multiplicative relationship. In P. Clarkson, A. Dowton, D. Gronn, A. McDonough, R. Pierce, & A. Roche (Eds.), *Building connections: Theory, research and practice* (Proceedings of the 28th annual conference of the Mathematics Education Research Association of Australasia, Melbourne, pp. 783-790). Sydney: MERGA.

White, P., Wilson, S., Faragher, R., & Mitchelmore, M. (2007). Percentages as part whole relationships. In J. Watson & K. Beswick (Eds.), *Mathematics: Essential research, essential practice* (Proceedings of the 30th annual conference of the Mathematics Education Research Association of Australasia, Melbourne). Sydney: MERGA.

Setting a Good Example: Teachers' Choice of Examples and their Contribution to Effective Teaching of Numeracy

Tracey Muir
University of Tasmania
<tracey.muir@utas.edu.au>

This paper reports on teachers' choice of examples and the role they play in students' construction of knowledge. Selecting an appropriate example is a challenging task for teachers, with both the teacher's content and pedagogical content knowledge being a determining factor in the selection process. A case study approach was used to document the nature of three different teachers' choice of examples. Qualitative descriptions illustrate the types of examples selected and the understandings the students constructed from these examples. The findings indicate that teachers need to consider carefully their choice of examples to avoid the likelihood of students forming misconceptions about important mathematical concepts.

Background

According to Askew (2005) effective teaching of numeracy involves helping students acquire knowledge of and facility with numbers, number relations, and number operations and assisting them with building an integrated network of understanding, techniques, strategies and application skills. In assisting students to construct understanding, teachers often select examples to illustrate particular principles, concepts and techniques. The selection of examples can be an indicator of effective teaching for numeracy, with both the teacher's content and pedagogical content knowledge being a determining factor in the selection process. There has been considerable research into what constitutes effective teaching of numeracy (Groves, Mousley, & Forgasz, 2006) including the Effective Teaching of Numeracy project (Askew, Brown, Rhodes, Johnson, & Wiliam, 1997a), which identified effective teachers of numeracy based on rigorous evidence of increases in pupil attainment, not on presumptions of "good practice". Their findings identified a number of characteristics which were common among effective numeracy teachers. Other studies (e.g., Jones, Tanner, & Treadaway, 2000; Clarke & Clarke, 2002; Saunders, 2004) supported these findings which indicated that effective teachers of numeracy:

- Maintained a focus on and taught for conceptual understanding of important mathematical ideas
- Used a variety of teaching approaches which foster connections between both different areas of mathematics and previous mathematical experiences
- Encouraged purposeful discussion through the use of question types to probe and challenge children's thinking and reasoning and encouraging children to explain their mathematical thinking
- Possessed knowledge and awareness of conceptual connections between the areas which they taught of the primary mathematics curriculum and confidence in their own knowledge of mathematics

Based on the commonalities identified among the various studies, the author devised a set of six "Principles of Practice" which provided part of the theoretical framework for conducting the study and involved the teacher's ability to: make connections, challenge all pupils, develop conceptual understanding, focus on the key ideas of mathematics, engage the students in purposeful discussion, and possess a positive attitude towards mathematics. This type of teaching places high demands on teachers' subject matter and pedagogic content knowledge (Commonwealth of Australia, 2004) and this became evident when particular teaching behaviours were examined. The author termed these behaviours, "observable numeracy practices" and they included choice of examples, teachable moments, modeling, questioning, use of a variety of representations, and choice of task. In this paper, teachers' choice of examples is specifically discussed along with the impact this choice has on students' construction of understanding.

Theoretical Framework

Constructivism

The basic tenet of constructivism is that the learner constructs his/her own knowledge; each learner constructs a unique mental representation of the material to be learned and the task to be performed, selects information perceived to be relevant, and interprets that information on the basis of his or her existing knowledge (Shuell, 1996). The process is an active one and according to Shuell (1996) the most important determiner of what is learned. The construction of an idea will therefore vary from individual to individual even with the same teacher and within the same classroom (Van de Walle, 2007). The teacher's role is to ensure that students engage with the material to be learned and particularly to foster the connections between both different areas of mathematics and previous mathematics learning. The connectionist teachers identified in the Askew et al., (1997a) study were found to hold beliefs that supported this premise, including the need to explicitly recognise and work on misunderstandings (Askew, Brown, Rhodes, Wiliam, & Johnson, 1997b).

Teacher Knowledge

In order for a teacher to practice within a constructivist paradigm, knowledge of the subject matter being taught, along with knowledge of the pedagogical principles needed to impart this knowledge to students, is required. There is a general lack of agreement over what exactly teachers need to know to teach mathematics (Hill, Schilling, & Ball, 2004), but teachers should possess a sufficient depth of understanding in order to communicate what is essential about a subject and be able to impart alternative explanations of the same concepts or principles (Shulman, 1987). Knowing more mathematics, however, does not ensure that one can teach it in ways that are meaningful for students (Mewborn, 2001). Individuals may have well-developed common knowledge, yet lack the specific kinds of knowledge needed to teach it (Hill et al., 2004). Pedagogical content knowledge (PCK) (Shulman, 1987) is of special interest because it represents "the blending of content and pedagogy into an understanding of how particular topics, problems or issues are organised, represented and adapted to the diverse interests and abilities of learners, and presented for instruction" (p. 8). PCK involves preparation, representation of ideas and instructional selections from an array of teaching methods and models (Shulman, 1987) and as a mathematics teacher, "one needs to know the location of each piece of knowledge in the whole mathematical system, its relation with previous knowledge" (Ma, 1999, p. 115). The study of teachers' PCK has been the focus of recent research in mathematics education (e.g., Baker & Chick, 2006; Southwell, White, & Klein, 2004) and the highly successful

Cognitively Guided Instruction (CGI) program (Carpenter, Fennema, & Franke, 1996) focused on developing teachers' PCK through the provision of a framework that teachers could use to represent and explain a subject to make it comprehensible. Ma's (1999) comprehensive study found considerable differences in both the subject knowledge and PCK between Chinese and American teachers and used the term "profound understanding of fundamental mathematics" (PUFM) to define understanding a topic with depth. She argued that elementary mathematics is not a simple collection of disconnected numbers facts and algorithms, and therefore elementary teachers require PUFM in order to approach a topic in multiple ways, supporting the findings by Askew et al., (1997a), which indicated that the highly connectionist teachers were the most effective teachers of numeracy.

Choice of Example

One instructional strategy that teachers can use to help students construct meaning and one that plays a central role in the learning of mathematics is the use of examples. Examples may include illustrations of concepts and principles, contexts that illustrate or motivate a particular topic in mathematics and particular solutions where several are possible (Watson & Mason, 2002). Because examples are chosen from a range of possibilities (Watson & Mason, 2002), teachers need to recognise that some examples are "better" than others (Huckstep, Rowland, & Thwaites, 2003). A good instructional example is one which is transparent to the learner, helpful in clarifying and resolving mathematical subtleties and generalisable (Bills, Dreyfus, Mason, Tsamir, Watson, & Zavlavsky, 2006). Bills et al., (2006) maintain that the specific representation of an example or set of examples and the respective focus of attention facilitated by the teacher, have bearing on what students notice, and consequently on their mathematical understanding. Inappropriate examples can lead to a construction of understanding that was not the intention of the teacher. For example, when teaching analogue time to students, Huckstep et al. (2003) noticed a pre-service teacher using the example of "half past six" to demonstrate "half past". When the students she was teaching were subsequently asked to show "half past seven" on their clocks, one child put both hands on the "7". As the authors note, of the twelve possible examples available to exemplify "half past", "half past six" is arguably the least helpful (Huckstep et al., 2003).

Clearly the extent to which an example is transparent or useful is subjective, requiring the teacher to offer learning opportunities that involve a large variety of "useful examples" (Bills et al., 2006, p. 9). Ball (1990) questioned her own choice of examples when teaching fraction concepts to a third grade class. Presenting a scenario involving sharing a dozen cookies among family members appeared to be a legitimate example, based on a context familiar to students. Ball (1990) however, found problems with the social and cultural appropriateness of her choice, and the fact that the problem entailed cookies encouraged the use of a circle representation, making the drawing of equal parts inside the circle technically difficult. Similarly, Askew (2004) found that pupils will always interpret classroom tasks in the light of their previous experiences and that, "however carefully a teacher sets up a task, one cannot assume that the individual pupils' interpretations of that task … are either similar to each other's, or fit with the activity expectations of the teacher" (p. 74).

Methodology

A case study approach (Stake, 1995) was used to document the numeracy practice of three teachers. The researcher observed and videotaped between four and seven numeracy lessons (one each week) for each teacher. The transcripts of these lessons were then analysed, initially used the "principles of practice" and "observable teaching behaviours" to identify instances of occurrence. Data analysis was flexible, however, and allowed for other themes to emerge. Lesson and interview transcripts were analysed manually and instances of particular behaviours highlighted. Observable numeracy practices, including choice of examples, were identified and analysed. A lesson transcript, for example, may have included six instances where the teacher chose examples. Each of these examples was then examined for effectiveness in terms of its transparency and generalisability (Bills et al., 2006). In relation to this paper, the following research questions were identified:

- What is the nature of the examples chosen by teachers in the study?
- To what extent are these examples useful in students' construction of understanding?

It must be acknowledged that although the researcher attempted to evaluate the appropriateness of the examples based on classroom observations and her own pedagogical knowledge, it was not possible to ascertain whether or not the example was perceived to be equally appropriate (or not appropriate) for all participants. Three teaching episodes in which examples were used to demonstrate and develop strategies and concepts are discussed.

Results and Discussion

Problem Solving Examples

In the first lesson excerpt described, the teacher, Sue, introduced the problem solving strategy of "guess and check"; it was one in a series of lessons based on problem solving observed by the researcher. The whole class of grade 5/6 students was seated on the floor in front of Sue. Two examples were presented to the class, with the emphasis being on using a table to record the guesses. The researcher interpreted the teacher's intention as being primarily to introduce the guess and check problem solving strategy and then providing students with an efficient method of recording their working out through the use of a table. Students took turns to volunteer their "guesses" and showed their working out on the whiteboard using a table. The first example presented to the class was:

> Jenny collected 45 stickers over a 5 day period. Each day she was given 3 more stickers than the day before. How many was she given each day?

The example was interpreted by the researcher as being a good example in that guess and check was an appropriate strategy to be used and following some clarification as to what the problem was actually asking, students were able to use the strategy of guess and check correctly to eventually solve it. They also adjusted their "guesses" accordingly, based on the information gained from other students' attempts. The drawing of a table initially caused confusion for at least one student, who volunteered to draw up a table on the whiteboard and actually drew a dining table with four legs, but subsequent modeling by students established what the teacher intended by "make a table".

The second example involved larger numbers and although it was set out in a table, and could have been solved by using guess and check (it could also have been solved by working backwards), the larger numbers made it more difficult for children:

> A family set out on a 5-day trek. Each day they traveled 50 kilometres less than the day they had before. Total distance that they traveled was exactly 1500 kilometres. How far did they travel each day?

The following exchange highlights some of the difficulties posed by the numbers:

> Tr:[1] All right Mandy, have a go.
>
> M: (goes to board) I thought it was, they started on 900.
>
> Tr: OK, so you're going to say they traveled 900 on Monday.
>
> Tr: 900 – that says 90 (Mandy has written 90 in the first column – she then adds another 0).
>
> Tr: OK, so how far would they travel on Tuesday?
>
> M: 850.
>
> Tr: OK.
>
> (Mandy writes down 850)
>
> Simon: They have to get to 0.
>
> Tr: Do we have to get to 0 Simon? What does it have to add up to? What do they all have to add up to?
>
> S: Oh, 1500.
>
> Tr: So looking at that, who's going to have another guess and see what they can come with? Randall? Just use that (referring to table drawn) and put a line down. So which one are you going to use Randall?
>
> R: 200.
>
> Tr: 200, all right.
>
> (Randall starts filling in table, beginning with 200)
>
> Tr: So they're not traveling – whoops – they're not traveling anywhere on Friday? They're going to stay at home. OK, so is that going to add up to 1500? Is it Randall? What does it add up to? 3, 4, 500?

As the process of "guess and check" was being introduced as a new strategy, it was unfortunate that the inclusion of larger numbers in the second problem created confusion and detracted from the modeling or consolidation of the "guess and check" process. The students did not have a written copy of the problem to refer to and Sue later reflected that this would have been beneficial.

Following the sharing of these two problems, students were issued with problems to complete individually. One of the problems involved identifying how many lizards and spiders there would be if one counted 60 legs and 10 heads – this may have been a preferable example to model with the whole class as it involved smaller numbers and arguably better suited the "guess and check" process. Although it is not possible to generalise that the whole class understood the process and used a table to record their guesses and checks, Figure 1 shows a typical response to the problem and indicated that this student did construct the understanding intended.

[1] Tr. refers to teacher throughout

Figure 1. Student's use of table to record guess and check strategy.

Sue's lesson highlighted the need for teachers to consider their choice of example in the context of students' previous mathematical experiences. If the aim was for students to construct an understanding of the guess and check process, then it was unfortunate that some students may have been excluded from the process because of their lack of confidence in operating with larger numbers. Examples need to be selected that are suitably challenging and motivating enough in order to engage students, yet still provide for the desired construction of understanding. The teacher's judgement is vital here, with appropriate selection likely being influenced by both the teacher's content and pedagogical content knowledge.

Percentage Examples

The following lesson details a second teacher, Ronald's, selection of examples with relation to percentages. This was the first "formal" introduction that the grade 5/6 had to percentages and it followed on from work on decimals and fractions. Following a general brainstorming discussion about percentages, students worked in small groups to identify where, why and how percentages were used. Several authentic examples were then shared and connections were made with real life, such as sport and discounts, and links were made within mathematics to decimals and fractions. Ronald then moved on to teach how to calculate percentages explicitly. The initial example selected was 20% of 100, and this was recorded on the board:

Tr: Just looking at that, can somebody tell me what 20% of 100 might be?

N: 20.

Tr: Why do you think the answer is 20 Nigel?

N: 'Cause it's out of 100.

Ronald reminded students about the process they used to multiply fractions and related this to the process used to find percentages of numbers using the above example. This example was worked through with the whole class with a variety of students contributing answers. The example chosen was deemed to be appropriate in that the numbers were 'friendly' to work with and the process could be used to demonstrate that the same answer was obtained as the original response. Further examples were also given, including 10% of 90 and 5% of $5.00 and the process was worked through again with the whole class. Again these examples were considered appropriate as they included a diversity of numbers yet were still reasonably straightforward to operate with (the 5% of $5.00 had the potential to be problematic, but the students appeared comfortable with the division of decimals).

Through the selection of the examples and the order in which they were presented the teacher provided scaffolding for the students to construct their understanding of the process. The use of different amounts and the money context may have also counteracted students' inclination to form the construction that percentages could only be calculated with amounts of 100.

Students then worked individually to calculate the example, 10% of 110. The choice of this example demonstrated that percentages could involve numbers more than 100, yet the numbers were easy to operate with. Students appeared to be comfortable with the process and the solution was again shared with the whole class. Ronald then wrote the following examples on the board which the students were expected to complete individually: 25% of 100, 15% of 200, 30% of 96, 60% of 110 and 24% of $48; 30% of 96, and 24% of $48 proved problematic for some students and Ronald later reflected on his choice of examples:

> They picked it up really quite quickly – once I did a couple of more complicated ones on the board, probably three quarters of the class were picking it up, and that one quarter who were still struggling, they were struggling with the numbers not with the actual process – I probably stuffed up with the last example that I used – I put 24% of something, when I should have put 25% - that's just one of those errors that can just happen, but in another way it was an advantage because it showed me those kids who would persevere through something when they come to a problem that wasn't straight forward … and it also showed me the limitations of some kids at this point in time

Ronald's choice of examples generally indicated he possessed a strong content knowledge of mathematics and PCK – the examples were mostly appropriate in terms of the numbers involved and the order in which they were presented provided for scaffolding of students' understanding to occur. He recognised that 24% was not a good example, but then interpreted it as a positive and used it as a subsequent teaching point. The excerpt also illustrates the value of using a variety of examples and Ronald's awareness of the links between different aspects of the mathematics curriculum (Askew et al., 1997b).

Decimal and Money Examples

The next lesson differs from the previous ones in that it documents a teaching episode involving a group of four grade 8 students. The lesson was conducted by a specialist mathematics teacher and the small group focus allowed for more interaction between teacher and students to occur than probably would under normal classroom conditions. The aim of the session was to gauge where students were at in terms of their understanding of place value involving whole numbers, then expand on this knowledge to include decimals. The students and teacher were seated around a table and the students had access to paper, pens and bundling sticks. Following a discussion and some demonstration involving bundling the sticks into groups of ten and a hundred, and feeling confident that the students could accurately represent a four-digit number using the materials, Jeff asked one of the students, John, to cut one of the sticks into parts to represent tenths. As he began randomly cutting the stick, a discussion occurred on whether or not the parts needed to be equal. To demonstrate this point, Jeff used the example of the bundling sticks and stated,

> OK, suppose I ask you this – see that number there 5345 – now with that 5, would it be all right do you think if we had 1000 in this pack and 995 in another pack and a group of 1500 in another pack, or is it important that all the group sizes are the same when we write a number like that?

John still was not convinced and stated that, "It doesn't matter what size the things are as long as you've got ten of them there."

Jeff continued to question John and tried to use the materials to demonstrate that the size of the pieces were important, but John remained confused. Jeff then decided to use the example of money and this choice proved to be quite problematic. Lampert (1989, as cited in Ball, 1990) argues that money may provide a useful familiar context to develop students' understanding of decimals, but the particular way in which this example was presented led to further confusion.

> Tr: If you were working in the work place and you were getting paid … you guys might get paid \$5.00 an hour and let's say you worked three hours for \$15.00 one night – now let's say, Cara, you work Wednesdays and Sarah worked Tuesdays and Sarah was actually getting more money than you on Tuesdays because \$15.00 actually meant a bit more than \$15.00 on Wednesdays – would you be very happy with that?

Jeff was using a context that he thought students would relate to, but did not anticipate the following response from Cara, "But some days, like Saturday and Sunday you do get paid more because it's like the weekend."

Jeff immediately recognised the logic behind this, and stated that,

> "Yeah, well this is an example of how my question hasn't worked there because the understanding I'm trying to get out of you – you've actually gone off on another whole track…"

The students still seem confused but agreed that it should be worth the same each day. Although not mentioned by any of the students, further confusion could have been created by the use of this particular example, as the value of the dollar does fluctuate and can indeed be seen to be worth more on a particular day as it varies in exchange rates for different countries.

The money example further proved problematic when Jeff asked students to write down \$1.05. Although Cara wrote the amount correctly, Adam wrote it as \$1.5. After some discussion and when there was no general agreement in the group about which was correct, Jeff returned to the bundling sticks:

> Tr: So (if we say) this is a dollar coin (holds up one bundling stick), this is a ten dollar note (holds up a bundle of ten sticks) and this is a hundred dollar note (holds up bundle of 100) and pretend just for today that we have a thousand dollar note (holds up the bag of 1000 sticks), so what would that bit there be worth (points to one of the chopped up pieces of bundling stick)?
>
> J: A five cent coin.
>
> Tr: A five cent coin – and how do you know that?
>
> J: Because five cents is the smallest and that (piece) is smaller than the rest of them.

John's answer shows a clear example of pupils interpreting information in light of their previous experiences (Askew, 2004); because the one cent coin is no longer in circulation, his response is quite a logical one. Jeff then reminded the students that we did actually once use one cent coins (although the piece of stick would actually represent ten cents) and made the comment that the five cent coin will probably be the next coin to go, making ten cents the smallest denomination. The exchange illustrates that again, although money is a common example to use when teaching decimals, it was proving to be too abstract for these students to construct a meaningful understanding of what the numbers on the right of the decimal point actually represented. Furthermore, during the plenary session when Jeff was encouraging the students to reflect on what they had learned from the lesson, one of the students, Sarah, responded that, "I learnt that five cent pieces will go…"

Jeff's reflection on the lesson revealed that he had to abandon the plan he originally made for the lesson because of the students' lack of conceptual understanding about the place value system, but that the lesson was useful in that it identified the misconceptions that they did hold. He stated:

> If you really believe in the constructivist view of learning, then it's really about the students' understanding and that's where the questioning is so important and you have to keep questioning – it's like the teacher is a mathematical doctor and the questions are like a scalpel and you keep probing at what it is they're picking up and leading them towards that desired understanding so they go, yes, I see what you're getting at.

Conclusions

In order to help students develop mathematical understanding, the teacher must select examples that enable students to construct accurate knowledge about the concepts presented. Effective teaching for numeracy involves many aspects, but through the careful consideration of selecting which examples to choose and then reflecting on their effectiveness, teachers can reduce the likelihood of students forming misconceptions about important mathematical concepts. The treatment of examples presents the teacher with a complex challenge and the specific choice and manner of working with examples can either facilitate or impede learning (Bills et al., 2006). This paper has provided descriptions about the types of examples teachers select and the way in which students construct understanding, based on these examples. It supports the findings of other research (e.g., Huckstep, et al., 2003; Ball, 1990) and acknowledges the role that teacher content knowledge and PCK play in selecting and presenting these examples. The discussion of Jeff's lesson also shows that even teachers considered to hold both sound content knowledge and PCK, can still select examples that do not lead to accurate construction of meaning. Further research may be needed to look at the link between teachers' PCK and the selection of good examples, along with research on comparing teachers' and students' perceptions of what constitutes a good example and documentation of individual students' interpretations of the same example.

References

Askew, M. (2004). Mediation and interpretation: Exploring the interpersonal and the intrapersonal in primary mathematics lessons. *Proceedings of the 28th Conference of the International Group for the Psychology of Mathematics Education* (pp. 71-78). Bergan, Norway: PME.

Askew, M. (2005). It ain't (just) what you do: effective teachers of numeracy. In I. Thompson (Ed.), *Issues in teaching numeracy in primary schools* (pp. 91-102). Berkshire, UK: Open University Press.

Askew, M., Brown, M., Rhodes, V., Johnson, D., & William, D. (1997a). *Effective teachers of numeracy*. London: School of Education, King's College.

Askew, M., Brown, M., Rhodes, V., Wiliam, D., & Johnson, D. (1997b). *Effective teachers of numeracy in primary schools: Teachers' beliefs, practices and pupils' learning.* Paper presented at the British Educational Research Association Annual Conference, University of York.

Baker, M., & Chick, H. (2006). Pedagogical content knowledge for teaching primary mathematics: A case study of two teachers. In P. Grootenboer, R. Zevenbergen & M. Chinnappan (Eds.), *Identities, Cultures and Learning Spaces* (Proceedings of the 29th annual conference of the Mathematics Education Research Group of Australasia, Canberra, pp. 60-67). Adelaide: MERGA.

Ball, D. L. (1990). Halves, pieces and twoths: Constructing representational contexts in teaching fractions, (Craft Paper 90-2). East Lansing: Michigan State University, National Center for Research on Teacher Learning.

Bills, L., Dreyfus, T., Mason, J., Tsamir, P., Watson, A., & Zaslavsky, O. (2006). Exemplification in mathematics education. In J. Novotna (Ed.), *Proceedings of the 30th Conference of the International Group for the Psychology of Mathematics Education.* Prague, Czech Republic: PME.

Carpenter, T. P., Fennema, E., & Franke, M.L. (1996). Cognitively Guided Instruction: A knowledge base for reform in primary mathematics instruction. *The Elementary School Journal, 97*(1), 3-20.

Clarke, D., & Clarke, B. (2002). Challenging and effective teaching in junior primary mathematics: What does it look like? In M. Goos & T. Spencer (Eds.), *Proceedings of the 19th Biennial Conference of AAMT* (pp. 309-318). Brisbane: AAMT.

Commonwealth of Australia. (2004). *Teachers enhancing numeracy*. Canberra, ACT: Clearinghouse for National Literacy and Numeracy Research.

Groves, S., Mousley, J., & Forgasz, H. (2006). *Primary numeracy compendium*. Canberra, ACT: Clearinghouse for National Literacy and Numeracy Research.

Hill, H.C., Schilling, S. G., & Ball, D.L. (2004). Developing measures of teachers' mathematics knowledge for teaching. *The Elementary School Journal, 105(1),* 11-30.

Huckstep, P., Rowland, T., & Thwaites, A. (2003). *Primary teachers' mathematics content knowledge: what does it look like in the classroom?* Retrieved 12 October 2005, from http://www.leeds.ac.uk/educol/documents/00002534.htm

Jones, S., Tanner, H., & Treadaway, M. (2000). *Raising standards in mathematics through effective classroom practice.* Paper presented at the annual meeting of the Australian Association for Research in Education, Sydney.

Ma, L. (1999). *Knowing and teaching elementary mathematics*. Mahwah, NJ: Lawrence Erlbaum Associates, Inc.

Mewborn, D. (2001). Teachers content knowledge, teacher education, and their effects on the preparation of elementary teachers in the United States. *Mathematics Teacher Education and Development, 3*, 28-36.

Saunders, P. (2004). *Characteristics of effective teaching*. Retrieved 18 April, 2005 from http://www.wmich.edu/teachlearn/new/char_effect_tch.html

Shuell, T. J. (1996). Teaching and learning in a classroom context. In D. C. Berliner & R. C. Calfee (Eds.), *Handbook of educational psychology* (pp. 726-764). NY: Simon & Schuster Macmillan.

Shulman, L. S. (1987). Knowledge and Teaching: Foundations of the new reform. *Harvard Educational Review, 57*(1), 1-22.

Southwell, B., White, A., & Klein, M. (2004). Learning to teach mathematics. In B. Perry, G. Anthony & C. Diezmann (Eds.), *Research in mathematics education in Australasia 2000-2003* (pp. 197-218). Flaxton, Qld:: MERGA.

Stake, R. E. (1995). *The art of case study research*. Thousand Oaks, CA: Sage Publications.

Van de Walle, J. A. (2007). *Elementary and middle school mathematics* (4th ed.). Boston, MA: Pearson.

Watson, A., & Mason, J. (2002). Student-generated examples in the learning of mathematics. *Canadian Journal of Science, Mathematics and Technology Education, 2*(2), 237-249.

Developing the Concept of Place Value

Mala Saraswathy Nataraj
The University of Auckland
<mala@math.auckland.ac.nz>

Michael O. J. Thomas
The University of Auckland
<m.thomas@math.auckland.ac.nz>

What a study of the historical development of mathematical concepts can offer teaching is still being debated. This study examines use of a combination of the historical development of number systems and modelling, with concrete materials as a way of deepening students' understanding of positional notation. It looks at place value in different number bases as a way of enhancing students' understanding of the decimal number system. The results suggest that the combination of a historical and a concrete approach helped the students to understand the place value system to the extent that they could generalise it to other bases.

Background

The understanding of the concept of a positional, or place value, system is central to developing number sense and is also the basis for the four fundamental operations on numbers, as confirmed by the concept map research of Schmittau and Vagliardo (2006, p. 7), who have shown "the centrality of positional system in the conceptually dense system of concepts that comprise elementary school mathematics. Not only does it connect to many important concepts … it is also a prerequisite for any real understanding of the base ten system". However, anecdotal and other evidence (Thomas, 2004) suggest that this vital and central concept is not well understood by students. One reason is that the concept of positional system cannot be developed through the teaching of base ten alone, and students cannot completely understand the decimal system unless it is seen as a particular case of a more general concept of positional notation. Thus this stresses the need for teaching of multiple bases to help students develop the concept of positional system. In addition, since a positional system is a superordinate concept, founded on multiple basic concepts, in order to understand it one must have rich foundational schemas. Unfortunately, one cannot just define such a concept into existence for students since "concepts of a higher order than those which a person already has cannot be communicated to him by a definition, but only by arranging for him to encounter a suitable collection of examples" (Skemp, 1971, p. 32).

Not only is the knowledge of multiple bases vital for understanding the concept of place value, but it also serves as a foundation for the development of other crucial concepts, such as variable, exponent, polynomial, and polynomial operations, amongst others. Students' difficulties in algebra and these areas have been well documented (e.g., Kieran, 1992; MacGregor & Stacey, 1994; Warren, 2003) and educators' views on the various approaches to beginning algebra, such as generalisation, problem solving, and function/modelling, are also clear in the literature (e.g., Mason, 1996; Radford, 1996; Ursini, 2001). According to Mason (1996), generalisation is the heartbeat of mathematics and that "expressing generality is central to all mathematics, including arithmetic" (Mason, Graham, & Johnston-Wilder, 2005, p. 95). He goes on to state that one of the most important sources of generalization is the domain of number and, in detecting and expressing number patterns, general number can be seen as a pre-cursor of variable, the central concept of algebra. Hence a good knowledge of positional notation could assist in a

Proceedings of the 30th annual conference of the Mathematics Education Research Group of Australasia
J. Watson & K. Beswick (Eds), © MERGA Inc. 2007

smoother transition to algebra through a consideration of multiple bases to the notion of a general number base *n*.

In this research study we considered the importance of understanding of positional notation and how it might be improved using a combination of concrete materials, multiple representations and historical perspectives. The first of these has been appreciated since the time of Piaget's description of the concrete operational stage of learning, because for these students one must recognise that "in sum, concrete thought remains essentially attached to empirical reality" (Inhelder & Piaget, 1958, p. 250). At one time materials such as Dienes blocks (Dienes, 1960) were widely used but have since grown unfashionable. Secondly, representational versatility (Thomas, 2006) lies at the heart of much of what mathematics is. Students may interact with a representation by observing it, for example by noticing properties of the representation itself or of the conceptual processes or object(s) represented, or acting on it. The versatility arises in the ability to translate between representations of the same concept and to interact with these representations in qualitatively different ways. The third aspect is the use of history to inform practice. In recent years, there has been a continuing tradition of using history of mathematics in the teaching and learning of mathematics. Educators and researchers (Fauvel & van Maanen, 2000; Gupta, 1995; Katz, 2001) have asserted that the history of mathematics is an excellent resource for motivating students to learn mathematics, and one of the greatest benefits is in enhancing the understanding of mathematics itself. Of course there are different ways in which historical material may be incorporated in the classroom, with the history implicit or explicit in the teaching situation (Fauvel & van Maanen, 2000). Either way it can bring about a global change in the teacher's approach. This is because a historical and epistemological analysis (Puig & Rojano, 2004) may help the teacher to understand stages in learning (Barbin, 2000) and why a certain concept is difficult for the student. In turn this can help with teaching strategy and development. A specific example of the implicit use of history is the historical development of the present day decimal number system.

A review of some texts (Datta & Singh, 2001; Joseph, 2000; Srinivasaiengar, 1967) reveals that the decimal number system with place value and zero used today originated in India, and this system was passed on to the Arab mathematicians who then carried the system to Europe. A study of this history reveals that the "perfection" of the number system was preceded by centuries of experience of working with very large numbers (as part of solving problems in astronomy). The ancient Indian mathematicians developed a scientific vocabulary of number names including names for powers of 10, even going up to 10^{53} and this consideration of large numbers and exponential multiplication and its symbolisation seems to have prompted the creation of zero and the number system with place value (Datta & Singh, 2001). Although the *rhetorical, syncopated,* and *symbolic* stages are usually associated with algebra (Kieran, 1992), they seem to have also been present in the realm of number in Indian history of mathematics. In addition, studying different number systems from history provides students with the opportunity of developing an understanding of the concept of numerals as number symbols, as well as the principles that were used with these symbols. Moreover, the study of number systems from history presents mathematics as a human endeavour with twists and turns, false paths, and dead-ends, and helps learners towards a more realistic appreciation of their own attempts.

In some countries, including New Zealand, the teaching of multiple bases is no longer present in most mathematics textbooks at the Primary and Intermediate level and so is not taught in schools. Hence this research study sought to use concrete materials, the theory of representations, and both explicit and implicit historical analysis, in the classroom for the concept of place value. We addressed the question of whether such an approach could help to improve students' understanding of this positional system of representing number.

Method

The research study comprised a case study of a class of 27 Year 9 (age 13 years) students at a decile 5 (middle socio-economic level) at a secondary school in Auckland, New Zealand. This class, called the "Global" class, was a new concept in 2005, with students from many different cultures and ethnicities, and a "global" approach to core subjects. The class used in the research thus represents a wide variety of cultural backgrounds, including Indonesian, Russian, Hungarian, Dutch, American, Malaysian, Zimbabwean, Chinese, Korean, Japanese, Cypriot, Swedish, Maori, Pacific Island, and New Zealand European students. However, most of the students had their intermediate schooling in New Zealand and hence were proficient in English. The exceptions to this were two Korean students and a Chinese student who had only recently arrived in the country, who were taking ESOL classes. Possibly due to a positive attitude the class was performing above average for the year group in the school. The teacher explained to the students what was going to be taught and why it was important to their learning. The classroom process was very much task oriented, and all the class lessons were taught by the first-named researcher. The first task was intended to get students to think about the need for a number system and how it might have been constructed. To accomplish this they were encouraged to work in groups of 2, 3, or 4 and try to create a number system of their own. This included deciding on the grouping size, the number of symbols needed, and how they would represent and add numbers. The students were given a large number of coloured sticks to help with their thinking and sheets of paper on which to write their ideas. Following this the students were given a pre-test comprising questions that addressed their current understanding of place value. A sample of the kind of questions used is given in Figure 1.

Following the test the students' second task was to investigate the number systems of past civilisations to see what could be learned from them. Having considered the numbers 0-10 in their own languages, including writing down the number symbols in their language on the board, and saying the numbers, they then spent five to six lessons of 60 minutes each working through worksheets on different number systems from around the world and from different time periods. These included Primitive, Egyptian, Babylonian, Roman, Greek, and Mayan, and finally the present Indian decimal system. The tasks involved them writing numbers in the different systems, only two of which had a place value (Babylonian and Mayan). Following the investigation of each number system the students discussed the symbols in the system, along with general features such as place value and zero, its advantages and disadvantages, and then wrote down their observations on the system.

The third task, comprising two lessons, was to use concrete materials to analyse base 10 numbers. The students were given large numbers of coloured sticks and were asked to group the sticks in tens and then hundreds, thousands (they managed one ten thousand!) etc., tying the sticks with elastic bands that they were given (parts of sticks were used for tenths and hundredths) and they used them to model numbers, such as 12386. Keeping in mind that the historical development through the rhetorical stage was in place for a long time, the sticks were used to model numbers in the same way that we say the number, that is, one set of ten thousand and two sets of one thousand (1 set of $10\times10\times10\times10$ and 2 sets of $10\times10\times10$).

Section A

1. Write the following in words.

   a) 7905 ____________________

   b) 100005 ____________________

2. Write the following in numerals

   a) fifty three thousand eight hundred and ninety two ____________

   b) sixty two thousand and nine____________

3. For the following numbers, what is the actual value for each of the digits?

   a) 35275 b) 6008 c) 7658.32

4. What is the meaning of zero in question 3b?
5. How many symbols do we have in the number system that we use? ________
6. What is the base of the number system that we use?________
7. How many symbols do we need for a number system with base six? ______
8. How many symbols do we need for a number system with base forty three?_____

Section B

1. Suppose we consider a number system with base **six**. Write the following numbers in words.

   a) 3524 ____________________

   b) 40035____________________

   c) 324.15____________________

2. Suppose we consider a number system with base **seven** where

   1 is written as 1
   2 is written as Γ
   3 is written as Δ
   4 is written as □
   5 is written as ∃
   6 is written as ϖ
   0 is written as 0

   Then write the following numbers in words.

   a) □ϖΔΓ b) Δ0□ϖ c) ΓΔ0.∃□

Figure 1. Some of the pre- and post-test questions.

In the next stage of this task, only a single bundle of sticks was placed to represent the place value. For example, only one bundle of 10 was placed and 8 sticks were placed underneath it to represent 80. During the final stage of the task the bundle of 10 was removed and students had to imagine the value of the place. Examples of two of these representations of the number 234.23 are given in Figure 2 (the decimal point is represented by a band). The cognitive linking of these representations is a key step in the construction of the system. There was also discussion surrounding the need for a symbol for zero when we consider a number such as 407.

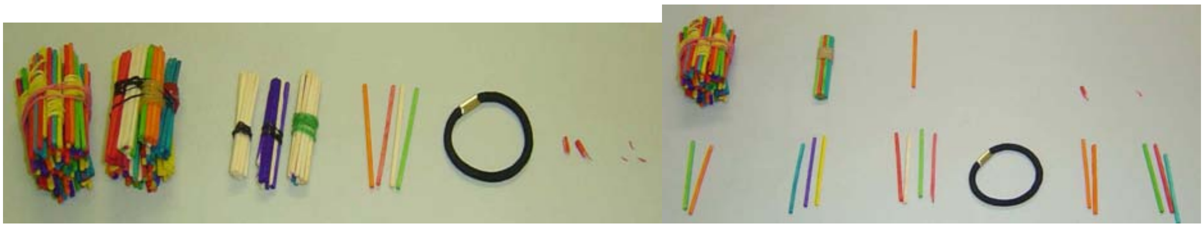


Figure 2. Two different representations of 234.23 used in task 3.

Following these tasks the students were given a post-test, along with extra questions on generalisation (see Figure 3 for some of these questions) involving bases 6, 7, 8, and 29 as well as base 10, and also asking for a generalisation.

4a) Write the values of the places for numbers with base 8 on top of the given boxes.

□ □ □ □ □ • □ □

b) Now generalise and write the place value for numbers with base 8.____________

5a) Write place values for number base 29 on top of the boxes.

□ □ □ □ • □

b) Generalise and write the place value for base 29. ____________________

6a) Make a generalization and write the place value for <u>any number</u> base. __________

Figure 3. Some of the "extra" post-test questions.

Due to the difficulty of the extra questions some students requested further explanation on the idea of generalization, so the teacher used half a lesson to put up some patterns on the board that the students had to generalise. She explained that she wanted them to look at the patterns, say what they saw and then write a sentence with symbols that would represent any one or all of the lines. They discussed what "make a generalisation", "in general" and "generalise" means. The patterns below were put up on the board and students had to verbalise as to what was the same and what was changing across any line and generalise. Then they had to look at the vertical line on the right and generalise further for any base *a*. Finally, the students were allowed to answer the extra questions one more time.

$$\begin{array}{llllllll}
7^4 & 7^3 & 7^2 & 7^1 & 7^0\,. & 7^{-1} & \ldots\ldots\ldots\ldots & 7^k \\
5^4 & 5^3 & 5^2 & 5^1 & 5^0\,. & 5^{-1} & \ldots\ldots\ldots\ldots & 5^k \\
34^4 & 34^3 & 34^2 & 34^1 & 34^0. & 34^{-1} & \ldots\ldots\ldots & 34^k \\
 & & & & & & & a^k
\end{array}$$

Results

In order to establish some comparative baseline data on Year 8 students' understanding of place value we accessed the results of the Assessment Tools for Teaching and Learning (asTTle) (Hattie, Brown, & Keegan, 2005) standardised tests for the whole Year 9 group at the school the Global class attended. Five of these questions were on the topic of place value and hence exposed areas of difficulty for students of this age and background, forming a comparative population. The students all sat the tests on the first day of their school year, before the research study took place. Results on questions 8 and 22 were combined on the test, since both address the skill "Explain the meaning of digits in numbers up to 3 decimal places", and could not be separated. Question 8 essentially asked whether 1.35 or 1.342 is larger, and question 22 asked students to write a number with 1 in the hundredths column, 2 in the tens, 5 in the thousandths, 6 in the ones and 9 in the tenths. Similarly questions 11 and 23 considered "Order decimals up to 3 decimal places" and question 13, "Explain the meaning of the digits in any whole number". Table 1 shows the comparison of the Global class results with those of the rest of the year group. These show that on questions 8 and 22 (χ^2=9.95, p<0.01) the Global class performed significantly better than the year group. However, on question 13 (χ^2=0.45, ns) and questions 11 and 23 (χ^2=3.06, ns), there was no significant difference in performance. Two comments may be made on this. Firstly it confirmed the view that the Global class was performing a little above average for their year group, and secondly that these place value skills are a problem for many students of this age.

Table 1

A Comparison of Year 9 Students with Global Class on Place Value

Question	Year 9 Group (N=125) % Wrong	Year 9 Global Class (N=27) % Wrong
8 and 22	72%	54%
13	72%	66%
11 and 23	51%	29%

Work on the Tasks

When we look at what the students produced for their number systems on the first task, most simply took the base 10 system and created their own symbols (Figure 4, row 1). Others (Figure 4, row 2) employed an additive system using a symbol for ten as their base to get 39. The only group who tried to do anything differently is shown in Figure 4 row 3. They used a system of merging two symbols together into a partial multiplicative arrangement, but they still have a new symbol for 36 and are not using place value. However, this was the first task that the students worked on and it accomplished its purpose of getting them to think about number systems and how they are constructed.

The second task on considering how the different number systems developed historically proved interesting to the students, for differing reasons. Some liked particular symbols such as the Egyptian and the Roman for aesthetic reasons, and others felt that some systems, such as Roman and Primitive systems, were easier to use, whereas others found the Mayan system difficult and confusing. However, when asked to represent large numbers students realised they had to repeat symbols many times and also had to create more and more symbols (see the sample comments of S1 and S24 in Figure 5). When asked why they were able to write large numbers with only ten symbols in the present decimal system, students found the question quite challenging and one student said “it was because of all the zeros”.

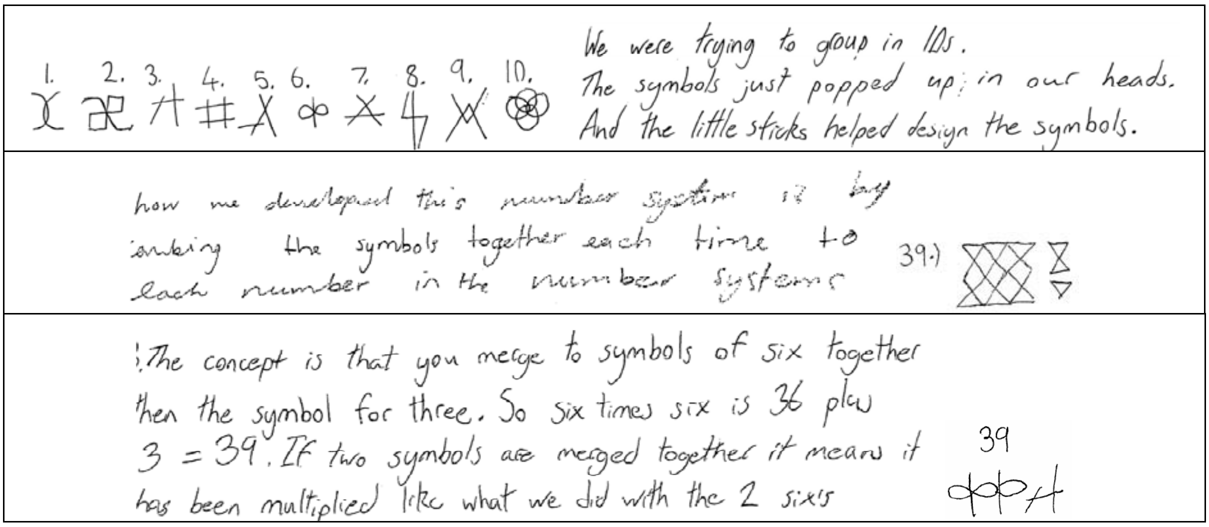


Figure 4. Students’ work on creating their own number system.

During episodes of teacher intervention during the work with the groups of coloured sticks, different numbers were modelled on the board in base 10 (for example 10^3 was also written as $10\times10\times10$, $10^2\times10$, 1000, and in words), leading to a discussion of exponential multiplication and place value. This was done so that students not only see one thousand as a thousand ones, but also as 10 groups of 10 groups of 10. The following was written up on the board for each one of the positions.

Thousand 1000 $10^2\times10$ $10\times10\times10$	10^4	10^3	10^2	10^1	10^0		10^{-1}
	$\boxed{1}$	$\boxed{4}$	$\boxed{8}$	$\boxed{6}$	$\boxed{7}$	●	$\boxed{2}$

S24 Babylonian	Sort of ok, sort of hard They had place value but not a real zero Grouping in large numbers is hard . This sign was used as a separator symbol.
S24 Roman	8. Difficult to write large numbers but they didn't large numbers
S1 Babelonean	7. Even though only two symbols were used, we had to write a lot of ~~numbers~~ symbols to express large number.
S1 Hindu-Arabic	3. We didn't have to keep making up new symbols — because the system has place value and zero.

Figure 5. Two students’ observations on historical number systems.

The tasks gave students an opportunity to construct other concepts, such as the relative sizes of numbers like 10^4 and 10^{-2}. That they were engaging with these ideas was shown by comments about the "bigness" of something like 10^{12} and 10^{53} and the smallness of 10^{-23}! In the second session of this task exactly the same procedure was followed, but this time the students grouped the sticks in sets of 6s, 36s, 216s, etc., and hence different numbers were represented in base six. Again this was written on the board as above in different representations: in words, exponential forms, and full forms, (e.g., 4 lots of 216 (6^3), 5 lots of 36 (6^2)). There was discussion on the word *base* and how many symbols were needed for a particular base. When working with the groups of coloured sticks and by looking at the patterns, students came up with 10^0 as 1 and one tenth as 10^{-1}. It was brought to the attention of students that in a number such as 12796.34 the three sticks used represented 3 lots of the tiny bits of sticks, or 10^{-1}.

Some of the students commented that they found the work on the tasks, and especially the "project" to create their own number system, enjoyable and fun, stating: "This is lots of fun. Got us thinking about funny names and symbols" (S3); "This is fun. We like working together and bounce ideas off each other but it is hard. It is like making your own language up" (S5); "It was fun. Kind of interesting figuring out what symbols to use. A great way to get creative" (S23); and "Very interesting. Sticks helped us to think. I felt I was designing something for the future" (S24). As S5 observed, it was also challenging for them, to the point that some found it very difficult and others felt out of their depth. This was occasionally, according to S9 and S12 because their group did not work so well together: "Extremely hard to create own number system. The group were not communicating very well as all of us were thinking differently and it was hard to co-ordinate our ideas and write them down"(S12); and "The group was confused. Different opinions in the group and they all wanted different things/symbols"(S9).

Others also said how they found the work "challenging" (S13) or "quite hard" (S21), or they were "Confused. Concerned I was not doing anything" (S27). Student S21 was the only one who was negative throughout the whole unit of work and it was very difficult to help her. She felt she was not good at mathematics and she said she did not care about mathematics anyway. In summary, we can say that the task was stimulating but not easy for this group of students.

Test Results

From the pre-test to the post-test all students except for S13 improved their scores, and overall there was a significant improvement in the mean score on the test ($Mean_{pre}$=7.41, $Mean_{post}$=13.63, t=6.22, p<0.0001). There was improvement on every question on the tests (sections A and B), but especially on section A, questions 7 and 8 (from 5 and 3 correct to 23 and 20, respectively), and every question in section B (from 0 on every question to scores from 15 to 17 correct). Questions 7 and 8 asked how many symbols are need for bases 6 and 43, and this generalisation was clearly better understood after the module of work. Two students, S6 and S19, are attending ESOL classes and were very hindered by language difficulties. Although they only attempted to answer some of the questions they did both improve, from 0 each on the pre-test to 6 and 7 respectively on the post-test. It was pleasing to see that by the end of the module of work 23 of the students could answer Q4a) for base 8 and 19 of these could generalise the place value to 8^x (Q4b)), or equivalent. Similarly 24 students could do the same for base 29 (Q5a)), 21 of these could generalise the place value here too to 29^x, and the same number could even take this to any base and write n^x (Figure 6).

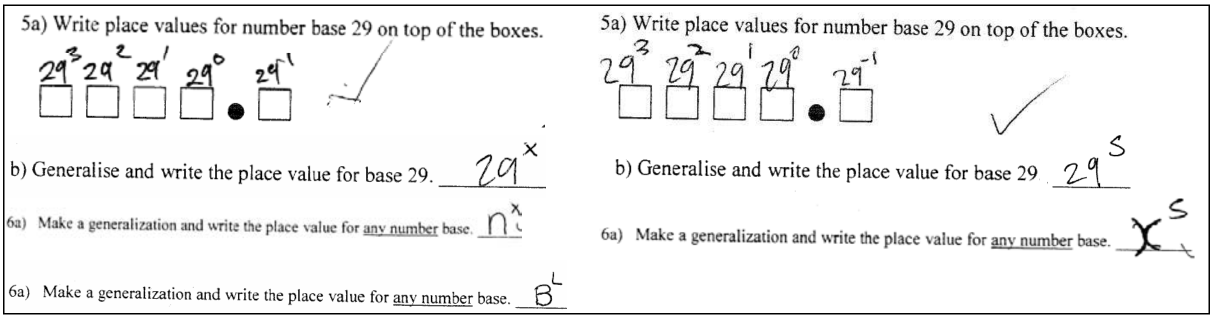


Figure 6. The generalisations of two students.

A number of students, S2, S3, S5, S9, S16, S17, S26, and S27, all expressed the thought that they had found the use of the sticks helpful to formulate their thinking, commenting that "I think the sticks helped me learn about doing place value in different bases" (S2), "The sticks helped me visualise the challenge" (S3), "With the sticks it was easier because we saw what we were doing not just hearing it" (S5), "When you do it with the sticks it helped because you learn better when you do stuff in person, using your hands" (S16). Only a couple of students (S22 and S25) mentioned negative aspects of the sticks, saying how "the sticks didn't help me much" (S22) or how they found the sticks "confusing" (S25). Some also mentioned that they had enjoyed and benefited from the historical ideas they had engaged with: "the different systems were quite fun because we now know how some other cultures write and do their systems" (S5); "the different number systems have made me realise how [much] easier our number system is" (S11); "I learnt how much they struggled to accomplish these historic number systems" (S12); and "Using the other number systems was fun" (S20).

Conclusions

We suggest that the importance of the understanding of place value cannot be underestimated, as Schmittau and Vagliardo's (2006) research on concept mapping confirms. This study attempted to develop in students a meaningful understanding of place value and a structure of the number system through: considerations of large numbers and exponential multiplication; use of concrete materials, multiple bases, multiple representations; and a review of development of historical number systems. The focus was on students' understanding of structure and recognition that the numerals that they deal with on a daily basis are number symbols forming part of a system. The results show that students achieved a certain measure of success and were able to generalise the multiplicative (including exponential) structure of the number system. The study also shows that students respond well when extended beyond what they are responsible for in terms of learning in order to conceptualise what they *have* to learn in the curriculum. This may have implications for mathematics curriculum development, as the positional system receives superficial treatment from most mathematical textbooks. The research suggests that if students are to develop meaning for place value then the topic should be included in the curriculum, since a failure to develop understanding of positional notation adequately will restrict future learning in mathematics.

References

Barbin, E. (2000). Integrating history: research perspectives. In J. Fauvel & J. van Maanen (Eds.), *History in mathematics education: The ICMI study* (pp. 63-90). Dordrecht: Kluwer Academic Publishers.

Datta, B. B., & Singh, A. N. (2001). *History of Hindu mathematics: A source book. Parts I and II*. New Delhi: Bharatiya Kala Prakashan.

Dienes, Z. P. (1960). *Building up mathematics*. London: Hutchinson Educational.

Fauvel, J., & van Maanen, J. (Eds.) (2000). *History in mathematics education: The ICMI study*. Dordrecht: Kluwer Academic Publishers.

Gupta, R. C. (1995). Why study history of mathematics? *Bulletin of the Indian Society for History of Mathematics, 17,* 1-4, 10-28.

Hattie, J. A., Brown, G. T. L., & Keegan, P. (2005). *asTTle V4 Manual 1.1*. Auckland: University of Auckland.

Inhelder, B., & Piaget, J. (1958). *The growth of logical thinking*. London: Routledge and Kegan Paul.

Joseph, G. G. (2000). *The crest of the peacock: Non-European roots of mathematics.* Harmondsworth: Penguin.

Katz, V. J. (2001). Using the history of algebra in teaching algebra. In H. Chick, K. Stacey, J. Vincent, & J. Vincent (Eds.), *Proceedings of the 12th ICMI Study Conference: The future of the teaching and learning of algebra,* (Vol. 2, pp. 353-359). Melbourne: University of Melbourne.

Kieran, C. (1992). The learning and teaching of school algebra. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 390–419). New York: Macmillan.

MacGregor, M., & Stacey, K. (1994). Progress in learning algebra: Temporary and persistent difficulties. In G. Bell, B. Wright, N. Leeson, & J. Geake (Eds.), *Challenges in Mathematics Education: Constraints on Construction* (Proceedings of the 17th annual conference of the Mathematics Education Research Group of Australasia, Vol 2, pp. 303-410) Lismore, NSW: MERGA.

Mason, J. (1996). Expressing generality and roots of algebra. In N. Bednarz, C. Kieran, & L. Lee (Eds.), *Approaches to algebra: Perspectives for research and teaching* (pp. 65-86). Dordrecht: Kluwer Academic Publishers.

Mason, J., Graham, A., & Johnston-Wilder, S. (2005). *Developing thinking in algebra.* Milton Keynes: The Open University.

Puig, L., & Rojano, T. (2004). The history of algebra in mathematics education. In K. Stacey, H. Chick, & M. Kendal (Eds.) *The future of the teaching and learning of algebra: The 12th ICMI Study* (pp. 189-223). Dordrecht: Kluwer Academic Publishers.

Radford, L. (1996). Some reflections on teaching algebra through generalisation. In N. Bednarz, C. Kieran & L. Lee (Eds.), *Approaches to algebra: Perspectives for research and teaching* (pp. 107-112). Dordrecht: Kluwer Academic Publishers.

Schmittau, J., & Vagliardo, J. J. (2006). Using concept mapping in the development of the concept of positional system. In A. J. Canas & J. D. Novak (Eds.), /Concept maps: Theory, methodology, technology/. (Proceedings of the 2nd international conference on Concept Mapping, pp. 590-597). San Jose, Costa Rica.

Skemp, R. R. (1971). *The psychology of learning mathematics*. Middlesex, UK: Penguin.

Srinivasaiengar, C. N. (1967). *The history of ancient Indian mathematics*. Calcutta: World Press Private Ltd.

Thomas, M. O. J. (2006). Developing versatility in mathematical thinking. In A. Simpson (Ed.) *Retirement as process and object: A festschrift for Eddie Gray and David Tall* (pp. 223-241). Prague: Charles University.

Thomas, N. (2004). The development of structure in the number system. In M. J. Hoines & A. B. Fuglestad (Eds.), *Proceedings of the 28th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 305-312). Bergen, Norway: Bergen University College.

Ursini, S. (2001). General methods: A way of entering the world of algebra. In R. Sutherland, T. Rojano, A. Bell, & R. Lins (Eds.), *Perspectives on school algebra* (pp. 13-36). Dordrecht: Kluwer Academic Publishers.

Warren, E. (2003). The role of arithmetic structure in the transition from arithmetic to algebra. *Mathematics Education Research Journal*, *15*(2), 122-137.

Interdisciplinary Learning: Development of Mathematical Confidence, Value, and the Interconnectedness of Mathematics Scales

Dawn Kit Ee Ng
University of Melbourne
<k.ng27@pgrad.unimelb.edu.au>

Gloria Stillman
University of Melbourne
<g.stillman@unimelb.edu.au>

This paper describes the process of developing a survey instrument aimed at measuring aspects of mathematical confidence, value, and the interconnectedness of mathematics as part of a larger study investigating the thinking processes and attitudes towards mathematics of Singaporean secondary school students (aged 12-14) during interdisciplinary learning. Results from exploratory and confirmatory factor analyses on scale items tested revealed six scales with sound validity and reliability properties. The scales are intended for measuring attitudes towards mathematics particularly during interdisciplinary education.

Background

Interdisciplinary and integrated curricula are present in education systems in the United States (Berlin & Lee, 2005) and Australia (VCAA, 2006; Norton, 2006). Interdisciplinary projects were introduced in Singapore schools in 2000 to provide opportunities for students to engage in holistic learning (Curriculum Planning and Development Division, 2001). This paper describes the process of developing a survey instrument aimed at measuring aspects of mathematical confidence, value of mathematics, and the interconnectedness of mathematics for Singaporean secondary students before and after participation in an interdisciplinary project undertaken over approximately 15 weeks.

It is assumed that mathematical confidence, value of mathematics, and the interconnectedness of mathematics are three affective domains directly associated with interdisciplinary learning involving mathematics. Such interdisciplinary tasks require integrating relevant mathematical knowledge with other school subject knowledge for decision making and problem solving within real-world contexts.

A review of literature revealed that different aspects were considered in the definitions of mathematical confidence and the perception of the value of mathematics. Hence, the decision was made to develop the scales for these domains in the study instead of adopting established ones so as to explore aspects of the constructs proposed by others, especially within the Singaporean context. The perception of the interconnectedness of mathematics, nonetheless, is a new contribution to literature by the first author. Though empirical studies on the impact of integrated learning on mathematical confidence and perception of the value of mathematics exist (e.g., Austin, Hirstein, & Walen, 1997), none was found measuring the effect of interdisciplinary learning on perceptions of the interconnectedness of mathematics. Empirical investigations into students' perceptions of the interconnectedness of mathematics pave the way for statistical generalisations on the impact of mathematically-based interdisciplinary work for secondary schools in Singapore that, on the average, conduct one interdisciplinary task per year level annually. Moreover, these scales could be useful for future research involving interdisciplinary learning in different education contexts.

Proceedings of the 30th annual conference of the Mathematics Education Research Group of Australasia
J. Watson & K. Beswick (Eds), © MERGA Inc. 2007

Literature Review on Theoretical Components of Domains

For this study, mathematical confidence consists of three components: students' perceptions of their (a) abilities to carry out mathematical tasks (Barnes, 2003), (b) confidence in learning and succeeding in mathematics with and without making comparisons with their peers (Fennema & Sherman, 1986; Lester, Garofalo, & Kroll, 1989), and (c) determination and effort in mathematics (Schunk, 1984). Items measuring mathematical confidence were adapted from confidence in mathematics scales of Fennema and Sherman (1986), Tapia and Marsh II (2002), and Mittelberg and Lev-Ari (1999), together with Sandman's (1979) self-concept in mathematics scale and Barnes' (2003) items measuring self-efficacy as part of mathematical confidence. Some items were also created by the first author according to the definition presented.

Perception of the value of mathematics is considered from three aspects: (a) current relevance or usefulness of mathematics (Meece, Parsons, Kaczala, Goff, & Futterman, 1982), (b) importance of mathematics for further education and career choice (Barnes, 2003), and (c) value of mathematics in society (Bishop, 2001). Initial items measuring the perception of the value of mathematics were adapted from Barnes' (2003) and Sandman's (1979) value of mathematics scales.

Interconnectedness of mathematics involves students' perceptions about (a) the possible links between mathematics with other subject areas (Jacobs, 1989), (b) usefulness of mathematics in understanding and learning other subjects (Boix Mansilla, Miller, & Gardner, 2000), and (c) complementary relationships between mathematics and other subjects in problem solving (Boix Mansilla et al., 2000). Items measuring this domain were created by the first author from a synthesis of literature about interdisciplinary education. The three components espoused in the definition can be represented on a continuum, ranging from awareness of interconnectedness knowledge through consideration of possible action upon this awareness to concrete use of relevant interconnectedness understanding.

Every item included in the initial item pool was examined carefully to determine if it needed rephrasing to suit Singaporean students between the ages of 12 and 14 who are non-native speakers of English. It was expected that subsequent piloting phases would reduce the number of items to critical representations of the three domains.

Scale Development, Analysis, and Results

Ten experts from mathematics education in Australia and Singapore, and 292 students (aged 12-14) with varying English competencies from seven Singaporean government co-educational secondary schools were involved in the pilot. Participating students had yet to encounter interdisciplinary projects at secondary level. An initial pool of 45 items was piloted in four phases consisting of student interviews, a large scale trial with exploratory factor analysis, confirmatory factor analysis, and test-retest reliability checks. The items were ordered differently without any section headings in the various versions of the scales used during the first two pilot phases to avoid presentation bias. A five-point Likert scale was used to elicit students' responses to the items.

Validity of Scales

The first author employed three approaches to address the content validity of the scales. Firstly, the theoretical components of the three affective constructs established or discussed in existing research were investigated. Some of these theoretical components were validated by extensive empirical research. Secondly, items measuring mathematical confidence and perception of the value of mathematics were chosen from item pools of established scales. The first author used professional experience as a secondary Mathematics and English teacher at a Singapore school to rephrase selected items to suit non-native speakers of English. For the scale measuring perception of the interconnectedness of mathematics, however, literature pertaining to interdisciplinary education was relied upon for creating the initial items. Lastly, two expert panels of mathematics educators from Singapore and Australia vetted the phrasing of each item and checked item appropriateness of the scales. The experts also commented on whether the scale items were grouped appropriately according to the identified theoretical components. Construct validity of the scales was further established through factor analysis techniques.

Phase I: Individual Student Interviews

The first pilot phase was conducted in stages. In the first stage, items from the three scales were reviewed by nine students (aged 12-14) of varying English language abilities from three educational streams in six schools. During face-to-face individual interviews, students selected their responses from the options and explained their choice to the researcher. Particular attention was paid to the selection of the neutral option in order to confirm if the option was chosen because of ambiguity in phasing or an informed reflection on the statement. Occasionally, students were asked to rephrase problematic items in their own words to check if they had interpreted them as intended. Rephrased versions of difficult items were re-tested immediately on subsequent interviewees for clarity.

In the second interview stage, all 45 items (reworded or otherwise) were administered to another group of 36 students (aged 13-14) from an average-ability stream in one school to attempt on two separate occasions one week apart. Their responses to each item both times were compared qualitatively to identify items of high response inconsistency. The first author then selected 13 students who had inconsistent responses to the majority of the tested items for individual face-to-face interviews to explain their response differences. Special attention was paid to the phrasing of items with general high response inconsistency in order to identify any confusing statements for deletion.

The scales were reduced to 41 items here. One example of deletion was an item from the mathematical confidence scale, “I can usually come up with good approaches for solving problems”. This item was highly ambiguous for the students because the phrase “good approaches” was misleading. Even mathematically confident students may “disagree” with the statement if they were not sure if they came up with “good” approaches most of the time during problem solving.

Tables 1 and 2 present the list of items measuring the three affective domains retained for large scale trial after reduction based on student interview feedback and item sources. Negatively phrased items are marked with “#” and scored in reverse during analysis. Items that were subsequently deleted after the large scale trial and confirmatory factor analysis are in italics. The items are arranged according to the theoretical components identified in the definitions of the three domains. For the component, “Perceiving Links between Mathematics and Other Subjects” under the interconnectedness of mathematics domain, a high score on “Math may share some common topics and skills with other subjects” indicated high personal sensitivity to the interconnectedness of mathematics.

Table 1
Mathematical Confidence: Items for Large Scale Trial and Sources

Item Code	Item	Item Source
Mathematical Confidence 1: Confidence in Learning and Succeeding in Mathematics		
CS1	I feel good when I am doing math.	FS[a]
CS2#	Math is my weakest subject.	FS[a]
CS5#	I am not good in math.	FS[t]
CS10	I am sure I can learn math.	FS[t]
CS11#	I will always find math difficult no matter how I hard I study.	FS[a]
CS12	I want to learn higher-level math.	FS[a]
CS13	I usually understand what is going on in my math class.	SM[a]
CS16#	I'm not the type to do well in math.	FS[t]
CA1#	Studying math makes me feel nervous.	TM[a]
CA2#	I am scared of math.	TM[a]
Mathematical Confidence 2: Confidence in Ability to Carry Out Mathematical Tasks		
CS3#	I am afraid to use math because I am not good at it.	R
CS4	I have a lot of self-confidence when it comes to doing math.	FS[a]
CS6	I am good at working with math problems.	SM[a]
CS9	I am ready to try more difficult math problems.	FS[a]
CS14	I'm confident I can understand even the most difficult material in my math class if it is explained clearly.	BN[a]
Mathematical Confidence 3: Determination and Effort in Mathematics		
CS8	I like to think how to solve the difficult math problem first before asking for help.	ML[a]
CS17#	*If I don't get an idea how to solve a math problem right away, I will never solve it.*	SM[t]
CS18#	*I often think, "I can't do it," when a math problem seems hard.*	SM[t]
CS19	When I meet a difficult math problem, I do not give up until I solve it.	ML[t]
Mathematical Confidence 4: Confidence in Mathematical Performance in Relation to Peers		
CR1	*Overall, I feel I am better than some of my friends in math.*	R

Note. FS = Fennema & Sherman (1986), SM = Sandman (1979), BN = Barnes (2003), ML = Mittelberg & Lev-Ari (1999), TM = Tapia & Marsh II (2002), a = adapted, t = taken, R = researcher-created, # = negatively phrased item.

Phase II: Large Scale Trial and Exploratory Factor Analysis

The second phase consisted of a large-scale trial (n = 204) using 41 scale items with students (aged 12-14) from two schools. Statistical analysis was conducted using SPSS (Noonan, 2001). The Kaiser-Meyer-Olkin measure of sampling adequacy was 0.833, implying that exploratory factor analysis was necessary to ascertain the minimum number of hypothetical factors. Initial solution to exploratory factor analysis using principal component extraction with eigen values more than one and varimax rotation revealed 12 orthogonal components accounting for 66.6% of variance. However, inspection of the scree plot (Figure 1) derived indicated the possibility of fewer components as the graph levelled off to form a straight line with an almost horizontal slope beginning at the fifth component.

Table 2
Value and Interconnectedness of Mathematics: Items for Large Scale Trial and Sources

Item Code	Item	Item Source
Value of Mathematics 1: Current Relevance/Usefulness of Mathematics		
VA1#	The math I am studying is useless to me now.	BN[a]
VA2#	The math I am learning won't be useful to me later in my life.	BN[a]
Value of Mathematics 2: Importance of Mathematics for Further Education and Career		
VE1#	The math I am learning won't be important in my future studies.	BN[a]
VE2	I expect to be able to use the math I am studying in my future job.	BN[a]
VE3	Being good in math will help me get a job more easily.	BN[a]
VC1	*I will choose to do math after secondary school because I will need it to get a job next time.*	BN[a]
VC2	*Getting high marks for math will get me more respect from family and friends.*	BN[a]
Value of Mathematics 3: Value of Mathematics to Society		
VS1#	*Math cannot help me understand my surrounding world.*	SM[a]
VS2	Math is of great importance to a country's development.	SM[a]
Interconnectedness of Mathematics 1: Perceiving Links Between Mathematics and Other Subjects		
IR1	Math may share some common topics and skills with other subjects.	R
IR2	I can see links between some math topics and other subjects.	R
IR3	I find learning more meaningful when math and other subjects have common topics.	R
IR4#	I don't try to make connections between math and other subjects when I learn.	R
IR5#	Math has no connections with the other subjects I am studying.	R
IR6	It is important to relate math to other subjects when learning.	R
Interconnectedness of Mathematics 2: Perceiving the Usefulness of Mathematics in the Learning of Other Subjects		
IU1	I can use math to help me learn another subject better.	R
IU2#	*We can't use another subject to help understand some math topics better.*	R
IU3	Sometimes I use math to help me understand another subject.	R
IU4	I use another subject to help me learn math sometimes.	R
IU6	I have used math while working in another subject before.	R
Interconnectedness of Mathematics 3: Perceiving the Complementary Relationship of Mathematics and Other Subjects in Problem Solving		
IC2	Sometimes, I combine what I know from math and other subjects to solve problems.	R

Note. SM = Sandman (1979), BN = Barnes (2003), a = adapted, t = taken, R = researcher-created, # = negatively phrased item.

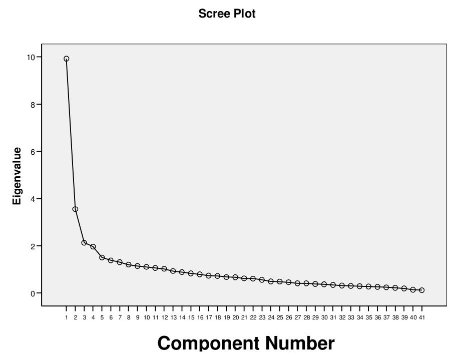


Figure 1. Scree plot showing initial 12-factor solution.

Having 12 factors for 41 items meant that small scales were formed with possibly low validity and reliability. Although there were ten theoretical components to start with, some were made up of single items which could either be deleted or grouped in stronger components. Factor models consisting of fewer components were then investigated to see if these could fit the data set. More solutions were thus generated using principal component analysis, in particular two to ten factor models, judging from the marked changes in the slope of the scree plot. Item factor loadings of less than 0.3 were suppressed. The results of selected models generated by exploratory factor analysis were analysed with the theoretical components defined for the three affective domains in mind. For each model, items purported to belong statistically to the same component were checked if they also fitted in meaningfully as part of a coherent construct. Allocation of items with similar factor loadings to two or more components was based on theoretical decisions.

Initial scale reliability checks and decisions about item deletions were based on an eight factor model. This was because the components of this model were closest in alignment with the theoretical components first envisioned. In this model, items from the mathematical confidence domain were grouped into four scales whereas those from value and the interconnectedness of mathematics domains were categorised into two scales each. The model explained 56% of total variance in the sample data, with the first two components accounting for the highest percentage of variance. Five relatively small scales were derived from the model. Four of the scales had Cronbach's alpha values of less than 0.6.

The process of scale reduction was cyclical, consisting of reiterated tests. Firstly, items with low communalities and low factor loadings within the component were marked for possible deletions. Secondly, student interview records of the marked items were examined for whether the item had appeared ambiguous to some students at times. Thirdly, the frequencies of neutral responses to the marked items were examined because items with high frequencies of such responses would not be helpful in future analyses. Fourthly, the internal consistency reliabilities of the scales generated in the eight factor model were assessed. Some items increased alpha values of the scales when deleted. Fifthly, items with low corrected item-total correlation values were considered for deletion. For scales with more than one item considered for deletion, repeated scale reliability checks with various combinations of items or single items deleted were carried out to choose the best option. Lastly, exploratory factor analysis was conducted again on the remaining items to check if they remained intact within the eight components generated earlier.

Five out of 41 items were deleted in the process of scale reduction. A deletion from the value of mathematics domain was, "Math cannot help me understand my surrounding world". Compared to others, this item had the lowest communality value of 0.317. It did not have any factor loadings greater than 0.3 to any of the eight components. Some student interviewees were puzzled about what the item meant. A high 42.6% of respondents chose the neutral response to this item. Its corrected item-total correlation in the scale was 0.269. Deleting this item raised the alpha value of the scale to 0.735. In addition, some items had similar factor loadings to more than one scale. For example, the item, "I feel good when I am doing math", had factor loadings of 0.469 and 0.508 to two scales. A model involving fewer components could be more best-fitting to the data. Confirmatory factor analysis was conducted next on a different sample to test this hypothesis.

Phase III: Confirmatory Factor Analysis

Data collection with the remaining 36 items was conducted from another three schools. The items were grouped under three headings during data collection, namely, (a) your feelings when doing mathematics, (b) mathematics in relation to other subjects, and (c) your feelings about school mathematics. It was not necessary to divide further the three sections consisting of items from the three domains into eight components.

Confirmatory factor analysis was conducted on a total of 398 questionnaire responses using AMOS (Noonan, 2001). The best-fitting model resulting from confirmatory factor analysis using data here could further establish scale validity because by then, the scales would be exposed to at least two implementations involving separate student samples. Results revealed that a six factor model consisting of 34 items and six correlated scales was best-fitting to the data. Two items (i.e., CS17# and CS18#) were deleted in this process. The six factor model (Tables 3 and 4) classified items from the three affective domains into two scales for each domain. There were still items having dual factor loadings to components under their theoretical scales. In such cases, item allocation was based on the standardised regression weights of these items to their scales.

This model explained about 50% of variance in the sample and had internal consistency reliability values of more than 0.7 in at least four of the scales. The AMOS run yielded a goodness of fit index (GFI) of 0.876. The adjusted goodness of fit value was close to this (0.855). Tabachnick and Fidell (2001) postulate that the GFI should be close to 100% for the model to be a good fit. In this case, the six factor model was a comparatively better fit compared to other models according to GFI values. The choice of the six factor model was further substantiated by its root mean square error of approximation value of 0.048, which indicated a good fit using standards proposed by Hu and Bentler (1999). Moreover, the root-mean square residual value was 0.044, an ideal fit according to Tabachnick and Fidell. Taken together, these statistics indicated the six factor model was a good fit to the data.

Phase IV: Test-Retest Reliability

The last piloting phase checked the test-retest reliability of scales from the six-factor model consisting of 34 items. The scale items were administered on two occasions one month apart to 34 students (aged 12-13) from a non-related sample who had not undergone interdisciplinary projects at secondary level. Correlations between the mean scores to the six scales from both administrations were calculated. Except for the smaller scales of usefulness of mathematics and prospects with mathematics, the test-retest reliabilities of the remaining scales were relatively high, ranging from 0.596 (Beliefs and Efforts at Making Connections) to 0.854 (Self-Concept in Mathematics) (Tables 3 and 4).

Table 3
From Six Factor Model: Mathematical Confidence

Item	Subscale/ Item Statement	Corrected Item-Total Correlation	F1	F2
Scale 1: Self-Concept in Mathematics (SCM)				
Cronbach's α = 0.880; Test-retest correlation r = 0.854**				
CS5#	I am not good in math.	0.820	*0.728*	
CS2#	Math is my weakest subject.	0.695	*0.798*	
CS3#	I am afraid to use math because I am not good at it.	0.645	*0.792*	
CA2#	I am scared of math.	0.668	*0.752*	
CS11#	I will always find math difficult no matter how I hard I study.	0.625	*0.767*	
CA1#	Studying math makes me feel nervous.	0.565	*0.728*	
CS16#	I'm not the type to do well in math.	0.639	*0.740*	
Scale 2: Confidence in Ability and Motivation in Mathematics (CMM)				
Cronbach's α = 0.850; Test-retest correlation r = 0.772**				
CS6	I am good at working with math problems.	0.544	0.542	*0.531*
CS12	I want to learn higher-level math.	0.567	0.397	*0.603*
CS9	I am ready to try more difficult math problems.	0.709	0.343	*0.701*
CS1	I feel good when I am doing math.	0.658	0.449	*0.514*
CS4	I have a lot of self-confidence when it comes to doing math.	0.628	0.420	*0.648*
CS10	I am sure I can learn math.	0.599		*0.638*
CS13	I usually understand what is going on in my math class.	0.517	0.366	*0.553*
CS14	I'm confident I can understand even the most difficult material in my math class if it is explained clearly.	0.391		*0.611*
CS8	I like to think how to solve the difficult math problem first before asking for help.	0.476		*0.562*
CS19	When I meet a difficult math problem, I do not give up until I solve it.	0.458		*0.590*

Note. # represents item in reverse coding. Factor loadings for stated scale in italics. *$p < 0.05$. **$p < 0.01$.

Discussion and Conclusion

Scale development requires a delicate balance between theory and statistical evaluation. Although the theoretical components conceptualised were assessed during factor analyses, the selection of factor models generated for further testing also depended on theoretical considerations. A limitation in this study is that the scales were only tested at high school level. To further validate the scales, the scale instrument could be administered to students from other levels of schooling in various educational settings where interdisciplinary learning takes place. The two small scales consisting of three to four items generated by both factor analyses had comparatively lower internal consistency values. An extension to this study would be to reassess the item composition of these scales, possibly adding parallel items for testing.

Table 4

From Six Factor Model: Value and Interconnectedness of Mathematics

Item	Subscale/ Item Statement	Corrected Item-Total Correlation	F3	F4	F5	F6
Scale 3: Usefulness of Mathematics (UOM)						
Cronbach's α = 0.735; Test-retest correlation r = 0.540**						
VA2#	The math I am learning won't be useful to me later in my life.	0.629	*0.844*			
VE1#	The math I am learning won't be important in my future studies.	0.575	*0.836*			
VA1#	The math I am studying is useless to me now.	0.484	*0.787*			
Scale 4: Prospects with Mathematics (PWM)						
Cronbach's α = 0.584; Test-retest correlation r = 0.445**						
VE3	Being good in math will help me get a job more easily.	0.436		*0.805*		
VS2	Math is of great importance to a country's development.	0.364		*0.752*		
VE2	I expect to be able to use the math I am studying in my future job.	0.381	0.379	*0.487*		
Scale 5: Inter-subject Learning (ISL)						
Cronbach's α = 0.735; Test-retest correlation r = 0.608**						
IU3	Sometimes I use math to help me understand another subject.	0.551			*0.687*	
IU1	I can use math to help me learn another subject better	0.560			*0.750*	
IU4	I use another subject to help me learn math sometimes.	0.503			*0.667*	
IC2	Sometimes, I combine what I know from math and other subjects to solve problems.	0.396			*0.615*	0.326
IR4#	I don't try to make connections between math and other subjects when I learn.	0.367			*0.554*	
IR3	I find learning more meaningful when math and other subjects have common topics.	0.351			*0.478*	
IR6	It is important to relate math to other subjects when learning.	0.408			*0.547*	
Scale 6: Beliefs and Efforts in making Connections (BEC)						
Cronbach's α = 0.622; Test-retest correlation r = 0.596**						
IU6	I have used math while working in another subject before.	0.442				*0.638*
IR2	I can see links between some math topics and other subjects.	0.382				*0.725*
IR1	Math may share some common topics and skills with other subjects.	0.424			0.472	*0.536*
IR5#	Math has no connections with the other subjects I am studying.	0.374			0.388	*0.528*

Note. # represents item in reverse coding, factor loadings for stated scale in italics. $*p < 0.05$. $**p < 0.01$.

In summary, items from the three affective domains, mathematical confidence, value of mathematics, and the interconnectedness of mathematics were classified into six scales, with two scales representing each domain during scale development. All items and their scales have been tested rigorously and the scales were found to have sound validity and reliability properties. Nevertheless, this study recognises that the scales especially purporting to measure perceptions of the interconnectedness of mathematics are new contributions to research on interdisciplinary learning, and that there were limitations to interpretations using the scales. However, information generated through the scales is useful in facilitating interdisciplinary learning. Hence, the scales are recommended for use in future research involving interdisciplinary education.

References

Austin, J. D., Hirstein, J., & Walen, S. (1997). Integrated mathematics interfaced with science. *School Science and Mathematics, 97*(1), 45-49.

Barnes, M. S. (2003). *Collaborative learning in senior mathematics classrooms: Issues of gender and power in student-student interactions.* Unpublished doctoral dissertation, University of Melbourne, Australia.

Berlin, D. F., & Lee, H. (2005). Integrating science and mathematics education: Historical analysis. *School Science and Mathematics, 105*(1), 15-24.

Boix Mansilla, V., Miller, W. C., & Gardner, H. (2000). On disciplinary lenses and interdisciplinary work. In S. Wineburg & P. Grossman (Eds.), *Interdisciplinary curriculum: Challenges to implementation* (pp. 17-38). NY: Teachers College.

Bishop, A. J. (2001). What values do you teach when you teach mathematics? In P. Gates (Ed.), *Issues in mathematics teaching* (pp. 93-104). London: Routledge Falmer.

Curriculum Planning and Development Division. (2001). *Crafting PW project tasks (Secondary)*. Singapore: Ministry of Education.

Fennema, E., & Sherman, J. A. (1986). *Fennema-Sherman mathematics attitude scales.* WI: Wisconsin Centre for Education Research, School of Education, University of Wisconsin-Madison.

Hu, L., & Bentler, P. M. (1999). Cutoff criteria for fit indexes in covariance structure analysis: Conventional criteria versus new alternatives. *Structural Equation Modeling*, 6(1), 1-55.

Jacobs, H. H. (1989). The interdisciplinary concept model: A step-by-step approach for developing integrated units of study. In H. H. Jacobs (Ed.), *Interdisciplinary curriculum: Design and implementation* (pp. 53-65). Alexandria, VA: Association for Supervision and Curriculum Development.

Lester, F. K., Garofalo, J., & Kroll, D. L. (1989). Self-confidence, interest, beliefs, and metacognition: Key influences on problem solving behaviour. In D. B. McLeod & V. M. Adam (Eds.), *Affect and mathematical problem solving: A new perspective* (pp. 75-88). New York: Springer-Verlag.

Meece, J. L., Parsons, J. E., Kaczala, C. M., Goff, S. B., & Futterman, R. (1982). Sex differences in math achievement: Towards a model of academic choice. *Psychological Bulletin, 91*, 324-348.

Mittelberg, D., & Lev-Ari, L. (1999). Confidence in mathematics and its consequences: Gender differences among Israeli Jewish and Arab youths (Part 1). *Gender and Education, 11*(1), 75-92.

Noonan, J. (2001). SPSS for Windows 98, 2000 or Windows NT (Student Version 12.0) [Computer software]. Upper Saddle River, N.J.: Prentice Hall.

Norton, S. (2006). Pedagogies for the engagement of girls in the learning of proportional reasoning through technology practice. *Mathematics Education Research Journal, 18*(3), 69-99.

Sandman, R. S. (1979). *Mathematics attitude and MAI user's manual*. Minneapolis, MN: Minnesota Research and Evaluation Centre, University of Minnesota.

Schunk, D. (1984). Self-efficacy perspective on achievement behavior. *Educational Psychologist, 19*, 48-58.

Tabachnick, B. G., & Fidell, L. S. (2001). *Using multivariate statistics*: Allyn and Bacon.

Tapia, M., & Marsh, G. E., II. (2002, November). *Confirmatory factor analysis of the attitudes towards mathematics inventory.* Paper presented at the annual meeting of the Mid-South Education Research Association, Chattanooga, TN. (ERIC Document Reproduction Service No. ED 471 301)

VCAA. (2006). *Victorian essential learning standards (VELS).* Retrieved March 18, 2007, from http://www.vcaa.vic.edu.au/prep10/vels/index.html

Mathematical Methods and Mathematical Methods Computer Algebra System (CAS) 2006 - Concurrent Implementation with a Common Technology Free Examination

Pam Norton
Monash University
<pam.norton@sci.monash.edu.au>

David Leigh-Lancaster
Victorian Curriculum and Assessment Authority
<leigh-lancaster.david.d@edumail.vic.gov.au>

Peter Jones
Swinburne University of Technology
<pjones@swin.edu.au>

Michael Evans
Australian Mathematical Sciences Institute
<michael@amsi.org.au>

Analyses and commentary for 2002-2005 Mathematical Methods (CAS) pilot examinations in Victoria, on student performance with respect to common items with the standard course have been reported at previous MERGA conferences. In 2006, both Mathematical Methods and Mathematical Methods (CAS) were available to all Victorian schools as equivalent subjects with a new examination structure that comprised a 1-hour *common technology-free* examination and a 2-hour approved technology active examination. This paper provides some analysis of student performance on the technology free examination, and also with respect to common items in both the multiple choice and extended response components of the technology-active examination.

Mathematical Methods and Mathematical Methods (CAS) are equivalent (in terms of curriculum and assessment) but alternative mainstream function, algebra, calculus and probability courses accredited 2006-2009 (Victorian Curriculum and Assessment Authority (VCAA), 2006a). Units 1 and 2 are typically studied at Year 11, and Units 3 and 4 are typically studied at Year 12 with corresponding end-of-final-year external examinations. Mathematical Methods was first accredited in 1993 and has been re-accredited several times, most recently in 2005. Student access to an approved graphics calculator (with stored material in calculator memory such as notes and supplementary programs allowed) both for learning and assessment, including examinations, has been assumed since 1998 (the use of graphics calculators was permitted but not assumed for the 1997 examinations). Mathematical Methods (CAS) was an accredited pilot study of the VCAA 2001-2005 and is now a fully accredited study available to all Victorian schools. Mathematical Methods (CAS) assumes student access to an approved CAS (calculator or software). For the first time in Australia it is now possible to carry out comparative analysis of student performance on two such studies with respect to a common technology-free examination.

During the most recent review of the Victorian Certificate of Education (VCE) Mathematics studies, the areas of study (content) and outcomes (expectations) for Mathematical Methods effectively converged to those for Mathematical Methods (CAS) – the latter essentially a progressive development from its parent study. In part this process was due to Mathematical Methods (CAS) being a more recently developed study of the mainstream function, algebra, calculus, and probability kind, but also it acknowledged the convergence between graphics calculator plus supplementary program and CAS functionality in several key regards. Thus, Mathematical Methods (CAS) encompasses

Proceedings of the 30th annual conference of the Mathematics Education Research Group of Australasia
J. Watson & K. Beswick (Eds), © MERGA Inc. 2007

Mathematical Methods, and includes some additional curriculum content related principally to the use of matrices with respect to the solution of systems of simultaneous linear equations, transformations of the plane, two state Markov sequences, and an elementary introduction to functional relations. Mathematical Methods (CAS) also involves a more general treatment of families of functions defined using parameters and related algebra, and a greater emphasis on exact value representations. The VCAA has foreshadowed that the two studies will be merged into a single CAS-enabled study from 2010.

Aspects of research related to the use of CAS in senior secondary mathematics from Australia and around the world has been noted in Evans, Norton, and Leigh-Lancaster (2005). This included a summary of those systems and jurisdictions that have some CAS permitted or assumed components of examination assessment. In particular, by 2008, Denmark will have moved from several years of a situation similar to that which now applies in Victoria, to a technology-free and CAS-assumed examination structure for its *Baccalaureat Mathematics* examination.

The emergence over the past few years of hand-held enabling technologies (at comparable cost to graphics calculators) such as the *Classpad* 300 and TI-*n*spire (with corresponding software versions) that readily support integrated numerical, graphical, statistical, dynamic geometry, symbolic, and text functionality in a single platform, provides an opportunity for the related research agenda to move beyond the context (senior secondary, function, algebra, calculus, and probability) in which much of this, and earlier, work of the authors has been predicated. That is, it is now possible to go beyond a conceptualisation of CAS calculators as essentially graphics calculator devices with symbolic manipulation capability, to one where the relevant enabling technology is understood to provide a selection of mathematical functionalities that may be deployed, and of which symbolic manipulation is just one such functionality.

The Common Technology Free Examination

Mathematical Methods (denoted MM) and Mathematical Methods (CAS) (denoted MM CAS) Examination 1 is a common 1-hour technology-free examination comprising short answer questions and some extended-answer questions worth a total of 40 marks (see VCAA, 2006b). It is designed to assess students' knowledge of mathematical concepts, their skills in carrying out mathematical algorithms and their ability to apply concepts and skills in standard ways without the use of technology.

A comparison of the mean performance of the two groups on the technology-free paper showed that the MM CAS group (M = 21.22, n = 538) performed at an almost identical level to the MM non-CAS group (M = 21.12, n = 16 057). This is also evident from Figure 1, which displays for each group the mean mark obtained for each question part on the examination. A non-significant result obtained by applying a sign test to these data is consistent with this conclusion ($n = 22$, $x = 11$, $p > 0.05$).

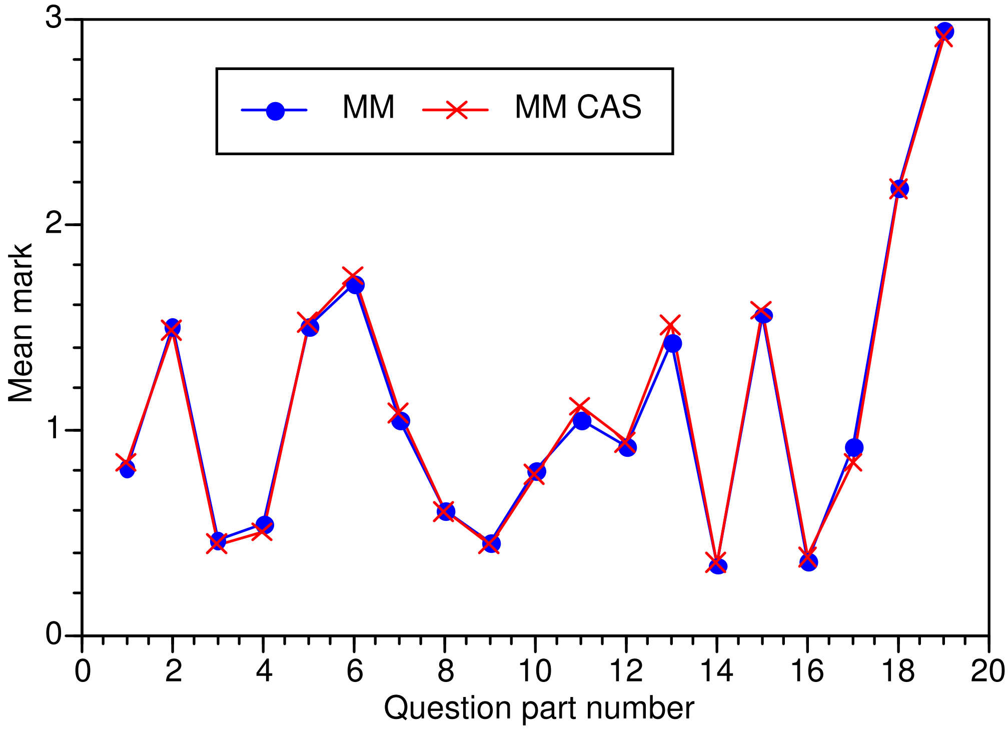


Figure 1. Mean question part marks by group (MM CAS and MM).

The virtually identical performance of the two groups on the technology-free examination does not appear to support the concern that students learning with the aid of CAS would potentially not develop the same level of symbolic facility as those learning without the support of a CAS. It should, however, be recognised that the group of students taking Mathematical Methods CAS in 2006 is not necessarily a representative sample of all students undertaking the Mathematical Methods study in 2006. The Victorian Tertiary Admissions Committee (VTAC, 2007) scaling report, which compares the performance of all students in a given study with the rest of the student cohort across studies, indicates that the overall level of ability of the two Mathematical Methods cohorts (the standard and CAS studies) is effectively the same. It would seem likely that the common curriculum requirements for *both* studies (in terms of key knowledge and key skills specified in the study designs) with respect to mental and by-hands skills of the type tested on the common examination 1, provides a robust basis for very similar levels of performance when students from either cohort do not have access to the relevant enabling technology. Indeed, given the slightly greater curriculum content for Mathematical Methods (CAS), it could be argued that these students have achieved very similar performance to the Mathematical Methods students, with slightly less available time.

Common Multiple Choice Items on the Technology Active Examinations

Mathematical Methods Examination 2 and Mathematical Methods (CAS) Examination 2 are separate two-hour approved technology-assumed access examinations worth a total of 80 marks each (VCAA 2006c, 2006d). They are designed to assess students' ability to understand and communicate mathematical ideas, and to interpret, analyse, and solve both routine and non-routine problems. Examination 2 comprises 22 multiple choice questions, worth a total of 22 marks, and several extended-answer questions (four in 2006) worth a total of 58 marks. Although there are some distinctive questions and/or parts of questions between the two examinations, much is common or very similar (roughly 70 - 80 % of material). Here we only look at the 17 *common* multiple choice items.

Discussion of Multiple Choice Questions

A comparison of the mean performance of the two groups on the common multiple-choice questions showed that the MM CAS group (M = 12.13, n = 538) out-performed the MM group (M =11.50, n = 16 057). This is also evident from Figure 2 which displays, for each group, the percentage of students correctly answering each multiple choice question. The superior performance of the MM CAS group is confirmed by a sign test (n = 15, x = 13, p = 0.004).

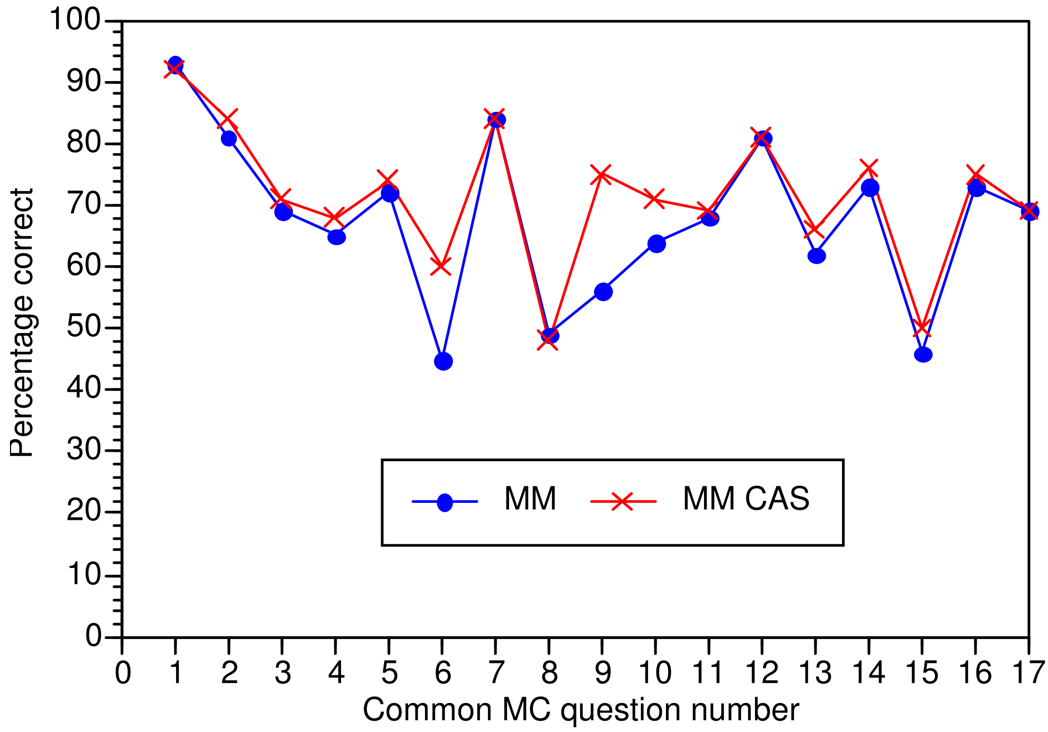


Figure 2. Percentage of students correctly answering each common multiple choice question by group (MM CAS and MM).

A comparison of the group mark profiles suggests that the MM CAS group outperformed the MM group on common questions 6 (by 15%), 9 (by 19%) and 10 (by 7%). A statistical test of these differences, conservatively corrected for the effects of repeated testing, shows all of these differences to be statistically significant (p < 0.001). There were no multiple-choice questions on which the MM group statistically outperformed the MM CAS group.

The questions have again been classified as technology independent (I); technology of assistance but neutral with respect to graphics calculators or CAS (N); or use of CAS likely to be advantageous (C). This classification scheme has now been used for several years in previous reports (Evans, Leigh-Lancaster, & Norton, 2005) and is similar to other schemes used by researchers. Table 1 lists the stems of the multiple choice questions for which the MM CAS group outperformed the MM group and the classification of the questions.

Table 1
Classification of Multiple Choice Questions for Which the MM CAS Group Clearly Outperformed the MM Group

Question number (% difference)	Question stem	Classification
6 (15)	The function g has rule $g = \log_e \lvert x - b \rvert$, where b is a real constant. The maximal domain is	I
9 (19)	The value(s) of k for which $\lvert 2k + 1 \rvert = k + 1$ are	C
10 (7)	A fair coin is tossed 10 times. The probability, correct to four decimal places, of getting 8 or more heads is	N

Question 6 is a classic pencil-and-paper problem. Computational technology has no direct role to play in its solution, although an intelligent student could look at one or more graphs with technology where the b was replaced by a number to assist in answering the question. Both the absolute value function and the term “maximal domain” appeared for the first time in the MM curriculum, but had been in the MM CAS curriculum for the previous four years. Question 9 also uses the absolute value function, the equation can be directly solved by a student with a CAS by simply entering a command like “solve (abs(2k + 1) = k + 1, k)”. The two required solutions, 0 and $-\frac{2}{3}$ are then automatically generated. In contrast, a non-algebraic graphics calculator only has a numerically-based equation solver that generates one solution at a time. This could potentially mislead a student into thinking that there is only a single solution. However, by drawing the graphs of either $y = \text{abs}(2x + 1) - x - 1$ or both $y = \text{abs}(2x + 1)$ and $y = x + 1$ a MM student could have arrived at the correct alternative. Moreover, this is an example of a question for which the correct answer could be obtained by substituting each of the given alternatives into the equation to determine the correct selection. In answering question 10, the use of computational technology is highly advantageous. However, a CAS offers no advantage over a non-CAS enabled graphics calculator in this situation.

Extended Answer Questions

Twenty-two question parts on the extended answer section of the MM CAS Examination 2 and the MM Examination 2 paper were both common in content and equally weighted in terms of marks. In terms of the marks obtained on these common questions, the MM CAS group ($M = 21.99$, $n = 538$) out performed the MM group ($M = 19.91$, $n = 16\,057$). This is also evident from Figure 3, which displays, for each group, the mean mark obtained for each question part. The superior performance of the MM CAS group is confirmed by a sign test ($n = 21$, $x = 19$, $p = 0.0007$).

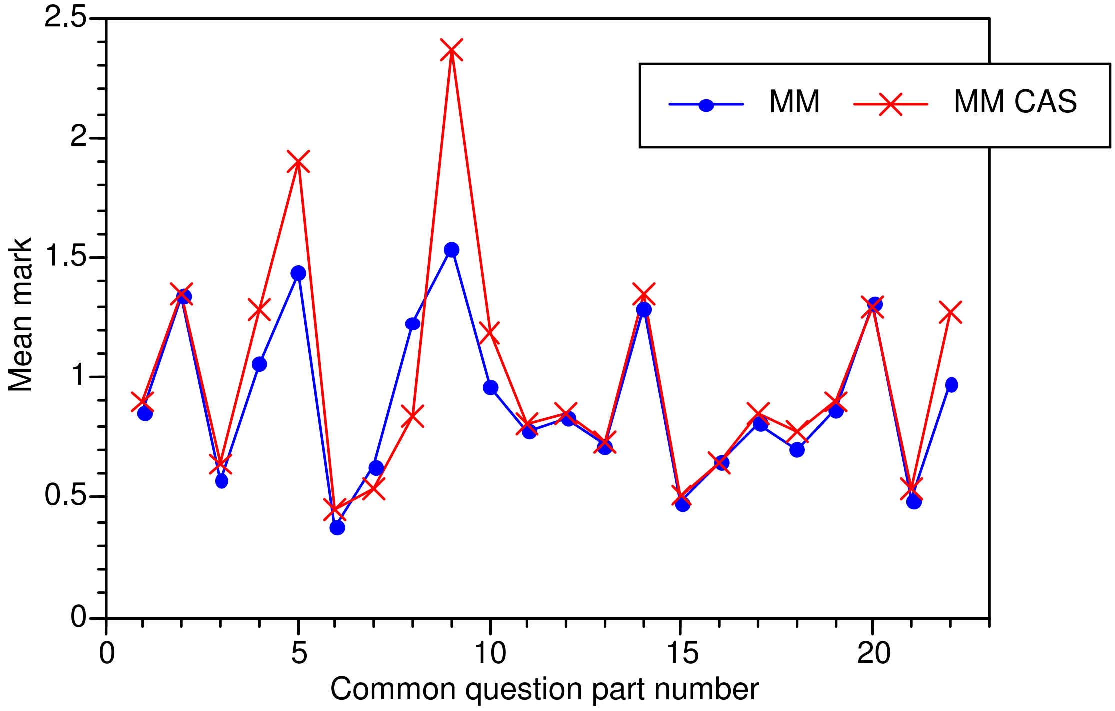


Figure 3. Mean question part marks by group (MM CAS and MM).

A comparison of the group mark profiles coupled with a statistical test of the observed differences, conservatively corrected for the effects of repeated testing, showed 11 questions on which the mean question marks differed between the two groups. All of these differences were found to be to be statistically significant ($p < 0.001$). For each of these questions, the mean difference in percentage terms (positive if in favour of the MM CAS group) and their classification in terms of technology independent (I), neutral (N) or CAS active (C) are displayed in Table 2. In addition, those items for which technology is of assistance but that are likely to be answered efficiently by conceptual understanding, pattern recognition or mental and/or by hand approaches have been indicated by an asterisk.

On nine of these questions the MM CAS group outperformed the MM group. On the remaining two questions, the situation was reversed.

Questions 7 and 8, where the MM group outperformed the CAS group, are clearly technology neutral (and asterisked), in that technology may be required to multiply and add fractions. There is evidence to suggest that the observed differences reflect the influences of curricula differences. These questions involved condition probabilities and their solution was best facilitated through the use of tree diagrams. This was consistent with the MM curriculum. In contrast, in the MM CAS curriculum, conditional probability is also introduced in the context of Markov chains in which problems are formulated in matrix terms. Using a matrix formulation to answer Questions 7 and 8 increases their difficulty level.

The other two questions appearing in Table 2 that are technology neutral, Questions 9 and 10, require a sketch of a density function and the calculation of an integral of a density function numerically, respectively. This area of continuous probability distributions is new to the Mathematical Methods curriculum.

Table 2
Classification of Extended Answer Question Parts for Which the MM CAS Group Clearly Outperformed the MM Group on Nine

Question	Mean difference (%)	Classification	Notes
3	12	C*	Solve $f'(x) = 0$
4	21	C	Equation of tangent
5	32	C*	Find axis intercepts of line
6	18	C*	Analysis skills required
7	–17	N*	Probability calculation based on conditional probabilities
8	–46	N*	Same as 7
9	54	N	Sketch of continuous density function
10	24	N	Numerical integral
18	11	I	Substitution of $x = 0$ into polynomial equation
21	10	C	Solve $f'(x) = 0$; find value of *f* at this point
22	10	C	Solve simultaneous equations, one arising from a derivative

All but one of the other questions mentioned are classified as being CAS-advantaged. Question 3 asked for the exact value of the other solution to 2 cos(x) = 1 over the domain [0, 2π]. (The solution $\frac{\pi}{3}$ had already been given.) It should be noted that not all CAS will find this answer. Questions 21 and 22 were easily done using CAS. For question 22, students would simply define the function $g(x) = \frac{a}{1-bx}$, and then simply issue a command such as "solve {g(0) = 7, $g'(0) = 4.25$ } for {a, b}".

Conclusions

The virtually identical performance of the two groups on the technology-free examination does not appear to support the concern that students learning with the aid of CAS would potentially not develop the same level of algebraic skills as those learning with an ordinary graphing calculator. This is the first time that such a comparison has been able to be made. Follow up studies will be possible for the next few years while the Mathematical Methods and Mathematical Methods (CAS) examinations continue in their present form, with a technology-free examination.

As has been observed in previous studies of Evans et al. (2005) MM CAS students generally perform better overall than MM students on common multiple choice items and on common parts of extended response questions. One advantage of using CAS is that once a solution method has been formulated, it is often simple to carry out the method using CAS thus avoiding trivial algebraic errors. This then allows the student with CAS to engage easily with further parts of the question.

References

Evans, M., Leigh-Lancaster, D., & Norton, P. (2005). Mathematical methods computer algebra system (CAS) 2004 pilot examinations and links to a broader research agenda In P. Clarkson, A. Downton, D. Gron, M. Horne, A. McDonough, R. Pierce and M. McLean (Eds.) *Building connections: Theory, research and practice* (Proceedings of the 28th annual conference of the Mathematics Education Research Group of Australasia, Melbourne, pp. 329-336). Sydney: MERGA.

Victorian Curriculum and Assessment Authority. (2006a). *Mathematics Victorian Certificate of Education Study Design*. Melbourne: VCAA. Retrieved 20 March 2007 from the World Wide Web: http://www.vcaa.vic.edu.au/vce/studies/index.html

Victorian Curriculum and Assessment Authority. (2006b). *Mathematical Methods Written Examination 1*.. Melbourne: VCAA. Retrieved 20 March 2007 from the World Wide Web: http://www.vcaa.vic.edu.au/vce/studies/mathematics/cas/pastexams/2006mmcas1-w.pdf

Victorian Curriculum and Assessment Authority. (2006c). *Mathematical Methods Written Examination 2*. Melbourne: VCAA. Retrieved 20 March 2007 from the World Wide Web: http://www.vcaa.vic.edu.au/vce/studies/mathematics/methods/pastexams/2006mm1-w.pdf

Victorian Curriculum and Assessment Authority. (2006d). *Mathematical Methods (CAS) Written Examination 2*. Melbourne: VCAA. Retrieved 20 March 2007 from the World Wide Web: http://www.vcaa.vic.edu.au/vce/studies/mathematics/cas/pastexams/2006mmcas2-w.pdf

Victorian Tertiary Admissions Committee (2007). 2006 *Scaling Report.* Melbourne: VTAC.. Retrieved 30 March 2007 from the World Wide Web: http://www.vtac.edu.au/pdf/scaling_report.pdf

A Concrete Approach to Teaching Symbolic Algebra

Stephen Norton
Giffith University
<s.norton@griffith.edu.au>

Jane Irvin
Griffith University

Student difficulties with the study of algebra have been well documented. The inability of many students to understand variables and formal symbolic manipulation act as a barrier to success in mathematics study. This report documents an intervention that uses a concrete approach to teaching algebra in a Year 9 class. Results indicate that much of the student struggle was associated with a lack of understanding of arithmetic concepts including those associated with equivalence, operations with negative integers, and the distributive law and fraction concepts. Once these difficulties were addressed through the explicit teaching of the links between materials and symbols, materials and language, language and symbols, students made considerable progress in writing, simplifying expressions, and solving equations with variables on both sides.

Introduction and Background

This paper reports on a teacher's (Jane) attempts to teach critical algebra understandings, in particular, how to solve equations with variables on both sides. Jane is the mathematics subject head of department in a suburban high school situated in a middle to lower class outer Brisbane suburb. Historically very few students in the school opted to study intermediate or advanced mathematics and Jane hoped to increase the proportion of students enrolling these courses (Mathematics B and Mathematics C). To this end Jane devised an algebra intervention for Year 9, in which she hoped that student success in middle school algebra would encourage a higher proportion of the students to enrol in the more advanced senior mathematics subjects. This paper describes the intervention (in brief) and reports on the barriers to, and successes in, student learning of algebra when a verbal and concrete approach to teaching was undertaken. Stacey and Chick (2004) noted "The algebra teacher has a crucial role to play both in bringing algebraic representations to the fore and in making their manipulation by students a venue for epistemic growth" (p. 31).

Many students come to the study of early algebra with poor understandings of arithmetic (Thompson & Fleming, 2003). The use of calculators can account for some of the difficulties associated with number computation (MacGregor, 2004), however, it is likely that failure to understand the structures of arithmetic (e.g., commutative law, distributive law, fractions, integers and operations) will place an added cognitive load on students when it comes to the study of algebra. Kieran and Yerushalmy (2004, p. 21) described algebra as "Generalization of numerical and geometric pattens and the laws governing numerical relationships" and Sfard (1994) discussed algebra as "generalised arithmetic" consisting of the "operational" and "structural" phases. Sfard's (1994) definition of "operational algebra" can be summed up as being tied to arithmetic operations, for example, the use of backtracking to solve simple linear equations can be seen as the reversal of arithmetic operations. "Structural algebra" can be seen in solving an equation with variables on both sides, however, simple reversal of operations such as in backtracking does not suffice. The solution requires the suspension of operational thinking to view the overall structure of the equation, that is, "structural" thinking. Stacey and MacGregor (1999) regarded students' ability to solve equations with variables on both sides as an indicator of "formal algebra" or what Sfard (1994) regarded as "structural algebra". The ability of students to solve such equations can be seen as a marker between arithmetic and algebraic thinking.

Proceedings of the 30th annual conference of the Mathematics Education Research Group of Australasia
J. Watson & K. Beswick (Eds), © MERGA Inc. 2007

Stacey and Chick (2004) noted an important part of algebra learning is transformational processes. Clearly, without the transformational tools of arithmetic, students are likely to be burdened with added cognitive load and struggle to move from operational to the structural phase of algebra thinking. Another way of putting this is to say that without a foundation of numeracy the "generalization" of it would seem to be a more difficult task, some would say an impossible task, unless the structures of arithmetic were made explicit and taught simultaneously with algebra, at least as far as can be done. In addition, Lins and Kaput (2004) support this position emphasising the parallels between fundamental processes of arithmetic and algebra.

Jane's concerns about the proportion of students undertaking more advanced mathematics are shared by the broader mathematics community (e.g., Barrington, 2006). It has previously been reported that traditional school algebra is not appropriate for students with weak literacy and numeracy skills and that these students may prefer to acquire knowledge through increased verbal interaction and concrete activity, and that failure in early algebra is likely to lead to passive withdrawal from further study or active rebellion (MacGregor, 2004). In this way algebra study acts as a filter to the study of more advanced mathematics (e.g., MacGregor, 2004; Stacey & Chick, 2004). Similarly, Jane's focus on equivalence, expressions, variables and solving with variables on both sides of the equal sign have been described as critical to algebra (e.g., Bazzini, Boero, & Garuti, 2001; Herscovics & Linchevski, 1994; MacGregor & Stacey, 1997; Stacey & Chick, 2004; Stacey & MacGregor, 1999). It is generally recognised that traditional approaches to teaching algebra have failed. Booker (1987) summed up the difficulties with problems associated with the introduction of symbolic values as being a result of changes in language and nuances with respect to operations when students attempt to move from operating arithmetically to algebraically. Kaput (1987) puts the issues more bluntly, pointing out the perceived meaninglessness of school mathematics in general, and algebra in particular, as being at the heart of the problem. Kaput (1995, p. 4) reported that most students see algebra as "little more than many different types of rules about how to write and rewrite strings of letters and numerals, rules that must be remembered for the next quiz or test." In short, algebra makes little sense to many children. Solutions to the problem of algebra failure are many and frequently interconnected, and include the following:

- Making explicit algebraic thinking inherent in arithmetic in children's earlier learning (e.g., Lins & Kaput, 2004; Warren & Cooper, 2006).
- Explicit teaching of nuances and processes of algebra in an algebraic and symbolic setting (e.g., Kirshner & Awtry, 2004; Sleeman, 1986; Stacey & MacGregor, 1997, 1999; Stacey & Chick, 2004), especially in transformational activities (e.g., Kieran & Yerushalmy, 2004; Stacey & Chick, 2004).
- Using multiple representations including the use of technology (e.g., Kieran & Yerushalmy, 2004; Van de Walle, 2006).
- Recognising the importance of embedding algebra into contextual themes (National Council of Teachers of Mathematics, 1998; Stacey & Chick, 2004).

Clearly, much more can be said about the scope of algebra research, however, this is a brief paper. A review of the literature reveals that as more and more is written the terminology becomes increasing specialised, but the problems have persisted over 20 years of algebra teaching reform. One explanation is that top down reform recommendations have been difficult to implement in the classroom. In this study, the reforms reported have been generated from a teacher's perceptions of student needs and implemented as a reform of pedagogy in her classroom.

Method

The overall design is a case study that uses design based research, in so much as cycles of design, enactment, analysis and re-enactment, analysis, and further design take place. As in all design-experiments, the specific research questions investigated in each iteration are conjured out of analysis of recent failures of previous iterations (Bereiter, 2002). This study reports on Jane's third iteration of the intervention in 2006, but each iteration was essentially identical in terms of teaching approach. This iteration was the beginning of the researcher's engagement with the school algebra project. Future iterations will reflect what has been learnt from the analysis reported in this paper. The involvement of the researcher as an active participant in this process gave the research design a participatory collaborative action research element (Kemmis & McTaggard, 2000).

Participants

The participants in this study were the classroom teacher, Jane, and the 18 students engaged in a 6 week algebra course. The school was a State School located in a middle to low socio-economic status suburb. In recent years between 4% and 7% of the senior school had enrolled in Mathematics C (Advanced Mathematics). Although approximately 25% enrolled in Mathematics B (Intermediate Mathematics), half of these students failed and or withdrew in Year 11, leaving approximately 12% entering Year 12. In comparison, the national average enrolment in 2004 for Advanced Mathematics was 11.7% and for Intermediate Mathematics it was 22.7% (Barrington, 2006). The students in the study were drawn from the 180 students in the Year 9 cohort. All 180 students were tested for general numeracy and more specifically to determine those who were "comfortable with the use of symbols to describe patterns" (Jane, personal communication, 2007). Students who scored in the top 1/3 on the pre-test were offered the algebra extension. There were three cohorts of about 20 students each. The intervention occurred in 18 one-hour lessons over 6 weeks.

Data collection and Analysis

All 18 lessons were observed and video recorded over the 6 weeks, including recording of class discussions, examples of student working on tasks in small groups, and examples of the teacher and researcher scaffolding student learning. Student work samples including workbooks, tests, and scripts were collected. Students were asked to explain their reasons for making mathematical decisions throughout the duration of the study. Student work was analysed for error patterns. In the case of their test scripts errors in computation and transformation could be seen in their recording of their mathematical processes. This also occurred in examining their class work. Additionally, in class students asked why they made mathematical decisions. Finally, the nature of student difficulties could be deduced from the questions they asked Jane and the discussions they had with their peers during group work.

Results and Discussion

Description of Instructional Discourse.

Instructional discourse refers to the rules for selecting and organising instructional content (Bernstein, 2000). Jane articulated her intentions as follows, "They needed to experience mathematics study in an academic and rigorous way." The instructional discourse was based on an underpinning theoretical framework put forward by Booker, Bond, Sparrow, and Swan (2004, p. 20).

> While the role of materials and patterns they develop is fundamental, materials by themselves do not literally carry meaning…it is language that communicated ideas, not only in describing concepts but also helping them take shape in each learner's mind.

Jane's selection of activity sources was based on helping students make connections between materials, verbal language initially, and then symbolic language. The primary sources of activity were *A Concrete Approach to Algebra* (Quinlan, Low, Sawyer, White, & Llewellyn, 1987) and *Access to Algebra Book 2* (Lowe, Johnston, Kissane, & Willis, 1993). These resources used unmarked cups with hidden counters (blobs), envelopes with hidden counters to help develop the concept of variables, and extensive use of other concrete materials including patterns made from counters or match sticks. Both resources emphasised the use of language and logic to connect patterns modelled with material to verbal descriptions of the patterns, tabular summaries of the patterns and symbolic representations. Jane used match stick patterning to introduce variables and activities with cups, counters (blobs) and envelopes to explore writing expressions, equivalent expressions, simplifying expressions, expanding expressions and writing equivalent equations. Equations were created and solved using the balance model, initially with the concrete materials and, then, linking to traditional recording using symbols. Activities from Lowe et al. (1993), were selected that emphasised the links between materials and symbols. In this way students saw the meaning of the equals sign in the context of an algebraic equation. They also learnt the careful recording of transformations on both sides of the equation. The third source of student activities was based on the symbolic recognition and manipulations of algebra terms covered above embedded in algebra games that Jane had devised. The algebra games were constructed according to principles outlined by Booker (2000), some were track or strategy grid board games in which diagrammatic representations of concrete materials needed to be matched with symbolic expressions. Other games included concept games in which randomness of question was introduced by throwing dice of various configurations. For example, a concept game required players to write an algebraic equation from a scenario given in words and then solve the equation: A *number* is multiplied by Δ (a ten sided die is rolled to provide this number), then ۞ is added to it (a second 10 sided die is rolled to provide this number), the answer is ◯ (a 36 sided die is rolled to provide this number), what is the *number*? Such an equation is linear with a variable on one side of the equals sign. It can be solved using the balance model and frequently results in a fraction solution. The games could be played by two or three students, and enabled them to consolidate and attain competency in the mathematics learnt in prior activities.

Description of Regulatory Discourse

Regulative discourse refers to the models of the teacher, learner and, pedagogic relations that underpin the selection and organisation of content within learning activities (Bernstein, 2000). Typically, the 1-hour lessons were divided into three segments. In an introductory segment, Jane used the white board and an activity selected from Quinlan et al. (1987) or Lowe et al. (2001) as the basis to conduct a class discussion on the key concepts. During the segment she kept a careful record of the discourse on the white board. In this discourse, Jane emphasised the links between materials, natural language which she extended to the nuances of algebraic language, and symbols. Typically, in the second segment, students worked in pairs or threes on activities selected from Quinlan et al. (1987) or Lowe et al. (1993) and Jane helped individuals or pairs of students when they requested assistance. Sometimes this activity continued to the end of the class. Generally, the third segment was used by students to play the algebra games designed to give students an opportunity to apply and consolidate the algebra learning that had occurred earlier.

Results of Discourse

The results of this discourse are presented in two sections. First, the types of errors that limited student completion of the algebra tasks are presented. Second, the success or otherwise of students on a written test and an analysis of their errors is presented.

Video analysis of teacher/student discussion indicated that the following difficulties and/or errors were most common in limiting student understanding and completion of the algebra based activities.

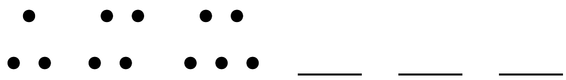
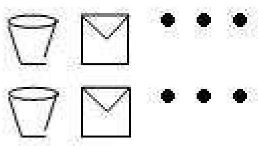
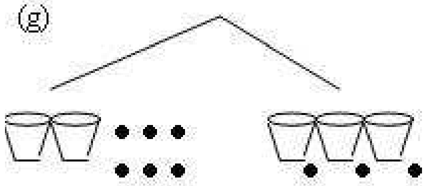
1. Difficulties associated with operations with negative integers (e.g., 4 – - 3; - 4 + - 2; -3 – -7). Students did not know how to complete these computations. In addition students experienced difficulties with subtraction signs when expanding, for example 2(4 – 5), students ignoring the – sign and treating it as an addition obtaining an answer of 18; and 3(2x – 4) expanded to 6x + 12.
2. Difficulties associated with solving equations of the form 3x + 3 = 15. In particular, students not treating the equal sign as an indication that equivalence must be maintained. For example, students removed the 3 from the left hand side but not the right hand side, thus solving for x as equal to 5. Similar mistakes were made on equations such as x – 2 = 2x + 3 where students would add 2 to the LHS but not to the RHS. When students were first challenged with problems of this structure, some attempted to use "backtracking" and simply reported it could not be done.
3. Difficulties associated with number facts, such as students not knowing their multiplication facts and making computational errors.
4. Difficulties associated with fractions, such as errors in solving equations of the form 3y + 18 = 6y + 6; students responding with y + 18 = 2y + 6 indicating that students had generalised inappropriately about cancelling. In this instance the error has its roots in arithmetic where students are taught to simplify fraction computations by cancelling. For example, in operating upon the fraction below (e.g., (2 + 3) divided by 2), students simply cancelled the 2s and answered 3.

This over generalisation in regard to fraction cancelling results from an inadequate understanding of fractions, and the application of this limited understanding to the algebra solving problem above fails the student irrespective of the student's understanding of symbolism. One of the goals of the teaching program was to address these difficulties within the teaching of the algebraic skills. Jane and the researcher's approach when confronted with such problems in the context of algebra was to re-teach the concepts in arithmetic contexts (e.g., students adding $^1/_2$ to $^3/_3$ equal $^4/_5$); Jane would revise the concept of equivalence of fractions using paper fraction strips to display a visual model of equality or in equality, in this case one half is not equal to one third, before linking this to multiplication by unity (e.g, $^1/_2 \times ^3/_3 = ^3/_6$ to enable the formation of fractions with the same name or denominator). The approach of teaching arithmetic and algebra concurrently with the aid of concrete materials has found favour in those who recommend the teaching of algebra early in students study (e.g, Lins & Kaput, 2004; Warren & Cooper, 2006).

Summary of Written Test Results

A written post test consisting of 25 separate questions was completed by 15 students. One of the students missed many of the algebra lessons and her results were consistently incorrect. A sample of the questions and the number of students who answered them correctly are listed in Table 1. All students were able to recognise the pattern, complete the table of ordered pairs and represent it symbolically as equivalent to p + 2 = n. One student did not complete the equation. Seven of the students were able to correctly graph the function. Little class time was spent on graphing of variables. A number of authors have noted that multiple representations of functions including the generation of tables and graphs assist student understanding of algebraic relationships (e.g., French, 2002; Kieran & Yerushalmy, 2004). French (2002, p. 81) commented that “students need to understand the links between the equation, the table of values or set of co-ordinates and the graph, and to be able to move fluently between these representation.” In this regard the use of technologies such as excel spread sheets and graphing calculators has been recommended (e.g, Kieran & Yerushalmy, 2004; Kissane, 1999). Clearly, this was an instructional discourse issue to be addressed in future algebra teaching in this school.

Table 1
Summary of Test Results for 15 Students

<table>
<tr><th>Concept</th><th>Typical question</th><th>Correct responses</th></tr>
<tr><td>Completing a pattern, table, and describing the pattern algebraically.</td><td>
1 2 3 4 5 6
<table><tr><th>P</th><th>1</th><th>2</th><th>3</th><th>4</th><th>5</th><th>6</th><th>20</th><th></th></tr><tr><td>N</td><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td>102</td></tr></table></td><td>14/15
1 partial</td></tr>
<tr><td>Writing expressions and equations</td><td>(f)
</td><td>14/15</td></tr>
<tr><td></td><td>(g)
</td><td>13 /15
1 partial</td></tr>
<tr><td>Simplify expressions</td><td>7x – 2x +5y – 3y</td><td>14/15</td></tr>
<tr><td>Expand and simplify</td><td>3(3x – 2y)
b(x + 2y)</td><td>14/15
8/15</td></tr>
<tr><td>Solving equations with model</td><td>3y + 2 = y + 6</td><td>10/15, 2 partial answer.</td></tr>
<tr><td>Solving equations without a model</td><td>5x + 2 = 7x – 9</td><td>5/15 correct, 2 partial correct.</td></tr>
</table>

Almost all students were able to write the symbolic expression given a pictorial representation. For example, all but two students could transform an equation represented with cups and counters into an algebraic equation (see question g; 2y + 6 = 3y + 3). These findings suggest that student understanding of the variable concept was progressing, in that students used symbols to represent variables in an unknown context. These findings are in contrast with those of MacGregor and Stacey (1997) who reported that the majority of students in a broad Australian study up to age 15 seemed unable to interpret algebraic letters as generalised numbers or even specific unknowns. MacGregor and Stacey found students ignored letters, replaced them with numerical numbers or regarded them as short hand names. For example, some students viewed letters in algebra as abbreviated words, whereas others the letter with its place in the alphabet (as occurs in some puzzles and code translations). In addition, MacGregor and Stacey noted that students writing of letters in contexts such as $h10$ meant add "10 to h" and 1y meant take one from y, indicative of the Roman subtraction principle. Clearly, some of these errors arise out of the inappropriate transfer of generalisations. Of additional concern to MacGregor and Stacey was the prevalence of students being unable to distinguish the name of the object (e.g., the person Con) from the name of the attribute (e.g., Con's height). Such errors are a serious obstacle to writing expressions and equations. Such errors were not evident in the final written tests or during class in the latter stages of the intervention in this study. The findings that almost all the students could interpret and simplify the cups and counters equation representations correctly is encouraging and in contrast to the results reported by MacGregor and Stacey (1997). Essentially, this meant that the students recognised that x and y were symbolic representations of a variable (generally) and could complete simple arithmetic computations involving the symbols.

Almost all students expanded 3(3x – 2y) correctly, but less than half of these students were able to expand b(x + 2y) appropriately. This suggests that the students might not have an understanding of multiplication separate from repeated addition. Subsequent to reviewing these results Jane reported that she had believed that the way she taught expansion by using concrete materials encouraged the students to use repeated addition at first. She had hoped for them to then establish a pattern which would mature to the full understanding of the distributive law. Jane said she was attempting to assist the students to develop a full understanding rather than a superficial procedural knowledge likely to be generated by the usual approach to expansion such as drawing arrows from the 3 to the 3x and -2y. The test scripts supported her preferred approach for treating 3(3x – 2y). However, those students who could not expand b(x + 2y) expanded 3(3x – 2y) using the repeated addition algorithm as follows (Figure 1):

$3(2x-2y)=$	$6x-6y$
$$\begin{array}{r} 2x-2y \\ +\,2x-2y \\ 2x-2y \\ \hline 6x-6y \end{array}$$	

Figure 1. Teaching expansion

When the variable in front is included, as in b(x + 2y), the repeated addition model is no longer an available strategy. However, students with a good understanding of the distributive law, for example, being able to view 14 x 3 as (10 + 4) multiplied by 3, which can be taught with a focus on place value (i.e., 4 ones multiplied by 3 ones is 12 ones, renamed as 2 ones and 1 ten; 1 ten multiplied by 3 ones is 3 tens, added the renamed ten gives a total of 4 tens and 2 ones or 42 ones), ought to have been able to make the transition. Most did not. When this early number teaching is linked to the array model and the application of the distributive law, the number multiplication 3(10 + 4) has exactly the same structure b(10 + 4) and the similarity in structure can be extended to b(x + 2y). This example illustrates the opportunity to capitalise on an understanding of arithmetic structures in the learning of algebra. In this study the use of the array model in linking the application of the distributive law in number and algebra was not made explicit, hence it might reasonably be argued that the student results reflected this omission.

Almost all students solved an algebraic equation with unknowns on both sides using materials (Table 1 – Solving equation with model), and one third of the students solved a similar structured equation without the use of materials (Table 1 – Solving equation without model). It could be said that those students who completed the solving task without materials had developed an abstract schema of variables while those who solved the equation with materials but not without, were at an intermediate stage. Ability to equation solve such as that above has been described as achieving beyond a didactic cut or cognitive gap (Herscovics & Linchevski, 1994) and is a critical indicator of algebraic thinking. Similarly, Stacey and MacGregor (1999) regard this type of problem solving as an indicator of formal algebra capacity. This is the case since the equation cannot be easily solved arithmetically, algebraic competence is required (Stacey & MacGregor, 1999). Stacey and MacGregor reported that only about 8% of Year 10 students made this cut, those failing tending not to use logical reasoning in relation to inverse operations, instead using guess and check methods or attempting to use numerical methods; that is, they could be described as not reasoning algebraically.

Encouragingly, there was no evidence at the end of the study that students retained misconceptions about symbolism including confounding with place value, letters standing for abbreviations or for specific numbers, misuse of conventions (e.g., work from left to right), and false analogies with ordinary language such as that described by Stacey and MacGregor (1997) and Sleeman (1986).

Conclusions and Recommendations

The activities in this intervention were not applied or linked to authentic contexts or real world situations. This was almost pure algebra with a heavy focus upon the development of symbolic meaning and symbolic manipulation through the use of concrete materials. The results cause us to qualify the recommendations of the NCTM (1998) that the teaching of algebra be tied to contextual themes. The relative success of students in writing expressions and solving equations reported in this study prompt us to reconsider what "contextual" really means. The use of concrete materials and student discussion such as that recommended by Quinlan et al. (1987) and Lowe et al. (1993), and also reflected in algebra games, was sufficient to engage and help students make sense of algebra processes.

The results support the notion that the essence of learning algebra like that of arithmetic is to make connections between materials, patterns and symbolic meaning through the medium of language (e.g., Booker et al., 2001). In this instance, the use of materials was guided by resources that have been available to Australian teachers since the late 1980s (e.g., Quinlan et al., 1987) and early 1990s (e.g., Lowe et al., 1993). These resources place emphasis on students making meaning through the use of materials, discussion and students' articulation of their mathematical thinking, through natural language initially, then subsequently through the specialised language of algebra conventions. The results support the explicit teaching of the nuances and processes of algebra in an algebraic and symbolic setting (e.g., Kirschner & Awtry, 2004; Sleeman, 1986; Stacey & MacGregor, 1999). The findings should encourage teachers and researchers to look again at multiple representational techniques and the use of concrete material resources as an alternative to the way algebra is traditionally taught in middle school.

An examination of student needs in needing the links between representations to be made explicit throughout the trial and, to a less extent the error patterns exhibited in the final test, indicate that much of the "trouble" for students was not associated with algebra but rather had its roots in incomplete understanding of arithmetic structures. The error patterns associated with doing operations with integers (operating with negative integers), lack of understanding of the equal sign, over generalisation of cancelling procedures (fraction errors), and an incomplete understanding of the distributive law, have their roots in arithmetic misconceptions, and incomplete understandings and inability to transfer arithmetic understandings to algebraic contexts.

In this small and "streamed" class most of the misconceptions usually could be addressed through the intervention of the teacher and researcher. Subsequent to this analysis, the use of more explicit linking of arithmetic and algebraic structures will be investigated in future iterations of the research study (e.g., the application of the distributive law in two digit multiplications and expansion of algebra expressions). In a larger and heterogeneous class it is easy to envision that a limited understanding of the structures of arithmetic and inability to see their relevance to algebra could spell the end of algebra competency and confidence among students. We concur with the assertions of previous authors (e.g., Lins & Kaput, 2004; Warren & Cooper, 2006) that critical concepts underpinning algebra (e.g., equal concepts, integer study, fractions, the distributive law and general arithmetic computational competency) need to be emphasised in the primary years. For example, younger students can be taught with the aid of materials in order to help them solve simple equations (Warren & Cooper, 2006). This process helps students understand the structures of arithmetic in that the unknown is seen as a quasi variable to be solved by backtracking, or arithmetic operations based about the balance model, and reverse operations that emphasise the meaning of equals. With the careful use of materials the balance model thinking can be extended to understanding how to solve equations with variables on both sides.

With an understanding of arithmetic, upon the beginning of formal algebra study, when arithmetic processes including "do the same to both sides", "use a graph", "guess and check", and "backtracking", do not work (Stacey & MacGregor, 1999), students would be equipped with an operational and structural understanding of arithmetic such that they can transfer the understanding to the "operational" then "structural" phases of algebra, and to "value" the study of algebra. The importance of valuing algebra is that usually arithmetic means do not work efficiently with "real algebra" problems, whereas algebra enables an efficient solution to be found (Stacey & MacGregor, 1999).

References

Barrington, F. (2006). *Participation in Year 12 Mathematics Across Australia 1995-2004*. Melbourne International Centre of Excellence in Mathematics and Australian Mathematical Science Institute. University of Melbourne. Available: www.boardofstudies.nsw.edu.au/manuals/pdf_doc/maths_st6_lit_curr_rev_pt1.pdf

Bazzini, L., Boero, P., & Garuti, R. (2001). Moving symbols around or developing understanding: the case of algebraic expressions. In M. den Huevel-Panhuizen (Ed.), *Proceedings of the 25th Conference of the International Group for the Pyschology of Mathematics Education* (Vol. 2, pp. 121-128). Utrech, The Netherlands, PME.

Bereiter, C. (2002). Design Research for Sustained Innovation. *Cognitive Studies, Bulletin of the Japanese Cognitive Science Society, 9*(3), 321-327.

Bernstein, B. (2000). *Pedagogy, Symbolic Control and Identity.* (Rev. ed.). New York and Oxford: Rowman & Littlefield Publishers, Inc.

Booker, G. (1987). Conceptual obstacles to the development of algebraic thinking. In J. Bergeron, N. Herscovics, & C. Kieran (Eds.), *Proceedings of the eleventh international conference Psychology of Mathematics Education* (Vol. 1, pp. 275-281). Montreal: PME.

Booker, G. (2000). *The Maths Game: Using Instructional Games to Teach Mathematics.* Wellington: New Zealand Council for Educational Research.

Booker, G., Bond, D., Sparrow, L., & Swan, P. (2004). *Teaching Primary Mathematics.* Frenchs Forest: Pearson Education.

French, D. (2002). *Teaching and Learning Algebra.* London: Continuum.

Herscovics, N., & Linchevski, L. (1994). A cognitive gap between arithmetic and algebra. *Educational Studies in Mathematics, 27*, 59-78.

Kaput, J. (1987). PME XI Algebra papers: A representational Framework. In J. Bergeron, N., Herscovics, & C. Kieran (Eds.), *Proceedings of the eleventh international conference Psychology of Mathematics Education*. (Vol. 1, pp. 345-354). Montreal: PME.

Kaput, J. (1995, October). *A research base supporting long term algebra reform?* Paper presented at the Seventeenth Annual Meeting for the Psychology of Mathematics Education (North American Chapter). Office of Educational Research and Improvement, Washington, D.C.

Kemmis, S., & McTaggart, R. (2000). Participatory action research. In N. K. Denzin & Y. S. Lincoln (Eds.), *Handbook of qualitative research* (pp. 567-606). Thousand Oaks, CA: Sage.

Kieran. C., & Yerushalmy, M. (2004). Research on the role of technology environments in algebra learning and teaching. In K. Stacey., H. Chick, & M. Kendal (Eds.), *The Future of Teaching and Learning of Algebra*. *The 12th ICMI Study* (pp. 99-152). Boston: Kluwer.

Kissane, B. (1999). Graphics calculator and algebra. In E. B. Ogena & E. F. Golla (Eds.), *8th South East Asian Conference on Mathematics Education, Technical Papers*: *Mathematics for the 21st Century* (pp. 211-219). Manila: Ateneo de Manilia University.

Kirshner, D., & Awtry, T. (2004). Visual Salience of Algebraic Transformations. *Journal for Research in Mathematics Education, 35*(4), 224-257.

Lins, R., & Kaput, J. (2004). The early development of algebraic reasoning: The current state of the field. In K. Stacey, H. Chick, & M. Kendal (Eds.), *The Future of Teaching and Learning of Algebra*. *The 12th ICMI Study*. (pp. 73-96). Boston: Kluwer.

Lowe, I., Johnson, J., Kissane, B, & Willis, S. (1993). *Access to Algebra: Book 2*. Carlton: Curriculum Corporation.

MacGregor, M. (2004). Goals and content of an algebra curriculum for the compulsory years of schooling. In K. Stacey, H. Chick, & M. Kendal (Eds.), *The Future of Teaching and Learning of Algebra*. *The 12th ICMI Study* (pp. 313-328). Boston: Kluwer.

MacGregor, M., & Stacey, K. (1997). Students' understanding of algebraic notation. *Educational Studies in Mathematics, 33*, 1-19.

National Council of Teachers of Mathematics, Algebra Working Group. (1998). A framework for constructing a vision of algebra: A discussion document. In *National Research Council, The nature and role of algebra in the K-14 curriculum: Proceedings of a national symposium,* May 27 and 28, 1997 (Appendix E, pp. 145-190). Washington, DC: National Academy Press. Available on the Web at http://books.nap.edu/catalog/6286.html.

Quinlan, C., Low, B., Sawyer, T., White, P., & Llewellyn, B. (1987). *A concrete Approach to Algebra*. N.S.W. Darlinghurst: Algebra Research Group.

Sfard, A. (1994). The gains and pitfalls of reification - the case of algebra. *Educational Studies in Mathematics, 26*, 191-228.

Sleeman, D. (1986). Introductory algebra: A case study of student misconceptions. *Journal of Mathematical Behavior, 5*(1), 25-52.

Stacey, K, & Chick, H. (2004). Solving the problem with algebra. In K. Stacey., H. Chick, & M. Kendal (Eds.), *The Future of Teaching and Learning of Algebra*. *The 12th ICMI Study* (pp. 1-20). Boston: Kluwer.

Stacey, K., & MacGregor, M. (1999). Taking the algebraic thinking out of algebra. *Mathematics Education Research Journal, 1*; 24-38.

Stacey, K., & MacGregor, M. (1997). Ideas about symbolism that students bring to algebra. *The Mathematics Teacher, 90*(2), 110-113.

Thompson, S., & Fleming, N. (2003). *Summing it up: Mathematics Achievement in Australian Schools, TIMSS 2002*. Australian Council for Educational Research. Camberwell. Available online: http://www.timss.acer.edu.au/documents/TIMSS_02_Maths_ES.pdf.

Van de Walle, J. (2006). *Elementary and Middle School Mathematics: Teaching Developmentally*. Boston: Pearson International.

Warren, E., & Cooper, T. J. (2006). Young children's ability to use the balance strategy to solve for unknowns. *Mathematics Education Research Journal, 17*(1), 58-72.

Developing Positive Attitudes Towards Algebra

Stephen Norton
Griffith University
<s.norton@griffith.edu.au>

Jane Irvin
Griffith University

This paper reports on one teacher's attempts to teach critical algebra understandings to a Year 9 class in ways that engage the students and help them to develop positive perceptions of their ability to learn algebra in a "rigorous and symbolic way". This paper describes a 6-week algebra intervention based upon connecting concrete representations with symbolic expressions and equations through the careful use of formal algebra language. The teacher had expressed her aspirations that interventions such as this would encourage more students to undertake intermediate and advanced mathematics courses in senior secondary years. The study collected data on student perceptions about their learning experiences including perceptions about mathematics as a subject domain, engagement with the activities, development of algebraic understanding, and the quality of discourse within the classroom. This study found that the students valued the classroom discourse much more than they did the normal mathematics learning experiences. These findings have implications for in-service and pre-service teacher education.

Introduction and Background

This paper analyses one teacher's (Jane) attempts to teach critical algebra understandings, in particular, how to solve equations with variables on both sides. Jane is the mathematics subject head of department in a large suburban school situated in a middle to lower class outer Brisbane suburb. Historically very few students in the school opted to study Advanced Mathematics and Jane's goal was to increase the proportion of students enrolling in these subjects (Mathematics B - Intermediate Mathematics and Mathematics C - Advanced Mathematics). In recent years between 4% and 7% of senior students enrolled in Advanced Mathematics, much lower than the national average of 11.7% in 2004 Barrington, 2006). Approximately 25% enrolled in Intermediate Mathematics, which is on a par with the national average (Barrington, 2006). With this goal in mind, Jane devised an algebra intervention for Year 9 to be extended to Year 10 that she considered would foster student success in middle school algebra and, consequently, would encourage a higher proportion of the students to enrol in the more advanced senior mathematics subjects. That is, she hoped that if students had success and formed positive perceptions about mathematics they would be more inclined to tackle Mathematics B and C.

The research literature indicates that declining student perceptions and participation in mathematics study is of broad concern. For example, a decline in student perceptions of the worth of mathematics study has been reported from about Year 4 onward (e.g., Thompson, & Fleming, 2003), and Barrington (2006) has reported the declining participation in intermediate and advanced mathematics between 1995 and 2004. He concluded that:

> There has been a significant nett loss of students taking the Year 12 mathematics options in which higher-level mathematical skills are taught. This has implications for the recruitment of students to undertake tertiary studies in the quantitative sciences, and for the national capacity for innovation in engineering and technology. The effects are much wider: fields such as finance and molecular biology are developing into quantitative and sophisticated areas. (p. 4)

Student attitudes to algebra are central to this process because early failure in algebra is likely to result in passive withdrawal from further study in the area or active rebellion (MacGregor, 2004). Hence, algebra study may act as a filter for further study in mathematics (e.g., MacGregor, 2004; Stacey & Chick, 2004), therefore, the development of positive attitudes to the subject are essential to increase student enrolments in advanced mathematics subjects.

Researchers have noted that early educational and socialisation processes are critical to children's learning and perceptions and subsequent participation in education (Khoon & Ainley, 2005). Student perceptions, which include their expectations of success and the value that they attribute to particular tasks, have been found to correlate strongly with later participation in study (Ethington, 1992; Wigfield & Eccles, 2000). The analysis of this relationship in the TIMSS data for Australian school students by Thomson and Fleming (2003) supports theorised connections between perceptions, participation and performance. Perceptions shape the information individuals attend to and how it is interpreted (De Bono, 2004). In summary, the decreasing participation of students in mathematics can be related to the interaction of three perceptions held by an increasing proportion of students:

1. Algebra is perceived as uninteresting and based upon symbolic manipulation with limited meaning and little relevance to every day life (e.g., Boaler, 2000; Kaput, 1995; MacGregor, 2004; Stacey & Chick, 2004).
2. Algebra is perceived as difficult (e.g., MacGregor, 2004).

Some critical aspects that have led students to see mathematics in these ways include an over-reliance on textbook work with a procedural focus, teacher dominated discourse, and closed learning activities that result in a lack of understanding and capacity to transfer knowledge (e.g., Hollingsworth, Lokan, & McRae, 2003). Gregg (1995) described this "school mathematics tradition", as a tradition that is well entrenched and resilient (Perry, Howard, & Tracey, 1999). Repeatedly, students report that they neither understand important mathematical concepts nor appreciate why they are worth the effort of learning (Watt, 2005). What is true for mathematics in general is especially true of algebra since its understanding assumes knowledge of the specialised processes and language nuances associated with symbolic representations (e.g., Stacey & Chick, 2004). The student perceptions that they cannot understand mathematics and that it is a hard subject is linked to an image students have of mathematics as an abstract collection of rules and processes (Boaler, 2000; Kaput, 1995). This is particularly in the case of algebra where resources found in standard texts frequently do not encourage teachers to enact appropriate pedagogy to foster algebraic thinking (Kaput, 1995; Stacey & MacGregor, 1999). Further, it has been reported that if students engage in "extensive symbolic manipulation before they have developed a solid conceptual foundation for their work, they will be unable to do more than mechanical manipulation" (Kirshner & Awtry, 2004, p. 39). That is, they did not think deeply about mathematical concepts and structures and were not challenged to think about solving problems, rather, classroom discourse was dominated by the practice of routine operations. In terms of the difficulty students have with algebra study, Stacey, and MacGregor (1999) found that only 8% of 116 Year 10 students (16 years old) could solve an equation with variables on both sides if it included fraction operations. Given this lack of success in algebra study it is hardly surprising that many students developed the perception that algebra was a hard subject and that they had little confidence in succeeding. In addition, MacGregor (2004, p. 315) noted:

> Academic learning based upon reading and writing, such as the traditional school algebra of symbolic manipulation and word-problems, is not appropriate for students with weak literacy and numeracy skills.

MacGregor (2004), citing Marks and Ainley (1997), indicated that only about 20% of 14-year-olds had the literacy and numeracy skills to cope with algebra study. Stacey and MacGregor (2004) have reported that the major reason for student difficulties with using algebraic methods for problem solving is that they do not understand its underpinning logic. Students tend to wish to calculate in the first instance, a behaviour that is consistent with their arithmetic learning. However, algebra requires an analysis of the problem and transforming it into algebraic equations. That is, students need to recognise, construct and manipulate algebraic expressions before applying their computational skills. Many students struggle with this change in operating and have had little support to make the transition.

Teachers are central to any model of effective educational reform and renewal (e.g., Doerr, 2004). Consequently, it is imperative to analyse systematically all aspects of teachers' classroom practices including the intended curriculum (curriculum guidelines, lesson plans), implemented/enacted curriculum (co-construction of classroom knowledge), and attained curriculum (what students actually learn) (Taylor, Muller, & Vinjevold, 2003). Thus the purposes of this paper are as follows:

1. To describe briefly an algebra intervention designed to immerse students in active learning through engagement with concrete materials and careful use of language.
2. To describe the students' perceptions about the algebra intervention including: whether they were engaged to think deeply and understand mathematical ideas, perceptions of fun, availability of teacher support, how hard they worked and how much they were challenged, the collaborative nature of tasks and perceptions about the nature of mathematics, in particular, whether they viewed algebra as essentially symbolic manipulation or about mathematical ideas.

The cognitive gains of students have been analysed and described in a companion paper (Norton & Irwin, 2007).

Method

The overall methodology is a case study that uses design based research, in so much as cycles of design, enactment, analysis and re-enactment, analysis and further design take place. As in all design-experiments, the specific research questions investigated in each cycle are conjured out of analysis of recent successes and/or failures of previous cycles of the research (Bereiter, 2002). This paper reports on Jane's third iteration of her teaching intervention commenced in 2006. Each iteration was essentially identical in terms of teaching approach. The third iteration was concurrent with the researcher's engagement with the school algebra project. Future iterations of the intervention will reflect what has been learnt from the analysis reported in this paper. The involvement of the researcher as an active participant in this process gave the approach a participatory collaborative action research element (Kemmis & McTaggard, 2000).

Participants

The participants in this study were the classroom teacher, Jane, and the 18 students engaged in a 6-week algebra course. The students in the study were drawn from the 180 students in the Year 9 cohort. All 180 students were tested for general numeracy and more specifically to determine those who were "comfortable with the use of symbols to describe patterns" (Jane, field notes). Students who scored in the top 1/3 on the pre-test were offered the algebra extension. There were three cohorts of about 20 students in each group. These students were drawn from the 7 mixed ability classes in Year 9. Although the students had done some patterning, backtracking and work with quazi variables, this was their first real exposure to algebraic symbolism. Since the selected students came from most of the 7 mixed ability mathematics classes, we can conclude that the students' descriptions of the mathematics classes that they were drawn from are representative of middle school teaching in that school.

Description of Intervention

The intervention will be described in terms of the concept Instructional Discourse, which refers to the rules for selecting and organising instructional content (Bernstein, 2000). The instructional discourse was based on an underpinning theoretical framework put forward by Booker, Bond, Sparrow, and Swan (2004, p. 20).

> While the role of materials and patterns they develop is fundamental, materials by themselves do not literally carry meaning … it is language that communicated ideas, not only in describing concepts but also helping them take shape in each learner's mind.

Jane's selection of activity sources was, therefore, based on helping students make connections between materials, verbal language initially, and then symbolic language. The primary sources of activity were *A Concrete Approach to Algebra* (Quinlan, Low, Sawyer, White, & Llewellyn, 1987) and *Access to Algebra Book 2* (Lowe, Johnston, Kissane, & Willis, 1993). Jane used match stick patterning to introduce variables and activities with cups, counters (blobs), and envelopes to explore writing expressions, equivalent expressions, simplifying expressions, expanding expressions, and writing equivalent equations. Activities from Lowe et al. (1993) were selected that emphasised the links between materials, patterns, and variables, and used the balance model for representing and solving equations including those with variables on both sides. The third source of student activities was based on the symbolic recognition and manipulations of algebra terms covered above embedded in algebra games that Jane had devised. The algebra games were constructed according to principles outlined in Booker (2000). The games were played by two or three students, and enabled them to consolidate and attain competency in the mathematics learnt in prior activities. A companion article describes the teaching approach in more detail (Norton & Irwin, 2007).

Typically the 1-hour lessons were divided into three sessions. In the introductory session Jane used the white board and an activity selected from Quinlan et al. (1987) or Lowe et al. (1993) as the basis to conduct a class discussion on the key concepts. In the second session students worked in pairs or threes on activities selected from Quinlan et al. (1987) or Lowe et al. (1993) and Jane scaffolded the learning of individuals or groups of students. Sometimes this activity continued to the end of the class. Generally, the third session was spent by the students playing the algebra games.

Data Collection

Eighteen one-hour classes were observed and recorded on video. At the end of the intervention the students completed a 5-point Likert perceptions survey, developed by the first author, consisting of 40 questions related to eight attributes. The selection of eight attributes was informed by the literature on student attitudes towards mathematics and, in particular, to learning of algebra (e.g., Boaler, 2000; Kaput, 1995; MacGregor, 2004;

Stacey & Chick, 2004). These attributes of mathematics learning are thought to be important to the formation of student perceptions and subsequent participation in mathematics study. The items were positive or negatively worded and each question started with the phrase, *Compared to the way I usually study (or study) maths ….* The eight attributes and a relevant sample question are shown below,

1. Student perceptions on the depth of their mathematical thinking. E.g., … *the activities in this class help me to think deeply about mathematical ideas.*
2. Student perceptions of fun and interest associated with the algebra learning. E.g., … *the learning in this class is more fun.*
3. Student perceptions of their confidence to develop mathematical understanding underpinning mathematical processes. … *the learning in this class has encouraged me to believe I can understand mathematics better.*
4. Student perceptions of support for learning provided by the teacher. E.g., … *the teacher in this class helps me more.*
5. Student perceptions of how hard they worked in class. … *I work harder in this class.*
6. Student perceptions about how challenging the activities were. *In this class I am challenged to figure out how to solve problems.*
7. Student perceptions of collaborative learning. … *I and the people I sit with help each other more.*
8. Student perceptions about the nature of mathematics … *Before I thought maths was mostly about operations and symbols, this class has helped me see it is about ideas.*

Subsequently, each student was interviewed about her/his responses, a process designed to increase the validity of the perceptions survey. For example, students were asked explain and expand upon their responses to the survey and, in addition, compare these responses with their perceptions of mathematics in their normal classrooms. These interviews were video recorded. Finally, the students were asked to draw two pictures, one to represent their perceptions with regard to their normal mathematics classroom experiences and the second to represent their perceptions with respect to their algebra learning experiences in the intervention class. The students then briefly interpreted their diagrams for the author and these were audio recorded.

Results

The results of the analysis of students' perceptions of the intervention are presented in three parts. The first part describes student responses to the perceptions survey. Second is the description and analysis of student interview data. Finally, an analysis of the pictures students were asked to draw about activities and feelings in the different classes is presented. The results of the survey are summarised in Table 1.

The responses indicate that students responded either strongly agree (5) or agree (4) on each attribute gauging their perceptions of Jane's intervention. In short, the high mean values for each attribute imply that on completion of their engagement in Jane's intervention all students perceived that they thought more deeply about mathematical ideas, perceived that the activities were more interesting and fun; that they had developed greater confidence in their capacity to understand mathematics; that they had the perception that the teacher had a greater role in helping them learn; that they worked harder and spent more time on task and perceived that the tasks were more challenging; that there was more collaborative learning; and that they had developed the perception that their concept of mathematics had shifted from one predominately associated with computations and symbolism towards one aligned with problem solving and mathematical ideas.

Table 1
Summary of Student Responses to the Perceptions Survey

Learning attribute	Mean n=18	Standard Deviation
1 Depth of thinking	4.48	0.44
2 Fun and interest	4.48	0.63
3 Confidence to understand	4.57	0.43
4 Teacher support	4.62	0.53
5 Hard working	4.57	0.46
6 Task challenge	4.22	0.47
7 Collaborative learning	4.39	0.37
8 Nature of mathematics	4.22	0.54

Interview data supported the key findings from the survey analysis described above. The interview data did reveal that, in stark contrast to positive outcomes from the teaching intervention, all 18 student participants reported that in their normal mathematics class they did little mathematics work. Several indicated that in one class a small group of boys was disruptive and diverted the teacher from teaching students who wanted to learn. It was also reported that classroom activities were usually boring and consisted mostly of repetitious work from a textbook or worksheets that the students found confusing. The students reported that – frequently – they did not understand the mathematics work and received little assistance from the teacher. The following comment can be described as a typical example of students' perceptions of their normal classrooms: "Text books are a bad thing, because, you are just doing the same thing from the same maths book over and over again. It gets really boring."

Explanations for the behaviour of peers who disrupted the normal classes included that these students had "given up trying to learn mathematics" and it was unlikely that any form of mathematics teaching "would interest them". For example Lorry reported:

> They just do not want to do it, or they can't, or they are just too lazy or they have just given up hope.

The students were asked "If the average student who 'mucks up' received the teaching that they had experienced in the algebra class, could they understand the algebra work?" Most students responded "Probably".

It was clear that student responses to questions about the learning environment could be grouped into two themes: those related to teacher scaffolding and student collaborative learning, and those related to the use of the concrete materials and games. The following responses made during the final interview were typical of student comments on teacher and peer support in the algebra class.

> This teacher helps me understand a lot more.
>
> The teacher explains it more and my friend can help me as well.
>
> We help each other a lot more in this class.

These student comments above are substantiated by data collected through prolonged video observation of the class during the intervention. An increased level of cooperative learning increased over time was observed as the students became more familiar with other and the teacher's ways of providing appropriate assistance.

When asked about the role of the materials in facilitating their understanding of algebra all but two students (Tammy and Simon) responded with comments similar to those shown below (Interview data).

> It shows you what is actually happening and what is going on, it is not all in your head. The cups and counters helped us to make sense when we had to write down the x and y's.
>
> The activities in the algebra class are a lot harder than the ones in the textbook and it actually makes you have to think. And it is fun.
>
> With the text book I had to work harder to find the information. With the cups and counters it makes it easier to understand.
>
> The cups and counters are very important. Without them it is just "What the!" but with the cups and counters and envelopes you can see what you are doing and so you can learn heaps more.
>
> You have to think about what does that equal before you can do it.
>
> I feel like a maths nerd, which is good (Girl).

Tammy reported that she did not really use the materials. When asked to explain this she responded,

> I used them at the start, but I did not really understand them. But then the teacher drew them on the board, and I did not need them (physical materials). The diagram was enough.

Simon who was asked to explain why he had responded on his survey that he worked less in the algebra class explained,

> Well, with the cups and counters and games, it is easier to understand, that's why I voted I worked less in this class.

The student comments with respect to the algebra games were similar to those above. The following comment by Kingsley summed up the class evaluation of the algebra games.

> I understood it more with the games. It is actually showing you how to put it into action. It is showing you things. You have to try harder to find it rather than just finding an answer by adding or multiplying. You have to find the equation, and you have to do it with cups and counters and a diagram.

Classroom observations supported student comments in that they demonstrated a high proportion of time on task while working with the games and much of the discussion between students centred around the underpinning mathematics.

With respect to student drawings of their activities and feelings in the different classes some common themes emerged and were subsequently expanded upon by the students. Figure 1 represents one student's (Harry) report of his perceptions about learning in the two classrooms. Harry explained that in his usual mathematics class he started with enthusiasm. This enthusiasm waned over time.

> This is like the path of a ball, it bounces around. It loses momentum and eventually just sits and does nothing. I was trying to do my work that was just confusing, so eventually you just lose motivation. You just do nothing.

Harry explained his feelings about the algebra class as follows, "It is like the opposite of the box. These (lines) are clear straight and easy to understand. It is like the algebra class." The underlying themes evident in Harry's drawings were common to all student sketches.

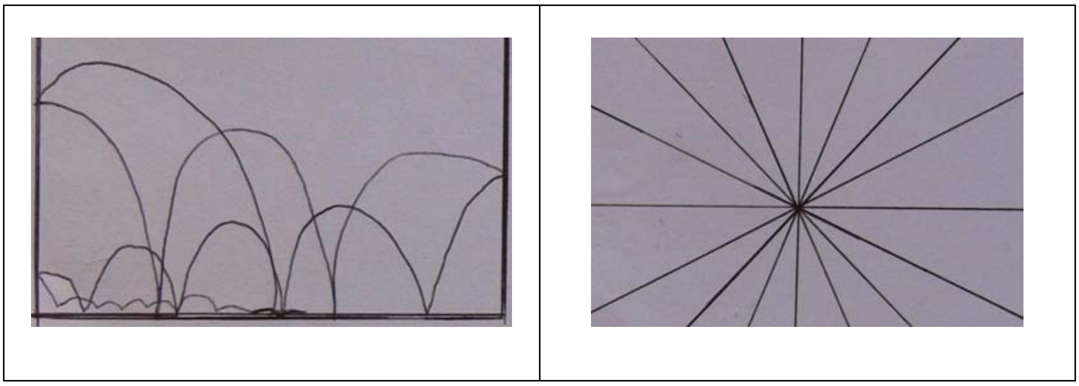


Figure 1. One Student's (Harry) Comparison of his usual Mathematics Lessons (LHS) with the Algebra Lessons (RHS).

Discussion and Conclusions

Survey responses, student diagrams, interview data, and sustained classroom observations in the specialist algebra class indicated high student engagement on mathematical tasks, with productive discourse between students and teacher, and high quality collaborative learning behaviours. Students helped each other and discussed activities and games. Sfard, Nesher, Streefland, Cobb, and Mason (1998) have reported that getting students to talk about mathematics in a meaningful way is challenging. The results of this study indicate that these students placed high value upon the instructional discourse. This discourse was based on the use of materials to build algebraic concepts, as recommended by a variety of researchers (e.g., Becker & Rivera, 2005; Booker et al., 2004; Quinlan et al., 1987; Lowe et al., 1993), and emphasised the links between concrete and visual representations and explicit algebraic language. No students reported that mathematics was inconsistent with their identity formation, an issue reported by other researchers (e.g., Khoon & Ainley, 2005; Watt, 2005). Rather, students reported an increased confidence in their capacity to understand algebra. This is an educationally significant finding because expectation models (e.g., Ethington, 1992; Wigfield & Eccles, 2000) indicate that success and positive perceptions about mathematical study are likely to encourage students to undertake studies in advanced mathematics.

The results from the interview data where students described and commented upon their learning in the non-streamed classes confirmed what many authors have reported, that the school mathematics tradition of talk and chalk from the front of the room and reliance on worksheets and textbooks with a focus upon repetitious symbolic manipulation played a significant role in their perception that mathematics is dull, boring, and hard and was a collection of rules that frequently made little sense (e.g., Barrington, 2006; Boaler, 2000; Kaput, 1996; Thompson & Fleming, 2003; Watt, 2005). In addition, the students reported that this pedagogy did little to foster their deep thinking about the mathematical ideas. Such findings support those of other authors with respect to standard algebra activities (e.g., Kaput, 1995; Stacey & Chick, 2004). Some students also reported that their disenchantment with the learning activities in normal classes was linked to issues of behaviour management and consequently limited help from the teacher for their learning. Although these results raise the issues of classroom management strategies and streaming of mathematics classes according to ability, such issues are largely beyond the scope of this paper. The authors are aware that the intervention class was smaller than normal classes that have about 25 students and that this would have impacted upon teaching dynamics. The students commented that the absence of students who were overt in disrupting the normal classes made the class "much better". Without exception they did not want to go back to their mixed ability classes. However, it is noteworthy that most students acknowledged that the choice of activities in this class would "probably" have helped most students in unstreamed classes to learn algebra. This finding encourages the authors to extend the teaching models to the mixed ability classes. The students and Jane reported that the pedagogy enacted in unstreamed classes appeared to condemn students to failure in mathematics.

The students in this study were highly articulate in explaining why they valued the instructional discourse. First, it was apparent that the learning activities helped create a classroom environment in which the teacher was able to provide learning support to individuals and small groups of students. This was the case because the students found the activities valuable and engaging and being on task, did not disrupt the class or their peers. Second, student comments emphasised the importance of physically manipulating the materials and linking the material representations, pictorial displays and symbolic representations. Some students reported the use of materials to be useful in early phases of learning, but once the procedural rules were understood, no longer needed to manipulate the materials physically. These results are consistent with those of researchers who recommend this approach to teaching (as above) and in contrast to the descriptions of teaching of algebra in most classrooms (e.g., Kirshner & Awtry, 2004; MacGregor, 2004; Stacey & MacGregor, 1999) and the teaching of mathematics in general (e.g., Barrington, 2006; Gregg, 1995; Hollingsworth et al., 2003; Perry et al., 1999; Thompson & Fleming, 2003).

Teachers are central to effective reform (e.g., Doerr, 2004) and this study indicated that there are good instructional discourse models upon which to build engaging conversations that would help students to develop perceptions that they can learn algebra, that it is not mostly a collection of unrelated rules and symbolic manipulations but rather an inquiry based upon ideas. It also provided data that students can improve in their perceptions that mathematics learning can be fun. This conclusion highlights the potential importance of the nature of the instructional pedagogical discourse used by Jane in her intervention for the professional development of pre-service and in-service teachers. In particular, the use of concrete materials, games and explicit language should underpin middle school mathematics teaching and learning in order to foster students' positive perceptions of algebra.

References

Barrington, F. (2006). *Participation in Year 12 Mathematics Across Australia 1995-2004*. Melbourne International Centre of Excellence in Mathematics and Australian Mathematical Science Institute. University of Melbourne. Available: www.boardofstudies.nsw.edu.au/manuals/pdf_doc/maths_st6_lit_curr_rev_pt1.pdf

Becker, J., & Rivera, F. (2005). Generalization strategies of beginning high school algebra students. In H. Chick & J. Vincent (Eds.), *Proceedings of the 29th annual conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 121-128). Melbourne: PME.

Bereiter, C. (2002). Design Research for Sustained Innovation. *Cognitive Studies, Bulletin of the Japanese Cognitive Science Society, 9*(3), 321-327.

Bernstein, B. (2000). *Pedagogy, Symbolic Control and Identity.* (Rev. ed.). New York and Oxford: Rowman & Littlefield Publishers, Inc.

Boaler, J. (2000). Mathematics from another world: Traditional communities and the alienation of learners. *Journal of Mathematical Behaviour, 18*(4), 379-397.

Booker, G. (2000*). The Maths Game: Using Instructional Games to Teach Mathematics.* Wellington: New Zealand Council for Educational Research.

Booker, G., Bond, D., Sparrow, L., & Swan, P. (2004). *Teaching Primary Mathematics.* Frenchs Forest: Pearson Education.

De Bono, E. (2004). *Thinking course: Powerful tools to transform your thinking*. London: BBC Books.

Doerr, H. (2004). Teachers' knowledge and the teaching of algebra. In K. Stacey, H. Chick & M. Kendal (Eds.), *The Future of Teaching and Learning of Algebra*. *The 12th ICMI Study* (pp. 267-290). Boston: Kluwer.

Ethington, C. (1992). Gender differences in a psychological model of mathematics achievement. *Journal for Research in Mathematics Education, 23*(2), 166-181.

Hollingsworth, H; Lokan, J., & McRae, B. (2003). *Teaching Mathematics in Australia: Results from the TIMSS 1999 Video study*. Melbourne: Australian Council for Educational Research.

Gregg, J. (1995). The tensions and contradictions of the school mathematics tradition. *Journal for Research in Mathematics Education, 26*(5), 442-466.

Kaput, J. (1995, October). *A research base supporting long term algebra reform?* Paper presented at the Seventeenth Annual Meeting for the Psychology of Mathematics Education (North American Chapter). Office of Educational Research and Improvement, Washington, D.C.

Kemmis, S., & McTaggart, R. (2000). Participatory action research. In N. K. Denzin & Y. S. Lincoln (Eds.), *Handbook of qualitative research* (pp. 567-606). Thousand Oaks, CA: Sage.

Kirshner, D., & Awtry, T. (2004). Visual Salience of Algebraic Transformations. *Journal for Research in Mathematics Education, 35*(4), 224-257.

Khoon, S., & Ainley, J. (2005). *Longitudinal surveys of Australian Youth; Research Report 41: Attitudes, Intentions and Participation*. Camberwell, Victoria: Australians Council for Educational Research.

Lowe, I., Johnson, J., Kissane, B, & Willis, S. (1993). *Access to Algebra: Book 2*. Carlton: Curriculum Corporation.

MacGregor, M, (2004). Goals and content of an algebra curriculum for the compulsory years of schooling. In K. Stacey., H. Chick, & M. Kendal (Eds.), *The Future of Teaching and Learning of Algebra*. *The 12th ICMI Study*. (pp. 313-328). Boston: Kluwer.

Norton, S., & Irwin, J. (2007). A concrete approach to teaching algebra. In J. M. Watson & K. Beswick (Eds.), *Mathematics: Essential Research, Essential Practice* (Proceedings of the 30th annual conference of the Mathematics Education Research Group of Australasia), Hobart. Sydney: MERGA.

Perry, B., Howard, P., & Tracey, D. (1999). Head mathematics teachers' beliefs about the learning and teaching of mathematics. *Mathematics Education Research Journal, 11*(1), 39-53.

Sfard, A., Nesher, P., Streefland, L., Cobb, P., & Mason, J. (1998). Learning mathematics through conversation: Is it is as good as they say? *For the Learning of Mathematics, 18*(1), 41-51.

Stacey, K, & Chick, H. (2004). Solving the problem with algebra. In K. Stacey., H. Chick, & M. Kendal (Eds.), *The Future of Teaching and Learning of Algebra*. *The 12th ICMI Study* (pp. 1-20). Boston: Kluwer.

Stacey, K., & MacGregor, M. (1999). Taking the algebraic thinking out of algebra. *Mathematics Education Research Journal, 1*, 24-38.

Stacey, K., & MacGregor, M. (1997). Ideas about symbolism that students bring to algebra. *The Mathematics Teacher, 90*(2), 110-113.

Taylor, N., Muller, J., & Vinjevold, P. (2003). *Getting Schools Working*. Cape Town: Pearson Education, South Africa.

Thompson, S., & Fleming, N. (2003). *Summing it up: Mathematics Achievement in Australian Schools, TIMSS 2002*. Australian Council for Educational Research. Camberwell. Available online: http://www.timss.acer.edu.au/documents/TIMSS_02_Maths_ES.pdf.

Watt, H. (2005). Exploring adolescent motivations for pursuing maths-related careers. *Australian Journal for Educational Psychology, 5*, 107-116.

Wigfield, A., & Eccles, J. (2000). Expectancy-value theory of achievement motivation. *Contemporary Educational Psychology, 25*, 68-81.

Changing Our Perspective on Measurement: A Cultural Case Study

Kay Owens
Charles Sturt University
<kowens@csu.edu.au>

Wilfred Kaleva
University of Goroka
<kalevaw@uog.ac.pg

Papua New Guinea has hundreds of languages and cultures and each group measures in different ways. This report discusses the informal measurement and contexts for measuring by a range of cultural groups as obtained from a survey. Intuitive approaches traditionally used in villages indicate an interesting use of length for deciding areas. People seem to visualise the areas and rely on lengths for comparing or counting to compare these areas. The use of informal measurement has implications for schooling in that it is a valuable place to begin measurement education rather than smaller formal units. Concepts, such as area, and the structure of measurement units, such as placing length units end to end, can be ascertained and established from these informal measures as a transition to more formal school measurement.

Students enter school with a wealth of home experiences. Teachers often make assumptions about the nature of these experiences based on their own familial situation rather than knowing what alternative experiences may be available to their students. Sensitive teachers realise there are differences when teaching students from socio-economic backgrounds or cultures that differ from their own but may not have a framework for exploring these differences or making use of these diverse experiences when teaching. Other teachers rely on the textbook to guide their teaching of a mathematics topic without realising that it might have little significance for the students. They presume mathematics is the same for all students. However, home cultural background can be very important in how and what a student learns in mathematics (D'Ambrosio & Gomes, 2006; de Abreu, Bishop, & Presmeg, 2006; Gerdes, 1996; Kaleva, 2003; Matang & Owens, 2004; Nunes, 1992). According to Bishop (1988), all cultures are involved in the mathematical activity of measuring and he showed that tertiary students from Papua New Guinean (PNG) societies thought differently about mathematics. He concluded "there is more than one way of viewing the world, the mathematician's view is a particular one … shaped by a particular culture, it assumes many cultural 'supports', and increasing our own awareness of these cultural supports will improve the ways we introduce learners to the mathematician's world" (Bishop, 1978, p. 90).

Recent studies on home-school transitions have focussed on the conflict of the school system and the socio-economically disadvantaged cultural groups (Civil & Andrade, 2006; de Abreu et al., 2006). Disassociated knowledge can be rationalised "At home I add, at school I multiply," said Bishop's (1978, p. 90) PNG interviewee when confronted with conflicting choices of ways of finding area in his two sociocultural contexts. Bishop interpreted the student's explanation of pacing up the side and across the width as finding a semi-perimeter. However, the student's rationalisation did not generate a coexistence productive of a strong understanding of area. One recent PNG study in culture and mathematics illustrated the continuities and discontinuities of out-of-school mathematics and school mathematics for counting and currency by showing a two-way influence of school and community (Esmonde & Saxe, 2004) but not as a focus of how to assist schooling.

More productive transitions are expected when teachers and students understand their cultural capital in terms that link to school mathematics. The possibilities are widened by associations that belong to another language and culture; they are curtailed by unresolved conflict (Presmeg, 2006; Valsiner, 2000). Students and teachers must recognise and value cultural mathematics for this knowledge to be related effectively in school mathematics (Gorgorió, Planas, & Vilella, 2006; Owens, 1999). The Yupiaq in Alaska have improved their performance on standard mathematics test questions from their use of cultural mathematics topics (Lipka & Adams, 2004). One cultural topic, fish racks, involved the use of measurement.

Complementarity underlying explicit cultural interaction may be dependent on cultural immersion in the first language, which is supported by schools recognising cultural ways of measuring. "(These) ways of acting, interacting, talking, valuing, and thinking, with associated objects, settings, and events (impact on) … the mental networks" that constitute meaning but can only be determined by ethnographic study (Gee, 1992, p. 141) because of their implicit manifestation (de Abreu et al., 2006; Thomas & Collier, 1997). Explicating tacit knowledge and visualisation requires careful observation, discussion, and reflection on practice (Frade & Borges, 2005). Cultural capital is a powerful tool for learning and social justice (de Abreu et al, 2006; Fowler, 1997) but it is embedded in cultural relationships. In PNG societies, respect impacts on the language of everyday activity and on communication, knowledge is embodied in actions that are often observed and not described, and certain people may have particular knowledge (Owens, 2006). Furthermore, researchers must provide educators with theory to consider how to make the tacit knowledge of the student explicit in school learning (Gee, 1992).

This paper presents recently collated data from tertiary students illustrating other important aspects of their thinking about area and clarifying the discontinuity that Bishop had observed during his visit to PNG. These data have informed our knowledge of contextual learning about area and hence our pedagogical knowledge of teaching and learning measurement, especially area.

Background on Papua New Guinea

Papua New Guinea is comprised of 800 distinct language groups living in mountainous regions, large valleys, coastal swamps, and plains, and many differently-sized islands. There are large towns with people from many language groups often communicating in the lingua franca Tok Pisin. These towns have modern buildings, simple dwellings, and self-help housing with or without basic amenities such as electricity, sewerage, and water. Most people live in rural areas in villages with bush-material housing often without piped water or electricity. Children now begin school in schools supported by the community with the government providing minimal, flexibly delivered training and salaries for teachers. These elementary schools begin in the children's home language as far as possible and gradually transfer to the English curriculum. Vernacular languages and cultures are encouraged throughout the primary school and later. These elementary schools have a syllabus called Culture and Mathematics that provides broad guidelines but because of the diversity of cultures does not give specific details. The teachers use some group work, the village as a resource and some basic equipment like a slate, an exercise book, pencil, and stones and sticks to assist with counting. Assisting the teachers and teacher educators to understand the continuities and discontinuities between cultural mathematics and western school mathematics is critical for improved education. Previous, extensive work on the diversity of counting systems by Lean (1993) has been linked recently to schooling (Matang & Owens, 2004; Owens, 2000). This study extends it for measurement.

Papua New Guinea Studies on Measurement

Earlier studies in Papua New Guinea referred mainly to Piagetian stages and in particular the time lag for conservation of quantity, length, area, volume, and mass compared to western students (Jones, 1973; Price, 1978; Shea, 1978). Prince's (1968) study of teachers college students indicated such a result. Prince (1968) commented that the rate of conceptual development was due to lack of manipulative skills, problems in logical operations, causality problems, and conceptual problems, particularly in conservation of physical quantities. However, in cross-cultural Piagetian studies, testing processes use unfamiliar circumstances and language, and schooling impacts on the formal operational level indicating the bias in the assessment processes (Dasen, 1972). However, although some of these studies were Piagetian style clinical studies as well as paper-and-pencil studies, they did not actually consider the cultural development of the students. Some mathematics tests and some of the Piagetian and spatial tests were given a cultural context for the questions but they did not consider cultural thinking. Although cultural issues were recognised by the Indigenous Mathematics Project (1979), and some continuing research by the Mathematics Education Centre at the PNG University of Technology (Philip Clarkson studied the language issues and Glen Lean carried out his now famous research on counting systems), cultural processes for measuring were not covered. Current doctoral research studies by Charly Mupe and Patricia Paraide are on their own cultural mathematics whereas Rex Matang is focussing on influences of his cultural counting on learning arithmetic strategies in school. Wilfred Kaleva (2003) and Francis Kari showed a strong interest in ethnomathematics and a need to pursue this area of research for improving mathematics education in PNG. A study of multiple systems should throw more light on the diversity of ways of thinking about measurement.

Current Knowledge about the Development of Measurement Concepts

Early psychological studies on measurement by Gal'perin and Georgiev (1960) showed that students need to learn that a length may be treated as a whole, that orientation and visual comparison, and rearrangement may be used to compare. Identification of the attribute, of units with parts, a unit's size, and the unit as a tool are important measurement knowledge. The ability to conserve, reason (Hiebert & Carpenter, 1980), and recognise the structure of repeated units (Curry, Mitchelmore, & Outhred, 2006) assists development. Willis (2005) pointed out that students and teachers may restrict their concepts and images of the abstract units for area by using concrete material tiles, and Owens and Outhred (1998) illustrated students have difficulties representing tiling of areas. From international studies, only 29% of students at the end of primary school could complete a diagram on grid paper to represent 13 square centimetres (Australian Council for Educational Research [ACER], 2002). Many students will calculate areas as a product of the length and breadth regardless of the shape being considered and many will not understand the concepts of area and an area unit (Clements, 1995; Hart, 1981; Willis, 2005). However, two separate studies have shown experiences that included both formal and informal units of measurement and self-made composite units (e.g., five paces) increased students' taken-as-shared understanding of measurement, units, and instruments (Maranhãa & Campos, 2000; Stephan & Cobb, 1998). Nevertheless, there is still a gap in our understanding of how intuitive thinking about area and home cultural experiences can enhance formal schooling. The study reported in this paper provides new insights into the diversity intuitive thinking and how people can successfully move from intuitive understandings of area to formal understandings.

The Current Study

By investigating how a range of different cultural groups think about measurement (especially of length and area), it is anticipated that our understanding of intuitive thinking about length and area will provide mathematics educators with a new perspective on learning about area and how to measure area. This knowledge will improve the teaching of area by illustrating how to bridge the intuitive and formal understandings, and the out-of-school and school views of area. This paper is based on survey data enhanced by some questioning of the participants as they completed the survey and some previously collected reports on culture and mathematics by teacher education (secondary and postgraduate primary) students at the University of Goroka, PNG. The majority of students are from the highlands region and northern mainland region (known as Momase). Most students were in their late twenties or older.

The survey was distributed either electronically or in paper copy to students. Currently 74 surveys from students from different language groups (some from the same language group) have been summarised. The surveys were introduced by explaining that the research was a joint project between the researchers from Australia and Papua New Guinea. Examples of measurement in different Pacific cultures were described briefly. The survey asked demographic questions on language, dialect, village, district, and subdistrict, and the following questions to be answered on length (including possible associations with area) and other kinds of measurement. These questions and the survey format developed after its initial use with a few students. The focus was on length and area. In addition, the authors had records from projects prepared by many students over the years linking their community and culture to secondary mathematics topics and reporting on comparisons of cultural differences in mathematics. The survey questions began with reference to western mathematics but also encouraged significant consideration of cultural mathematical activities. The questions were:

1. During which activities in your language community have you noticed people using traditional ways of measuring?
   For each activity, note what was being compared or measured? (e.g., length, area, volume (size), mass (weight), other, something specific to your community)
2. Select an activity in which people were using length.
3. Do specific people in your community carry out this measurement in certain activities?
4. Describe the processes in detail of how they compare or measure for each activity? E.g., what units do they use, what do they do with these units or tools.
5. Do people use a unit that combines smaller units? If so, how many and how do they join them? Why might this be done?
6. Do they use body parts? Explain and give an example of how the body part might be used in describing the measurement.
7. Is there a standard unit kept for comparing from one time to the next?
8. Talk about how much people think about accuracy when measuring. How do they achieve this?
9. What neighbouring language groups use this practice?
10. Is there another thing they might measure that might be closely linked to this measurement? (For example, some people associate bamboo lengths with the area of land it can water. Some people associate the plan area of a house with the needed wall area.)
11. Is there anything else that you think is important about this measurement activity?

In addition, students were asked to provide their language words for a range of words commonly used in relation to measuring such as *big, heavy, long*. They were asked to repeat these questions for one other activity involving another measurable attribute.

The data were analysed in terms of western perceptions of education in measurement (Owens & Outhred, 2006) and cultural capital (de Abreu et al., 2006).

Results

A number of students only provided language words or descriptions that were insufficient for us to analyse in terms of the measurement activity. However, these surveys did indicate that concepts related to measurement were used in their cultures.

The descriptions by 30 students from the highlands provinces indicated that paces, foot-sized steps, and ropes were commonly used for measuring lengths. A wide range of activities involved measuring lengths and some indicated links to area. These included house building, drains, and gardens but they also included smaller three-dimensional objects such as wigs in which small lengths of string and finger parts were used to ensure symmetry and a good fit on the person's head. The wig-makers provided words to indicate the finger width unit that could be used to mark off lengths on the wig.

Informal discussions with students indicated that many villages used a length measure to determine area. The students confirmed that they visualised the garden area width for the plot. One student indicated that the garden plots were generally a certain width so the total size could be determined by pacing out the lengths. Another student from a different area pointed out the garden plot was generally a fixed length as well as width and the plots would grow different vegetables. Several plots might contain one kind of plant. Another student commented that the gardens were long and thin running in long strips down the hillside. Each garden strip had a particular vegetable. Gardens owned by different people could also be compared as the widths were roughly the same. Sometimes a long rope was used to measure each of the lengths. The length of rope may or may not have been equal to a fixed number but it was common for it to equal 20 paces or arm spans as most languages have 20 cycle counting systems (without a specific word for 10, which is denoted as two fives) (Lean, 1993; Owens, 2000). In the cases where people counted paces to a certain number, marked the place, and then repeated the count, a long line for a garden would be marked off in 20 paces with a tankard plant or stick that also acted as a boundary marker (Simbu province languages, from Charly Muke and students). Twenty paces illustrated an intuitive understanding of a composite unit for measuring lengths but it also indicated a garden plot or area as western mathematics might consider an area unit like a hectare. In this way, the person was using a form of composite unit for length primarily but coincidentally marking out an area unit. The width of area unfolds in the mind as the length is paced out. The image did not appear to be that of blocks of narrow area one pace long but of the whole area determined by the counting. It is like the footballer who instinctively has an image of the size of a football field. If those fields were together, they would image the total area as units of a football field.

Volume measures in this region generally linked to feasts such as bride-price ceremonies, pig exchanges and *mumu* of large quantities of food cooked in a ground pit covered by leaves and heated by hot stones. In exchange and other recognition ceremonies, the number of pigs was important but the sizes of the pigs were also considered. This was frequently decided by the height of the pig but the girth of the pig was also considered. This was measured and compared using rope and much discussion. There was recognition of the idea of volume in taking the girth. One student indicated that this was a relatively new practice in his place. Cooking food, for example, a *mumu* in the ground requires a certain amount of water for steam and this will be determined by the size of the pit and the type of wood used to heat the stones. The amounts needed are decided by experience.

The 20 students from the coastal mainland provinces (Momase region) described how measures were used for making houses, bridges, gardens, holes, canoes, and bows and arrows. The depth of holes and heights were often found by a long cane. In some cases, marks were made on sticks or cane. These were used for smaller lengths such as canoe building. The marks were not necessarily showing a unit. They may have been developed for a particular canoe so that lengths can be assessed as equal for symmetry whereas other marks provided the necessary curvature. Further field work will assist with exploring the details of this aspect of measurement. In some cases the stick or cane was used for more than one measuring task. Some students made connections between these shorter lengths and long distances.

Other examples of length being used to determine areas included that of the round shell money, *maprik*. It was measured using a string around the circumference. It could be argued that the shells are generally of a similar shape and so the circumference, although not necessarily linearly related to area, could be used to compare the relative area (size) of shells. We also found composite units being used such as those given from the Ambulas language area. One bamboo length called, *Kama nak* is equivalent to five bamboo internodes called *ndik nak tamba*. About 5 x 7 bamboo length (that is 7 lengths of 5-internodes) is equal to one garden area or *tumbu.* The use of the multiplication sign to indicate the composite length is confusing with the use of multiplication for rectangular area. Records from Lean (1993) of languages given this name by students in the 1970s and 1980s indicated more than one kind of counting system including a (2, 5 cycle) system, that is, counting words were made up of the frame words 1, 2, and 5 only (note that counting numbers above five were built on five rather than the decimal system of 10). It would be interesting to investigate the use of seven as well as five. Our personal experience with men whose languages use a (2, 5 cycle) counting system is that they frequently stop to think after seven when asked to count. From another language, it was said that 1 bamboo stick = 10 arm spans. If so, it is difficult to visualise this length suggesting further information needs to be sought from this student or his language group. It is possible the stick was representing ten arm spans in a similar way to a stick representing 10 rather than a very long split or whole bamboo.

Measurement was linked to purpose. The data indicated that the cultural activity – bride price, canoe making, garden building, or hole making – influenced the comparison and measurement technique. The unit size, if used, was appropriate for the kind of length.

Similar information on the use of legs and hands was given by the 11 students from the Papuan regions. In the Bamu area of Western Province, the student reported "They use a long cane which has marks indicating length of different things, for example, men's house, garden, war canoe, men's carving etc." Previously, these people built large communal houses requiring a degree of accuracy as well as communication between a number of people. The student whose language is Vula'a (Hula) from the Central Province around Port Moresby, said, "In measuring fish they use hand span. In measuring depth of the ocean they use a long stick. In gardens they take steps of the same pace." This quote illustrates the use of different units for different reasons.

The data given by 13 students from the New Guinea island languages covered some of the activities above like *mumu* cooking, canoe and garden making, and bride-price ceremonies but they added the use of length measurement for making fishing nets, bat nets and graters, as well as weaving and cutting up pigs. The nets had different lengths for the holes as needed. Graters are used for sago and for coconut and the spacings were carefully measured. Elaborate details of the importance of lengths of shell money *tabu* (arm span) were also discussed.

Discussion

In often subtle ways, there was variation between cultural groups. For example, for the same task of house-building, paces and foot-sized steps were used by some whereas others just compared with a rope or cane without counting units. Many places used both methods from time to time. Although terrain may have determined when measurements were taken, for example upland valleys with high rainfall required drainage, other measures were linked to marriage and other exchange ceremonies that are fairly ubiquitous but again ways of measuring and the value placed on measurements varied. These measurements were also influenced by the role that size of shell money or pigs played in the exchanges.

Lengths were often compared indirectly using a length of rope or a stick and some cultures did not use units to measure or compare. Body parts were commonly used as units. Feet and paces, arm spans, and hand spans were used extensively. Although some places did measure certain lengths with counted units, it was not always the case that the same unit was used for measuring different lengths in different activities (e.g., house and canoe). In other cases, they did use the same unit for length (pace for garden and house). Furthermore, not all language reports used more than one kind of length unit. These data would indicate that measuring more accurately was needed in only some activities. Small length units were evident in making nets and graters, wig making and canoe building. In general, these smaller units were not related to larger units as they were used in different activities (house and canoe building) whereas 20 paces may have been a length used as a composite unit for measuring gardens. Our data come from less than a hundred of the hundreds of available languages and we may find that in other groups there is more of an association of larger and smaller units.

The counting system structure sometimes influenced the composite units used in a community. For example, 20 paces were marked off or a rope used for that length. Variance and change were also recognised for the area of a net hole especially when the hole area increased in size when it was stretched out between poles or trees to catch bats.

The use of composite units arises from either the practical use of a rope or stick length, the counting system, or the natural environment such as a bamboo or stick node. Other practical considerations are also made such as the height or width of the biggest man in the village. His width was used to determine the diameter of a hole in a special door for entering the spirit house in one village whereas it was used for the height of a house in another village. There is also some evidence that the lengths used in a canoe depend on the size of the tree but are usually determined by previous constructions of canoes. Canoes for rivers are usually a single hollowed out trunk but the canoes used for the sea and large rivers are usually one-sided outrigger canoes with a sail.

Informal interviews with students on garden measuring by pacing have indicated a visualising of area that coincided with the pacing of the length. In other words, the students were not measuring area as a semi-perimeter but they determined the width and then established the area plots by counting lengths. Some visualised fixed plots by counting 20 paces and repeating this composite unit. Different places had differently sized plots and arrangements so that some formed a long line whereas others formed wider rectangles and the units formed a grid or were kept as separate plots.

The students were expressing their answers in English, which was sometimes mixed with the lingua franca Tok Pisin. This has made it difficult to be sure what students were intending if they said “leg”. After the first class completed the survey, we encouraged students to be more precise in how they described their unit.

Numerous students hesitated in completing the survey as they were not sure how to explain how much the use of the eye, estimates and “logic” (as they called it) dominated the process rather than what they perceived as a western idea of exact unit.

Conclusion

These survey data have indicated that estimation is commonly used for comparing lengths and areas in different PNG cultural groups. However, it is also clear that informal units are used extensively with varying degrees of emphasis on accuracy using these measures. Making gardens and drains were the main areas for discussion, especially in the highlands. Food gathering and preparation is also an area where measurement takes place. This might be for nets or graters or for cooking-pit size and water for steam.

The idea of gardens being compared by pacing out the length and width (a semi-perimeter) (Bishop, 1978) now seems to be only part of the story. Although one student did say they did this and then gave the measures to someone else to calculate the size of the land, it would seem that in any village area, the people also consider the space taken up by the garden of a relatively standard width. The length is then used to decide how many of those garden plots will be used or compared. Garden area is only one of the variables that are important in comparing gardens. Fertility of the soil, closeness to the village and to water, natural drainage, and the direction it faces are all considered. The purpose for valuing also positions the discussion and may indicate relationships to family members. This study has begun to make the implicit and visual explicit (Frade & Borges, 2005).

The informal cultural approach to measurement allows students to grasp more easily the meaning of measurement and how units are structured (e.g., end to end when measuring length). The cultural practices have elicited the structure of the units (Curry, Mitchelmore, & Outhred, 2006). Moreover, the area unit such as the garden plot or the hole of the net is recognised even though they are counted by lengths. It is wise for a teacher to use the cultural or out-of-school experiences of students for measuring rather than textbook suggestions that may have been written in a different context emphasising calculations and giving small visuals of shapes. It is no wonder that experiences of large areas from out-of-school contexts were not related to small diagrams drawn on the board or in textbooks.

This ethnomathematics study is rich in itself in changing our perspectives on measurement in a cultural context and informs us of the importance of visualisation and out-of-school experience for learning in the classroom. It also suggests alternative ways by which we can introduce area in Australasian schools. For example, experiencing larger informal area units might assist students to recognise an area unit. These units might then be associated with sets of paces for length and then the larger areas imagined by pacing out these sets. In some schools, these might be garden plots but they are likely to be maps of a block of houses or classrooms, netball or handball courts, provide the context. Estimations of areas in terms of the larger units may take on more meaning. Informal measurements made by paces may solve problems related to determining the number of area units to fill a space like the netball court.

This research has only just tapped the potential wealth of Indigenous knowledge expected to be generated by further research in PNG. Although the western notions of units, structure of the units, and the notions of estimation, comparison, order, and size can be linked to traditional measurement systems, it is also clear that the measurement systems have their own specific non-western methods, purposes, and indeed strengths in introducing students to the idea of measurement.

Acknowledgements. Charly Muke and other staff of the University of Goroka for their assistance with the research process and discussions; permission and encouragement of the Acting Vice-Chancellor and the Dean of Science of the University of Goroka to carry out the research; students who completed the survey and staff who permitted the surveys to be completed during class time; and Kuipe Geda for data entry.

References

Australian Council for Educational Research. (2002). *Summing it up: Mathematics achievement in Australian schools in TIMMS 2002. TIMMS Australia Monograph No. 6*. Melbourne: ACER. Retrieved 10 February, 2005 http://www.acer.edu.au/research/TIMSS/documents/TIMSS_Maths.pdf

Bishop, A. (1978). Visualising and mathematics in a pre-technological culture. In E. Cohors-Fresenborg & I. Wachsmuth (Eds.), *Proceedings of the Second International Conference for the Psychology of Mathematics Education* (pp. 79-90). Osnabrück: PME.

Bishop, A. (1988). *Mathematical enculturation: A cultural perspective on mathematics education.* Dordrecht, Neterlands: Kluwer.

Civil, M., & Andrade, R. (2006). Transitions between home and school mathematics: Rays of hope amidst the passing clouds. In G. de Abreu, A. Bishop, & N. Presmeg (Eds.), *Transitions between contexts of mathematical practices* (pp. 148-168). Dordrecht, Netherlands: Kluwer.

Clements, M. (2005). The rhetoric/reality gap in school mathematics. *Reflections, 20* (1), 2–9.

Curry, M., Mitchelmore, M., & Outhred, L. (2006). Development of children's understanding of length, area, and volume measurement principles. In H. Novotná, M. Krátká, & N. Stehliková (Eds.) *Proceedings 30th annual conference of the International Group for the Psychology of Mathematics Education.* (Vol. 2, pp. 377-384). Prague: PME.

D'Ambrosio, U., & Gomes, M. (2006). D'Ambrosio on ethnomathematics*, HPM Newsletter 62,* 1-5. http://www.clab.edc.uoc.gr/hpm

Dasen, P. (1972). Cross-cultural Piagetian research: A summary. *Journal of Cross-cultural Psychology, 3*(1), 23-29.

de Abreu, G., Bishop, A., & Presmeg, N. (2006). Mathematics learners in transition. In G. de Abreu, A. Bishop, & N. Presmeg (Eds.), *Transitions between contexts of mathematical practices* (pp. 6-21). Dordrecht, Netherlands: Kluwer.

Esmonde, I., & Saxe, G. (2004). 'Cultural mathematics' in the Oksapmin curriculum: Continuities and discontinuities. *Proceedings of Sixth International Conference on Learning Sciences* (pp. 174-181) Available: http://portal.acm.org/

Fowler, B. (1997). *Pierre Bourdieu and cultural theory: Critical investigations*. London: Sage.

Frade, K., & Borges, O. (2005). The tacit-explicit dimension of the learning of mathematics: An investigation report. *International Journal of Science and Mathematics Education.* www.springerlink.com/content/t480t1v7v769u5r

Gal'perin, P., & Georgiev, L. (1960). The formation of elementary mathematical notions. In D. Henderson (Translator), *Reports of the Academy of Pedagogical Sciences of the USSR, 1,* 189-215.

Gee, J. (1992). *The social mind: language, ideology and social practice*. New York: Bergin and Garvey.

Gerdes, P. (1996). Ethnomathematics and mathematics education. In A. Bishop, K. Clements, C. Keitel, J. Kilpatrick, & C. Laborde (Eds.), *International handbook of mathematics education* (pp. 909-944). Netherlands: Kluwer.

Gorgorió, N., Planas, N., & Vilella, X. (2006). In G. de Abreu, A. Bishop, & N. Presmeg (Eds.), *Transitions between contexts of mathematical practices* (pp. 22-53). Dordrecht, Netherlands: Kluwer.

Hart, K. (1981). *Children's understanding of mathematics: 11-16.* London: Murray.

Hiebert,J., & Carpenter, T. (1980). *Information processing capacity, logical reasoning ability, and the development of measurement concepts: Working Paper No. 299*. Washington, DC: National Institute of Education.

Indigenous Mathematics Project. (1979). *PNG Journal of Education: Special Edition, 14.*

Jones, J. (1973). The concept of proportionality as a predictor of success at the University of Papua New Guinea, *UPNG Educational Research Report, No. 6.* Port Moresby: University of Papau New Guinea.

Kaleva, W. (2003). Secondary teacher beliefs and practices about mathematics in the Papua New Guinea Context. In A. Maha & T. Flaherty (Eds), *Education for the 21st century in Papua New Guinea and the South Pacific. University of Goroka. Available: http://www.uog.ac.pg/glec/Key/Kaleva/SECONDARY.htm*

Lean, G. (1993). *Counting systems of Papua New Guinea and Oceania. Doctoral dissertation, PNG University of Technology.* Retrieved 17 January, 2007. http://www.uog.ac.pg/glec/index.htm

Lipka, J., & Adams, B. (2004). *Culturally based math education as a way to improve Alaska native students' math performance.* Appalachian Collaborative Center for Learning, Working Paper No. 20. (ED484849)

Maranhãa, C., & Campos, T. (2000). Length measurement: Conventional units articulated with arbitrary ones. In T. Nakahara & M. Koyama (Eds.) *Proceedings of the 24thannualconference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 255-262). Hiroshima, Japan: PME.

Matang, R., & Owens, K. (2004). Rich transitions from Indigenous counting systems to English arithmetic strategies: Implications for mathematics education in Papua New Guinea. In F. Favilli (Ed.), *Ethnomathematics and mathematics education* (Proceedings of the 10th International Congress on Mathematical Education Copenhagen, Denmark, Discussion Group 15 Ethnomathematics, pp. 107-117). Pisa, Italy: Tipografia Editrice Pisana.

Nunes, T. (1992). Ethnomathematics and everyday cognition. In D. Grouws (Ed.) *Handbook of research on mathematics teaching and learning* (pp. 557-574). Toronto: Macmillan.

Owens, K. (1999). The role of culture and mathematics in a creative design activity in Papua New Guinea. In E. Ogena & E. Golla (Eds.), *8th South-East Asia Conference on Mathematics Education: Technical papers* (pp. 289-302). Manila: SouthEast Asian Mathematical Society.

Owens, K., (2000). Traditional counting systems and their relevance for elementary schools in Papua New Guinea, *PNG Journal of Education, 36*(1 & 2), 62-72.

Owens, K. (2006). Rethinking cultural mathematics. *Proceedings of Third International Conference on Ethnomathematics*, Auckland, New Zealand. http://www.math.auckland.ac.nz/~poisard/ICEm3/

Owens, K., & Outhred, L. (1998). Primary school students' visualising and understanding of tiling areas of different shapes. *Mathematics Education Research Journal, 10*(3), 28-41.

Owens, K. & Outhred, L. (2006). The complexity of learning geometry and measurement. In A. Gutiérrez & P. Boero (Eds.), *Handbook of research on the psychology of mathematics education: Past, present and future* (pp. 83-115). Rotterdam, Netherlands: Sense Publishers.

Presmeg, N. (2006). Shifts in meaning during transitions. In G. de Abreu, A. Bishop, & N. Presmeg (Eds.), *Transitions between contexts of mathematical practices* (pp. 211-227). Dordrecht, Netherlands: Kluwer.

Price, J. (1978). Conservation studies in Papua New Guinea, *International Journal of Psychology, 13*(1), 1-24.

Prince, J. (1968). The effect of western education on science conceptualisation in New Guinea. *British Journal of Educational Psychology, 38* ,64-73.

Shea, J. (1978). The study of cognitive development in The Indigenous Mathematics Project, *PNG Journal of Education, 14, Special Issue,* 85-112.

Stephan, M., & Cobb, P. (1998). The evolution of mathematical practices: How one first-grade classroom learned to measure. In A. Olivier & K. Newstead (Eds.), *Proceedings of the 22nd annual conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 97-104). Stellenbosch, South Africa: PME.

Thomas, W., & Collier, V. (1997). *School effectiveness for language minority students.* Resource Collection Series No. 9. Washington, DC: George Mason University, National Clearing House for Bilingual Education.

Valsiner, J. (2000). *Culture and human development*. London: Sage.

Willis, S. (2005). Oversights and insights: Mathematics teaching and learning. In M. Coupland, J. Anderson, T. Spencer (Eds.), *Proceedings of 20th biennial conference of the Australian Association of Mathematics Education* (pp. 32-42). Adelaide: AAMT.

Enhancing Student Achievement in Mathematics: Identifying the Needs of Rural and Regional Teachers in Australia

Debra Panizzon
University of New England, Australia
<dpanizzo@une.edu.au>

John Pegg
University of New England, Australia
<jpegg@une.edu.au>

This paper presents results from a survey of secondary mathematics teachers in rural, regional and metropolitan schools across Australia. The purpose of the survey was to compare the major needs of teachers in relation to the attraction and retention of qualified staff, professional development, availability of material resources and support personnel, and the accessibility of a range of student learning opportunities across the three geographical areas. Although differences emerged for some of these factors, the most significant findings were identified in schools with Indigenous populations of greater than 20%.

A review of the 2003 Programme for International Student Assessment (PISA) results indicates that Australian students achieved comparably with a mean of 525 points to the OECD mean of 500 points, with similar results emerging for PISA 2000. However, when these results are deconstructed further, variations in student achievement across geographical divisions are identifiable. Table 1 presents data for PISA 2003 and illustrates that the mean score for students in remote schools for scientific and mathematical literacy was below the international mean of 500. Further, the standard error bars demonstrate that Australian students in metropolitan schools significantly outperformed ($p < 0.05$) those in provincial schools, who in turn had a higher mean achievement than students in remote schools (Thomson, Cresswell, & DeBortoli, 2004).

Table 1
Means and Standard Errors Across Location of Schools for PISA 2003

Geographic Location	*Mathematical Literacy*		*Scientific Literacy*		*Problem Solving*	
	Mean	*SE*	*Mean*	*SE*	*Mean*	*SE*
Metropolitan	528	2.5	529	2.6	533	2.2
Provincial	515	4.4	516	4.2	522	4.4
Remote	493	9.6	489	6.8	503	8.4

(Source: Thomson, Cresswell, & De Bortoli, 2004)

Further evidence of the gap between student achievement across geographical regions is provided from the National Numeracy Benchmarks, which represent agreed minimal standards for numeracy at particular year levels. Figure 1 identifies the percentages of students in Years 3, 5 and 7 across geographical locations in Australia achieving these standards in 2004. Clearly, there are differences between the achievement of students with particularly lower numbers of students in Remote and Very Remote schools achieving the benchmarks.

The factors driving this geographical divide in mathematics have not been explored to any great extent although studies on rural education (Roberts, 2005; Vinson, 2002) have identified several areas for investigation, including the attraction and retention of teachers, accessibility to professional development, provision of adequate teaching resources

(Cresswell & Underwood, 2004; Vinson, 2002), and the provision of learning opportunities for students.

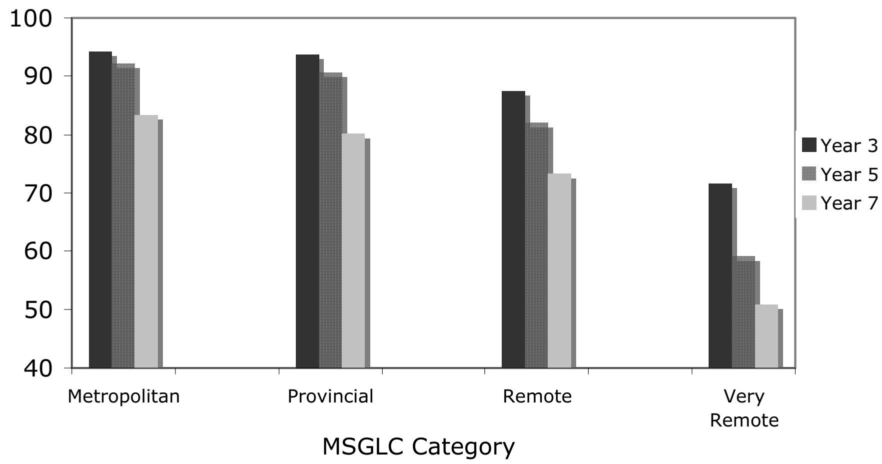


Figure 1. Percentages of Year 3, 5, and 7 students achieving the National Numeracy Benchmarks in 2004 across geographical locations (MCEETYA, 2006).

Clearly, a key factor when considering these research studies is the impact of socio-economic status. Williams (2005) reported that much of the rural-urban variation in the mathematics results for PISA 2000 could be explained by the socio-economic backgrounds of students and schools in the different regions. Importantly, this is not just the case in Australia with many international studies recognising socio-economic status as a confounding variable (Canadian Council on Learning, 2006; Howley, 2003) when investigating student achievement in this manner.

To explore the issues impacting secondary mathematics, science and ICT teachers in rural and regional, a National Survey was conducted in 2005. This paper discusses the findings of this survey (Lyons, Cooksey, Panizzon, Parnell, & Pegg, 2006) as it related to mathematics teachers.

Method

The National Survey consisted of five questionnaire surveys designed for primary teachers, secondary science, ICT and mathematics teachers, and parents. Each of the teacher surveys sought views about the difficulties in attracting and retaining qualified teachers, the degree of access to professional development, the material resources, and support personnel available with each school context, along with student accessibility to a range of learning opportunities.

Definitions of Rural and Metropolitan

Schools in the study were categorised according to the MCEETYA Schools Geographic Location Classification (MSGLC), which considers population size and accessibility to a range of facilities and services. The MSGLC has four main categories of location: Metropolitan Areas, Provincial Cities, Provincial Areas, and Remote Areas (Jones, 2004). Table 2 provides details regarding the category criteria.

Research Sample

Mathematics teacher surveys were distributed to 1998 secondary departments, including all provincial and remote secondary departments across Australia along with a stratified random sample of 20% (N=291) of metropolitan secondary departments. Teachers were invited to complete the survey online if they preferred using an identifiable code for the school. Responses were received from 547 secondary mathematics teachers representing Government, Catholic and Independent schools (Table 2).

Table 2
Secondary Mathematics Teacher Respondents by MSGLC Category

	Main MSGLC categories				
	Metropolitan Area	*Provincial City*	*Provincial Area*	*Remote Area*	
Criteria	Major cities pop. ≥ 100 000	Cities with pop. 25 000 – 99 999	Pop. < 25 000 and ARIA* Plus score ≤ 5.92	Pop. < 25 000 and ARIA* Plus score > 5.92	*Total*
Number of mathematics respondents (%)	142 (26%)	132 (24.1%)	240 (43.9%)	33 (6%)	547 (100%)
Total teacher respondents (%)	580 (19.7%)	661 (22.5%)	1425 (48.5%)	274 (9.3%)	2940 (100%)

* ARIA = Accessibility and Remoteness Index of Australia (ARIA). Locations are given a value for each of these criteria between 0-15 based on road distance to the nearest town or service centre.

Data Analysis

The analytical strategies altered depending on the research questions and the characteristics of the data sets. For example, categorical data (teacher background information) were explored through frequency analyses, cross-tabulations, and chi-squared significance tests. To minimise inaccurate claims about significance the convention of p = 0.05 was reset to a much stricter level of p = 0.001. However, statistical tests achieving a level of significance of p = 0.01 were identified as suggestive and worthy of further exploration.

Rating importance and availability of need items. The mathematics teacher survey consisted of two Likert scales with teachers rating the *Importance* and *Availability* of a range of items related to professional development opportunities, resources, and learning experiences in their school. The Importance scales ranged from 1 (Not at all Important) to 5 (Extremely Important) whereas the Availability scales ranged from 1 (Never Available) to 4 (Always Available). The Importance and Availability ratings were then combined to produce an “Unmet Need” scores, where higher values indicated a greater unmet need for the resource or opportunity. This score was calculated using the transformation “need” = I x (5 – A), where ‘I’ was the Importance rating and ‘A’ the Availability rating. An item considered extremely important (5) but unavailable (1) generated the highest unmet need score (20). Alternatively, items that were unimportant and always available attracted the lowest score (1). More detail about this approach is found in the full technical report (Lyons et al., 2006).

Principal components and multivariate analysis of covariance (MANCOVA). As the mathematics teacher survey contained several items addressing an overarching theme (e.g., professional development) Principal Components analysis was undertaken to identify subsets of items measuring common sub-themes. Once the components were identified in each analysis, respondents were given a score for each component with subsequent statistical tests focused on these component scores. In particular, MANCOVAs were conducted to compare the component scores across various respondent categories including, sex, MSGLC of school, and Indigenous population. Only those MANCOVAs revealing a significant result were pursued by undertaking univariate tests on each component separately, an analytical flow consistent with the logic set out by Tabachnick and Fidell (2001). Importantly, the MANCOVAs controlled for the effects of school size and socio-economic status of the school location, thus minimising any confounding effects of these variables on the results (Lyons et al., 2006).

Results and Discussion

Within this section the major findings from the survey are presented for each of the four main factors. Given that identical analyses were undertaken for the professional development, material resources, and student learning experiences items, full details are provided for the first analysis with reference made to this in later discussions.

Attraction and Retention of Qualified Mathematics Teachers

Teachers were asked initially to consider staff turnover rates by selecting the percentage of teachers leaving the school each year. Choices included: 0-10%, 11-20%, 21-30%, 31-50% and greater than 50%. Compared to their metropolitan colleagues, almost twice as many respondents from Provincial Area schools, and about six times as many from Remote Area schools reported a turnover rate of >20% p.a. These results were highly significant ($p < .001$).

In the next item, teachers rated the degree of difficulty experienced in filling secondary mathematics positions. Options included: Not difficult, Somewhat difficult, Moderately difficult and Very difficult. Significant differences ($p < .001$) emerged with secondary mathematics teachers in Provincial Areas twice as likely and those in Remote Areas about four times as likely as those in Metropolitan Areas to be working in a school in which it was "very difficult" to fill vacant teaching positions in mathematics (Table 3).

Table 3
Percentage of Mathematics Teachers in MSGLC Categories Selecting "Very Difficult"

		MSGLC categories				
		Metropolitan	*Provincial City*	*Provincial Area*	*Remote Area*	*Total*
Secondary Mathematics Teachers	Count	18	29	78	20	132
	% within Row item	12.4	20.0	53.8	13.8	100
	% within MSGLC	14.0	24.6	33.8	64.5	28.5

Subsequently, mathematics teachers were asked whether they were teaching subjects for which they were not qualified. Results were significant ($p < 0.001$) with twice as many teachers in Provincial Areas and four times as many in Remote Areas identifying the need to teach outside of their subject expertise (Table 4). However, when compared to the science and ICT results, mathematics teachers were least likely to be required to teach outside of their subject area. This finding probably relates to the national shortage of qualified secondary mathematics teachers.

Table 4
Percentage of Mathematics Teachers in MSGLC Categories Required to Teach Subjects for which they are not Qualified

		MSGLC categories				
		Metropolitan	*Provincial City*	*Provincial Area*	*Remote Area*	*Total*
Secondary Mathematics Teachers	Count	17	24	75	16	132
	% within Row item	12.9	18.2	56.8	12.1	100
	% within MSGLC	12.2	18.9	31.5	50.0	24.6

Professional Development Opportunities

When teachers rated items within this construct the areas of greatest need were professional development opportunities for teaching higher-order thinking, classroom management, organization and alternative teaching methods, and release from face-to-face teaching for in-school collaborative activities (Table 5).

A principal components analysis of these "need"-transformed items produced four substantive components: Mathematics Teaching Professional Development, General Professional Development, Development for Teaching to Targeted Groups, and Professional Relationships Development. Scores on these four components were analysed using a series of MANCOVAs in order to make specific group comparisons. Two MANCOVAs were conducted comparing mean component "need" scores by MSGLC categories and percentage of students with Indigenous backgrounds. Although the multivariate test for MSGLC category differences across the four professional development components was not significant, the multivariate test in relation to Indigenous students was significant (p<0.001).

A subsequent test revealed that the reasons for this result were due to a significant univariate difference for the Development for Teaching to Targeted Groups ($p < 0.001$) component and a suggestive difference for the Mathematics Teaching Professional Development component (p<0.01). Teachers from schools with more than 40% Indigenous students, and to a lesser extent from schools where the percentage was between 21% and 40%, indicated substantially greater levels of "need" for these two components than other teachers. These differences are identifiable in Figure 2 with a display of the profile plot of the original professional development "need" transformed items (ordered by component and labelled across the top of the graph) by percentage of students with Indigenous backgrounds.

Material Resources and Support Personnel

The average scores on the "need"-transformed items dealing with material resources and support personnel are provided in Table 6. Clearly, the areas of greatest overall "need" included having a suitably skilled assistant to help integrate ICT in the classroom, having appropriate numbers of computers for student use, having suitable learning support assistant(s), and having other computer hardware for teaching and learning mathematics.

Table 5
Summary of Average "Need" Scores, Standard Deviations and Valid N for Mathematics Teachers' Ratings for Professional Development*

Professional Development Items	*Mean*	*SD*	*Valid N*
Professional development opportunities: teaching of higher-order skills	10.70	3.91	492
Professional development opportunities: classroom management & organisation	10.47	4.04	496
Professional development opportunities: alternative teaching methods	10.34	3.98	494
Release from face-to-face teaching for collaborative activities	10.33	4.25	499
Effective communication between education authorities & teachers	9.92	3.72	492
Professional development opportunities: teach mathematics to gift/talented students	9.89	3.72	490
Professional development opportunities: integrating technology into math lessons	9.89	3.85	497
Professional development opportunities: teaching math to special needs students	9.77	3.96	493
Collaboration with mathematics teachers in other schools	9.65	3.61	501
Professional development opportunities: methods for using group teaching strategies	9.60	3.80	489
Opportunities for observing teaching techniques of colleagues	9.49	3.97	499
Workshops to develop your ICT skills	9.47	3.82	492
Involvement in region/state-wide syllabus development/research projects	9.29	3.90	493
Financial support to attend external in-services/conferences	9.04	4.00	498
Opportunities for mentoring new staff	8.90	3.68	501
Opportunities to attend external in-services/conferences related to T&L math	8.76	3.57	502
Professional development opportunities: use of graphics calculators	8.75	3.82	495
Professional development opportunities: outcomes/standards-based teaching	8.72	3.87	495
Opportunities to mark/mod external mathematics assessments	8.62	3.99	488
Professional development opportunities: teaching mathematics to Indigenous students	8.40	4.31	480
Professional development opportunities teaching mathematics to NESB students	8.29	3.99	459
Collaboration between mathematics teachers in your school	7.86	3.44	500

*Items arranged in descending order of mean "need" score between 1-20 (Adapted: Lyons, et al., 2006)

A principal components analysis of "need"-transformed material resources produced three components: ICT Resources and Support, Mathematics Teaching Resources and Support, and Teaching Resources for Targeted Groups. As with the earlier analysis, scores for the three components were analysed using a series of MANCOVAs. The multivariate test for MSGLC category differences across the three material resources components was not significant. However, the test comparing the three components across schools with different percentages of student with Indigenous backgrounds was significant (p<0.001).

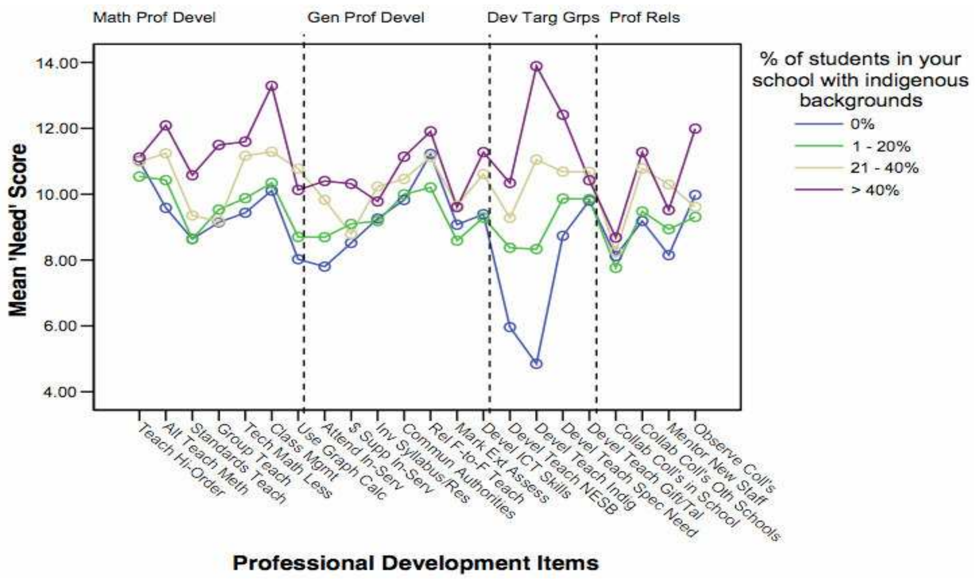


Figure 2. Profile plot of mean “need” scores of mathematics teachers for professional development components compared by percentage of students from Indigenous backgrounds (Table 5 lists full item names) (Source: Lyons et al., 2006).

A follow-up test identified that this difference was due to significant univariate differences on the Mathematics Teaching Resources and Support (p<0.001) and Teaching Resources for Targeted Groups components (p<0.001). Essentially, teachers from schools having more than 21% of students with Indigenous backgrounds indicated substantially greater levels of “need” for the two components when compared to teachers from remaining schools. Figure 3 illustrates that “needs” are greatest in the specific areas of resources for teaching mathematics to Indigenous students, having suitable Indigenous Education Assistants, students having access to scientific calculators, and having suitably skilled personnel to assist in integrating ICT in the classroom from schools having more than 40% of students with Indigenous backgrounds. In schools where the percentage of students with Indigenous backgrounds was between 21% and 40%, “needs” were greatest in the specific areas of resources for teaching to gifted and talented students and having concrete materials for mathematics teaching. Overall, it is clear that where the percentage of students in a school with Indigenous backgrounds exceeds 20%, “needs” are greater in most of these areas (Lyons et. al., 2006).

Student Learning Experiences

The areas of greatest overall “need” identified by mathematics teachers for these items (Table 7) included students having opportunities to visit mathematics-related educational sites, alternative/extension activities in mathematics teaching programs for gifted and talented and for special needs students. Interestingly, the results of this component was lower for mathematics teachers than science and ICT teachers suggesting that this was a moderate rather than high need.

A principal components analysis of these Student Learning Experience items highlighted three substantive components: Alternative and Extension Activities for

Targeted Groups, Teaching Context in the School, and Student Learning Opportunities. Subsequent analyses of these components using MANCOVAs identified that differences for the three Student Learning Experience components across MSGLC categories was not significant. Alternatively, the multivariate test between schools having different percentages of students with Indigenous backgrounds was significant (p<0.001).

Table 6

Summary of Average "Need" Scores, Standard Deviations and Valid N for Mathematics Teachers' Ratings of the Material Resources and Support Personnel items*

Mathematics Resource and Support Items	*Mean*	*SD*	*Valid N*
Suitably skilled personnel to assist in integrating ICT in your classroom	9.72	4.34	517
Appropriate number of computers for student use	9.44	3.69	520
Suitable learning support assistant(s)	9.24	3.61	523
Other computer hardware for teaching & learning mathematics	9.06	3.76	512
Suitable software for teaching & learning mathematics	8.91	3.69	520
Suitably skilled ICT support staff	8.87	3.75	518
Mathematical resources that address the needs of gifted/talented students	8.59	3.48	511
Suitable computer resources for teacher use	8.58	3.63	523
Mathematical resources that address the needs of special needs students	8.57	3.72	514
Suitable Indigenous Education assistant(s)	8.21	4.05	501
Effective maintenance & repair of teaching equipment	8.07	3.21	515
Sufficient mathematics equipment & materials	8.02	3.03	525
Fast, reliable internet connection	7.98	3.68	523
Mathematical resources that address the needs of Indigenous students	7.91	4.24	488
Concrete materials for mathematics teaching	7.85	3.11	524
Mathematical resources that address the needs of NESB students	7.80	4.05	462
Access range of internet mathematics resources	7.78	3.45	517
Student access to scientific calculators	7.55	3.30	520
Student access to graphics calculators for in class	6.84	3.41	519
Class sets of suitable texts	6.50	3.22	518
Suitable library resources for teaching & learning mathematics	6.46	2.97	515
Suitable AV equipment	6.39	3.24	520
Worksheets for classroom teaching	6.14	2.77	526

*Items arranged in descending order of mean "need" score between 1-20 (Adapted: Lyons, et al., 2006).

Further testing revealed significant univariate differences on the Teaching Context in the School (p<0.001) and Student Learning Opportunities (p<0.001) components as well as a suggestive difference on the Alternative and Extension Activities for Targeted Groups (p<0.01) component. The greatest level of "need" in the Teaching Context in the School component was demonstrated by teachers from schools having a percentage of Indigenous students between 21% and 40% while the lowest level of "need" was expressed by teachers in schools with no Indigenous students.

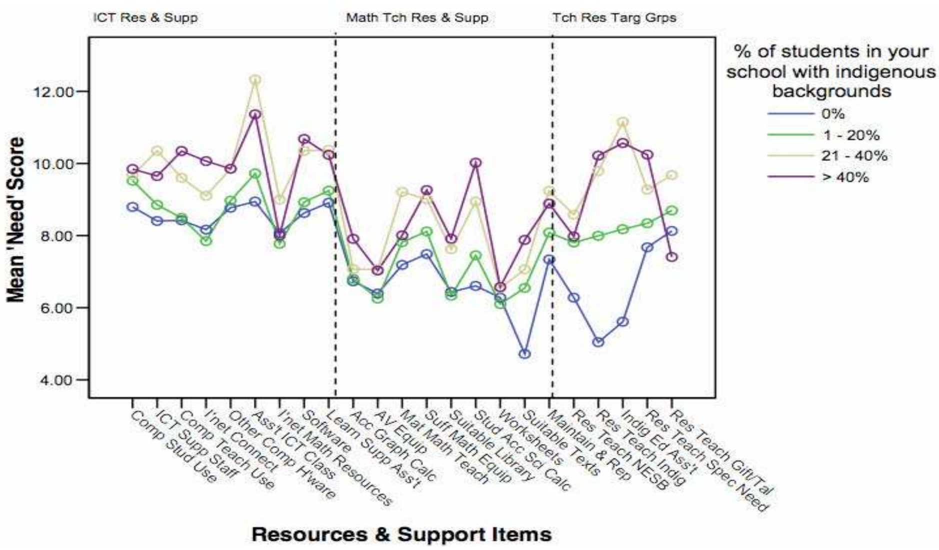


Figure 3. Profile plot of mean "need" scores of mathematics teachers for the Material Resources and Support Personnel components compared by percentage of students from Indigenous backgrounds (Table 6 lists full item names) (Source: Lyons et al., 2006).

Table 7

Summary of Average "need" scores, Standard Deviations and Valid N for Mathematics Teachers' Ratings of the Student Learning Experience*

Student Learning Need Items	*Mean*	*SD*	*Valid N*
Opportunities for students to visit mathematics related educational sites	9.36	3.70	505
Alternative/extension activities in mathematics teaching programs for gifted & talented students	9.22	3.58	500
Alternative/extension activities in mathematics teaching programs for special needs students	8.86	3.64	496
Alternative/extension activities in mathematics teaching programs for Indigenous students	8.47	4.16	474
Alternative/extension activities in mathematics teaching programs for NESB students	8.43	4.05	455
Teachers qualified to teach the mathematics courses offered in your school	8.15	3.06	505
Having the total indicative hours allocated to face-to-face teaching	8.12	3.48	492
Having the full range of senior mathematics courses available in your school	7.14	3.24	506
Student participation in external mathematics competitions and activities	5.92	2.49	510

*Items are arranged in descending order of mean "need" score between 1-20 (Adapted: Lyons, et al., 2006).

Teachers from schools with Indigenous populations of between 21-40% of students indicated a high "need" for alternative or extension activities with respect to all four targeted groups. Within the Teaching Context component, having a full range of mathematics courses on offer with total indicative hours allocated to face-to-face teaching reflected a markedly higher level of "need" from respondents from schools where 21-40% of students were from Indigenous backgrounds; having qualified teachers was at a high level of need for respondents from schools where the percentage of student with Indigenous backgrounds exceeded 20%. Within the Student Learning Opportunities component, teachers from schools where greater than 20% of students were from

Indigenous backgrounds indicated a substantially greater level of “need” in the area of opportunities for students to visit mathematics related educational sites.

Conclusion

The results from the survey suggest that teachers in Remote Area and to a lesser extent Provincial Area schools are likely to experience the effects of teacher shortages, a lack of opportunity to access professional development, and difficulties in providing resources for their students to a greater extent than teachers in Metropolitan and Provincial schools. However, it was interesting that significant differences did not emerge consistently for these components across MSGLC categories for mathematics teachers whereas this was the case for science and ICT teachers. Alternatively, significant differences emerged across the MSGLC categories when the percentage of Indigenous Students higher than 20% was considered as a variable. Addressing the needs of our Indigenous Students highlights a critical area for which our mathematics teachers seek major support.

Acknowledgements. Appreciation to Terry Lyons, Ray Cooksey, and Anne Parnell from SiMERR Australia and the financial support provided by the Department of Education, Science and Training (DEST) and the Department of Transport and Regional Services (DOTARS).

References

Canadian Council on Learning. (2006). *Lessons in learning*. Retrieved 20th May 2006, from www.ccl-cca.ca/CCL/Reports/LessonsInLearning/LiL1March2006.htm

Howley, C. (2003). Mathematics achievement in rural schools. *ERIC Digest*. Retrieved 10th April 2006, from www.ael.org/digests/edorc03-3.pdf

Jones, R. (2004). *Geolocation Questions and Coding Index. A technical report submitted to the MCEETYA Performance Measurement and Reporting Taskforce*. Retrieved 17th July 2005, from www.mceetya.edu.au/mceetya/default.asp?id=11968

Lyons, T., Cooksey, R., Panizzon, D., Parnell, A., & Pegg, J. (2006). *Science, ICT and mathematics education in rural and regional Australia: Report from the SiMERR National Survey*. Canberra: Department of Education, Science and Training.

MCEETYA. (2006). *National report on schooling in Austalia.* Retrieved 2nd February 2007, from www.mceetya.edu.au/mceetya/default.asp?id=13342

Roberts, P. (2005). *Staffing an empty schoolhouse: Attracting and retaining teachers in rural, remote and isolated communities*. Sydney: NSW Teachers Federation.

Tabachnick, B.G., & Fidell, L.S. (2001). *Using multivariate statistics,* (4th ed.). Boston, MA: Allyn & Bacon.

Thomson, S., Cresswell, J., & De Bortoli, L. (2004). *Facing the future: A focus on mathematical literacy among Australian 15-year-old students in PISA*. Camberwell, Vic: Australian Council for Educational Research.

Williams, J.H. (2005). Cross-national variations in rural mathematics achievement: A descriptive overview. *Journal of Research in Rural Education, 20*(5). Retrieved 24th May 2006, from www.umaine.edu/jrre/20-5.pdf

Vinson, A. (2002). Inquiry into public education in New South Wales: Second Report. Retrieved 18th August 2005, from www.pub-edinquiry.org/reports/final_reports/03/

The Growth of Early Mathematical Patterning: An Intervention Study

Marina Papic
Macquarie University, Sydney
<marina.papic@mq.edu.au>

Joanne Mulligan
Macquarie University, Sydney
<joanne.mulligan@mq.edu.au>

A case study monitored the development of 53 preschoolers' mathematical patterning skills in two similar preschools, one of which implemented a 6-month Intervention promoting patterning concepts. Pre- and post-Intervention assessment data and follow-up data evaluated the impact of the Intervention on the growth of Repeating and Spatial Patterns. Intervention children outperformed Non-Intervention children across a range of patterning tasks and this trend was maintained 12 months after formal schooling. Intervention children readily identified the *unit of repeat* and the structure of spatial patterns. Without exposure to Growing Patterns, Intervention children identified, extended, represented and justified triangular and squared number patterns.

Background to the Study

Despite recent research interest in early algebra (Kieran, 2006), there is little known about the role of young children's mathematical patterning in the development of algebraic reasoning. Studies conducted before the 1990s contributed to the belief that algebra was best left for the later years of schooling. The 1990s saw a shift in research to children's mathematics reasoning and problem solving which included the development of combinatorial thinking. This was paralleled by studies with children aged 4 to 9 years supporting the idea that young children could learn more complex mathematics than previously thought. Further, much of the research in the 1980s and 1990s on early numeracy that focused on the development of arithmetic strategies influenced research on the relationship between arithmetic structure and algebraic thinking. However, there were few studies focused on underlying processes of patterning and abstraction with very young children.

Research on Patterning in Early Mathematics

Recently, mathematics education researchers have focused more seriously on the early development of patterning and its role in early mathematical thinking. Some studies have incorporated patterning as one component of investigation in early mathematical development. A series of studies have indicated that first and second graders' use of *pattern and structure* generalises across a wide range of mathematical content domains and this can be described as a general cognitive characteristic (Mulligan, Mitchelmore, & Prescott, 2006). Children's identification and representation of the structure of patterns was critical to successful task solution and the level of sophistication of structural awareness. Children's patterning knowledge has also been found to influence the development of analogical reasoning and the ability to identify, extend, and generalise patterns important to inductive reasoning (English, 2004).

Studies of preschoolers have found that they are capable of symbolic and abstract thought far beyond traditional expectations (Ginsburg, 2002). Young children have been observed developing skills in argumentation (Dockett & Perry, 2001) and algebraic reasoning (Blanton & Kaput, 2004). Some studies have included aspects of patterning such as simple repetition, part-whole thinking, spatial and geometric patterns, subitising, and

Proceedings of the 30th annual conference of the Mathematics Education Research Group of Australasia
J. Watson & K. Beswick (Eds), © MERGA Inc. 2007

counting patterns using calculators. However, few studies have focused explicitly on young children's development of patterning skills in early childcare settings. One recent observational study by Waters (2004) found that preschool children initiated and described their own patterns, ranging from simple repetition to geometric forms. Waters highlighted the limited pedagogical content knowledge of preschool teachers who needed to become more aware of the types, level and complexity of patterns. Her study suggests that more research is needed to support the inclusion of patterning in early childhood programs, and to develop a more coherent understanding of how early patterning skills develop. The study of patterning has also been explored through early childhood programs designed to enhance mathematical development generally (e.g., Ginsburg, 2002). Although it appears that patterning forms an integral part of these types of programs, the scope and complexity of patterning has not necessarily been informed by research that describes explicitly, the informal development of mathematical patterning. It is not yet clear how simple repeating patterns are extended to other mathematical contexts or how they are linked to growing patterns and functional thinking. Although contemporary studies of children's early algebraic thinking, such as exploring repeating and growing patterns, and functional thinking are mainly concerned with children in the 6-8 years age range, there remains unanswered questions about how and when early algebraic thinking develops in the years prior-to-formal schooling.

A case study was therefore designed to describe the development of patterning skills from preschool through to formal schooling and to investigate the role patterning plays in the development of early mathematical concepts and processes. Four key research questions were addressed: What are the characteristics of mathematical patterning young children develop naturally prior-to-school? In what ways does an intervention promoting mathematical patterning impact on the complexity of children's patterning concepts and skills and the development of other mathematical processes such as multiplicative thinking? Is the influence of such an intervention maintained after one year of formal schooling? If so, in what ways? What is the role of patterning in the development of early algebraic thinking?

In an earlier report, Papic and Mulligan (2005) presented preliminary findings of initial assessment data from the study. This paper describes the assessment data focusing on changes in children's patterning skills at pre- and post- Intervention and following 12 months of formal schooling.

Method

The study was designed as an intervention employing a mixed-method approach: integrating a traditional constructivist-based teaching experiment with more contemporary aspects of a design study. Following pilot work, an interview-based assessment of children informed the development of an instructional framework implemented through the Intervention. The Intervention provided explicit opportunities for children to explore and develop their patterning skills through problem-based tasks. The researcher (as *participant observer*) collaborated with teachers to model opportunities for the development of *Repeating Patterns* and *Spatial Patterns*. Observations included data showing how children constructed and justified patterns in a variety of modes. Further, the Intervention included on-going professional development on the importance of pattern and structure in early mathematical learning, which assisted teachers in modifying the emergent curriculum to incorporate patterning skills.

Setting and Participants

A large long-day care centre in the South-Western area of Sydney that operated a preschool program was selected as a case study for the Intervention (for details see Papic & Mulligan, 2005). A similar long-day care centre was identified within the region as a "contrast" group (Non-intervention preschool). It was not intended to generalise the results from this case study but every attempt was made to select two similar preschools that were considered to be typical of centres in this region. The sample comprised 53 preschoolers, balanced for gender and broadly representative of the children in the final year of each preschool. Thirty-five of the initial sample were reassessed on completion of the preschool year and 32 of these on completion of the first year of formal schooling. Despite the substantial attrition, there was no indication that the final sample was biased. Analysis of the data collected at each assessment showed that, for both groups, the children who were not retained had given a fair distribution of responses at the first assessment.

Data Collection and Analysis

Data collection included three interview-based assessments on children's patterning skills and an additional numeracy assessment at the third assessment (Schedule for Early Number Assessment 1, NSW Department of Education & Training, 2001). A systematic interview protocol was employed to elicit each child's explanations and strategies used to solve each assessment task. A range of data sources collected throughout the Intervention included photographs, video recording and observations of children's patterning in structured and play situations. Work samples were compiled in individual portfolios. Figure 1 provides a summary of the data collection points. Preschool and Kindergarten teacher surveys were conducted at the conclusion of the study. The first researcher conducted all interview-based assessments and teacher surveys.

Intervention Preschool	June 2003 Interview 1 (n = 27)	→	July – Dec 03 Intervention Program	→	Dec 2003 Interview 2 (n = 19)	→	Dec 2004 Interview 3 (n = 20)	Dec 2004 Teacher Survey (n = 18)
Non-intervention Preschool	June 2003 Interview 1 (n = 26)	→	Regular Preschool Program	→	Dec 2003 Interview 2 (n = 16)	→	Dec 2004 Interview 3 (n = 12)	Dec 2004 Teacher Survey (n = 12)

Figure 1. Data collection points.

The classification of children's responses to assessment tasks was supported by other data: drawn representations, photographs of children's patterns and solution processes, interview transcripts, observation notes and digital recordings (20% of interviews). The analysis of assessment data involved initial coding of responses for accuracy, followed by classification of solution processes focused on the level of complexity of pattern recognition. Initial coding was verified by an independent coder (intercoder reliability calculated at 89%).

Three key aspects of patterning were identified from the research literature and initial analyses (Papic & Mulligan, 2005): *Repeating Patterns*, *Spatial Structure Patterns*, and *Growing Patterns*. Eleven task categories were derived from these key aspects (see Table

1). These tasks were devised to investigate children's ability to create, identify, extend, and copy from memory patterns, in a variety of modes. Tasks administered at Assessments 1 and 2 (for task descriptors see Papic & Mulligan, 2005) were identical, but tasks at Assessment 3 increased in complexity to accommodate Growing Patterns and children's growth in patterning concepts and skills.

Interview-Based Assessment Tasks

Table 1
Key Aspects of Patterning and Related Task Categories

Key Aspect	Task Category	Descriptor
Repeating Patterns	Tower Border Hopscotch Number	Repeating Patterns contain an element that continuously recurs. In these tasks patterns contained single or dual variable, simple and complex repetitions using coloured blocks, tiles or numerals.
Spatial Structure Patterns	Array Block Grid Subitising Triangular 1	Spatial Structure is the mental organisation of objects or groups of objects and their components. In these tasks the organisation of patterns was presented in the form of triangular patterns of dots and square and rectangular patterns of dots, arrays and grids.
Growing Patterns	Triangular 2 Square Tiles	Growing patterns increase (or decrease) systematically. Spatial Structure tasks were reformulated to explore the idea of more complex, growing patterns presented as the pattern of triangular numbers (triangular dots) and the pattern of squared numbers (square tiles).

The Intervention

The researcher, in collaboration with the preschool staff, developed, implemented, and monitored an intervention program. The Intervention was designed on the basis of children's existing patterning knowledge to: provide explicit opportunities to explore and develop patterning skills through problem-based tasks; develop children's mathematical reasoning in order to provide a foundation for later mathematical learning particularly in early algebraic thinking; provide a framework of assessment and learning experiences to guide emergent curriculum and scaffold individual children's learning; describe the development of patterning in both play situations as well as structured situations; and provide professional development for staff on the importance of pattern and structure in early mathematical learning to assist them in modifying their emergent curriculum to incorporate patterning.

The Intervention comprised three distinct components: structured individual and small group work on pattern-eliciting tasks, *Patternising* the regular preschool program, and observing children's patterning in free play. Structured pattern-eliciting tasks were based on the *Tower*, *Subitising* and *Hopscotch* tasks administered in the first assessment because they provided critical opportunities for developing patterning concepts. A *Framework of Assessment and Learning* that guided instruction and highlighted children's development was designed for both the *Tower* and *Subitising* tasks.

Discussion of Results

The following results compare Intervention (I) and Non-intervention (NI) children's responses across three assessment points. A discussion of the growth in children's acquisition of patterning skills is provided, supported by excerpts from interview transcripts and children's drawn and constructed representations. When interpreting data it must be noted that small differences in percentages, particularly NI at Assessment 3, are insignificant due to the size of each sample group.

Table 2 indicates the percentage of correct responses for the eleven task categories (data show the average score, as a percentage of correct responses on sub tasks within each category). The NI group was moderately more successful across most task categories at Assessment 1, but by Assessment 2, the I group was more successful across all task categories. This success was particularly evident in the task categories *Number*, *Grid*, *Subitising*, and *Triangular 1*. Number tasks were more challenging than other Repeating Pattern tasks because children were not provided with concrete materials and the tasks involved two variables, colour and number. NI children showed no improvement on Number tasks between Assessments 1 and 2, whereas I children improved substantially. Between the first two assessments, I children participated in various games and activities using dice and regular dot patterns as part of the 6-month Intervention. This may have impacted on I children's responses at the second assessment where their performance on *Subitising* tasks improved. Conversely NI children showed no improvement on *Subitising* tasks. It was observed that NI children were more focused on counting the individual dots or blocks in the patterns. For example, the simple three-dot pattern, which children immediately recognised at Assessment 1 was instead counted one-by-one at Assessment 2. This unitary counting strategy may have been attributed to the overemphasis on counting by ones in their preschool program.

Table 2
Percentage of Correct Responses for Task Categories at Three Assessment Points

Task Category	Assessment 1		Assessment 2		Assessment 3	
	I n = 27	NI n = 26	I n = 19	NI n = 16	I n = 20	NI n = 12
Repeating patterns						
Tower	34	47	85	73	93	47
Border	74	81	100	88	53	22
Hopscotch	16	28	55	45	65	8
Number	11	19	58	19	83	17
Spatial structure patterns						
Array	47	42	79	72		
Block	47	46				
Grid	33	27	79	25		
Subitising	15	20	58	16		
Triangular 1	7	8	50	13		
Growing patterns						
Triangular 2					38	0
Square Tiles					48	0

At Assessment 2, the *Array* proved to be the easiest of the Spatial Structure tasks. It was inferred that arrangements of dots in this task (e.g., 2 x 3 array of dots) made spatial structure explicit. In comparison, *Triangular 1* proved to be the most difficult of the Spatial Structure tasks. NI children found it difficult to identify the number, shape, size, orientation, spatial and numerical structure of the triangles when copying with counters and drawing triangular dot patterns. However, even without intervention, the NI group showed some progress at Assessment 2: *Tower*, *Border*, and *Array* tasks. However, there were marked differences between the two groups in terms of the patterning strategies employed to solve the tasks.

The increase in task complexity renders any comparison between Assessment 2 and Assessment 3 invalid. However, it is valid to compare performance between the I and NI children at Assessment 3. There were striking differences across all task categories in favour of the I children. Intervention children continued to show improvement across the more complex Repeating Pattern tasks at Assessment 3. However, the NI children found the tasks more challenging and performed well below the I group, particularly on *Hopscotch* and *Number* tasks.

Spatial Structure tasks were reformulated into more complex Growing Pattern tasks at Assessment 3. Neither I nor NI children had been exposed to Growing Patterns in the first year of schooling and these tasks had not comprised part of the Intervention. Nevertheless, many of the I children could construct, extend, represent, and justify these patterns. It appeared that about half these children depicted some underlying structure in the pattern. Forty-five percent could successfully continue a growing triangular number pattern "1, 3, 6", presented as a triangular dot pattern and 55% could successfully continue a growing squared number pattern "1, 4, 9", made with square tiles (see Figures 6 and 7 following). In comparison, Growing Patterns proved to be extremely difficult for all NI children, with no NI child giving a correct response.

Patterning Strategies

Repeating Patterns. By Assessment 2, I children developed a sound understanding of pattern as unit of repeat that appeared to lead to growth in the abstraction and complexity of patterning skills. Intervention children could successfully identify, construct and abstract the unit of repeat and calculate the number of repetitions. This was the dominant strategy used by I children at Assessment 2 and sustained at Assessment 3 (12 months later). Many I children were able to draw complex repetitions from memory, identify the pattern element, and number of repetitions as exemplified in the following excerpt.

Researcher: How do you know that you have finished making your tower?
Why didn't you keep adding some more blocks?
Child I 19: I remembered red, blue, blue, black, three times.

In comparison, NI children relied on an alternating colours strategy to complete Repeating Pattern tasks. For example, when copying an ABABAB tower, NI children remembered the tower pattern as single alternating colours of "red, blue, red, blue, red, blue" rather than the element "red, blue" and the number of repetitions. For example, one NI child continued to add alternating colours of blocks, red then blue, and then after making a 9-block tower measured it against the tower that had been modelled to establish height. At Assessment 3, when the complexity of the tower was increased, (e.g., an ABBC repetition), and when asked to complete the task from memory, NI children's alternating colours strategies became ineffective. Most NI children tried to remember the order of the coloured blocks and at times, the height of the tower. However, due to the complexity of the tower pattern they could not remember the sequence and thus made errors.

At Assessments 1 and 2 a simple repetition was presented in a vertical and horizontal hopscotch pattern with a unit of repeat created with four squares: Two vertical, two horizontal (see Figure 2). The *Hopscotch* category differed from other Repeating Patterns tasks in that it investigated changes in orientation of the pattern and children's transformation skills. At Assessment 1, both I and NI found it difficult to visualise the *Hopscotch* pattern when it had been rotated by 90º. At Assessment 2 both groups improved on the *Hopscotch* tasks. It could be assumed that exposure to a variety of concrete materials and viewing objects from different perspectives in the children's regular program assisted in developing these skills. For example, by the second assessment children had been exposed to a variety of activities such as block play and puzzles that encouraged transformation skills and this was critical to the completion of the *Hopscotch* rotation tasks. However, I children were more confident at drawing the rotated hopscotch from memory than the NI children. Figure 3 shows an I child's drawing of the hopscotch template rotated by 90° (on the left hand side) at Assessment 1 and her drawing 6 months later at Assessment 2 (on the right hand side).

Figure 2. Hopscotch patterns.

Figure 3. Drawing of a Hopscotch pattern rotated at 90º, Child I 17.

At Assessment 2 children were also given an extension task where they were asked to design their own hopscotch pattern. Sixty-three percent of I children successfully designed their own hopscotch that showed repetition of elements. Many I children could additionally integrate a second variable, colour, in their hopscotch pattern and could extend the number of tiles that formed the pattern element. For example, in Figure 4 the child created a complex pattern element, "two horizontal, one vertical, two horizontal, two vertical, four horizontal" using a systematic arrangement of colours, and replicated it once. In Figure 5 the child created a pattern element of "three, two, one", creating a descending row of steps. In contrast, only 25% of NI children designed a hopscotch pattern that showed a single variable repetition and there were no examples of complex patterns; rather they were restricted to AB repetitions. All NI children attempted to make their own hopscotch but they seemed unaware of the need to create and replicate a pattern element.

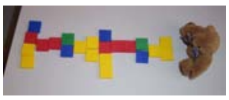


Figure 4. Design own Hopscotch pattern,
I 25, 5.1 yrs.

Figure 5. Design own Hopscotch pattern,
I 18, 5.0 yrs.

At Assessment 3, the Hopscotch task required the children to complete a cyclic pattern where they needed to identify the pattern as a sequence of 90° turns. Sixty-five percent of I children successfully modelled and predicted the pattern as a sequence of 90° turns. In contrast, the NI children did not identify the pattern as a sequence of 90° turns but saw the three hopscotch templates as an ABC pattern element to repeat.

At Assessment 3, children's ability to identify a pattern beyond a linear form was also explored. One of the *Border* tasks required children to identify an ABC repetition (3 x 5 border pattern of red, blue, green tiles) from multiple starting points. The task proved very difficult for both groups, with only a small number of children from each group accurately completing this task. The majority of children identified the pattern with a starting point in the top left hand corner. It may be inferred that this was due to the children's limited exposure to patterns presented as different spatial arrangements. This response could also be explained by the children's classroom experience of making patterns that were limited to horizontal and vertical linear forms that begin in a designated position, using left-to-right or bottom-to-top directions.

In another *Border* task, children were asked to identify the number of green tiles required to complete the ABC pattern. Structuring the task in this way allowed the researcher to observe whether children determined the number of times the pattern element could fit into the remaining spaces. Intervention children outperformed NI children on this task. This may have occurred because the children were more aware that the pattern element contained three colours and they needed only to count every third tile. Such a strategy would suggest a sophisticated understanding of pattern as repetition and reflect early multiplicative thinking. Many of the I children immediately identified every third position in the border by placing their fingers on the square where the missing green tile needed to be placed. It appeared that these children visualised the pattern element accurately; some *skip counted* every third position in the pattern, translating the repetition of colours into a number pattern of multiples. In contrast, most NI children attempted to complete the pattern by verbalising alternating colours to determine how many greens were required.

Spatial Structure Patterns. Intervention children outperformed NI children on all Spatial Structure tasks at the second assessment where almost all I children represented the structure of the patterns. For example, one *Grid* task required children to copy a grid of three connected squares. Most I children were able to draw the correct number of equal-sized squares in correct formation. Those who made errors, made counting errors rather than those related to the spatial arrangement. In another example, when presented with an *array* of dots (e.g., 2 x 3) a number of children clearly represented the structure of two rows of counters forming a rectangular shape however, there were two rows of four counters, rather than two rows of three counters presented. It seemed that the I children focused their attention on the spatial structure of the patterns. This is not surprising since teachers encouraged children to look for similarity and difference in the structure of patterns throughout the Intervention. In comparison, many NI children's incorrect responses lacked any structural features. For example, in *Array* tasks, children's responses did not represent the shape of the array and frequently included an incorrect number of counters. It was inferred that the children did not "see" the structure of the array or the rows of dots in alignment.

Growing Patterns. A number of I children, although not exposed to Growing Patterns throughout the Intervention or in the first year of schooling, were able to extend a growing triangular number pattern (see Figure 6) and a growing square number pattern (see Figure 7). Most of the I children who made errors in constructing the Growing Patterns were still able to observe holistically the increasing size of the triangles or squares, and attempted to make the pattern larger.

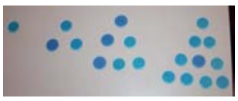


Figure 6. Triangular Growing Pattern, I 11, 6.3 yrs.

Figure 7. Square Tile Growing Pattern, I 21, 5.5 yrs.

Of particular importance was the I children's use of spatial structure to explain the pattern as an extension of the previous pattern element. This showed early signs of co-variational thinking where children were required to deal with a change in the structure of the pattern. This result supports the findings of Blanton and Kaput (2002), which highlight the importance of quantitative relationships in developing algebraic thinking.

Researcher: Can you tell me what is happening each time we make the triangle bigger.
Child I 18: It gets bigger.
Researcher: Can you tell me how it is getting bigger?
Child I 18: It's going one, two, three, four.
Researcher: What's going one, two, three, four?
Child I 18: See the bottom of the triangle, here it is one, then here it is two, then three, here it's four (*outlines each successive triangle when explaining it).*

Most I children who could successfully extend Growing Patterns could also justify the pattern. The following excerpt demonstrates one I child's justification of the pattern as growing systematically in two dimensions.

Researcher: Can you tell me what is happening each time we make the square bigger.
Child I 4: Yeh, here it has one, then it has 2 and 2 lines and it's bigger. Then this one has three and three lines and then four and four lines.
Researcher: What do you mean four and four lines.
Child I 4 See there's four in each line.
Researcher: So what would the next one in my pattern be?
Child I 4: Umm … five and five lines.

In contrast, NI children were unable to identify or extend Growing Patterns. Many saw the triangles and squares exclusively as *items* in simple repetitions in the same way as the simple repetitions that they were familiar with. Many successfully created an ABC repetition however, they did not see the pattern as a growing pattern.

Conclusions and Implications

Interview-based assessment of children's patterning skills identified that young children can develop complex patterning concepts prior-to-formal schooling. It appears that the Intervention experiences encouraged children to see the structure of simple repetition using a unit of repeat, and to represent patterns in different spatial forms such as borders, grids, arrays, subitising patterns, and numerical sequences. It was also apparent that the development of pattern as a unit of repeat promoted other mathematical processes such as multiplicative thinking and transformation skills.

Warren (2005), in her study with 9-year-olds, questioned whether growing patterns were cognitively more difficult, or whether the real difficulty could be traced to over-emphasis on repeating patterns in early mathematics curricula. The findings of this present study showed that the difficulty with growing patterns was not necessarily the absence, or predominance of repeating patterns in early mathematics curricula. Rather, the inadequate or inappropriate development of repeating patterns without a sound understanding of the unit of repeat, limited and possibly impeded the development of growing patterns. Commonly, when teachers are dealing with repeating patterns, the structure of the pattern is ignored or misinterpreted. Therefore, expecting children to observe other pattern structures such as growing patterns is unreasonable.

Algebra has at times been considered developmentally inappropriate for young children, lying well beyond their developmental capabilities. However, the findings of this study suggest that this is not the case. It can be inferred that older students' difficulties may not be a result of developmental constraints after all, but rather, traced to the limited opportunities and/or limited or inaccurate approaches experienced in the early years. These include a lack of awareness of unit of repeat and inadequate attention to structure. The results indicated that the predominant strategy used by NI children to solve patterning problems was an alternating colours strategy. In comparison, I children were able to identify the unit of repeat and use this to solve various complex patterning tasks. Therefore it might be questioned whether the approach to teaching patterns and algebra used in mathematics curricula encourages an alternating colours strategy rather than the identification of pattern elements and number of repetitions. Could teachers' lack of understanding and their approach to teaching repeating patterns limit children's development of patterning? Further research is needed to explore the impact on children's mathematical development if changes to curriculum and teacher pedagogy were to occur that explicitly encourage representation, abstraction, and generalisation of repeating and growing patterns in the early years.

References

Blanton, M. L., & Kaput, J. (2002). Design principles for tasks that support algebraic thinking in elementary school classrooms. In A. Cockburn & E. Nardi (Eds.), *Proceedings of the 26th annual conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 105-111). Norwich, UK: PME.

Blanton, M., & Kaput, J. (2004). Elementary grades students' capacity for functional thinking. In M. Jonsen, M.J. Høines & A. Fuglestad (Eds.), *Proceedings of the 28th annual conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp.135-142). Bergen, Norway: PME.

Dockett, S., & Perry, B. (2001). "Air is a kind of wind": Argumentation and the construction of knowledge. In S. Reifel & M. Brown (Eds.), *Early education and care, and reconceptualizing play,* (pp. 227-256). New York: Elsevier Science.

English, L. D. (2004). Promoting the development of young children's mathematical and analogical reasoning. In L. English (Ed.), *Mathematical and analogical reasoning of young learners.* Mahwah, NJ: Lawrence Erlbaum.

Ginsburg, H. P. (2002). Little children, big mathematics: Learning and teaching in the preschool. In A. D. Cockburn & E. Nardi (Eds.), *Proceedings of the 26th annual conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 003-014). Norwich, UK: PME.

Kieran, C. (2006). Research on the learning and teaching of algebra. In A. Gutiérrez & P. Boero (Eds.), *Handbook of research on the psychology of mathematics education: past, present and future* (pp. 11-49). Rotterdam, The Netherlands: Sense Publishers.

Mulligan, J.T., Mitchelmore, M., & Prescott, A. (2006). Integrating concepts and processes in early mathematics: The Australian Pattern and Structure Awareness Project (PASMAP). In Novotná, J., Moraová, H., Kratká, M., & Stehliková, N. (Eds.) *Proceedings of the 30th annual conference of the International Group for the Psychology of Mathematics Education, Prague* (Vol. 4, pp. 209-216). Prague: PME.

NSW Department of Education & Training (2001). *Count Me In Too Professional Development Package (CMIT).* Sydney: Author.

Papic, M., & Mulligan, J. (2005). Pre-schoolers' mathematical patterning. In P. Clarkson, A. Downton, D. Gronn, M. Horne, A. McDonough, R. Pierce, & A. Roche (Eds.), *Building connections: Research, theory and practice.* (Proceedings of the 28th annual conference of the Mathematics Education Research Group of Australasia, Melbourne, pp. 609-616). Sydney: MERGA.

Warren, E. (2005). Patterns supporting the development of early algebraic thinking. In P. Clarkson, A. Downton, D. Gronn, M. Horne, A. McDonough, R. Pierce, & A. Roche (Eds.), *Building connections: Research, theory and practice* (Proceedings of the 28th annual conference of the Mathematics Education Research Group of Australasia, Melbourne, pp. 759-766). Sydney: MERGA.

Waters, J. (2004). Mathematical patterning in early childhood settings. In I. Putt, R. Faragher, & M. Mclean (Eds.), *Mathematics Education for the Third Millennium: Towards 2010* (Proceedings of the 27th annual conference of the Mathematics Education Research Group of Australasia, Townsville, Vol. 2, pp. 565-572). Sydney: MERGA.

Whole Number Knowledge and Number Lines Help to Develop Fraction Concepts

Catherine Pearn
The University of Melbourne
<cpearn@unimelb.edu.au>

Max Stephens
The University of Melbourne
<m.stephens@unimelb.edu.au>

Many researchers have noted that students' whole number knowledge can interfere with their efforts to learn fractions. In this paper we discuss a teaching experiment conducted with students in Years 5 and 6 from an eastern suburban school in Melbourne. The focus of the teaching experiment was to use number lines to highlight students' understanding of whole numbers then fractions. This research showed that successful students had easily accessible whole number knowledge and recognised the relationship between the whole and the parts whereas the weakest students had poor number knowledge and could not see the connections.

Research Background

Over the past 20 years research on rational number learning has focused on the development of basic fraction concepts. This has included partitioning of a whole into fractional parts, naming of fractional parts, and order and equivalence (Behr, Wachsmuth, Post, & Lesh, 1984; Kieren, 1983; Streefland, 1984). Kieren (1976) distinguished seven interpretations of rational number that were necessary to enable the learner to acquire sound rational number knowledge, but subsequently (Kieren, 1980; 1988) condensed these into five: whole-part relations, ratios, quotients, measures, and operators. Kieren suggested that difficulties experienced by children solving rational number tasks arise because rational number ideas are sophisticated and different from natural number ideas and that children have to develop the appropriate images, actions, and language to precede the formal work with fractions, decimals, and rational algebraic forms.

Several researchers have noted how children's whole number schemes can interfere with their efforts to learn fractions (Behr et al., 1984; Bezuk, 1988; Hunting, 1986; Streefland, 1984). Behr and Post (1988) indicated that children need to be competent in the four operations of whole numbers, along with an understanding of measurement, to enable them to understand rational numbers. They suggested that rational numbers are the first set of numbers experienced by children that are not dependent on a counting algorithm. The required shift of thinking causes difficulty for many students.

Mack (1990) found that where students possessed knowledge of rote procedures they focused on symbolic manipulations. Mack's study suggested that if a strand of rational number is developed based on partitioning, using the students' informal knowledge, then other strands of rational number could be developed more easily.

Steffe and Olive (1990) showed that concepts and operations represented by children's natural language are used in their construction of fraction knowledge. Two distinct fraction schemes emerged from their research. In the iterative scheme, children established a unit fraction as part of a continuous but segmented unit. From this, children developed their own fraction knowledge by iterating unit fractions. The foundation of a measurement scheme occurred when the children's number sequence was modified to form a connected number sequence.

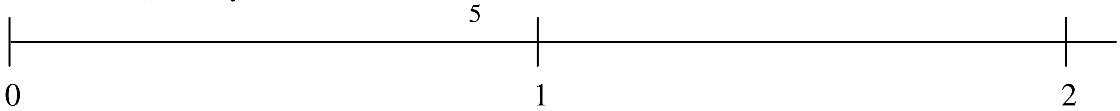
Saenz-Ludlow (1994) maintained that students need to conceptualise fractions as quantities before being introduced to standard fractional symbolic computational algorithms. Streefland (1984) discussed the importance of students constructing their own understanding of fractions by constructing the procedures of the operations, rules, and language of fractions. This research focuses on students' use of number lines firstly to probe students' understanding of fractions as numbers capable of being represented on a number line, and then to look at how number lines involving whole numbers and fractions can be used to develop fractional language and to articulate fractional concepts.

Previous Studies

In previous research (Pearn & Stephens, 2004; Pearn, Stephens, & Lewis, 2002; Stephens & Pearn, 2003) analysis of results from the Fraction Screening Test A (Pearn & Stephens, 2002) has highlighted students' difficulties with fraction concepts. The Fraction Screening Test is a paper and pencil test designed mainly for students in Years 5 and 6 and for weaker students in Years 7 and 8. The tasks include contexts such as discrete items, lengths, fraction walls, and number lines. Analysis of the results from the Fraction Screening Test highlighted the difficulties that many students experienced with number lines. The three number line tasks from the Screening Test are shown in Figure 1.

9. Here is a number line 2 units long.

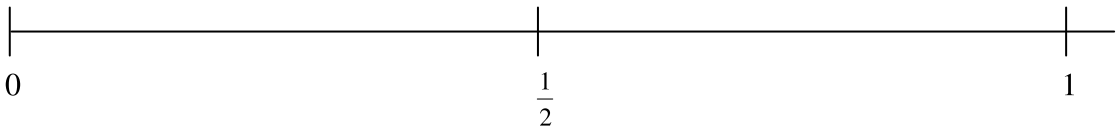
Put a cross (x) where you think the number $\frac{3}{5}$ would be on the number line below.



10. This number line shows where the numbers $\frac{1}{2}$ and 1 are.

Write any fraction that fits between 0 and $\frac{1}{2}$. _______

Place your fraction as accurately as you can on the number line below.



11. This number line shows where the number $\frac{1}{3}$ is.

Put a cross (x) where you think the number 1 would be on the number line.

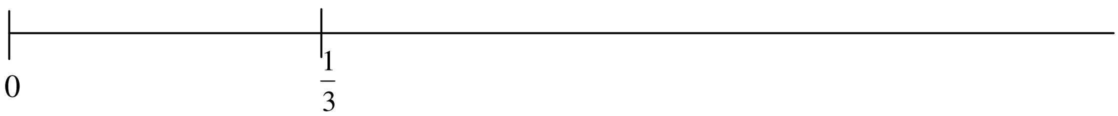


Figure 1. The three number line tasks (Fraction Screening Test A).

Many students in Question 9 confused three-fifths of the number line with the number three-fifths. In Question 10 many students who chose one-quarter represented it correctly. Other fractions seemed to be placed using guess work rather than any systematic division of the number line. A similar tendency to use guess work was evident in Question 11. Table 1 compares the results of 288 students in four year levels from four different Victorian schools on the above three number line tasks. These results highlight the difficulties that students have with the notion of fractions as numbers and with placing the fractions on number lines accurately.

Table 1
Success with Tasks from the Fraction Screening Test A (n = 288)

Task from Fraction Screening Test A	Year 5 (n = 84)	Year 6 (n = 66)	Year 7 (n = 89)	Year 8 (n = 49)
Marks 3/5 of the number line	23%	32%	32%	20%
Chooses then marks number between 0 and 1/2	44%	31%	52%	19%
Marks 1 given 1/3	46%	41%	56%	25%

Subsequent Interviews

In a previous study (Pearn & Stephens, 2004), several students who had completed the Screening Test were asked to compare two fractions and then place them on number lines marked zero to one. We observed that some students just "placed" the fractions on the number lines without using any referents to other known fractions, for example, one-half. For example, one student randomly placed the fraction three-quarters close to the number one on the number line then placed three-fifths the same distance from three-quarters as she had placed three-quarters from one (Figure 2). This was because, "three-quarters is only one away from a whole and three-fifths is two away from a whole". Pearn and Stephens (2004) refer to this as gap thinking, illustrating how whole number thinking can interfere with fraction knowledge.

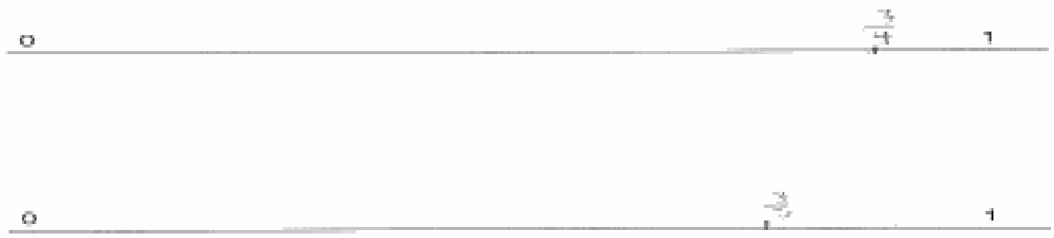


Figure 2. Three-quarters and three-fifths.

Another student when comparing three-quarters and three-fifths correctly converted both fractions to twentieths concluding that three-quarters was bigger (Pearn & Stephens, 2004). When invited to use number lines to compare these two fractions he divided the first number line (below) by eye into quarters and marked one-half and three-quarters. He then placed one-half on the number line below corresponding to its position on the first number line. He said that "three-fifths is smaller than three-quarters" and marked three-fifths to the right of one-half and to the left of three-quarters on the first number line with no attempt to divide the line into fifths (Figure 3).

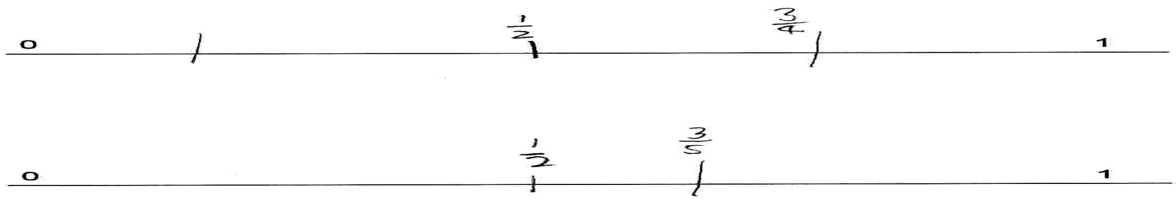


Figure 3. One-half, three-quarters and three-fifths.

When the interviewer asked where the fraction one-fifth would be the student responded with “One-fifth is more than one-half, I think.” He then used a new number line and placed one-fifth to the right of one-half. The interviewer then asked where he thought one-third and one-quarter would be on the number line. The student then placed these two fractions in between one-half and one-fifth as shown in figure 4. Despite apparent correct thinking in the previous example, this student unexpectedly lapsed into *larger-is-bigger thinking* – another example of incorrect whole number thinking.

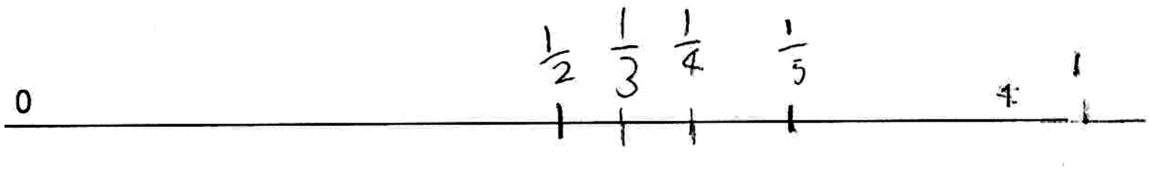


Figure 4. Larger denominator is bigger.

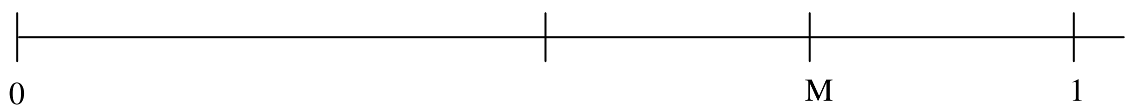
These instances demonstrate the importance of asking students in a probing interview to represent their fractional thinking using a number line. On the other hand, asking other students to represent fractions on a number line assisted them to identify and correct their misconceptions. However the study did not set out to explore remedial strategies with the students interviewed.

The current study also uses a screening test and interview using number lines to probe students’ understanding of fractions as numbers. The interview commenced by looking at how number lines involving whole numbers can be used to develop fractional language and to articulate fractional concepts.

Initial Testing

All students from Years 5 and 6 from School A in the eastern suburbs of Melbourne were given Fraction Screening Test A (Pearn & Stephens, 2002). The tasks used contexts such as discrete items, lengths, fraction walls, and number lines. One fraction task based on area was replaced in this study with an extra number line task. Figure 5 shows the additional number line task added specifically for this group of students.

12. On this number line 0 and 1 are shown.



What fraction number do you think M represents? _______

Figure 5. Additional number line task (Fraction Screening Test A).

Results

The students’ results on the Fraction Screening Test A reflected the types of responses achieved previously from other groups of students. Results shown in Figure 6 show that these students were more successful with tasks presented in conventional contexts such as shading three-fifths of an unmarked rectangle and with the fraction one-third, for example, finding the whole given a third using discrete objects. They were less successful with tasks that involved fractions as numbers, for example “Put a cross (x) where you think the number $\frac{3}{5}$ would be on the number line". Many students interpreted this question as requiring them to find three-fifths of the entire line ignoring the numbers 0, 1, and 2 marked on the number line.

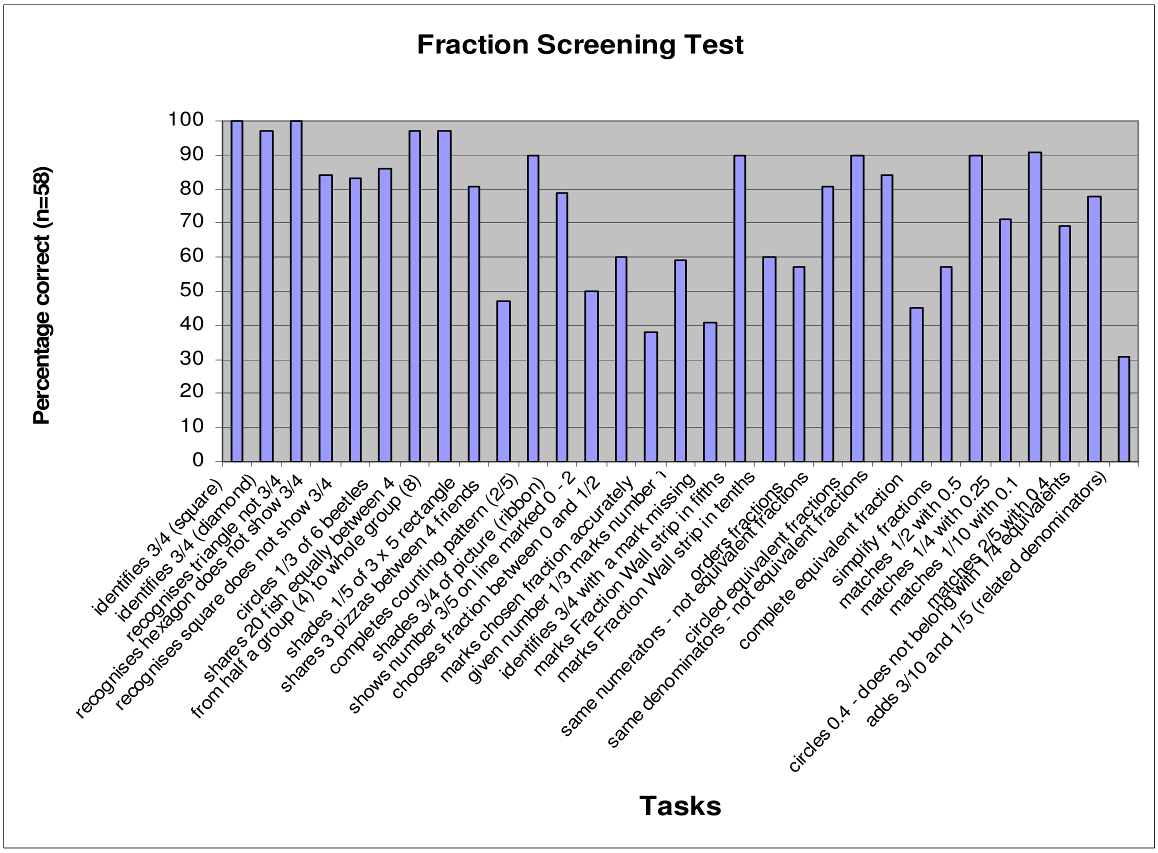


Figure 6. Success with tasks from Fraction Screening Test A.

Teachers from School A had undertaken considerable professional development presented by the authors. In Table 2 we compared the combined results of Years 5 and 6 in School A with results on the same three questions from other schools (see Table 1) where teachers had not had the same level of professional development. Students at School A were more successful with the first and third tasks. In the second task, while 60% of School A's students were able to state a fraction between 0 and ½, only 38% could place the fraction they chose accurately on the number line.

Table 2
Comparative Success of Students from School A on Fraction Screening Test

Number line tasks (Fraction Screening Test A)	*School A (n = 58)*	*Other Year 5 (n = 84)*	*Other Year 6 (n = 66)*
Marks 3/5 of the number line	*50%*	23%	32%
Chooses then marks number between 0 and 1/2	38%	44%	31%
Marks number 1 given 1/3	*59%*	46%	41%

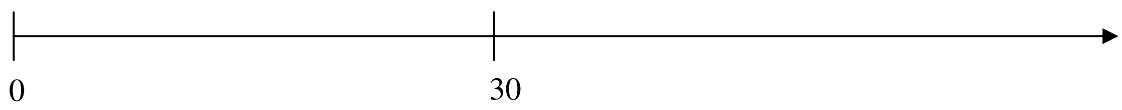
Analysis of the additional number line question (Figure 5) revealed that only 41% of the students from School A were able to identify the number denoted by M (3/4) on the number line. A few students thought the letter M should represent a letter so responses included words like “million”, “middle”, and “mixed number”.

Fraction Number Line Interview

The authors developed an interview protocol called *Working with number lines to probe fraction concepts* (Pearn & Stephens, 2006). The interview required students to complete number line tasks while describing what they were thinking or how they worked it out. Students were initially required to place whole numbers on number lines, then fractions on number lines and finally, to review their responses to the four number line questions from the Fraction Screening Test. Figure 7 is an example of one question that requires students to place a number between two given whole numbers and then place another number relative to one of the given whole numbers. Following research by Behr and Post (1988) and Mack (1990), questions like this were designed to see how well students could connect their whole number knowledge in a fraction context.

4. This number line shows 0 to 30.

a) Where would you put the number 10? How could you be sure?



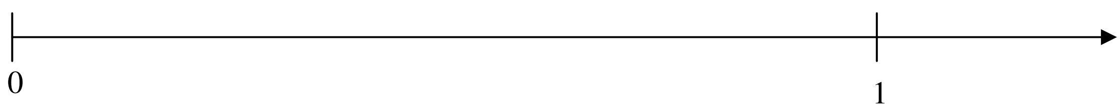
b) If you want to mark the number 40 on this line how could you do that? How could you be sure?

Figure 7. Marking whole numbers on a number line.

After working with whole numbers students were asked to place proper fractions and mixed numbers on number lines. Figure 8 gives an example of a question involving fractions. For this task the interviewers were looking for evidence that students could place fractions accurately by using points of reference rather than just “placing” the fraction randomly on the line. The second part of this task requires students to use previous information to assist them to decide the most appropriate point for the number.

8. This number line is marked 0 and 1.

a) Where would you put the number ¾? How could you be sure?



b) If you want to mark the number $1\frac{1}{8}$ on this line how would you do that? How could you be sure?

Figure 8. Marking fractions on a number line.

The Interviews

Students were individually interviewed by the authors. In Task 1, students were shown a number line marked 0 and 100. They were then asked to show where the number 50 would be placed. Students justified their answers by saying things like:

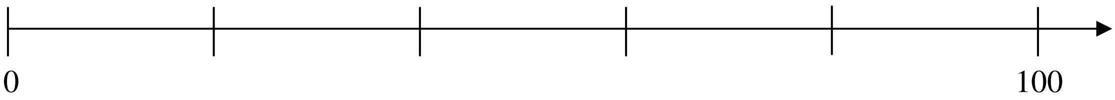
- 50. It’s in between. Half of a 100 is 50.

- Another student placed 50 correctly and said: "It's in the middle (of the line)."

Many students found Task 2 (Figure 9) more difficult where, unlike the previous task, the midpoint of the line was unmarked. Some students' responses to this task highlighted the lack of understanding of the relationship between the number of marks used to divide the line and the numbers parts so formed. Despite giving a correct answer, Student S could not connect her numbers to the parts. Even when students, like Students R and T, were helped to identify the number of parts their lack of number knowledge prevented them from giving a confident correct response.

2. This number line is marked 0 to 100 and has been divided up.

Can you work out what numbers should go on the marks?



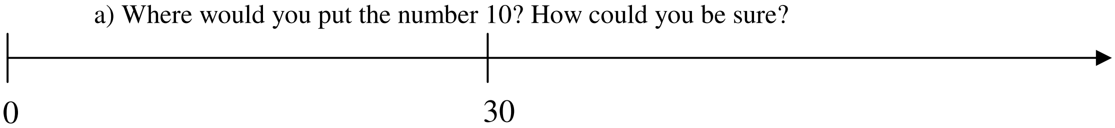
Student R (Year 5)	Student S (Year 5)	Student T (Year 5)
Pointing to the last mark (where 80 should be) she said: "Maybe this should be 75". Interviewer: How many parts? A: four … six I: Count the parts. A: Five I: Five people to share 100 lollies. How many each? A: Fifteen … 15, 30, 45, 60 … No. Maybe 30 … maybe 25 … maybe 20 I: Please check for 20. A: correctly marked the line 20, 40, 60 … to 100	Wrote 20, 40, 60, 80 S: I just know. I: How many parts are there? S: Four I: Does it help to know the parts? S: Not really.	Placed the numbers 15, 20, 60, 75 on the marks provided. I: Is 15 going to work? A: 15, 30, 45, 60 … No. He then placed 20, 40, 60, 80. I: How many spaces? A: Five I: Share 100 between five people. I: 2 … 20

Figure 9. Examples of students' responses for Task 2.

Those students who knew that 30 consisted of three 10s, or that 10 was one third of 30, dividing the number line into three equal parts was easy. For students like Student T the process of halving and then partitioning again proved problematic (see Figure 10).

4. This number line shows 0 to 30.

a) Where would you put the number 10? How could you be sure?



b) If you want to mark the number 40 on this line how could you do that? How could you be sure?

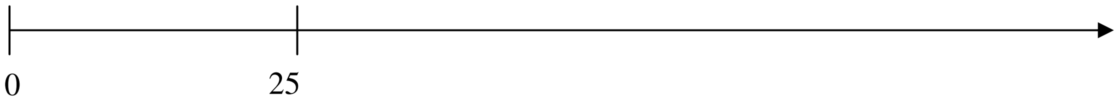
Student S: About there. (Placing 10 correctly). It's about a third. I: "Could you check." She marked in 10, 20, and 30 correctly. Placed 40 correctly. "Because it's the same distance (10) up from 30".	Student T first placed 15 half way. Then said: "Twenty would be about there." He then estimated where 10 would be (no partitioning) and decided that 40 would be the same distance from 30.

Figure 10. Examples of students' responses for Task 4.

For Task 5, (Figure 11), several students, including Students U and V, assumed the arrow at the end of the drawn line was the mark for 100. These students used this assumption rather than the information given on the number line.

5. This number line shows 0 to 25.

a) Where would you put the number 75? How could you be sure?



b) If you want to mark the number 5 on this line how could you do that?

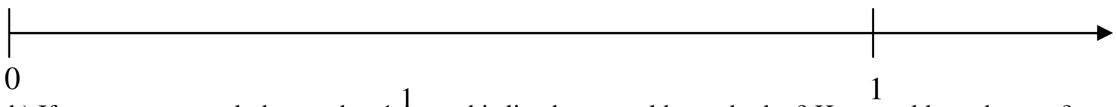
Student S Put in two marks to represent 50 and 75 but very inaccurate increments of 25. I: Could you use your pencil to measure? A used an accurate measure to place 50 and 75 but didn't know how to place 5.	Student U marked 50 then 75. "I think here is about 100 (end of drawn line). "Three-quarters is 75. Because 25, 50, 75". Placed 5 about half way between 0 and 25, then rethought. I: Half of 25? S: 12½ I: Half of 12 is …? S: "Six". Placed 5 a bit to the left of where 6 would be.	Student V (Year 6) marked in two more intervals to correctly place 75. He appeared puzzled because he assumed the end (arrow) was 100. He initially subdivided 0 to 25 too small. Self corrected to get fifths quite accurately.

Figure 11. Examples of students' responses for Task 5.

In Figure 12 the interviewer assisted students by asking them to focus on the interim fractional points (¼, ½, and ¾). Some students thought the arrow was the mark for the number two but once they had focussed on the interim fractional points were able to correctly place 1⅛ by subdividing correctly the line between 1 and 1¼.

8. This number line is marked 0 and 1.

a) Where would you put the number ¾? How could you be sure?



b) If you want to mark the number $1\frac{1}{8}$ on this line how would you do that? How could you be sure?

Student S marked ½ then ¾ correctly by eye. Not sure about 1⅛. I: What's the distance between ¾ and 1? S: ¼ I: Where is 1¼? She identified 1¼ and then said "Half of that (distance between 1 and 1¼) is 1⅛".	Student T said: ¾ is about here (placed it but didn't use ½ or ¼ as reference points). I: Where is ½ and ¼? He subdivided and then was able to place 1¼ correctly and halved the distance from 1 to 1¼ to get 1⅛

Figure 12. Examples of student responses for Task 8.

Analysis of Interview Results

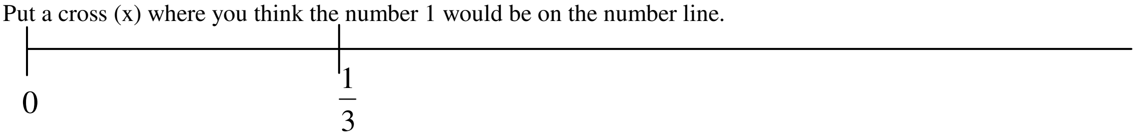
Successful students used number knowledge, accurate skip-counting, and multiplication facts to partition the number line. They confidently related halves, quarters, and three-quarters to the numbers being used. For example they could relate eighths to quarters. Some students needed help to identify the number of spaces (parts) instead of focussing only on the vertical division marks. The number line questions allowed those students who had confident whole number knowledge to apply fractional concepts to their subdivisions of the number line. Other students who were unable to draw on whole number knowledge frequently used guesses to place numbers on the number line using "Where I think it should be" rather than accurate "by-eye" partitioning. These students were rarely able to apply the language of fractions to subdivisions of the number line, and often needed assistance to see connections between halves, quarters, and eighths.

Students Reviewing their Written Responses to the Screening Test

On the initial Screening Test, Student S correctly marked the number one but showed no evidence of the strategy she used. Student T's response showed no understanding of equal intervals. However after being interviewed Students S and T applied correct subdivision strategies to this task that they had used for their whole number questions (Figure 13).

This number line shows where the number $\frac{1}{3}$ is.

Put a cross (x) where you think the number 1 would be on the number line.



On the Screening Test Student S had placed the number 1 correctly with no interim marks. In interview she doubled 1/3 to give 2/3 and added a further 1/3 to get 1 then explained that she could also count in sixths by halving the line 0 to 1/3.	On the Screening Test Student T had placed the number one incorrectly (too close with two incorrect interim marks that were not equivalent to 1/3). I: "Is your old one correct?" He then used the distance from 0 to 1/3 to create 2/3 and added 1/3 accurately to get 3/3.

Figure 13. Comparison of Task 3 responses before and after the interview.

When asked to review their earlier written responses, many students showed evidence of being able to recognise errors and to self correct, as shown in Figure 14, for the fraction task using the letter M. Both Students S and T were now able to see that the letter M represented the fraction ¾.

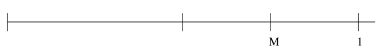
Task	Student S	Student T
On this number line 0 and 1 are shown. What fraction number do you think M represents?  	S wrote ¼ as the value of M on the Screening Test. In interview she said: "M is ¼ … ? Oh no, it's ¾."	A wrote that M was 2/3 but in interview he said: "It should be ¾." I: Why did you choose 2/3? A: Because there were three parts. He then added by pointing: "That part (0 → ½) is bigger than this (½ → 2/3)"

Figure 14. Comparison of Task 4 responses before and after the interview.

Conclusions

Successful students demonstrated easily accessible and correct whole number knowledge and knew relationships between whole and parts. They attended to equal parts not the vertical lines used to create the parts. They could apply fractional terms to the equal parts. Less successful students tended to look at lines and needed help to focus on equal parts. These students often had difficulties with number lines marked without a midpoint. Sometimes these students assumed that arrows at the end of lines represented "the next" whole number. Due to their poor whole number knowledge, the weakest students could not see connections between whole numbers and fractional parts of the number line. Also, they appeared dependent on guess work to place numbers on number lines.

By using whole numbers on number lines first, the interview questions clearly helped many students to connect whole number and fraction knowledge. The interviews also helped students to recognise and correct their own misconceptions in previous assessment tasks.

References

Behr, M. J. & Post, T. R. (1988). Teaching rational number and decimal concepts. In T. R. Post (Ed.), *Teaching Mathematics in Grades K-8* (pp. 201-248). Boston, MA: Allyn and Bacon.

Behr, M., Wachsmuth, I., Post, T., & Lesh, R. (1984). Order and equivalence of rational numbers. *Journal for Research in Mathematics Education, 15,* 323-341.

Bezuk, N. S. (1988). Fractions in early childhood mathematics curriculum. *Arithmetic Teacher, 35,* 56-59.

Hunting, R. P. (1986). Rachel's schemes for constructing fraction knowledge. *Educational Studies in Mathematics, 17*, 49-66.

Kieren, T. E. (1976). On the mathematical, cognitive and instructional foundations of rational numbers. In R. A. Lesh (Ed.), *Number and measurement: Papers from a research workshop* (pp. 101-144). Columbus: ERIC/SMEAC.

Kieren, T. E. (1980). The rational number construct – Its elements and mechanisms. In T. E. Kieren (Ed.), *Recent research on number learning* (pp. 125-150). Columbus: ERIC/SMEAC.

Kieren, T. E. (1983). Partitioning, equivalence, and the construction of rational number ideas. In M. Zweng (Ed.)., *Proceedings of the 4th International Congress on Mathematical Education* (pp. 506-508). Boston: Birkhauser.

Kieren, T. E. (1988). Personal knowledge of rational numbers – Its intuitive and formal development. In J. Hiebert & M. Behr (Eds.), *Number concepts and operations in the middle grades* (pp. 162-181). Hillsdale, NJ: Lawrence Erlbaum.

Mack, N. K. (1990). Learning fractions with understanding: Building on informal knowledge. *Journal for Research in Mathematics Education, 21,* 16-33.

Pearn, C., Stephens, M., & Lewis, G. (2002). Assessing rational number knowledge in the middle years of schooling. In M. Goos & T. Spencer (Eds.), *Mathematics making waves.* (Proceedings of the 19th Biennial Conference of the Australian Association of Mathematics Teachers, CD-ROM pp. 170-178). Adelaide: AAMT.

Pearn, C., & Stephens, M. (2002). *Success in Numeracy Education (Years 5 – 8): Fraction Screening Test A*. Melbourne: Catholic Education Commission of Victoria.

Pearn, C., & Stephens, M. (2004). Why you have to probe to discover what Year 8 students really think about fractions. In I. Putt, R. Faragher, & M. McLean (Eds.), *Mathematics education for the third millennium: Towards 2010.* (*Proceedings of the 27th annual conference of the Mathematics Education Research Group of Australasia*, pp. 430-437), Sydney: MERGA.

Pearn, C., & Stephens, M. (2006). *Working with number lines to probe fraction concepts*. Melbourne: Catholic Education Commission of Victoria.

Saenz-Ludlow, A. (1994). Michael's fraction schemes. *Journal of Research in Mathematics Education*, *25*, 50-85.

Steffe, L. P., & Olive, J. (1990). Constructing fractions in computer microworlds. In G. Booker, P. Cobb, & T. N. de Mendicutti (Eds.), *Proceedings of the fourteenth annual conference of the International Group for the Psychology of Mathematics Education* (pp. 59-66). Mexico City: PME.

Stephens, M., & Pearn, C. (2003). Probing whole number dominance with fractions. In L. Bragg, C. Campbell, G. Herbert, & J. Mousley (Eds.), *Mathematics Education Research: Innovation, Networking, Opportunity*. (*Proceedings of the Twenty-sixth Annual Conference of the Mathematics Education Research Group of Australasia*, pp. 650-657). Sydney: MERGA.

Streefland, L. (1984). Unmasking N-distractors as a source of failures in learning fractions. In B. Southwell, R. Eyland, M. Cooper, J. Conroy, & K Collis (Eds.), *Proceedings of the eighth annual conference of the International Conference for the Psychology of Mathematics Education,* (pp. 142-152). Sydney: Mathematical Association of New South Wales.

Identifying and Analysing Processes in NSW Public Schooling Producing Outstanding Educational Outcomes in Mathematics

John Pegg
University of New England, Australia
<jpegg@une.edu.au>

Debra Panizzon
University of New England, Australia
<dpanizzo@une.edu.au>

Trevor Lynch
University of New England, Australia

This paper reports on a project that identified and explored the factors leading to outstanding mathematics outcomes in junior secondary public education in NSW for students across the ability spectrum. Once a sample of mathematics faculties was identified by drawing upon the extensive quantitative and qualitative data-bases within the NSW Department of Education and Training (DET), seven intensive case studies were conducted to identify faculty-level factors. Seven common themes are reported and these are the strong sense of team, staff qualifications and experience, teaching style, time on task, assessment practices, expectations of students, and teachers caring for students.

An Exceptional Schooling Outcomes Project (ÆSOP) was designed to investigate the principles, processes, and practices in a sample of sites in NSW Years 7-10 Department of Education and Training (DET) schools producing outstanding educational outcomes. The research focus was on teams of teachers (i.e., mathematics faculties). The nature of "outstanding educational outcomes" was determined using the *Adelaide Declaration on National Goals for Schooling in the Twenty-first Century*, approved by all State, Territory and Commonwealth Ministers of Education in 1999. They stated that schooling should:

- Develop fully the talents and capacities of all students;
- Enable high standards of knowledge, skills and understanding through a comprehensive and balanced curriculum; and
- Be "socially just" (MCEETYA, 1999).

There is growing evidence in the research literature of the importance of a research focus on faculties in secondary schools. Although there is an extensive body of research highlighting the important roles played by the school Principal at one end of the spectrum, and the individual classroom teacher at the other, in advancing the quality of students' educational outcomes as they proceed through school, there is comparatively little research on the significance of the roles played by subject faculties as groupings of teachers working towards a common agenda. Yet, as Goodson and Marsh (1996, p. 54) stated "the subject department provides the most common organisational vehicle for school subject knowledge, certainly in secondary schools, but unlike 'the curriculum' it has not been widely researched or much noted in our studies of schools." Bennett (1999, p. 289) supported this perspective suggesting that the latest school effectiveness and school improvement research recognised the different levels of school structure and practice, and the "resurgence of interest in sub-units of schools" – in particular, subject faculties and their organisation and leadership (Busher & Harris, 1999; Sammons, Thomas, & Mortimore, 1997).

Other evidence from school improvement research has also emphasised the growing importance of focusing efforts at changing practices at various levels within an organisation. For example, the largest study of differential school effectiveness in the

Proceedings of the 30th annual conference of the Mathematics Education Research Group of Australasia
J. Watson & K. Beswick (Eds), © MERGA Inc. 2007

United Kingdom identified the differences between faculties as a means of explaining school performance (Busher & Harris, 1999; Sammons et al., 1997). As Hannay and Ross (1999, p. 346) concluded, "we need far more research on the micro-processes involved in secondary schools."

In a report on the *Investigation of Effective Mathematics Teaching and Learning in Australian Secondary Schools* (ACER, 2004) one of the main findings of the study was that the effectiveness of mathematics teaching in a school is related to the strength of professional community in the school's mathematics department. Ayres, Sawyer, and Dinham (2004) came to a similar conclusion in their study that focused on characteristics of effective teachers at the Higher School Certificate (HSC) level. The researchers found that the subject faculty was one of seven factors deemed to contribute towards HSC teaching success and warranting further investigation.

This paper reports on seven mathematics faculties in which the past 4 years of student cohorts had either scored consistently highly on value-added measures or demonstrated consistent improvement on the same scores. Importantly, sites had to demonstrate their ability to "value add" for students in low, middle and high achievement bands. Sites were selected to cover as wide a socio-economic and geographical cross-section of schools as possible. In particular, the more influential themes emerging from the analysis of processes and procedures of secondary mathematics faculties visited are discussed.

Research Design and Methods

Overall, the ÆSOP study involved a series of approximately 50 intensive case studies in a variety of "sites" across NSW. These sites were generally faculty-based although some other teacher groupings were explored in some schools (e.g., learning support teachers). Paramount to the project was a valid and justifiable method for selecting schools given that students had to be achieving outstanding educational outcomes.

Selection of Sites in Schools

The process for selecting schools for inclusion in the project was complex, involving a matrix of data. The basic source was value-added data collected for all students attending DET schools in NSW. The data were prepared by the DET School Accountability and Assessment Directorate by profiling student learning outcomes as measured in standardised tests commencing with the Year 5 Basic Skills Test, the Year 7 ELLA and SNAP tests, and the English/literacy, mathematics, science, Australian history, geography, civics and citizenship tests in the School Certificate. The criteria for selection of a site in a school were as follows:

- Cohorts of students consistently, i.e., over the past four years, scoring high on value-added data, across the low, middle and high achievement bands, or
- Cohorts of students consistently, i.e., over the past four years, improving their value added scores across the low, middle and high achievement bands.

With the emphasis on the three bands of students, selective schools in NSW were automatically excluded as potential sites due to the lack of low and often middle achieving students.

Selection also included qualitative data as part of a triangulation process. Nominations of sites were sought from DET staff at the central, district, and school levels, as well as key education groups, such as the NSW Teachers' Federation, the NSW Federation of Parents and Citizens, the NSW Secondary Principals' Council, the Professional Teachers' Associations and the NSW Student Representative Council. In all cases, nominations had to be substantiated by evidence. Consideration was also given to HSC data in the relevant subject area in terms of the numbers of students pursuing the subject and overall student results. Finally, District Superintendents and school Principals were contacted by phone to discuss the appropriateness of the selection of sites particular to their district and school, respectively. Once the initial selection of the *sample* sites was verified as potentially outstanding, agreement was reached with Principals of schools for the research visits to 35 schools in 23 districts throughout the state. Site visits were made to seven schools for mathematics representing a cross-section of socio-economic and geographical locations (Table 1).

Table 1
Profile of Sites Visited for ÆSOP Mathematics

School	Location	Student Population	% Indigenous Students	% NESB Students	Other Characteristics
1	Western NSW	900-1000	2	4	High proportion of students with disabilities Middle socio-economic status
2	Northern Coast NSW	900	< 1	< 1	Few students with disabilities Middle socio-economic status
3	Northern Sydney	1300-1500	< 1	< 1	Few students with disabilities High socio-economic status
4	Western Sydney	1000-1200	< 1	85	Middle socio-economic status
5	South Western Sydney (Female only)	1000-1100	< 1	92	Low to middle socio-economic status
6	South Western Sydney (male only)	900	< 1	50	Low to middle socio-economic status
7	South Sydney	1100	< 1	85	Middle socio-economic status

Study Design

ÆSOP was guided by four research questions:

- What are the variables and processes leading to outstanding educational outcomes in terms of the goals specified in *The Adelaide Declaration* – personal identify, academic success, and social attainment?
- Is it possible to identify the relationship(s), if any, between the different types of goals specified in *The Adelaide Declaration* as achieved through subject departments and/or other formal groups and special programmes and initiatives?

- What organisational and institutional factors – NSW DET, district, school, leadership, community, faculty, other groups and individuals – contribute to and/or constrain this success?
- To what degree and through what means, if any, can the outstanding educational outcomes identified be shared with others within and beyond the schools investigated?

With sites selected intensive case studies were undertaken in each of the schools. This involved a Project Team consisting of a university researcher in a related discipline (i.e., mathematics education), a university researcher with expertise in case study methodology, the local Chief Education Officer (School Improvement), and a Head Teacher from a nearby school. Each team spent up to 5 days investigating the policies, programs, and practices that lead to the outstanding schooling outcomes being achieved in sites.

In the school the team collected a range of data. This included semi-structured interviews with Principals, Deputy Principal(s), Head Teachers, subject teachers, school advisors, students (Year 7-11), and parents. Lesson observations using a specified protocol were conducted with teachers who agreed to the involvement of the research team. Finally, a document analysis was undertaken of school reports, results, subject programmes, school and faculty policy documents, and any other documents deemed appropriate (e.g., media coverage). All interviews were taped with approval of interviewees.

Once the protocols and procedures for ensuring consistency across Project Teams were developed, four site visits were conducted to pilot the techniques for the main study. This resulted in an interim report with changes made to the conceptual framework guiding the study, alterations to the protocols, and variation to the overall design. The main study was conducted over the next 2 years of the project.

Data Analysis

At the completion of each site visit the research team prepared a report using the data available. The work of the writing teams was assisted by two key activities – consultations/workshops and detailed, qualitative analysis of each relevant site report. Frequent combined meetings of the writing teams were held so that experiences during visits could be shared with preliminary findings discussed and compared. Further analysis of each site report was assisted through the use of NUD*IST software. This facilitated analysis through a tree-node system as a hierarchical index of topics, themes, concepts, and ideas emerged (Richards, 2002).

Results and Discussion

Analysis of the mathematics data identified seven major elements in relation to the school, faculty, characteristics of teachers, pedagogical practices, and parents and students. The findings discussed in this section represent a number of the major themes that emerged as being particularly influential from these elements.

Strong Sense of Team

> We are working in a friendly environment, staff are helpful. Good teamwork. Keen to help each other. We have similar views … like correct Mathematics … We use different methods. Our department has a staff room that is good for working together. I am very happy to teach here. (Teacher)

This quote encapsulates many of the comments made by mathematics teachers about their faculties as teachers invariably likened the experience to working as a "family". Interestingly, this included agreements and disagreements, good times and bad, and friendships of varying intensities. Through it all, however, there was a unifying sense of purpose and collegiality. It was common to hear teachers speak about how much they gained from sharing with their colleagues and how much they appreciated their own opinions being valued. The ability of mathematics teachers to function collaboratively was evident from the policy changes (e.g., registers and programmes), continued changes in assessment practices, and improved classroom approaches aimed at enhancing student understanding.

At the individual level, teachers were cognisant of issues their mathematics colleagues faced and were supportive of one another's challenges and achievements. Teachers had established good working relationships with their peers and used their initiative to determine ways to help colleagues maintain a high-quality learning environment for their students. The focus of this support was evident in various ways such as ensuring colleagues' classes were not disadvantaged by covering absences and ensuring equity in the provision of resources.

Each team of mathematics teachers exhibited a clear sense of pride in the culture of success they helped create and this was disseminated to newly appointed teachers. New teachers who came to the school spoke of encountering an established faculty culture with an expectation for conformity to meet relatively high standards of performance. The enculturation of new staff was implicit and/or explicit ensuring that members of these faculties were able to advocate and share a common vision that encouraged a consistent staff approach.

Importantly, leadership qualities were admired and respected by the mathematics staff. Although leadership was usually the province of the Head Teacher this was not always the case with a distributed leadership (Spillane, Halverson, & Diamond, 2001) style evident in some instances. In general, the leaders of these exceptional faculties exhibited common characteristics including a commitment to keeping abreast of the latest developments in teaching mathematics, a strong subject and syllabus knowledge that would enable them to support other faculty members, and sound classroom practice. These leaders appeared pivotal in establishing and/or maintaining the culture of the faculty.

Qualified Staff with a Breadth and Depth of Experience

Subject knowledge and experience in teaching mathematics were two important features of the staff in the faculties visited. First, the University training of the staff was at a high level with the majority of teachers holding third-year majors or an equivalent in mathematics in their degrees. Second, teachers in these faculties had many years of successful teaching experience, often in several schools. Subsequently, they brought a wealth of different experiences to their current positions.

These faculties could be described as communities of scholars with deep knowledge of the subject and a special pride in teaching mathematics clearly evident. Their work was well recognised by people outside the faculty who were aware that the Mathematics teachers always exhibited a high degree of professionalism. As one Year Adviser remarked:

> Staff members here are confident about mathematics. We sit here and talk Mathematics and exchange ideas. When we put in a request for what classes we want next year there are a number of us who automatically put up our hands for the lower classes … I think that is unusual.

Solid Teaching

All teachers interviewed referred to their style as "traditional" meaning it involved a "standard" approach to classroom instruction. Although there were variations to the meaning of a standard approach there was a great deal of commonality in approaches across schools. In particular, there was a clear and consistent structure to lessons.

In practice, this common structure related to similarities in the way teachers started lessons, how lessons proceeded, and how lessons ended. This structure gave a sense of security to students in their learning. Nevertheless, within this structure, there was still variety in these lessons. For students, lessons were not dull, repetitive, or boring.

At some stage in the lessons observed students were given practice exercises. Students who finished the work were given additional activities, usually from another source. Teachers made every effort to ensure that students were given an opportunity to learn, or to practise skills, in each lesson. A feature of the lessons observed was that teachers were aware of the need for appropriate revision before proceeding, careful explanation of new concepts, appropriate practice and follow-up.

Common to many lessons observed was an underlying rigour appropriate to the ability of the students. Teachers were conscious of helping and encouraging all students to achieve. Numerous conversations with teachers revealed the importance of "bringing students up to a level rather than pitching the work down". Every effort was made to ensure that students achieved syllabus outcomes.

Faculty members established supportive classroom environments for their students using an array of teaching aids or interesting approaches to topics. They accepted the need for some change and appeared willing to try new ideas, but did so in an environment of scrutiny. They were skeptical of educational fads and felt that they had been "burnt" many times before through change for change's sake. They spoke about being prepared to put in place whatever was needed to ensure that their students were placed in the best position to benefit from changes.

> We have battled away with all these new approaches in teaching, group work and so forth … and mathematics-wise we have found it very hard to really move away from set maths lessons … you know your structured maths lessons. … As soon as you get the unstructured happening the students are not comfortable. (Head Teacher Mathematics)

Time on Task

Time on-task was maximised by the teachers and students at the schools visited. Emphasis around "on-task" time and a commitment to a cooperative and supportive environment were high on the teachers' agendas. Classroom teachers made every effort to ensure that students were actively engaged in the learning process. When asked about discipline in Mathematics a Year 8 student said: "In Mathematics we are too busy to muck up."

The value of on-task time was also apparent in more subtle ways in the schools. An example from among many help exemplify intrinsic aspects of this feature. In one school visited, the staffroom was located at some distance from the demountable village teaching rooms allocated to mathematics. Despite this geographic arrangement, the mathematics teachers were invariably punctual to lessons and got down to productive teaching and learning in minimum time. Further, when they had consecutive lessons it was noted that they took resources for all lessons with them so as to save time and not have students waiting while they returned to the staffroom.

In this and other cases there was a clear message being directed to students: their teachers valued mathematics, valued teaching mathematics, and valued the time provided to mathematics. Further, this implied that the time spent on Mathematics was important and teachers would do all they could to maximise this time. Students came to accept the importance of time. At the staff forum a teacher commented on this:

> When I came here my first problem was I'd walk in and would run out of work–what took 40 minutes at my former school took 20 minutes here so the implications of the students being good is that you have to change your style of teaching and I think that is characteristic of us here. We probably all come from different backgrounds and changed our style of teaching to suit the school … Kids come with a lot and we have to add to it. (Teacher)

Time-on-task was considered a vital factor in helping students achieve their best. This was communicated to students in many ways both explicitly and implicitly. Nothing seemed more powerful in getting the message across to students than the teachers' role-modelling this practice.

Assessment as a Catalyst for Teacher Cohesion

The faculties invariably had a well-developed testing regime. Some had formal half-yearly and yearly examinations that commenced with students in Year 7. Regardless of the type and formality of the testing, the faculties appeared to use the testing/assessment process for a variety of purposes. For students, the testing regime served to provide a catalyst to assist them in developing and consolidating their understanding. It also enhanced their skills, expectations, and preparation for examinations, revision techniques for examinations, and the establishment of regular patterns of study. Interestingly, students viewed this positively.

For teachers the testing process was different. It was to identify students' abilities, what they had understood, and how they were proceeding in comparison with their peers. However, tests were also used as a basis to discuss with colleagues the effectiveness of teaching various topics. They helped provide a focus on pacing lessons and illuminating different emphases that teachers had placed in their teaching. These tests were seen as helping identify and better understand the major issues, what were important subsidiary ideas, and the development of questions that would elicit greater student understanding.

Most faculties had elaborate and collaborative setting and marking plans for tests. Sometimes teachers not currently teaching a course were required to produce tests. In other faculties, teams of teachers teaching a particular course would collaborate, often with teachers within a team taking on different roles. Regardless, of the organisational structure, tests were carefully scrutinised. This would involve a focus on the wording, the breadth of content, the overall standard, and the marking scale. It was important in these faculties that there was consistency and that all tests were set to a high standard.

Quick feedback on student performance was also a feature. Papers were invariably returned very soon after the test. Feedback varied from school to school, but in general there was a focus on how the test had addressed syllabus outcomes and also what students needed to do to ensure a maximum score for each question. As classes were invariably streamed, this enabled students to see where they were tracking in comparison with their peers. Usually, in the case of substantial tests, this ranking resulted in some students being allocated to a different class. The argument was that this reorganisation allowed students to work with peers at their level. This would encourage greater and more relevant on-task learning time as students within the same class would more likely be at similar learning points.

In practice there was some flexibility in this process and students were often moved or retained within a specific class out of consideration of social and/or personal factors. The overriding consideration was: "What was in the best educational interests for a particular student." In reaching decisions most Faculties involved parents and students.

Students appeared to respond positively to class movement based on test results, supporting teachers' view that this action had a motivating effect on students. They saw the outcomes as "fair" and in the case of those demoted, they spoke how they had the chance to return to their class if their results improved.

Clear Mission of High Expectations

> A lot has to do with the kids. The kids are on the whole studious, value an education, and they're concerned about their progress and that makes a big difference … The support you get from the families … If they are away for a day and miss something then they worry about what they have missed. Not like other kids who say "hooray I have missed something", they worry about it. (Head Teacher)

> The environment provided by the maths department teachers helps her (daughter) learn. (Parent)

These two quotes are illustrative of how teachers and parents attributed reasons for the exceptional mathematics performances in their school. A common theme associated with these observations was mutual respect among all parties. Teachers acknowledged that students and their parents were a central reason for the results obtained, whereas parents and students saw the teachers as being the key.

> Every time you deal with a parent here it is usually a very pleasant experience because they are interested in their kids. We might send letters home saying the standard sort of thing, "We are worried about your child in this area" or "They haven't been doing their homework" … Their response is usually positive. (Head Teacher)

Teachers spoke of students in their school being well motivated and that they came to school and to class willing to learn. The students appreciated the care and support provided by the mathematics teachers and they cooperated accordingly.

It was evident that a situation was established in these schools whereby teachers were assisted by students' commitment to learning and their desire to achieve. Students at all schools generally agreed that it was easy to get help from their mathematics teachers. One Year 8 student, when asked if she could go to the staffroom to ask for help, commented: "In class she asks if we have problems you only have to put your hand up."

Who has contributed most to success – the teacher, student or parent? The answer to this question is irrelevant. The key feature is that all three stakeholders are moving in the same direction. In these exceptional faculties there have developed over time a culture of success and achievement for students at all ability levels. Teachers, students and parents are all swept along with it. That each group takes pride in recognising the efforts of others is simply one manifestation on this shared commitment.

Caring for Students in their Learning

> In the maths area, I just think good pedagogy goes on up there. They have a fantastic concern for kids, all of them. In the school we have very few problems in the Maths area because they get on with kids and they work very well with them. I mean their programming and all that sort of stuff would be similar to what might occur in other Faculties. There is nothing innovative from what I have seen anyway, it is sequential as I would have seen other Maths Faculties. But I just think the personnel and the leadership and the way they get on with kids and their care for kids are very important factors as far as I am concerned. (Principal)

The Project Team was impressed by the strong student focus of the mathematics faculties visited. Policies and actions had a clear student focus. These policies, developed through ongoing discussion about student matters in the staffrooms were extensive. Clearly, teachers saw their role as helping students whenever they could.

Teachers reported genuine enjoyment in teaching their classes. They had developed a strong rapport and what appeared to be healthy relationships with students. There was a nice balance of formality and informality. At a personal level, the students saw their mathematics teachers as approachable and available to offer assistance.

It was obvious that teachers cared for their students' learning and encouraged students to approach them if they were having difficulties. Teachers were happy to make themselves available at breaks to assist students who came to the staff room. The staffrooms were welcoming places for students and many commented upon how they were encouraged when they went there for additional help. When they did ask for help, they found the teachers to be supportive, patient, and helpful.

At these schools it was common to see students from all years and at different ability levels at the mathematics staffroom seeking help. Seeing students across the full age range requesting help and being supported seemed to have a positive effect on all students. In particular, it was seen as beneficial to those with low self-esteem and belief in themselves. The practice of having staff readily available for help meant that students were aware that there was the backup in the faculty to support them and help them believe that they could be successful.

In considering the findings presented in this section it is important to interpret them in relation to the research questions and design. Essentially, only schools that demonstrated a sustained record (over 4 years) of outstanding achievement for students across all ability levels were targeted. Subsequently, there are particular findings of the study that are a direct consequence of the methodology employed. For example, there is no implication that the teaching practices observed were always "cutting edge", innovative, or exemplary and that the sites visited could not improve their practices. Answers to questions concerning what teaching practices display these qualities and how schools, who are achieving outstanding outcomes, improve further are interesting but they lie outside the scope of this particular study. Consequently, caution is required in generalising or extending the findings beyond what the research set out to achieve.

Conclusion

It is clear from these findings that these outstanding faculties have evolved over time and have developed a strong academic and educational culture in their schools. The mathematics teachers in these sites realised there was no opportunity for "resting on their laurels" with continued effort required to maintain these high standards.

An Exceptional Schoolings Outcomes Project (ÆSOP) has provided substantial evidence of excellent mathematics teaching in NSW public secondary schools in Years 7-10. The overriding challenge is how the insights generated by this study can improve the educational achievement of students across the public education system. It also highlights a number of potential important issues for schooling into the future around the need:

- To provide opportunities to help teachers develop the knowledge and skills necessary to exercise effective leadership in the role of Head Teacher;
- For early career teachers to work with and learn from experienced mid and later career teachers;
- To facilitate strong group interaction within faculties;
- For relevant professional development;
- For high subject-knowledge standards for new and current teachers;
- To create a culture in which teaching and learning, rather than behaviour management, dominates all classrooms; and
- To develop common goals among teachers, students, and families.

Acknowledgements. This project is supported through funding by the Australian Research Council and the NSW Department of Education and Training No. C00107053. The views expressed are those of the authors and do not necessarily represent the views of the Australian Government Department of Education, Science and Training (DEST) or the NSW Department of Education and Training (DET).

References

Australian Council for Educational Research. (2004). *Investigation of Effective mathematics teaching and learning in Australian Secondary Schools.* Camberwell: ACER Press.

Ayres, P., Sawyer, W., & Dinham, S. (2004). Effective teaching in the context of a Grade 12 high-stakes external examination in New South Wales, Australia. *British Educational Research Journal, 30*(1), 141-165.

Bennett, N. (1999). Middle management in secondary schools: Introduction. *School Leadership and Management, 19*(3), 289-292).

Busher, H., & Harris, A. (1999). Leadership of school subject areas: Tensions and dimensions of managing in the middle. *School Leadership and Management*, *19*(3), 305-317.

Goodson, I., & Marsh, C. (1996). *Studying school subjects.* Washington, DC: The Falmer Press.

Hannay, L. M., & Ross, J. A. (1999). Department heads as middle managers? Questioning the black box. *School Leadership and Management*, *19*(3), 345-358.

MCEETYA (1999). *The Adelaide Declaration on national goals for schooling in the twenty-first century.* Retrieved 9th April 2006, from http://www.dest.gov.au/sectors/school_education/policy_initiatives

Richards, L. (2002). Using N6 in qualitative research. Doncaster, VIC: QSR International Pty.

Sammons, P., Thomas, S., & Mortimore, P. (1997). *Forging links: Effective schools and effective departments*. London: Paul Chapman.

Spillane, J. P., Halverson, R., & Diamond, J. B. (2001). Investigating school leadership practice: A distributed perspective. *Educational Researcher, 30*(3), 23-28.

Teachers Research their Practice: Developing Methodologies that Reflect Teachers' Perspectives

Ruth Pritchard
Victoria University of Wellington
Te Whare Wānanga o te Ūpoko o te Ika a Maui
<ruth.pritchard@vuw.ac.nz>

Linda Bonne
Redwood School, Wellington
<lindabonne@redwood.school.nz>

In this study eight primary school teachers formed partnerships with researchers to investigate the use of questioning during two numeracy lessons. The teacher researchers were encouraged to act as reflective practitioners (Schön, 1995) and contribute to formulating their own "interpretive frames" (Cochran-Smith & Lytle, 1990). Methods of data-gathering, analysis and interpretation were developed to allow the teacher researchers to have control over the research and contribute to the direction of the project as it evolved. This paper describes some of the challenges faced by both the teacher researchers and the researchers in developing partnerships. It also discusses how the methodologies allowed teachers' views about practice to be made explicit. Teachers gained insights into the complexity of their teaching practices and described ways in which the experience impacted on their views of research as a bridge between theory and practice.

Theoretical Perspectives

The Nature of Teacher Research

Many teachers have the perception that research in teaching is "an esoteric activity having little to do with their practical concerns" (Carr & Kemmis, 1986, p. 8) and regard the creation of a knowledge base for teaching as belonging to the domain of university academic researchers. Recent prevailing concepts of the teacher "as technician, consumer, receiver, transmitter, and implementor of other people's knowledge" (Cochran-Smith & Lytle, 1999, p. 16) have perpetuated this perception, fuelled by the way in which "some consider the kind of knowledge that teacher research produces to be inferior to, and less valuable than, other kinds of academic work" (Roulston, Legette, DeLoach, & Buckhalter Pitman, 2005, p. 182). Cochran-Smith and Lytle (1990) describe the gap that has occurred:

> What is missing from the knowledge base for teaching, therefore, are the voices of the teachers themselves, the questions teachers ask, the ways teachers use writing and intentional talk in their work lives, and the interpretive frames teachers use to understand and improve their own classroom practice. (p. 2)

Gould (2005) has identified the need to reduce the "gap" that exists between research and practice in classrooms. Approaches that encourage teachers to carry out their own research in the context of their own classrooms, with the support of researchers, serve to validate their perspectives and enable greater insights into the complexities of teaching and learning. Teacher research, defined as "a systematic and intentional inquiry carried out by teachers", represents a "significant way of knowing" about teaching (Cochran-Smith & Lytle, 1993, p. 43). Traditional views about the relationships of knowledge and practice, and the roles of teachers in educational change are challenged, "blurring the boundaries between teachers and researchers, knowers and doers, and experts and novices" (Cochran-Smith & Lytle, 1999, p. 22). Such approaches can produce opportunities for a "hybrid discourse" between practitioners and university researchers based on "democratic research relationships" (Paugh, 2004) resulting in increased learning for both partners, and significant contributions to the knowledge base of teaching.

By participating more substantially in research, teachers develop their own skills as researchers, which are more likely to impact on their practice: "Experienced teacher-researchers become the high risk-takers we need to develop innovative practice" (Mitchell, 2002, p. 253). This may, in turn, encourage other teachers to examine more closely their own pedagogical practice: "Teachers may be influenced to change their practices more readily by reading reports of research by other teachers … rather than university researchers" (van Zee & Minstrell, 1998, p. 792). However, establishing suitable conditions and productive partnerships for effective teacher research is problematic. Difficulties with teacher research are discussed by a variety of writers and include: issues of power and ownership, access to resources, isolation, and possibilities for manipulation and exploitation (Cochran-Smith & Lytle, 1993, 1999; Mitchell, 2002; Paugh, 2004). Dissemination of teacher research has also presented problems. In their investigation of the ways such research had impacted on schools, Berger, Boles, and Troen (2004) found it difficult to find schools where teacher research was making a difference to the teaching and learning culture of an entire school.

Methods of Research into Questioning

Much of the research undertaken to investigate teachers' questioning has been synthesised from data gathered by researchers observing in classrooms, rather than from teachers themselves. A review of comprehensive research syntheses (Houston, Haberman, & Sikula, 1990; Richardson, 2001; Sikula, Buttery, & Guyton, 1996; Wittrock, 1986) did not reveal any studies deeply grounded in teachers' perspectives. The existing knowledge base reflects a looking from the "outside in". A search of the literature located studies that reported teachers' questions and questioning, but few investigations were identified that looked from the "inside out". Up until now, categorisations of teachers' questions in mathematics have predominantly been undertaken by researchers focussing on only a selection of the questions asked by teachers during a lesson. Perry, VanderStoep, and Yu (1993) coded questions about addition and subtraction asked in 311 lessons in Japan, Taiwan, and the United States. They deliberately excluded questions they deemed nonmathematical or questions that were asking for agreement. Vale (2003) devised question categories to accommodate the question types teachers nominated they used most often. Some research has allowed for categorisation of questions by general intention rather than "type" (Morgan & Saxton, 1991), allowing for a focus on the function of a question rather than form (Cazden, 2001). Other researchers have observed "expert" teachers and synthesised how questions can be used in mathematics lessons to develop students' thinking (Fraivillig, Murphy, & Fuson, 1999; Jacobs & Ambrose, 2003). Each of these categorisations was devised by researchers or observers rather than by the teachers from within the lesson.

Formulating questions within a lesson is a complex process driven by a range of variables, and analysis of this process requires more than categorising and counting by researchers: "Real insight into questioning needs to take on board contextual factors which are too subtle for the classification systems to handle" (Kerry, 2002, p. 71).

Method

The eight teacher researchers (TRs) gathered data in two cycles, each taking five consecutive days in each of the middle two terms of the four-term school year. In each cycle they recorded, categorised and analysed their use of questioning within a numeracy lesson. To assist the analysis and interpretation of their findings, the teachers discussed aspects of their findings in individual interviews with a researcher, and then as part of a forum with the other TRs. They also had opportunities to examine current research in this area, reflect on aspects of the research process and contribute to report-writing. Over the course of the project, data were also collected by the researchers who took the role of Research Team Leaders (RTLs). These data related to the TRs' involvement and experience of the research, with the processes for its collection emerging as the project unfolded.

Overall, the data were analysed using the three main stages of data reduction, data display, and drawing and verifying conclusions (Miles & Huberman, 1994). Most of the data collected were qualitative. The qualitative information was considered alongside the quantitative data to identify similarities and differences. The RTLs met following interviews with the TRs to share and compare findings, sorting responses using the same sorting process that the TRs had used in their initial data analysis. This enabled themes to emerge and helped to reduce the collected data to its key elements. The reduced data were then displayed to help identify trends. Responses to various questionnaires given to the TRs throughout the project were compiled to support the identification of key ideas. TRs contributed to the process of interpreting findings at all stages of the project by responding to summaries of emerging ideas presented by the RTLs. The TRs also interpreted their findings in light of current research, which they discussed in a group meeting. The RTLs verified their interpretations of the data with the TRs by feeding speculations back to them at research team meetings for discussion and comment. These methods reflected a grounded theory approach, such as that described by Strauss and Corbin (1998).

Processes for Data-gathering and Analysis

At the introductory meeting of the research team, the roles of team members were clarified, the research aims for the project were shared, and interview questions were negotiated. Processes for data-gathering were discussed by the team, and the "F-sort" (Miller, Wylie, & Wolfe, 1986) data categorisation method was examined. This method allowed teachers freely to generate their own categories for their questions, and provided access to the teachers' ideas and language about categories of questions from the outset of the project.

Within each of the cycles, the TRs recorded two consecutive mathematics lessons, and chose one to analyse. To enable the TRs to have maximum control over the data-gathering process, the TRs themselves were responsible for setting up the technology for the recording procedures. This ensured ownership of the process – no one else was "present" in their classroom. The technology comprised a video camera that remained in one position throughout the lesson, and a "Notetaker" cassette recorder with built-in microphone, which they wore around their necks. After the second lesson the TRs sent the audiotapes of their chosen lessons to be transcribed, which were returned a day later. Only the audio recordings were transcribed and access to these transcripts was restricted to the teachers concerned, the transcriber, and the two RTLs. The TRs were subsequently released from teaching for 2 days to analyse their lesson using their reading of the transcript, assisted by viewing the videotape footage, alongside their recent recollections of the lesson.

The main activity in the analysis phase involved the identification and categorisation of questions within the lesson. This was achieved by extracting the TRs identified questions from hard copies of their transcripts, then sorting them into groups of similar questions for which they devised labels (Miller et al., 1986). At the end of the second day of analysis, the TRs discussed their findings with one of the RTLs in semi-structured, one-to-one interviews (Denscombe, 1999). Summaries of the interviews were later sent to the TRs for verification, and findings were shared in subsequent group meetings.

Group meetings were a key aspect in distilling meaning from findings as they emerged throughout the project. Members of the team brought aspects of their findings to share, and similarities and differences were explored and debated. The Cycle 1 group discussion began the process of establishing common categories with which to analyse the lesson in Cycle 2. The TRs were also asked to record any questions and issues arising from the analysis of their first transcript. Their responses were to be used to inform the future direction of the project. Throughout the project teachers responded to questionnaires that explored their perspectives on aspects of the research process. The TRs were unable to be involved fully in writing the final report of the research project. Instead, they wrote reflective responses to the final questionnaire, and these responses were used to amplify the TR's voice in sections of the report.

Findings

Ownership of the Research and Roles of the Research Team

In the initial stages of data-gathering and analysis, some of the TRs described difficulty with the sorting of questions into categories. At this early stage, the TRs tended to draw on frameworks and language about questioning that were familiar to them. In some cases they struggled to produce efficient descriptors from their own language to label groups of questions. Perhaps this indicated the TRs' doubts that what they had to say would have validity or authority in the research project. The TRs may have seen the research in traditional terms such as those described by Cochran-Smith and Lytle (1993) as "outside-in", or as research that "constructs and pre-determines teachers' roles in the research process" (p. 7). The process of sorting their questions had meant that the TRs were encouraged to take responsibility for generating language and ideas, and the commonly agreed categories developed within the forum reflected their own language, which promoted a sense of ownership.

An important principle of teacher research is that teachers have a "sense of ownership and control of their research" (Mitchell, 2002, p. 250). Current definitions of teacher research describe the selection and development of research questions as emerging from the teachers' own practices (Cochran-Smith & Lytle, 1993). Although each of the TRs joined the team with an awareness of the field they were to research, the requirements for the funding for this research had meant that the research questions and aims were established before they met together as a team. However, the research questions had emerged from close links to teaching practice that the RTLs had developed, both in their current and recent classroom teaching experience, and in the considerable number of mathematics lessons they had observed as numeracy advisers.

The RTLs' sense of ownership was strong at the onset of the proposal process as initiators of the research questions and the methodology. This diminished as the proposal progressed and as the three institutions involved established areas of territory and accountability. Ownership was further dispersed as the RTLs continued to work with the TRs. It became apparent that the RTLs had begun the project expecting significant but limited input from the TRs rather than an authentic partnership. Thus, to ensure the development of research capabilities of the TRs, and to increase validity of findings, it was felt necessary to share aspects of control of the project. This was not easily achieved, as the TRs demonstrated differences in perceptions of their role and the RTLs' role. Perceptions of roles were further complicated by the relationships previously established by the RTLs as mentors and advisers within the context of in-depth professional development. It would seem that the co-researcher relationship "was infiltrated by the discursive positionings more in common in relationships between academics and teachers, or teachers and students" (Honan, 2007, p. 622).

The Changing Nature of the Methodology

Aspects of the methodology were continually adjusted to allow the TRs to develop a greater sense of control within the project.

> The approach was good because it was flexible and allowed the group to have true ownership. The "organic" nature of the form of our meetings allowed researchers to listen without taking over with pre-determined paths. (Erin, Final questionnaire)

In some respects this flexibility paralleled the way the TRs responded to their students, changing direction and transferring power within their classroom practice:

> One thing I've really enjoyed about the research, is that it's just confirmed for me a lot of good teaching practice … It's made me be a little bit more relaxed about letting the children take control. (Erin, Interview 2)

It was originally intended that the RTLs would conduct an analysis of each lesson at the same time as the TRs, reading the transcript and viewing a video of the lesson. Their analysis would then be compared with the TR's findings. However after the initial trial phase, it was decided that the TRs would be solely in charge of the analysis process. This meant that the TRs' own observations and views on their lessons were paramount. Feedback from the Trial teacher shifted the focus of the interview from a comparison of findings to a vehicle for assisting the TRs' reflective processes.

An important aspect of developing the teachers' capability as researchers was introduced between the two cycles of data gathering. At the suggestion of the research consultant, relevant research readings were sent to the TRs for discussion at the upcoming meeting. The themes for these readings were established in response to ideas emerging throughout the interviews and in the second research team meeting, and were also directly indicated by the TRs in their responses to questions and issues arising from the analysis of Transcript 1. An additional day was allocated to discuss these and other relevant themes, to enable the TRs to see their current research in the context of other research in this area.

Moves to incorporate the TRs' voices more prominently in the writing aspects of the research included the use of a final questionnaire. This allowed them opportunity to review the research outcomes and processes, and contribute reflective and crafted responses that could be incorporated into the report. The style of the report reflected the partnerships developed in the project, by aligning the RTLs' contributions, observations and interpretations alongside those of the TRs'. This made visible the key role the TRs had throughout the project by anchoring interpretations of findings in their statements. A draft of the findings was shared with the TRs for their editorial comment before publication.

Developing Community and Accessing Support

The research team meetings were important in refining the methodology and allowing the research team to discuss and interpret findings. They contributed toward establishing a shared understanding of the research question, served to generate common categories for coding questions, and assisted the TRs to establish a common interpretation of findings. These forums also provided the collaborative support necessary for such projects as described in Mitchell

(2002). Mitchell notes the loneliness often experienced in such studies, which was also identified within our project.

> For the first release days I felt isolated and completely lost. (Ingrid, Final questionnaire)

At times, interactions at the research team meetings caused concern. The fact that three of the teachers were drawn from one school, and knew each other well, may have impacted on the group dynamics. Moves to incorporate the views of all the team members more fully included the provision of extra meetings and the use of strategic groupings and activities within group meetings.

An awareness of the issue for the TRs of managing their research project commitments along with teaching workloads was evident throughout.

> The amount of time involved was underestimated and at times it got stressful with other demands of work. (Stephanie, Final questionnaire)

It was often apparent that the teachers felt a tension between the demands of undertaking the research and being present in their classrooms. Cochran-Smith and Lytle (1993) note that:

> Participation in teacher research requires considerable effort by innovative and dedicated teachers to stay in their classrooms and at the same time carve out opportunities to enquire and reflect on their own practice. (p. 20)

Oliver (2005) found that school support was a significant factor in the success of teacher research projects. Responses to a questionnaire given to the TRs midway through the research described a full range of support from the teachers' schools. External systemic support (Osler & Flack, 2002) was also essential to the project. Money allocated from funding provided through the research funding allowed the teachers to have release time to analyse their lessons in detail, and to attend meetings.

Links to Practice

The research process was seen as providing significant relevance and immediate impact on the TRs' own classroom practice.

> I have developed an awareness of the types of questions that I can use ... the research has helped to identify a specific area of focus and thought and therefore it must have an impact back in the classroom. (Quentin, Final questionnaire)

> This has identified needs and gaps in my questioning and there have been surprises in other areas. (Olivia, Final questionnaire)

The TRs also described possible directions for further research about their own practice.

> Maybe the biggest question for me personally is how to take the information I have now about my questioning and find practical ways to implement change in the class. Maybe I need to do more reading about that. (Olivia, Final questionnaire)

> It would be interesting to look again at the types of questions asked at which part of the lesson. ... Are there any significant shifts in the types of questions asked? (Stephanie, Final questionnaire)

Although early on in the project the TRs recognised that this research should be able to inform the wider teaching community, at the conclusion of the project it was felt that the research process itself, rather than their findings about their use of questioning, was what they considered significant.

> Having the opportunity to micro-analyse within a subject area has heightened my awareness of the strengths and weaknesses of my own classroom practice. This in turn has challenged me to either strengthen those practices that are valuable and to adjust/ improve those practices that are weak. (Erin, Final questionnaire)

The TRs found it difficult to be specific about exactly how the research findings relating to questioning might be applicable to teachers in general. The categories were seen as useful to the teachers involved in the project, as they had created them and "owned" them. There was a lack of confidence that other teachers would find them useful.

> We need to be careful with transferring research to their [other teachers'] situations – qualify it with the fact that it is for "here and now" and may be less relevant when different factors are taken into account. (Ursula, Final questionnaire)

> This research was done by a small group of teachers. What are the implications for other teachers? How would it transfer across to other teachers? (Natalie, Final questionnaire)

Perhaps this reflects findings from Mitchell (2002) who noted: "TRs are more interested, at least initially, in finding what may appear to be context-specific solutions in their own classrooms" and that many aspects of the research process are personal: "in some important ways, the journey is experiential – some parts of the story cannot be told, they must also be experienced" (pp. 262-263).

Changing views of research

Osler and Flack (2002) found that skills to be developed by TRs included: "reflection, articulation, familiarity with research literature, linking their own work to the work of others, writing, and presentations" (p. 243). The development of each of these skills was in evidence in various forms throughout the project. The developing capability of the teachers as researchers was reflected in their changing views about the nature of research. The ability to reflect on and articulate their practice was evident.

> It is a huge learning curve because you see things from a different perspective. (Quentin, Final questionnaire)

Research was seen as a vehicle for sharing, challenging, or confirming existing ideas and introducing new ones. One aspect described by the TRs was the complexity and scale of the research process.

> Research is fascinating when you are involved in it!! It is really difficult to do. [There are] heaps of factors to consider. It doesn't always give us answers. (Ursula, Final questionnaire)

> It has been fun, scary, challenging and time consuming... I realise how much work goes into these projects. (Olivia, Final questionnaire)

Throughout the research, areas for future investigation continually arose. At the completion of the study, a range of diverse questions for further research had emerged from the group. Some major shifts in understanding about research were also evident.

> When we first started out I was not sure of what I was getting into and therefore my mind was a bit of a blank slate. I think there is a definite need for teacher research to continue as it informs practice and changes views and brings together your own personal experiences which must be better for your classroom. (Quentin, Final questionnaire)

The TRs have been encouraged to present and discuss the findings and methods of the research with their staff to contribute to developing a culture of inquiry within their schools.

Aspects of the research process were presented by TRs and teacher researchers at one regional and two national conferences. This has further contributed to the development of teacher researcher skills and enabled the research partnerships fostered during the project to be made visible.

Conclusions

> Research doesn't always provide you with answers. It often provides more questions. There isn't always a neat, tidy conclusion that can be drawn. (Natalie, final questionnaire)

Participation in this project impacted on the teacher researchers' views of the relationship between research and practice and provided opportunities to reduce the gap between them. Throughout the project, the teacher researchers encountered authentic research problems regarding methodology, analysis, and interpretation of data as they sought to make meaning from data gathered. The process of researching their own teaching practices served to transform the apparent simplicity of the task of identifying and categorizing questions, to a complex undertaking that confronted the teacher researchers with some of the essential elements implicit in their everyday teaching. This acted to problematise rather than simplify the teaching process.

The unique perspectives of these teacher researchers about questioning provide a valuable contribution to the knowledge base about teaching in this area. The use of the interview and team forums compelled the teacher researchers to articulate their practice more precisely, and to discuss and debate related issues. The process of close analysis and discussion of their teaching practice was an outcome valued by the teacher researchers, which they saw as useful for other teachers. However, it was difficult for them to assess the value of their observations about the questions they asked and the categorisations they devised; they seemed unsure of the validity of their findings, perhaps due to the lack of sufficient time to explore fully patterns and commonalities that may have been present in their questioning practices.

The structure of the initial research design was significant in developing the TRs' confidence and capabilities in research, as it scaffolded the data-gathering and analysis process. This structure allowed the TRs maximum control over the selection of the primary level of data to be analysed, and opportunities for in-depth reflection. Important features that contributed to the success of this process were:

- the use of accessible technology, which the TRs controlled,
- the lesson transcript being made available to them within a short timeframe,
- the interaction between the printed transcript and the video,
- the inductive categorising process used,
- having immediate and concentrated time for analysis, and
- discussing their findings with a RTL in a reflective interview.

A key feature of this study was the ability for the Research Team Leaders to be responsive to the input of the team members as the research progressed. Respecting their contributions and interpretations was imperative, and this was firmly established by making teacher researchers solely responsible for the initial stages of data-gathering and analysis. This ensured their interpretation of data was central to the project and established a sense of trust in the developing research partnerships. The researchers had greater time for reflection and interpretation of findings which meant they initiated much of the direction for the research. Although this was necessary, it created a tension within the project, as the teacher researchers had only a limited time available for these activities. This meant that the balance of "power" within the partnerships, and the responsibility for the direction and the interpretations of findings were aspects of the project that were constantly negotiated.

Implications

Support for further research that includes the teachers' perspectives in the analysis of teaching practice is vital. To allow teachers to develop the research skills necessary to contribute their perspective in a meaningful and rigorous manner, teacher researchers need to be provided with:

- sufficient release time to examine their practice in depth, and to attend research meetings,
- access to experienced researchers for support and guidance,
- research forums for discussing ideas with other teacher researchers, and
- interest and encouragement from management and colleagues within their schools.

Research questions that originate from teachers themselves can contribute to a closer alignment between research and practice. To enable them to have authentic ownership of research questions, involvement in the earliest stages of a research project needs to be encouraged. Teacher initiation of such proposals could be promoted by the inclusion of a research component into teachers' job descriptions. Consideration also needs to be given to methods that enable teachers to have maximum ownership of processes throughout.

> I had the impression research was often done by a researcher to you, however this has shown that it can be embedded in your practice and the research can be for you. (Natalie, Final questionnaire)

Acknowledgements. The project was reliant on our team of teacher researchers working collaboratively with us and with each other. We are grateful to them for their professional approach throughout the project and to their schools for supporting their participation. This research project was funded by the Teaching and Learning Research Initiative at The New Zealand Council for Educational Research.

References

Berger, J. G., Boles, K. C., Troen, V. (2004). Teacher research and school change: Paradoxes, problems and possibilities. *Teaching and Teacher Education, 21*(1), 93-105.

Carr, W., & Kemmis, S. (1986). *Becoming critical: Education, knowledge, and action research.* London: Farmer Press.

Cazden, C.B. (2001). *Classroom discourse: The language of teaching and learning* (2nd ed.). Portsmouth, NH: Heinemann.

Cochran-Smith, M., & Lytle, S. L. (1990). Research on teaching and teacher research: The issues that divide. *Educational Researcher, 19*(2), 2-11.

Cochran-Smith, M., & Lytle, S.L. (Eds.). (1993). *Inside outside*. New York: Teachers' College Press.

Cochran-Smith, M., & Lytle, S. L. (1999). The teacher research movement: A decade later. *Educational Researcher, 28*(7), 15-25.

Denscombe, M. (1999). *The good research guide*. Buckingham: Open University Press.

Fraivillig, J. L., Murphy, L. A., & Fuson, K. C. (1999). Advancing children's mathematical thinking. In J. Sowder & B. Schappelle (Eds.), *Lessons learned from research* (pp.37-38). Reston, VA: National Council of Teachers of Mathematics.

Gould, P. (2005). From the hill to the swamp. In P.Clarkson, A. Downton, D. Gronn, M. Horne, A. McDonough, R. Pierce, & A. Roche (Eds.), *Building connections: Research theory and practice.* (Proceedings of the 28th annual conference of the Mathematics Education Research Group of Australasia, Melbourne, Vol 1, pp. 50-53). Sydney: MERGA.

Houston, W. R., Haberman, M., & Sikula, J. (Eds.). (1990). *Handbook of research on teacher education*. New York, Macmillan.

Kerry, T. (2002). *Explaining and questioning*. Cheltenham: Nelson Thornes.

Jacobs, V.R., & Ambrose, R.C. (2003, April). *Individual interviews as a window into teachers' practice: A framework for understanding teacher-student interactions during mathematical problem solving.* Paper presented at American Educational Research Association Annual Meeting, Chicago, Il.

Honan, E. (2007). Teachers engaging in research as professional development. In T. Townsend & R. Bates (Eds.), *Handbook of teacher education: Globalisation, standards and professionalism in times of change* (pp. 613-624). Dordrecht: Springer.

Miles, M., & Huberman, A. (1994). *Qualitative data analysis* (2nd ed.). Thousand Oaks, CA: Sage.

Miller, D.M., Wiley, D. E., & Wolfe, R. G. (1986). Categorization methodology: An approach to the collection and analysis of certain classes of qualitative information. *Multivariate Behavioral Research, 21*(2), 135-167.

Mitchell, I. (2002). Learning from teacher research for teacher research. In J. Loughran, I. Mitchell, & J. Mitchell (Eds.), *Learning from teacher research* (pp. 249-266). New York: Teachers' College Press.

Morgan, N., & Saxton, J. (1991). *Teaching, questioning and learning*. London: Routledge.

Oliver, A. (2005). *The TLRI: Teachers' perspectives on partnership and research*. Wellington: New Zealand Council for Educational Research.

Osler, J., & Flack, J. (2002). Tales from the poppy patch. In J. Loughran, I. Mitchell, & J. Mitchell (Eds.), *Learning from teacher research* (pp. 222-245). New York: Teachers' College Press.

Paugh, P.C. (2004). "Making room": Negotiating democratic research relationships between school and academy. *Teaching Education, 15*(2), 215-227.

Perry, M., VanderStoep, S. W., & Yu, S. L. (1993). Asking questions in first-grade mathematics classes: Potential influences on mathematical thought. *Journal of Educational Psychology, 85*(1), 31-40.

Richardson, V., (Ed.). (2001). *Handbook of research on teaching* (4th ed.). Washington, DC: American Educational Research Association.

Roulston, K., Legette, R., DeLoach, M., & Buckhalter Pitman, C. (2005). What is 'research' for teacher-researchers? *Educational Action Research*, *13*(2), 169-189.

Schön, D. (1995). *Reflective practitioner: How professionals think in action*. Aldershot, England: Arena.

Sikula, J., Buttery, T. J., & Guyton, E. (Eds.). (1996). *Handbook of research on teacher education*. New York: Macmillan.

Strauss, A. L., & Corbin, J. (1998). *Basics of qualitative research: Techniques and procedures for developing grounded theory* (2nd Ed.). Thousand Oaks, CA: Sage.

Vale, C. (2003). Questions in primary mathematics classrooms. In L. Bragg, C. Campbell, G. Herbert & J. Mousley (Eds.), *Mathematics education research: Innovation, networking, opportunity* (Proceedings of the 26th annual conference of the Mathematics Education Research Group of Australasia, Geelong, Vol II, pp.688-695). Pymble, NSW: MERGA.

van Zee, E. H., & Minstrell, J. (1997). Using questioning to guide student thinking. *The Journal of the Learning Sciences, 6*(2), 227-269.

Wittrock, M. C. (Ed.). (1986). *Handbook of research on teaching* (3rd ed.). New York: Macmillan.

Teacher Professional Learning in Mathematics: An Example of a Change Process

Pauline Rogers
Monash University
<oonpared@bigpond.com>

Debate about changes in teachers' beliefs and attitudes about mathematics teaching leads us to understand that these changes result from a teacher's personal experience. Professional learning in its various forms is an attempt to change teachers' practices in the classroom, and hence influence student learning outcomes. The paper uses the responses of one mathematics teacher involved in a professional learning project to examine the relationship among the professional learning, classroom practices, and teacher beliefs and attitudes.

The Australian Government Quality Teacher Programme (AGQTP) is a federal government flagship initiative for supporting quality teaching and school leadership, with $300 million allocated to the programme to the end of 2009. The Department of Education, Science and Training (DEST) website states that the programme's "primary function is to fund professional learning activities for teachers under agreements with state and territory government and non-government education authorities" (DEST, 2007). The programme's two objectives are:

1. to update and improve teachers' skills and understanding in priority areas (literacy, numeracy, mathematics, science, information technology and vocational education); and
2. to enhance the status of teaching in government and non-government schools.

The programme was established in 2000 and since then more than "240 000 professional development opportunities have been taken up by teachers" (DEST, 2007, p. 1). This paper examines the impact of the project of one of those teachers involved in a professional learning project funded within the AGQTP programme.

The State of Victoria, Department of Education and Training states that "teacher professional learning can be defined as those processes and activities; formal and informal, designed to enhance the knowledge, skills and capacity of staff" (2007). This can include on-site or at school learning, which may involve formal activities such as mentoring and working in project teams or informal activities such as the involvement in school discussions about policy. Teacher professional learning may also take place off-site or as outside school learning such as conferences, workshops, on-line training, and modular programs over a period of time or network activities. The State of Victoria, Department of Education and Training conducts programs such as the Principles of Learning and Teaching (PoLT) and state "the initiative provides a structure to help teachers focus their professional learning" (2007). It aims to "capture the essence" of effective learning and teaching as well as providing a basis for teachers to review and develop their teaching practices. The teacher who is the focus of this report participated in a teacher professional learning model that occurred on-site at her school. The model is based around a mathematician in residence (in this case the author who acted in a role as a mathematics educator) providing a form of mentoring. The professional learning model consisted of three week-long visits spaced throughout a year in a rural Victorian primary school. Some background to the two key elements underpinning the professional learning, teacher change and reflective practices, are described in the following sections.

Teacher Change

Teacher professional learning programs are an attempt to bring about "change in the classroom practices of teachers, change in their beliefs and attitude, and change in the learning outcomes of students" (Guskey, 1986, p. 5). As presented in Figure 1, Guskey proposed an ordered framework to help better understand trends that appear to "typify the dynamics of the teacher change process" (p. 7). This model proposes that change is a learning process for teachers largely determined by their experiences in the classroom. If their experience results in a change in student learning outcomes Guskey proposes that the teacher then uses this to make a judgement about the effectiveness of their teaching. Guskey found when teachers see students attaining higher levels of achievement as a result of a new programme or professional learning then possibly, although not always, there is significant change in the teachers' beliefs and attitudes.

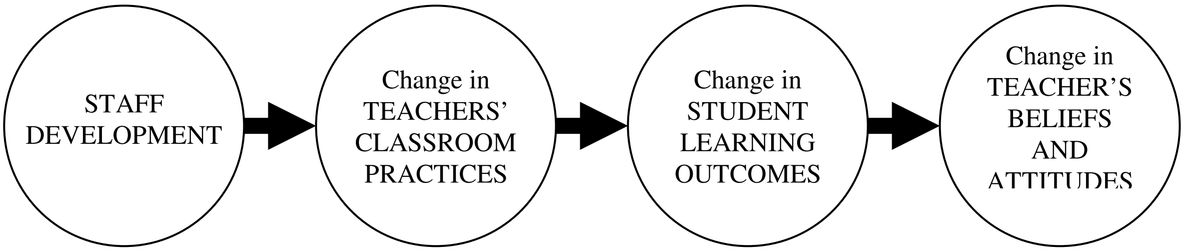


Figure 1. A model of the process of teacher change (Guskey, 1986).

This developmental sequence is reflected in other studies of teacher learning. Brown and Renshaw (2006) argue that change in teaching practices requires teachers to negotiate with past pedagogy, while maintaining the useful skills and techniques that work, and dispensing with the techniques that do not work. This change in teachers' pedagogical practice takes time (Guskey, 1986; Higgins, 2002, 2003; King & Newmann, 2001, 2004; MacGilchrist, Myers, & Reed, 1997). It seems that for change to occur in teaching, pedagogical professional learning needs to be on-going and requires continual support to be sustained.

This model, in which Guskey argues that teacher's beliefs and attitudes are developed largely from classroom experience, fits the process developed by the school in the presented case study. In the case of the school, it was felt by the principal that there was a need for teacher professional learning as a vehicle to improve students' Achievement Improvement Mentoring (AIM) test results and student engagement in mathematics. The teacher professional learning model developed by the principal and the mathematician in residence presented new ideas, theories, and activities for the teachers to try with the support of the mathematician in residence. The teachers tried these in their classrooms to see what happened, and in some cases there were perceived improvements in student learning and motivation. More ideas were tried, and classrooms as well as students observed. It was hoped that after a period of time (initially it was thought in excess of three years) change in students learning outcomes and perhaps teachers' beliefs and attitudes may be observed.

Reflective Practice

The second component of the process adopted by the school as key to the teacher professional learning model, relates to the reflective practice by the staff in professional partners with the support of the mathematician in residence. Pritchard and McDiarmid (2005) state that one of the key components within effective teaching and professional development is reflective practice. This is the deliberate act of "reviewing and critically thinking about practice with the purpose of increasing learning opportunities for students and teachers" (p. 433). McDuffie (2004) argued that reflective practice is distinguished from "thinking back" to a process which requires teachers acting on their reflections resulting from when a difficulty or a problem has been experienced. In the case study of the school, reflective practice was implemented with teachers, their professional partners and the mathematician in residence. It is these reflections that can lead toward the development of a professional learning community as teachers critically examine and reflect on their practice individually, in groups and as a whole staff. Reflective practice provided the support for teachers in this case study to make changes in their classrooms, and in this paper the case study of one teacher illustrates that it was through this reflective practice that the teacher found a tool in facilitating change. Willis (2002) states that "teachers need to learn how to analyse practice – both other teachers' practice and their own" (p. 2). The teachers in the project were provided with the opportunity to view other teachers and classes, as well as spending time discussing and reflecting with professional partners and the mathematician in residence about their own learning and teaching experience. Stigler (2002, as cited in Willis, 2002), argued that to analyse means one needs to think about the relationship between teaching and learning in a cause-and-effect kind of way. This is compatible with the Guskey (1986) model in which change in student learning outcomes is a result of a change in teacher's practices which in turn is a result of staff development or a teacher's own learning. Hence it can be interpreted that student learning is related to teachers and teacher learning.

The Professional Learning Project

The teacher professional learning project at the basis of this report is centred on an external critical friend termed in the project "mathematician" in residence, conducting teacher professional learning, visiting classes, observing specific lessons, teaching model lessons and team teaching with staff as required. The role expanded to include attending staff meetings and conducting professional learning sessions at these meetings. In this case mathematics was the focus, but as the project developed it is clear that it could be implemented in any or all subject areas. This particular AGQTP project involved three week-long visits to a school in rural Victoria spaced throughout the year. Rather than a random and ad-hoc approach during each of the visit weeks, a timetable was developed and teachers were paired with professional partners, which they selected from their peers. For each teacher, a half hour was spent with the external critical friend prior to each lesson to be observed, discussing the lesson and other concerns or interests regarding the teaching of mathematics. Then a lesson of approximately one hour was taught, with the external critical friend and professional partner viewing and participating as appropriate. After the lesson, a half an hour (or more) was spent reflecting on the lesson with the external critical friend and professional partner. The aim of the model was to promote teacher learning, with an individual focus on goal setting. This would hopefully lead to changes in the classroom which result in changes in student learning outcomes. A longer term goal was a change in teachers' beliefs and attitude may also be attained.

During the initial phase of the project one of the teachers called Belinda indicated her desire to change her teaching methods and practices in mathematics through her written and verbal comments. It is her journey of her perceived change that is presented in this paper – a single study within a larger study of ten primary school teachers. It is noted that there are significant practical and ethical issues associated with methods of researching change in teachers such as, what is the actual change, how has this change occurred and was this change permanent.

The data collected and presented in this paper are predominately written and verbal responses, as well as comments made by Belinda throughout the project as her process of change is examined. Methods included written observations, written comments and reflections by Belinda and transcripts from video footage as the author was involved in the methodology of design research. Sometimes these responses were prompted with questioning whereas other responses were of Belinda's own reflections. A survey of 25 statements adapted from Barell (2003), which required teachers to respond with a ranking of 1 – 5 (hardly ever – often) on a Likert scale, was implemented at the beginning and at the end of the project. In this paper the survey results are only used to support Belinda's comments.

One Teacher's Response to the Professional Learning Initiative

Belinda is an experienced teacher who has been teaching in excess of 15 years. She has taught at all levels from Prep to Year 6 and Belinda has been a leading teacher and acting principal at different times during her teaching career. She seems confident and involved in school life as is exemplified by her involvement in another project at the school that focused on rich assessment tasks and students expressing their learning and understanding. It is through this involvement that Belinda seems willing to learn and seeks opportunity to do so. From observation, she is a quiet and thoughtful member of staff, and is well respected by the staff and the principal who readily seek her advice.

The case study of Belinda attempts to examine the changes in Belinda's beliefs and attitudes in response to this particular AGQTP project. The data collected focussed on:

- Belinda's initial feelings about the project and how a mathematician in residence may impact on her teaching.
- What impact the professional learning had in terms of changing classroom practices?
- Was there a change in student learning outcomes?

Although the project had only been running for a year, an open mind was kept to see whether there was an indication of the above factors leading to a change in Belinda's beliefs.

Initial Response

At the beginning of the project, three questions were posed:

- How do you feel about having a mathematician in residence?
- How do you feel about having someone coming to view your teaching?
- Would you prefer to attend external professional development?

Belinda's responses to these initial three questions indicated that she supported the concept of having a mathematician in residence and was looking forward to the experience. Belinda wrote:

> a mathematician in residence would get to the nitty gritty of what was happening [in her classroom] and what was needed and hopefully real change and progress in both teaching and learning would be attained.

Belinda saw the process as being an advantage for the students by having "an expert on hand" and she was looking forward to "having someone who could watch the kids with me and help me evaluate both their needs and my teaching practices". She was not worried about having someone in her classroom viewing her teaching, although she did admit that she would be a "little uncomfortable, nervous and apprehensive" as she was not as confident in teaching mathematics and she was also returning to Year 5/6 after a number of years with younger students. Belinda felt that having someone in the classroom would allow the person to work with her and her students. She felt that "a mathematician in residence would get to the nitty gritty of what was happening and what was needed and hopefully real change and progress in both teaching and learning would be attained".

All teachers were asked to respond via email to the question "After the first week of the mathematician in residence, how do you feel about the project?" Belinda responded with:

> After a week working with Pauline [the mathematician in residence] I feel extremely positive about the project. Any apprehensions I had re Pauline watching my teaching proved false as she always concentrated on the positives and had heaps of suggestions on anything I asked about. She also followed up on things immediately and has already emailed suggestions. I have tried several already and can't wait to share them on her return. She was very insightful about the kids learning and was aware of the direction our school wanted to head, as directed in our charter. Pauline also took the lead from the teachers' concerns and needs. She became very much part of our team during the week, which was appreciated by everyone. I found it interesting to go into another teacher's classroom, to participate in a lesson as that is not a possibility often afforded to teachers. The discussion from that experience was also valuable for my teaching.

Belinda's initial feelings of apprehension were dispelled as she seemed to see the advantages of the mathematician in residence for her teaching. The professional learning Belinda had experienced in the form of a full staff development day, as well as the professional partner experience, particularly the reflection and discussion, had already resulted in change in Belinda's classroom practices as she attempted new ideas. These changes were positive as she was keen to share her experiences and continue the learning process. In other words, the discussion about practice, the practical ideas and the observations of the other teachers, all created an apparent openness for Belinda to consider her practice.

Developing Ideas

Throughout the project Belinda seemed open to ideas and tried many new activities with her class. She introduced partner tables games and table tests that the students make up themselves. She wrote “they love (tables) Bingo. I think they are improving their skills also.” Belinda was experiencing positive changes with the students as a result of changing her classroom practices. This was also reflected in her comment “their favourite topic for the term was BODMAS”. One student wrote “It was a whole new thing for me and I was good at it.” Belinda also wrote “I still need to work on negative attitude and motivation to Maths. Comments from the students such as ‘that wasn’t Maths it was fun!’ were offered.” Although some changes in classroom practice had occurred, Belinda still felt that changes in student outcomes were only developing. She wanted to see greater improvements in student attitudes across the whole class. Although Belinda had examples of positive student attitudes, she still felt that overall student motivation and attitude needed more time and perseverance. This is where Guskey’s model could be seen as cyclic as Belinda sought more ideas and strategies through professional learning as she set about making changes in her classroom practice in an attempt to improve student attitudes and motivation. Belinda’s personal goal for the year was student attitudes and this became part of the drive in Belinda’s professional learning. Belinda maintained focus on this goal throughout the year, and this is reflected in Belinda’s evaluations and reflection of her involvement in the project.

Reflection

Some of the best insights into Belinda’s experience were gained at the end of the project therefore, the final reflection component of the project is introduced here. The last day of the project was one of reflection and looking forward. Each staff member was asked to reflect on the project and their own learning. Teachers were asked to prepare a written piece to bring to the last session. Sentence starters and questions were provided or teachers could just write about their experience. Belinda wrote a piece not based on any of the prompters, and this proved to be a great insight into Belinda’s learning. Two main components of this reflective piece will be explored in this section: goal setting and Belinda’s reflection on her own teaching.

Goal Setting

Belinda found the goal setting component of the project helpful.

> It made me think about my teaching by setting goals, talking about why I set them and then putting them into practice.

This was a new experience for many of the teachers in the project as they were asked to set large project goals about their own learning, and Belinda’s was to motivate her students. She indicated that this was “still a work in progress”. This perhaps indicates that not all Belinda’s attempts at change were positive and new ideas were continually being tried. This is where Guskey’s model could be cyclic as staff development is on-going as teachers’ classroom practices are continually adapted and refined, leading towards a particular or desired change in student outcomes. As well as project goals, teachers were also asked to set lesson goals and Belinda found the lesson goals to be useful as “I feel I consistently set small goals for each lesson and am achieving them more consistently.” This goal setting is a change in Belinda’s practice as prior to the project: lesson and personal goals were not being set. The goal setting component of the project continued to provide aims and direction for teachers, providing them with motivation within the project. It was also hoped goal setting would establish some of the sustainability of the project, which would be a change in teachers' practice and perhaps their beliefs as goal setting would be seen as a worthwhile tool in teaching.

Belinda's Reflection: My Teaching

Belinda explained that the professional learning "enabled me to see other teachers in their classrooms, it allowed me to see an expert model lessons at my level and other levels, and most important of all it pushed me to improve my teaching and achieve my goals." When asked how she improved her teaching, Belinda indicated that she learnt that mathematics needed to be more "real" and that "students need to see a purpose in all that they do" in the mathematics classroom. This was also evident in the survey in which Belinda had ranked the statement "One of my goals is ensuring that students understand and can apply mathematical concepts to life experiences" as high. She felt that students "need to have ownership of the activities" and "that many activities need to have a 'fun' element". Belinda's perception about improvements in her teaching resulted from positive experiences in her classroom due to new ideas and changes she made. According to Guskey's model this would result in a change in student learning outcomes. Belinda felt that she had seen an overall improvement in student attitude and motivation. This also came in the form of feedback from her professional partner viewing her class. This acted to reinforce the changes in Belinda's beliefs and attitudes, particularly her involvement in the project as it continued strongly for another year, and she expanded it into other subject areas such as Science. Belinda noted that:

> my own maths teaching is changing in that I try to make activities more real, I involve the students more often in composing and assessing the tasks, I try to include games on a regular basis, I am trying to set more open tasks that students of all levels can tackle and I am becoming more of a facilitator rather than a stand out the front teacher.

When viewed in the first teaching session at the beginning of the initiative, Belinda modelled the "stand out the front" style of teaching where she was driving the lesson, questioning and reflection. The lesson on patterning, felt like a "one off" and no references were made to previous lessons or prior knowledge or learning. During the last teaching session of the project Belinda facilitated a lesson on graphing, in which previous learning was brought into the lesson. The task of drawing a graph was open-ended and students shared their learning during the session and at the end of the lesson they completed a self assessment rubric. Although it could be argued that the final lesson could have been carefully planned to exhibit the "correct" elements of a lesson, Belinda taught her lesson at short notice due to a change in the time table for the week. Also, the questions that students asked could have been responded to with single word answers, however Belinda guided the students to find their own answers by referring to previous lessons in their maths books and looking up information in a "big book" the class had created. Belinda was exhibiting many of the changes she felt had occurred, such as acting as a facilitator and linking the lessons so that the students could see the purpose of the different lessons within the mathematics classroom.

Reworking the Model

Guskey's model argues that change in teacher practice occurs before change in beliefs, and indeed change in practice occurs before changes in orientation. It seems, however, that the model presented by Guskey (Figure 1) is a simplistic representation of a much more complex process. Indeed, although the model presented by Guskey is linear, the actual process seems to be cyclic (see Figure 2). The model presented in Figure 2 is an alternative for the conception process of teacher change. This proposed model shows the change in teachers' classroom practices is a result of on-going teacher learning. This teacher learning has the aim of a change in student learning outcomes. This is a process which is slow and on-going and requires time and the continual input of teacher learning. It is this more complex model that appears to apply to Belinda's situation, as it is the on-going teacher learning that is contributing to change in student learning and a resulting change in Belinda's beliefs. Like Guskey's linear model (Figure 1) it is after a significant or desired change in student learning outcomes is attained, that perhaps a change in the beliefs and attitudes of teachers may be observed. Guskey (1986) mentioned that it was only when teachers used new ideas *and* gained evidence of positive change that changes occurred in their beliefs and attitudes. Belinda continued to try new ideas to attain goals set for student outcomes as she reflected on her own practice. It was after much professional exploration of her teaching that Belinda began to see results in student outcomes and hence felt a shift in her own teaching practice and beliefs. So it seems staff professional learning needs to be ongoing and changes in teachers' classroom practices supported before positive changes are seen in student learning outcomes, which then may result in a change in teacher's beliefs and attitudes.

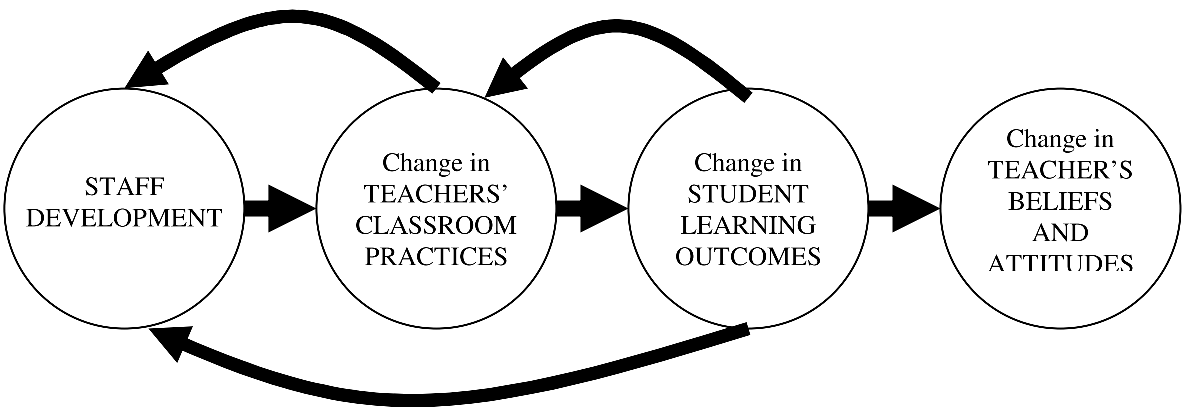


Figure 2. Alternative model of the process of teacher change.

This proposed model has implications for teacher professional learning. It implies that some teacher professional learning needs to be on-going over a period of time and may not reflect an immediate change in student learning outcomes. Teachers need time to implement changes in their classrooms and critically reflect on these changes and those in student learning. This implementation of professional learning and reflective practices can be supported with a person such as a mathematician in residence, who can offer a different view on the learning taking place. On-going professional learning may need to occur before desired or positive changes in student learning outcomes take place. It appears that it is only when these changes are seen to be positive or having results in student learning that a change in teacher beliefs and attitudes may take place (Guskey, 1986).

Belinda's story is a single study within a larger study of ten primary school teachers. Recommendations for further study include more case study analysis to test the rigour of the proposed alternative model presented in Figure 2 and further case study work with Belinda to examine if there has been a true and on-going change in beliefs and attitudes.

Conclusion

Belinda had identified areas of her teaching that she wished to change such as making activities for her students more "real". From the beginning of the project, Belinda supported the concept of a mathematician in residence, and she looked forward to examining her teaching practices in her classroom from both a teaching and learning perspective. Belinda took ideas from the professional learning discussions, as well as observations of other classes and adapted suggestions from the reflective discussions about practice and implemented these in her classroom. These resulted in a change in Belinda's classroom practices as she attempted new ideas.

Belinda also found the goal setting component of the project useful as she set both lesson and personal goals, a practice she had not previously used. This goal setting aided in the reflective practice of the professional learning project as Belinda and the other teachers involved in the project analysed both their own practice and that of their peers. Belinda found through this professional learning experience, a tool that helped her facilitate change in her own classroom. The implementation of these new classroom practices, such as the goal setting, led to slow and gradual changes in student attitudes and motivation. As this change in student learning outcomes was gradual, Belinda continued to re-evaluate her own teaching.

Guskey (1986) argued that changes in practice precede changes in beliefs, and it may be that changes in practice precede changes in orientation. It seems Belinda feels that she has changed. Belinda now sees herself more as a facilitator than a "stand out the front teacher". She has made changes to her teaching and classroom practices that have provided some positive results in both students' attitudes and motivation. This change has been gradual and has encouraged Belinda to continue her own learning and to make different changes in her practice as she looks for improvements in student learning and attitudes. This indeed supports Guskey's model that a change in beliefs occurs from a positive change in student learning outcomes resulting from changes in classroom practices due to professional learning. However, this process appears to be cyclic in nature rather than linear as many changes in practice may need to be made and the professional learning ongoing before a change in student learning outcomes observed.

It is proposed that Guskey's model is cyclic rather than linear as it appears that continual professional learning needs to be experienced to allow teachers to try new ideas in their classrooms and time to reflect and evaluate the resulting student learning outcomes. If these student outcomes are not of or to a teacher's expectations, then further strategies may need to be implemented as a result of further professional learning. In the case of Belinda, she attempted new ideas in her classroom and after a period of time some change in student attitudes was noted; however, Belinda wanted a change across her class, so she tried new strategies as a result of ongoing professional learning. This process continued throughout the year and as the results in the student outcomes in Belinda's class improved, she found her own teaching and beliefs were also changing. The responses from Belinda supports Guskey's argument that it is when teachers use new ideas *and* gain evidence of positive change that a change may occur in their beliefs. It appears that these changes in student learning outcomes need to be positive before a change in teacher's beliefs and attitudes is observed, and it appears that this may be a result of on-going professional learning and a cyclic interpretation of Guskey's model.

References

Australian Government Department of Education, Science and Training, Australian Government Quality Teaching Programme. http://www.qualityteaching.dest.gov.au/default.htm accessed 17/01/07

Barell, J. (2003). *Developing more curious minds.* Alexandria, VA: Association for Supervision and Curriculum Development.

Brown, R., & Renshaw, P. (2006). Transforming practice: Using collective argumentation to bring about teacher change in a year 7 mathematics classroom. In P. Grootenboer, R. Zevenbergen, & M. Chinnappan (Eds.), *Identities cultures and learning spaces* (Proceedings of the 29th Annual Conference of the Mathematics Education Research Group of Australasia, 99-106). Sydney: MERGA.

Guskey, T. R. (1986). Staff development and the process of teacher change. *Educational Researcher*, *15*(5), 5-12.

Higgins, J. (2002). *Evaluation of Advanced Numeracy Project 2001.* Wellington: Ministry of Education. Te Kete Ipurangi website: http://www.tki.org.nz/r/literacy_numeracy/professional/evaluation_anp.pdf.

Higgins, J. (2003). *Evaluation of Advanced Numeracy Project 2002.*. Wellington: Ministry of Education. Te Kete Ipurangi website: http://www.tki.org.nz/r/literacy_numeracy/professional/evaluation_anp2002.pdf

King, M. B., & Newmann, F. M. (2001). Building school capacity through professional development: conceptual and empirical considerations. *The International Journal of Educational Management*, *15*(2), 86-93.

King, M. B., & Newmann, F. M. (2004). Key link. *Journal of Staff Development*, *25*(1), 26-28.

MacGilchrist, B., Myers, K., & Reed, J. (1997). *The Intelligent School.* London: Paul Chapman.

McDuffie, A. R., (2004). Mathematics teaching as a deliberate practice: An investigation of elementary pre-service teachers' reflective thinking during student teaching. *Journal of Mathematics Teacher Education, 7,* 33-61.

Pritchard, R., & McDiarmid, F. (2005). Promoting change in teacher practices: Investigating factors which contribute to sustainability. In P. Grootenboer, R. Zevenbergen & M. Chinnappan (Eds.), *Identities cultures and learning spaces (Proceedings of the 29th annual conference of the Mathematics Education Research Group of Australasia,* pp. 432 - 439). Sydney: MERGA.

State of Victoria, Department of Education and Training, *Principles of learning and teaching P – 12 (PoLT).* http://www.sofweb.vic.edu.au/blueprint/fs1/polt/default/.htm accessed 4/01/07

State of Victoria, Department of Education and Training. *Professional Learning*. http://www.sofweb.vic.edu.au/pd/tchdev/index.htm accessed 4/01/07

Willis, S. (2002). Creating a knowledge base for teaching: A conversation with James Stigler. *Educational Leadership*, *59*(6), 1-3.

Seeking Evidence of Thinking and Mathematical Understandings in Students' Writing

Anne Scott
Australian Catholic University
<a.scott@patrick.acu.edu.au>

This paper reports the use of three questions to guide students' discussions and reflective writing in a year 5/6 mathematics class. Journal entries and work samples were examined for evidence of students making sense of their thoughts and processes used during the completion of Space-based tasks. Reflective writings were inspected for evidence of the three functions of metacognition and Bloom's Taxonomy was used to note changes in students' levels of understanding of the content. Preliminary findings suggest that the approach and questions used in this study warrant further investigation.

Directives in curriculum that require teachers to assess and report students' thinking are complex. One approach is explained in this paper, which commences with background information about the change in emphases in recent curriculum. This is followed with an overview of the literature that informed the approach used and provided the basis for the data analysis in the investigation. Then preliminary findings are discussed.

Trends in Curriculum for Developing Thinking and Understandings in Mathematics

New directions in curriculum across Australia share a focus on preparing students for further education, work, and life (Department of Education and Children's Services, 2001; Department of Education Tasmania, 2007; Department of Education Training and the Arts, 2004; Victorian Curriculum Assessment Authority (VCAA), 2006). In 2005, the Victorian government introduced the *Victorian Essential Learning Standards* (VELS) (VCAA, 2004), a framework for planning whole school curriculum from Preparatory – Year 10. The *Learning Standards* are developed within three interrelated strands: *Physical, personal and social learning; Discipline-based learning; and, Interdisciplinary learning*. These three strands seek "to equip students with capacities to manage themselves and their relations with others, to understand the world, and to act effectively in that world" (p. 3). Each strand has a number of domains. In each domain, the essential knowledge, skills, and behaviours are identified in subcategories called dimensions. Specific standards are written for each dimension according to three broad stages of learning: P-4, Years 5-8, and Years 9-10. These standards define essential and developmentally appropriate expectations for teaching and learning programs (VCAA, 2004). The *Learning Standards* may be addressed in programs either "through explicit teaching focused on a particular strand [or] … by creating units of work which address a number of standards at the same time" (p. 3).

Since the implementation of VELS teachers have been grappling with the complex task "for ensuring that all three strands, and their domains are addressed by all schools in their teaching programs and in their assessment and reporting practices" (VCAA, 2004, p. 3). The complexity of the task is not necessarily in the planning or implementation stages but in the mandate to assess and report each of the domains. For example, Table 1 lists a possible set of domains and dimensions from the three strands included in a mathematics-based unit of work.

Proceedings of the 30th annual conference of the Mathematics Education Research Group of Australasia
J. Watson & K. Beswick (Eds), © MERGA Inc. 2007

Table 1
Strands, Domains, and Dimensions in a Mathematics-based Unit of Work

Strand	Domain	Dimension
Physical, Personal and Social Learning	Personal Learning	The individual learner Managing personal learning
Discipline-based Learning	English Mathematics	Writing Space
Interdisciplinary Learning	ICT Thinking	ICT for visualising thinking Reflection, evaluation and metacognition

A mathematics-based unit comprising these domains and dimensions may produce worthwhile experiences for students learning not only in the content but also possibly in those generic skills and strategies applicable in various contexts. Yet, one might ask: Which tools and strategies will teachers use to measure and report students' progress in the domain of thinking?

This paper reports one approach for assessing and reporting student progress given the expectations of teachers in Victorian schools using three questions addressing the three strands in VELS. The key question addressed in this study is:

- Does the use of three specific questions at the commencement of reflective writing sessions provide evidence of the development in children's thinking and mathematical understandings?

Gaining Insights into Students' Thinking and Understandings of Mathematics

A scan of proceedings at MERGA conferences suggests that teacher educators not only share a desire to help students articulate their ideas during mathematics lessons but also have various ways of encouraging the exchange of thoughts either orally and/or in writing (Beswick & Muir, 2004; Brown & Renshaw, 2004; English & Doerr, 2004; Falle, 2005). Although not necessarily building on the same theme, insights from each of these studies shaped and informed the study discussed in this paper.

In a study by Beswick and Muir (2004) comprising 20 year 6 students from five primary schools researchers examined participants' abilities to communicate their problem solving strategies and mathematical thinking. Using semi-structured interviews, each problem was read to the student by the interviewer. Students were asked to solve the problem and record the process used in writing. Concrete materials were available for students' use. On completion of the task, students were asked to explain verbally what they had done. Beswick and Muir reported that, regardless of students' abilities, students expressed their mathematical thinking more effectively in verbal than in written forms.

Beswick and Muir (2004) concluded that learners would benefit from instruction that encouraged visualisation of their thinking and "efficient and meaningful ways of recording their thinking in writing" (p. 101) and this is one of the goals of the study discussed in this paper.

Another source shaped the approach and the design of the tasks used. Brown and Renshaw (2004) argued that "success in school mathematics is often measured in terms of a student's capacity to reproduce others' inventions and justifications" (p. 135) and advocated the need to link students' experiences and processes with the more formal content knowledge in the domain of mathematics. They proposed an alternative format to teachers for initiating class discussions and for developing deeper understandings of mathematics by incorporating both everyday and scientific notions of mathematics into their discussions.

Two terms, *replacement* and *interweaving*, were recommended as ways for students "to make sense of the mathematics being presented to them and about linking students' inventions to the conventions of mathematics rather than about teacher and/or textbook evaluations of student answers" (Brown & Renshaw, 2004, p. 142). *Replacement* referred to using "an everyday understanding with a more sophisticated conventionalised understanding" (p. 135). *Interweaving* seemed to refer to an acceptance of and interchange between informal and scientific concepts and/or language.

Also contributing to the teaching approach, English and Doerr (2004) claimed that recent research necessitates teachers to be "more attentive and responsive to their students' mathematical reasoning" (p. 222). Teachers who display a hermeneutic disposition in their teaching tend to use tasks that provide opportunities for students to explore mathematical ideas, carefully listen to students' ways of thinking, and adopt various roles in their interactions with students. Such teachers observe, listen, and ask students questions for further clarification.

Similarly, Falle (2005) reported that students' explanations reveal not only the degree of their mathematical thinking but also the linguistic features used by students in their responses that may serve as indicators of their level of understanding. Falle noted that less successful students resort to "parroting" mathematical rules even though they may not be able to use them. In contrast, students who are more mathematically capable tend to experiment with logic and have greater control over the language needed to express themselves. This provided further justification for the attention to developing students' expressive skills in mathematics.

Monitoring Metacognition

An overview of processes for monitoring students' thinking processes is also relevant to the discussion given the focus on developing thinking skills in several curriculum policies. Wilson and Clarke (2004) referred to metacognition as the "awareness individuals have of their own thinking; the evaluation of that thinking; and the regulation of that thinking" (p. 26). Given this definition Wilson and Clarke noted three functions of metacognition: awareness, evaluation, and regulation. "Metacognitive awareness relates to individuals' awareness of where they are in learning process or in the process of solving the problem, of their content-specific knowledge, and of their knowledge about the personal learning contexts or problem solving strategies" (p. 27). "Metacognitive evaluation refers to judgments made regarding one's thinking processes, capacities and limitations as these are employed in a particular situation or as self-attributes" (p. 27). "Metacognitive regulation occurs when individuals make use of the metacognitive skills to direct their knowledge and thinking" (p. 27).

Wilson and Clarke (2004) assumed that promoting metacognition was a valuable exercise in mathematical learning contexts and that some strategies encouraged metacognitive acts. To address the known difficulties with monitoring metacognition, they refined a multi-method clinical interview that involved self-reporting and a think-aloud technique, observation, and audio and video recording. The clinical interview involved a card-sorting procedure enabling the participant to reconstruct his/her "thought processes during a problem solving episode just completed" (p. 29).

Wilson and Clarke's study (2004) comprised 90 one-on-one interviews with year six students from six different classes across Victoria using three different types of tasks: numerical, spatial, and logical. A series of metacognitive action statement cards varied according to the task but were categorised according to the three functions of metacognition identified in their earlier definition: awareness, evaluation, and regulation. For example, statements from the awareness category included: I thought about what I already know; I had tried to remember if I had ever done a problem like this before; I thought "I know this sort of problem". Sample statements from the evaluation category included: I thought about how I was going; I checked my work; I thought "is this right?" In the regulation category some statements included: I thought about what I would do next; I made a plan to work out; I changed the way I was working.

Wilson and Clarke (2004) reported that it seemed reasonable to expect a particular pattern in these three functions: awareness first, followed by an evaluation and finally a regulatory act. However, students used various sequences, many of which were non-linear. Generally, sequences commenced with awareness. Regulatory and evaluative statements were often arranged in different combinations. Students concluded tasks with an evaluation statement regardless of whether the task was completed successfully.

Analysing Levels of Understandings

Although not specifically from the mathematics education field of research, some reference to the levels of understanding using Bloom's Taxonomy (Anderson, 1999) is helpful with the data analysis in this investigation. Bloom's Taxonomy was first published in 1956 with six categories knowledge, comprehension, application, analysis, synthesis, and evaluation. These were considered to be increasingly complex behaviours that cumulated in a hierarchical structure (Anderson, 1999).

Over the past 50 years there have been variations of the original model (Houghton, 2003). Changes in the new taxonomy include the recognition of the role of social learning and "cultural-specificity of knowledge" (Anderson, 1999, p. 7). Another is the qualification of the premise "that the categories form a cumulative hierarchy in all cases … depends on a series of factors" (Anderson, 1999, p. 8). For example, Anderson (1999) reported that an individual may use several cognitive processes such as recall, understand, analyse, synthesise, and evaluate in selecting an appropriate strategy to solve a problem. However, there are other cases in which one may apply a given or known strategy in a routine manner. The difference is that metacognition is evident in the former but not necessarily in the latter.

Houghton (2003) compared models of the taxonomy. The version that listed verbs for each category was helpful for inspecting and assessing student work samples in this investigation.

In summary, over recent years various authors cited in this paper, have suggested ways in which teachers may link students' experiences with mathematical learning through their interactions and discussions with students. Some offered a way to help identify the functions of an individual's metacognitive processes in completing a task. Others suggested that teachers provide guidance to help students visualise and record their thoughts in writing, or ask questions so that students may clarify their ideas. Insights from such authors provide the basis for the analysis. Yet, perhaps more is needed for gathering and analysing children's written records of their thinking and mathematical understandings. One approach is to use three specific questions as prompts for children's reflective writing within mathematics lessons and examine these for evidence of cognitive processes used and mathematical understandings gained.

Investigation of the Effectiveness of Three Questions

The study examined the changes in year 5 and 6 students' perceptions of themselves as learners, their knowledge and skills in an aspect of Space, and their ability to use specific tools and strategies over a 2-week period. The specific research question addressed in this paper is:

- Does the use of three specific questions at the commencement of reflective writing sessions give evidence of the development in children's thinking and mathematical understandings?

Participants

Twenty-three students in the year 5/6 class attended a small inner city school where 94% of the student population came from Culturally and Linguistically Diverse (CALD) backgrounds and 67% of the families received financial assistance. One student had recently arrived in Australia with limited English skills and several had learning disabilities.

The classroom teacher had 2 years teaching experience and chose to work along side the researcher. The researcher had taught for 14 years in primary schools.

Overview of the Planning, Lesson Format, Tasks and Approaches used

In the week prior to the study commencing the classroom teacher collected students' prior knowledge of the content and discussed these and the content to be taught with the researcher. The researcher planned and delivered four lessons. The classroom teacher was always present in the room and interacted with students as they worked on the activities.

Each lesson was between 60 – 80 minutes in duration and followed a similar format. The researcher introduced the focus of lesson to the whole class and invited the students to accept a challenge posed in tasks. Students investigated the open-ended tasks for ten minutes, were asked to share their ideas and strategies, and then were directed to resume working on the tasks being mindful of shared insights. Lessons concluded with the researcher summing up key points and students reflected on their experiences of the lesson and wrote personal reactions in their workbooks.

During the fourth lesson, students were invited to consider what knowledge, skills, and feelings they had that were somewhat different to those which they had prior to these lessons. Students wrote for approximately 40 minutes in response to three specific questions:

- What have you learnt which is somewhat different to what you already knew about **mathematics**? Give examples.
- What have you learnt which is somewhat different to what you already knew about the **program, tools and games** used?
- What have you learnt which is somewhat different to what you already knew about **yourself or the way you learn**?

Although 40 minutes is not realistic in many classrooms these students predominantly from non-English speaking families needed the time to reflect and write.

Because there were only three desktop computers in the classroom students worked in pairs rotating through planned tasks. These involved either the manipulation of concrete materials and discussions at students' tables or completing a computer-based task. The series of lessons were designed to link students' everyday experiences with mathematical content. Three additional aims were:

1. to draw on students' interests in arcade-type computer games and in programs such as *MS Powerpoint* (Microsoft Corporation, 1995),
2. to develop students' use and understandings of mathematical language when transforming two-dimensional shapes such as flip/reflection, slide/translation, turn/rotation, resize/enlarge/reduce/dilation,
3. to provide opportunities for students to discuss, reflect and write their thoughts at the conclusion of each lesson.

Table 2 summarises the tasks completed.

Table 2
Computer-based and Table Tasks Completed over Four Lessons

Type of task	Brief description of activity
Computer-based	Individuals play two games of *Tetris* (2M Games, 2004).
Table	Using multi-link cubes make 12 shapes which could be used in a game like *Tetris*. In pairs, one person plays the game and fills as many whole lines as possible gaining 10 points each time. The other person provides the pieces one by one. (No flipping allowed). Is the game better or worse if you are allowed to flip the pieces?
Table	Create a picture using 7 tangram pieces. Trace around the outline. Make a small scale drawing of your solution. Recreate another person's picture. Check the answer sheet.
Table	Groups of three complete a barrier game using tangram pieces/picture. A tells B how to make his/her picture by giving verbal instructions only. C acts as observer and records the language used.
Table	Create mosaic picture/pattern using pattern blocks. Then using grid paper, create a tiled floor. After a few attempts create a piece of art work using Escher's style.
Table	Make a picture flick note pad to show an image moving.
Computer-based	In pairs, create a series of four/five slides which show shapes moving (flipping, sliding, rotating, resizing).
Table	Draw a simple picture onto grid paper. Enlarge and reduce the picture according to a scale.

Data Collection and Analyses Techniques, Tools, and Approaches

Prior to the series of lessons commencing the classroom teacher asked students to write what they knew about the topic and in which situations one might use the content or related terms. During the four lessons students' computer-based work files were saved on the class server and samples of their book work were collected. Researcher took anecdotal notes of significant events and discussions with students. Researcher and classroom teacher each kept journals with their reflections of each lesson and later shared their thoughts via email communication.

After the lessons, dated work samples were examined in two ways. First, for evidence of levels of understandings about concepts in transforming 2D shapes using Bloom's Taxonomy from written responses to questions in pre-lesson and from the fourth lesson. Analyses of data were tabulated to provide an overview of the levels of understandings about concepts in transforming shapes for each student. Second, the work samples were inspected for evidence of the three functions of metacognition (Wilson & Clarke, 2004). The *Monitoring Metacognition Interview* (MMI) multi-method interview technique (Wilson & Clarke, 2004, p. 29) was not used in this study.

Results and Discussion

A small group of participants attending a professional development session were provided with the adapted version of Bloom's Taxonomy used in this investigation and asked to look for evidence of understandings in students' work samples. Their responses were similar to those independently categorised by both the researcher and teacher. Table 3 presents frequencies of level of understanding of concepts and vocabulary related to transforming 2D shapes using Bloom's Taxonomy in students' responses recorded pre-lessons and in fourth lesson. This was the first way work samples were examined.

Table 3
Students' Levels of Understanding of Topic using Adapted Version of Bloom's Taxonomy in Written Responses

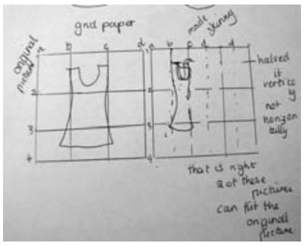
		Remembering	*Understanding*	*Applying*	*Analysing*	*Creating*	*Evaluating*
Pre-lesson Student (n = 14)	No evidence			1			
	Limited example	3	4	9			
	Multiple examples	11	10	4			
Fourth Lesson Student (n = 20)	No evidence				2	4	11
	Limited example	6	7	6	6	6	7
	Multiple examples	14	13	14	12	10	2

The figure 14 in the bottom left hand cell indicates that 14 of the 20 students either listed or described two or more examples related to the topic in their reflective writing from the fourth lesson. There was evidence of students' increased levels of understandings about concepts in transforming 2D shapes using the adapted version of Bloom's Taxonomy from written responses to questions in pre-lesson 3 Nov (n = 14) and from journal entries dated 17 Nov (n = 20). For example, although a group of 10 or 11 students began the series of lessons with a reasonable knowledge of the terms and were able to describe or define the terms, only four gave examples of when the terms were used in both mathematical and everyday settings. In contrast, by the fourth lesson there was evidence that 14 students saw applications for these terms. There was also evidence that students (n = 12) were synthesising their understandings that went beyond the tasks or saw connections between them.

An excerpt from student N1's fourth lesson written response provides a sample of the evidence identified for the *creating* category.

> I also learnt that by stretching a picture, the picture would look very different because your (sic) only changing the width, but if you change both height and width the picture will look the same but bigger.

It seems that this student is developing a generalisation about ratio and proportion.An excerpt from another student N2's fourth lesson written response provides a sample of the evidence identified for the *evaluating* category.



> I learn't (sic) how to draw a particular picture on grid paper and then making it skinny. Since my original picture was drawn on a 2cm scale I wanted to make it skinny. First I halved the 2cm which would be 1cm but I didn't halve it horizontally only vertically and drew my picture (sic). That is right 2 of these pictures can fit the original picture.

There are two comments added to the diagram in which student N2 justifies her thoughts: "halved it vertically not horizontally" and "That is right two of these pictures can fit the original picture".

Insights from Wilson and Clarke's (2004) three functions of metacognition and action card statements provided the basis for the second form for data analysis. The culturally diverse group of students, who refrained from participating in class discussions, were willing to write in journals at the end of the fourth lesson. Written responses from ten students indicated that they noted changes in their own thinking, skill level and/or attitude towards aspects related to these activities.

Many students wrote about increased awareness of the applications of the mathematics being studied in everyday activities. For example, student A wrote:

> I never knew that I was using mathematics when on (sic) powerpoint but I [now know] that I was estimating sizes when [I was] changing [resizing] pictures [to use in slides] for [creating] animations. When I play tetris, I play it for fun but I was using flip, slide and rotate to fit shapes into gaps.

Although this student had some difficulties with clear expression, the entry provides evidence of the awareness the student gained as a result of these lessons. Without the opportunity for writing such insights would be more difficult to capture.

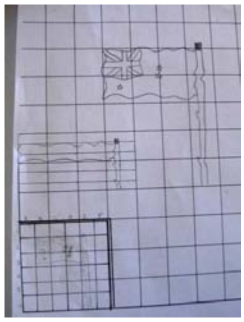
The following excerpts are all from student D's fourth-lesson written response:

> I learnt that the game *Tetris* involves maths because when we use the shapes to make lines/rows, we are using tessellation.

Similarly, the student seems to be reflecting on the activity and drawing on the metacognitive function, awareness.

> I also learnt that when allowing the person to *flip* in the game, it is sometimes easier [to get higher scores].

This sentence could be within the evaluation category.



> Drawing on grid paper also involves maths because we use scales when either enlarging or reducing the size of images/pictures. When we make our drawings flatter, we divide the grid that goes horizontally, smaller.
>
> I've learnt that when doing an animation on powerpoint (on the computer), you only move each picture a bit on each slide to make it moving when the entire slide was played. It was one of the best things I learnt because I have never done it before.

Again, in both of these sentences there is evidence of some metacognitive awareness and regulation occurring. The student is aware of the new knowledge and indicates that he will use the knowledge to plan and complete similar tasks in the future.

Conclusion

To an extent the goal of the investigation was successful. The approach and the three specific questions provided students with opportunities to discuss and write responses gathering evidence of students' progress in the three interwoven strands central to the Victorian Essential Learning Standards (VCAA, 2004). Even students with limited skills in English were able to communicate their thought processes and some deepened their mathematical understandings about aspects of Space over four lessons.

There were also limitations to using the approach. Reflective writing is a text-type and a generic skill that needs to be explicitly taught. As with other text-types teachers need to model the language features used in such forms of writing (Derewianka, 1990). For this group of students reflective writing was a new text-type and skill. Part of the mathematics session was spent explaining the questions and expectations of the writing which was non-mathematics specific learning.

The tools and techniques used for data analyses seemed helpful in identifying changes in students' written responses. Having said that, it might be useful to expand the list of verbs in the table of the version of Bloom's Taxonomy used.

Given these preliminary findings, it would be useful to replicate this investigation or conduct further research using these three questions with students and teachers P-10 classrooms to check whether similar trends emerge.

References

2M Games. (2004). *Eureka: 50 Blockbuster Games: Volume 2.* Nodtronics Pty Ltd.

Anderson, L. (1999). *Rethinking Bloom's Taxonomy: Implications for Testing and Assessment* (Report). Columbia: US Department of Education.

Beswick, K., & Muir, T. (2004). Talking and writing about the problem solving process. In I. Putt, R. Faragher, & M. McLean (Eds.), *Mathematics Education For the Third Millennium: Towards 2010.* (Proceedings of the twenty-seventh annual conference of the Mathematics Education Research Group of Australasia). (Vol. 1, pp. 95-102). Sydney: MERGA.

Brown, R., & Renshaw, P. (2004). Integrating everyday and scientific ways of knowing mathematics through forms of participation in classroom talk. In I. Putt, R. Faragher & M. McLean (Eds.), *Mathematics Education For the Third Millennium: Towards 2010.* (Proceedings of the twenty-seventh annual conference of the Mathematics Education Research Group of Australasia.) (Vol. 1, pp. 135-142). Sydney: MERGA.

Department of Education and Children's Services. (2001). *South Australian Curriculum Standards and Accountability Framework*. Retrieved 18 March 2007. from http://www.sacsa.sa.edu.au/index_fsrc.asp?t=EL.

Department of Education Tasmania. (2007). *Essential Learnings Curriculum*. Retrieved 18 March 2007. from http://www.education.tas.gov.au/school/educators/curriculum.

Department of Education Training and the Arts. (2004). *The New Basics Project*. Retrieved 18 March 2007. from http://education.qld.gov.au/corporate/newbasics/.

Derewianka, B. (1990). *Exploring how texts work*. Newtown: Primary English Teaching Association.

English, L., & Doerr, H. (2004). Learning through interacting with students' ways of thinking. In I. Putt, R. Faragher & M. McLean (Eds.), *Mathematics Education For the Third Millennium: Towards 2010.* Proceedings of the twenty-seventh annual conference of the Mathematics Education Research Group of Australasia. (Vol. 1, pp. 215-222). Townsville: MERGA.

Falle, J. (2005). Towards a language-based model of students' early algebraic understandings: Some preliminary findings. In P. Clarkson, A. Downton, D. Gronn, M. Horne, A. McDonough, R. Pierce & A. Roche (Eds.), (Proceedings of the 28th annual conference of the Mathematics Education Research Group of Australasia, Mathematics Education Research: *Building connections: Theory, Research and Practice* Vol. 1, pp. 345-352). Sydney: MERGA.

Houghton, B. (2003). *Comparing Thinking Skills Models.* Retrieved 23 March, 2007, from http://www.ceap.wcu.edu/Houghton/Learner/Think94/NCmarzanoThink.html

Microsoft Corporation. (1995). *Microsoft Office PowerPoint97.* USA: Microsoft Corporation.

Victorian Curriculum and Assessment Authority. (2004). *Introducing the Victorian Essential Learning Standards*. Melbourne: Victorian Curriculum and Assessment Authority.

Victorian Curriculum Assessment Authority. (2006). *Victorian Essential Learning Standards.* Retrieved 9 February 2006, from http://vels.vcaa.vic.edu.au/index.html

Wilson, J., & Clarke, D. (2004). Towards the modelling of mathematical metacognition. *Mathematics Education Research Journal, 16*(2), 25 - 48.

Utilising the Rasch Model to Gain Insight into Students' Understandings of Class Inclusion Concepts in Geometry

Penelope Serow
University of New England
<pserow2@une.edu.au>

This study extends research into the van Hiele Theory by narrowing the microscopic lens and providing a focused analysis on the understanding and development of class inclusion concepts in Geometry. This paper integrates two qualitative frameworks, identified through the utilisation of the SOLO model, that indicate developmental growth in understanding of relationships among figures, and relationships among properties. This is considered via a quantitative approach, using a Rasch analysis model, which provides a comparison of the complexity of seven different interview tasks within the context of triangles and quadrilaterals.

This study is part of a larger study that extends research into the van Hiele Theory by narrowing the microscopic lens and providing a focused analysis on the understanding and development of class inclusion concepts in Geometry. Pertinent to this study, the level associated with a student who accepts and utilises notions of class inclusion is described as Level 3 (van Hiele, 1986). This aspect of Level 3 is regarded as both a difficult concept to acquire and a prerequisite for formal deductive reasoning (De Villiers, 1998; Heinze, 2002). The networks of relations, which are the students' focus when exhibiting Level 3 thinking, can be described as those that deal with the relationships among properties within figures, and relationships among figures (van Hiele, 1986). In an attempt to refine the characteristics of the development of this concept, an initial qualitative study (Currie & Pegg, 1998; Serow, 2006) utilised the SOLO (Structure of the Observed Learning Outcomes) model (Biggs & Collis, 1982) to provide deeper insights into the van Hiele levels. A central finding of this initial study was the identification of two frameworks that describe developmental pathways leading to a) an understanding of relationships among figures, and, b) an understanding of relationships among properties. This study is a quantitative analysis of the results using a Rasch analysis model with the aim of providing further insights into students' understandings of class inclusion. Rasch measurement has been described as permitting "the identification and examination of developmental pathways, such as those inherent in the development of mathematical concepts" (Callingham & Bond, 2006).

Background

This study provides a quantitative synthesis of the developmental pathways described in Table 1, based upon the application of ACER's QUEST analysis program, using the partial credit modelling process, provided by Masters (1982). This analysis program enabled the plotting of item difficulties for the seven tasks upon a single scale and provides some initial insights into a comparison of item/category difficulty concerning tasks that target geometrical relationships within the contexts of triangles and quadrilaterals.

In addition to the van Hiele Theory, the SOLO model was utilised in the initial qualitative study. This model is comprised of two main components, these being: the modes of functioning, and, the cycles of levels. There are two modes of functioning relevant to this paper, namely, concrete symbolic (CS) and formal (F). The concrete symbolic mode involves the application and use of a system of symbols, for example, written language and number problems, which can be related to real world experiences. The formal mode is characterised by a focus upon an abstract system, based upon principles, in which concepts are imbedded. Within each mode development occurs described in terms of levels. General descriptions of the levels are the following.

1. Unistructural (U): response is characterised by a focus on a single aspect of the problem/task.
2. Multistructural (M): response is characterised by a focus on more than one independent aspect of the problem/task.
3. Relational (R): response is characterised by a focus on the integration of the components of the problem/task.

Studies (Campbell, Watson, & Collis, 1992) have extended the SOLO model through the suggestion that more than one cycle of levels exist within each mode. As a result, two cycles of levels in the concrete symbolic mode have been identified. The pathways that were identified in the earlier qualitative study were characterised by two cycles of responses of the concrete symbolic mode (SOLO), and two cycles of responses of the formal mode (SOLO).

In the initial qualitative study, the developmental frameworks that emerged through the application of the SOLO model are detailed in Table 1 below. Descriptors of the tasks used, within the contexts of triangles and quadrilaterals, are outlined in Table 2 in the Methodology section.

Table 1
Developmental Frameworks Concerning Relationships Among Properties and Relationships Among Figures

Coding	Properties	Figures
R_1(CS)	The focus of the task is upon the figure in question from which all known properties are derived. A specific example of the figure is utilised from which each property is determined. There is a strong reliance on ikonic support. The properties are perceived as features.	A single property or feature is identified to link the figures. The focus of the response is upon the identification of an observed single quantifiable aspect, which places figures into spontaneous groups. There is a strong reliance on visual cues.
U_2(CS)	The reference for the response is the figure in question. The figure determines a single property. Minimisation is understood to be "less" and is based upon the uniqueness of a single property to the figure.	Classes of figures are known by name and are characterised by a single property. The class represents an identifiable unit. Links do not exist between classes, unless supported by visual cues. Observed differences play a significant role.
M_2(CS)	The single reference remains the figure in question. The figure determines two or more unique properties, which are utilised to represent the figure. Properties remain in isolation. Minimisation is understood to be "less".	M_2 responses incorporate classes of figures, which are known by name. These classes are characterised by more than one property. Links are not made between classes where differences in properties are accentuated by visual differences.
R_2(CS) response	The focus of the response is upon a link or ordering between a pair of properties, or a pair of figures within the same context. The link is characterised by a single dominant property that precludes the utilisation of a relationship in both directions.	Relationships exist between classes of figures, which are based upon similar properties. Inclusive language is used to describe the classes of figures; hence, property descriptions allow for similarities to be acknowledged.
U_1(F) response	This type of response incorporates a relationship between two properties, or between	When prompted, tentative statements are made concerning the possibility of subsets

	two figures, and is justified accurately. Two properties are now perceived to work together, and as a result determine the figure. This single relationship has become a workable unit.	within a class of figures. There is no acceptance of this notion, however, it is able to be discussed tentatively.
M_1(F)	The response is based upon the existence of multiple pairs of relationships between properties or relationships between figures. While the focus of the response is on more than one relationship, they are treated in isolation. Minimisations remain in conflict with the need to distinguish a certain figure from other figures within the same global class.	There is an unprompted acceptance of a class of figures containing subsets. While this notion of class inclusion is accepted and utilised, it is not justified adequately.
R_1(F) response	The response includes a focus upon the network of relationships among known property and figure relationships. The interrelationships may not incorporate all property relationships.	The notion of class inclusion is an integrating feature of the response. A class of figures incorporates subsets, which are inclusive of generic categories identified by other names. Each class maintains a workable identity while the focus is upon the network of relationships.
U_2(F) response	A network of relationships is the focus of the response. There is an understanding of the general overview, which utilises relationships among groups of properties and figures. The notion of minimisation can be held in more than one circumstance spontaneously.	The notion of class inclusion acquires further development. Conditions are placed upon the classes of figures, which acknowledge more than one system of relationships. This requires an overview of the interrelationships.

The study reported here was designed to provide a quantitative synthesis of the developmental frameworks that described students' understandings of the relationships among figures and properties. The research questions addressed are the following.

1. How do the identified response categories reflect the hierarchical framework of the SOLO model?

2. Is there an order of difficulty among the item responses, which can assist in interpreting the complexity of students' responses to tasks concerning relationships among figures and relationships among properties?

3. Which response categories to tasks had relatively larger increases in complexity from the prior response category, concerning students' understandings of relationships among figures, and relationships among properties?

Methodology

The previous qualitative study involved in-depth interviews with 24 students of higher mathematical ability, purposely selected, within Years 8–12 (ages 13–18 years) in two secondary schools. There were equal numbers of males and females. Twelve of these students repeated the interview tasks two years later, and hence the data set to be analysed comprises a total of 36 sets of student responses.

The nature of the qualitative study was to have the students complete seven tasks that focused upon known relationships among figures and among properties within the contexts of triangles and quadrilaterals. Seven items were included in the interview protocol. The tasks provided a catalyst for discussion that enabled prompts and probes as appropriate. The duration of each interview was approximately 1 hour. Further details of the interview are presented in Serow (2006) and Currie and Pegg (1998). An outline of interview tasks (items) is contained in Table 2 below.

Table 2.
Item Focus and Descriptors

Item	Focus of the Item	Item Descriptors
1	Relationships among triangle figures	Design a tree diagram that links the different triangles (equilateral, right isosceles, acute isosceles, obtuse isosceles, right scalene, acute scalene, and obtuse scalene). Discussion follows concerning the reasons for links and/or lack of links.
2	Relationships among quadrilateral figures	Design a tree diagram that links the different quadrilaterals (trapezium, square, rectangle, rhombus, parallelogram, kite). Discussion follows concerning the reasons for links and/or lack of links.
3	Relationships among equilateral triangle properties	After selection of known property cards for the equilateral triangle, the student is asked to provide a minimum combination of cards to enable a friend to identify the shape with accuracy. Multiple combinations were then requested.
4	Relationships among right isosceles triangle properties.	Task above repeated for the right isosceles triangle.
5	Relationships among square properties.	Task above repeated for the square.
6	Relationships among parallelogram properties.	Task above repeated for the parallelogram.
7	Relationships among rhombus properties.	Task above repeated for the rhombus.

Each of the responses to the seven tasks was coded according to the SOLO codings described in Table 1. The results presented in this paper are a review of the Rasch results across the seven items and 36 student response sets. With the categories of each item being of an ordinal nature, the data assumptions of the QUEST application of the Rasch modelling process are consistent with the data of this study. The partial credit model was used to provide data concerning the relatively larger distances between response categories and clusters of response categories. The data set is combined to allow a conservative comparison of the item response categories on a single hierarchical line of inquiry (Bond & Fox, 2001).

Results

Reliability

Item separation reliability statistics produced by the QUEST software are described by Adams and Khoo (1993) as the proportion of the observed variance that is considered true. In this study, the relatively small sample size across a limited number of grades meant that the item separation reliability was low, due to larger measurement error. Due to this factor, the item estimates are to be interpreted conservatively and the results are presented in clusters of response categories. Even though the item separation reliability was low, there are some points of interest in terms of the relative difficulties among the response categories and this is the focus of the paper.

Fit Statistics

Fit statistics are the means and standard deviations of the infit (weighted) and outfit (unweighted) fit statistics in the mean square form. When the observed data and estimates are compatible, the expected value of the infit mean square is close to 1 (1.02) with a small standard deviation (0.17), and the transformed infit (Infit t) is close to zero (0.12). Hence, the items come from the same underlying construct, namely, relationships among figures and relationships among properties.

The component infit mean square values are presented in graphical form in Figure 1 to assist in interpretation. The infit statistic for each item is the weighted residual based statistic, which indicates quantitatively how appropriately each item fits the model (Fisher, 1993). This comparison can be used to confirm the unidimensionality of the items, confirming construct validity of the items. Fit is acceptable if the mean lies between 0.77 ad 1.3 (Keeves & Alagumalai, 1999), in this case the infit mean is 1.02.

The figures on the horizontal scale represent the infit mean square scale and the asterisks indicate the magnitude of the fit statistic for each item on the same line. Fit statistics that lie within the two dotted vertical lines are considered acceptable. The well-fitting nature of the items to the model indicates that the items represent aspects of a latent trait. The infit mean square map for the seven items, which appears below in Figure 1, indicates that six of the seven items are within the acceptable limits. Item 4, which concerns students' understanding of the relationships among properties of the right isosceles triangle, is only slightly to the right-hand side of the acceptable limits. This indicates that for Item 4, there is an element of randomness in coding.

```
INFIT MNSQ
           .63     .71       .83      1.00      1.20      1.40
-----------+--------+---------+---------+---------+---------+-----
  1 item 1               .             *|              .
  2 item 2               .         *    |              .
  3 item 3               .   *          |              .
  4 item 4               .              |              .*
  5 item 5               .              *              .
  6 item 6               .          *   |              .
  7 item 7               .              |    *         .
==================================================================
```

Figure 1. Item map.

Item Difficulty

The information pertinent to item estimates is displayed in the variable map in Figure 2. There are seven tasks in total, and 36 sets of student responses represented. The chart includes a logit scale on the left of the diagram on which both items (n=7) and cases (n=36) are calibrated. The distribution of students is represented by XXXs on the left-hand side of the chart. The seven tasks are identified in Figure 2 as:

1. Relationships among triangle figures.
2. Relationships among quadrilateral figures.
3. Relationships among equilateral triangle properties.
4. Relationships among right isosceles triangle properties.
5. Relationships among square properties.
6. Relationships among parallelogram properties.
7. Relationships among rhombus properties.

LOGIT	CASES	ITEMS			
3					
	X				
				5.U_2(F)	
2					
				5.R_1(F)	
		1.U_2(F)			
				6.U_2(F)	7.U_2(F)
		2.U_2(F)			
1	X			3.U_2(F)	
		1.R_1(F)		3.R_1(F)	4.U_2(F)
	XXX	1.M_1(F)		6.R_1(F)	
		1.U_1(F)	2.R_1(F)	4.M_1(F)	
	X			3.M_1(F)	
	XXX			7.M_1(F)	
	XXX				
	XX	2.R_2(CS)			
	XX			5.M_1(F)	6.M_1(F)
0	XX	1.R_2(CS)			
	XXX				
	X			3.U_1(F)	4.U_1(F)
	XX			6.U_1(F)	7.U_1(F)
	XXXX			3.R_2(CS)	5.U_1(F)
	XXXX			4.R_2(CS)	
	XX				
				7.R_2(CS)	
				6.R_2(CS)	
		2.M_2(CS)			
-1	X			3.M_2(CS)	5.R_2(CS)
		1.M_2(CS)			
	X	2.U_2(CS)		6.M_2(CS)	
				4.M_2(CS)	5.M_2(CS)
				6.U_2(CS)	7.M_2(CS)
-2		1.U2(CS)			

Figure 2. Item and case estimates (thresholds).

Item Analysis

The following discussion addresses the patterns that have emerged concerning item difficulty across item response categories. A comparison of item difficulties across items concerning figures and property relationships follows. The comparison involves individual response category item difficulties, which appear in Table 3.

Table 3
Item Response Category Difficulty Levels

	Concrete Symbolic			Formal			
FIGURES	U_2	M_2	R_2	U_1	M_1	R_1	U_2
Triangles	-2.06	-1.10	0.07	0.74	0.82	0.94	1.42
Quadrilaterals	-1.28	-0.97	0.22			0.70	1.16
PROPERTIES							
Equilateral		-1.08	-0.35	-0.20	0.58	0.91	1.08
Right Isosceles		-1.44	-0.50	-0.12	0.72		0.88
Square		-1.44	-1.03	-0.36	0.19	1.72	2.09
Parallelogram	-1.63	-1.26	-0.77	-0.32	0.16	0.81	1.23
Rhombus		-1.63	-0.74	-0.28	0.48		1.24

Similarities and differences in relation to degree of difficulty and characteristics of the responses form the basis of the comparison. This is considered in clusters of item responses beginning with the lower level SOLO responses, which also appear at the lower end of the item estimate threshold.

In the tasks concerning relationships among figures, and those concerning relationships among properties, a hierarchical framework emerged that is evident in the SOLO categorisations and is reinforced by the application of the Rasch analysis. Each of the items followed the SOLO sequence of levels within cycles without exception. The following discussion provides a comparison of item estimate thresholds when comparing item difficulty across clusters of response categories concerning relationships among figures, and item responses concerning relationships among properties.

The U_2(CS) response category concerning relationships among triangle figures was found by the sample of students to be of the lowest degree of difficulty. This was followed by other groups of U_2(CS) and M_2(CS) responses concerning relationships among figures, and relationships among properties. Hence, the students found the utilisation of the three mutually exclusive classes of triangles at a similar degree of difficulty to focusing upon unique property signifiers of figures with reference to the figure only. It appears that the progression to finding multiple properties that are unique to a figure assists in the formation of minimum combinations to encapsulate multiple properties to form generic categories. Although restrictive language, which does not facilitate the inclusive nature of properties, is utilised in U_2(CS) responses concerning figures and properties, this level is a necessary precursor for developing notions of minimum property combinations.

Next on the logit scale is a cluster of R_2(CS) responses including all five tasks concerning relationships among properties. Hence, the students found ordering between two properties to be at a similar degree of difficulty in both the triangle and quadrilateral contexts. Although the U_1(F) responses are grouped together when addressing tasks concerning the relationships among properties, these appear before the R_2(CS) responses in the context of relationships among figures, thus indicating that the students found a focus upon relationships between pairs of properties and/or figures, and making property links across classes, of a similar degree of difficulty in both triangles and quadrilaterals contexts. The U_1(F) response concerning property relationships appears to be a precursor to focusing upon relationships among classes of figures, which are not supported by visual cues. The remaining first cycle formal responses are clustered at a similar degree of difficulty, thus indicating that the utilisation of a single network of relationships among figures, utilising multiple relationships among properties, and an attempt to focus upon the interrelationships among property relationships are at a similar degree of difficulty.

The U_2(F) responses have a greater range in terms of degree of difficulty. This final cluster indicates that the students found the focus upon more than one network of relationships involving notions of class inclusion, and the focus upon the network of relationships to form minimisations, the most difficult groups of responses. In the context of property relationships the students found the right isosceles triangle and parallelogram items the least difficult at this SOLO level. Class inclusion notions requiring the acknowledgment of multiple subsets when relating figures were at a similar degree of difficulty to the utilisation of the network of relationships among properties of the equilateral triangle. This was closely followed by the rhombus task.

It is interesting to note the high degree of difficulty found by the sample of students when forming minimisations of square properties based upon the network of property relationships. Although this indicates that the lower SOLO categories indicated a comparatively lower degree of difficulty for the square item compared with other items of the same SOLO level, the shift required to move from M_1(F) to R_1(F) is relatively difficult in the context of the square. The responses indicated that this is due to factors such as visual cues assisting links, and multiple unique properties of the square that assist understanding at lower SOLO levels. In contrast, at the formal mode the student must leave the real world referent behind and focus upon the network of relationships among the properties, as opposed to concrete symbolic justifications.

The degree of difficulties between item response categories, known as step difficulties, further clarifies the similarities and differences among the SOLO categorisations. The step difficulties describe the change in degree of difficulty, found by the sample of students, between one SOLO level and the subsequent SOLO level. These appear in Table 4, and also include the mean step difficulty for each SOLO response category.

Table 4
Step Difficulties

	Concrete Symbolic		Formal			
	U_2 to M_2	M_2 to R_2	R_2 to U_1	U_1 to M_1	M_1 to R_1	R_1 to U_2
Item 1	0.96	1.17	0.67	0.08	0.12	0.48
Item 2	0.31	1.19				0.46
FIGURES MEAN	0.64	1.18	0.67	0.08	0.12	0.47
Item 3		0.73	0.15	0.78	0.33	0.17
Item 4		0.94	0.38	0.84	0.16	0.25
Item 5		0.41	0.67	0.55	1.53	0.37
Item 6	0.37	0.49	0.45	0.48	0.65	0.42
Item 7		0.89	0.46	0.76		
PROPERTIES MEAN	0.37	0.70	0.42	0.68	0.67	0.30

Of particular interest, are the higher and lower step difficulties. The step difficulty between a U_2(CS) response and an M_2(CS) response concerning relationships among figures has a mean of 0.64. It was also found to be difficult by the sample of students to respond at R_2(CS) compared with M_2(CS) concerning relationships among figures (mean 1.18). This was similar to the step difficulties concerning relationships among properties, where M_2(CS) to R_2(CS) (0.70) was found to have a comparatively high step difficulty.

In addition, movement through the first cycle of the formal mode is a difficult progression concerning relationships among properties. This is evident by: U_1(F) to M_1(F) (mean 0.68) and M_1(F) to R_1(F) (mean 0.67). It is interesting to note that the highest individual step difficulty concerns the shift from M_1(F) to R_1(F) in regards to relationships among square properties (1.53). Overall the progression from U_1(F) to M_1(F) concerning relationships among figures has the least step difficulty (0.08).

Discussion

The study was designed to complement and extend a qualitative analysis of results, through a procedure that provided comparative qualitative results across relationships among figures, relationships among properties, and different contexts. Of particular interest was the finding that despite the quadrilateral context being chosen in the study due to an increase in complexity, this was not mirrored by the analysis. The degree of difficulty was found to be similar within the triangle and quadrilateral contexts. The application of the Rasch model supported the developmental sequence that evolved through the SOLO categorisations. The results also highlighted a number of interesting trends. The first of these is the consistency of the groupings evident in the item estimate thresholds when comparing student responses across figure tasks, property tasks, and different contexts. Secondly, the fit statistics and item estimates indicate that the items came from the same underlying construct. This provides confirmation of the appropriateness of the SOLO model.

The concrete symbolic responses indicate that a focus upon a single property to encapsulate separate classes of figures is a prerequisite to focusing upon a single unique property of a figure when asked to provide a minimum description of a figure. The M_2(CS) responses indicate that the shift in moving from multiple properties to form individual classes of figures is at the same level as identifying multiple unique property signifiers while maintaining a real world referent. Thus, the figure determines the properties.

The identification of a link between two properties, and the shift to utilising the relationship as a workable unit, are necessary precursors to the utilisation of relationships among classes of figures without the need for a real world referent. This progression is a shift into the formal mode in terms of relationships among properties, and is characterised by the property relationships determining the figure in both contexts. When the formal mode is entered, concerning relationships among properties, the degree of difficulty is the same in regards to linking properties or figures despite the bifurcation. The focus upon perceiving the property relationships as determining the figures and utilising inclusive language to describe properties begins at a lower level than focusing upon links across classes of figures. This sequence flows through to a focus upon the network of relationships among figures and properties where there is greater variation in degree of difficulty found by the students across the seven tasks when providing a U_2(F) response.

The higher and lower step difficulties between SOLO response categories assist in the interpretation of the more difficult, and less difficult progressions from one SOLO level to the subsequent SOLO level. The highest increases, or “hard boundaries”, were found to be in the second cycle of the concrete symbolic mode concerning relationships among figures. These increases concerned the progression from a focus upon single properties to form individual classes of figures, to multiple properties while maintaining mutually exclusive classes, with the hardest boundary being the shift to a focus upon relationships between classes that are not supported by dominant visual differences. Similarly, in the context of property relationships, hard boundaries exist in the shift from a focus upon multiple properties as unique signifiers of a figure, to a focus upon an ordering between two properties. To a lesser extent, the boundary is relatively difficult when moving from an understanding that the figure determines the property, to a shift into the formal mode where relationships among properties determine the figure. Another boundary exists in the progression from a focus upon multiple relationships among properties, to an overview of the network of relationships among properties.

Of particular interest are the supporting influences between relationships among figures, and relationships among properties. These include the encapsulation of properties to form classes, a shift to perceiving the properties as determining the figure, the dominance of recognised similarities and differences across classes of figures and among properties, and the utilisation of inclusive or exclusive, class, or property descriptions. The identification of differing boundaries between the categories provides insight into the difficulties found by students when encountering notions of class inclusion in Geometry.

References

Adams, R. J., & Khoo, S. (1993). *Quest – The Interactive Test Analysis System.* Melbourne: Australian Council for Educational Research.

Biggs, J. B., & Collis, K. F. (1982). *Evaluating the Quality of Learning: The SOLO Taxonomy*. New York: Academic Press.

Bond, T. G., & Fox, C. M. (2001). *Applying the Rasch Model: Fundamental Measurement in the Human Sciences*. Mahwah, NJ: Lawrence Erlbaum Associates.

Callingham, R., & Bond, T. (2006). Research in Mathematics Education and Rasch Measurement (Editorial), *Mathematics Education Research Journal*, *18*(2), 1-10.

Campbell, K. J., Watson, J. M., & Collis, K. F. (1992). Volume measurement and intellectual development, *Journal of Structural Learning*, *11*(3), 279-298.

Currie, P., & Pegg, J. (1998). Three sides equal means it is not isosceles. In A. Oliver & K. Newstead (Eds.), *Proceedings of the 22nd annual conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 216-223). Stellenbosch: PME.

De Villiers, M. (1998). To teach definitions in geometry or teach to define? In A. Oliver & K. Newstead (Eds.), *Proceedings of the 22nd annual conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 248-255). Stellenbosch: PME.

Fisher, W. (1993). Measurement-related problems in functional assessment. *The American Journal of Occupational Therapy*, *47*(4), 331-337.

Heinze, A. (2002). …because a square is not a rectangle – Students' knowledge of simple geometrical concepts when starting to learn proof. In A. D. Cockburn & E. Nardi (Eds), *Proceedings of the annual conference of the 26th International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 81-88). Hiroshima: PME.

Keeves, J. P., & Alagumalai, S. (1999). New approaches to measurement. In G. N. Masters & J. P. Keeves (Eds.), *Advances in measurement in educational research and assessment* (pp. 23-42). Oxford: Pergamon.

Masters, G. N. (1982). A Rasch model for partial credit scoring. *Psychometrica*, *47*(2), 149-174.

Serow, P. (2006). Triangle property relationships: Making the connections. In Novotná, J., Moraová, H., Krátká, M., & Stehliková, N. (Eds.), *Proceedings of the 30th annual conference of the International Group for the Psychology of Mathematics Education* (Vol. 5, pp. 89-96). Prague: PME.

van Hiele, P. M. (1986). *Structure and insight: a theory of mathematics education*. New York: Academic Press.

Exploring Teachers' Numeracy Pedagogies and Subsequent Student Learning across Five Dimensions of Numeracy

Jane Skalicky
University of Tasmania
<jane.skalicky@utas.edu.au>

This paper reports the case of two teachers with respect to the positioning of numeracy in a reform curriculum and subsequent student learning across five dimensions of numeracy. By analysing the conversations of these two teachers, their underlying beliefs about numeracy and its value and role in the curriculum were able to be explored. These beliefs were further reflected in the learning outcomes of the six students in this study. The paper describes examples of how the five dimensions of numeracy were evident in the thinking and practice of both the teachers and their students.

Recent curriculum reform in Tasmania has been guided by a consideration of the knowledge, skills, and attributes required of students living in the twenty-first century. Tasmania's Essential Learnings Framework (Department *of* Education, Tasmania [D*o*ET], 2002) places thinking skills and strategies at the core of the curriculum and encourages the connection of knowledge and concepts across the curriculum. It emphasises the importance of being numerate rather than purely of knowing and doing mathematics. An ability to understand and apply mathematical concepts is valued alongside the development of students' abilities to problem solve, reason, communicate, and reflect upon their learning.

Teachers and Students Negotiating Curriculum in the Classroom

Innovative and reform curricula are filtered through teachers' beliefs and practices (Wilson & Lloyd, 2000). Although researchers are aware of the broader contexts and policy-driven environments that influence curriculum construction, it is the curriculum that is enacted in the classroom that drives the research from which this study is taken. Teachers add a pedagogical dimension to curriculum to create daily learning experiences for their students. It is that knowledge that equips teachers to "lift the curriculum away from texts and materials [and] to give it an independent existence" (Doyle, 1992, p. 499).

The role of the student in curriculum is also acknowledged. Students determine their own level of engagement and interest in classroom activity and therefore exert some control over their learning and knowledge construction. Snyder, Acker-Hocevar, and Snyder (1992) suggest that "curriculum enactment" appropriately describes the process of implementation and educational experience that teachers and students jointly undertake as they negotiate and determine what the curriculum will be like in each classroom.

With respect to the teaching of mathematics, teachers' knowledge, beliefs, and practices play a significant role in the learning of their students (Hill, Rowan, & Ball, 2005). It is therefore important to look at the beliefs and practices of teachers in relation to the learning of students in Tasmania's curriculum context. This study aims to deepen understanding of the construct of numeracy through considering two ideas:

- The way in which two teachers position numeracy in a values-focused curriculum, and
- The way in which their students experience numeracy.

Theoretical Framework

Numeracy has become an essential capability for any individual who wishes to participate fully in a democratic society and to apply not only knowledge and skills, but also critical reasoning capabilities, to learning and to everyday life. “Whereas mathematics is a well-established discipline, numeracy is necessarily interdisciplinary … numeracy must permeate the curriculum. When it does … it will enhance students’ understanding of all subjects and their capacity to lead informed lives” (Steen, 2001, p. 115). The concepts and skills required to meet the numeracy demands of everyday life are defined and examined under various names, including quantitative literacy (Steen, 2001), mathematical literacy (Organisation for Economic Cooperation and Development [OECD], 2006), critical numeracy (Johnston, 1994), mathemacy (Skovsmose, 2004), and numeracy (Australian Association of Mathematics Teachers [AAMT], 1998). Each definition has particular theoretical underpinnings, whether it be an emphasis on the psychological, social, or cultural nature of learning, or as is the case of more recent terms, such as mathemacy and criticalmathematical [sic] literacy, whether it be informed by critical theory where the role of politics and power within social and cultural contexts is placed at the fore.

Green (2002), in considering the role of literacy in the English classroom, and the wider curriculum, acknowledges the different discourses of language, meaning, and power that play a role in the development of literacy. He advocates the synthesis of these dimensions in forming a three-dimensional model of literacy where “the most worthwhile robust understanding of literacy is one that brings together the “operational”, “cultural”, and “critical” dimensions of literate practice and learning” (Green, 2002, p. 27). Although Green acknowledges the political nature of literacy as a social practice, he calls for a balance between all the important dimensions of literacy with the aim being to support students in meaning-making in context.

It is equally important for mathematics educators to acknowledge the different dimensions that are necessary for the development of competent and effective numeracy practice. Mathematical language, skills, and functions are required for students to make sense of, and critically evaluate, the contexts in which the mathematics is embedded. The socio-cultural and critical aspects of knowledge construction enable the selection of appropriate mathematical tools and informed critique of both mathematics and society. This study acknowledges the important contribution each element brings to a comprehensive definition of numeracy. Numeracy is about making meaning of mathematics, at whatever level of mathematical skill. It is not inferior to mathematics, but rather is about understanding and using mathematics, in all of its representations, for making sense of the world, for considering critically information presented, and for making informed decisions.

The view of numeracy adopted in this study is underpinned by social constructivist theory. Shepard (2001) expounds the principles of social constructivism as drawing from contemporary cognitive, constructivist, and socio-cultural theories. Although valuing the sense-making and active process of mental construction that individuals undergo to construct their own knowledge, the importance of the social and cultural interactions is not neglected.

Table 1 contains a summary of five dimensions of numeracy, based upon the aforementioned principles together with a comprehensive review of the literature as it pertains to numeracy education. In particular, the work of AAMT (1998), Steen (2001) and

Queensland School Curriculum Council (1999) was considered in presenting a comprehensive balanced view of numeracy extending across foundational mathematical concepts and skills, strategic thinking, disposition, recognition of context, and critical practice. Numeracy is a complex construct with many aspects, beyond mathematical skill, contributing to a high level of numerate behaviour.

Table 1
Dimensions of Numeracy

Aspects of knowledge construction	Dimensions of numeracy	Description
FOUNDATIONAL	Mathematics	The foundational understanding and use of the language, concepts, and skills of mathematics, as they relate to number, measurement, space, data and chance, and pattern and algebra.
PSYCHOLOGICAL	Reasoning	The use of (i) mathematical thinking strategies to question, identify, represent, explain, and justify mathematical approaches relevant to a given context, and (ii) general thinking strategies to support the problem solving process, from lower level cognitive processes, such as recall and application, to higher level critical thinking processes involved in evaluation, judgment, decision making, and creativity.
AFFECTIVE	Attitude	The confidence and disposition to choose and use mathematical understandings wherever required. Willingness to take risks and persevere in approaching new mathematics and new contexts.
SOCIO-CULTURAL	Context	The ability to select and apply the appropriate mathematical tools for sense-making in a given context and understanding how the context impacts on the mathematics. Contexts related to school and everyday life, public and social issues, and an awareness of mathematics connected to history and culture.
CRITICAL	Equity	Awareness that mathematics can be used inappropriately, can be represented to promote bias, and can therefore promote inequities in society. The ability to question assumptions and use mathematics in an analytical and critical manner to make decisions and resolve problems and investigations.

Method

The research reported in this paper was part of a larger qualitative study designed to investigate the positioning of numeracy by teachers of middle grade classrooms (Grades 5-8) in Tasmania's reform environment and student experiences of numeracy in these classrooms. The larger study adopted a collective case study approach with five participant teachers and their students. All the teachers had an interest in numeracy and planned and implemented units of work informed by Tasmania's Essential Learnings curriculum framework (D*o*ET, 2002). In addition, a representation of middle years' grades was sought across a range of schools. The research used a combination of interview, observation, document, and photographic data to provide insight into the unique positioning of numeracy as enacted in the classroom by each teacher and the experiences of their students.

In this study the case of two Grade 8 teachers, interviewed together, and six of their students is reported. Teacher interviews were semi-structured and lasted approximately 40-50 minutes. They were designed to gain an insight into teacher beliefs and practices with respect to current curriculum reforms; views concerning the place of numeracy within these reforms; and their planning, teaching, and assessment practices.

After completing the unit of work with the 34 Grade 8 students, six students were invited to participate in an individual in-depth interview. These six students were chosen by the teachers, in consultation with the researcher, as representing a spread of mathematical ability. The student interviews were semi-structured and were 30-40 minutes in length. The students brought relevant work samples to the interview to support discussion about the tasks they completed.

As a qualitative study, cluster analysis (Miles & Huberman, 1994) was used to code the teacher and student interviews. They were analysed according to the five dimensions of numeracy as detailed in Table 1 of the theoretical framework. Excerpts from both the teacher and student interviews are included in this study to illustrate each of the dimensions of numeracy as they were exhibited.

Context of the Study

The School Setting

Tanglefoot School, an independent girls' school, was the setting for the study. Although part of the wider Tasmanian educational community, as an independent school, Tanglefoot did not fall directly under control of the state government education system. The Essential Learnings Framework (D*o*ET, 2002) was included in Tanglefoot's Middle School Handbook as the underpinning framework that informed a curriculum incorporating three main aspects: traditional subject disciplines, interdisciplinary units of work, and six week mini-courses providing options in academic, life skills, and recreational areas of student interest.

At Tanglefoot School the discipline of mathematics was a core subject for students in the middle school. It involved the explicit teaching of the five strands of mathematics: number, space, measurement, chance and data, and algebra, in addition to thinking, acting, and communicating mathematically. Each Grade 8 class had four 50-minute mathematics lessons timetabled each week with students' numeracy capabilities encouraged through the discipline of mathematics. At times, however, students were also required to draw upon their knowledge and skills developed in mathematics for use in other subject areas and in their interdisciplinary units of work. This was the situation with the unit of work observed during the case study.

The Unit of Work: Live 8

Ange and Jen (pseudonyms), the two Grade 8 teachers at Tanglefoot School, worked collaboratively to implement a five week integrated unit of work, *Live 8*, inspired by music concerts held across the world in 2005, by prominent musicians, to highlight the issue of world poverty. Ange and Jen were motivated by a belief that the *Live 8* concerts would provide the Grade 8 students with an engaging, real-world context in which to learn about the contrasting nature of developed and under-developed countries and issues related to the broader concept of poverty. The unit of work brought together the disciplines of Mathematics and Studies of Society and the Environment (SOSE) with the aim of enhancing students' numeracy capabilities, their abilities to work collaboratively, their skills in information literacy and communication, and ultimately their understandings of the concept of poverty. The context of poverty was used to develop further the students' skills in graphing and data analysis. The unit of work culminated with the girls completing a major assignment requiring them to investigate one country and compare it to life in Australia. Students were specifically asked to consider aspects such as population, mortality rates, literacy levels, income, government systems, economies, water supply, and aid programs.

Results

The Teachers: Ange and Jen

Ange, the teacher of Mathematics and Science for the middle school had been teaching for seven years and commenced at Tanglefoot School in 2003. Jen, the teacher of English and SOSE, had been teaching for three years. Tasmania's Essential Learnings Framework (D*o*ET, 2002) informed Ange and Jen's teaching since it had been incorporated into Tanglefoot's construction of middle school curriculum in 2003. Ange and Jen were interviewed together, with Ange playing a predominant role and Jen contributing where she felt comfortable and where she wished to add a comment.

Ange and Jen felt that the Essential Learnings (ELs), Tasmania's curriculum, supported "real-life learning goals" and "[sat] nicely with integrated units of work". They gave examples of where they had planned for mathematical learning in previous integrated units of work. In addition, Ange and Jen spoke of the inclusive nature of the ELs catering for the "different learning needs" of students and allowing students to "go in and show exactly what they do know and what they can achieve", contributing to development of students' "self-esteem". Their personal views were aligned with the school's construction of curriculum and they also had autonomy over how they implemented integrated units of work with their students.

Ange and Jen expressed a view toward numeracy that did not place the role of numeracy across the curriculum above the role of mathematics as a "discrete subject". Ange, in particular, spoke of the importance of mathematics for providing some students with "pathways" for their future learning and that the ELs enabled a focus to "get the girls interested in maths" and learning to be numerate across disciplines. The following section details how the two teachers' conversations about their teaching practice could be described according to the five dimensions as state earlier in the Theoretical Framework.

Mathematics. The foundational role that mathematics plays in developing numeracy was evident when Ange described numeracy. "I think numeracy is applying, the application of those mathematics skills into different areas". Her content knowledge was evident through many of the comments she made as she discussed her teaching. For example, aspects of the content of algebra and number were mentioned in describing the importance of teaching and assessing for numeracy.

As the SOSE teacher, Jen mentioned the importance of students having the opportunity to apply their knowledge of concepts related to culture, community, society, and the environment to build their understanding of important mathematical concepts.

> In SOSE we use numeracy in graphing, reading tables, analysing statistics and things like that. I make sure they can relate it to [life] … If they have to apply it they can actually grasp the concept.

Reasoning. In discussing the role of numeracy in the middle school curriculum, Ange referred to the language of "thinking" as forming an important part of student assessment. Both teachers referred to the middle school assessment booklet (Tanglefoot, 2005) on numerous occasions and the important role it had in informing their teaching and assessment practices. In this booklet, strategies such as posing questions, recalling strategies and relationships, conjecturing, justifying, explaining, and drawing conclusions were listed as important elements of working mathematically.

The teachers talked about wanting to see evidence of how the students were thinking and problem solving. Ange highlighted the value of students "showing their working out" as it helped the teachers to "really know how they [the students] are going" as opposed to "working in class out of books". She felt that text books did not provide her with information on how students were thinking when solving problems.

Attitude. The importance of a positive disposition toward numeracy in contributing to positive numeracy outcomes, although not explicitly mentioned by either of the two teachers, was implicit in their comments. Ange mentioned her aim to "get the girls interested in maths" through the teaching of numeracy. Jen said she wanted to "make sure the girls can relate to it" and tried to engage the students with tasks that would be of interest to them. For Ange and Jen, the role of numeracy, as mathematics in context, was the key to developing this engagement, "interest", and positive disposition.

Context. Both Ange and Jen expressed a belief that numeracy was very much about using mathematics in context. They saw numeracy as "something that is taught in lots of subjects" and involving the "application of mathematical skills into different areas".

Jen gave examples from when she spent a short time teaching Grade 7 mathematics, not her usual teaching area, where she would provide the students with opportunities to "try to apply that knowledge too… to real-life situations". The contexts valued by the teachers were authentic, real-world contexts, as evidenced in this comment by Ange.

> I think that is the way that maths will probably be going in the future. It is going to be real-world context and I think that is important. … and I think the ELs, with Being Numerate as a focus, will sit quite nicely with integrated units of work. Hopefully that will develop over the years.

When talking about their teaching they provided examples of contexts they had used with students. Contexts such as crime, health, design, and decorating were mentioned.

Equity. Ange discussed how important it was for mathematics education to cater for "the needs of all students". She described numeracy in its role across the curriculum as being the way "to get the people who struggle". Although neither Ange nor Jen expressed in the interview aspects of numeracy teaching that would equip students with the ability to consider information critically, or consider inequities in society, the *Live 8* unit of work implemented after the interview provided an example of their underlying beliefs in this area.

The Students

The six students interviewed in this study were asked to describe and discuss specific graphs they had completed during the unit of work. In particular, the graphs included in students' major assignments on poverty formed the focus for the interviews. The students were happy to participate in the interview and were forthcoming in telling the stories of their graphs. The conversations started with specific mathematical content displayed in the graphs but as the interview progressed the comments encompassed broader issues about how the graphs helped them understand poverty. The following excerpts provide examples of the students' learning across the five dimensions of numeracy.

Mathematics. All the students demonstrated specific mathematics understandings in explaining their graphs and used the mathematics to help them when comparing their country of investigation to Australia. They also used the language of mathematics, specifically as it related to chance and data, and number. The range of their responses is evident in the following excerpts.

> The big difference in the way that Afghanistan live to the way Australia live, like the average income, people in Australia can earn $30,000 a year easily, and in Afghanistan it is $280 a month (emphasised). [Student 5]

> The gini index for Australia is 35.2 and then for Rwanda it is 28.9. Basically with the gini index, zero is all the money in the country is completely fairly distributed and 100 means completely unevenly distributed. [Student 6]

> It shows that pretty much everyone in Australia can read that are over the age of fifteen, but in Nepal they don't have much literacy, or options to read. The literacy rates for men are just over 60% and the females just over 25%. [Student 3]

Some students also mentioned how their previous mathematics learning had helped during this unit of work, as exemplified by Student 6's comment about frequency tables.

> We have done frequency tables before in maths and it helped here because I understood. With graphs I always forget which way the x and y axes go. It helped me to remember and how to set it out and what they're for. [Student 6]

Reasoning. The thinking strategies of students were identified when they were describing their graphs. Student 4's comments about literacy rates in Sudan was typical of the students as they became engaged in the discussion and moved beyond the mathematics in their graphs to the reasons for the results and considering other information they had researched about their country.

> Ah well the literacy rate. Here I suppose it shows that the females, as in probably most countries in that region or area are less educated than males, probably because of priorities in the system and religious beliefs. The literacy in Australia is obviously amazingly higher than Sudan but in Sudan they have a program, I can't remember but they give free education and I think it is for the first six years and the government is focusing on eliminating illiteracy in the country. [Student 4]

The ability of the students to make comparisons and explain their work was shown by one student when she demonstrated a distinct engagement with the issue by her surprise and shock.

> When I looked at this it really shocked me a bit because you don't really realise how much money goes in and out of your house and for Australia $800 a week is really a lot of money and when I saw Somalia which is one dollar it was really amazing. The graph when you look at it you can really see the difference between the two countries. [Student 2]

At times the students' thinking moved to a focus on the impact of poverty on the context of their particular country of investigation.

> The other countries, since they are so rich they shouldn't worry about it because they have a lot of money and the aid programs are good, but they probably need to do more to help the country out like bring in more food supplies and more fresh water. [Student 3]

> Comparing the data makes it more personal and thinking about children there who can't read and write when they're fifteen and stuff like that. [Student 1]

Attitude. Students' personal disposition toward numeracy became evident when they were discussing their work. Five of the six students expressed a preference for using mathematics in real-life settings. Student 6, for example, focused on the application of skills in engaging her in learning.

> It is more interesting and you actually put it to use rather than just learning it so we actually put it there and have to come back to the skills we've learnt and stick it on. I prefer to use it in real life because it's interesting and it's so much better because it is for something you're learning about and not learning how to do it. [Student 6]

By "stick it on" the student here was referring to applying the mathematics. Three of the six students mentioned the value of having to find the relevant information for themselves and making decisions about how to represent their data.

Students' willingness to persevere and engage with the task was exemplified in different ways. Examples include a comment by one student about the time put into gathering correct statistics, and by another student about the discussions she had with family members.

> Well it was actually quite hard to compile all the correct statistics and data. I had to go to several different websites and collect different numbers for each year and then I had to put them all together. [Student 4]

> I talked to my sister about it on the phone and she is in London, so it helped me to understand how *Live 8* was working and what was happening and that had an influence on the way I did my assignment. [Student 5]

Context. The context of student learning, in this case the country of investigation and wider issues of poverty, featured prominently when students were explaining their graphs. As discussed above in the dimension of "Attitude" students expressed a preference for applying their learning of mathematics to real-life contexts. There were many times, in explaining their graphs, that the students focused on the context of their country.

> They're one of the poorest countries in the world and most of the people live under the poverty line and children under five die of malnourishment before they reach the age of five. [Student 3]

> They are probably not living past the age of 50 because of all the violence, heaps and heaps of people were killed, and the water and disease and stuff like cholera and dysentery. [Student 6]

Student 5 also noted the value of situating her learning in the context of the country that formed the focus of her investigation, and the importance of comparing living conditions in that country to life in Australia.

> It was good to realise how much of a difference there is to the way we live to the way other countries live, because if we hadn't done this we wouldn't have known. It has made us more aware of the way we live to the way for example that Afghanistan lives. [Student 5]

Equity. In this particular unit of work, the mathematics enabled the students to question societal structures. Student 1 described the tensions between the importance of wealthier countries providing financial support and the difficulty in ensuring the money goes to where it is needed.

> I think that the richer and more developed countries in the world need to offer money and support, like they are at the moment but I also think like in Africa they have got corrupt governments and so they give them money and all that sort of thing but often the government takes it for themselves rather than using it to help the people. I think something needs to be done about the governments, but even if their government is overthrown they are still going to need support from richer countries. [Student 1]

Student 2 focused on the basic needs and important resources needed in underdeveloped countries.

> Just getting aid into countries and helping get clean water and clean food and resources and I think all the countries like America and Australia and Asia should really put in to help out these countries that are not as well off because Africa is a struggling country and I think they really do need some help and we are being a big selfish with our resources. [Student 2]

All six students considered personal contributions that they felt would, in some small way, assist in alleviating poverty. The following excerpts represent the range of suggestions, from supporting local fundraising opportunities and raising awareness, to considering a career in overseas aid work.

> I think a lot of countries do help out, but people in general could help more. Like everyday people, we do try to help and we may see an ad on tv about sponsoring children or fundraising but I think we need to focus more on ourselves and what we can do. [Student 4]

> Well ever since I was little I have wanted to make a difference in those kinds of places, like I want to study medicine and help out there. … I was thinking of going and being a doctor of an anaesthetist or paediatrician in somewhere like Somalia or probably in Iraq. [Student 2]

Discussion and Conclusion

Although the two teachers in this study were not explicit about what contributes to high level numeracy, when discussing their teaching practice the five dimensions of numeracy were evident in their discourse and impacted upon the learning outcomes of their students. The positioning of numeracy, as a cross-curricular construct in the curriculum, informed Ange and Jen's teaching practice. They saw the foundational role of mathematics as crucial for students when tasks required them to apply their understanding of mathematical concepts in subjects other than mathematics and in integrated units of work. The teachers encouraged their students to show their reasoning when solving problems and when discussing their work in order to inform Ange and Jen's assessment of student learning.

Real-world contexts were described by the teachers as important for the learning of students. These included a combination of school and everyday contexts that the girls could relate to and also wider social, cultural, and political contexts. These real-world contexts were viewed as being central to the development of positive student attitudes toward numeracy. Planning for numeracy outcomes in interdisciplinary settings was seen by the teachers as supporting not only the learning of mathematics, but also of other important concepts. Students were encouraged to use mathematics to consider and reflect upon society and its structures and inequities.

By analysing the conversations of these two teachers, their underlying beliefs about numeracy and its value and role in the curriculum were able to be explored. These beliefs were further reflected in the discourse of the six students in this study. The students were able to identify appropriate mathematics in describing their work. Their mathematical understandings also enabled them to engage positively with the context, to consider many issues related to the context of the country they investigated, and finally to move toward an informed critique of poverty.

As Australia grapples with the re-conceptualisation of curriculum it is crucial that the place of numeracy is considered. This study has considered the positioning of numeracy by two teachers in a reform environment and the numeracy experiences of their students. It has described examples of how the five dimensions of numeracy were evident in the thinking and practice of both the teachers and the students. The results demonstrate the possibilities for student learning across all the dimensions of numeracy, when mathematics is purposefully embedded within interdisciplinary frameworks. Further analysis of the data that forms the larger research project, from which this study was reported, will contribute further to understanding the complex nature of the construct of numeracy.

Acknowledgements. The author wishes to thank Jane Watson and Noleine Fitzallen for their support and feedback in the preparation of this paper.

References

Australian Association of Mathematics Teachers, Inc. (1998). *Policy on numeracy education in schools*. Retrieved 16/03/04, from http://www.aamt.edu.au/about/policy/numpol/pdf

Department *of* Education Tasmania. (2002). *Essential learnings framework 1*. Hobart: Author.

Doyle, W. (1992). Curriculum and pedagogy. In P. Jackson (Ed.), *Handbook of research on curriculum*. (pp. 486-516). New York: Macmillan.

Green, B. (2002). A literacy project of our own? *English in Australia*, *134*, 25-32.

Hill, H.D., Rowan, R., & Ball, D.L. (2005). Effects of teachers' mathematical knowledge for teaching on student achievement. *American Educational Research Journal*. *42*(2), 371-406.

Johnston, B. (1994). Critical numeracy. *FinePrint*, *16*(4), 32-35. Melbourne, Australia: VALBEC.

Miles, M. B., & Huberman, A. M. (1994). *Qualitative data analysis* (2nd ed.). Thousand Oaks, CA: Sage Publications, Inc.

Organisation for Economic Cooperation and Development. (2006). *Assessing scientific, reading and mathematical literacy: A framework for PISA 2006*. OECD: Paris. Retrieved 5/01/06, from http://www.pisa.oecd.org/dataoecd/63/35/37464175.pdf

Queensland School Curriculum Council. (1999). *Numeracy position paper*. Queensland: The Office of the Queensland School Curriculum Council. Retrieved 24/12/05 from, www.qsa.qld.edu.au/yrs1to10/oia/papers/numeracy.pdf

Shepard, L. A. (2001). The role of classroom assessment in teaching and learning. In V. Richardson (Ed.), *Handbook on research on teaching* (4th ed.), pp.1066-1101. Washington, DC: American Educational Research Association.

Skovsmose, O. (2004). *Critical mathematics education for the future*. Retrieved from, www.lfd.learning.aau.dk/resources/CME-for-the-Future.pdf

Snyder, K.J., Acker-Hocevar, M. & Snyder, K.M. (1994). *Organisational development in transition: The schooling perspective*. Paper presented at the Annual Conference of the American Educational Research Association. New Orleans.

Steen, L. A. (Ed.). (2001). *Mathematics and democracy: The case for quantitative literacy*. Princeton, NJ: National Council on Education and the Disciplines.

[Tanglefoot]. 2005. *Middle School assessment booklet*. Tasmania: Author.

Wilson, M., & Lloyd, G. (2000). The challenge to share mathematical authority with students: High school teachers reforming classroom roles. *Journal of Curriculum and Supervision*, *15*, 146-169.

The Complexities for New Graduates Planning Mathematics Based on Student Need

Carole Steketee
The University of Notre Dame Australia
<csteketee@nd.edu.au>

Keith McNaught
The University of Notre Dame Australia
<kmcnaught@nd.edu.au>

During 2006, two teams of preservice teachers spent a week in three rural schools and completed diagnostic assessment tasks in mathematics using the Nelson Numeracy Assessment Kit. The classes that were assessed were all being taught by newly graduated teachers. The results were collated into detailed profiles, which enabled these teachers to identify whole class, small group, and individual strengths and weaknesses. It was anticipated that the new graduates would find these profiles of great benefit in planning for mathematics. However, the teacher-educators who continued to work with the new graduates discovered that this assumption was flawed, and that the new graduates experienced difficulty in planning curriculum based on identified needs. This paper discusses the typical approaches to curriculum planning adopted by the teachers, which were largely teacher-centred.

Introduction

The challenges faced by newly graduated teachers working in "hard to staff" rural locations are well recognised. Western Australia has a number of rural locations where schools find it difficult to attract and retain staff. Although coastal locations are highly sought after by teachers, including new graduates, less desirable locations frequently attract a limited pool of applicants, and those applicants are often uncompetitive in a large field of applicants. The poorer academic performance of students in rural areas, compared to their metropolitan counterparts is well recognised (Pegg, 2005). Further, the more isolated the location, the more pronounced the negative impact on student learning is (Cresswell & Underwood, 2004).

During 2006, as part of an ASISTM project designed to support newly graduated teachers, a university-school partnership was established with three "hard to staff" locations (Northville, Eastville, and Westville Primary Schools). The project goal was to support newly graduated teachers with mathematics teaching and learning. Using final year undergraduate students, all of whom were completing a mathematics "specialisation" pathway, the plan involved administering diagnostic assessment to build detailed profiles of student needs. Given the physical isolation factors, video conferencing was used to provide ongoing support throughout the year. Fifteen final year students at the University were trained to administer the diagnostic tasks within the Nelson Numeracy Assessment Kit. The kit provides assessment tasks for four strands of mathematics: Number, Measurement, Space, and Chance and Data. The Number test was administered to a total of 14 classes across the three target schools, and each class was being taught by a newly graduated teacher.

Prior to testing occurring within the schools, a full day of professional development was provided on site for the teachers involved in the project. The teachers were trained in diagnostic assessment procedures and trained in the use of the Nelson kit. All three schools were independent schools, and needed to source their own staff. Northville was the least desirable of the three locations. All the classes at Northville were "split grades", with a total enrolment of less than 90 students. Although some schools chose to operate with mixed-age/multi-age groupings, the use of composite grades at Northville was related to small student numbers within year levels and was an administrative rather than an educational decision. Both Eastville and Westville were hard to staff, but offered a range of social and recreational opportunities for staff, and school sizes not dissimilar to regular metropolitan schools. Eastville and Westville had staff with a range of experience and it was usual for a new graduate to stay 2 or 3 years before returning to the metropolitan area. However, Northville's most experienced staff member in 2006 was in her second year of teaching. In 2004 and 2005, Northville had experienced 100% staff turnover for class teachers; a support teacher and principal were the only two to remain on staff.

All three schools had identified that this lack of experienced teachers on staff, and small staff numbers, limited the capacity of the school to offer a mentoring program on site. The need for mentors for new graduates is well recognised and the benefits of "buddy teacher" on staff can provide invaluable assistance to a new graduate (Kyle, Moore, & Sanders, 1999).

Developing Profiles of Students' Mathematics Learning Needs

During term two, 2006, the final year students spent a week in residence in each school. They administered the diagnostic assessment tasks from the Nelson kit. With the tests administered, in collaboration with their teacher-educator also in residence, a whole class profile was created. Each child within each class was plotted on the full range of tasks within Number for each year level. In a small number of cases, children were assessed and plotted on tests from different year levels, in most cases to cater better for students who were working at least two years below their current grade level. For example, in Northville, two students in Year 7 were assessed on the tasks from the second grade battery of tests, and this provided valuable data about their performance level.

With the whole class profiles created, hand-over meetings were conducted. The pre-service teachers had assessed classes in collaboration with a peer, and both were present to hand over the profile and discuss the various components and results, with the teacher-educator facilitating the meeting. In all three schools, the principal was present for the handover meeting, and took an active role in the analysis of each profile. The class teachers responded to the profiles in range of ways. Most common was delight that this detailed profile had been prepared largely "for them", and they appreciated that a serious and sustained time commitment had been required. In most cases, the individual class profile appeared to confirm their understandings and sense of how individual performance would be shown. In all classes, there were at least some students who were a surprise to the principal and/or class teacher, either with better than expected, or worse than expected, performance.

The Year 4 class profile from Northville provides an example of the results of 13 students within that class (Table 1).

Armed with these profiles, and knowing the considerable amount of work that had gone into their creation, it was a clearly conveyed expectation that these profiles would provide the basis for future programming and planning in mathematics for each class. Each of the principals was explicit in this expectation, and the teachers were encouraged to use each other, physically and via video conferencing, and the two teacher-educators (via video conferencing, email, and telephone support) to do this.

Table 1
Northville Year 4 Class Profile

Topics in which there was an average student score of 75% and over	Topics in which there was an average score of 50% - 74%	Topics in which there was an average score of 26% - 49%	Topics in which there was an average score of less than 25%
Whole number	Mental strategies	Multiplication	Division
Addition	Subtraction	Problem solving	Mathematical laws
	Patterns		Computation with decimals
	Place value of decimal numbers		Computation with fractions

Using the Profiles for Curriculum Planning in Mathematics

The teacher-educators coordinating the project assumed that this planning process would occur quite naturally. That is, they assumed, that armed with the detailed class and individual profiles, the teachers would be able to identify the topics and skills that required whole class teaching focus. Additionally, it was expected that individual needs would be catered for, and that support programs would target specific skills for specific individuals within the classes. For example, it was expected that the Year 4 teacher at Northville's plan for the coming term would focus on multiplication, division, computation with decimals and fractions, problem solving, and mathematical laws.

Evidence suggests that expert teachers base instruction on student need. Hattie (1992) identifies that effective feedback, based on recognising student strength and weaknesses, is the variable that provides the most impact on improved student learning. The "coach" metaphor is regularly applied to teachers who are highly skilled at effective feedback. They provide specific, not generalised, feedback and implement a teaching (coaching) plan based on addressing specific sub-skills to improve performance (Wiggins, 1998). Although the teacher-educators did not expect this level of intervention, their aim was to introduce the graduate teachers to the concept of curriculum planning based on student needs – a form of practice they hoped would become common practice with experience.

However, this assumption was flawed and it was apparent almost immediately that the graduate teachers were largely overlooking the profiles and basing their curriculum planning on past practices. The fortnightly video conference sessions that commenced at the beginning of Term 3 were intended to support the teachers in their implementation of their mathematics program. The graduate teachers were expected to "drive" these sessions based on questions and concerns that arose in the course of their instruction. Instead, however, much of the conversation centred around trying to extract from the teachers what they were teaching, how they were teaching it, and what their purpose was in taking this approach (if not in the light of students' learning needs). It was becoming increasingly evident that the graduate teachers were not able to use the profiles as the basis of their planning and that a range of different methods of planning were emerging and ultimately affecting the success of the project.

As such, it was necessary to gain a deeper insight into these planning methods if the graduate teachers were to be supported in making the link between the profiles and effective planning. Consequently, the data that emerged from the video conference session transcriptions were explored using a constant comparative method of data analysis (Glaser & Strauss, cited in Lincoln & Guba, 1985). Categories of meaning in relation to the various methods the teachers used for curriculum planning were described and refined into the following five themes.

Findings

Teachers' Preferences

A number of teachers openly discussed the fact that they mostly taught mathematics concepts that they enjoyed and/or were personally confident in teaching. Curriculum planning was in relation to their identification of concepts in the curriculum that they had sufficient content knowledge of, and those that they did not fully grasp. As the following quote below suggests, some teachers purposefully excluded certain concepts if they felt they were unable to understand it themselves.

> I have to admit maths is not my best personal area. I am good with things like times-tables, but when it gets technical, things like fractions and decimals, I have to revise all the work before I teach the class.

Ball (1997) argues that primary school teachers' self efficacy about their mathematical content knowledge and pedagogical ability is low.

Text-book Teaching

Text-book teaching was possibly the most common type of planning discussed by teachers in the project. Although this theme indicates that some form of forward planning is occurring, it is largely in relation to the order in which certain aspects of a text should be taught over the course of a term and year. The ideas of what to teach are extracted from the text along with the suggestions of how these concepts should be taught. As is indicated in the following quote, the ideas espoused by the book are often supplemented by the use of manipluatives and concrete objects, as well as worksheets.

> Yeah I use a few text-books. We've got some good ones at our school. I like how it helps you understand the sequencing of how the kids should learn how to do something. Yeah, I don't only use the text-book though … I get the kids working with lots of different materials …we use a lot of different worksheets, not just the ones from the text.

Research indicates that both experienced and beginning teachers rely heavily on commercially published materials to plan and deliver their mathematics instruction (Woodward & Elliott, 1990). The actual extent to which teachers use these materials, however, is possibly related to their level of confidence and experience in the classroom. More experienced teachers might use them to make decisions about what instruction to implement in the classroom whereas beginning teachers might use them to prescribe regimented, page by page activity.

Curriculum Driven Planning

Curriculum driven planning was also very common. Teachers identified a variety of mandatory curriculum documents as being integral to the mathematics programs they developed. The perception is that these documents are benchmarks of what students should be able to do/know at a certain year level. These benchmarks are subsequently used to source pre-made activities and worksheets from text-books and other commercially produced products. Typical of this theme is the notion that mathematics concepts are planned to be taught on the basis that the students "have not done them yet", as is indicated in the following quote.

> The progress maps tell you what your kids should be doing at their age. Not all the kids can do the same thing so I have a lot of group work going in my class with kids doing different things at the same time. We've spent a lot of time on measurement and time last term so this term we're going to do number … we haven't covered a lot of it yet.

This theme is closely linked to the previous one but differs in that use of commercially published text-books is guided by the Western Australian Curriculum Framework (1998), which all schools must use to base their curriculum planning on. Although this document was the first point of reference for these teachers, they mainly used it to discern the level of complexity at which students should be performing. Most of the teachers stated that it was of little use beyond that as it did not provide much detail and specification about what to teach and how to teach it.

School Focus Planning

School focus planning was prevalent due to the fact that the project coincided with the West Australian Literacy and Numeracy Assessment (WALNA) testing that all students in Years 3, 5, 7, and 9 in Western Australia must sit annually. WALNA is a curriculum-based assessment that tests students' knowledge and skills in numeracy, reading, spelling, and writing. The results provide schools with insight into their overall performance in these curriculum areas and, if used correctly, also assist teachers in setting improvement targets for their students for the following planning cycle. Teachers in this project did not discuss WALNA as a tool to make judgments about their students' learning needs. Rather, they saw it as something that they had to do given that it was a school focus, and something that would ultimately be used to evaluate the school as a whole. At least half of the term's planning was devoted to preparing students for WALNA, and then implementing it.

> We haven't got much time to do anything else just now. We've got WALNA this term so that's pretty much all we're doing in class at the moment.

Teaching Intuitively

As the term progressed and it became evident that the profiles of student numeracy learning needs had not been consulted by the teachers to plan their mathematics instruction, the project leaders began to question how mathematics classes were being taught and on what basis. In response to requests to see written mathematics forward planning documents and lesson plans, at least three of the teachers commented that they did not prepare handwritten programs. They stated that they were able to assess what the students needed to learn instinctively and usually had a mental outline of what they would implement over the term. Decisions about what to teach and how to teach were usually made in conjunction with other curriculum documents such as the progress maps and other commercially published texts.

> No I haven't done a [hand-written] program since Uni. They're such a waste of time … I just know what the kids have and haven't done and have a good idea of what I want to do each term …then I use lots of resources to give to the students.

Although it is feasible that experienced teachers are able to plan intuitively and spontaneously (Jones & Smith, 1997), it is unlikely that beginning teachers would be able to do so successfully without a considerable amount of practice across a range of different contexts. Jones and Smith write, "In constructing [curriculum], an experienced teacher is able to draw on a range of experiences and knowledge in an attempt to fit the anticipated and observed needs of a particular lesson or set of lessons" (p. 3). This practice comes after repeated opportunities to structure series of lessons around explicit learning objectives in the light of a particular context and available resources.

Discussion

The five themes that emerged in this study represent the participating teachers' methods of curriculum planning in relation to mathematics instruction. A common link among these themes is that planning is largely teacher-centred and based on factors that are external to the students. The teachers' decisions to teach particular content, and their instructional method were influenced by their perceived mathematical ability, the schools' mandated priorities, system enforced curriculum documents and/or other commercially published curriculum documents. At no stage did the teachers identify students' learning needs as being the starting point for their planning, despite the fact that they were armed with the profiles.

This outcome was unexpected. The teacher-educators anticipated that the teachers would have little experience and expertise in identifying comprehensive overviews of their students' mathematical learning needs. However, it was assumed that if they were supported in producing this information they would intuitively use it as the basis for their planning. Surprisingly, the teachers overlooked these profiles and instead reverted to their typical approach to planning.

By far the most common method was the use of text-books and other curricular materials. In a case study of four beginning teachers, Kauffman (2002) also found that text-books were central to new teachers' planning. He suggests the reasons behind this are related to the teachers' perceptions of the superior quality of the materials, the extent to which aspects of the text can be used to fit their own purposes and the ease with which the text can be used. Certainly, the stresses placed on graduate teachers during their first year would warrant them turning to curricular materials that alleviated the pressure to some degree. This is problematic, however, if teachers develop an over reliance on prescriptive teaching materials rather than teaching to clearly identified learning needs. What is even more concerning is when the teachers believe they are capable of teaching intuitively and in such a way that their lessons are loosely guided by mental plans of what should be taught.

Consequently, the overall goal of this project was modified and plans have been implemented to support these teachers to develop methods of curriculum planning that are based on clearly identified learning needs. Given that these same teachers are working in areas of recognised student-disadvantage, it is imperative that they are able to plan based on student need to maximise learning. Furthermore, the findings have been used by the teacher-educators to consider the extent to which curriculum planning is effectively taught in their pre-service courses. It may well be that curriculum planning taught during these courses is too hypothetical and the opportunities that pre-service teachers have to plan for real groups of students during their final internships is simply not sufficient and does not adequately prepare them for their first year of teaching.

Conclusion

The findings from this phase of the project suggest that the participating graduate teachers are not proficient curriculum planners. Even when made aware of their students' learning needs they chose to plan as they have in the past, adopting methods that were largely teacher-centred. Although there is a range of possible reasons, the fact remains that, if left unchecked, these methods could become common practice for these teachers. Consequently, the teacher-educators have entered into a new phase of the project and aim to support these teachers in the identification of the importance of basing their mathematics planning on their students' learning needs.

References

Ball, D.L. (1997). Developing mathematics reform: What don't we know about teacher learning – but would make good hypotheses? In S. N. Friel & G. W. Bright (Eds.), *Reflecting on our work: NSF teacher enhancement in K-6 mathematics* (pp. 11-112). Latham, MD: University Press of America.

Cresswell, J. & Underwood, C. (2004). *Location, location, location: Implications of geographic situation on Australian student performance in PISA 2000. ACER Research Monograph No 58.* Camberwell, VIC: Australian Council for Educational Research.

Curriculum Council of Western Australia. (1998). *Curriculum Framework.* Perth.

Hattie, J. (2002). What are the attributes of excellent teachers? in *Teachers Make a Difference: What is the Research Evidence.* (pp. 3-26).Wellington: New Zealand Council for Educational Research.

Jones, K. & Smith, K. (1997, May). *Student teachers learning to plan mathematics lessons*. Paper presented at the 1997 Annual Conference of the Association of Mathematics Education Teachers (AMET1997), Leicester.

Kauffman, D. (2002). *A search for support: Beginning elementary teachers' use of mathematics curriculum materials.* Paper presented at the Annual Meeting of the American Educational Research Association, New Orleans, LA.

Kyle, D. W., Moore, G. H., & Sanders, J. L. (1999). The role of the mentor teacher: Insights, challenges, and implications. *Peabody Journal of Education*, 74(3,4), 109-122.

Lincoln, Y., & Guba, E.G. (1985). *Naturalistic Inquiry*. Beverley Hills, CA: Sage.

Pegg, J. (2004). SiMERR National Summit at Australian Science and Mathematics School, Adelaide, Australia.

Wiggins, G. (1998). *Educative assessment: Designing assessments to inform and improve student performance.* San Francisco, CA: Jossey-Bass Inc.

Woodward, A., & Elliott, D.L. (1990). Textbook use and teacher professionalism. In D.L. Elliott & W. Arthur (Eds.), *Texbooks and schooling in the United States* (pp. 178-193). Chicago: University of Chicago Press.

Students' Emerging Algebraic Thinking in the Middle School Years

Max Stephens
The University of Melbourne
<m.stephens@unimelb.edu.au>

There is a strong case for arguing that the application of relational thinking to solve number sentences embodies features of mathematical thinking that are centrally important to algebra. This study investigates how well students in Years 5, 6, and 7 in three countries were able to use relational thinking to solve different types of number sentences. There were other students who appeared to rely solely on computational method to solve the same number sentences. The study then examined whether those who had shown clear evidence of relational strategies to solve the number sentences were better placed to solve symbolic sentences than those who had used only computational methods on these number sentences.

Relational Thinking

In their study, "The algebraic nature of students' numerical manipulation in the New Zealand Numeracy Project", Irwin and Britt (2005) argue that the methods of compensating and equivalence that some students use in solving number sentences may provide a foundation for algebraic thinking (p. 169). These authors give as an example the number sentence 47 + 25 which can be transformed into 50 + 22 by "adding 3" to 47 and "subtracting 3" from 25. They claim (p. 171) "that when students apply this strategy to sensibly solve different numerical problems they disclose an understanding of the relationships of the numbers involved. They show, without recourse to literal symbols, that the strategy is generalisable." Several authors, including Stephens (2006) and Carpenter and Franke (2001), refer to the thinking underpinning this kind of strategy as relational thinking.

Solving number sentences successfully using relational thinking certainly calls on a deep understanding of equivalence. Students need to know the direction in which compensation has to be carried out in order to maintain equivalence (Kieran, 1981; Irwin & Britt, 2005; Stephens, 2006). Some children who correctly transform number sentences involving addition reason incorrectly that a number sentence such as 87 – 48 can be transformed to be equivalent to 90 – 45. These children do not understand the direction in which compensation must take place when using subtraction or difference. They fail to recognise that the relationship of difference is fundamentally different from addition. Other children, however, recognise this feature explaining that in order for the difference to remain the same, the same number has to be added to (or subtracted from) each number to the left of the equal sign. These children write correctly 87 – 48 = 89 – 50. The first part of this study probed children's thinking with number sentences.

The Study

Three groups of number tasks shown in Figure 1 were given to students in Years 5, 6, and 7 using a pencil-and-paper questionnaire administered in regular class time. In introducing the questionnaire, classroom teachers told students that:

> This is not a test. It is a questionnaire prepared by researchers … looking at how students read interpret and understand number sentences. For most of the questions there is more than one way of

giving a correct answer. Please write your thinking as clearly as you can in the space provided after each question and don't feel that you have to write a lot.

The questionnaire and the teacher's introduction were translated into Japanese and Thai. Each group of problems, shown in Figure 1, was introduced with the words: "Write a number in each of the boxes to make a true statement. Explain your working".

Group A (on one page)	Group B (on one page)	Group C (on two pages)
23 + 15 = 26 + □	39 – 15 = 41 – □	746 – 262 + □ = 747
73 + 49 = 72 + □	99 – □ = 90 – 59	746 + □ – 262 = 747
43 + □ = 48 + 76	104 – 45 = □ – 46	
□ + 17 = 15 + 24		

Figure 1. Three groups of missing number sentences.

The study involved three cohorts of students Japan (277 students), Australia (301 students) and Thailand (194 students). Two schools were used in each country with students in Years 5, 6, and 7 approximately the same age (10 years old to 13 years old). In all schools involved in the study the teaching of computational algorithms forms a key part of the curriculum. Even if relational approaches are taught in some schools, they are not given the same time or emphasis as computational approaches. In Australia and Thailand, the study was carried across all year levels at the one time. In the case of Japan, Year 5 was tested at the end of one school year and Year 6 and Year 7 at the start of the next school Year. For this reason, the Japan results for Year 5 and Year 6 are considered together, whereas Year level results for Thailand and Australia are separated.

Evidence of Relational Thinking

Relational thinking is evident when, for example, verbal descriptions, arrows, or diagrams are used to compare the size of numbers either side of the equal sign, and where these verbal descriptions, arrows or diagrams are used in chain of argument, based on uncalculated pairs, using compensation and equivalence to find the value of a missing number. By contrast, computational thinking follows a fixed pattern. These features were discussed more fully in Stephens (2004, 2006).

In Group A and B questions, students must complete the calculation on the opposite side to where the □ is shown, and use this result to find the value of the missing number. For example, in the first problem of Group B, students must first find 39 – 15; and having found this to be 24, they then need to find the number which taken from 41 gives a result of 24 (or which added to 24 gives 41) for which the result is 17. In Group C, students must first subtract 262 from 746 giving 484, before proceeding to find the missing number by subtracting 484 from 747.

For each group of questions a benchmark sample was prepared, illustrating each score. Each student's work was checked independently by two markers. A high degree of consistency was evident across markers in all three countries. Whenever there was disagreement between markers, this was usually resolved by the markers themselves – usually one had missed an important clue. Very rarely, such disagreements were referred to a supervising researcher. Two student responses showing very clear relational thinking are given for each group of items in Figure 2.

Group A

- If I take 2 from 17 and add 2 to 22, it is the same as the number sentence after it. (Year 6 student)
- In 43 + □ = 48 + 76, 43 to 48 is + 5, 81 to 76 is – 5. These are equivalent, as you've done the same action to both sides. (Year 7 student)

Group B:

- As 99 is 9 more than 90, the missing number must be 9 more than 59. Therefore the answer is 68. (Year 5 student)
- I added 1 to 104 and 45. As long as I add the same number to both, it (104 – 45) will stay equivalent. (Year 6 student)

Group C:

- 746 is one less than 747, so 262 is one less than the answer. My answer is 263. (Year 5 student)
- 746 is 1 unit less than 747, so if you add 263 you will only need to minus 1 unit less than 263 for the equation to be equal on both sides. (Year 7 student)

Figure 2. Selected students' responses showing relational thinking.

Scoring procedures. Each group of problems was scored using a five-point scale shown in Figure 3. Thus, a single score was assigned to each group of questions even if children did not solve each question in the same way. This scoring scheme which had been validated for an earlier study (Stephens, 2004) was applied to Groups A, B, and C.

0 – arithmetical thinking evident for all questions; for example, through evidence of progressive calculations and use of algorithms to obtain results for additions and subtractions, even where these approaches resulted in incorrect answers, and no evidence of any relational thinking; also where an answer only has been given with no working shown to indicate what method has been used

1– a clear attempt to use relational thinking in at least one question, but not successfully executed (e.g., in Group B by giving answers of 13, 50 and 103)

2 – relational thinking clearly and successfully executed in one question, even if other problems are solved computationally or by incorrect relational thinking

3 – relational thinking clearly and successfully executed in at least two questions, but where the remaining question or questions are not solved relationally or solved using incorrect relational thinking

4 – all questions are solved clearly and successfully using relational thinking, even if computational solutions are also provided in parallel.

Figure 3. Scoring rubric.

Results of the Questionnaire

Clear evidence of relational thinking was present across all three Groups of questions among Japanese and Australian students. In the Japanese Year 5 and 6 cohort, almost 40% of students achieved a Score 4 (accomplished relational thinking) on Group A. The proportion of Score 4 was nearly 25% for Group B, and a little less than 20% for Group C. On the other hand, the proportion of Year 5 and 6 students who obtained Score 0, by using clear computational approaches or providing no evidence of relational thinking, ranged from about 35% for Group A, to 40% for Group B, and 65% for Group C. By Year 7, the proportion of Score 4 performances increased for all three groups of questions. This increase was not offset by an equivalent fall in the proportion of Score 0 performances that fell only slightly from Years 5 and 6 to Year 7. The increase in Score 4 performances in the Year 7 cohort was matched by reductions in the proportion of Scores 2 and 3. Although the Japanese mathematics curriculum seems to favour the development of relational approaches among many students, many other students still seem unable to or prefer not to use them.

The vast majority of Thai students used computational approaches in all three Groups of questions. In Year 5, no Thai student scored 4 on any group of questions. In Year 5, the proportions of Score 0 were 83% (Group A), 92% (Group B) and 98% (Group C). In Year 6 the average of Score 0 across the three groups of questions was 90%. In Year 7, it was 84% with gradual increases in the proportion of students in Years 6 and 7 achieving between Score 2 and Score 4. The gradual emergence of relational thinking in the Thai cohort seems more likely to be the result of individual student insight rather than an intended result of the mathematics curriculum.

The two Australian schools showed wide variation in the use of relational strategies. Looking only at the Year 6 cohorts in the two schools, the proportion of Score 0 results for Group A, B, and C questions in School 1 was 60%, 64%, and 78% respectively, compared to 34%, 32%, and 48% in School 2. Similarly, the proportion of Score 4 results for Group A, B, and C questions in School 1 was 25%, 9%, and 16% respectively, compared to 48%, 30%, and 46% in School 2. The reason for this marked difference is that in School 2 relational approaches are featured explicitly in the mathematics curriculum, whereas in School 1 they seem not to be emphasised.

Stability of Thinking

How consistent were students in their use of relational or computational approaches across the three groups of problems? Students were classified into three groups: those students who used relational strategies across all three groups of problems (SR–Stable Relational); those students who used only arithmetical or computational approaches across all three groups of problems (SA–Stable Arithmetical); and those students whose thinking was not consistent across the three groups (NS–Not Stable). The following rule was used.

SR: if student scored ≥ 1 on each of Group A, B, and C
SA: if student scored 0 on each of Group A, B, and C
NS: if student scored ≥ 1 on one or two of Group A, B, or C; and 0 on other(s).

A criterion of ≥ 1, instead of ≥ 2, across the three groups as evidence of stable relational thinking was justified because a score of 1 on Group B was without exception associated with successful relational thinking (≥ 2) for Group A and/or Group C questions. Aside from responses to Group B where students compensated in the wrong direction, a score of 1, indicating incorrect relational thinking, was very rarely given for responses to Group A and Group C questions.

How do Relational Thinkers Deal Successfully with Symbolic Sentences?

What evidence is there that students who successfully apply relational thinking to solve number sentences are able to extend these processes to solve sentences that are explicitly algebraic? Linchevski and Livneh (1999) point to the structural relations that students need to understand from arithmetic if they are to move successfully into algebra. MacGregor and Stacey (1999) also contend that deeper understanding of numerical operations is linked to later success in algebra. Using symbolic terms makes it more difficult for students to use computational checks. Some students solve symbolic expressions, such as $x + 3 = 21$, by drawing on their knowledge of number facts, or using guess-and-check methods. But another type of symbolic sentence, true for all values of the literal symbol, can be used to probe students' understanding of the meaning of symbolic expressions. This type of question, shown in Figure 4, was used to probe whether there is a clear link between successful application of relational thinking applied to number sentences and students' ability to understand the structure of symbolic sentences.

Place the four numbers $n - 1$, $n + 5$, 7 and 1 in the four boxes below so that the statement is always true.

$\square + \square = \square + \square$

Box A Box B Box C Box D

Explain why your answer is correct.

Figure 4. Making a sentence that is always true.

Students in Years 5, 6, and 7 in three countries had not been introduced to "always true" symbolic expressions. (Of course, many students by Year 5 have met single-value missing number sentences, such as $\square + 3 = 21$.) Some students did not attempt the question, or they wrote a sentence which is not true for all values of n, for example by writing a sentence which has the four numbers in boxes in the order in which they appear in the question. Some other students wrote a correct sentence but could not explain why it was true for all values of n. On the other hand, several possible approaches were used by students to explain why their sentence is always true. These various possibilities informed the partial-credit scoring rubric shown in Figure 5 used to grade students' responses.

NR – no response to the question involving literal symbols and number terms

Score 0 – incorrect or inadequate relation, no evidence of relational thinking

Score 1 – correct relation shown but no explanation given

Score 2 – correct relation shown, and correctly illustrated with one or more numerical values

Score 3 – correct relationship shown, and successfully illustrated by showing a balance with respect to the numbers, "ignoring" n terms; or by generally referring to balance among terms

Score 4 – correct relationship shown, and explained by explicit reference to the numbers and the n terms being equivalent on both sides, whatever the value of n, or by showing that the same algebraic structure exists on both sides.

Figure 5. Rubric used for scoring question involving literal symbols.

The following responses formed a benchmark sample for a score of 4, 3, or 2 for this question.

Exemplifying Score 4. A Year 6 student, having written, $7 + n - 1 = 1 + n + 5$, said:

"This answer is correct because you will always get an answer 6 more than n, because n less 1 plus 7 will give us 6 more than n. Also because n more than 5 plus 1 will give 6 more than n. This will have a lot of different answers but you will always get an answer 6 more than n."

A Year 7 student wrote $n - 1 + 7 = n + 5 + 1$, and explained:

"My answer is correct as no matter what n is, $n - 1$ is 6 units less than $n + 5$. This is balanced as 7 is six units more than 1."

Exemplifying Score 3. A Year 7 student wrote $n - 1 + 7 = n + 5 + 1$, and wrote:

"7 and $n - 1$ become 6; $n + 5$ and 1 become 6. Both sides are equivalent to 6".

Exemplifying Score 2. A Year 5 student wrote $1 + n + 5 = 7 + n - 1$, and then let $n = 5$ showing that

$1 + 5 + 5 = 7 + 5 - 1$. No reason was offered to show why the statement is always true.

There were some clear associations between highly accomplished explanations (Score 4) given to this question involving literal symbols and accomplished relational thinking used on the number sentences. For example, in Japan in Years 5 and 6, all 6 students who scored 4 on the question involving literal symbols also scored at least one 4 on the number questions. In Japan, where 54 Year 7 students scored 4 on this question, 44 showed very clear relational thinking on the number sentences, even if this was not always scored as high as a 4. In Australian School 2, the same applied to all 10 students in Years 5 and 6 who scored 4 on this question. Further, no student in Years 5 and 6 in any of the three countries who scored 0 on all three groups of number sentences scored 4 on the question involving literal symbols. This pattern was almost perfectly replicated in Year 7 cohorts.

Very many students who gave highly accomplished responses (Score 4) to this question applied compensation to the two terms involving literal symbols and to the two number terms, showing equivalence, whatever the value of *n*. Is there a clear connection between relational thinking on number sentences and success on the question involving literal symbols? Put most simply, one might expect a strong connection between those students who were classified as Stable Relational (SR) thinkers on the three groups of number sentences and their success in dealing with the question using literal symbols. A consequence of this "strong" position, if it were true, is that students who were classified as Stable Arithmetical (SA) on the three groups of number sentences would be less likely to deal successfully with the question involving literal symbols. These positions are now analysed.

Using Relational Thinking on Number Sentences (SR) as a Predictor

The following table gives the numbers of students who were classified as SR who also obtained a score of ≥ 1 on the question involving literal symbols (SR/LS). Their success rate is then compared to the percentage of their cohort in dealing successfully (i.e., obtained a score of ≥ 1) with the question involving literal symbols (LS).

Table 1
Using Stable Relational Thinking (SR) as a Predictor

Country	Cohort		Number of SR students	Number (%) SR/LS		Number (%) of LS in cohort	
Japan	Year 5/6	N = 133	41	32	78%	70	53%
	Year 7	N = 144	56	55	98%	127	88%
Australia (School 1)	Year 5	N = 41	8	4	50%	13	32%
	Year 6	N = 45	8	0	0%	3	7%
	Year 7	N = 44	9	7	77%	27	61%
Australia (School 2)	Year 5	N = 50	13	7	54%	17	34%
	Year 6	N = 50	27	22	81%	31	62%
	Year 7	N = 71	49	44	92%	58	82%
Thailand	Year 5	N = 53	N/A	N/A		N/A	
	Year 6	N = 64	N/A	N/A		N/A	
	Year 7	N = 77	4	2	50%	21	31%

This criterion seems to work well in Years 5 and 6 in Japan and Australian School 2 where the number of students classified as SR is comparatively high. In these two groups, the success rate of students who showed stable relational (SR) performance on the three groups of number sentences was almost 20% higher in obtaining a score ≥ 1 on the question involving literal symbols than the general success rate. The strength of connection is not as strong in both groups in Year 7 where the success rate of the SR performers on the number sentences is only 10% higher than the general success rate. Ceiling effects begin to emerge in the Year 7 in Japan and in Australian School 2 where 88% and 82% respectively of students in Year 7 were able to deal successfully (Score ≥ 1) with the question involving literal symbols.

However, serious difficulties exist in the application of the criterion in Year 5 and Year 6 the Australian School 1 and in the Thai cohort where few students were able to be classified as SR on the number questions, and where few were also successful on the question involving literal symbols. The criterion could not reasonably be applied in the case of Years 5 and Year 6 in the Thai cohort where only one student was classified as SR; and where in Year 5 only two students scored ≥ 1 on the question involving literal symbols. In Thailand in Year 6, however, 12 students scored ≥ 1 on the question involving literal symbols, despite the paucity of stable relational (SR) thinkers on the number sentences. Even in the Year 7 Thai cohort, the number of students classified SR was too small (4) to allow any reliable predictions. Similar difficulties also occur in Australian School 1 where only 3 students in the entire Year 6 sample scored ≥ 1 on the question involving literal symbols.

Using Arithmetic Thinking (SA) as a Predictor

How well did those students who met the criterion for Stable Arithmetic (SA) – that is, those who scored 0, 0, 0 on all three Groups of number sentences – perform on the question involving literal symbols? Given the difficulties applying the preceding test to the entire Thailand cohort and to Australian School 1, this test becomes more important. In Australian School 1 in Year 6, 23 students scored 0 on all three groups of number sentences. Of these 23, 21 were graded either NR or 0 on the question involving literal symbols, with only one of the 23 obtaining a 1 for this question, and one other obtaining a 2. In Year 5, 28 students got a 0 on all three groups of number sentences. Of these 21 got either NR or 0 (no success) on the question involving literal symbols, with four obtaining 1 for this question, and three obtaining a 2.

Likewise, in the Thailand cohort, there is a strong connection at each Year level between SA thinking on number sentences and failure to deal successfully with the question involving literal symbols. However, even for this cohort, the strength of this connection declines with each additional Year level. With each successive year level, more students classified as SA on the number sentences are able to score ≥ 1 on the question involving literal symbols. These results across all cohorts of Years 5, 6, and 7 students are given in Table 2.

The predictive value of this criterion seems to be strongest in Years 5 and 6 in all three country samples. Its predictive force is still quite strong in Thailand in Year 7; much less so in Year 7 in the Australian schools; and not at all in Year 7 in Japan. It may be argued that by Year 7 more students are familiar with literal symbols and so are able to deal successfully with the question involving literal symbols.

Table 2
Using Stable Arithmetical Thinking (SA) as a Predictor

Country	Cohort		Number of SA students.	SA students with no success on literal symbol question	
Japan	Year 5/6	N = 133	37	25	68%
	Year 7	N = 144	43	12	28%
Australia (School 1)	Year 5	N = 41	28	21	75%
	Year 6	N = 45	23	21	93%
	Year 7	N = 44	24	13	54%
Australia (School 2)	Year 5	N = 50	18	16	89%
	Year 6	N = 50	15	9	60%
	Year 7	N = 71	11	5	45%
Thailand	Year 5	N = 53	43	41	95%
	Year 6	N = 64	51	44	86%
	Year 7	N = 77	55	44	80%

There are some students, more in Japan and Australia than in Thailand, who are able to adopt relational thinking for the question involving literal symbols, even though they showed no evidence of relational thinking on the number sentences. These students can exercise choice; they are able to apply relational strategies when required in the case of the sentence involving literal symbols. For example, in the Japanese Year 5 and 6 cohort, 37 students obtained 0 on all three groups of number sentences, with 25 of these receiving either NR or 0 for the question involving literal symbols. Of the remaining 12 students, five received a score of 1, four a score of 2, and three a score of 3. The competent performances (Score 2 and Score 3) of these 5 students had not been preceded by any relational thinking in their work on number sentences. In Australian School 2 in Year 6, of the 15 students who scored 0 on all three groups of number sentences, 9 of these received either NR or 0, but three students received a score of 1 on the question involving literal symbols, and a further three also obtained a score of 2. By Year 7 in Thailand, 11 students classified as SA (0, 0, 0) on the number sentences achieved scores ranging from 1 to 3 on the question involving literal symbols.

Discussion of Limitations and Future Directions

In statistical analyses where some clear associations are present but not definitive, it is important to ask why this is so. The first and most obvious comment is that the three groups of number sentences may not have separated those who were capable only of thinking computationally from those who chose to solve the number sentences computationally but who could have used relational approaches to solve these sentences if pressed to do so. Some of these "computational" students applied relational approaches to deal more or less successfully with the expression involving literal symbols. Students who are competent calculators may prefer that approach even though it is much more demanding than relational thinking in the case of Group C, and somewhat more demanding in the case of Group A and B questions.

It should be remembered that no student who consistently solved the number sentences computationally was able to achieve the highest score (Score 4) on the question involving literal symbols, although there were quite a few who produced an expression in the correct form but with no explanation (Score 1) and others who were able to justify their choice of a correct literal expression by using one or more values of the literal symbol (Score 2). Those with Score 1 who produced an expression in the correct form – without explanation or justification – may have used strategies such as "guess-and-check" that fall a long way short of deep relational thinking.

It is also clear that some students who appeared to be stable relational thinkers (SR) did not deal successfully with the question involving literal symbols. Among this latter group might be those who solved only some of each group of number sentences relationally. It is a big jump from being able to apply relational thinking to complete an already formed number sentence to being able to construct and justify an "always true" sentence involving literal symbols and numbers in an equivalence relation. Although the findings of this study support the view of Linchevski and Livneh (1999) that many of algebraic relations met by students inherit the structural properties associated with number sentences with which students are, or should be, familiar, it is clear that the missing number questions were not sufficiently sensitive to elicit and identify the kind of relational thinking that students needed in order to solve the question involving literal symbols.

Some students may have used grouping and simplification techniques to deal with the question involving literal symbols even if they had chosen to solve by computation all the number sentences. From our study of the curriculum documents of the three countries we were confident that students in Year 7 had not been taught these techniques, but this cannot be ruled out for every student.

Is it possible to introduce an extra question that would press those who chose to solve the number sentences computationally to disclose any latent relational understanding, and at the same time to discriminate among relational thinkers? To these purposes, a question modelled after the research programme, Concepts in Secondary Mathematics and Science, (CSMS, see Hart, 1981) might ask students:

What can you say about c and d in the following mathematical sentence?

$$c + 2 = d + 10$$

If equivalence and compensation are at the heart of relational thinking, the goal of this question is to have students say that this sentence will be true for *any* values of c and d provided c is 8 more than d. But there are intermediate responses that fall short of this understanding. Computational thinkers are likely to be able to give several values of c and d for which the sentence is true. They may even offer several pairs without seeing that the values are part of a pattern. More developed responses could be expected to give pairs in a systematic list such that (c, d) could be (9, 1), (10, 2), (11, 3), (12, 4). In this case, are students able to generalise a rule connecting c and d? It might also be possible to probe whether students can give a clear mathematical sense to sentence being true for "*any* values of c and d provided c is 8 more than d". Such responses might make it clear that, for example, fractional or decimal values are possible – or even negative numbers. Being able to derive a correct mathematical generalisation from numerical examples is key element of algebraic reasoning (Carpenter & Franke, 2001; Lee, 2001; Zazkis & Liljedahl, 2002).

A similar question could be constructed for probing relational thinking about subtraction. The value of questions such as these is that they can be given a limited meaning by computational thinkers, but they can only be answered in any depth using relational thinking. A fully elaborated response needs to show that the relationship is determined by the operation as well as the specific numbers involved, and that the sentence can be true for *any* values of c and d where the given condition is met. This kind of question is likely to be a better predictor of success in dealing with literal expressions.

Conclusion

Students' use of relational thinking to solve number sentences is evident in all three countries by the end of elementary school. The extent of its acquisition varies between countries and between schools. Even where it appears to be strong, there are still many students who seem unable to use it. Those who were consistent relational thinkers on number sentences were more likely to deal successfully with a sentence involving literal symbols and number terms than those who showed only arithmetical thinking on the number sentences. In all three countries, particularly in Years 5 and 6, the majority of this latter group was unable to deal successfully with the sentence involving literal symbols. This group especially should concern teachers. They may obtain perfectly correct answers to number questions through careful use of computational based approaches, but these approaches are clearly deficient when students are confronted with questions using literal symbols where computation will not work. Their inability to use relational thinking means that they are not well prepared to deal with the kinds of thinking – in particular, those involving equivalence and compensation – that they will need in high school algebra. More importantly, one should ask how much better their understanding of number and arithmetical operations might have been in primary school if they had been introduced to and were able to use relational strategies to solve number sentences.

References

Carpenter, T. P., & Franke, M. L. (2001). Developing algebraic reasoning in the elementary school: Generalization and proof. In H. Chick, K. Stacey, J. Vincent, & J. Vincent (Eds.), *Proceedings of the 12th ICMI Study Conference. The Future of the Teaching and Learning of Algebra* (pp. 155-162). Melbourne: University of Melbourne.

Hart, K. M. (1981). *Children's understanding of Mathematics: 11-16*. London: John Murray.

Irwin, K., & Britt, M. (2005). The algebraic nature of students' numerical manipulation in the New Zealand Numeracy Project. *Educational Studies in Mathematics 58*, 169-188.

Kieran, C. (1981). Concepts associated with the equality symbol. *Educational Studies in Mathematics, 12,* 317-326.

Lee, L. (2001). Early algebra – But which algebra? In H. Chick, K. Stacey, J. Vincent, & J. Vincent (Eds.), *Proceedings of the 12th ICMI Study Conference. The Future of the Teaching and Learning of Algebra* (pp. 392-399). Melbourne: University of Melbourne.

Linchevski, L., & Livneh, D. (1999). Structure sense: The relationship between algebraic and numerical contexts. *Educational Studies in Mathematics 40,* 173-196.

MacGregor, M., & Stacey, K. (1999). A flying start to algebra. *Teaching Children Mathematics*, *6(2)*, 78-85.

Stephens, M. (2006). Describing and exploring the power of relational thinking. In P. Grootenboer, R. Zevenbergen and M. Chinnappan (eds.), *Identities, Cultures and Learning Spaces,* (Proceedings of the 29th annual conference of the Mathematics Education Research Group of Australasia, pp. 479-486). Sydney: MERGA.

Stephens, M. (2004). *Researching relational thinking: Technical report of research. Program report by Visiting Foreign Research Fellows, May 31, 2004*. Tsukuba, Japan: University of Tsukuba, Centre for Research on International Cooperation in Educational Development.

Zazkis, I., & Liljedhal, P. (2002). Generalisation of patterns: The tension between algebraic thinking and algebraic notation. *Educational Studies in Mathematics, 49,* 379-402.

A Framework for Success in Implementing Mathematical Modelling in the Secondary Classroom

Gloria Stillman
University of Melbourne
<g.stillman@unimelb.edu.au>
Peter Galbraith
University of Queensland
<p.galbraith@mailbox.uq.edu.au>
Jill Brown
Australian Catholic University
<jill.brown@acu.edu.au>
Ian Edwards
Luther College
<ie@luther.edu.au>

A framework to support successful implementation of mathematical modelling in the secondary classroom was developed from transitions between stages in the modelling process and the cognitive activities associated with these. This framework is used to analyse implementation of a task with a Year 9 class. Cognitive activities engaged in during the task and competencies and technological knowledge required to complete the task successfully are identified. This framework can be used by teachers, researchers, and curriculum designers to design tasks and predict where in a given task blockages occur, hence allowing advance consideration of scaffolding for in-the-moment classroom decisions.

Internationally the field of applications and mathematical modelling in education features prominently in every continent and research into teaching and learning through applications and mathematical modelling is currently being pursued with renewed vigour in many parts of the world (Kaiser, Blomhoj, & Sriraman, 2006), boosted by the 14th International Commission on Mathematical Instruction (ICMI) study on applications and modelling in mathematics education held in 2004. The recently published volume by Blum, Galbraith, Henn, and Niss (2007) from this study contains an up-to-the-minute account of progress and challenges within the field. International initiatives currently addressing these challenges include, for example, the Organisation for Economic Co-operation and Development Programme for International Student Assessment (OECD PISA) project, which includes the following within its assessment domains.

> An individual's capacity to identify and understand the role that mathematics plays in the world, to make well-founded judgements and to use and engage in mathematics in ways that meet the needs of that individual's life as a constructive, concerned, and reflective citizen. (OECD, 2003, p. 15)

This implies engaging with mathematics across a variety of situations and contexts. In countries both within and outside the OECD such statements are associated with ongoing discussion about the design of curricula, and in particular the role of mathematical modelling, applications, and relations to the real world in the teaching and learning of mathematics (Blum et al., 2007). However, Turner (2007, p. 440) raises concerns about the extent to which the mathematical thinking that underpin such mathematical modelling tasks is really valued by those overseeing curriculum and instruction in various countries considering "the level of complexity of the mathematical modelling activities that 15-year-old students can cope with … seems to be rather low". Turner also asks: "How can teachers be more effectively empowered to explore and promote the mathematical thinking underlying these tasks, and what kinds of teaching and learning activities will be most effective in facilitating this kind of mathematical thinking among 15-year-old students?"

Within Australasia modelling is advocated in curriculum documents from the primary years (e.g., Victorian Curriculum and Assessment Authority (VCAA), 2005) through to the upper end of secondary (e.g., Ministry of Education, 1992; Queensland Board of Senior Secondary School Studies (QBSSSS), 2000). Evaluations of curricular initiatives become confused when there are idiosyncratic interpretations, which muddy notions of authentic practice in the field. It is of continuing importance that initiatives claiming *mathematical modelling* as their focus, are presented in terms of frameworks, criteria, and alternatives that are endorsed by the international community of practice.

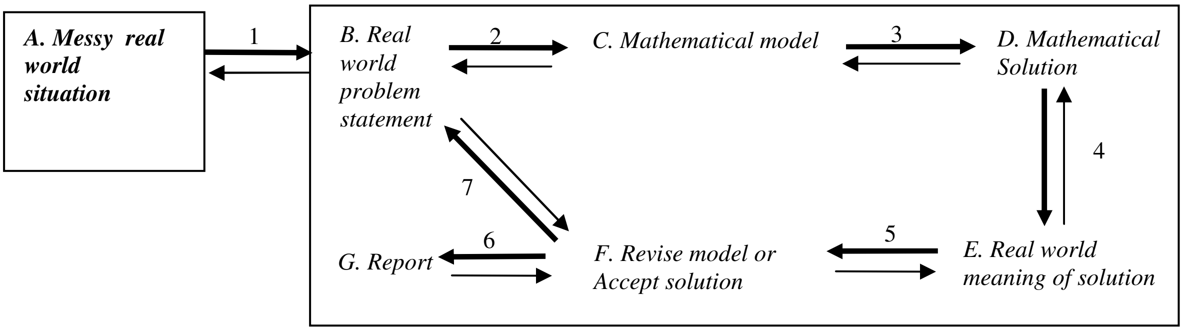
Given the various idiosyncrasies associated with some localised curricular initiatives (including Australian) we wish to be clear about meanings and interpretations ascribed to terms such as *applications* and *mathematical modelling* in our work. Our meanings are consistent with those adopted by the International Community for the Teaching of Mathematical Modelling and Applications (ICTMA), which is an Affiliated Study Group of the ICMI. Simply put, with *applications* we tend to focus on the direction (mathematics → reality). "Where can I use this particular piece of mathematical knowledge?" On the other hand with *mathematical modelling* we focus on the reverse direction (reality → mathematics). "Where can I find some mathematics to help me with this problem?"

The term mathematical modelling itself, as it is used in curricular discussions and implementations has different, although clearly delineated, interpretations. One interpretation sees mathematical modelling as motivating, developing, and illustrating the relevance of particular mathematical content (e.g., Chinnappan & Thomas, 2003). A second perspective views use of applications and modelling as an end in itself for educational purposes not a means for achieving some other mathematical learning end. The models and modelling perspectives of Lesh and English (2005), for example, although clearly associated with the first interpretation, extend beyond to include elements of the second. Our own approach sees the second interpretation as encompassing the first. Both approaches agree that modelling involves some total process that encompasses formulation, solution, interpretation, and evaluation as essential components.

The Modelling Process and Modelling Competencies

As interests in teaching and learning are central, our theoretical framework for studying modelling is oriented towards the problem solving individual to give not only a better understanding of what students do when solving (or failing to solve) modelling problems, but also a better basis for teachers' decision making and interventions. Figure 1, modified from Galbraith and Stillman (2006), encompasses both the task orientation of many diagrammatic representations of the modelling cycle and the need to capture what is going on in the minds of individuals as they work on modelling tasks. This latter focus has also led to a reduction in the number of stages identified specifically as other researchers (e.g., Borromeo Ferri, 2006) have pointed out that fewer are of more use in a schooling context.

The respective entries A-G represent stages in the modelling process, where the thicker arrows signify *transitions* between the stages, and the total solution process is described by following these arrows clockwise around the diagram from the top left. It culminates either in the report of a successful modelling outcome, or a further cycle of modelling if evaluation indicates that the solution is unsatisfactory in some way. The kinds of mental activity that individuals engage in as modellers attempt to make the transition from one modelling stage to the next are given by the broad descriptors of cognitive activity 1 to 7 in Figure 1. The light arrows that are in the reverse direction to the modelling cycle are included to emphasise that the modelling process is far from linear, or unidirectional, and to indicate the presence of reflective metacognitive activity (Maaß, 2006).



1. Understanding, structuring, simplifying, interpreting context
2. Assuming, formulating, mathematising
3. Working mathematically
4. Interpreting mathematical output
5. Comparing, critiquing, validating
6. Communicating, justifying (if model is deemed satisfactory)
7. Revisiting the modelling process (if model is deemed unsatisfactory).

Figure 1. Modelling Process.

It is imperative that we identify specifically activities with which modellers need to have competence in order to apply mathematics successfully particularly in settings where there is increasing access to electronic technologies. By "competency" is meant the capacity of an individual to make relevant decisions, and perform appropriate actions in situations where those decisions and actions are necessary to enable success.

> *Mathematical modelling competency* means the ability to identify relevant questions, variables, relations or assumptions in a given real world situation, to translate these into mathematics and to interpret and validate the solution of the resulting mathematical problem in relation to the given situation, as well as the ability to analyse or compare given models by investigating the assumptions being made, checking properties and scope of a given model (Niss, Blum, & Galbraith, 2007, p. 12).

We elaborate how these components of modelling competency are realised within the research settings that have provided data for this paper.

From Theory to Empiricism

The transitions arising from our theoretical framework (Figure 1) served as a structural framework for indentifying student blockages in transitions as students undertook various modelling and application tasks. Initially the contents of the respective transition sections were empty, except for the bold headings of Figure 2. Intensive data were generated from implementations of two teacher designed tasks at one school where modelling and the use of technology were an integral part of classroom practice in order to develop our first result, an "emergent framework" (Galbraith, Stillman, Brown, & Edwards, 2007), from empirical study. The resulting emergent framework was then refined and tested by examining the implementations of a different task and a revised version of one of the first tasks in a second school (Galbraith & Stillman, 2006). The task was revised by the researchers in collaboration with the teacher to suit the different motivation towards real world tasks and technology use and time frame of the teacher in a different school setting. The resulting refined transitions framework is shown in Figure 2. The empirics gave rise to case specific categories and generalisations of these from the various elements in each transition section. Our research indicates there is potential for blockages to occur when any of these component activities have to be undertaken.

1. MESSY REAL WORLD SITUATION → REAL WORLD PROBLEM STATEMENT:
1.1 Clarifying context of problem
1.2 Making simplifying assumptions
1.3 Identifying strategic entit(ies)
1.4 Specifying the correct elements of strategic entit(ies)
2. REAL WORLD PROBLEM STATEMENT → MATHEMATICAL MODEL:
2.1 Identifying dependent and independent variables for inclusion in algebraic model
2.2 Realising independent variable must be uniquely defined
2.3 Representing elements mathematically so formulae can be applied
2.4 Making relevant assumptions
2.5 Choosing technology/mathematical tables to enable calculation
2.6 Choosing technology to automate application of formulae to multiple cases
2.7 Choosing technology to produce graphical representation of model
2.8 Choosing to use technology to verify algebraic equation
2.9 Perceiving a graph can be used on function graphers but not data plotters to verify an algebraic equation
3. MATHEMATICAL MODEL → MATHEMATICAL SOLUTION:
3.1 Applying appropriate symbolic formulae
3.2 Applying algebraic simplification processes to formulae to produce more sophisticated functions
3.3 Using technology/mathematical tables to perform calculation
3.4 Using technology to automate extension of formulae application to multiple cases
3.5 Using technology to produce graphical representations
3.6 Using correctly the rules of notational syntax (whether mathematical or technological)
3.7 Verifying of algebraic model using technology
3.8 Obtaining additional results to enable interpretation of solutions
4. MATHEMATICAL SOLUTION → REAL WORLD MEANING OF SOLUTION:
4.1 Identifying mathematical results with their real world counterparts
4.2 Contextualising interim and final mathematical results in terms of RW situation (routine → complex versions)
4.3 Integrating arguments to justify interpretations
4.4 Relaxing of prior constraints to produce results needed to support a new interpretation
4.5 Realising the need to involve mathematics before addressing an interpretive question
5. REAL WORLD MEANING OF SOLUTION→ REVISE MODEL OR ACCEPT SOLUTION:
5.1 Reconciling unexpected interim results with real situation
5.2 Considering Real World implications of mathematical results
5.3 Reconciling mathematical and Real World aspects of the problem
5.4 Realising there is a limit to the relaxation of constraints that is acceptable for a valid solution
5.5 Considering real world adequacy of model output globally

Figure 2. Refined framework for identifying student blockages in transitions.

Practical Applications: Using the Framework

With respect to the questions raised by Turner (2007) about how we can promote the mathematical thinking underlying modelling tasks, our attention turns to the use of this transitions framework and Figure 1 to examine the implementation by a teacher and experience by students of a real world task. In order to identify the mathematical thinking that is being promoted by the task and the competencies required for a successful experience, we answer the following questions:

- What kinds of cognitive activites are students engaging in when the task is structured and implemented in this manner?
- With respect to the task as a modelling experience, what competencies (mathematical/modelling/technological) must students have to complete the task successfully?

The implementation of the task, *The Bungee Experience*, was chosen to illustrate the utility of the framework for three reasons. Firstly, it has fewer transition elements than more complex tasks. It is one of a series of modelling and applications tasks used in this year level at the school concerned so it is not necessary for all tasks to promote all elements of Figure 2 but it is important for pedagogical decisions to be based on informed judgements about the nature of the elements it does include or exclude. Secondly, the task, in various forms, has been used as both a teaching task and an assessment task as in this instance. The possibility arises that the teacher's purpose for the task also affects the elements that are promoted and the competencies required. Thirdly, this particular implementation was followed by a lesson reflecting on the model the students had used.

The Bungee Experience: Barbie has turned 40. Her friend Ken has given her an extreme sports experience, part of which is an afternoon's bungee jumping. Your task as the operator is to CALCULATE the length of Bungee Cord Barbie will need to jump from the given height, off the Bungee tower. Remember there is concrete below and we don't want to mess up Barbie's hair.
During the next two maths periods your team will:
1. Conduct measurements in the classroom to determine a model that links the fall distance to the number of rubber bands used for a shock cord.
2. Record your data, the graph for the data, and your linear equation.
3. Test your model by predicting the requirements for a fall from an unknown height.
(This height was announced later. Suggestions were provided as to how to collect data and display results. Students used a doll, usually Barbie or a toy such as Poombah the Warthog for the Bungee Jumper.)

Method

The task was implemented in a Year 9 class of 21 students during one 100-minute double period. The reflection lesson comprised the next 50 minute mathematics lesson two days later. Two video cameras were set up in the classroom. These were mainly focused on the class as a whole at the beginning of the double lesson and then on the collaborative activity of two focus pairs of students. At times, critical incidents involving other students were also videotaped. A third focus pair of students were audio-recorded. One camera was used to record the the reflection lesson and the jump phase of the implementation lessons when the class went outside and tested their bungee chords. Scripts from the 10 groups and rough working sheets from focus groups were collected. Five students participated in post-task interviews. Only one of these students was from the focus groups. Field notes were also made by the researchers during and immediately after the lessons.

All audio and video data were transcribed for analysis. The transcripts, in conjunction with the video recordings, were analysed at a macrolevel to identify episodes where students encountered and resolved (or otherwise) blockages between the identified transitions. Each episode was coded using elements of the transitions framework in Figure 2 then subjected to intense microanalysis to see if it shared the same characteristics as the elements of the framework identified previously or if further elements needed to be added. There were none. At the end of this process a framework showing the potential blockages was produced. Finally, typical instances of the cognitive actvities engaged in by students in the task and the competencies that underpin successful transitions from one modelling phase to the next were identified.

Results

Figure 3 shows the elements in each transition that were identified in this implementation of the task. Each element has two parts where key (generic) categories in the transitions between phases of the modelling cycle are indicated (in regular type), and illustrated (in capitals) with reference to the task. Cognitive activities associated with the elements of transitions identified in Figure 3 are: understanding, simplifying, interpreting context; assuming, formulating, mathematising; working mathematically; interpreting mathematical output; and comparing, critiquing, validating. Evidence for selected examples of these activities is presented in the analysis of transitions that follows. Finally, the competencies for carrying these out successfully are identified.

1. MESSY REAL WORLD SITUATION → REAL WORLD PROBLEM STATEMENT:
1.1 Clarifying context of problem [WATCHING DEMONSTRATION & DISCUSSING PROBLEM SITUATION]
1.2 Making simplifying assumptions [ELASTIC LIMIT NOT EXCEEDED; AERODYNAMICS OF TOYS CAN BE IGNORED]
2. REAL WORLD PROBLEM STATEMENT → MATHEMATICAL MODEL:
2.1 Identifying dependent and independent variables for inclusion in algebraic model [*FALL DISTANCE* AND *NUMBER OF ELASTIC BANDS* – WHAT CONTROLS WHAT]
2.3 Representing elements mathematically so formulae can be applied [POINTS]
2.4 Making relevant assumptions [LINEAR MODEL APPROPRIATE EVEN WHEN DATA POINTS APPEAR TO FOLLOW CURVE]
2.5 Choosing technology to enable calculation [RECOGNISING HAND METHODS ARE NOT SUFFICIENT]
2.7 Choosing technology to produce graphical representation of model [GRAPHING CALCULATOR WILL GENERATE PLOT OF *FALL* DISTANCE FOR DIFFERENT *NUMBERS OF RUBBER* BANDS]
3. MATHEMATICAL MODEL → MATHEMATICAL SOLUTION:
3.1 Applying appropriate formulae [EG. LINEAR MODEL TO FIND PREDICTED NUMBER OF BANDS]
3.3 Using technology/mathematical tables to perform calculation [SUCCESSFUL CALCULATION OF GRADIENT]
3.5 Using technology to produce graphical representations [EFFECTIVE USE OF GRAPHING CALCULATOR STATPLOT]
3.8 Obtaining additional results to enable interpretation [PLOTTING EXTRA VALUES TO TEST HUNCHES]
4. MATHEMATICAL SOLUTION → REAL WORLD MEANING OF SOLUTION:
4.1 Identifying mathematical results with their real world counterparts [INTERPRETING PREDICTION VALUE]
4.2 Contextualising interim and final mathematical results in terms of RW situation (routine versions) [GRADIENT MEASURES HOW FAR IT WILL FALL PER BAND]
4.3 Integrating arguments to justify interpretations [PRESENTING REASONED CHOICE FOR METHOD OF FINDING EQUATION OF LINEAR MODEL]
4.4 Relaxing of prior constraints to produce results needed to support a new interpretation [CAN USE POINTS INVOLVING HALF BANDS TO FIND GRADIENT OF LINEAR MODEL]
5. REAL WORLD MEANING OF SOLUTION → REVISE MODEL OR ACCEPT SOLUTION:
5.1 Reconciling unexpected interim results with real situation [RECONCILING THE RESULTS OF TESTING THEIR PREDICTIONS 26 BANDS WITH BARBIE VERSUS 26 WITH POOMBAH]
5.2 Considering Real World implications of mathematical results [LOCAL – DO INDIVIDUAL CALCULATIONS/GRAPHS ETC MAKE SENSE WHEN TRANSLATED TO REAL WORLD MEANINGS?]
5.3 Reconciling mathematical and Real World aspects of the problem [SIGNIFICANCE OF Y-INTERCPET IN LINEAR MODEL & HOW IT COULD BE USED TO PARTIALLY EVALUATE MATHEMATICAL EQUATION CONSTRUCTED]
5.5 Considering real world adequacy of model output globally [MODEL PROVIDES ALL ANSWERS TO RW PROBLEM & EXTENDS TO OTHER SITUATIONS]

Figure 3. Framework showing transition elements in Barbie Bungee implementation.

Transitions

Messy real world situation → Real world problem statement. In this implementation this transition presented no blockages to students' progress. The teacher demonstrated how to topple the doll and the attaching of bands to the doll's ankles (1.1). In other implementations viewed by the researchers, difficulties arose when students attempted data collection with the doll upside down hanging from her toes or they threw the doll rather than toppled her from a standing position. A major assumption (1.2) is that the bands will stretch at a constant rate and not exceed their elastic limit when the model is used to extrapolate well beyond the set of collected data (maximum drops of around 2m in a classroom). This assumption was debated by students during the reflection lesson.

Ray: But eventually shouldn't the rubber bands snap therefore it can't [interrupted]

Teacher: It would if the weight component enables it, and ultimately the weight of the rubber band itself causes problems. There are some problems in the linear model …

Dale: Instead of snapping it could get 0.3 cm longer by bending so our calculations

Ray: [interrupting] But eventually it is going to bend. It is going to snap.
Dale: Stretch!
Ray: It is not a linear model!
Tine: Rubber bands stretch.
Dale: That would make our distances vary.
Teacher: It has to stretch. It gets to what is called its elastic limit and then the linear part changes, okay? … but in terms of 8 bands we would probably get away with a good approximation.

In this implementation it was also assumed that the aerodynamic characteristics of toys such as Poombah would have negligible impact (1.2). When Poombah dropped about half the distance of a Barbie doll with the same length Bungee cord, one student suggested that perhaps it was a weight difference. No students raised its aerodynamic characteristics. From a modelling perspective, some of the responsibility for elements of formulation such as identifying the strategic entity and specifying its elements were removed from the students as they were told they were to collect fall distance data.

Real world problem statement → Mathematical model. Even though students were told to collect fall distances for 3 to 8 rubber bands, not all students easily recognised which of these was the dependent and which the independent variable in the situation (2.1). Bea and Sue for example, had them reversed initially and remained unsure they were correct when they swapped them. Students choice of points for calculating gradients for lines of best fit (2.3) caused some delays in moving on from their models when algebraic manipulation produced equations with intercepts that clearly were too large. Although the task setter had already made many decisions in this transition for students by specifying a linear model be made and choosing the technology to use for a plot (graph paper), some students (e.g., Evan) decided to question whether they should use a linear model as their plot showed their data were curved (2.4) or chose to use a graphing calculator to check their hand drawn plot (2.7). It did not occur to Evan to check his data rather than merely question the model.

Mathematical model → Mathematical solution. Sue and Bea did not use an appropriate linear model when they calculated the number of bands for their prediction (3.1) believing that they should choose plotted points in such a manner that the line would pass through the origin. They later told the teacher this was because they expected a y-intercept of zero as Barbie would stand and not fall at all if the length of the Bungee cord was zero. Unlike previous implementations of the task, no students obtained additional results or attempted in some way to test their models before the Jump Phase, although Sue suggested they test their model using 9 bands but they failed to do so (3.8).

Mathematical solution → Real world meaning of solution. Possible dilemmas for students in this transition occur when students do not identify mathematical results such as the gradient and the y-intercept with their real world counterparts (4.1) and when they need to contextualise interim and final mathematical results in terms of the real world situation (4.2) for example, when predicting a shock cord length for their test jump outside the classroom. The doll was to be dropped so as to stop as close to the ground as possible. When the students finally found their mathematical result for the predicted number of bands, decisions had to be made about whether they should round up, truncate their answer, or over or under estimate. The real world implication (5.2), that rounding up or over estimating would mean the doll would hit the ground was foreseen by four groups.

Tony: So we need 24 rubber bands.
Reg: Yeah. Should we go with 23 just to be safe? Or should we just go 24.

Tony: No 24 because look point [Calculator shows 24.3950762]
Reg: All right. So we rounded down because if it hits the ground we have to clean the tennis courts.

Two other groups rounded their result up, in keeping with expected classroom practice, clearly not considering the implications in the context, with Tine, Lil, and Ally showing their prediction resulted in a fall distance of 441.3 cm exceeding the jump distance of 440 cm. Others such as Ray fortuitously rounded down giving no thought to the context.

Researcher: 27.16 recurring. So why did you round down? Do you just always do that?
Ray: I don't know I just rounded down. Should I have rounded up? Well, even if it was a 9 it would have made it 27.2 and that is still not enough to round up anyway.

Real world meaning of solution→ Revise model or accept solution. As students tested their predictions many were puzzled as to why these were wrong. They had difficulty reconciling the jump results with the mathematics of the situation (5.1), which they did not use to evaluate their models. Ray and Joe's Barbie made an almost perfect jump with 27 bands whereas Di and Ash's Barbie with 28 bands was about 70 cm short. There was no discussion of the difference in their models, $y = 15x + 32.5$ and *distance* (cm) = 14 × *rubber bands* + 38.3, although the teacher brought to the students' attention the role of the gradient in determining how many more bands to add after he allowed them a third jump. In the reflection lesson student discussion teased out the meaning of the gradient.

Teacher: What does the gradient measure? In Barbie's case what does the gradient measure?
Dale: Rise divided by run.
Teacher: Yeah, that's how you calculate it. What does it actually physically mean?
Tony: How many rubber bands.
Ray: How far it would fall per rubber band.

No students showed evidence of realising the significance of the y-intercept in their mathematical model and how it could be used to evaluate partially the mathematical equation they had constructed (5.3). This was discussed in the following lesson when the students reflected on the model they had made. In the exchange below they are discussing a linear equation of the form $y = \text{m}x + \text{c}$ where c is 25.4.

Teacher: What was the 25.4 in Barbie or, if you had Poombah (the Warthog) it was less.
Tine: The length of Barbie.
Teacher: It was the length of Barbie. Now I had this discussion with a couple of people [Bea snd Sue] about how many rubber bands, how far Poombah or it would fall. Some said, "Zero", but the thing is that Barbie would fall … So she would still be hanging by her toenails… a lot of the times you get c values that don't make a lot of sense. In Barbie's case it did.

The reflection lesson continued with students suggesting and discussing many other applications of linear models (5.5) such as mobile phone plan charges, cost of purchasing concert tickets over the phone, council service charges, and electricity costs.

Competencies

Modelling and mathematical competencies identified as being required for the task, and graphing calculator technological knowledge required for the task are presented in Figure 4.

Modelling and mathematical competencies:

To identify from the available information what is relevant and what is irrelevant,

To make simplifying assumptions about the situation,

To recognise relevant variables and to mathematise these,

To make relevant assumptions to enable mathematics to be applied,
To select technology where needed to enable or check calculations,
To chose appropriate methods of representing, checking and testing the model,
To select and apply appropriate formulae (e.g., general form of linear model: $y = ax + b$., gradient: $\Delta y \div \Delta x$),
To use technology appropriately to perform calculations,
To use mathematical knowledge to solve the problem,
To obtain additional results to enable interpretation,
To link mathematical results with their corresponding real world components,
To generalise or extend solution,
To critically check results with the real situation,
To consider implications of decisions and results.

Technological knowledge needed for effective use of a graphing calculator:
To know data can be entered into LISTS and LIST data can be plotted,
To use of Homescreen of a calculator to perform calculations,
To know how to Plot correctly and effectively.

Figure 4. Modelling and mathematical competencies and technological knowledge for *Barbie Experience*.

Practical Implications for the Curriculum and Classroom

The framework we have devised allows a researcher, teacher, or curriculum designer to identify the particular competencies that students would need in order to complete a particular modelling task successfully. By identifying difficulties with generic properties the possibility arises of teachers/researchers/curriculum designers being able to predict where in given problems, blockages of different types might be expected. This understanding then contributes to the planning of teaching, in particular the identification of necessary prerequisite knowledge and skills (including modelling competencies), preparation of interventions for introduction at key points if required, and the scaffolding of significant learning episodes. As well as identified blockages showing teachers what they may need to address to help students overcome blockages the framework also informs the teacher who is trying to move from dependent to independent modelling by students (Leiβ, 2005).

Although it is acknowledged in many curricula documents (e.g., Ministry of Education, 1992; OECD, 2003; QBSSSS, 2000; VCAA, 2005) that mathematical modelling is an essential component of secondary schooling, implementing this is no simple task. Considering how mathematics can be used to solve real problems, requires students to make decisions about many aspects of the task. Whilst this is an important part of the learning process, it can place the teacher in the position of needing to provide appropriate scaffolding "on the spot" when some unforseen blockage is encountered by one or more students. This can be a challenge for the most experienced teacher. Thus, both practising and pre-service teachers could benefit from the use of a tool.

By mapping the task and its intended implementation to the transitions Framework (Figure 3), prior to the actual implementation, teachers can identify the specific activities with which the student modellers need to have competence in order to apply their mathematical and technological knowledge successfully to the problem. Identifying potential blockages can inform planning of teaching. This does not mean making decisions for students to avoid their confronting blockages, rather it allows the teacher to be well prepared, expecting particular blockages and better supporting students to overcome these. It also gives teachers information on which to base decisions about the preparedness of their students to complete a particular task.

One task can be easily modified to suit a range of purposes using the Framework. These purposes vary and include: the intent of the mathematical modelling (e.g., as a vehicle to teach modelling competencies or to legitimate mathematical content), the time that can be allocated to the task, the purpose being assessment or learning focused, previous experience of the teacher in implementing modelling tasks, previous experiences of students with modelling tasks, the technological expertise of the students, and the mathematical knowledge of the students. By using the Framework, teachers and others can modify a task to suit their particular purpose and constraints.

Teachers and curricula designers wishing to implement a series of modelling tasks over the course of a year can use the Framework to ensure that, although not all elements will be addressed in every task, every element is included in as many tasks as necessary to develop students' modelling competencies. The incorporation of formulation and reflection activities are critical to developing modelling skills.

Acknowledgement: This research was funded by Australian Research Council linkage project, RITEMATHS (LP0453701), industry partner secondary schools and Texas Instruments.

References

Blum, W. Galbraith, P.L. Henn, H.-W., Niss, M. (Eds.). (2007). *Modelling and applications in mathematics education: The 14th ICMI study (New ICMI Study Series, Vol. 10)*. New York: Springer.

Borromeo Ferri, R. (2006). Theoretical and empirical differentiations of phases in the modelling process. *ZDM, 38*(2), 86-95.

Chinnappan, M., & Thomas, M. (2003). Teachers' function schemas and their role in modelling. *Mathematics Education Research Journal, 15*(2), 151-170.

Galbraith, P., & Stillman, G. (2006). A framework for identifying student blockages during transitions in the modelling process. *ZDM, 38*(2), 143-162.

Galbraith, P., Stillman, G., Brown, J., & Edwards, I. (2007). Facilitating middle secondary modelling competencies. In C. Haines, P., Galbraith, W., Blum, & S. Khan, (Eds.), *Mathematical modelling: Education, engineering and economics* (pp. 130-140*)*. Chichester, UK: Horwood.

Kaiser, G., Blomhoj, M., & Sriraman, B. (2006). Towards a didactical theory for mathematical modeling. *ZDM, 38*(2), 82-85.

Leiβ, D. (2005). Teacher intervention versus self-regulated learning? *Teaching Mathematics and its Applications, 24*(2-3), 75-89.

Lesh, R., & English, L. D. (2005). Trends in the evolution of models and modeling perspectives on mathematical learning and problem solving. *ZDM, 37*(6), 487-489.

Maaß, K. (2006). What are modelling competencies? *ZDM, 38(2), 113-142.*

Niss, M., Blum, W., & Galbraith, P. (2007). How to replace the word problems. In W. Blum, P. Galbraith, H-W. Henn, & M. Niss (Eds.), *Modelling and applications in mathematics education: The 14th ICMI study.* (pp. 3-32). New York: Springer.

Ministry of Education. (1992). *Mathematics in the NZ curriculum.* Wellington: Author.

OECD. (2003). *The PISA 2003 assessment framework: Mathematics, reading, science and problem solving knowledge and skills.* Paris: Author.

Queensland Board of Senior Secondary School Studies [QBSSSS]. (2000). *Mathematics B senior syllabus 2001*. Brisbane, QLD: Author.

Turner, R. (2007). Modelling and applications in PISA. In W. Blum, P. Galbraith, H-W. Henn, & M. Niss (Eds.), *Modelling and applications in mathematics education: The 14th ICMI study,* (pp. 433-440). New York: Springer.

Victorian Curriculum and Assessment Authority [VCAA]. (2005, December). *Victorian essential learning standards: Discipline-based learning strand mathematics* (Rev. ed.). Melbourne: Author.

Eliciting Positive Student Motivation for Learning Mathematics

Peter Sullivan
Monash University
<Peter.Sullivan@Education.monash.edu.au>

Andrea McDonough
Australian Catholic University
<andrea.mcdonough@acu.edu.au>

Responding to an instrument we developed to give insights into students' orientation to, and motivation for, learning mathematics, Year 8 students showed more confidence in their ability to learn mathematics and in their persistence than observations of their classes would indicate is warranted. They identified a negative influence of peers for some classmates but less for themselves, and had modest career aspirations. We believe teachers can assist students by becoming aware of their orientations to learning, their perceptions of the value of schooling, and their further vocational aspirations, and by finding ways to overcome factors inhibiting their engagement in school.

Introduction

The under-participation of students in learning in the middle years in Australia (students aged 10 to 14) is both widely reported and persistent (e.g., Hill, Holmes-Smith, & Rowe, 1993; Russell, Mackay, & Jane, 2003). This under-participation, in the case of learning mathematics, may be a product of some students: lacking confidence and giving up readily (e.g., Dweck, 2000); not connecting current learning opportunities with their future goals (e.g., Sfard & Prusak, 2005); and experiencing discontinuities between the curriculum, the pedagogy, assessment regimes, and their own culture and family influenced expectations (e.g., Delpit, 1988).

Ideally, to promote student engagement in learning, two sets of factors must align. The first set of factors include that the students have the requisite prior knowledge, the curriculum is relevant to them, the classroom tasks interest them, and the pedagogies and assessment regimes match their expectations. The second set of factors relate to their goals for learning, their willingness to persist, and the extent to which they see participation in schooling as creating opportunities. The focus of this paper is on assisting teachers to address the second set of factors, even though the challenge for teachers of middle years classes, in particular, is to address both sets of factors, more or less simultaneously.

In an earlier study, we investigated individual students' perceptions of the extent to which their own efforts contribute to success in mathematics (Sullivan, Tobias, & McDonough, 2006) and English (Sullivan, McDonough, & Prain, 2005) through two separate interviews where Year 8 students encountered increasingly difficult tasks. The intention was that eventually nearly all students would confront the challenge of a task that was difficult for them. The students were asked how they felt about the challenge they experienced, and the type of support they needed to solve the problem. The survey included items adapted from three instruments proposed by Dweck (2000), and asked students to rate their self confidence and achievement, their persistence, their perception of the value of schooling, and what constitutes successful learning.

We found that the students were surprisingly confident in their own ability, they perceived effort as important and themselves as trying hard, and saw these as linked. The students seemed to have short term goals, aiming to please the teacher by getting questions correct and scoring well on tests. We further found that a significant minority of responses referred to the negative influence of classmates. In such cases, a lack of observable effort could be a result of a desire to be popular or a fear of censure by peers. The present study extends this to examine ways that teachers might be able to support students to overcome inappropriate orientations and negative influences.

The Orientation of Students to Learning

Both the previous work and the current study draw on the work of Ames (1992) and Dweck (2000) who categorized students' orientation to learning in terms of whether they hold either *mastery* goals or *performance* goals. Students with *mastery* goals seek to understand the content, and evaluate their success by whether they feel they can use and transfer their knowledge. They tend to have a resilient response to failure, they remain focused on mastering skills and knowledge even when challenged, they do not see failure as an indictment on themselves, and they believe that effort leads to success. Students with *performance* goals are interested predominantly in whether they can perform assigned tasks correctly, as defined by the endorsement of the teacher. Such students seek success but mainly on tasks with which they are familiar, they avoid or give up quickly on challenging tasks, they derive their perception of ability from their capacity to attract recognition, and they feel threats to self worth when effort does not lead to recognition.

Dweck (2000) connected these goals with two perspectives on intelligence: a fixed perspective termed *entity* theory that refers to students who believe that their intelligence is genetically predetermined and remains fixed through life; and an *incremental* perspective in which students believe that they can change their intelligence and/or achievement by manipulating factors over which they have some control. Students with *incremental* perspectives tend to hold *mastery* goals, whereas an *entity* view can result in *performance* goals.

Of course, most students hold a mix of these types of goals, and there is considerable complexity within each type. For example, performance goals to please a teacher can motivate students to complete tasks satisfactorily as long as the teacher's endorsement is forthcoming (Elliot, 1999). Such goals can also lead to performance avoidance in which students choose not to engage in tasks for fear of failure and the risk of teacher censure.

More recently researchers have recognised the complexity of factors influencing students' orientations to learning. Watt (2004), for example, argued that course choices and achievement are related to students' self-perceptions, including their rating of their ability, and their expectations of success, the value they attribute to the particular content, such as its intrinsic value and its usefulness, and their evaluations of a particular task, such as its difficulty and the amount of effort required to complete it. Similarly, Martin and Marsh (2006) described adaptive or helpful characteristics of students' orientation to learning as the extent to which they feel they can succeed at a task, their valuing of school, mastery orientation, persistence, planning, and self management.

Drawing on these approaches, this research examines students' self perceptions of confidence and effort, aspects aligned with Watt's rating of ability, expectations of success, and effort to complete the tasks, and to Martin and Marsh's self efficacy and persistence, as well as Dweck's (2000) entity/incremental distinction, which is connected to Martin and Marsh's (2006) mastery orientation.

We are also interested in examining external influences on effort. For this we draw on Hannula (2004) who explained that potentially negative influences on effort are derived from adolescents' need for identity, autonomy, and social connectedness that are often enacted in negative ways, such as by challenging the authority of the teacher, and by conforming to peer pressure to under-perform.

Further, we see potentially positive influences as including the extent to which students connect current schooling with future opportunities or their *possible selves,* which is "the future-oriented component of self-concept" (Oyserman, Terry, & Bybee, 2002, p. 313).

Research Context and Data Collection

As a first step, we used a questionnaire to seek students' responses to items addressing such issues, with the intention of subsequently using their collective responses as a prompt for discussion with their class on the implications of trends in the results.

The questionnaire used in the earlier study was based on items proposed by Dweck (2000), predominantly seeking to discriminate between students who had *incremental* or *entity* views on intelligence, and *mastery* or *performance* goals. For this current study, we chose the items from the earlier questionnaire that discriminated between the responses of the students, and added open response items on students' job aspirations, their perception of the effort of others in their class, and their ideas about causes of other students' lack of effort. We removed most negatively worded statements, since we found that these were difficult for weaker readers to interpret. Overall, our intention was for the instrument to be brief, clear, unambiguous, individually completed, easily analysed, and completed in under 15 minutes requiring minimal assistance or explanation. The new instrument was piloted with similar students to the target population, one on one, with the students talking aloud as they responded, and resultant changes were made to clarify wording. In this piloting we found that the items were clear for students who were fluent readers, although we were surprised with some wordings that proved difficult (e.g., suburb) for weaker readers. We adjusted the protocol for administering the tool to allow explanations of words that were not clear. Subsequent interviews with a selection of students, including weak readers, indicated that those students comprehended the questions.

Responses to the final instrument were sought from students in year 8 (age 13) in three government secondary schools, and one Catholic school, in a regional city in Australia. There were a total of 205 responses, 101 male and 104 female, with 15, 41, and 39 respectively from the government schools and 110 from the Catholic school. The schools served predominantly lower socio-economic families. The regional city is prosperous, overall community infrastructure is good, and there are ample further education and employment possibilities for school leavers.

In each school, we interviewed three students in each class following the completion of the questionnaire: one student identified by the teachers as a high achiever, one as a low achiever, and the other in between. Generally the students interviewed endorsed their responses on the questionnaire, and it seems that the instrument does give insights into most students' thinking.

We also conducted a class discussion with one group of students who responded to the questionnaire in which we presented their results in the form of a "research seminar" and sought their responses. The rationale was that if students become more aware of their respective individual responses in comparison with the group responses overall, and if they consider possible implications of their responses, this might allow more active decisions on the connections between their current effort, their learning and future opportunities. Bar graphs were presented to the class, the interpretation of the graphs was clarified, the students discussed the graphs in groups, the groups reported to the whole class, and these reports were recorded, and transcribed.

Results

The results presented here are from items addressing the students' confidence and perceptions of their own effort, their reported commitment to an incremental perspective on intelligence, the influences on their effort including the negative influence of their peers, and their future career aspirations. We also present data on the students' suggestions of what can be done to address the issues raised, and including responses from a particular "research seminar" style intervention.

The tables present the frequency of each of the six response options (strongly agree, agree, mostly agree, etc.). There was no attempt to quantify confidence in the instrument overall, such as using Cronbach Alpha, since the items were addressing quite different constructs, and the items and responses can be taken on face value.

Students' Confidence and Effort

Table 1 presents responses of students to the item relating to confidence. Nearly all of the students report confidence in their ability to learn mathematics.

Table 1
Self Perceptions of Confidence (n=205)

	Strongly agree	Agree	Mostly agree	Mostly disagree	Disagree	Strongly disagree
I feel confident I can learn most things in maths	56	87	50	10	1	1

Table 2 presents the items seeking their self-perception of how they felt their effort would be reported by friends. The stems were phrased in this way to create a sense of distance for the students, and perhaps get more accurate responses. In the previous study, the students' responses to more direct items seemed unrealistic.

Table 2
Perceptions of How Effort would be Seen by Friends (n=205)

	Strongly agree	Agree	Mostly agree	Mostly disagree	Disagree	Strongly disagree
My friends would say that I keep trying when our maths work gets hard	24	60	69	28	15	9
My friends probably think I give up quickly when maths gets hard	9	16	24	34	76	46

These items were designed to get two perspectives on the same variable. Even though the distribution of responses seems similar (with one reversed), the responses were not significantly correlated. The majority of the students report that they consider that their friends would think they try hard, although there is a substantial minority who do not think so. Overall we can infer that most students are satisfied with their level of effort.

It is interesting to contrast these responses with the comments by their teachers who report low levels of persistence and significant difficulties in engaging students in learning mathematics. Based on our observations in mathematics classes, the students overall seem neither confident in their learning nor do they try hard. This is discussed further below.

Entity vs Incremental Views of Mathematics Ability

We were also interested in students' responses to items seeking their views on the nature of ability for mathematics learning. The items are presented in Table 3.

Table 3
Commitment to Incremental or Entity Perspectives of Ability (n=205)

	Strongly agree	Agree	Mostly agree	Mostly disagree	Disagree	Strongly disagree
Anyone can be good at maths if they put their mind to it	81	70	42	10	2	0
People are either good at maths or not. They cannot get better by trying	7	12	16	23	62	85

The responses to the two items are significantly correlated ($r = -0.2$, $p < .05$), and the distributions are similar. Across each of the schools, the responses of these students indicate a strong commitment to an *incremental* view of intelligence.

Influences on Effort

Part of the rationale for the questionnaire is to offer teachers prompts that they can discuss with their students. One key focus could be the effort of the class and the influences on that effort. The following presents some questionnaire responses and some responses of students during an intervention with one class. Table 4 summarises the responses to the prompt "Tick the statement that best describes your maths class".

Table 4
Student Perceptions of the Effort of their Maths Class (n=205)

	Frequency
All try their best	18
Most try their hardest, a few could try harder	108
A few try their hardest, most could try harder	55
All students could try much harder	24

Over half of the students report that most in their class try hard. Overall the students report a positive orientation to effort for their class, although a significant minority think that students could try harder.

There was an open response item, "If there are any students who do not try their hardest in maths, why do you think this is?" The responses were categorized to simplify reporting. The more frequently cited categories, using our words, can be summarized as: lack of motivation or laziness; dislike of mathematics; boredom; difficulties with understanding of the mathematics; desire to be popular; and lack of sense of future. These categories could perhaps have been anticipated, nevertheless the responses give teachers some indication of ways that they might address the engagement of their students.

In the earlier study, the responses to open items and interview questions suggested that there was a significant minority of students whose effort and participation were negatively influenced by peers. Since this was an important and unanticipated result, we included a number of further items that sought responses related to influence of the class on the effort of others or themselves, the results of which are in Table 5. The first item in the table seeks a response about other students, and the others refer to the influences of the class on themselves as individuals.

Table 5
Influences of Other Students on Effort (n=205)

	Strongly agree	Agree	Mostly agree	Mostly disagree	Disagree	Strongly disagree
In my maths class, some students don't try hard because they are afraid of what other students might think of them	35	52	44	39	24	10
I would try much harder in a different maths class. This class holds me back	8	11	28	44	76	38
I am able to try my hardest in maths. The rest of the class doesn't make any difference to me	54	68	57	16	7	3
In maths, I try my hardest in maths no matter what the other students think	51	78	56	13	4	3

The negatively worded items were retained because they seem to offer an additional perspective, and it also seemed that weak readers could interpret them. The trend is clear across the items, and the responses to each are significantly correlated with the others. Most affirm their own effort irrespective of other class members, and they deny that the other class members have a negative influence on their own effort. There is a minority who acknowledge a negative influence of peers.

An intervention seeking to explore this further consisted of presenting these tabulated results to students as column graphs, clarifying that they could interpret the graphs, inviting them to discuss, in groups, the reasons for the responses of the classes as portrayed in the graphs, and then facilitating reporting back by the groups with some whole class discussion. The following are representative extracts from two groups of students reporting on their discussion in response to the first item in Table 5.

> Most people try their hardest because they don't want bad marks, but some people didn't try because they didn't want to look like nerds, and some people are sitting next to smart people so they felt like being smart and doing it, but sometimes there's a dumb group and they don't want to look like a nerd in front of everyone.
>
> We just talked about how people try to get good marks but some people don't try to become nerds and stuff, to get kicked out of social groups and things like that.

In other words, the students are reflecting the results in Table 5 with a recognition that most students try hard, but there are some who are negatively influenced by others. Another group more explicitly connected effort with criticism.

> Because if you try hard in maths, people think you're a nerd and then you get teased. Because if you're smart usually no one likes you, as in they don't not like you but they just call you names because you're smart, and when you're not smart they just…

This illustrates the subtlety of the effect. It appears there is not a direct correlation between not being liked and effort, although effort is likely to draw comment. In a similar way, another of the groups noted:

> There are loser nerds that are losers, and if they're nerds, they are…if you're popular and you are a nerd you're going to still have all your friends around you, and if you're a loser you're going to have no friends around you and no one defending you.

As did the class teacher, this group noted that some students who try hard, even if considered "nerds", are still popular, and so presumably willing to try and achieve despite any criticism whereas the "loser nerd" seems vulnerable. These are issues on which teachers could build further discussion. Again the comments confirm the tabulated results and suggest that the influence of others is indeed an issue that classes could productively discuss.

Future Aspirations

It is assumed that students who have future career aspirations that might include tertiary education would be more orientated to positive participation in school. To explore this, the following open question was posed "What type of job do you want to do after you leave school?" Not all students responded to this item.

We categorised 68 responses as "professional": medicine/health (20), ICT (9), veterinarian (14), lawyer (7), science (2), small business (5), architect (6), and teacher (5).

Eighty-two responses, described as "non professional", included entertainment (13), beauty (16), sports (11), and military/police (9). The 13 responses that indicated they did not know what career they would pursue were included as "non professional" in that it is assumed that lack of a specific career aspiration would not be a positive motivating influence. Likewise, the 20 students who indicated a particular trade were included. Even though trades require post-school study, an aspiration to be a plumber, for example, is not usually associated with greater attention to learning mathematics. The item from the questionnaire that addressed career aspirations is presented in Table 6.

Table 6
Effort and Job Opportunities (n=205)

	Strongly agree	Agree	Mostly agree	Mostly disagree	Disagree	Strongly disagree
Trying hard in maths will give me more future job opportunities	120	66	12	5	1	1

There were no significant differences between the responses of the students from different schools. The majority of the students connected effort with increased job opportunities. It is interesting to compare this with Beavis, Curtis, and Curtis (2005) who reported that students were more likely to have not planned post-school education if they had below average levels of achievement, and if they had educational aspirations, these were more likely to be apprenticeships for trades.

To investigate this further, we cross tabulated the responses of the 30 students who strongly agreed with the proposition, "I feel confident I can learn most things in maths" with whether their aspirations were professional, as described above, or not. We found that it was more likely that students with professional aspirations would be part of this "more confident" group than those without professional aspirations. We also cross tabulated the career responses with those who strongly agreed with the statement, "My friends would say that I keep trying when our maths work gets hard", but students with professional aspirations were not more likely to choose the "strongly agree" category than the others.

In the class discussion, a question was posed contrasting their responses to the questionnaire item that effort in mathematics class is connected to job opportunities with the earlier discussion of influences on effort. Two of the groups responded as follows.

> Because we don't think that it's important. We're not like really thinking of what we want to be right now, and we're not thinking of how failing a subject … is going to affect our jobs and stuff.

Another group responded:

> If you're at school and you sit down and you have to do maths or something, you're not really thinking…like, if someone asks you a question, "Will maths affect what job you get when you're older", you can like sit down and think about it, you go, of course it's going to affect it. But when you sit down in maths, to do your maths after recess or lunch or whatever, you don't really think what I do right now is going to affect what I'm doing in 15 or 20 years.

Yet another group compared the influence of friends and consideration of the future:

> When you're in school, you don't really think about, like, that work is important, you only think that your friends are important and what you do at recess and lunch and not in classes and that.

It seemed that the students took the point, and that they are also both reflective and honest in their assessment. Such responses would provide an opportunity for teachers to pursue the issue further with the students, and perhaps find ways to connect current efforts with future opportunities more explicitly.

Discussion and Conclusion

The limited number of responses is due to stringent procedures for seeking parental approval, and this process may have biased the sample. Nevertheless this potential bias would make the results more severe. In other words, the students who did return the ethics approval forms were presumably those more positively oriented to schooling.

One of the results of interest is the positive self ratings of the students' confidence that they can learn mathematics. It is possible that the students' self perceptions are accurate, and there are other factors constraining their participation in learning. It is also possible that the items do not allow students to communicate their actual confidence and self perception of effort. It is also possible that the students' self perceptions may be inaccurate, in which case some attention to these unrealistic perceptions is necessary. This explanation is favoured by Dweck (2000) who argues that some teachers give students unrealistic positive evaluations of their achievement, and even conspire to reduce challenge to produce success. In other words, it seems important that assessments of students' performance are realistic, and that teachers should be encouraged to affirm effort more than achievement.

Another result relates to the influence of peers. Most students acknowledged this negative influence of peers on some others, but denied that it influenced them. The instrument and the results also raise the possibility that there is a significant minority of students for whom this factor is a negative influence. Recently the first author was teaching a year 8 class. The class was asked to work out what percentage calculations expressed in the form "50% of 200" they could work out in their head. The intent of the task was that the children would realise that it is possible to calculate some percentage calculations in your head, and then make generalisations about what type of percentages are straightforward and can be calculated mentally and for which types it is more appropriate to use a calculating device. After a group discussion, a spokesperson for one of the groups of students in reporting back said, "Well you can get half of anything, and quarter of …" at which stage there was a chorus of derision from some other students about the nature of this response. The students from the group that had formulated the response then refused to articulate their answer further even though it was clear that they were satisfying part of the task by seeking to form generalisations. Because some other class members were critical of students who were either seen to be trying hard or seeking to intellectualise their engagement with the task, these students then not only stopped working, but could not be encouraged to re-engage with the task. In other words, it seems that this negative influence of peers would be particularly detrimental if the teacher is hoping to promote argumentation as a pedagogical tool.

The third result relates to the career aspirations of these students, in that only one third of the students listed a career aspiration that would be associated with success in mathematics at school. Even though the students discounted the motivational impact of their future aspirations, it seems that helping students to become aware that a decision not to work in order to please the peer group, or for some other reason, does have longer term consequences. It seems that teachers could assist students by making this connection between current effort and future opportunities more explicit.

For each of these aspects of learning it is important that teachers are aware of the responses that their students would give. We conclude that a simple instrument similar to the one used in this case can provide a prompt for discussion and consideration of these potentially important issues. The hypothesis is that if teachers are aware of the orientations of their students they can intervene positively. Dweck (2000), for example, argued that teachers can teach self regulatory behaviours such as decoding tasks, perseverance, seeing difficulties as opportunities, and learning from mistakes. This capacity is evident in quite separate research strands on self fulfilling prophecy (e.g., Brophy, 1983), and motivation (e.g., Middleton, 1995).

We suspect that, concurrent with considering ways of overcoming any difficulties their students may be experiencing with learning, teachers could well develop awareness of connections between study and career opportunities, encourage students to keep future options open (by studying), make tasks relevant to their lives, illustrate utility of learning mathematics to all, especially to those who do not aspire to continue with further study, and develop greater awareness of effort expended and required, and ways of overcoming negative influence of peers.

A Possible Continuation of the Class Discussion

To illustrate the way that the instrument and the ensuing discussion of results might be used, the following suggests some ways that teachers build on the students' comments.

With this class, there were a number of occasions that the students made responses wise for their age. For example, one group, in discussing the influence of peers, said:

> It's good to be smart because then you know stuff, and if you're dumb just so your friends like you then it's really bad. Obviously they're not your friends if they make you be dumb to be their friend.

This response could be used by the teacher as the basis of further discussion on the potential for peers to be both a positive and a negative influence, and on ways that students could respond to negative peer pressure. For example, the teacher could: create a story scenario using photographs or drawing and invite the students to work out the sequence of the events; invite the students to write story about a time that they underperformed for fear of censure by their friends; have the students create a role play of a scenario; or ask them what how they might encourage a friend who was not trying.

In the same discussion, another student made a similarly interesting comment using a sporting perspective.

> …if you're playing and you mess up or something and you have a kick and it falls short or it goes out of bounds on the full where it shouldn't, if you have someone on your team that says, 'You'll get the next one,' you're more confident to keep playing, but if someone is like, 'What are you doing?'

Even though students probably see sport and school as different, this response would also serve as a useful prompt for further discussion on ways that peers have the potential to be both supportive and critical, and on the positive influence that peers can have on effort and achievement. For example, the teacher might follow up by asking:

- How might "you'll get the next one" help?
- What would "you'll get the next one" look like in a mathematics class?

Responses of Year 8 students in the present study reveal awareness that perhaps is not expected of students who are less engaged in learning mathematics than would be desirable. We have argued that the insights provided by students can be a powerful starting point for addressing under-participation of students in middle years learning.

References

Ames, C. (1992). Classrooms: Goals structures and student motivation. *Journal of Educational Psychology, 84*(3), 261-271.

Beavis, A., Curtis, D., & Curtis, N. (2005). *What do students think of work? Are they on the right page? Junior secondary school students' perception of the world of work.* Sydney: The Smith Family.

Brophy, J. (1983). Research on the self-fulfilling prophecy and teacher expectations. *Journal of Educational Psychology, 75*(5), 631-661.

Delpit, L. (1988). The silenced dialogue: Power and pedagogy in educating other people's children. *Harvard Educational Review, 58*(3), 280–298.

Dweck, C. S. (2000). *Self theories: Their role in motivation, personality, and development*. Philadelphia: Psychology Press.

Elliot, A. J. (1999). Approach and avoidance motivation and achievement goals. *Educational Psychologist, 34*(3), 169-189.

Hannula, M. (2004). *Affect in mathematical thinking and learning.* Turku: Turun Yliopisto.

Hill, P., Holmes-Smith, P., & Rowe, K. (1993). *School and teacher effectiveness in Victoria: Key findings from Phase 1 of the Victorian Quality Schools Project.* Melbourne: Centre for Applied Educational Research.

Martin, A., & Marsh, H. (2006). Academic resilience and its psychological and educational correlates: A construct validity approach. *Psychology in the Schools, 43*(3), 267-281.

Middleton, J. A. (1995). A study of intrinsic motivation in the mathematics classroom: A personal construct approach. *Journal for Research in Mathematics Education, 26*(3), 254-279.

Oyserman, D., Terry, K., & Bybee, D. (2002.) A possible selves intervention to enhance school involvement. *Journal of Adolescence, 25*, 313-316.

Russell, V. J., Mackay, T., & Jane, G. (2003). *Messages from MYRAD (Middle Years Research and Development) - Improving the middle years of Schooling*, Melbourne: IARTV.

Sfard, A., & Prusak, A. (2005). Telling identities: The missing link between culture and learning mathematics. In H. L. Chick & J. L. Vincent (Eds), *Proceedings of the 29th Annual Conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 37 - 53), Melbourne: PME.

Sullivan, P., McDonough, A., & Prain, V. (2005, November). Student engagement in the Middle Years: Describing influences and possible teacher actions. *Proceedings of the Australian Association of Research in Education Annual Conference. Sydney*, http://www.aare.edu.au/05pap/sul05134.pdf

Sullivan, P., Tobias, S., & McDonough, A. (2006). Perhaps the decision of some students not to engage in learning mathematics in school is deliberate. *Educational Studies in Mathematics, 62*(1), 81-99.

Watt, H. (2004). Development of adolescents' self–perceptions, values, and task perception according to gender and domain in 7th though 11th grade Australian students. *Child Development, 73*(5), 1556-1574.

Learning from Children about their Learning with and without ICT using Video-Stimulated Reflective Dialogue

Howard Tanner
Swansea Institute of Higher Education
<Howard.Tanner@sihe.ac.uk>

Sonia Jones
Swansea Institute of Higher Education
<Sonia.Jones@sihe.ac.uk>

The Interactive Teaching and ICT project explored the process of interactive teaching and learning with and without ICT. A key technique in our methodology was the use of video-stimulated reflective dialogue to assist teachers to reflect on key episodes in their teaching. In this paper we discuss how this technique was extended to encourage pupils between the ages of 5 and 14 to reflect on their learning of mathematics. Analysis of the reflective dialogues indicates that even quite young children were able to articulate opinions about the ways in which they learned and the ways in which ICT supported this.

In recent years there has been significant investment in the provision of ICT resources for schools in England and Wales in the expectation that this will lead to improvements in teaching and learning. In particular, there has been significant investment in presentational tools such as data-projectors and *Interactive White Boards* (IWBs). However, research indicates that the impact of ICT on pedagogy and learning within mathematics classrooms has been limited (Becta, 2003; Smith, Higgins, Wall, & Miller, 2005; Smith, Hardman, & Higgins, 2006; Moss et al., 2007).

The Interactive Teaching and ICT (ITICT) project (Kennewell et al., 2005) is investigating the processes of effective, interactive teaching with and without ICT. A variety of ICT resources were used in the project schools, however all the teachers used presentational tools such as IWBs or data-projectors and the extent to which these tools could be used to support effective interactive teaching was a focus of the research.

Changes in teaching and learning practices impact directly on pupils, however, and the project privileges pupils' voices by ascribing to them a key role in the analysis of their own learning. A range of quantitative and qualitative data was collected over the course of the project, including pre- and post attainment tests, interviews with teachers and pupils and lesson observations. However, a key research technique for qualitative data collection and analysis was *video-stimulated reflective dialogue* (VSRD) (Hargreaves et al., 2003). The project extended the use of the VSRD technique to include pupils' voices in the discourse. Video-clips of lessons selected by their teachers were shown to focus groups of pupils as a prompt for generating a reflective dialogue

This paper examines the efficacy of VSRD as a research tool to stimulate children to reflect on their own learning of mathematics and expose their perceptions of teaching episodes. It examines the extent to which children are able to identify those pedagogies that are most effective in helping them to learn. In particular, the paper probes children's perceptions about pedagogies associated with interactivity and the use of ICT.

Interactive Teaching and ICT

Recent policy initiatives in England and Wales have been concerned with the development of whole class teaching approaches that are intended to be "oral, interactive and lively" (DfEE, 2001: 1.26). This was intended to be more dialogical than the traditional recitation script of Initiation, Response, Feedback (IRF) (Tharp & Gallimore, 1988).

However, the nature of interactive whole class teaching was not clearly defined in the Strategies and is widely interpreted in practice (Mroz, Smith, & Hardman, 2000; English, Hargreaves, & Hislam, 2002).

Although pedagogical interactivity may be seen as implying bi-directional communication, with children developing independent voices in discussion and experiencing higher levels of autonomy (Burns & Myhill, 2004), interactive whole class teaching has largely been implemented as pupil participation in fast, teacher-led question and answer sessions (Moyles, Hargreaves, & Merry, 2003; Hargreaves et al., 2003). Although teachers now ask more questions most pupil responses remain very short, just 5 seconds on average, and involve three or fewer words. There is little opportunity for pupils to engage in extended responses or to express and evaluate ideas of their own (Moyles et al., 2003; Hargreaves et al., 2003; Smith, Hardman, Wall, & Mroz, 2004).

The teacher-centred approaches encouraged by the Strategies contrast strongly with more pupil-centred approaches more often associated with the use of ICT. In the context of ICT, interactivity usually refers to its facility to provide rapid and dynamic feedback and response. Such technical interactivity has been shown to afford increased learner autonomy and effective independent learning by pupils (Harrison et al., 2002).

The use of interactive whiteboards (IWBs) in particular, is claimed to motivate students because of “the high level of interaction – students enjoy interacting physically with the board, manipulating text and images” (Becta, 2003). We should distinguish, however, between the technical interaction of the IWB as an interface with the computer and the pedagogical interaction that is required for effective learning.

Presentational tools such as data projectors and IWBs do not naturally afford an increase in learner autonomy in the way that, for example, individual or paired use of laptops to sustain interaction with learning resources does. In fact, IWBs may be used to “tame” ICT, bringing it more tightly under the control and mediation of the teacher.

A potential drawback of the introduction of IWBs is the reinforcement of a transmission style of teaching that reduces pupil autonomy and interaction, sometimes reducing the role of the pupil to that of “spectator” (Moss et al., 2007). Recent large scale research reports that in lessons involving IWBs, initially there is an increase in the pace of lessons but fewer uptake questions are used and pupils’ responses remain short. The traditional pattern of questioning (IRF) persists in spite of the Strategies and is more prevalent in IWB lessons (Smith et al., 2006).

Teachers were most likely to incorporate the more visible “surface features” of the Strategies, such as pupil engagement or inviting children out to the board into their pedagogy; “deeper features” including formative assessment, the co-construction of meaning through dialogue, and the development of thinking and learning skills tended to be less well developed (Hargreaves et al., 2003; Moyles et al., 2003; Smith et al., 2004). Similarly, when using IWBs, teachers sometimes focus on technical interactivity and over-value relatively mundane activities that pupils perform at the board (Moss et al., 2007).

It is perhaps unsurprising then that large scale studies report that the introduction of IWBs does not lead to general improvements in pupils’ attainment (Smith et al., 2006; Moss et al., 2007). The introduction of technology does not in itself encourage the development of more dialogical approaches.

Several thinking skills projects, which achieved significant improvements in pupils’ learning, have included dialogical teaching approaches as key aspects of their intervention strategies, Significantly, the most successful interventions also included metacognition as a major feature of their approaches (see McGuinness (2005) for an overview).

Metacognition

We are interested in metacognition in this paper for two reasons. First, because of its significance for learning – meta-studies of interventions based on metacognition report improved learning with large effect sizes (Hattie, Biggs, & Purdie, 1996). Second, we are interested in metacognition because our intention was to use VSRD to explore children's awareness of their own learning processes and the extent to which they considered the affordances of ICT could be used to support their learning.

Metacognition is a "fuzzy" and elusive term that refers loosely to the knowledge and control that individuals have of their own cognitive systems (Brown, 1987). This dual nature includes both (a) the awareness that individuals have of their own knowledge, their strengths and weaknesses and their capabilities and preferences as learners; and (b) their ability to regulate their own actions in the construction of new knowledge (Flavell, 1976).

The association between some aspects of metacognition and reflected abstraction has led to debate about whether primary age children are able to think metacognitively or benefit from metacognitively based interventions as reflected abstraction is characteristic of formal operational thinking (Georghiades, 2004). However, Adey, Robertson, and Venville (2002) have reported the success of a cognitive acceleration programme with 5- and 6-year-old pupils that included a significant metacognitive component. This accords with Kuhn's (1999) suggestion that metacognitive processes are developmental in character.

Metacognitive knowledge about one's own thinking and learning processes is often described as "late developing". It is usually stateable and requires a higher degree of understanding than does regulation of cognition. Metacognitive skills, used to regulate learning and problem solving, are less conscious processes which are often invoked in an implicit manner and rarely stateable; "knowing how to do something does not necessarily mean that the activities can be brought to the level of conscious awareness and reported on to others" (Brown, 1987, p. 68).

The literature is unclear, however, on whether metacognitive skills are age dependent. The lowest level of self regulation is to be found in quite young children but the capacity for reflected abstraction is suggested to develop between the ages of 11 and 15 (Piaget, 1978). It may be that the extent to which young children are aware of and are able to articulate their use of metacognitive strategies is limited.

Methodology

The ITICT project examined teacher controlled interventions in a number of subjects within a quasi-experimental design of control and intervention classes. This paper reports on the results of the 12 classes that focused on Mathematics. There were two matched pairs of classes in each of the first three *Key Stages* (KS1 to KS3) of the Welsh education system (KS1: 5-7 years, KS2: 7-11 years and KS3: 11-14 years).

In each pair of classes, one teacher used ICT as and when thought appropriate. The other teacher taught the same topics without ICT. The teachers who had volunteered to participate in the project had been selected on the recommendation of their head teachers as effective practitioners who wished to explore and develop their use of interactive teaching approaches and the extent to which the affordances of ICT supported these approaches.

Each research cycle included lesson observations by two members of the research team, and group meetings of teachers and researchers at which issues were discussed, tentative hypotheses formed and new focuses decided.

During lesson observations, the lead researcher took written notes in an open, but semi-structured framework for analysing teaching and learning activities (Kennewell et al., 2005). The second researcher was responsible for creating a video-recording of the lesson.

The teachers analysed their videos after the lesson and selected sections that they felt exemplified their best practice. In the case of the teacher using ICT, one focus was always on the ways in which the affordances of ICT were supporting interactive teaching. The lead researcher returned the following week and engaged in a reflective dialogue with the teacher that was stimulated by the teacher's selected clips. The dialogue was recorded for transcription and analysis.

Following the VSRD with the teacher, the lead researcher met with a focus group of pupils from the lesson. Focus groups generally consisted of between four and six boys and girls who had volunteered to participate. The focus group was shown the episodes of the video that the teacher had selected for their reflective dialogue and after each episode were engaged in semi-structured discussion about the learning that might be occurring.

Pupils were invited to comment on the features of the episode and setting that helped them to learn or inhibited their learning. Follow up questions were asked to probe why they thought that their learning had been helped or hindered by the approach taken in the episode. If it had not already arisen, they were invited to discuss whether the use of ICT had assisted their learning (or would have assisted their learning in the non-ICT lessons).

Results

Pupils of all ages were generally keen to participate in the VSRD. The types of classroom interactions identified by pupils were similar across subject areas and occurred in both ICT and non ICT classes. The following key themes emerged from the VSRD case studies.

Preferred Teaching and Learning Approaches

There was a clear preference for interactive oral work with a strong dislike of lessons where pupils were "writing all the time and copying off the board" or "teachers are talking all the time and you're just listening". Pupils preferred the use of pictures and animation rather than just writing and, with particular reference to IWBs, appreciated "bright, coloured displays that hold your attention". This preference for interaction was partly associated with the boredom that arose from a lack of variety in some teaching approaches but it also points to pupils' awareness that active participation may result in more effective learning.

In KS2 and KS3, pupils in the case studies could identify the value of discussing alternative viewpoints to challenge and clarify their learning. In one mathematics lesson, the teacher had deliberately designed questions to expose common misconceptions and generated a class discussion in which pupils argued through their solutions. During the VSRD, pupils commented on how this had made them reflect on their own thinking:

> P1: When the first couple of pupils said it [the misconception answer] I thought no, that's not right, but then after more pupils said it I'm thinking, hang on now, I used to think this but now they've made me confused.

P2: It does sway you a bit, doesn't it.
P3: That actually got me thinking, why are they thinking that?
R: So what do you do then?
P1: Well, I'd really check it through in my head and then after I did that I thought no, they are wrong…

The Importance of "Fun"

In nearly all the discussions, pupils commented on whether the classroom activities were "fun" or not. Many pupils recognised that their teachers were trying to make activities "more funner" for them in order to motivate them to learn. However, unpicking the nature of fun revealed a number of different factors. Often fun was equated with the variety and novelty of the tasks. Boredom was often associated with lack of variety, although it was also used to describe a lack of understanding.

Many "fun" activities were described as "games". Competition was a feature of many games, but it was not a necessary feature. Competition was sometimes seen as "fun" but this was not always the case and sometimes was viewed negatively. Sometimes activities were described as games because they included a random element and the fun arose from not knowing what would happen next. Children also valued an element of farce or silliness. Often activities were described as games because they involved a degree of personal control of strategy within a challenging context.

Older pupils, particularly at KS3, were able to distinguish between fun and the value of the task for their learning.

P: I don't really mind whether we use it (IWB) or not. I honestly think that, yeah, it's a bit of fun but I don't have my learning improved by it.

Affordances of ICT

The IWB was perceived to have clear advantages over a static board for presentation. They appreciated the accuracy of diagrams and the neatness of writing on IWBs.

P: You can actually understand the writing because you can't usually understand Mr X's writing.

Pupils claimed to be more motivated by working with the technology they saw as belonging to their generation such as IWBs instead of "old technology" such as OHPs.

The transitory, provisional nature of work done on IWBs was considered useful. Pupils also valued the use of mini-whiteboards for the same reason: "You can just rub it out. It's not untidy". Pupils seemed happier to "have a go" and to make errors in these transitory formats rather than in their exercise books which were seen as "best work" that ought to be a neat, finished product.

KS3 pupils distinguished between occasions when technology was used to present solutions as opposed to facilitating active participation with the support of a teacher.

R: So why aren't you convinced about IWBs?
P1: `Cos on the whiteboard they would just load up a calculator, they'd type it in and they'd hit the equals and it would come up with the answer and you don't know how it came out, so if you're not allowed a calculator you can't get it. Whereas, if it's just a plain whiteboard they have to show you how to work it out otherwise you can't just work it out.
P2: Yeah `cos Miss shows us how to work it out. We'll know what to do in a test then.
P1: `Cos the normal whiteboard, it isn't all like, you know, pre programmed, you have to work yourself step by step through it.

P1: Instead of just clicking a button then 'Ooh look, it's happened.

The Value of Feedback

Feedback was identified by many pupils in KS2 and KS3 as important for their learning. The ease with which computers could give instant and individual feedback was valued but a distinction was made by many pupils between being told merely whether their answer was correct or not, and the explanation that they would get from discussion with the teacher or with their friends.

Pupils reported liking learning from their friends not just the teacher. They described working with a partner as motivating. They liked to work collaboratively, often sharing the load, but they also recognised the value of the occasions when disagreements led to views being challenged and refined through discussion

In some contexts, teachers were seen as mediating information that had been originally taken from the internet. In such cases, the computer was seen as a more reliable source of information than the teacher. However in other contexts, such as dedicated teaching software, the computer was seen as a limited source of information and restricted in its teaching potential, with the teacher being viewed as having a more elaborate knowledge and being a source of alternative explanation. Pupils talked of computers and IWBs in an anthropomorphic fashion, for example claiming that "The board thinks that…" or "The board's method is…"

Pupil Interaction at the Front of the Class

In most classrooms, irrespective of whether an IWB was available, pupils were expected to go out to the board; how children felt about this depended on the classroom culture. In some classrooms, pupils said that they did not mind making a mistake at the board because they knew that no-one would laugh at them and that they could learn from their mistakes. In other classes, pupils were scared of making mistakes in public as they knew they would be laughed at. Some said that they would laugh at their own mistakes to get in first before someone else laughed at them.

Most pupils enjoyed going to the board to participate in the lesson, however, sometimes their contribution required only low cognitive demands. On the other hand, some teachers used the affordances of ICT to challenge and develop higher order thinking, using the board as a site for the co-construction of knowledge.

Pupils' Metacognitive Awareness of their Learning

Although these themes were common across subjects and Key Stages, the quality of pupils' comments about their learning differed according to their age and ability. Pupils' responses could be classified into four categories: affective comments, recall of lesson, description of intended learning, and metacognitive comments about their learning.

In KS1, pupils' responses were generally of the first two categories. When they watched the video, the children often re-lived the moment. They responded as if they were in the lesson, placing themselves back in the action again. They put their hands up as if to answer the teachers' questions or called out answers. Alternatively, they described superficial aspects of the lesson, e.g., "Simon's at the board now". Many pupils were able to comment about the importance of working together in social terms and needing to be kind to each other, but this was usually using forms of words that had been used explicitly by their teachers.

As Kuhn (1999) suggested, pupils were often not aware of how or what they had learned and struggled to describe their thinking. One lesson in KS1 focused on the use of a number square to subtract two 2-digit numbers, for example, 49 – 37. Initially, one pupil had been unable to calculate such questions yet in the VSRD appeared not to recognise what or how he had learned, claiming instead that he had always known how to do it.

No explicit metacognitive reflection on the learning process was observed with KS1 pupils. However some precursors to metacognition were seen from the most advanced pupils who were beginning to be able to pause and reflect on how they had performed a particular task. Some pupils were aware of some of what they knew and could indicate how they had learned it.

One KS1 pupil was asked a particularly challenging question. He sat in silence for several seconds then gave the correct answer. After congratulating him the researcher asked how he had arrived at the answer. The pupil replied that he had thought about it. At the end of the lesson, before the researcher left, the pupil came up unprompted and explained how he had worked out the solution. The delay suggests that he was not fully aware of his own knowledge but that he had chosen to reflect on his thought processes and had been able to reconstruct his thinking sufficiently to explain it.

In KS2 and KS3, pupils were more able to use the video to facilitate reflection. More children were able to comment on which learning activities they enjoyed and which motivated them to learn. They were able to talk about the value of working together in learning as well as social terms. Pupils were more able to talk explicitly about their learning processes in schools that had a focus on thinking skills and learning to learn.

At KS2, pupils' comments about learning often echoed phrases commonly used by their teacher, for example: "You have to make at least one mistake every lesson otherwise you aren't learning". However, although derivative, these aphorisms were applied in appropriate contexts, indicating a degree of internalisation or appropriation. However, in most cases, knowledge about their own learning processes was implicit rather than explicit.

At KS3, far more children were able to talk explicitly about their learning. Many were able to use the video as a prompt to reflect on their learning, not only during the specific episodes shown, but also in more general terms. All were able to describe their feelings about activities, which they enjoyed and which motivated them. Some were able to analyse which teaching and learning strategies worked for them, separating enjoyment from learning potential.

Some KS3 pupils commented on the value of VSRD for making them aware of how their own learning had progressed.

P1: Oh, I remember this lesson. It seems so obvious now when we look at it.
P2: It is.
R: What seems so obvious now?
P1: How we got it wrong!
P2: Yeah, when you said [wrong answer] but it wasn't, it was…

Conclusions

ICT and Interaction

In our case studies, children's views about the value of ICT for their learning often focused on the superficial features of presentational tools such as IWBs. They valued the big, bright, colourful display and the neatness of type. They considered themselves to be a technological generation and are motivated more by modern technology than older tools. ICT and presentational tools such as IWBs in particular, were often described as being fun and were valued for their potential to include an element of play or game into school life.

However, when they were asked about how they learn, children tended to talk about interaction rather than the technology. The pedagogical approaches they described were generally not ICT dependent, although the affordances of ICT could be used to support them.

They valued the social and affective aspects of school life, such as working with friends, having work explained by teachers and feeling safe to make mistakes. Interaction was highly valued for learning, both in a whole class context and on a one to one basis with teachers or other pupils. Oral work was preferred to writing or "copying down from the board". However, "listening to teachers talk" was disliked and distinguished from more interactive approaches. In KS2 and KS3, some of the children were able to distinguish between what they enjoyed and what helped them to learn. Interactive approaches were considered to be more enjoyable and more effective.

Metacognition

Our case studies are consistent with the position that stateable metacognitive knowledge is relatively late developing in comparison with metacognitive skills and strategies. Metacognitive skills are evident, in implicit forms at least, in quite young children. In our case studies, metacognitive skills were more apparent in classes where there was an emphasis on thinking skills, discussion and reflection.

This research is unable to make claims about the conditions for the development of the different forms of metacognition, but the results are consistent with the position that the development of metacognitive knowledge and skills is responsive to dialogical and reflective teaching approaches.

VSRD

Children can offer important insights into their learning processes that are of interest to us as researchers and teachers. They provide a perspective on learning that arguably could be viewed as central to the business of education. The use of VSRD as a research tool helped us to gain access to these insights by providing a focus for collective reflection.

References

Adey, P., Robertson, A., & Venville, G. (2002). Effects of a cognitive acceleration programme on Year 1 pupils. *British Journal of Educational Psychology, 72,* 1-75.

Becta (2003). *What research says about interactive white boards*. Retrieved 20 March 2004 from the World Wide Web: http://www.becta.org.uk/page_documents/research/wtrs_whiteboards.pdf.

Brown, A. L. (1987). Metacognition, executive control, self regulation and other more mysterious mechanisms. In F. E. Weinert & R. H. Kluwe (Eds.), *Metacognition, motivation and understanding* (pp. 65-116). Hillsdale, NJ: Lawrence Erlbaum.

Burns, C., & Myhill, D. (2004). Interactive or inactive? A consideration of the nature of interaction in whole class teaching. *Cambridge Journal of Education 34,* 35-49.

DfEE (2001). *Key Stage 3 National Strategy: framework for teaching mathematics*. London: Author.

English, E., Hargreaves, L., & Hislam, J. (2002). Pedagogical dilemmas in the National Literacy Strategy, *Cambridge Journal of Education, 32,* 9-26.

Flavell, J. H. (1976). Metacognitive aspects of problem solving. In L. B. Resnick (Ed.), *The nature of intelligence* (pp. 231-235). Hillsdale, NJ: Lawrence Erlbaum.

Georghiades, P. (2004). From the general to the situated: Three decades of metacognition, *International Journal of Science Education, 26,* 365-383.

Hargreaves, L., Moyles, J., Merry, R., Paterson, F., Pell, A., & Esarte-Sarries, V. (2003). How do primary school teachers define and implement 'interactive teaching' in the National Literacy Strategy in England. *Research Papers in Education 18,* 217-236.

Harrison, C., Comber, C., Fisher, T., Haw, K., Lewin, C., Linzer, E., McFarlane, A., Mavers, D., Scrimshaw, P., Somekh, B., & Watling, R. (2002). *ImpaCT2: The impact of information and communication technologies on pupil learning and attainment.* Coventry: Becta.

Hattie, J., Biggs, J., & Purdie, N. (1996). Effects of learning skills interventions on student learning: A meta analysis. *Review of educational research, 66,* 99-136.

Kennewell, S., Beauchamp, G., Jones, S., Norman, N., Parkinson, J., Thomas, G., & Tanner, H. (2005). *Interactive Teaching and ICT Project.* Retrieved 7 August 2006 from the World Wide Web: http://www.tlrp.org/proj/kennewell.html.

Kuhn, D. (1999) A developmental model of critical thinking. *Educational Researcher, 28*, 16-25, 46.

McGuinness, C. (2005). Teaching thinking: theory and practice. *British Journal of Educational Psychology, Monograph Series, II, 3,* 107-126.

Moss, G., Jewitt, C., Levaaic, R., Armstrong, V., Cardini, A., & Castle, F. (2007). *The interactive whiteboards, pedagogy and pupil performance evaluation: an evaluation of the schools whiteboard expansion (SWE) project: London challenge,* London: Institute of Education.

Moyles, J., Hargreaves, L., & Merry, R. (2003). Interactive teaching. In J. Moyles, L. Hargreaves, R. Merry, F. Paterson, & V. Esartes-Sarries (Eds.), *Interactive teaching in the primary school: Digging deeper into meanings* (pp. 171-192). Maidenhead: Open University Press.

Mroz, M., Smith, F., & Hardman, F. (2000). The discourse of the literacy hour. *Cambridge Journal of Education 30,* 379-390.

Piaget, J. (1978). *Success and understanding*. London: Routledge & Kegan Paul.

Smith, F., Hardman, F., Wall, K., & Mroz, M. (2004). Interactive whole class teaching in the National Literacy and Numeracy Strategies. *British Educational Research Journal, 30,* 395-411.

Smith, H., Higgins, S., Wall, K., & Miller, J. (2005). Interactive whiteboards: boon or bandwagon? A critical review of the literature. *Journal of Computer Assisted Learning, 21*, 91-101.

Smith, F., Hardman, F., & Higgins, S. (2006). The impact of interactive whiteboards on teacher-student interaction in the national literacy and numeracy strategies. *British Educational Research Journal, 32*, 443-457.

Tharp, R., & Gallimore, R. (1988). *Rousing minds to life.* New York: Cambridge University Press.

Dependency and Objectification in a Year 7 Mathematics Classroom: Insights from Sociolinguistics

Steve Thornton
Griffith University
<s.thornton@griffith.edu.au>

This paper examines how the activities, discourse, and artefacts in a mathematics classroom may serve to position students as dependents or to objectify them, rather than encouraging the development of subjectivity by apprenticing them into the valued discourse of the mathematics classroom. The paper uses three sociolinguistic approaches to interpret the interactions between Simon, the teacher, and Dean, a student, in a Year 7 mathematics classroom. Although they have very different goals and methodologies, each approach has the potential to reveal the social function of language in a mathematics classroom.

Introduction: Sociolinguistics

Sociolinguistics is the study of language in society. It asks questions such as why we speak differently in different social contexts, how language can be used to serve social functions, and how language is used to convey meaning. It focuses on issues such as gender, race, social class, power relations, and identity through looking at language choice and variations (Holmes, 1992). Although it is not possible in this short paper to do justice to the huge field of sociolinguistics, nor to any one of the three approaches described, the different sociolinguistic lenses used illustrate the potential of sociolinguistics as a tool for examining interactions in the mathematics classroom.

Critical Discourse Analysis (Fairclough, 1992)

Fairclough (1992) considers discourse as a mode of action in which people act on the world and each other, in addition to being a mode of representation. He stresses that there is a dialectic relationship between discourse and social structure, with discourse on the one hand being constrained by social structure, and on the other being socially constitutive. He sketches a three-dimensional framework for conceiving of and analysing discourse, considering "every discursive event as being simultaneously a piece of text, an instance of discursive practice and an instance of social practice" (p. 4).

The first dimension is discourse-as-text, i.e., the linguistic features and organization of concrete instances of discourse. Building on the work of Halliday (1978), Fairclough maintains that text analysis must include a consideration of vocabulary, grammar, cohesion, and text structure. Halliday describes the ideational function of language, which may be material processes, mental processes, or relational processes. This function is revealed by examining the field of the text and by looking at the use of active or passive voice and at the use of verbs such as "think" or "do". Aspects of language such as cohesion and the use of given/new structures are important in describing its textual function.

Fairclough's second dimension is discourse-as-discursive-practice, i.e. discourse as something that is produced, distributed, and consumed in society. He introduces the concepts of "force" to describe what the text is being used to do socially, "coherence" to describe the extent to which an interpreting subject is able to infer meaningful relationships

Proceedings of the 30th annual conference of the Mathematics Education Research Group of Australasia
J. Watson & K. Beswick (Eds), © MERGA Inc. 2007

and to make sense of the text as a whole, and "intertextuality" to describe how texts are related historically to other texts. The tenor and mode of the text, indicated through the use of personal pronouns and the degree of certainty conveyed by verbs, adverbs, or adjectives, reveals the interpersonal function of language.

Fairclough's third dimension is discourse-as-social-practice, drawing on the Marxist concepts of ideology and hegemony. He claims that ideology is located both in the structure of discourse and in the discourse events themselves. For example, he suggests that the turn-taking practice of a typical classroom implies particular ideological assumptions about the social identities of and relationships between teacher and pupils. Hegemony concerns power that is achieved through constructing alliances and integrating groups. Dominant groups exercise power through integrating rather than dominating subordinate groups, winning their consent, and establishing a "precarious equilibrium".

Morgan (2005) uses Halliday's (1978) systemic functional linguistics to explore the notion of definition within two school mathematics texts, one written for advanced students and one for intermediate students. Her analysis reveals that in the higher level text students are included in the community of mathematicians through the use of passive voice, a focus on relations rather than materials processes and through reduced modality, which allows for alternative ways of thinking about ideas.

Thornton and Reynolds (2006) use critical discourse analysis to examine one mathematics classroom in which students argue over the effect of changing the value of a in the graph of $y = ax + b$, suggesting that the discursive norms in the classroom led to heightened levels of personal agency. They conclude that a discourse that is exploratory, tentative and invitational, that contains emergent and unanticipated sequences, and that recognises alternative ideas even ones that are strange, enables students to see themselves as active participants in learning, having power over both the mathematics and the discursive practices of the classroom.

Symbolic Control and Cultural Reproduction (Bernstein, 1990)

Bernstein (1990) discusses what he terms the pedagogic device, considering the distributive rules, recontextualising rules, and rules of evaluation. The pedagogic device is the object of struggle for control, played out within a particular arena. Activities within that arena create pedagogic modalities or generating codes, which have strong or weak values and classificatory or framing functions.

Classification refers to the degree of insulation between categories of discourse, agents, practices, and contexts, and provides recognition rules for both transmitters and acquirers. It is concerned primarily with power. Where school mathematics focuses on the development of skills and concepts such as fractions or algebra it is strongly classified, as it is maintains strong boundaries between mathematics and the outside world. Strong classification legitimises and reproduces power relations, whereas weak classification will challenge the boundaries upon which the division of labour is based. Framing refers to the location of control over the selection, organization, sequencing, pacing, and criteria of the communication. Strong framing locates control with the transmitter, whereas weak framing locates it more with the acquirer.

Bernstein distinguishes between voice, which is a function of classification, and message, which is a function of framing. Voice refers to the limits of a category's legitimate communicative potential; it is what can be said or realised if the identity is to be seen as legitimate within the arena. Message is what is actually said and its form of contextual realisation. It is dependent both on voice and its potential instrument of change. The principle of the social division of labour necessarily limits the realisation of its practices, yet these practices also contain the possibility of change in the social division of labour.

Bernstein calls pedagogic discourse, the process of moving a practice from its original site to a new site, a process of recontextualisation. Within this process values and ideologies always play a part, thus particular classroom practices produce behaviours that legitimate or disrupt what might be considered appropriate knowing. He distinguishes between instructional discourse, which transmits specialised competencies, and their relation to each other, and regulative discourse, which creates order, relation, and identity.

Lerman and Tsatsaroni (1998) use Bernstein's ideas to look at the systemic failure of certain categories of pupil to engage with the pedagogic processes through which the pedagogic text is produced, acquired, and assessed. They conclude that the forms of school knowledge constructed by certain values of classification and framing will produce different recognition and realisation rules to different categories of students, such as those from different social classes.

Dowling (1998), building on the ideas of Bernstein, describes a "social activity theory", which he uses to analyse mathematical texts written for pupils categorised as of high or low ability. He concludes that the texts written for pupils of high ability invite these pupils into the valued discourse of school mathematics as apprentices, whereas those written for pupils of low ability cast them as dependents.

Evans, Morgan, and Tsatsaroni (2006) describe the link between discursive positioning and emotion in school mathematics. Drawing on Bernstein's concepts of classification and framing in pedagogic discourse, they analyse how the discourse of the classroom makes alternative positions available to students. They describe these contrasting positions as evaluator and evaluated, helper and seeker of help, collaborator and solitary worker, leader and follower, insider and outsider.

Ideology in Discourses (Gee, 1991)

Gee (1991) maintains that ideology underlies all human interactions and their use of language. He states that there are two major motivations underlying all uses of language: status and solidarity. All uses of language situate the speaker and hearer within fields of status and solidarity, which are inherent social goods to humans. Thus all language is always and everywhere ideological, containing and transmitting beliefs, values, and attitudes. It is spoken and written out of a particular social identity.

Gee discusses the notion of a Discourse, a combination of saying, doing, believing, valuing, and being. He distinguishes this notion of a Discourse from discourse, which is a connected stretch of language. A Discourse, for Gee, is an identity kit, coming complete with rules and resources on how to talk and act in order to take on a social role that others recognise. Discourses are effectively clubs with tacit rules about who is a member and who is not, and about how members ought to behave.

Gee distinguishes between acquisition, which is a process of acquiring something subconsciously by exposure to models, trial and error, and social practice, and learning, a process that involves conscious knowledge gained through teaching or conscious reflection. He maintains that Discourses can only be mastered through acquisition, not learning. However learning can facilitate the development of meta-knowledge, but only when the process of acquisition has begun. Gee argues that classrooms that do not properly balance acquisition and learning simply privilege those students who have begun the acquisition process at home and marginalise those who have not.

More recently Gee (2003) has analysed the structure and learning principles inherent in video games, identifying principles related to the semiotic domain, to learning and identity, to situational meaning, to telling and doing, to cultural models, and to the social mind. Learning principles such as low cost failure, strong identities, amplification of input, just-in-time information, and belonging to an affinity group, are all inherent in the structure of video games and lead to acquisition of a Discourse.

Thornton (2006) uses Gee's notion of Discourse to discuss the potential mismatch between students' "first space" (Moje, Ciechanowski, Kramer, Ellis, Carrillo, & Collazo, 2004), the historically and culturally accumulated funds of knowledge and skills that enable people to function effectively as individuals and in society, and the "second space" of the mathematics classroom, the valued knowledge and academic norms of the formal school environment. He suggests that rather than seeing the mismatch between first and second spaces as a problem, both students' home and community funds of knowledge and their school funds of knowledge should be seen as a resource through which to empower them as effective learners in the school situation.

Context of this Research and Data Collection

The research reported in this paper arose from a study (Thornton, 2006) that originally set out to examine high school students' funds of knowledge (Moll, Amanti, Neff, & Gonzalez, 1992) in mathematics. In an endeavour to obtain data relating to these funds of knowledge, eight case study students were given digital voice recorders and asked to reflect on any issues they felt strongly about, particularly as these issues affected their learning and participation in school mathematics lessons. The digital recordings were saved onto a secure computer, and then transcribed. Unfortunately the students in the study seldom used the recorders, often forgot to bring them to school so that the recordings could be downloaded and, with one exception, provided only one or two sentence recordings that merely stated the topic of the mathematics lesson.

To obtain data relating to second space, mathematics lessons were observed in each of three Year 7 classes at the school. Field notes were taken and transcribed as soon as possible after the lesson. Interviews with individual students and groups were then conducted, using the lesson observations as a stimulus for discussion. These interviews were recorded and transcribed.

Due to the difficulties of obtaining voice recordings that might illuminate the students' funds of knowledge, the original research became a pilot study, informing further research. This paper uses a portion of the data from the original research. Rather than using qualitative data analysis software or coding systems to study the data, extracts from the field notes and interviews are used to illustrate the potential contribution of sociolinguists to understanding how students are positioned and position themselves in the classroom.

Results

The data below are arranged in five stanzas, representing different activities during two lessons I observed. Stanzas 1 to 3 are taken from a lesson related to converting from 12 to 24 hour time, and differing time zones. Stanzas 4 and 5 are taken from a lesson introducing percentages a few weeks later in the year. The focus of the observations was on the interaction between the teacher, Simon, and one student, Dean. Rather than providing a transcript of a whole lesson I have selected extracts typical of the interactions between Simon and Dean. I spoke with Dean after the lessons to obtain his impressions of various incidents.

The class is a Year 7 class in a middle- to low-socioeconomic area of a capital city. It is a mixed ability class containing eleven girls and fourteen boys. The teacher in the study, Simon, is a young teacher who has been at the school for 3 years. He is trained in science, physical education, and mathematics, and teaches the Year 7 class for both science and mathematics. The case study student discussed in the paper, Dean, comes from a single parent family. He lives with his mother, but visits his father interstate. He frequently has a "blue card", which he takes to lessons and asks teachers to sign to report on his behaviour and participation. During the time I spent at the school it was not unusual to see Dean being brought to the year level coordinator for disciplinary action. Dean scored relatively low marks in the school's tests of mental computation.

Stanza 1: Cajoling and resisting. Simon handed the students a worksheet, explaining that it was revision for a forthcoming test on time.

Simon: Dean, get on with your work so you won't be staying in.

At this stage Dean had not picked up his pen to start the questions on the worksheet, and did not do so for a further 10 minutes. Fifteen minutes into the lesson Dean walked over to another boy, Matthew, to look at the picture on the wall behind him.

Simon: Dean, if you don't do your work, you will do it at lunch time.

Stanza 2: Helping and receiving.

Simon: Dean, sit up in your chair and I'll give you a hand.

Simon sat next to Dean, who responded to a question about 24-hour time. He nodded his head as Simon counted 12, 1pm, 2pm, 3pm, etc. Simon sat next to Dean for about 5 minutes, writing answers on Dean's worksheet.

Simon: Can you do the rest?
Dean: Yeah.

Stanza 3: Questioning and responding.

Simon: Class, face the front. We're gonna go through the answers to page 60 first.

Simon asked selected students by name, in rapid succession, to read out their answers to the worksheet. The students responded with single word or number answers.

Simon: What's the difference between the two planes, Dean?
James: 45 minutes.
Dean: 45 minutes.

Simon did not respond to Dean's answer and moved on to the next question.

Stanza 4: Eliciting and contributing.

Simon: Where might you have seen percentages?
James: At the shops where they have a discount. For example, a 20% off sale.
Dean: Biscuits are 97% fat free.
Mark: Home loan ads. Interest is 8%.
Simon What do they mean?

Dean:	I don't care.
Simon:	Taking 20% off, what does that mean?
Dean:	100% is full price. 50% is half price.
Simon:	What might 20% mean?
Dean:	A little less than 50%.
James:	0.2.
Simon:	What sort of fraction might 20% be?
Mark:	¼.
James:	That's 25%.
Simon:	What do you think 97% fat free means?
Dean:	That's only 3% fat.

Stanza 5: Summarising and copying. Following the above exchange Simon wrote notes on the whiteboard (Figure 1), and told the students to copy the notes into their books.

> ***Percentages***
> *Percentages are used all the time in the world around you – discounts, home loans, bank interest, etc.*
> *The term "percent" literally means "per hundred" or simply "out of a hundred".*
> *This means that when you see a percentage e.g. 30%, you can write it as a fraction by taking the number in front of the % sign and putting it as a fraction over 100.*
> *E.g.* $30\% = \frac{30}{100}$
> $5\% = \frac{5}{100}$

Figure 1: Text on whiteboard.

Discussion

The discussion below is neither a systematic unpacking of the observations line by line, nor is it a rigorous analysis of the data using a particular sociolinguistic perspective. Rather it is a discussion of the data using insights from each of the perspectives discussed above.

Stanza 1: Cajoling and resisting. Stanza 1 is a struggle for control. The appellation "Dean" at the beginning of each of Simon's statements presents a strong message about power relations. Dean uses subjunctive clauses, "get … so" and "if … you will". In doing so he locates responsibility with Dean, rather than with himself, suggesting that staying in is a natural consequence of responding inappropriately to work. In neither sentence does Simon use the personal pronoun "I", thus he locates himself as impartially present to fulfil his role of ensuring that the students do their work. The respective roles of the speaker and listener are thus made very clear. It is Simon's role to set work and Dean's job to "do it".

The verbs "get" and "do" have strong modality. They are concerned with material, rather than mental processes. It is not the purpose of mathematics classrooms to think or understand, but rather to "do". These verbs, together with the use of appellation present a strong social message. These discursive practices construct students as doers, not as learners or contributors. Simon exercises power through domination, sitting behind an ideology of work/test and the hegemonic assumption that if students do their work they will perform better in their test.

Simon's language and action is characterised by strong classification and framing. The situation does not permit alternative agents, practices, or contexts, thus the power relations are maintained through a clear division of labour between the teacher and students. Selection, organisation, and pacing are controlled by the teacher, who requires a given amount of work to be covered in a given time. At the same time Dean resists this control, setting up a situation marked by struggle. Simon's attempts to reproduce culture through pedagogic devices such as threat are ineffective.

The stanza illustrates a clash of Discourses. Simon's primary Discourse is one of school as a place in which students do work and teachers make sure that it is done. His solidarity is with students in the classroom who ascribe to his beliefs and values about schooling. Dean takes on the role of resistor, maintaining solidarity with Matthew rather than Simon.

Stanza 2: Helping and receiving. In stanza 2 Simon constructs a tenuous alliance with Dean. His physical positioning next to Dean suggests solidarity with a student who struggles to do the work. Dean nods while Simon writes on his page, suggesting at least a partial acceptance of this alliance. Rather than using subjunctive clauses that suggest staying in at lunch time as a logical consequence of not doing work, Simon uses the phrase "sit down … and". Simon uses the personal pronoun "I", suggesting that helping is a personal choice and that he is choosing to express solidarity with Dean's situation rather than an adversarial position of forcing Dean to stay in. Simon's use of the phrase "give you a hand" reinforces this expressed choice and solidarity. In this way Simon's conversation foregrounds a relational process.

Yet at the same time Simon continues to use the word "do", promoting mathematics as a material rather than a mental process. By audibly counting 12, 1pm, 2pm, he attempts to introduce Dean to the Discourse of school mathematics, yet he stresses a procedure rather than a relation. Dean is expected to become part of this Discourse by learning rather than by acquisition. Simon asks Dean if he can "do" the rest, to which Dean rapidly responds "Yeah". Gee (1991) calls this an example of "mushfaking", making do with a partial understanding through learning, rather than entering legitimately into the Discourse.

In this exchange Simon uses a distributing strategy which constructs Dean as a dependent[1]. By spending 5 minutes with Dean and by writing on his page he limits Dean's potential response to one of agreement with what Simon writes. There is no potential for Dean to realise his own voice. Simon maintains strong classification in that he maintains clear boundaries between his role and Dean's.

Stanza 3: Questioning and responding. This stanza is characterised by extremely strong classification and framing. Simon permits only single word or number answers to questions, and only those that relate to the questions on the worksheet. By rapidly asking questions of selected students in the class he is maintaining control over pacing. Voice is strictly limited, and the messaging strategies construct students as receivers. Simon selects only the boys at the back of the room, including Dean, and one girl as candidates to answer

[1] I asked Dean in a subsequent conversation if he could do the worksheet, and how often he needed help. He said he could do them now, but that he "always needed help in maths".

questions. In this way he was using the question and response technique as a regulatory discourse, rather than as an instructional discourse.[2]

In this stanza Simon is maintaining a hegemony of mathematics as being about correct answers. The pattern of response constructs the teacher as the arbiter of that correctness. He maintains strong boundaries between the roles of the teacher and the students, using appellation to add force to the discourse. Simon foregrounds the material process of getting correct answers, rather than a mental process of understanding. His use of the verb "go through" implies that correctness is a destination to be reached rather than a process of understanding.

The strategy of singling out students by name introduces high cost failure to the exchange. Unlike video games in which players learn through failure and can recommence at the point of "death", the students cannot redeem themselves. Dean reduces the risk of failure by repeating an answer given by James.

Stanza 4: Eliciting and contributing. Stanza 4 is the only stanza in which classification is weakened. Simon asks students to draw on their primary Discourse to suggest everyday situations in which percentages are used. In this way he weakens the boundary marking what is permitted as legitimate content in the mathematics lesson, allowing the students a measure of power. However he maintains tight control of the pacing and sequencing of the discourse. At no stage do more than three students take turns to speak, and in almost every other case each student utterance is a direct response to a question asked by Simon.

The discourse is marked by limited coherence. The three consecutive student utterances are disconnected examples of the use of percentages. With the exception of Simon's follow-up questions on the meaning of 20% off and 97% fat free, there are no given/new structures in the discourse. Although Simon asks for the meaning of these phrases, students provide only simple answers.

Simon's use of the verb "mean" suggests that the discourse focuses on mental, rather than material processes. In the initial question he uses the verb "see", which implies awareness rather than action. Simon's conversation in this stanza is marked by significantly reduced modality. He uses the word "might" three times, and asks students what they "think". In this way he permits a level of uncertainty and allows an apparent element of choice in how they answer. However the students make a succession of confident statements, and seem unable or unwilling to embrace that uncertainty or to exercise that choice. Dean's statement "I don't care" expresses his unwillingness to engage in a mental process. It is significant that this is the only example of a student using the personal pronoun in an utterance, again suggesting that students in the class are focused on the material process of giving answers rather than on mental processes such as thinking, which are more likely to be expressed using the personal pronoun I.

In this stanza Simon foregrounds students' primary Discourse of the real world. By linking mathematics and the real world he attempts to increase intertextuality and thus to construct an alliance with students, recruiting them into the Discourse of school

---

[2] I asked Dean why he thought Simon only asked the boys at the back questions, and especially why he asked Dean. He said it was because Simon knew he understood the work because he had helped him, so it was a strategy to give him confidence. I also asked Kath why she was the only girl to whom Simon had asked a question. She said it was to make sure she was paying attention, because she was often disruptive in class. My observation of Kath suggested that, unlike Dean, she was able to play the game of school by "switching on and off" at will.

mathematics. In contrast to students' knowledge of the world, the Discourse of school mathematics remains learned rather than acquired[3]. The concept of 20% as a fraction is an isolated piece of knowledge, unconnected to James' initial observation about a 20% sale as a use of percentages.

Stanza 5: Summarising and copying. In stanza 5 Simon recontextualises everyday language into the formal symbols of mathematics. The possibility for change afforded by the weakened classification of stanza 4 is not realised. The messaging strategy of text privileges a particular form of knowledge and expression. The structure of the text valorises mathematical understanding as being on a higher plane than everyday language, propagating what Dowling (1998) terms the "myth of reference".

The text contains strong modality. Simon writes phrases such as "all the time" and "when(ever) you see" and adverbs such as "literally", implying that the text contains universal truth. He uses verbs such as "write", "take" and "put" as actions that "you" do. This use of personal active voice is in stark contrast to the impersonal "this" or the noun "percent" that precede the verb "means". Meaning is thus cast as inherent in mathematics, but the role of the learner is to do things.

The use of the word "simply" makes a strong statement about the relative positions of students in relation to teachers or to mathematics itself. It suggests that the word "per" requires recontextualisation to become "out of". Thus students are cast as being incapable of accessing the strongly classified discourse of school mathematics.

The structure of the text reinforces a hegemony that casts teachers as authors and students as copiers. The bold heading "percentages" draws attention to the presumed importance of the notes, giving them priority over student generated text. The everyday context is quickly replaced by mathematical symbols, reinforcing the priority of the academic over the everyday. The message is that reading notes will promote true understanding. Whether the teacher writing notes and students copying is an act of instructional or regulative discourse is open to question.

Conclusions

The above discussion draws on ideas from three sociolinguistic frameworks to look at some episodes in two mathematics lessons. The discussion is neither rigorous nor systematic, but paints a vivid picture of the struggle between the teacher, Simon, and one student, Dean, in the arena of a mathematics classroom.

Critical discourse analysis shows a precarious equilibrium, with Simon alternately wielding power over Dean and constructing an uneasy alliance with him. Throughout the interchanges Simon emphasises mathematics as being a material rather than a mental process, in which correct answers are more valuable than thinking. The discursive practice casts the teacher as instigator and the students as responders. Classroom practices such as question and answer and writing notes are unquestioningly accepted by both the teacher and students as being an integral part of school mathematics.

The classroom interchanges are generally marked by strong classification and framing. Simon permits only certain content and allows students a limited voice in the classroom. He maintains tight control over the sequencing, pacing and evaluation of the activities of

[3] I later asked Dean how he knew so much about percentages and how he knew that 97% fat free meant 3% fat. He said that he hadn't learned it, he had just "picked it up".

the classroom, using what appear to be instructional practices as regulatory devices. He valorises mathematics over the everyday, recontextualising intuitive knowledge into formal symbols, thus placing student knowledge as of lesser value than teacher knowledge.

Students are invited or cajoled to learn the valued Discourse of school mathematics, rather than being permitted opportunities to acquire it. Within the classroom both the teacher and students take on clearly defined roles as members of a particular group. Yet this is also the site of struggle as Dean resists and expresses solidarity with another student rather than with the teacher. This resistance is also apparent when Dean claims that he "doesn't care".

Throughout the exchanges Simon objectifies the students as little more than producers of work and objectifies mathematics as little more than something to be done. Students are positioned as dependent on the teacher, and their own knowledge is positioned as subservient to mathematics. In turn, Dean casts himself as a dependent in the classroom.

References

Bernstein, B. (1990). *The structure of pedagogic discourse. Volume IV Class codes and control*. New York: Routledge.

Dowling, P. (1998). *The sociology of mathematics education: Mathematical myths/Pedagogic texts*. London: Falmer Press.

Evans, J.,Morgan, & Tsatsaroni, A. (2006). Discursive positioning and emotion in school mathematics practices. *Educational Studies in Mathematics, 63*, 209-226

Fairclough, N. (1992). *Discourse and social change*. Cambridge: Polity Press.

Gee, J. (1991). *Social linguistics and literacies: Ideology in discourses*. London: Falmer Press.

Gee, J. (2003). *What video games have to teach us about learning and literacy*. New York; Houndmills.

Halliday, M. A. K. (1978). *Language as social semiotic: The social interpretation of language and meaning*. London: Edward Arnold.

Holmes, J. (1992). *An introduction to sociolinguistics*. London: Longman.

Lerman S., & Tsatsaroni, A. (1998) Why children fail and what mathematics education studies can do about it: The role of sociology. In P. Gates (Ed)., *Proceedings of First International Conference on Mathematics, Education and Society (MEAS1).* (pp. 26-33) Nottingham: Centre for the Study of Mathematics Education, University of Nottingham.

Moje, E., Ciechanowski, K., Kramer, K., Ellis, L., Carrillo, R., & Collazo, T. (2004). Working toward third space in content area literacy: An examination of everyday funds of knowledge and discourse. *Reading Research Quarterly, 39*(1), 38-70.

Moll, L. C., Amanti, C., Neff, D., & Gonzalez, N. (1992). Funds of knowledge for teaching: Using a qualitative approach to connect homes and classrooms. *Theory into Practice, 31*(2), 132.

Morgan, C. (2005). Words, definitions and concepts in discourses of mathematics, teaching and learning. *Language and Education, 19*(2), 103-117.

Thornton, S., & Reynolds, N. (2006). Analysing classroom interactions using critical discourse analysis. In J. Novotná, H. Moraová, M. Krátká, & N. Stehlíková (Eds.), *Proceedings of 30th annual conference of the International Group for the Psychology of Mathematics Education*. (Vol. 5, 273-280) Prague: PME.

Thornton, S. (2006). Moving into third space – high school students' funds of knowledge in the mathematics classroom. In P. Grootenboer, R. Zevenbergen, & M. Chinnappan (Eds.), *Identities, cultures and learning spaces, (*Proceedings of the 29th annual conference of the Mathematics Education Research Group of Australasia, Vol. 2, 504-511). Sydney: MERGA

Pedagogical Practices with Digital Technologies: Pre-service and Practicing Teachers

Colleen Vale
Victoria University
<Colleen.Vale@vu.edu.au>

In this paper the pedagogical practices of practising teachers and pre-service teachers when using digital technologies are described and compared. Data were collected by observation of presentations about using digital technology in mathematics by teachers and pre-service teachers and practising teachers were interviewed. Teachers generally used pedagogical approaches involving student-centred activity whereas pre-service teachers were more likely to use technology to teach concepts by demonstration and were not inclined to use the more student-centred approaches, though many used guided tasks. The study enabled some analysis and reflection upon the promoted action in the learning environments of pre-service teachers.

Numerous researchers have reported the limited use of digital technology in secondary and primary mathematics classrooms (Forgasz, 2006; Goos & Cretchly, 2004; Ruthven & Hennessy, 2002; Sinclair, 2006). The use of digital technology in senior secondary mathematics classrooms where the assessment of students permits or assumes the use of digital technology are notable exceptions around the world (Forgasz, Griffith, & Tan, 2006). Responding to studies that suggest that pre-service teachers will have limited opportunity to develop innovative pedagogical practices that include the use of digital technology, mathematics teacher educators have developed and evaluated technology enriched programs and practices in the education of pre-service mathematics teachers (e.g., Goos, 2005; Sinclair, 2006). The research reported in this paper has a similar genesis and purpose. The pedagogical practices with respect to the use of digital technology of three cohorts of pre-service secondary mathematics teachers are compared with those of a small sample of secondary school teachers who use technology relatively frequently in junior secondary mathematics classrooms.

Background

Ruthven and Hennessy (2002) reported that mathematics teachers in the United Kingdom used computers in mathematics to enhance the classroom ambience, assist tinkering, facilitate routine processes, and accentuate features of mathematics. In Queensland, teachers agreed that technology enabled students to perform calculations more quickly, receive dynamic feedback, study real life applications, and make links between numeric, graphic, and algebraic representations (Goos, 2004). These positive affective and cognitive affects of the use of technology in mathematics learning contribute to teachers' likelihood to use digital technology in their mathematics lessons (Forgasz, 2006; Norton & Cooper, 2001). Teachers' knowledge of software and pedagogical approaches and their beliefs about mathematics and the teaching and learning of mathematics also influence their use digital technology (Forgasz, 2006; Norton & Cooper, 2001). Goos and Cretchley (2004) argued that theoretical frameworks that focussed on identifying factors that encourage or hinder teachers' use of digital technology were too deterministic.

Proceedings of the 30th annual conference of the Mathematics Education Research Group of Australasia
J. Watson & K. Beswick (Eds), © MERGA Inc. 2007

Using Valsiner's zone theory, Goos (2005) theorised the development of professional identity of beginning mathematics teachers as the negotiation of the constraints and affordances of their learning and professional environment. She analysed the elements of the zones of proximal development (ZPD), promoted action (ZPA) and free movement (ZFM) of a pre-service teacher. Pre-service teachers' skills in using digital technology, pedagogical knowledge using digital technology for teaching and pedagogical beliefs constituted their ZPD. They were influenced by two zones of promoted action, the first that of the university lecturer and program and the second created by the mentor or supervising teacher in the practical component of the training program. The university program was described as "technologically rich" as pre-service teachers had access to graphics calculators that could be readily used in university classroom settings, activities specifically devoted to develop technical and pedagogical skills with technology, and one assessment task that required students to work in pairs and present a technology-based activity for a secondary program. Goos also promoted "mathematical thinking, real world applications and collaborative inquiry" (p. 42). The ZFM included the resources available in the school, curriculum program requirements, and the students of the pre-service teacher's classroom. The way in which this pre-service teacher negotiated this environment illustrated the dynamic nature of learning to teach mathematics.

Sinclair (2006) on the other hand used an ecological framework of complexity theory to reflect upon her practices as a mathematics teacher educator. In this framework the systemic conditions for learning include internal diversity, internal redundancy, distributed control, organised randomness, and neighbour interactions. In her pre-service course rather than setting a specific assessment task on the use of technology, she left the range of tasks more open allowing pre-service teachers to take more responsibility for their learning. As well as specialised workshops she worked on illustrating the diversity of digital use and developing shared understandings for meaningful communication by imbedding technology in every session, thereby modelling the use of technology as "an extension of self" (Goos, Galbraith, Renshaw, & Geiger, 2003). She observed that her pre-service teachers initiated the use of technology in the various assessment and teaching activities of their program.

Pre-service teachers' pedagogical practices with technology was not the particular focus of either of these studies, though others have observed that teachers use technology in ways that are consistent with their pedagogical practices. For example, Ng and Teong (2003) observed that mathematics teachers in Singapore most frequently used digital technology for demonstration of mathematics. Teachers in Singapore who used dynamic geometry most often used the software to prepare worksheets and test papers and to use dragging and animation of pre-designed templates or sketches to show geometrical properties and aid students' visualisation.

Previous research shows digital technology is most effective when students are actively engaged in the constructing of meaning through and with the technology (Goos & Cretchley, 2004). Effective pedagogical approaches therefore involve students using technology as a "partner", or as an "extension of self", where students utilise the affordances of the technology to develop understanding of mathematical concepts and solve problems, rather than using technology as a "servant" to perform mathematical operations uncritically (Goos et al., 2003).

Ng and Teong (2003) developed the framework shown in Table 1 as part of a professional development program for teachers on the use of Geometers' Sketchpad. The level of instruction in this framework varies from teacher demonstration, the most structured and teacher-centred activity, to student-centred open-ended tasks that Ng and Teong call "black box tasks". They identified the alternate student learning objectives related to geometry as teaching (or learning) a concept, consolidating the concept, developing an informal proof of a geometric property or theorem, and problem solving. The consolidation of concepts could be interpreted as providing students with a range of other experiences of the particular concepts previously introduced or developed. Alternately it could be interpreted as practice exercises or the application of geometric skills and concepts to routine geometric problems.

Table 1
A Framework for Teaching Geometry with GSP (Ng & Teong, 2003)

Level No.	Purpose of instruction/Level	Teach concept	Consolidate concept	Informal proof	Problem solving
1	Teacher demonstration				
2	Templates/pre-made sketches				
3	Guided exploration/construction tasks				
4	Black box tasks				

In this paper the pedagogical approaches developed by pre-service teachers and the learning purposes of using digital technology are investigated. These approaches are compared with teachers of mathematics. The aim of this study is to consider the way in which the promoted action of a pre-service teacher education program and that of teachers in schools is reflected in the pedagogical practices of pre-service teachers.

Methods

Eight teachers, ranging in experience from 1 year to 25 years, who teach in socially disadvantaged schools in the western region of Melbourne and who reported that they used digital technology regularly participated in the first study that is reported in this paper. These teachers were selected following telephone interviews of mathematics teachers to identify teachers who used technology in junior secondary mathematics more than twice per term. Four of these teachers had previously supervised or mentored a pre-service teacher. The practicing teachers were interviewed using as semi-structured interview protocol and their responses tape-recorded. During the interview the teachers were asked to describe a successful mathematics lesson that integrated the use of digital technology and to explain why they thought that it was successful. These teachers also participated in a whole day workshop on the use of digital technology and presented examples of mathematics lessons that they found to be successful to their peers as part of the workshop program. Field notes were taken of these presentations and digital copies of some of these activities were gathered later.

In the second study, pre-service teachers who were enrolled in the mathematics pedagogy subject of a secondary teacher education program (Graduate Diploma of

Education (Secondary)) in the years from 2004–2006 were participants. The number of pre-service teachers enrolled varied: 30 in 2004, 25 in 2005, and 10 in 2006. As part of the course pre-service teachers are required to work in schools under the supervision of an experienced teacher of mathematics. The pre-service teachers worked in one school for 1-day per week throughout the year and for two 4-week periods during the year.

During the course pre-service teachers participated in workshops on the use of digital technology. The purposes of these workshops were two-fold: firstly to provide pre-service teachers with the opportunity to develop some knowledge of the software and the technical skills to operate the software or hardware, and secondly to model innovative practices in the implementation of digital technology in the classroom. In each year students participated in a three-hour workshop on Geometer's Sketchpad (dynamic geometry software) and another three-hour workshop on graphics calculators where the focus was on teaching and learning functions. In 2005 and 2006 students also participated in a three-hour workshop on a CAS (computer algebra system) calculator. Students in 2006 also used graphics calculators during a further session on senior secondary chance and data curriculum. Many students in each year of the course were mature age students or educated overseas and had not had an opportunity through their own secondary education to develop technical skills in the use of graphics calculators and other mathematics specific software. Furthermore, it cannot be assumed that all pre-service teachers in the course will have used computer algebra or statistical software in their undergraduate studies of mathematics (Lavicza, 2006).

In each year students were required to complete a technology assignment to fulfil the assessment requirements for the mathematics curriculum and pedagogy subject. For this assignment pre-service teachers conducted some research into the use of digital technology in the teaching of mathematics and then presented a teaching and learning activity to the rest of the group. The specific requirements varied slightly from year to year. In 2004 and 2005 students worked in pairs on this task, though some students chose to complete this assignment individually. In 2006 the students worked individually on the task and were required to evaluate the use of the resource based on their experience of observing or using it during the practical program of the course (partnership placement).

Descriptive accounts (field notes) of the pre-service teachers' presentations were kept by the researcher, who was also the lecturer for this subject, in each of these years. Digital copies of the materials that students' presented were collected for most students, especially in 2005 and 2006.

Data were analysed using an adapted version of the Ng and Teong's (2003) Framework to apply more generally to other areas of the mathematics curriculum and the range of digital resources that teachers of mathematics may choose to use. The purposes and objectives of teaching and learning were redefined and expanded to include practise and the application of mathematics to real world situation. The levels of teacher direction in the design of instructional activities were also re-interpreted. The category "templates and pre-made sketches" was modified to also include interactive learning objects. Many of the resources available on the Internet are in the form of interactive learning objects. "Black-box" tasks were re-labelled as "open-ended tasks". An additional category was added: "research project" for semi-structured tasks or assignments that involved student inquiry.

Findings

The digital learning activities that teachers reported in the interview or presented to peers during the workshop are categorised in Table 2. A letter identifies each teacher in the study. These data show that each teacher used either one or two pedagogical approaches. They used approaches of varying levels of student-centredness: templates or learning objects, guided tasks, or research projects. Teachers also used digital technology for a range of learning purposes, though informal proof was the least likely to be reported. In a previous paper I presented a more detailed analysis of one of these teacher's pedagogical practice (Vale, 2006). Here I provide a brief description of some of the activities reported by teachers.

Table 2
Teachers' use of Digital Technology

Purpose of instruction/Level	Teach/learn concept	Consolidate concept	Informal proof	Problem solving	Application
Teacher demonstration					
Templates/ interactive learning objects	G, H	A, E, F, H		A, C, F, G	
Guided exploration/ tasks	A, B, F, H		H		A, E
Research project Open-ended task	C, D, G	C, G			C, G

Teachers prepared templates for students to record results of investigations and to learn concepts such as π, and they used games and online interactive learning objects to practice number skills, consolidate understanding of the relationship between algebraic and graphic representations of linear equation, and develop skills such as estimation. They claimed that these activities enabled students to work at their own pace. Templates and interactive learning objects were also used for problem solving.

Guided tasks were also popular and used by teachers for exploring geometric properties and measurement concepts and learning about box-plots and statistics. At least two of the teachers who used this teaching approach were adamant these activities were most successful when students were provided very clear "step-by-step" instructions. These instructions were concerned with learning to use the tool but also helped students to focus on what to observe and scaffolded mathematical interpretation of dynamic visual media. Although two of these teachers preferred to provide students with guided tasks for the application of mathematically skills, such as presentation of data, others preferred to use integrated curriculum research projects where students worked collaboratively to use the Internet to gather data or information and to use digital tools to analyse or present their work. Teachers using this approach observed peer assistance, tutoring, and mentoring.

Over the 3 years in which pre-service teachers completed the technology assignments 35 presentations were analysed. In each year at least one pair of pre-service teachers chose to base their presentation on the use of technology to design assessment tasks and these presentations were not included in the data presented in Table 3. Also a few pre-service teachers did not complete the task or records of their presentation were not retained. The 35 presentations analysed and reported in Table 3 were the work of 53 pre-service teachers. Table 3 shows the percentage and number of pre-service teacher technology presentations in each category. Many of the activities presented by pre-service teachers were ones that they had used or observed during their practical experience in schools, though data on this was not always recorded for pre-service teachers in 2004 and 2005.

Table 3
Pre-service Teachers' use of Digital Technology

Purpose of instruction/Level	Teach/ learn concept	Consolid ate concept	Informal proof	Problem solving	Appl'n	Total
Teacher demonstration	14 % (5)	6% (2)				20% (7)
Templates/ ILOs	6% (2)	11% (4)		3% (1)	3% (1)	23% (8)
Guided exploration/ tasks	31% (11)	6% (2)		6% (2)	9% (3)	51% (18)
Research project					3% (1)	3% (1)
Open-ended task					3% (1)	3% (1)
Total	51% (18)	23% (8)		9% (3)	17% (6)	100% (35)

The data in Table 3 show that pre-service teachers were most likely to present a technology-based learning activity that was a guided exploration or task (51%) and they were also most likely to use digital technology to teach or learn a concept (51%). Moreover, the pre-service teachers were most likely to use a guided task to teach or learn a concept (31%). Typically these presentations involved the use of a function-grapher (graphics calculator, Excel, Geometers' Sketchpad, or Graphmatica) to investigate the effect of parameters in symbolic expressions on the graphs of functions (such as linear, quadratic, or exponential functions). The pre-service teachers were thus focussed on the interaction of symbolic, graphic, and numeric representations of concepts and, in particular, the use of visualisation of graphic images or numeric patterns to learn about a mathematical concept. A closer analysis of materials presented however found that one-third of these guided activities did not include questions that required students to compare and contrast graphic or numeric data or to require students to conduct further exploration through the use of "what if" type questions. The guided tasks were in essence designed for students to learn to use the tool to generate graphs or tables of data.

Pre-service teachers also showed a propensity to use templates or interactive learning objects (23%) or incorporate the use of technology through teacher demonstration (20%). Pre-service teachers were also likely to use technology to consolidate concepts or practice skills (23%) and for application to real world situations (17%). Activities for consolidating concepts and practising skills included online quiz and game sites, game software, and teacher designed games using spreadsheets. One presentation involved the uncritical and routine use of spreadsheet templates for presenting data.

Most of the teacher-centred demonstration methods occurred in 2006. It is not clear why this was the case. These demonstrations typically concerned instruction on the technical skills needed to use software for a particular task. For example, one of these demonstrations involved teaching students to use Excel for the calculation of statistics (mean, mode, and median). In other presentations pre-service teachers used software tools or authentic data to demonstrate a concept. For example one pre-service teacher used Excel to demonstrate an application in financial mathematics and another pre-service teacher used TinkerPlots as part of a PowerPoint presentation to demonstrate correlation. Perhaps because pre-service teachers worked individually they did not benefit from collaboration with peers that may have involved them in more pedagogical discussions of how to best use the material and resources available. In two of these cases, the presenters drew upon data and methods of using technology from their previous professional occupations.

The use of digital technology for problem solving or applications was more likely to be the focus of the presentation for pre-service teachers in 2005 and 2006. Pre-service teachers typically made use of commercially available learning objects and template for problem solving. Student surveys and other forms of data collection were typical application tasks. The only open-ended task involved the use of drawing software to explore and create tessellated designs.

Discussion and Conclusion

Pre-service teachers adopted the practice of guided tasks, popular with teachers to demonstrate their understanding of the role of technology in mathematic learning. However, the execution of these tasks indicated that rather than indicating a propensity to involve the use of technology as a “partner” for students in mathematics classrooms, the practice of pre-service teachers suggested they were more likely to use digital technology as a “servant” in mathematics classrooms. As the data gathered from practising teachers relied on self-report data that were not always accompanied by copies of material it is possible the pre-service teachers are observing guided instruction on the development of technical skills. This finding indicates that I need to work with pre-service teachers on planning structured inquiry with technology and in particular on the framing of questions that will scaffold and focus students’ learning. Some very good resources exist and these need to be shared with mentors and supervising teachers in partnership schools.

Teachers in the study were more likely than pre-service teachers to use research projects as a way of integrating technology in mathematics learning. These tasks provided for internal diversity, meaningful communication among students, collaborative inquiry and shared responsibility for learning (Sinclair, 2006). It would seem however that this form of learning activity is not commonly modelled in the range of schools in which pre-service teachers undertake their practical training. Effective models of this kind of task need to be incorporated into the pre-service program.

Pre-service teachers used demonstration whereas the teachers in the study did not. Indisputably this is a common and successful element of teachers’ pedagogical practice and pre-service teachers need to develop the communication and instructional skills required to use technology successfully for demonstration in their teaching too. However this finding suggests that the teachers in the study are more amenable to innovative practices than many of the pre-service teachers, and probably many of their peers who use digital technology less regularly, or rarely, in their teaching of mathematics. It suggests some differences in pedagogical beliefs or that pre-service teachers have not had sufficient opportunity to develop diverse pedagogical practices when using technology.

This project has provided me with information about the nature of "promoted action" with respect to the use of technology in secondary mathematics classrooms and to analyse and reflect upon pre-service teachers' developing pedagogical practice with technology. Although many pre-service teachers were afforded the opportunity to observe or practise the use of technology in mathematic teaching, others would appear to have less experience in classroom settings. Some mentor teachers encourage pre-service teachers to trial activities using digital tools and materials whereas others are constrained by the lack of encouragement or resources or by the curriculum requirements set by the supervising teacher. I need to model more regularly the various ways in which technology may be imbedded in mathematics teaching and to work with school colleagues to provide pre-service teachers with further opportunities for collaborative inquiry in university or school settings.

References

Forgasz, H.J., (2006). Factors that encourage or inhibit computer use for secondary mathematics teaching. *Journal of Computers in Mathematics and Science Teaching, 25*(1), 77-93.

Forgasz, H.J., Griffith, S., & Tan, H. (2006). Gender, equity, teachers, students and technology use in secondary mathematics classrooms. In C. Hoyles, J-B. Lagrange, L.H. Son & N. Sinclair (Eds.) *Proceedings of the Seventeenth ICMI Study Conference "Technology Revisited"*, (CD ROM) December 3-8, 2006. Hanoi: Hanoi University of Technology.

Goos, M. (2004). Teachers' use of technology in secondary school mathematics classrooms. Paper presented at annual conference of AARE, Melbourne, 28th November – 2nd December, 2004.

Goos, M. (2005). A socio-cultural analysis of the development of pre-service and beginning teachers' pedagogical identities as users of technology. *Journal of Mathematics Teacher Education, 8,* 35-59.

Goos, M., & Cretchley, P. (2004). Teaching and learning mathematics with computers, the internet, and multimedia. In B. Perry, G. Anthony, & C. Diezmann (Eds.), *Research in mathematics education in Australasia 2000-2003* (pp. 151-174). Flaxton, Qld: Post Pressed.

Goos, M., Galbraith, P., Renshaw, P., & Geiger, V. (2003) Perspectives on technology mediated learning in secondary school mathematics classrooms. *Journal of Mathematical Behavior*, *22*, 73-89.

Lavicza, Z. (2006). The examination of computer algebra systems (CAS) integration into university-level mathematics teaching. In C. Hoyles, J-B. Lagrange, L.H. Son, & N. Sinclair (Eds.) *Proceedings of the Seventeenth ICMI Study Conference "Technology Revisited"*, (CD ROM) December 3-8 2006. Hanoi: Hanoi University of Technology.

Norton, S., & Cooper, T. (2001). Factors influencing computer use in mathematics teaching in secondary schools. In J. Bobis, B. Perry & M. Mitchelmore (Eds.), *Numeracy and Beyond* (Proceedings of the 24th Annual Conference of the Mathematics Education Research Group of Australasia, pp. 386-393). Sydney: MERGA.

Ng, K.E.D., & Teong, S.K. (2003). Dynamic geometry in primary schools in Singapore. *MicroMath, 19*(3), 5-9.

Ruthven, K., & Hennessy, S. (2002). A practitioner model of the use of computer-based tools and resources to support mathematics teaching and learning, *Educational Studies in Mathematics, 49*(1), 47-88.

Sinclair, M. (2006). Teaching with technology: Complexity theory as a lens for reflecting on practice. In C. Hoyles, J-B. Lagrange, L.H. Son & N. Sinclair (Eds.) *Proceedings of the Seventeenth ICMI Study Conference "Technology Revisited"*, (CD ROM) December 3-8 2006. Hanoi: Hanoi University of Technology.

Vale, C. (2006). Equity and technology: A case study. In P. Grootenboer, R. Zevenbergen, & M. Chinnappan (Eds.), *Identities, cultures and learning Spaces* (Proceedings of the 29th annual conference of the Mathematics Education Research Group of Australasia, Canberra, pp. 512-518). Sydney: MERGA.

Procedural Complexity and Mathematical Solving Processes in Year 8 Mathematics Textbook Questions

Jill Vincent
University of Melbourne
<jlvinc@unimelb.edu.au>

Kaye Stacey
University of Melbourne
<k.stacey@unimelb.edu.au>

This study examines the procedural complexity and mathematical solving processes required by problems on two topics in seven Year 8 textbooks from four Australian states. The study used definitions from the 1999 TIMSS Video Study. Although variation existed between textbooks, the majority of problems were of low procedural complexity, requiring only the practising of procedures. The general picture was consistent with that painted by the Video Study, with a somewhat stronger emphasis on procedural work.

The 1999 Third International Mathematics and Science Study (TIMSS) Video Study described teaching practices in eighth-grade mathematics and science in the United States and in six countries where students performed well relative to the United States on the TIMSS 1995 assessments. Countries participating in the mathematics component of the TIMSS 1999 Video Study were: Australia, the Czech Republic, Hong Kong SAR, Japan, the Netherlands, Switzerland, and the United States.

Many common features were apparent across the seven countries, for example, teachers in all seven countries talked more than the students, at a ratio of at least 8:1; mathematics teachers in all countries organised the average lesson to include some public whole-class work and some private individual or small-group work; and on average at least 80% of lesson time was spent in solving mathematical problems. Almost 15 000 mathematics problems were analysed, with 82% of the problems focusing on number, geometry, and algebra.

There were some features of the 87 randomly selected Australian mathematics lessons that many mathematics educators would find disturbing. Three quarters of the problems presented in the Australian lessons were repetitions of the preceding problems, the highest proportion of the seven countries. The Australian lessons also included the highest proportion of problems of low procedural complexity (77%) and virtually no Australian lessons included verification of results by logical reasoning (Hiebert et al., 2003). This cluster of features of Australian lessons – *low complexity* of problems, which are undertaken with *excessive repetition*, and *absence of mathematical reasoning* in classroom discourse – together constitute what we have termed the “shallow teaching syndrome” (Stacey, 2003).

The Study

This paper presents findings from an early stage of an investigation into the shallow teaching syndrome – whether it is a real pattern or just an artifact of the definitions and procedures of the Video Study, (if real) whether it is indeed undesirable, and (if real) whether it is most evident in “textbook teaching”. With this motivation, we set our first goal to compare “textbook teaching” with the findings of the Video Study, asking if the general picture revealed by the Video Study would arise if all lessons followed textbooks exactly. This study is also intended to provide insight into the way in which the classifications of problems used for the Video Study operate in practice.

Three Classifications of Problems

Procedural complexity was defined in the Video Study in terms of the number of steps required to solve a problem by a standard method and whether the problem comprised sub-problems. (Details are given in Methodology.) Table 1 shows the average percentage of problems at each level of procedural complexity for Australia, Japan and the Netherlands. Other countries had from 63% to 68% of problems of low complexity.

Table 1
Average Percentage of Problems per Eighth-grade Mathematics Lesson at each Level of Procedural Complexity for Australia, Japan and the Netherlands

	Low	Moderate	High
Australia	77	16	8
Japan	17	45	39
The Netherlands	69	25	6

Note: The percentages do not all sum to 100 because of rounding.

Problems solved in the lessons were also classified according to the mathematical solving processes involved. Three categories were used: *using procedures*, *stating mathematical concepts,* and *making connections* (see Methodology for definitions). The majority of lessons in all countries except Japan were found to have a high proportion of problems per lesson that focused on *using procedures*, with smaller percentages of problems focusing on *stating concepts* or *making connections*. Table 2 gives the average percentage of problems per lesson in these three categories for Australia, Hong Kong SAR and The Netherlands (see Hiebert et al., 2003, p. 99). In addition to Japanese lessons having the highest percentage of problem statements focusing on *making connections* (54%), 39% of lessons contained a proof. Contrasting sharply with the Japanese lessons, virtually none of the lessons from Australia, the Netherlands, and the United States contained instances requiring verification or demonstration by reasoning that a result must be true (a sub-category of *making connections* problems).

Table 2
Average Percentage of Problem Statements per Eighth-grade Mathematics Lesson Focusing on Different Types of Mathematical Solving Processes for Three Countries

	Using procedures	Stating concepts	Making connections
Australia	61	24	15
Hong Kong SAR	84	4	13
The Netherlands	57	18	24

Note: The percentages do not all sum to 100 because of rounding

Where problems were solved publicly, the Video Study compared the implied solving process and the actual solving process. Problems that were intended to engage students in *stating concepts* or *making connections* frequently only exhibited *using procedures* when discussed publicly. In the Australian lessons, for only 8% of problems categorised as *making connections* did the public explanation explicitly draw attention to these connections. Problems were also classified as either exercises or applications (see Methodology for definitions). In the Australian lessons, 45% of problems were applications, compared with 74% for Japan and 34% for the United States.

Characteristics of Textbooks

Textbooks or worksheets were used in at least 90% of the mathematics lessons in all countries (Hiebert et al., 2003). The analysis of textbook questions therefore provides a useful indication of the procedural complexity to which students are likely to be exposed and the extent to which the majority of students are being challenged beyond the application of procedures. In a study of the use of mathematics textbooks in English, French, and German classrooms, Pepin and Haggarty (2001) analysed how textbooks vary, how they were used by teachers in the classroom and how this influenced the culture of the mathematics classroom. They note that in some textbooks, exercises predominated, with few connections made between the concepts practised. In others, student exploration, questioning, and autonomy were encouraged, and the posing of problems motivated the acquisition of new knowledge. Pepin and Haggarty claim that in the English textbooks "questions were mostly straightforward applications of the worked examples provided. They were the routine-type where a 'taught' method was applied in relatively impoverished and non-real contexts and they only rarely required deeper levels of thinking from pupils" (p. 172). By contrast, they found that the French textbooks contained "graduated exercises with many demanding questions requiring insights and understanding from pupils" (p. 173). In Germany, textbooks were differentiated for the perceived achievement level of students, with a relatively high level of complexity and coherence, particularly with respect to mathematical logic and structure.

Brändström (2005) analysed three different Swedish seventh-grade mathematics textbooks, focusing on how the textbooks provided opportunities for all students to learn. Each book catered for different ability levels by means of two or three alternative strands within each chapter. Brändström's analysis of the textbook tasks included a comparison of the number of operations, the cognitive processes involved (based on Bloom's taxonomy), and the level of cognitive demand on a four-point scale. Brändström found that the lower strands focused predominantly on the lower two levels of cognitive demand (memorisation and applying a procedure). Even in the higher strands, more than 85% of tasks were at the lower two levels. Tasks at the top level were identified only in the strands for more able students in two of the three textbooks, approximately 5% and 10% respectively. It appears, then, that even when textbooks are written specifically for students of different ability levels, only a small proportion of textbook questions challenge students beyond the application of procedures. In view of this literature and the fact that Australian mathematics textbooks are generally written for mixed ability classrooms, the levels of procedural complexity in questions, and the different types of mathematical processes included are important issues.

This paper focuses on the analysis of selected problem sets in a sample of Australian mathematics textbooks, addressing in particular the following research questions:

1. to what extent are the Video Study criteria for procedural complexity, types of mathematical processes, and the exercise/application distinction useful in analysing problem sets and associated tasks in Australian mathematics textbooks?
2. can differences between textbooks be identified using the Video Study criteria?
3. does the analysis of textbook problems align with the findings of the Video Study?

Methodology

In order to gain insight into the methods and findings of the Video Study, we needed to select problems that were typical of Year 8 work, and then analyse them using the Video Study criteria. In this study, we used three of the Video Study variables: procedural complexity, mathematical processes, and the exercise/application classification. Although the Video Study also classified aspects of lesson delivery, the selected variables were applicable to problem statements, and so could be used on textbook problems.

Selecting the Textbooks and Problem Sets

For this preliminary study, we investigated two topics from the 2006 best-selling Year 8 textbooks (textbooks A, B, C, and D) in four Australian states. Each was a clear market-leader. It should be kept in mind that for textbooks A and B, Year 8 is the first year of secondary school, whereas for textbooks C and D, Year 8 is the second year of secondary school. The best-selling textbooks were selected simply because this gave us the best "one book" picture of the problems that might be presented to Australian students. The same topics were also analysed in an additional sample of three different textbooks from one state for which Year 8 is the second year of secondary school (textbooks E, F and G). Because the results were limited to just two topics, and it is unclear whether these are representative, the textbooks are not named in this paper.

All problem sets from two mathematical topics were chosen: *addition and subtraction of fractions* and *solving linear equations*. For solving linear equations, we selected material related to "doing the same to both sides" (not guess and check or graphical solving). These topics were common to all states at this level and were also representative of two of the three most prevalent topic areas in the Video Study – number, geometry and algebra. The problem sets were drawn from the part of the textbook dedicated to that topic. We did not search the rest of the books to find problems that used knowledge from these topics.

Definitions from the Video Study

In each of the selected problem sets, the problems were classified using the Video Study descriptors for procedural complexity, the mathematical processes required in the solution, and as either exercises or applications. Here we describe these classifications.

In the Video Study lesson analysis, problems were defined in the following way: "Problems contain an explicit or implicit Problem Statement that includes an unknown aspect, something that must be determined by applying a mathematical operation, and they contain a Target Result". The Target Result is the answer to the Problem Statement and "may be a number, an algebraic expression, a geometric object, a strategy for solving problems, and even the creation of a new problem" (TIMSS 1999 Video Study Math Coding Manual, pp. 20, 21). A mathematical operation or decision that occurs between the problem statement and the target result is referred to as a *step*. Problems involve one or more steps to reach the target result. Examples of problems provided in the Coding Manual are:

1. Which of the following numbers is bigger?
2. Solve the following equations: (a) $3x + 1 = 8$ (b) $x - 7 = 42$ (2 problems)
3. Find the area of a parallelogram with a base of 8 cm and a height of 4 cm.
4. Make a table of values and graph the equation $3x = 2y - 1$ (problem with sub-problem)

Problems were categorised as either *exercises*, that is, practising a procedure on a set of similar problems, or *applications*, where students applied procedures they had learned in one context to solve problems about a different context. An example of an application problem based on the practised procedure of solving equations is: "The sum of three consecutive integers is 240. Find the integers." (Hiebert et al., 2003, p. 90). Under this definition, applications do not necessarily have real-world references. Problems were classified as being of low, moderate, or high procedural complexity according to the number of steps and sub-problems. The criteria and an example for each level of procedural complexity are shown in Table 3.

Table 3
TIMSS Classification for Problem Complexity [from Hiebert et al., 2003, p. 71]

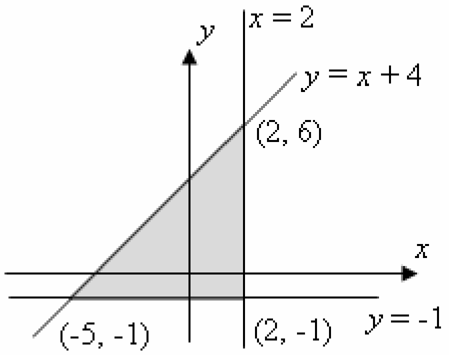
Complexity	Description
Low	Solving the problem, using conventional procedures, requires four or fewer decisions by the students (decisions to be considered small steps). The problem contains no sub-problems or tasks embedded in larger problems that themselves could be coded as problems. Example: Solve the equation: $2x+7=2$
Moderate	Solving the problem, using conventional procedures, requires more than four decisions by the students and can contain one sub-problem. Example: Solve the set of equations for x and y: $2y=3x$; $2x+y=5$
High	Solving the problem, using conventional procedures, requires more than four decisions by the students and contains two or more sub-problems. Example: Graph the following linear inequalities and find the area of intersection: $y \le x+4$; $x \le 2$; $y \ge -1$

As a check that we were applying the criteria in the intended way, we classified examples including those in Table 3 according to the Video Study criteria. As shown in Tables 4a and 4b, our classifications of complexity coincided with that of the Video Study, although we do not know if steps we identified coincided precisely with those identified by the Video Study, as their steps were not made explicit in the examples.

Table 4a
Examples of Applying the Video Study Criteria for Procedural Complexity

Example 1: Solve the set of equations for x and y: $2y=3x$; $2x+y=5$	
$2y=3x$ (1)	Step 1: Decide on an appropriate strategy
$2x+y=5$ (2)	Step 2: Multiply equation (1) by 2 to get equation (3)
$4x+2y=10$ (3)	
$4x+3x=10$ $\therefore 7x=10$	Step 3: Substitute equation (1) into equation (3)
$\frac{7x}{7}=\frac{10}{7}$, $x=\frac{10}{7}$	Step 4: Divide both sides by 7
$2y=3x$ $2y=\frac{3\times 10}{7}=\frac{30}{7}$	Step 5: Substitute $x=\frac{10}{7}$ in equation (1)
$y=\frac{15}{7}$	Step 6: Divide both sides by 2
More than four steps, but no sub-problem, so moderate procedural complexity.	

Table 4b
Examples of Applying the Video Study Criteria for Procedural Complexity

Example 2: Graph the following linear inequalities and find the area of intersection: $y \le x+4$; $x \le 2$; $y \ge -1$	
x-intercept (-4, 0), *y*-intercept (0, 4)	Step 1: Find intercepts for $y \le x+4$
	Steps 2- 4: Sketch graphs Sub-problem: Steps 5, 6: Find coordinates of intersections Step 7: Decide on required region Sub-problem: Steps 8, 9: Find base and height of right-angled triangle Step 10: Calculate area of triangle
More than four steps, and two sub-problems, so high procedural complexity.	

Problem statements were also categorised according to the implied mathematical processes: *using procedures*, *stating concepts,* or *making connections*. The criteria and an example for each category are shown in Table 5.

Table 5
Defining the Types of Mathematical Processes Implied by Problem Statements [from Hiebert et al., 2003, p. 98]

Mathematical process	Description
Using procedures	Problem statements that suggested the problem was typically solved by applying a procedure or set of procedures. These include arithmetic with whole numbers, fractions, decimals, manipulating algebraic symbols to simplify expressions and solve equations, finding areas and perimeters of simple plane figures, and so on. Example: Solve for x in the equation $2x + 5 = 6 - x$.
Stating concepts	Problem statements that called for a mathematical convention or an example of a mathematical concept. Examples: Plot the point (3, 2) on a coordinate plane. Draw an isosceles triangle.
Making connections	Problem statements that implied the problem would focus on constructing relationships among mathematical ideas, facts or procedures. Often, the problem statement suggested that students would engage in special forms of mathematical reasoning such as conjecturing, generalizing, and verifying. Examples: Graph the equations $y = 2x + 3$, $2y = x - 2$ and $y = -4x$, and examine the role played by the numbers in determining the position and slope of the associated lines.

Results

Variations occurred in the way the fractions problems were organised, with some textbooks including addition and subtraction together in a single problem set, and others presenting addition and subtraction separately. In some books, simple fractions were placed in a separate problem set from mixed numbers. In one book, students were directed to use calculators in the problems involving mixed number addition and subtraction. In states where Year 8 was the first year of secondary schooling, the Year 8 textbooks (A and B) included an extensive treatment of fractions, compared with the states where Year 8 was the second year of secondary schooling. Although textbook E provided substantial revision, textbooks C, D, and F included only a small number of problems and G had no fractions section. Textbook D focused only on very simple problems with no mixed numbers.

Table 6 shows the number of problems, procedural complexity, and type of solving process for "Addition and subtraction of fractions" problems in the sample of seven textbooks. The majority of problems in all books were of low complexity. The data in Table 6 show a tendency for the textbooks that regarded this as a revision topic to have relatively more problems of moderate complexity, although textbook D is an exception. Almost all of the problems required only *using procedures*. Although the relatively high percentage of *making connections* problems in textbook C represents only four problems from a small revision set, it does indicate a different approach to this revision than in the other books.

A similar pattern of procedural complexity was found in the problems relating to solving linear equations (see Table 7). One might expect that in states where Year 8 was the second year of secondary schooling a smaller percentage of low complexity problems would appear in the Year 8 textbooks. However, this was not the case. Textbook C, for example, contained the highest proportion of low complexity problems despite the inclusion of equation solving in the corresponding Year 7 book. All textbooks included at least some problems that required students to make connections (ranging from 2% for textbook A to 27% for textbook B) but the focus was still predominantly on using procedures. Wide variation in the number of problems was also evident, ranging from 87 problems in textbook A to 337 problems in textbook D. The last line of Tables 6 and 7 gives the Australian averages from the Video Study for comparison. The lessons of the Video Study had more problems of high complexity and more problems requiring stating concepts and making connections than these two sections of the textbooks.

Table 6
Procedural Complexity and Type of Solving Process for "Addition and Subtraction of Fractions" Problems for Sample of Australian Year 8 Mathematics Textbooks

Textbook	Number of problems	Procedural complexity (percentage of problems)			Solving process (percentage of problems)		
		Low	Moderate	High	Using procedures	Stating concepts	Making connections
A	114	76	24	0	93	2	5
B	116	76	24	0	95	0	5
C	16	56	44	0	75	0	25
D	12	83	17	0	100	0	0
E	74	69	31	0	95	0	5
F	18	61	39	0	100	0	0
G	no section	-	-	-	-	-	-
Video 99 Australia		77	16	8	61	24	15

Note: The percentages do not all sum to 100 because of rounding.

Table 7
Procedural Complexity and Type of Solving Process for "Solving Linear Equations" Problems for Sample of Australian Year 8 Mathematics Textbooks

Textbook	Number of problems	Procedural complexity (percentage of problems)			Solving process (percentage of problems)		
		Low	Moderate	High	Using procedures	Stating concepts	Making connections
A	87	79	21	0	84	14	2
B	132	85	15	0	64	9	27
C	213	88	12	0	72	5	23
D	337	85	15	0	91	1	8
E	298	73	26	1	89	1	10
F	172	62	38	0	77	7	16
G	250	84	16	0	91	4	6
Video 99 Australia		77	16	8	61	24	15

Classification of the problems as either exercises or applications indicated that the emphasis for both topics in all textbooks was on the practising of procedures (exercises) rather than on the application of those procedures (see Figure 1). This was particularly evident in the case of addition and subtraction of fractions, where only three books included application problems. Curiously, books C, D, and F, which were revising the topic from the previous year's work (recall that G had no section on this topic), had no application problems. It will be interesting to check with other topics whether revision focuses more strongly on procedures than is the case when the topics are first introduced. (Note that there is a methodological difficulty here that urges caution: only problems in the designated chapters have been analysed, but there may be many applications in later chapters).

For solving linear equations, the average proportion of application problems was higher (see Figure 1) with more variation. Textbook A had only exercises, but most books included a number of application word problems, sometimes in separate problem sets or as investigations. In textbook B (from a state where Year 8 is the first secondary school year) over 25% of the problems were applications involving the solving of word problems. However, when the total number of equation solving problems was considered, it could be seen that there was a high level of repetitive exercises. In textbook D, for example, there were only 43 application problems from a total of 337 problems.

Two further observations are of interest. First, the relative proportions of applications and exercises in the books vary between the two topics. It does not appear that some books have more applications in all chapters. Second, the proportions of applications for all of these textbooks for both topics are substantially below the Australian average of 45% of problems being applications in the Video Study lessons.

Discussion

As in the Video Study, textbook problems were overwhelmingly low complexity problems and they focussed on using procedures. There was a broad similarity in the proportions of problems in each category in this and the Video Study, although it will be useful to test this on a further sample of textbooks and topics. In fact, the results of the Video Study showed more variation than the textbook problems, having more problems of high complexity, more applications and fewer problems that only required using procedures. This may indicate that much of this variation in lessons came from resources other than textbooks, and the Video Study data can be examined in future to test this.

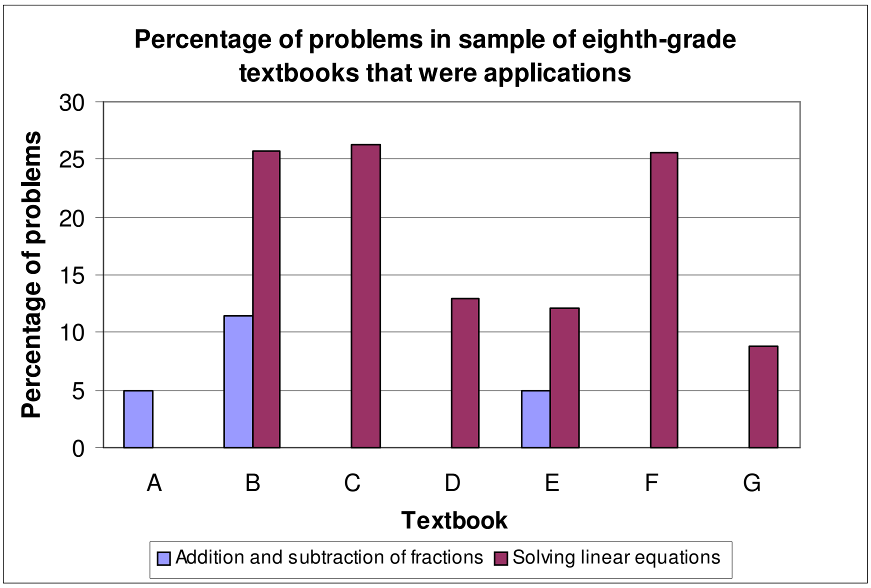


Figure 1. Percentage of problems in the sample of Year 8 textbooks that were applications.

Choosing topics that were comparable across the different states was complicated by the slightly different curriculum emphases, and by whether Year 8 was the first or the second year of secondary schooling. Although addition and subtraction of fractions was a common curriculum element, in states where Year 8 was the second year of secondary school, most textbooks included only a brief revision set of problems. However, contrary to expectations, these revision problems were generally low complexity exercises, with few application problems or problems that required students to make connections or consider underlying mathematical concepts. Consequently, students with conceptual difficulties after first exposure to a topic are less likely to have them addressed in later years.

A major aim of this study was to explore the use of the definitions and constructs of the Video Study, and their suitability for capturing the essence of the mathematical work on which students spend their time. In general, the classification procedures seemed reasonably robust. For example, in determining problem complexity, it was sometimes difficult to decide whether to count a particular operation as one or two steps. At Year 8 level, students are likely to be still gaining confidence with addition and subtraction involving negative integers. Hence we classified solving the equation $-2x-5=-11$ as requiring three steps: deciding to add 5 to both sides, calculating $-11+5$, and dividing both sides by -2. However, the equation $2x+5=11$ was classified as having only two steps: subtracting 5 from both sides to give 6 on the right side, and dividing both sides by 2. With either 2 or 3 steps, though, both these equations are classified as low complexity.

Different types of problems play different pedagogical roles. It is important that textbooks provide students with sufficient exercises so that procedures may be practised and become a secure part of a student's mathematical toolbox. Likewise there should be sufficient problems for students to learn to apply those practised skills, for making connections between different aspects of mathematics, for recognising underlying mathematical concepts, and for reasoning. Having two classifications, one for complexity and one for mathematical processes, highlights the fact that higher procedural complexity does not indicate higher quality of problems in terms of challenging students to make connections or to reason. In the case of the equation solving problems, for example, many problems qualified as moderate complexity because the solving required more steps, for example, $7(x-3)-2(5-x)+25=4(x+3)-8$. However, apart from deciding upon the order of steps, the student simply repeats the same types of operations: expanding brackets, dealing with positive and negative signs, collecting like terms, etc. It is important that students should be able to solve equations involving multiple steps. Mathematicians have to be able to sustain a chain of reasoning without error. However, the textbooks tended to include these moderately complex equations at the expense of including high complexity problems, where students must plan a path through sub-problems in order to reach the target result. In several books, investigations were included that would have been classified as one high-complexity problem, except that the investigation was broken down into a number of clearly stated sub-problems, each of which became a separate problem of generally low complexity.

It was also evident during the classification process, that the classifications do not show which are “good” problems, and that there are problems that provoke and do not provoke mathematical thought in all categories. A problem such as “plot the point (3, 2)”, for example, is classified as “stating concepts”, but it may stimulate less learning than a simple “using procedures” problem. It is not that “using procedures” problems and problems of low complexity are “bad” of themselves, but that their dominance curtails the experiences that students have of mathematical thinking. It is also the case that using the percentage of problems in each category as the basic measure is problematic (providing a few more exercises will put up the percentage of low complexity problems), especially as problems of higher complexity and those requiring connections may each take more students’ time.

References

Brändström, A. (2005). *Differentiated tasks in mathematics textbooks: An analysis of the levels of difficulty*. Licentiate Thesis, Luleå University of Technology, Sweden.

Hiebert, J., Gallimore, R., Garnier, H., Givvin, K. B., Hollingsworth, H., Jacobs, J., Chui, A. M-Y., Wearne, D., Smith, M., Kersting, N., Manaster, A., Tseng., E., Etterbeck, W., Manaster, C., Gonzales, P., & Stigler, J. (2003). *Teaching mathematics in seven countries: Results from the TIMSS 1999 Video Study*. Washington, DC: National Centre for Education Statistics, U.S. Department of Education.

Pepin, B., & Haggarty, L. (2001). Mathematics textbooks and their use in English, French and German classrooms: A way to understand teaching and learning cultures. *Zentralblatt für Didaktik der Mathematik, 33*(5), 158 – 175.

Stacey, K. (2003). The need to increase attention to mathematical reasoning. In H. Hollingsworth, J. Lokan & B. McCrae, *Teaching Mathematics in Australia: Results from the TIMSS 1999 Video Study*. (pp. 119-122) Melbourne: Australian Council for Educational Research.

Designing Effective Professional Development: How do We Understand Teachers' Current Instructional Practices?

Jana Višňovská
Vanderbilt University
<jana.visnovska@vanderbilt.edu>

Drawing on a review conducted of the resources that the mathematics education research community has developed while learning to support teacher learning, I direct attention to researchers' understanding of teachers' current practices. In particular, I argue that designers, facilitators, and researchers of professional development alike would benefit from understanding teachers' practices (a) as reasonable from teachers' perspectives, (b) in a way that can directly feed into the efforts of supporting teacher learning, and (c) as shaped by the institutional context of teachers' work.

Introduction

Designing effective professional development (PD) programs for mathematics teachers is a complex endeavour about which a lot remains to be learned (Borko, 2004). To explicate the complexity, I first discuss how it is deeply rooted in the demands related to teaching mathematics for understanding. I then argue that for PD interventions to be effective, the facilitators need to have both an understanding of teachers' current instructional practices and a way to build on those PD designs. Lastly, I use illustrations from several PD research studies to build an image of what might be involved in understanding teachers' practices in useful ways for the purposes of designing and facilitating effective PD programs.

For the past 15 years, an important goal for mathematics educators in the US has been to change the nature of mathematics teaching and learning in classrooms. Reformers have proposed substantial changes in the content and pedagogy of the K–12 mathematics curriculum, so that all students have the opportunity to learn more intellectually demanding mathematics. Among the important contributions of the reform efforts to this point is that they "shed light on the vital role played by teachers in educational change" (Llinares & Krainer, 2006, p. 439). The broad consensus about the critical role of teachers fuelled studies of classroom instructional practices that would support all students' development of the kinds of mathematical understanding that are the aim of the reform. A number of these studies suggest that the effective practices require that teachers build from their students' current reasoning while, at the same time, keeping in mind significant mathematical ideas that are the goal of instruction (e.g., Ball, 1993; Gravemeijer, 2004; Hiebert et al., 1997; Lampert, 2001). The forms of the envisioned instructional practices emphasise students' opportunities to engage in mathematically challenging tasks, maintaining the level of challenge as tasks are enacted in the classroom (e.g., Stein & Lane, 1996), and students' opportunities to communicate their mathematical thinking (e.g., Lampert, 2001).

The complexity of supporting mathematics teachers to develop such instructional practices has been documented by numerous investigations that focused on teacher PD (e.g., Cobb & McClain, 2001; Fennema, Carpenter, Franke, & Carey, 1993; Franke & Kazemi, 2001; Simon & Tzur, 1999). Researchers reported that even in cases when teachers were willing to collaborate and seemed engaged in the work-session setting, understanding children's reasoning was not always easy (Ball, 2001; Schifter, 2001). In

Proceedings of the 30th annual conference of the Mathematics Education Research Group of Australasia
J. Watson & K. Beswick (Eds), © MERGA Inc. 2007

addition, teachers did not always see the use of their new knowledge in their classrooms as immediately obvious (Fennema et al., 1993; Zhao, Visnovska, Cobb, & McClain, 2006). Part of this complexity resides in the nature of the required teacher learning that targets changes in what Elmore (1996) called "the core of educational practice" – that is, the ways teachers think about the nature of knowledge, the nature of mathematics that would be beneficial for students to learn, as well as about their own and their students' roles in teaching and learning (cf. Carpenter et al., 2004). Those conducting PD thus face a challenge in finding ways to support the teachers to revise the core assumptions of their practice and help them develop a need to change their classroom instruction. This is where understanding teachers' current practices in a useful way comes to the foreground in the process of designing effective PD.

Intervening to Support Mathematics Teachers' Learning

Designing PD programs that build on and benefit from teachers' current instructional practices and, at the same time, are effective in pursuing a PD agenda is important for reasons parallel to those of building on students' reasoning towards an instructional agenda in mathematics classrooms. On the one hand, there is little doubt that PD interventions should pursue their agendas, such as to focus on the key learning goals for teachers. On the other hand, my experiences when working with a group of middle school mathematics teachers convinced me that linking these goals to the participating teachers' current practices so that the teachers could come to see them as beneficial was as important (e.g., Zhao et al., 2006).

The issues I discuss in this paper arose when I reviewed the research on teacher PD in mathematics, with a goal of gaining better insights into understanding teachers' current practices and how they can be used effectively as a resource in designing and facilitating PD. Pragmatically, I concentrated on interventionist studies with the goal of supporting teachers to develop instructional practices centred in student's mathematical reasoning. In particular, I tried to understand what guidance asking different questions and adopting different perspectives bring to an endeavour of supporting and understanding teacher learning. The studies I discuss in this paper are intended to serve as paradigmatic cases of pursuing specific types of research goals while drawing on a specific set of assumptions and perspectives. They enable me to raise issues of importance with respect to *understanding* teachers' current practices, specifically understanding them (a) as reasonable from teachers' perspective, (b) in a way that can directly feed into the efforts of supporting teacher learning, and (c) as shaped by institutional context of teachers' work.

Understanding Teachers' Practices as Reasonable from their Perspective

Although recommendations to view teachers' instruction as reasonable are a repeating theme in teacher education literature (e.g., Leatham, 2006; McIntyre & Hagger, 1992; Thompson, 1992), developing such a view might often seem counterintuitive. This is true especially in cases when teachers' instructional practices differ significantly from those advocated by the reform proponents. However, if we do not commit to see teachers' current instruction as reasonable from their perspective we risk both (a) overlooking opportunities for supporting teachers in making their perspectives a worthwhile topic of guided reflection, and (b) positioning teachers as deficient, having little to bring to the new instructional practices they are expected to develop. The professional developers' job then becomes filling the gaps between teachers' current – "deficient" – instructional practices and the envisioned ones. The problematic nature of this approach is well documented by the frustrations of both teachers who ended up participating in PD programs that were not justifiable within their current understanding of teaching and learning (e.g., Putnam & Borko, 2000), and professional developers who struggled to earn participating teachers' compliance and enthusiasm (e.g., Franke, Kazemi, Carpenter, Battey, & Deneroff, 2002). The resulting mismatch in professional developers' and participating teachers' views of ways to improve classroom mathematics instruction has been discussed in terms of incongruence in beliefs (e.g., Tillema, 1995) and changing teachers' beliefs has been repeatedly reported a challenging task (e.g., Thompson, 1992).

Simon and colleagues (Simon, 2000; Simon, Tzur, Heinz, Kinzel, & Smith, 2000) illustrated that if we want to take teachers' current instructional practices "as a valuable starting point, not as something to be replaced, but a useful platform on which to build" (McIntyre & Hagger, 1992, p. 271), understanding these practices as a coherent system, rather than a random conglomerate of teaching moves, is valuable. Their Mathematics Teacher Development (MTD) Project experiences suggest that approaches that succeed in taking teachers' current instructional practices as a PD starting point might significantly reduce problematic mismatches between researchers' expectations and teachers' actual participation in PD activities. The phenomenon of teachers' "constraining" beliefs might then be tackled by re-conceptualising the issue as a problem of PD design. To investigate whether this indeed is the case, two related issues arise for those interested in supporting teachers' development of new instructional practices: (a) How to see and explain the teachers' actions as reasonable from their perspective, and (b) How to design for PD that builds on teachers' current instructional practices towards a PD agenda rather than pursuing a gap-filling approach. In the work I reviewed, Simon and colleagues productively contributed to addressing the first question by generating *accounts of practice* (Simon & Tzur, 1999) – an adaptation of a case study methodology tailored to yield insights into an individual teacher's current perspective on teaching and learning while seeing this perspective as reasonable from the teacher's point of view.

Understanding Teachers' Practices *in order to* Support Teacher Learning

Simon and colleagues' focus in their study was on documenting perspectives that mathematics teachers held about teaching and learning and theorising these perspectives developmentally. This focus, as any particular focus, highlighted some aspects of teacher learning while it chose not to address other aspects. My goal in this section is to discuss the guidance that MTD Project research provided for both the design of further intervention and analysis of actual teachers' learning. As the researchers (Tzur, Simon, Heinz, & Kinzel, 2001) point out, we can think of guidance at three different levels.

On the broad level, categorising teachers with respect to their perspective on learning can help to highlight some of the key characteristics of instructional practices, development of which might be worth supporting. In this sense, the distinction between *perception-based* and *conception-based* perspectives that the researchers explicated provided a general direction for teacher development. Specifically, conception-based perspective stands for a common core of emergent and constructivist perspectives and its development requires a difficult shift from "we understand what we see" to "*we see what we understand*" (Simon et al., 2000, p. 585), a shift that can be counterintuitive to many teachers. On the other hand,

> (a) perception-based perspective is grounded in a view of mathematics as a connected, logical, and universally accessible part of an ontological reality. From this perspective, learning mathematics

> with understanding requires learner's direct (firsthand) perception of relevant mathematical relationships. … teaching involves creating opportunities for students to apprehend (perceive) the mathematical relationships that exist around them (Simon et al., 2000, pp. 579, 594).

This perspective is problematic in that the teachers often do not consider what students already have to know and be able to do *in order to* gain the valued insights. With respect to a perspective that underlies "traditional" teaching practices, developing a perception-based perspective suggests an important accomplishment. With respect to developing instructional practices that would support students' learning mathematics with understanding, further support of teachers' development of a conception-based perspective would be needed.

Fine-grained understanding of teachers' instruction as reasonable from teachers' perspectives is especially useful in both anticipating and analysing teachers' interpretations of designed activities. Explication of a perception-based perspective helped Simon and colleagues corroborate their observations. In the researchers' view, the teachers were not inquiring into the nature of their students' understanding in their daily instruction. Portraying teachers' decisions as reasonable from their perspective, however, helped the researchers to understand that from the teachers' perspective, they *were* basing their instruction on their students' reasoning. However, they were only doing it as long as students' reasoning corresponded – in teachers' view – to observable mathematical reality. Simon and colleagues stressed that the sense that the teachers were making of opportunities to explore students' reasoning both in their classrooms and in PD sessions was constrained by their current perspectives on teaching and learning. Promoting MTD Project teachers' inquiry into their students' reasoning would be likely interpreted by the teachers as something they already do in their classrooms and would therefore not lead to the envisioned changes in teachers' instructional practices.

In order to guide professional developers' decisions when planning specific interventions in response to teachers' actual participation a yet different grain-level of understanding teachers' actions is beneficial. I will refer to this as a *meso-level* of PD design. To guide the design effectively, this meso-level should, in my view, be specific enough to help developers discern aspects of teachers' current practices that might provide a springboard for further intervention. At the same time, it is an advantage if the grain size allows for consideration of how patterns in practices of the group, rather than individual teachers, are shaped. I now discuss each of these two points in more detail.

First, researchers working within constructivist, emergent, and situated paradigms concur that teachers' current instructional practices can and should serve as a basis on which to build in supporting teachers' further learning (Ball & Cohen, 1999; Kazemi & Franke, 2004; McIntyre & Hagger, 1992; Simon et al., 2000; Wilson & Berne, 1999). Pragmatically, they aim to design PD activities to promote participation that both engages teachers' current professional expertise and supports its transformation. MTD Project experiences illustrate that this is not a trivial task. For the teachers, further learning would involve a shift in paradigm with respect to development of mathematical knowledge. In what ways could teachers' current practices, oriented by a paradigm we want them to overcome, serve as a leverage in supporting the envisioned shift? I suggest that as designers of teacher PD with an ultimate goal of improving students' mathematical learning we need to understand teachers' current practices in ways that will allow us to answer this question. A systematic view of teachers' practices that would enable us to formulate revisable conjectures about ways of supporting teachers' learning on an ongoing basis would be of both theoretical and pragmatic value.

Second, although the usefulness of researchers' understanding was also a priority for Simon and colleagues, I would like to point to what I see as possible limitations of understanding teachers' practices solely in terms of individual teachers' underlying perspectives of mathematics teaching and learning. It has been documented that other aspects significantly influence teaching from teachers' point of view, often by shaping the setting in which teachers work. Aspects of teaching, like available instructional resources for use in classrooms (Cobb, McClain, Lamberg, & Dean, 2003; Remillard, 2005), teachers' views of student motivation and classroom misbehaviour (Dean, 2006; Visnovska, 2005; Zhao et al., 2006), and overall organizational aspects of the institutional contexts in which teachers work (Cobb et al., 2003; Elmore, 2000; Gamoran et al., 2003), all significantly shape how teachers approach teaching and learning. Each of these aspects constitutes a source of explanation to understand the rationality of teachers' instructional practices (Zhao, 2005) that remain in the background when the focus is on teachers' conceptions. More importantly, each of these may serve as a resource in designing starting points for PD that would capitalise on the teachers' current instructional practices. Several of these aspects of teachers' work point our attention to influences on teaching that are common across the participating teachers. From a perspective of a designer, this would allow for planning PD activities where current concerns of all teachers could become a topic of discussion. The teachers' individual responses to these common concerns could then provide the facilitator with the diversity of ideas on which to build in supporting teacher learning.

I would like to clarify that this broadening of the scope within which to understand teachers' practices is not motivated by a quest for an ultimate theoretical account. Others' PD experiences that I review suggest that we cannot expect that all teachers characterised as having developed a certain perspective on teaching and learning could be further supported in the same way. That is, in a way that would be independent of the institutional context of their work, instructional resources available in their schools, or major impediments to instruction as seen from teachers' perspective. As I illustrate in the following discussion of the Cognitively Guided Instruction (CGI) project experiences, this broadening of the scope has long been implicitly present, across the spectrum of adopted theoretical perspectives, in the designs of PD that could be claimed effective in supporting teacher learning.

Understanding Teachers' Practices as Profoundly Shaped by Institutional Context of their Work

I first introduce a CGI study (Fennema et al., 1996) conducted under a cognitive research paradigm. I chose the study based on a rich picture that the researchers provided of the concerns that played an important role in their design and research efforts. Concerns that related to the institutional context of teachers' school were treated as background issues and were not accounted for within the cognitive framework adopted for the study. Nevertheless, it would be hard to overlook the design efforts explicitly devoted to shaping the institutional context in which the teachers worked.

CGI: Research-based Knowledge for Teaching

CGI researchers developed their program in the mid 1980s to investigate how mathematics teachers may capitalise upon research-based knowledge in their classroom instruction. In terms of content, most of the CGI research work was grounded in a substantial body of research that provided a consistent and coherent picture of the development of basic number concepts (Carpenter, 1985; Carpenter, Fennema, Franke, Levi, & Empson, 1999; Fuson, 1992). Over the years, CGI researchers engaged in a number of research and PD projects in which they collaborated with a variety of mathematics teacher groups. The teachers' active part in the PD was in deciding how to make use of the knowledge in the context of their own classroom instruction. The researchers conjectured that by providing teachers with an operationalised model of how children's thinking develops the teachers would become competent in identifying different forms of students' mathematical reasoning in their classrooms, as well as in planning appropriate follow up instruction that would capitalise on identified forms of reasoning.

The success of the PD efforts was framed in terms of changes in the individual teachers' beliefs and instruction. Findings from case studies led the researchers to conclude that "developing an understanding of children's mathematical thinking *can be* a productive basis for helping teachers to make the fundamental changes called for in current reform recommendations" (p. 403, emphasis added). Such studies served as an existence proof of what could be achieved with teachers through focusing on a research-based framework of student thinking, and provided insights into the specifics of achieved instructional changes. Teachers' knowledge of students' developmental processes and their ability to understand their students' reasoning were both framed as instrumental to the documented changes.

In terms of means that supported the discussed developments, the early CGI reports accordingly focused on two issues (a) a research-based model of student thinking, and (b) teachers' use of that model in their classrooms. It is important to clarify that *supporting collaborating teachers' learning* also included the following.

> A CGI staff member and a mentor teacher were assigned to each school. Their responsibilities included participating in the workshops, visiting classrooms, engaging the teachers in discussions, and generally providing support as the teachers learned to base instruction on their students' thinking. Both staff members and the mentor teachers were trained to focus most of their interactions with teachers directly on children's thinking and its use. Insofar as possible, these interactions concerned specific children (Fennema et al., 1996, p. 409).

In its plan of action, the CGI program did not focus solely on cognitive aspects of teachers' learning. It involved significant interventions with both school principals and mathematics support staff based in the teachers' schools. In order to generate the proof of the usefulness of research-based knowledge to teachers' instruction, the researchers took seriously the *institutional context* within which teachers worked. In a very real sense, the CGI work involved designing for a particular institutional context that the researchers envisioned as supportive of teachers' learning. Yet, at this point, these considerations were conceptualised as a background for the project, rather than as key support for teachers' developing practices. The distinction is critical with respect to generalizability of the research findings, that is, with respect to the orientation the findings provide to designing and facilitating teacher PD programs. I clarify this issue when I discuss one of the more recent CGI studies, in which the researchers drew on situated theories of learning and used considerations related to institutional setting as resources for understanding teachers' current instructional practices. I draw on this study to corroborate further what I mean by usefulness of understanding teachers' practices on the *meso-level* of PD design.

CGI: The Case of Algebraic Reasoning

After years of experience with PD in context of early number concepts, Franke and colleagues (Franke, Carpenter, & Battey, in press; Franke et al., 2002) engaged in PD and research efforts focusing on early algebraic thinking. Using their intimate understanding of CGI principles and findings, they aimed to support elementary teachers in enhancing students' ability to generate, use, represent, and justify generalizations about fundamental properties of arithmetic. As in their previous work, the researchers intended to do this by both supporting teachers in developing a model of students' development of algebraic reasoning, and by supporting teachers' development of practices that place their students' reasoning in the centre of classroom instruction. However, they came to view teachers' cognition as being inherently social, inseparable from the cultural and institutional aspects of teachers' work.

The case I discuss comes from a CGI collaboration with a group of teachers in one of the lowest achieving elementary schools in the state of California (Franke et al., 2002). The researchers intended to use discussions of student work as leverage in supporting teachers' appreciation of understanding students' algebraic reasoning in instruction. To the researchers' surprise and frustration, even after many work-sessions, student reasoning did not become something teachers wanted to learn about and use in their instruction: "All the teachers … see is the answer and while this occurred initially in our earlier work the teachers quickly began to see on the paper and in their questioning what students did to solve the problem" (p. 28). The teachers continued to check for correctness of responses and did not find it useful to discuss in classrooms how different students arrived at their solutions. Instead, they requested that the researchers provide them with more "worksheets" for students to practice until they ceased making mistakes.

To support these teachers' learning effectively, the researchers needed to understand *why*, despite CGI efforts, it continued to be reasonable from the teachers' perspective to support their students' learning of early algebra by providing them with abundant opportunities to practice, and by correcting their mistakes. Simon and colleagues' focus on teachers' conceptions locates the source of the reasonableness of teachers' actions within individual teachers' cognition. According to analysis from such a viewpoint, the California teachers could be characterised as making instructional decisions within a traditional perspective, based on a view of algebra as a collection of rules and facts that can be best learned by repetition. Although such characterization might capture quite accurately teachers' actions at the time, it does not clarify why sustained efforts at supporting these teachers' change were not viable. This point is critical because, according to Franke and colleagues (2002), teachers initially focused on correctness and practice in the earlier CGI collaborations as well. However, supported by the CGI team, they soon came to appreciate student reasoning as an instructional resource. It appears that although providing a useful and specific orientation in terms of goals for teacher learning, Simon and colleagues' characterization of teachers' perspectives is not specific enough to guide the ongoing process of designing for teacher learning. The exclusively cognitive focus of this characterization seems insufficient to explain why the means of support that had proven effective earlier were not effective with the California teachers.

Franke and colleagues' (in press) analysis instead located the encountered PD difficulties in both content-specific demands on teachers' learning, and the institutional setting of teachers' work. This allowed the researchers to propose specific adaptations to the PD design that took into account the unique characteristics of the PD context. As an example, consider the content-specific dimension related to the institutional setting of teachers' work. It concerned the extent to which the content area in the focus of teachers' PD was central (or peripheral) within the curriculum used in the teachers' schools. The researchers documented that the emphasis that the curriculum put on a specific content area had consequences for development of teacher's expertise in that area. Specifically, the differences manifested in both (a) the resources for PD work available in form of the teachers' current practices in the content area, and (b) the opportunities afforded for teachers' further learning in that area in their classrooms. To elaborate the first point, the number development directly related to the early grades curricula that were in place in the collaborating schools. However, the ideas of relational thinking and formulating conjectures that were central to the CGI model of development of students' early algebraic reasoning were not explicit aspects of the typical mathematics curricula. As such, these were not areas where teachers had many opportunities to hear their students work with the ideas, or to deepen their own algebraic understanding. Consequently, the teachers often lacked the confidence that they could master the content issues that might arise in their classrooms, and productively engage students in algebraic thinking.

To address the second point, the central position of early number development content provided teachers with plenty of opportunities to pose CGI word problems and ponder student solutions. In contrast, to make seemingly "extracurricular" algebraic reasoning an instructional focus in their classrooms, the teachers would have to develop ways to coordinate the mathematical content addressed explicitly in the required curriculum with supporting students in making generalizations, noticing relations, and justifying conjectures. Not surprisingly, this presented additional challenges for teachers' development of new instructional practices.

Researchers' understanding of critical content-related demands on teachers' developing instructional practices and how they relate to the institutional context of teachers' daily instruction oriented researchers' conjectures about viable means of supporting teachers' further learning. For example, the researchers reported that to help teachers develop knowledge about identifying opportunities for algebraic thinking, they brought examples of interactions they observed in teachers' classrooms to the group for discussion. In addition, they started to create structured opportunities for teachers to reflect on "where their own students are in their understanding of the various ideas of algebraic thinking" (Franke et al., in press), as students' progress in this content area did not feature on the district quarterly benchmark assessment. These adaptations, although open for further testing and modifications, serve as an example of the flexibility that understanding teachers' practices as situated in the cultural and institutional aspects of teachers' work affords those working with groups of mathematics teachers. Adopting this perspective seemed to enhance the CGI researchers' capacity to manoeuvre on the *meso-level* of PD design, where pragmatic decisions of how to proceed are informed by systematic ongoing analyses.

Summary

Although developmental approaches can help us delineate worthwhile end points for teacher learning, it appears that studies conducted under a situated paradigm are especially well positioned to develop valuable means for supporting teacher learning on the meso-level of design. On this level, understanding of teachers' practices yields resources that can directly feed back to PD designs. In this paper, I outlined an argument for usefulness of this level of understanding teachers' practices when designing and facilitating PD interventions. As an example, I discussed how CGI researchers adapted their PD design based on their ongoing analysis of the institutional context of teachers' work. However, detailed analysis would be required to understand how means of support based on these design resources contribute to teachers' development of new instructional practices. In addition, understanding which aspects of teachers' practices would be most useful in feeding back to designs is an important question to answer.

References

Ball, D. L. (1993). With an eye on the mathematical horizon: Dilemmas of teaching elementary school mathematics. *Elementary School Journal*, *93*(4), 373-397.

Ball, D. L. (2001). Teaching, with respect to mathematics and students. In T. Wood, B. S. Nelson & J. Warfield (Eds.), *Beyond classical pedagogy: Teaching elementary school mathematics* (pp. 11–22). Mahwah, NJ: Lawrence Erlbaum.

Ball, D. L., & Cohen, D. (1999). Developing practice, developing practitioners: Towards a practice-based theory of professional education. In G. Sykes & L. Darling-Hammond (Eds.), *Teaching as the learning profession: Handbook of policy and practice* (pp. 3-32). San Francisco: Jossey-Bass.

Borko, H. (2004). Professional development and teacher learning: Mapping the terrain. *Educational Researcher*, *33*(8), 3-15.

Carpenter, T. P. (1985). Learning to add and subtract: An exercise in problem solving. In E. A. Silver (Ed.), *Teaching and learning mathematical problem solving: Multiple research perspectives* (pp. 17-40). Hillsdale, NJ: Lawrence Erlbaum.

Carpenter, T. P., Blanton, M. L., Cobb, P., Franke, M., Kaput, J. J., & McClain, K. (2004). *Scaling up innovative practices in mathematics and science*. Madison, WI: NCISLA, Wisconsin Center for Education Research. Retrieved 2006, from http://www.wcer.wisc.edu/NCISLA/publications/reports/NCISLAReport1.pdf

Carpenter, T. P., Fennema, E., Franke, M., Levi, L., & Empson, S. B. (1999). *Children's mathematics: Cognitively guided instruction*. Portsmouth, NH: Heinemann.

Cobb, P., & McClain, K. (2001). An approach for supporting teachers' learning in social context. In F. L. Lin & T. Cooney (Eds.), *Making sense of mathematics teacher education (pp. 207-232)*. Dordrecht, The Netherlands: Kluwer.

Cobb, P., McClain, K., Lamberg, T., & Dean, C. (2003). Situating teachers' instructional practices in the institutional setting of the school and school district. *Educational Researcher*, *32*(6), 13-24.

Dean, C. (2006, April). *The evolution of strategic reasoning norms in a professional teaching community*. Paper presented at the annual meeting of the American Educational Research Association Conference, San Francisco, CA.

Elmore, R. F. (1996). Getting to scale with successful educational practices. In S. W. Fuhr & J. A. O'Day (Eds.), *Rewards and reform: Creating educational incentives that work* (pp. 294-329). San Francisco: Jossey Bass.

Elmore, R. F. (2000). *Building a new structure for school leadership*. Washington, DC: Albert Shanker Institute.

Fennema, E., Carpenter, T. P., Franke, M., & Carey, D. A. (1993). Learning to use children's mathematics thinking: A case study. In R. B. Davis & C. A. Maher (Eds.), *Schools, mathematics, and the world of reality* (pp. 93-117). Boston, MA: Allyn and Bacon.

Fennema, E., Carpenter, T. P., Franke, M. L., Levi, L., Jacobs, V. R., & Empson, S. B. (1996). A longitudinal study of learning to use children's thinking in mathematics instruction. *Journal for Research in Mathematics Education*, *27*, 403-434.

Franke, M. L., Carpenter, T. P., & Battey, D. (in press). Content matters: The case of algebraic reasoning in teacher professional development. In J. J. Kaput & M. L. Blanton (Eds.), *Exploring early algebra*. Mahwah, NJ: Lawrence Erlbaum.

Franke, M. L., & Kazemi, E. (2001). Learning to teach mathematics: Developing a focus on students' mathematical thinking. *Theory Into Practice*, *40*, 102-109.

Franke, M. L., Kazemi, E., Carpenter, T. P., Battey, D., & Deneroff, V. (2002, April). *Articulating and capturing generative growth: Implications for professional development*. Paper presented at the annual meeting of the American Educational Research Association, San Diego, CA.

Fuson, K. C. (1992). Research on whole number addition and subtraction. In D. Grouws (Ed.), *Handbook for research on mathematics teaching and learning* (pp. 243–275). New York: Macmillan.

Gamoran, A., Anderson, C. W., Quiroz, P. A., Secada, W. G., Williams, T., & Ashman, S. (2003). *Transforming teaching in math and science: How schools and districts can support change*. New York: Teachers College Press.

Gravemeijer, K. (2004). Local instruction theories as means of support for teachers in reform mathematics education. *Mathematical Thinking and Learning*, *6*, 105-128.

Hiebert, J., Carpenter, T. P., Fennema, E., Fuson, K. C., Wearne, D., Murray, H., et al. (1997). *Making sense: Teaching and learning mathematics with understanding*. Portsmouth, NH: Heinemann.

Kazemi, E., & Franke, M. L. (2004). Teacher learning in mathematics: Using student work to promote collective inquiry. *Journal of Mathematics Teacher Education*, *7*, 203-225.

Lampert, M. (2001). *Teaching problems and the problems of teaching*. New Haven, CT: Yale University.

Leatham, K. (2006). Viewing mathematics teachers' beliefs as sensible systems. *Journal of Mathematics Teacher Education, 9*(2), 91-102.

Llinares, S., & Krainer, K. (2006). Mathematics (student) teachers and teacher educators as learners. In A. Gutiérrez & P. Boero (Eds.), *Handbook of research on the psychology of mathematics education: Past, present and future* (pp. 429-459). Rotterdam: Sense Publishers.

McIntyre, D., & Hagger, H. (1992). Professional-development through the Oxford internship model. *British Journal of Educational Studies*, *40*(3), 264-283.

Putnam, R. T., & Borko, H. (2000). What do new views of knowledge and thinking have to say about research on teacher learning? *Educational Researcher*, *29*(1), 4-15.

Remillard, J. T. (2005). Examining key concepts in research on teachers' use of mathematics curricula. *Review of Educational Research*, *75*(2), 211-246.

Schifter, D. (2001). Learning to see the invisible: What skills and knowledge are needed to engage with students' mathematical ideas? In T. Wood, B. S. Nelson, & J. Warfield (Eds.), *Beyond classical pedagogy: Teaching elementary school mathematics* (pp. 109–134). Mahwah, NJ: Lawrence Erlbaum.

Simon, M. A. (2000). Research on the development of mathematics teachers: The teacher development experiment. In A. E. Kelly & R. A. Lesh (Eds.), *Handbook of research design in mathematics and science education* (pp. 335-359). Mahwah, NJ: Lawrence Erlbaum.

Simon, M. A., & Tzur, R. (1999). Explicating the teacher's perspective from the researchers' perspective: Generating accounts of mathematics teachers' practice. *Journal for Research in Mathematics Education*, *30*, 252-264.

Simon, M. A., Tzur, R., Heinz, K., Kinzel, M., & Smith, M. S. (2000). Characterizing a perspective underlying the practices of mathematics teachers in transition. *Journal for Research in Mathematics Education*, *31*, 579-601.

Stein, M. K., & Lane, S. (1996). Instructional tasks and the development of student capacity to think and reason: An analysis of the relationship between teaching and learning in a reform mathematics project. *Educational Research and Evaluation*, *2*, 50-80.

Thompson, A. G. (1992). Teachers' beliefs and conceptions: A synthesis of the research. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 127-146). New York: Macmillan.

Tillema, H. H. (1995). Changing the professional knowledge and beliefs of teachers: A training study. *Learning and Instruction*, *5*, 291-318.

Tzur, R., Simon, M. A., Heinz, K., & Kinzel, M. (2001). An account of a teacher's perspective on learning and teaching mathematics: Implications for teacher development. *Journal of Mathematics Teacher Education*, *4*, 227-254.

Visnovska, J. (2005). Same video, different understandings: Co-construction of representations of classroom instructional practices within a professional teaching community. In G. M. Lloyd, M. R. Wilson, J. L. Wilkins, & S. L. Behm (Eds.), *Proceedings of the 27th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. [CD-ROM]. Eugene, OR: All Academic.

Wilson, S. M., & Berne, J. (1999). Teacher learning and the acquisition of professional knowledge: An examination of research on contemporary professional development. In A. Iran-Nejad & P. D. Pearson (Eds.), *Review of research in education* (Vol. 24, pp. 173-209). Washington, DC: American Educational Research Association.

Zhao, Q. (2005). Supporting teachers' learning through the use of students' work: Conceptualizing teacher learning across the setting of professional development and the classroom. In G. M. Lloyd, M. R. Wilson, J. L. Wilkins, & S. L. Behm (Eds.), *Proceedings of the 27th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. [CD-ROM]. Eugene, OR: All Academic.

Zhao, Q., Visnovska, J., Cobb, P., & McClain, K. (2006, April). *Supporting the mathematics learning of a professional teaching community: Focusing on teachers' instructional reality*. Paper presented at the annual meeting of the American Educational Research Association Conference, San Francisco, CA.

"Doing Maths": Children Talk About Their Classroom Experiences

Fiona Walls
James Cook University
<Fiona.Walls@jcu.edu.au>

From their everyday experiences of life in classrooms, children develop understandings of what is meant by "doing maths". This paper draws on the findings of an ongoing longitudinal study following the mathematical learning careers of ten children from the beginning of their third year at primary school as seven-year-olds to the end of their eleventh year as sixteen-year-olds. Over this time, "doing maths" has changed remarkably little for these students. Using the children's accounts of doing maths, the paper probes the connections among mathematical content, teaching, and learning, and considers the implications of their stories for teaching practice.

Setting the Research Scene

Researcher: So what things do you usually do in maths time?
Georgina: Get out our maths books and do our maths. (Early Year 4)

Over the past 20 years there has been a concerted effort on the part of curriculum designers and mathematics education researchers to describe and change the culture of teaching mathematics (e.g. Boaler, 1997; Davis, 1996; Yackel & Cobb, 1996). Transmission pedagogies in which the teacher positions her/himself in front of the class to explain new mathematical ideas followed by the children sitting at their desks completing written tasks from textbooks or worksheets, have been criticised for their failure to engage and motivate children, and their failure to invoke children's powerful mathematical thinking, reasoning and working (e.g., Yackel, 2000).

Eight-year-old Georgina's response to my question (above) was typical of the children who were participating in a longitudinal study of children's attitudes to mathematics. The study began in 1998 as the children were about to start their third year of primary schooling and followed their evolving relationships with mathematics and growing mathematical identities until the end of their fifth school year. The study asked, "How, beginning from a young age, a significant proportion of children experience a loss of interest in mathematics with a concomitant decline in their achievement?", a phenomenon revealed by research in many countries (Garden, 1997). The study focused on ten 7-year-old children, randomly selected from ten different schools in the Wellington region of New Zealand. It was hoped that a sample of young children of similar age from a range of school environments might provide a more complex understanding of how children experience mathematical learning and what features of their learning environments might be linked to the disaffection and alienation noted in large-scale quantitative studies such as TIMMS.

An ethnographic case study approach was used in order to construct an intimate picture of the children's lived experiences of learning mathematics, particularly through the words of child participants, their families and their teachers (Walls, 2001; Walls, 2003). In early 2007, it was decided to extend the study. These ten students, now 16 years old, were contacted again and asked to continue their case-narratives. In addition to the earlier research question, I was interested to find out: (1) whether the children's engagement in mathematics classrooms had changed over time (2) how the children's experiences had

Proceedings of the 30th annual conference of the Mathematics Education Research Group of Australasia
J. Watson & K. Beswick (Eds), © MERGA Inc. 2007

shaped their feelings about mathematics as a subject, and (3) whether these experiences had an impact on their feelings about and continuing participation in mathematics.

The study draws on the theory of *symbolic interactionism*, which suggests that we make meaning about the world from the everyday rituals and routines we experience. Blumer (1969), a key proponent of this theory, described symbolic interactionism as being founded on a number of *root images*, the most important of which is *social interaction.* He contended that "societies" or "cultures" exist only in action, and must therefore be viewed *in action*. By *action* he meant "the multitudinous activities that individuals perform in their life as they encounter one another and as they deal with the succession of situations confronting them" (p. 6). To reveal out how the children in the study might make meaning through everyday social interaction, data were gathered through a wide range of methods including classroom observations, interviews with the children, teachers and parents, informal discussion with classmates, questionnaire sheets, and examination of mathematics exercise books. Pictures began to emerge of how learning mathematics was typically experienced by these children.

"This is me Doing Maths": Gathering Young Children's Experiences

During their first interviews in early Year 3, the children were provided with a blank page headed "*This is a picture of me during maths time*". They were encouraged to draw themselves in any way that best showed what they usually did during this part of the school day. The drawings of these 7-year-olds revealed much about what they perceived as "doing maths" (Figures 1 to 10). Eight of the children drew themselves seated at a desk or table, pencil in hand and their maths book or worksheet in front of them. Liam was the only child to draw himself actively engaged with others. He depicted himself with his friends, naming each one as he drew, constructing a tower of wooden blocks (Figure 10*).* Dominic drew himself at a table with other children, all working individually in their maths books (Figure 6). Toby drew other (childless) desks with worksheets to indicate classmates, but showed himself to be working alone (Figure 3). Mitchell was the only child who was not able to distinguish "maths" from the other activities he was expected to do at school. He drew himself skipping, the activity in which he had been engaged a short time before the interview, and drawing, the activity he said he most liked (Figure 9). Jared's drawing is notable for its action and movement (Figure 1).

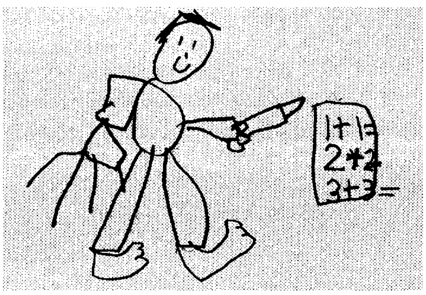


Figure 1. Jared (Early Year 3).

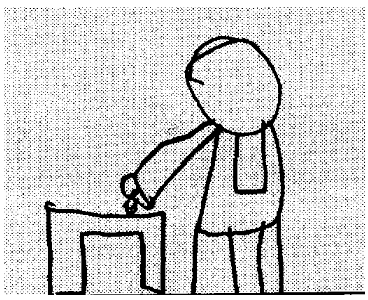


Figure 2. Peter (Early Year 3).

The children were asked to explain their drawings.

Toby: This is the table and that on there is the worksheet. (Early Year 3) (Figure 3).
Researcher: And what's that you have just drawn? (Figure 4).
Rochelle: It's my desk.

Researcher: So what's this here?
Rochelle: Book.
Researcher: Is that your maths book? (*Rochelle nods*) (Early Year 3)

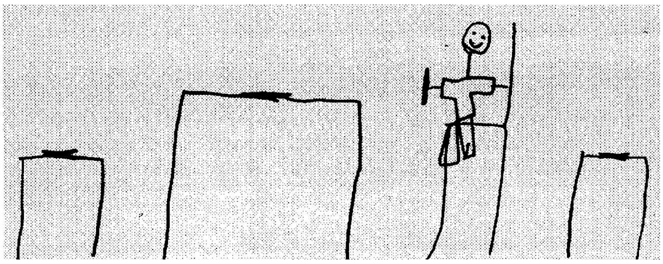


Figure 3. Toby (Early Year 3).

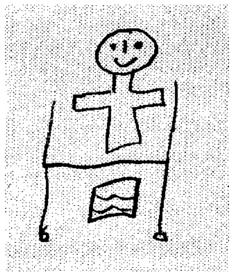


Figure 4. Rochelle (Early Year 3).

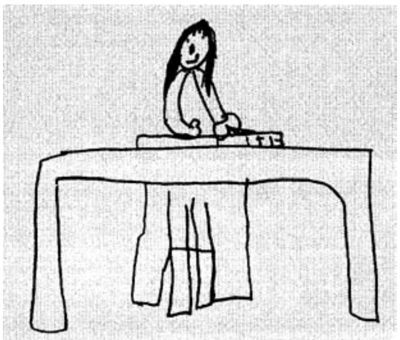


Figure 5. Georgina (Early Year 3).

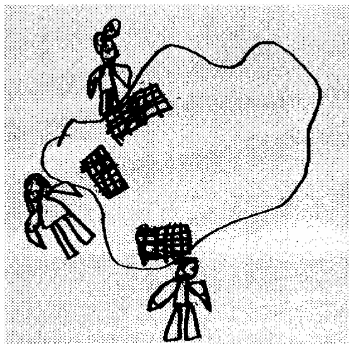


Figure 6. Dominic (Early Year 3).

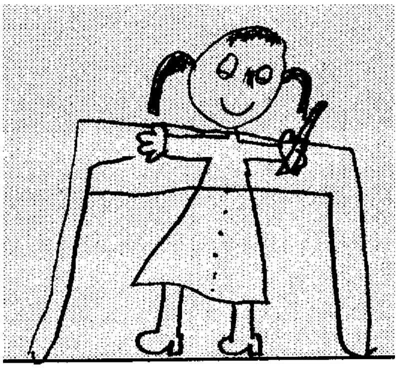


Figure 7. Fleur (Early Year 3).

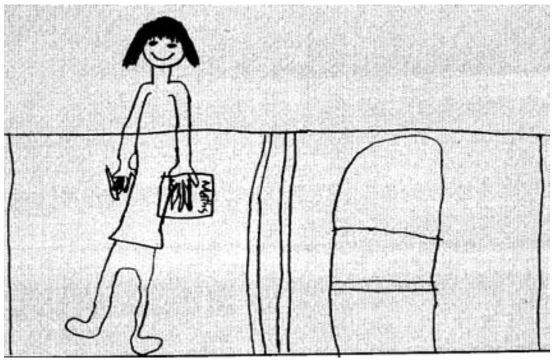


Figure 8. Jessica (Early Year 3).

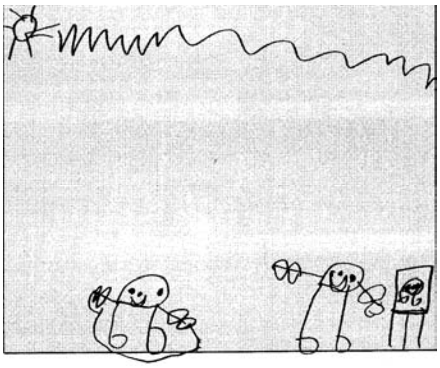


Figure 9. Mitchell (Early Year 3).

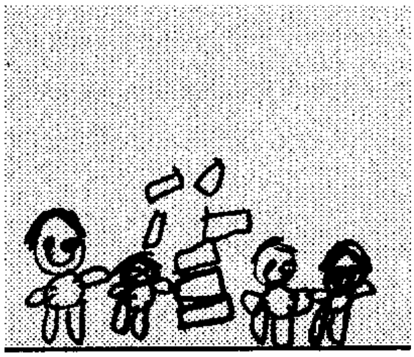


Figure 10. Liam (Early Year 3).

At the beginning of Year 4, the children were again asked to draw themselves during mathematics time (Figures 11 – 20). By this time, Mitchell was able to talk about what happened at mathematics time and how to identify mathematics as a distinct subject as the following conversation shows:

Researcher: How could you show me that you're doing maths on your picture?
Mitchell: I've got a desk.
Researcher: And what's that?
Mitchell: My maths book.
Researcher: And it's got a tick on it, has it?
Mitchell: No, it's a 'seven' [See Figure 11] (Early Year 4)

Figure 11. Mitchell (Early Year 4).

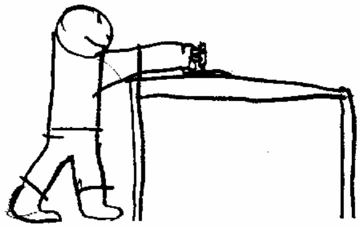


Figure 12. Peter (Early Year 4).

Figure 13. Dominic (Early Year 4).

Figure 14. Liam (Early Year 4).

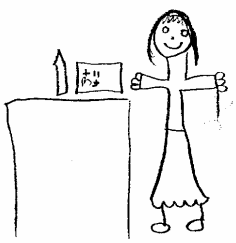


Figure 15. Fleur (Early Year 4).

Figure 16. Rochelle (Early Year 4).

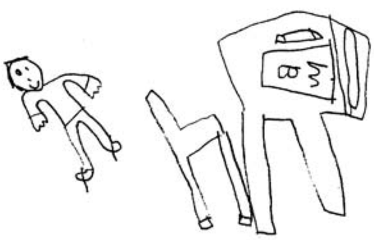


Figure 17. Jared (Early Year 4).

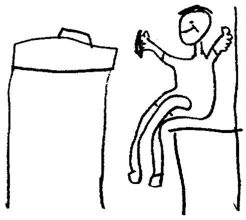


Figure 18. Toby (Early Year 4).

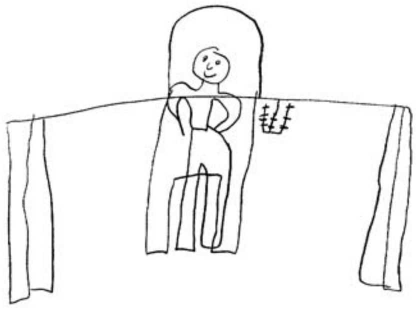


Figure 19. Georgina (Early Year 4).

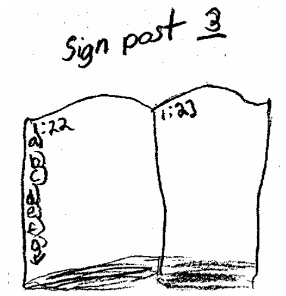


Figure 20. Jessica (Early Year 4).

Although nine of the ten children drew themselves engaged in a writing task, Georgina drew herself with a three-bar abacus, (Figure 19). Earlier in the interview she explained that using the abacus was one of the few mathematics activities she had really enjoyed. The fact that she drew this instead of what usually happened at mathematics time was the result of comments made during the drawing process:

Researcher: Here's a place for drawing a picture of yourself during maths time. So what would you usually do?
Georgina: Shall I draw a table?
Researcher: Yes. *(After Georgina has drawn herself with a big smile)* You're looking pretty happy. *(She has earlier rated herself at only 1.5 out of 10 on the self-rating scale for how happy she feels at maths time)*
Georgina: I'll put the abacus.
Researcher: So what things do you usually do in maths time?
Georgina: Get out our maths books and do our maths. (Early Year 4)

Jessica was not keen to draw herself so she drew her mathematics exercise book *(Figure 20).*

Jessica: Do I have to do it of me? Can I just do it of my maths book?
Researcher: It's hard drawing you is it? (*Jessica nods*) How would you want to draw yourself if you could? How would you imagine yourself, what would you be doing with the maths book?
Jessica: Um, well, what I could do is I could do us standing looking at the maths book and then you could see a little bit of the writing.
Researcher: Sounds great. Away you go.
Jessica: Then it would be the one we work out of. (*Draws the her maths exercise book opened at a page of exercises*)
Researcher: What's the book called?
Jessica: We usually put the label, Signpost 1, Signpost 2.
Researcher: Which one would you usually use?
Jessica: Signpost 3. (*Writes this label above her exercise book.*) (Early Year 4)

Liam's Year 3 and Year 4 pictures differ markedly. Classroom observations revealed why. In Year 3, his teacher conducted an activity-based programme using *Beginning School Mathematics*. Discussion and direct experience with concrete materials were the norm in this classroom, with children recording as necessary on worksheets or paper, while the teacher recorded on a small blackboard. When Liam moved on to Years 4 and 5, mathematics exercise books were introduced and used almost daily, whereas peer collaboration and the use of equipment became less and less frequent.

There was an overwhelming prevalence in the children's representations of "doing maths" as solitary deskwork, with an emphasis on written number tasks, such as completing equations. This distinctive common feature of their drawings indicated that individual written work was repeatedly experienced by the children at maths time, and what they most identified as "doing maths". Observations of mathematics sessions, teachers' and children's descriptions of a typical lesson, and examination of children's mathematics exercise books for evidence of frequency of written tasks, supported these suppositions. Written work as depicted in their drawings was the most common activity experienced by the children at mathematics time. Because of this, the children attached the most significance to it, so that less frequent kinds of mathematics activities such as using equipment for measuring, or gathering statistical data, were considered by the children as less typically "maths". Although the children were regularly seated on the mat at mathematics time either as a whole class listening to the teacher, or in a teacher-guided group learning situation, this did not feature in their drawings, and seldom in their verbal descriptions of doing maths. The teacher is notably absent from all of the children's drawings indicating that "doing maths" was not seen as a partnership between children and teachers.

A cumulative picture of the everyday experience of mathematics was established through the children's descriptions of typical lessons as the following excerpts show.

Fleur: We go into our book. Our green or red books. NCM[textbooks] (Mid Year 5)
Researcher: Does she explain it first or do you just go and do it?
Fleur: She explains it. (Mid Year 5)

Georgina: We get into our groups and do the worksheet. (Mid Year 4)

Jessica: It would usually be out of a textbook and once we've finished that we would do a sheet. (Late Year 5)

Rochelle: A group goes on the mat. Then the group that was on the mat does the group sheet. (Late Year 3)
Rochelle: We do these.*(Shows exercises in her maths book)* (Mid Year 5)

Dominic: Then we do NCM. Do you know what that is?
Researcher: Yes, one of those textbooks.
Dominic: Yeah, or *Figure it Outs*. (Late Year 5)
Jared: The teacher says, 'Go and get your maths books out.' And she writes stuff on the board for maths. (Mid Year 4)

Liam: We do sheets and we work with Miss Peake. (Early Year 3)

Mitchell: You have to sit down and do some times tables or pluses or take away. (Late Year 5)

Peter: Just do worksheets … finishing the worksheets and sticking it into your book. (Late year 4)

Toby: Then we mostly turn to the front of our book and do proper maths. Mrs Kyle gets the questions out of a book, and we have to get the answers.

Teachers' descriptions of an everyday lesson, verified by classroom observation, were consistent with the children's accounts as the following typical account illustrates:

Ms Fell: I'll bring everyone down on the mat and we'll talk about what we're doing that day. If it's something new, quite often we won't be doing anything in our books, we'll be talking about a lot of things, get in a circle, and you know, talk, and then send people off for ten or fifteen minutes to do some work in their books so I can get around and work with people individually … We've just

> purchased halfway through last year, that AWS[1] series of books where there's one for every strand and they've been excellent … we've been able to photocopy off class sets. (Mid Year 4)

From the children's and teachers' descriptions and 95 classroom observations over 3 years, it was found that a high proportion of mathematics time was spent on written tasks in the form of worksheets, textbook pages, or work from the board.

The Typical Maths Lesson: Stories from Secondary School

Six years later in early 2007, having just completed 3 years of secondary schooling, their 11th year at school, and their first major national mathematics exams, the students were again asked to talk about their experiences of learning mathematics including describing a typical mathematics lesson.

> Dominic: Um, well, we sort of learn a new kind of variation of what we were doing like say if we were doing linear equations another like step into it, like, adding brackets or that kind of thing, and then he'll allocate us some questions to you know, and it just gets slightly harder and harder and as soon as you get through and once you're done, usually that's it for the class because it takes us … he'll set about 10 or 15 questions, or so, it takes us the best part of half an hour. Yeah, out of a textbook usually, and whenever we come to a, you know, get stuck, Hans my teacher will go through it on the board and explain it and that kind of thing.

> Jessica: We have a "notes" book and an "exercise" book and we'll come into class and the teacher will be putting up the notes or we'll write up the notes and we'll copy down the notes … then you do a few exercises out of the book or whatever she's set us, there might be like a sheet instead of the exercise book, and then, depending on how difficult it is and stuff like that, we'll either keep doing it for the whole lesson and she'll just write up exercise after exercise and we'll have to do it, or we'll move on and have to write up more notes. And throughout the notes she'll sort of explain it to us and we'll sort of, kinda discuss it and that's where we'll do the questioning and that, discussing and all that and then we do the work. … I've never really thought about it before but it seems like maths might be the one [subject] that's sort of, every lesson's the same, even though the work is different, every lesson's the same and because it's like numbers it seems like it's always the same and when you look at English or Economics or Science you're always doing different topics, and to me maths, even though some of the topics are different is quite repetitive and stuff like that.

> Toby: The teacher gives us notes…if it gets dragged on for a long time it just gets boring.

> Georgina: I get bored having the same. It just gets so repetitive and boring, (I would like) going outside and something and diagrams not just notes all the time.

> Peter: We usually just do exercises and stuff and they tell us the formulas that we need to know and that doesn't change much throughout the year for different things … we've got like a quite a big text book and it just has all the exercises that we do in it and some, like, exam questions and stuff …

> Fleur: Every day it was the textbook … our class is like, for the first like, 20 minutes you just write down notes and then you'd have 20 minutes of doing the work and then you do it at home…3rd and 4th form we did a bit more practical. 5th form was real textbook and notes.

> Rochelle: When we walk into maths it's pretty much the work's on the board or the teacher just says, "Right do this page and when you're finished bring it up or go onto the next page," and stuff like that.

---

[1] AWS : A.W. Stark (1997 –2000) author of mathematics worksheets and teacher guides.

Mitchell: We just like get a bit of paper, a sheet of paper and like just write the answers on the piece of paper.

Researcher: If you drew a picture of yourself doing maths now, what would it look like do you think?
Jared: Um me sleeping on my desk … we had heaps of textbooks and stuff like that … That was boring too.

Liam: We'd just sit down and this year there'd be like a starter on the board just like, 10 questions, not on the same topic, just reminds us... mark those, go over any problems, if there's any problems with homework, just start on the work that we're doing that day and if it's like, a new thing the teacher would explain it on the board and that, if it's the same stuff just get the books out, the homework and work through them.

The students' verbal "pictures" once again placed them alone at desks engaged in written tasks such as taking notes, doing exercises from the textbook and answering and marking questions. Once again, classmates and the teacher are remarkably absent from these pictures. When present in these accounts, teachers are positioned as the setter of work, explainer of rules, formulas and procedures, and the rescuer when students become stuck. Their accounts emphasise the disengagement and boredom created by an unrelenting diet of textbook based written work.

It comes as little surprise then, that Fleur has already decided to drop mathematics as a subject in her penultimate year of schooling, and Jessica would have done so had there been an alternative option. Although Mitchell has been severely alienated and marginalized at school in general, he is continuing to take the Basic Maths option for Year 12, and Georgina the less demanding Mathematics Numeracy option. Rochelle explains that she is pursuing mathematics only as a means of entry into a nursing degree. Liam says he is taking General Maths rather than "higher maths" because his grades were too low. Dominic, who now lives in Melbourne, has decided to drop Maths A and Maths Methods along with his long-held dream of studying for a degree in aviation having been told by his teacher that Year 12 maths will be hard work for him because he lacks natural ability. Toby has made the cut for the "full Year 12 Maths with Algebra" but Peter has just missed out, much to his disappointment. Years of struggling to make sense of mathematics has taken its toll.

Discussion

For the children in this study whose school lives have spanned the years from 1996 to 2007, most have experienced only traditional modes of teaching and learning mathematics. Oakes and Lipton (2003) describe such modes of classroom interaction as follows:

> Most teachers striving for quiet and efficient classrooms organize their instruction to control or minimize activity and social interactions … after a short time in school, students decide that real learning is what they do by themselves … traditional modes of classroom interaction are supported by beliefs that each student must do his or her own learning and that the benefits of education accrue through individual accomplishment. These individualistic practices and norms reflect powerful cultural traditions and learning theories (p. 228).

Teachers are able to maintain tight control when teaching mathematics in this manner delivering powerful messages about what is meant by "doing maths". This management of classroom work is consistent with the observations of Doyle (1988) who described work in mathematics classes as a process in which, "teachers affect tasks, and thus students' learning, by defining and structuring the work that students do, that is, by setting specifications for products and explaining processes that can be used to accomplish work" (p. 169). He argues that much classroom mathematics work is of the structured and familiar variety, and that, "such work creates only minimal demands for students to interpret situations or make decisions within the content domain" (p. 173). Doyle expresses concern about the meaning of the work students do in mathematics classrooms, by arguing that teachers often emphasise production at the expense of understanding, claiming that "meaning itself is seldom at the heart of the work they [students] accomplish" (p. 177). In an earlier study Doyle (1983) explained "doing mathematics" as an induction into the world of academic work. He estimated that "in general, 60 to 70 percent of class time is spent in seatwork in which students complete assignments, check homework, or take tests" (p. 179).

Repeated daily routines are the social means by which we construct our senses of "reality" (Berger & Luckman, 1966; Yackel, 2000). When asked in their recent interviews how learning mathematics might be improved, the students in this study struggled to imagine alternative realities but pinpointed important features of lessons that they wished to change as the following comments illustrate:

Toby: I'm not sure, I don't think so, I think it's pretty good how they teach it here already, it's just a matter of having a good teacher really.

Rochelle: Not old teachers, teaching the old way … they only think the old way's easier because that's the way they were taught it but I think that yeah, we need to know the easiest way.

Jared: Make it more useful in life … then we'd have success because we wouldn't spend so much time working on stuff we don't need.

Dominic: I reckon it's probably smaller class sizes and sort of more emphasis on teacher-to-student relationship kind of thing, rather than just everything you can get your answers out of a textbook and you can get your questions out of a textbook and you can just live off a textbook because a textbook doesn't tell you how to do it, it has a few steps in writing, you know, a textbook doesn't talk back.

Conclusion

Starting from Year 3 of the children's schooling, and increasingly through subsequent years, mathematics exercise books, worksheets, textbooks and questions on the board became the everyday tools of trade for teachers at mathematics time. They represented to teachers and children alike, the solitary nature of "doing" of mathematics. Rather than fostering processes of exploration, experimentation and creativity as suggested in contemporary curricula, these tools obstructed such an approach to the teaching and learning of mathematics.

The sociomathematical worlds of the ten study children were rarely places where mathematics was taught or learned as a process through which ideas and possible solutions might be brainstormed, explored, trialed, presented, evaluated and recorded in a variety of ways. Instead, they were places that fostered a belief that mathematical knowledge and competence was to be gained primarily through conscientious application to solitary written work as defined through the authoritative directives of teacher, textbook and worksheet. Teacher emphasis on desirable work habits such as setting out, neatness, completion, and working "independently" indicated that these skills were highly valued, establishing a work ethic within classroom environments that superseded concerns about children's mathematical understanding. It was assumed that by a certain age, children would benefit from the "structure" of this kind of work.

These taken-for-granted customary practices of teaching and learning mathematics have formed a significant part of the everyday worlds of the children. For them, there has been no other way of "doing maths". As the children have become older, written work has increased, while active exploration and the use of concrete materials all but disappeared. As early as Year 5 use of concrete materials had become largely confined to small group instruction time with the teacher, or abolished altogether for all but the most "needy" of learners. Symbolic and abstract modes of working have been privileged over the use of real objects, working in silence over group discussion, and individual endeavour over collaboration.

For most of these children, the isolation, tedium, and inaccessibility of written mathematics tasks experienced on a daily basis over a long period of time, have been sufficiently off-putting to produce profound feelings of alienation and inadequacy. By upper secondary school, mathematics has become a subject they have chosen to study only as a means to a vocational end. If the experience of these children is typical, mathematics educators must be concerned. Such findings indicate that for many of our young learners "doing mathematics" in the spirit of contemporary curriculum frameworks within which mathematical learning is portrayed as social, dynamic, active, meaningful and purposeful, has failed to become a reality enacted through classroom practice.

References

Berger, P., and Luckmann, T. (1991). *The social construction of reality: A treatise in the sociology of knowledge.* London: Penguin Books. (Original work published 1966)

Blumer, H. (1969). *Symbolic interactionism: Perspective and method.* Englewood Cliffs, NJ: Prentice-Hall.

Boaler, J. (1997). *Experiencing school mathematics: Teaching styles, sex and setting*. Buckingham: Open University Press.

Davis, B. (1996). *Teaching mathematics: Toward a sound alternative*. New York: Garland.

Doyle, W. (1988). Work in mathematics classes: The context of students' thinking during instruction. *Educational Psychologist, 23*(2), 167-180.

Doyle, W. (1983). Academic work. *Review of Education Research*, 53(2), 159-199.

Garden, R. (Ed.) (1997). *Mathematics and science performance in the middle primary school: Results from New Zealand's participation in the Third International Mathematics and Science Study*. Wellington: Ministry of Education, Research and International Section.

Oakes, J., and Lipton, M. (2003). *Teaching to change the world.* New York: McGraw-Hill.

Walls, F. (2001). Children talk about their mathematics lives. In J. Bobis, B. Perry, & M. Mitchelmore (Eds.), *Numeracy and beyond* (Proceedings of the Twenty-fourth annual conference of the Mathematics Education Research Group of Australasia, Vol. 2, pp. 499-506). Sydney: MERGA.

Walls, F. (2003). *Sociomathematical worlds: Investigating children's developing relationships with mathematics.* In L. Bragg, C. Campbell, G. Herbert, & J. Mousley (Eds.), *Mathematics education research: Innovation, networking, opportunity* (Proceedings of the 26th annual conference of the Mathematics Education Research Group of Australasia, Vol. 2, pp. 704-711). Sydney: MERGA.

Yackel, E. (2000). *Creating a mathematics classroom environment that fosters the development of mathematical argumentation.* Paper prepared for Working Group for Action 1, International Congress of Mathematical Education, July 31 – August 6, 2000, Tokyo/Makuhari, Japan.

Yackel, E., & Cobb, B. (1996). Sociomathematical norms, argumentation, and autonomy in mathematics. *Journal for Research in Mathematics Education,* 27(4), 458-477.

The Role of Pedagogy in Classroom Discourse

Margaret Walshaw
Massey University
<m.a.walshaw@massey.ac.nz>

Glenda Anthony
Massey University
<g.j.anthony@massey.ac.nz>

Current curriculum initiatives in mathematics call for the development of classroom communities in which communication about mathematics is a central focus. In these proposals, mathematical discourse involving explanation, argumentation, and defense of mathematical ideas, becomes a defining feature of a quality classroom experience. In this paper we provide a comprehensive and critical review of how mathematics teachers deal with classroom discourse. Synthesising the literature around a number of key themes, we critically assess the kinds of human and material infrastructure that promote mathematical discourse in the classroom and that allow students to achieve desirable outcomes.

Introduction

Classroom mathematical discourse plays a central role in shaping mathematical capability and disposition (Ball, Lubienski, & Mewborn, 2001; Shulman & Shulman, 2004; Stein, 2001). Carpenter, Franke, and Levi (2003) maintain that the very nature of mathematics presupposes that students cannot learn mathematics with understanding without engaging in discussion. Initiatives like Principles and Standards for School Mathematics (PSSM) (National Council of Teachers of Mathematics, 2000) and the Numeracy Development Project (Ministry of Education, 2006) have replaced traditional classrooms by "learners talking to each other, [and] by groups of students voicing their opinions in whole class discussions" (Sfard, Forman, & Kieran, 2001, p. 1). In such classrooms, talking about mathematics becomes acceptable, indeed essential, and mathematical discussion, explanations, and defense of ideas becomes a defining feature of a quality mathematical experience.

In this paper we explore the sorts of pedagogies that, through classroom discourse, contribute to students' active engagement with mathematics. Our starting point is in the acknowledgement that effective classroom discourse is not as easy to implement as is often assumed. Although new initiatives have urged teachers to invite students to "develop explanations, make predictions, debate alternatives approaches to problems … [and] clarify or justify their assertions" (Brophy, 2001, p. 13), implementing such proposals with positive effect is often fraught with problems.

We look at what research has shown about effective classroom discourse and explore how those findings play out within mathematics pedagogy. We do this by critically investigating recent research on quality mathematics classroom pedagogy. Arguably, influences beyond the classroom also have a marked effect on teacher effectiveness and hence on learner outcomes. For example, a number of researchers (see McClain & Cobb, 2004; Millett, Brown, & Askew, 2004) have demonstrated that what is done in classrooms can be attributed in no small way to the human resources provided by others in the school. Other researchers (e.g., Sheldon & Epstein, 2005) have found that effective and sustainable relationships between the home, community, and school, significantly influence classroom teachers' enthusiasm for and success with enhancing learning. Findings, like these, that

Proceedings of the 30th annual conference of the Mathematics Education Research Group of Australasia
J. Watson & K. Beswick (Eds), © MERGA Inc. 2007

point to shared responsibilities and mutual investment in students' well-being, serve to underwrite our discussion on how teachers deal with classroom discourse in a way that enhances desirable student outcomes.

In reporting on the work undertaken on mathematical discourse we have conceptualised teaching as nested within an evolving network of systems. The system itself functions like an ecology, in which the activities of the students and the teacher, as well as the school community, the home, the processes involving the mandated curriculum, and education-at-large, are constituted mutually through their interactions with each other. From a bottom-up vantage point, the classroom is a central pivot within the system and, in this paper, creates the context for our discussion on discourse.

In the next section we outline the method we used to access our data. We then synthesise the literature, organising the discussion around a number of key themes, through which we critically assess the kinds of human and material infrastructure that allow students to achieve mathematical and social outcomes.

Method of Locating and Assembling Data

In this paper our objective is to report findings from research about communication in mathematics classrooms. Our review looks at research that addresses the following question: What are the characteristics of pedagogical approaches to classroom discourse that produce desirable outcomes for diverse students? It draws on data from the Effective Pedagogy in Mathematics/Pāngarau: Best Evidence Synthesis Iteration [BES] (Anthony & Walshaw, 2007). Confining our search to studies undertaken in English-speaking countries, the search took into account relevant publications within the mathematics education literature, first and foremost, and was complemented by general and specialist education literature.

In our first pass through the literature, we noted that many studies offered detailed explanations of student outcomes yet failed to draw conclusive evidence about how those outcomes related to specific teaching practices. Others provided detailed explanations of pedagogical practice yet made unsubstantiated claims about, or provided only inferential evidence for, how those practices connected with student outcomes. These particular studies did not satisfy our selection criteria, precisely because we were searching for studies that offered not just descriptions of pedagogy and outcomes but rigorous explanation for close associations between pedagogical practice and student academic and social outcomes.

Decisions over outcomes were guided by the National Research Council's (2001) understanding of mathematical proficiency. We included conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, and productive disposition. We added to these specific academic outcomes a range of other outcomes that relate to affect, behaviour, communication, and participation.

Included are many different kinds of evidence that take into account human volition, programme variability, cultural diversity, and multiple perspectives. Each study, characterised by its own way of looking at the world, has led to different kinds of truth claims and different ways of investigating the truth. Our assessments about the quality of research depended on the nature of the knowledge claims made and the degree of explanatory coherence between those claims and the evidence provided.

In reviewing the work undertaken in this area, we found that a number of critical aspects of pedagogical practice came to the fore. These included: (a) articulating thinking,

(b) fine-tuning mathematical thinking through language, (c) communicating within multilingual contexts, and d) shaping mathematical argumentation. We use these themes to organise the literature on classroom discourse. Each theme serves as a point of discussion, providing insight into definitions of effective domain-specific pedagogy relating to classroom discourse in mathematics classrooms.

Results

Articulating Thinking

There is now a large body of empirical and theoretical evidence that demonstrates the beneficial effects of participating in mathematical dialogue within the classroom (e.g., Clarke, Keitel, & Shimizu, 2006; Fraivillig, Murphy, & Fuson, 1999; Goos, 2004; Kazemi & Franke, 2004; McClain & Cobb, 2001; Mercer, 2000; O'Connor, 2001; Sfard & Kieran, 2001; White, 2003; Wood, Williams, & McNeal, 2006). However, many of the same researchers who elevate student articulation of mathematics thinking have, simultaneously, cautioned that providing comprehensible explanations about mathematical concepts is essentially a learned strategy. Sfard and Kieran (2001) emphasise that "the art of communicating has to be taught" (p. 70). It is a major challenge to make discourse integral to an overall strategy of teaching and learning.

A number of studies have found that, without pedagogical support, students are often not able to elaborate on their mathematical reasoning. Effective pedagogy focused on support, demands careful attention to students' articulation of ideas. Franke and Kazemi (2001) make the important claim that an effective teacher tries to delve into the minds of students by noticing and listening carefully to what students have to say. Yackel, Cobb, and Wood (1990) provide evidence to substantiate the claim. They report on the ways in which one Year 2 teacher listened to, reflected upon, and learned from her students' mathematical reasoning while they were involved in a discussion on relationships between numbers. Analyses of the discussion revealed that her mathematical subject knowledge and her focus on listening, observing, and questioning for understanding and clarification greatly enhanced her understanding of students' thinking.

Other researchers (e.g., Davies & Walker, 2005; Jaworski, 2004) have also drawn attention to the critical role of the teacher in listening to students and orchestrating mathematical discourse. In a study undertaken within a heterogeneously grouped seventh-grade mathematics classroom, Manouchehri and Enderson (1999) found that the teacher provided responsive rather than directive support, all the while monitoring student engagement and understanding. She did this through careful questioning, purposeful interventions, and with a view towards shifting students' reliance from her, towards the support and the challenge of peers. The teacher's primary objectives were to facilitate the establishment of situations in which students had to share ideas and elaborate on their thinking, to help students expand the boundary of their exploration, and to invite multiple representations of ideas.

Fraivillig, Murphy, and Fuson (1999) reported on the discursive exchange of ideas that took place within a Year 1/2 classroom. What was particularly effective was the way the teacher sustained the discussions. She developed a sensitivity about when to "step in and out" of the classroom interactions and had learned how to resolve competing student claims and address misunderstanding or confusion (theirs and hers). For their part, the students listened to others' ideas and participated in debates to establish common meanings.

Knowing when to "step in" is important for teachers focused on making a difference to students' learning. Turner and colleagues (2002) found that what distinguished high-involvement Year 5 and 6 classrooms was the engagement of the teachers in forms of instruction that allowed them to "step in" at significant moments during classroom discussions. In particular, the teachers negotiated meaning through "telling" tailored to students' current understandings. They shared and then transferred responsibility so that students could attain greater autonomy. In these classrooms, telling was followed by a pedagogical action that had the express intent of finding out students' understandings and interpretations of the given information.

Hill, Rowan, and Ball (2005) have found from observations in their Study of Instructional Improvement that effective practice requires a moment-by-moment synthesis of actions, thinking, theories, and principles. In their Leverhulme Numeracy Research Program, Askew and Millett (in press) observed that pedagogical practice that makes a difference for all learners requires professional reflecting-in-action. In particular, teachers who were able to develop student mathematical understanding applied sound subject knowledge to inform their on-the-spot decision making during classroom interactions. Subject knowledge informed decisions about the particular content that the students would learn, the activities they carried out, how they engaged with the content, and how they conveyed to the teacher their understanding of the content.

Fine-tuning Mathematical Thinking Through Language

Engagement in effective classroom discourse is "a complex process that combines doing, talking, thinking, feeling, and belonging" (Wenger, 1998, p. 56). As we have seen, engagement in discourse that successfully advances students' understanding, demands a respectful exchange of ideas, teacher listening, attentiveness, and reflection-in-action. It also involves familiarising students into mathematical convention. Effective teachers are able to bridge students' intuitive understandings with the mathematical understandings sanctioned by the world at large. Language plays a central role in building these bridges: it constructs meaning for students as they move towards modes of thinking and reasoning characterised by precision, brevity, and logical coherence (Marton & Tsui, 2004). In particular, the teacher who makes a difference for diverse learners is focused on shaping the development of novice mathematicians who speak the precise and generalisable language of mathematics.

McChesney (2005) explored students' contributions in low- and middle-band New Zealand classes at the junior secondary school level. She noted that teachers who established classroom communities, in which there was access to discursive resources, were able to support students' mathematical activity significantly. Her research demonstrated a direct relationship between the quality of teacher/student interaction and students' negotiation of mathematical meaning. The effective teachers in this research were able to set up an environment in which conventional mathematical language migrated from the teacher to the students. Over time, students' contributions, which were initially marked by informal understandings, began to appropriate the language and the understandings of the wider mathematical community. It was through the take-up of conventional language that mathematical ideas were seeded.

Khisty and Chval (2002), among others, have reported that the language that students use derives from the language used by their teacher. Hence the meanings that students construct ultimately descends from those captured through the kind of language the teacher uses. In order to enculturate students into the mathematics community, effective teachers share with their students the conventions and meanings associated with mathematical discourse, representation, and forms of argument.

Competency in mathematics demonstrates control over the specialised discourse (Gee & Clinton, 2000). But the specialised language of mathematics can be problematic for learners. Particular words, grammar, and vocabulary used in school mathematics can hinder access to the meaning sought and the objective for a given lesson. Words, phrases, and terms can take on completely different meanings from those that they have in the everyday context. Sullivan, Mousley, and Zevenbergen (2003) found that students with a familiarity of standard English (usually students from middle-class homes) had greater access to school mathematics. As the teachers in their study said, the students were able to “crack the code” of the language being spoken.

Lubienski (2002), as teacher-researcher, compared the learning experiences of students of diverse socio-economic status (SES) in a seventh-grade classroom. She reported that higher SES students believed that the patterns of interaction and discourse established within the classroom helped them learn other ways of thinking about ideas. The discussions helped them reflect, clarify, and modify their own thinking, and construct convincing arguments. However, in Lubienski’s study, the lower SES students were reluctant to contribute because they lacked confidence in their ability. They claimed that the wide range of ideas contributed in the discussions confused their efforts to produce correct answers. Their difficulty in distinguishing between mathematically appropriate solutions and nonsensical solutions influenced their decisions to give up trying. Pedagogy, in Lubienski’s analysis, tended to privilege the ways of being and doing of high SES students.

Communicating Within Multilingual Contexts

Mathematical language presents difficulties to students, in general, and presents certain tensions in multilingual classrooms, in particular. In our reading of the literature we found a number of studies that had investigated the specific challenges of teaching mathematics in multilingual contexts (Adler, 2001; Khisty, 1995; Moschkovich, 1999). Neville-Barton and Barton (2005) looked at these tensions as experienced by Chinese Mandarin-speaking students in New Zealand schools. Their investigation focused on the difficulties that could be attributable to limited proficiency with the English language. It also sought to identify language features that might create difficulties for students. Two tests were administered, seven weeks apart. In each, one half of the students sat the English version and the other half sat the Mandarin version, ensuring that each student experienced both versions. There was a noticeable difference in their performances on the two versions. On average, the students were disadvantaged in the English test by 15%. What created problems for them was the syntax of mathematical discourse. In particular, prepositions, word order, and interpretation of difficulties arising out of the contexts. Vocabulary did not appear to disadvantage the students to the same extent. Importantly, Neville-Barton and Barton found that the teachers in their study had not been aware of some of the student misunderstandings.

Similar difficulties were made evident in students from Sāmoa and Tonga, in Latu’s (2005) research. Latu noted that English words are sometimes phonetically translated into

Pasifika languages to express mathematical ideas when no suitable vocabulary is available in the home language. The same point was made by Fasi (1999) in his study with Tongan students. Concepts such as "absolute value", "standard deviation", and "simultaneous equations" and comparative terms like "very likely", "probable", and "almost certain" have no equivalent in Tongan culture, whereas some English words, such as "sikuea" (square), have multiple Tongan equivalents.

Fasi (1999) investigated the discursive approaches of two teachers, one Sāmoan and the other Tongan, both of whom had been educated in their native country before moving to New Zealand to complete their higher education. He found that the teachers switched between the language of instruction and the learners' main language in order to explain and clarify the concepts to students. Clarkson (1992) and Setati and Adler (2001) all found evidence of language switching (code switching) for bilingual students, particularly when students could not understand the mathematical concept or when the task level increased. Code switching involved words and phrases as well as sentences and tended to enhance student understanding.

Shaping Mathematical Argumentation

We have now looked at the approaches teachers take to fine tune thinking through language. But mathematical language involves more than technical vocabulary. It also encompasses the way it is used within mathematical argumentation. The positive effects of providing regular opportunities for students to engage in argumentation have been well documented (Carpenter & Lehrer, 1999; Cobb, Boufi, McClain, & Whitenack, 1997; Empson, 2003; Goos, 2004; Kazemi & Stipek, 2001; McClain & Cobb, 2001; O'Connor, 2001; Wood & McNeal, 2003; Zack & Graves, 2002). These researchers have provided evidence that students should have the opportunity and space, for example, to interpret, generalise, justify, and prove their ideas, as well as critique the ideas of others in the class.

Many researchers have found that pedagogical practices that allow students to engage in these activities greatly enhance the development of their mathematical thinking. Such practices also enhance the view that students hold of themselves as mathematics learners and doers. In particular, O'Connor and Michaels (1996) have highlighted the importance of shaping mathematical argumentation by fostering students' involvement in taking and defending a particular position against the claims of other students. They point out that this instructional process depends upon the skilful orchestration of classroom discussion by the teacher. The skill "provides a site for aligning students with each other and with the content of the academic work while simultaneously socialising them into particular ways of speaking and thinking" (p. 65).

As straightforward as it might seem, shaping students' mathematical thinking is, in fact, a highly complex activity. It is complex because teachers and students are "negotiating more than conceptual differences … they are building an understanding of what it means to think and speak mathematically" (Meyer & Turner, 2002, p. 19). Watson (2002) reported that teaching mathematics to low-attaining students in secondary school often involved "simplification of the mathematics until it becomes a sequence of small smooth steps which can be easily traversed" (p. 462). Frequently teachers took the student through the chain of reasoning and students merely filled in the gaps with the arithmetical answer, or low-level recall of facts. This "path smoothing", it was found, did not lead to sustained learning precisely because the strategy deliberately reduced a problem to what the learner could already do, with minimal opportunity for cognitive processing.

Fraivillig and colleagues (1999) observed teachers who did not simplify the task demands. Teachers in their research did more than sustain discussion – they moved conversations in mathematically enriching ways, they clarified mathematical conventions, and they arbitrated between competing conjectures. In short, they picked up on the critical moments in discursive interactions and took learning forward. Hiebert and colleagues (1997) have found that relevant and meaningful teacher responses to student talk involves drawing out the specific mathematical ideas set within students' methods, sharing other methods, and advancing students' understanding of appropriate mathematical conventions. Reframing student talk in mathematically acceptable language provides teachers with the opportunity to enhance connections between language and conceptual understanding.

Zack and Graves (2002) have reported that teachers who develop student argumentation and enhance learning are themselves active searchers and enquirers into mathematics. O'Connor's (2001) classroom research highlighted how one teacher, through purposeful listening, facilitated a group of students towards a mathematical solution. The research students took varying positions towards the solution and attempted to support those positions with evidence. The teacher made her contribution by challenging the students' claims through the use of counter-examples.

Goos (2004) described how a secondary school mathematics teacher developed his students' mathematical thinking through scaffolding the processes of inquiry. The teacher "call[ed] on students to clarify, elaborate, critique, and justify their assertions. The teacher structured students' thinking by leading them through strategic steps or linking ideas to previously or concurrently developed knowledge" (p. 269). In a series of lesson episodes, Goos provided evidence of how the teacher pulled learners "forward into mature participation in communities of mathematical practice" (p. 283), until they were able to engage independently with mathematical ideas.

On other studies Stein, Grover, and Henningsen (1996) and Kazemi and Franke, (2004) have found that a sustained press for justifications, explanations, and meaning, significantly contributed to high-level cognitive activity. When a teacher "presses a student to elaborate on an idea, attempts to encourage students to make their reasoning explicit, or follows up on a student's answer or question with encouragement to think more deeply" (Morrone, Harkness, D'Ambrosio, & Caulfield, 2004, p. 29), the teacher is not only providing an incentive for students to enrich their knowledge, but also socialising them into a larger mathematical world that honours standards of reasoning and rules of practice. In effect, by participating in a "microcosm of mathematical practice" (Schoenfeld, 1992), students are learning how to appropriate mathematical ideas, language, and methods and how to become apprentice mathematicians.

Conclusion

This review represents a systematic and credible evidence base about quality discourse in mathematics classrooms and explains the sort of pedagogical approaches that lead to improved engagement and desirable outcomes for learners from diverse social groups. Our search through the literature focused attention on different contexts, different communities, and to multiple ways of thinking and working. The evidence drew on the histories, cultures, language, and practices found in mathematics classroom contexts and considered a range of research evidence irrespective of regardless of methodological approaches.

Our focus on classroom discourse and scaffolding of student engagement has revealed specific pedagogical skills, knowledges and dispositions that make a difference to all students. These pedagogical factors shape how, and with what effect, mathematics is taught and learned. Student outcomes are contingent upon them, not as single entities, but as interrelated contingencies. Although our review has surveyed the literature on mathematics classroom discourse, it is important to note that classroom discourse will gain positive effect only when there is a strong cohesion between all the various elements of a teacher's work. In other words, the facilitation of productive classroom discourse is part of a larger matrix of the effective teacher's repertoire that allow students to develop habits of mind to engage with mathematics productively and to make use of appropriate mathematical tools to support understanding.

Our review has deepened our understanding of mathematics discursive practices in many ways. Teachers who set up communities of practice that are conducive to classroom discussion, come to understand their students better. Students benefit too and the ideas put forward in the classroom become rich resources for knowledge. Through students' purposeful involvement in discourse, through listening respectfully to other students' ideas, through arguing and defending their own position, and through receiving and providing a critique of ideas, students enhance their own knowledge and develop their mathematical identities. Teachers who are able to provide such contexts simultaneously increase students' sense of control, and develop valuable student mathematical dispositions.

Acknowledgment. The authors wish to acknowledge the Ministry of Education (New Zealand) for funding the project on which this paper is based.

References

Adler, J. (2001). *Teaching mathematics in multilingual classrooms*. Dordrecht: Kluwer.

Anthony, G., & Walshaw, M. (2007). *Effective pedagogy in Mathematics/Pāngarau: Best evidence synthesis iteration [BES]*. Wellington: Ministry of Education.

Askew, M., & Millet, A. (in press). Changing teaching in Year 4: The impact of a national intervention. In M. Brown, M. Askew, & A. Millet (Eds.), *Learning about number: Interactions and outcomes in primary classrooms.* Dordrecht: Kluwer.

Ball, D. L., Lubienski, S. T., Mewborn, D. S. (2001). Research on teaching mathematics: The unsolved problem of teachers' mathematical knowledge. In V. Richardson, V. (Ed.), *Handbook of research on teaching* (4th ed., pp. 433-456). Washington, DC: American Educational Research Association.

Brophy, J. (2001). Introduction. In J. Brophy (Ed.), *Subject-specific instructional methods and activities* (pp. 1-23). Amsterdam: JAI.

Carpenter, T., & Lehrer, R. (1999). Teaching and learning mathematics with understanding. In E. Fennema & T. Romberg (Eds.), *Mathematics classrooms that promote understanding* (pp. 19-32). Mahwah, NJ: Lawrence Erlbaum Associates.

Carpenter, T., Franke, M., & Levi, L. (2003). *Thinking mathematically:Iintegrating arithmetic and algebra in elementary school.* Portsmouth: Heinemann.

Clarke, D., Keitel, C., & Shimizu, Y. (2006). (Eds.). *Mathematics classrooms in twelve countries: The insider's perspective.* Rotterdam: Sense Publishers.

Clarkson, P. (1992). Language and mathematics: A comparison of bilingual and monolingual students of mathematics. *Educational Studies in Mathematics 23*(4), 417-430.

Cobb, P., Boufi, A., McClain, K., & Whitenack, J. (1997). Reflective discourse and collective reflection. *Journal for Research in Mathematics Education, 28*(3), 258-277

Davies, N., & Walker, K. (2005). Learning to notice: One aspect of teachers' content knowledge in numeracy classrooms. In P. Clarkson, A. Downton, M. Horne, A, McDonough, & R. Pierce (Eds.), *Building connections: Theory, research and practice* (Proceedings of the 28th annual conference of the Mathematics Education Research Group of Australasia, Vol. 1, pp. 273-280). Sydney: MERGA.

Empson, S. (2003). Low performing students and teaching fractions for understanding: An interactions analysis. *Journal for Research in Mathematics Education 34*(4), 305-343.

Fasi, U. M. L. (1999). *Bilingualism and learning mathematics in English as a second language in Tonga.* Unpublished master's thesis, Reading, England.

Fraivillig, J. L., Murphy, L. A., & Fuson, K. C. (1999). Advancing children's mathematical thinking in everyday mathematics classrooms. *Journal for Research in Mathematics Education, 30*(2), 148-170.

Franke, M. L., & Kazemi, E. (2001). Teaching as learning within a community of practice: Characterising generative growth. In T. Wood, B. S. Nelson-Scott, & J. Warfield (Eds.), *Beyond classical pedagogy: Teaching elementary school mathematics* (pp. 47-74). Mahwah, NJ: Lawrence Erlbaum Associates.

Gee, J. P., & Clinton, K. (2000). An African American child's science talk: Co-construction of meaning from the perspectives of multiple discourses. In M. A. Gallego & S. Hollingsworth (Eds.), *What counts as literacy: Challenging the school standard* (pp. 118-138). New York: Teachers College Press.

Goos, M. (2004). Learning mathematics a classroom community of inquiry. *Journal for Research in Mathematics Education, 35*(4), 258-291.

Hiebert, J., Carpenter, T., Fennema, E., Fuson, K. C., Wearne, D., Murray, H., Olivier, A., & Human, P. (1997). *Making sense: Teaching and learning mathematics with understanding*. Portsmouth, NH: Heinemann.

Hill, H., Rowan, B., & Ball, D. (2005). Effects of teachers' mathematical knowledge for teaching on student achievement. *American Educational Research Journal, 42*(2), 371-406.

Jaworski, B. (2004). Grappling with complexity: Co-learning in inquiry communities in mathematics teaching development. In M. Hoines & A. Fuglestad (Eds), *Proceedings of the 28th annual conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 17-36). Bergen: PME:

Kazemi, E., & Franke, M. (2004). Teacher learning in mathematics: Using student work to promote collective inquiry. *Journal of Mathematics Teacher Education, 7*(3), 203-235.

Kazemi, E., & Stipek, D. (2001). Promoting conceptual thinking in four upper-elementary mathematics classrooms. *The Elementary School Journal, 102*(1), 59-80.

Khisty, L. L. (1995). Making inequality: Issues of language and meaning in mathematics teaching with Hispanic students. In W. Secada, E. Fennema, & L. B. Adajian (Eds.), *New directions for equity in mathematics education* (pp. 279-297). Cambridge: Cambridge University Press.

Khisty, L., & Chval. K. (2002). Pedagogic discourse and equity in mathematics: When teachers' talk matters. *Mathematics Education Research Journal, 14*(3), 154-168.

Latu, V. (2005). Language factors that affect mathematics teaching and learning of Pasifika students. In A. D. P. Clarkson, D. Gronn, M. Horne, & A. McDonough, & R. Pierce. (Eds.), *Building connections: Research, theory and practice* (Proceedings of the 28th annual conference of the Mathematics Education Research Group of Australasia, pp. 483-490). Sydney: MERGA.

Lubienski, S. T. (2002). Research, reform, and equity in US mathematics education. *Mathematical Thinking and Learning, 4*(2&3), 103-125.

Manoucheri, A., & Enderson, M. (1999). Promoting mathematical discourse: Learning from classroom examples. *Mathematics Teaching in the Middle School, 4*(4), 10-19.

Marton, F., & Tsui, A. (2004). *Classroom discourse and the space of learning*. Mahwah, NJ: Lawrence Erlbaum Associates.

McChesney, J. (2005, April). *Student-initiated interactions: Student agency in mathematics classrooms.* Paper presented at the American Educational Research Association annual meeting, Montreal, Canada.

McClain, K., & Cobb, P. (2001). An analysis of development of sociomathematical norms in one first-grade classroom. *Journal for Research in Mathematics Education, 32*(3), 236-266.

McClain, K., & Cobb, P. (2004). The critical role of institutional context in teacher development. In M. Hoines & A. Fuglestad (Eds.), *Proceedings of the 28th annual conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 281-288). Bergen: PME.

Ministry of Education. (2006). *Numeracy Development Projects.* Wellington: Learning Media.

Mercer, N. (2000). *Words and minds*. London: Routledge.

Meyer, D. K., & Turner, J. C. (2002). Using instructional discourse analysis to study the scaffolding of student self-regulation. *Educational Psychologist, 37*(1), 17-25.

Millet, A., Brown, M., & Askew, M. (Eds.). (2004). *Primary mathematics and the developing professional.* Dordrecht: Kluwer Academic Publishers.

Morrone, A., Harkness, S., D'Ambrosio, B., & Caulfield, R (2004). Patterns of instructional discourse that promote the perception of mastery goals in a social constructivist mathematics course. *Educational Studies in Mathematics, 56*, 19-38.

Moschkovich, J. (1999). Supporting the participation of English language learners in mathematical discussions. *For the Learning of Mathematics, 19*, 11-19.

National Council of Teachers of Mathematics (2000). *Principles and standards for school mathematics*. Reston, VA: Author.

National Research Council. (2001). *Adding it up: Helping children learning mathematics*. Washington, DC: National Academy Press.

Neville-Barton, P., & Barton, B. (2005*). The relationship between English-language and mathematics learning for non-native speakers. Teaching and Learning Research Initiative (TLRI) Final Report.* Accessed August 2005 from http://www.tlri.org.nz/index.html

O'Connor, M. C. (2001). "Can any fraction be turned into a decimal?" A case study of the mathematical group discussion. *Educational Studies in Mathematics, 46*, 143-185.

O'Connor, M. C., & Michaels, S. (1996). Shifting participant frameworks: Orchestrating thinking practices in group discussion. In D. Hicks (Ed.), *Discourse, learning and schooling* (pp. 63-103). New York: Cambridge University Press.

Schoenfeld, A. H. (1992). Learning to think mathematically: Problem solving, metacognition, and sense making in mathematics. In D. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 334-370). New York, Macmillan.

Setati, M., & Adler, J. (2001). Code-switching in a senior primary class of secondary-language mathematics learners. *For the Learning of Mathematics, 18*, 34-42.

Sfard, A., Forman, E., & Kieran, C. (2001). Guest editorial. Learning discourse: Sociocultural approaches to research in mathematics education. *Educational Studies in Mathematics, 46*(1-3), 1-11.

Sfard, A., & Kieran, C. (2001). Cognition as communication: Rethinking learning-by-talking through multi-faceted analysis of students' mathematical interactions. *Mind, Culture, and Activity, 8*(1), 42-76.

Sheldon, S., & Epstein, J. (2005). Involvement counts: Family and community partnerships and mathematics achievement. *The Journal of Educational Research, 98*(4), 196-206.

Shulman, L., & Shulman, J. (2004). How and what teachers learn: A shifting perspective. *Journal of Curriculum Studies, 36*(2), 257-271.

Stein, M. (2001). Teaching and learning mathematics: How instruction can foster the knowing and understanding of number. In J. Brophy (Ed.), *Subject-specific instructional methods and activities* (Vol. 8, pp. 111-144). Amsterdam: JAI.

Stein, M., Grover, B., & Henningsen, M. (1996). Building student capacity for mathematical thinking and reasoning: An analysis of mathematical tasks used in reform classrooms. *American Educational Research Journal, 33*(2), 455-488.

Sullivan, P., Mousley, J., & Zevenbergen, R. (2003). Being explicit about aspects of mathematics pedagogy. In N. A. Pateman, B. Dougherty, & J. Zilliox (Eds.), *Proceedings of the 27th annual conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 267-273). Honolulu: PME & PMENA.

Turner, J. C., Midgley, C., Meyer, D. K., Gheen, M., Anderman, E. M., & Kang, J., et al. (2002). The classroom environment and students' reports of avoidance strategies in mathematics: A multimethod study. *Journal of Educational Psychology, 94*(1), 88-106.

Watson, A. (2002). Instances of mathematical thinking among low attaining students in an ordinary secondary classroom. *Journal of Mathematical Behavior 20*, 461-475.

Wenger, E. (1998). *Communities of practice: Learning, meaning and identity.* Cambridge: Cambridge University Press.

White, D. (2003). Promoting productive mathematical classroom discourse. *Journal of Mathematical Behavior 22,* 37-53.

Wood, T., & McNeal, B. (2003). Complexity in teaching and children's mathematical thinking. N. Pateman, B. Dougherty, & J. Zilliox (Eds.), *Proceedings of the 27th annual conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 435-441). Honolulu: PME.

Wood, T., Williams, G., & McNeal, B. (2006). Children's mathematical thinking in different classroom cultures. *Journal for Research in Mathematics Education, 37*(3), 222-252.

Yackel, E., Cobb, P., & Wood, T. (1990). The interactive constitution of mathematical meaning in one second grade classroom: An illustrative example. *Journal of Mathematical Behaviour, 11*(2), 469-488.

Zack, V., & Graves, B. (2002). Making mathematics meaning through dialogue: "Once you think of it, the z minus three seems pretty weird." *Educational Studies in Mathematics, 46*, 229-271.

Australian Indigenous Students: The Role of Oral Language and Representations in the Negotiation of Mathematical Understanding

Elizabeth Warren
Australian Catholic University
<E.Warren@mcauley.acu.edu.au>

Janelle Young
Australian Catholic University
<J.Young@mcauley.acu.edu.au>

Eva deVries
Independent Schools Queensland

This paper reports on a small pilot study conducted in an Indigenous P-13 school in North Queensland. This pilot study occurred over a two day period with the specific aim of exploring the role of oral language and representations in negotiating mathematical understanding. Implications are drawn for the implementation of a large study, commencing in 2007 with 4-year-old Indigenous students as they transition from home to school. All students in this context either speak Aboriginal English or Creole as their first language. The pilot study occurred in two classrooms, one with 15 Year 6/7 students and the other with fourteen Years 4/5/6 students. The preliminary results indicate that explicit consideration needs to be given to the development of precise mathematical language, strategies for linking school mathematics to home environments, the use of questioning in establishing classroom discourse, and the recognition that many of these classrooms are bilingual.

Introduction

This paper reports on a small pilot study that occurred at the commencement of a new project to be conducted in four schools in North Queensland. The main project, a longitudinal study, aims to explore the role that oral language and representations play in assisting Indigenous students reach an understanding of white mathematics, with a particular focus on Prep students as they transition into school from home. One of the schools, an Indigenous school has 465 students, with nearly all the students being either Indigenous Australians or from the Torres Strait Islands. The pilot study was conducted with older students and their teachers in this school with a specific aim of exploring oral language, representations and understanding mathematical concepts, drawing initial implications for the main project.

Many researchers have found there is a mismatch of conditions for learning for young Indigenous Australian children as they enter school (Bliss, 2004; Dunn, 1999; Simpson & Clancy, 2005; Simpson, Munns, & Clancy, 1999; Zevenbergen, 2000). Tension still exists between policy and suggested strategies for Indigenous students. The reality of responding to cultural differences and practices and adjusting the interactions and strategies for teaching and learning in classrooms is still far from ideal (Simpson & Clancy, 2005). The use of spoken language in school and the types of interactions teachers utilize can either advantage or disadvantage Indigenous Australian students. Furthermore, the importance of spoken language as the foundation for all learning is often not fully recognized and many young Indigenous Australian children are not able to make a strong start in the early years of schooling as the discourses of the family often do not match that of the school (Cairney, 2003). This mismatch of home and school language has been shown to disadvantage Indigenous students' achievements in literacy and numeracy in the long term (Dickinson, McCabe, & Essex, 2006; MCEETYA, 2004). Understanding and accepting

Aboriginal English (AE) as a dialect of spoken English used by most Aboriginal and Torres Strait Islander people is vital and knowing that there are variations across particular communities is important (Haig, Konisberg, & Collard, 2005). While Standard Australian English (SAE) is the discourse of the school, and it is conjectured that teachers need to create a bridge for young Indigenous students between AE and SAE as they grapple with both the new language and new concepts little is known about what this means in practice.

Patterns of classroom interactions have been shown to disadvantage some students particularly the interaction of teacher questioning as Indigenous students do not commonly experience this type of interaction at home or within their community (Galloway, 2003; Haig, Konisberg, & Collard, 2005). Unjustified blame has been placed upon Indigenous students in the past and absenteeism, disadvantaged social background and culture have all been viewed as contributing factors (Bourke & Rigby, 2000). This is seen as irresponsible (Cooper, Baturo, Doig, & Warren, 2004). Insufficient consideration has been given to the complexities that confront young Indigenous students as they enter school. Educators have not lifted the blame and given sufficient positive consideration to ways of adapting the conditions for learning for these students to prepare them for success rather than failure. Thus the dominant view of society in blaming aspects of culture, disadvantage and maintaining low expectations needs to be turned around so that a positive framework can be adopted in order to improve the educational outcomes for Indigenous Australian students (Matthews, Howard, & Perry, 2003; Sarra, 2003).

Theoretical Frameworks

Various broad theoretical fields are relevant in addressing the issues related to this research, for example, situated cognition (Kirshner & Whitson, 1997; Lave & Wenger, 1991; Watson, 1998), and cultural models (Holland & Quinn, 1987). As the focus of this pilot study was on two particular aspects of classroom interactions, namely, oral language and mathematical representations, the frameworks chosen in this initial study reflect these dimensions. The initial lenses chosen to view the classroom discourse were Duval's representations and Peirce's semiotics.

Duval (2002) argues that mathematics comprehension results from the coordination of at least two representational forms or *registers*; the multifunctional registers of *natural language*, and *figures/diagrams*, and the mono-functional registers of *notation systems* (symbols) and *graphs*. He contends that learning involves moving from *treatments* where students stay within one register (e.g., carrying out calculations while remaining strictly in the one notation system) to *conversions* where students change register without changing the objects being donated (e.g., passing from natural language of a relationship to using letters to represent it) and finally to *coordination* of registers. He argues that learning also requires building understanding of the mathematical processing performed in each register (Duval, 1999). One theory relating to communication in the classroom is semiotics.

The epistemological stance taken in this analysis is the science of semiotics; a means of addressing signs, their connections and meanings. In this instance signs refer to external representations. Presmeg (1997) suggests that when one recognizes the structure of the system he or she engages in, explains this structure to others by such means as encoding it in a diagram or applying some overarching framework, then mathematics exists. So while semiotics is commonly used to construct links between cultural and historical practices and mathematics (Presmeg, 1997; Radford, 1997) it also assists us to understand classroom discourse in mathematics (Saenz-Ludlow, 2001; Warren, 2003). Sign interpretation is a personal process with some students being unable to move beyond the physical characteristics of the sign (the external representation). Peirce (1960) believes that the sign relation is inherently triadic, linking an object, a representation and an interpretation so that the object determines the representation and in turn determines the interpretation. Semiosis involves the process of going beyond particular signs to more and more complex representations incorporating new signs and generalizations (Peirce, 1960); an evolving process. Vygotsky regarded signs as tools that were capable of influencing one's inward behaviour and the behaviour of another. Thus the teaching and learning process can be seen as a process of semiosis where the teacher and students become both contributors and interpreters.

Methods

Participants

This paper reports on how students and teachers use the language of mathematics and representations in their mathematical learning. The school chosen for this study is a P-13 school; a large boarding school catering exclusively for Indigenous and Torres Strait students. This school prides itself in offering quality education for Indigenous students in far north Queensland. In 2006 47% of Year 3 students, 69% of Year 5 students, and 17% of Year 7 students achieved above the national benchmark for numeracy. In addition, approximately 30 students successfully completed Year 12. Two teachers, David and Melissa, volunteered to participate in this pilot study. David teaches 15 Year 6/7 students whose ages range from 10 years to 12 years with eight being Australian Indigenous, 6 from Torres Strait and 1 from Papua New Guinea. Melisa's class consisted of 14 Year 3/4/5 students, with eight being Australian Indigenous and six of Torres Strait Island origin. Both of these teachers had been working in these types of environments for up to 5 years and were perceived by both the school community and local educational consultants as exemplary teachers of Indigenous students.

Data Sources and Analysis

The data was gathered from three main sources, namely, (a) open ended interviews with the two teachers before the teaching began (Pre Interview), (b) videotapes of two lessons especially constructed by the teachers to illustrate the adaptations they made to their teaching in these environments when teaching mathematics, and (c) a reflective interview with each teacher at the end of the teaching episode (Post Interview). All lessons were videotaped and field notes were taken. At the completion of the lessons, the researcher and teacher reflected on the researcher's field notes, endeavouring to minimise the distortions inherent in this form of data collection, and arrive at some common perspective of the instruction that occurred and the thinking exhibited by the students participating in the classroom discussions. The video-tapes were transcribed. The videos and participant observation scripts served to provide insights to the learning of the community and particularly identifying specific actions, specific use of representations and conversations that supported this learning.

Results and Discussion

Pre-interview

Both teachers commented on the difficulties they experienced on a day to day basis in these environments. These related to the language difficulties that they experienced, the need to relate all their examples to relevant real world contexts, the use of a variety of visual aides needed to allow access to the ideas, and the tension between what they perceived as "talking about mathematics in Australian Indigenous English" and precise mathematical language, for example, using "big" and "big up" for tall and taller, and the need to ensure that Indigenous Australian children had the opportunity to communicate in "proper mathematical language". This last issue relates to a notion of empowerment. They believed that "setting the benchmarks" too low was in fact an act of "keeping Indigenous Australians in their own class, denying them the opportunity to move out of their low socioeconomic circumstances and act as "activists for real social change". Both presented two lessons that they believed exhibited these characteristics. They perceived that teaching in these classrooms required a high use of oral language, hands on experiences, a range of representations and an ability to continually adapt the learning trajectory to maximise access of the participants to the mathematical concepts. The data reported in this paper is one excerpt from the Year 3/4/5 classroom and one short excerpt chosen from the Year 6/7 classroom. The first illustrates the use of different representations and contexts to assist students solve a problem involving comparing the heights of two children, and the second illustrates students "code switching" as they engage in an activity involving calculating volumes of a variety of shapes made from blocks. Figure 1 illustrates the particular representations utilised by Melissa as she discussed the problem with the students.

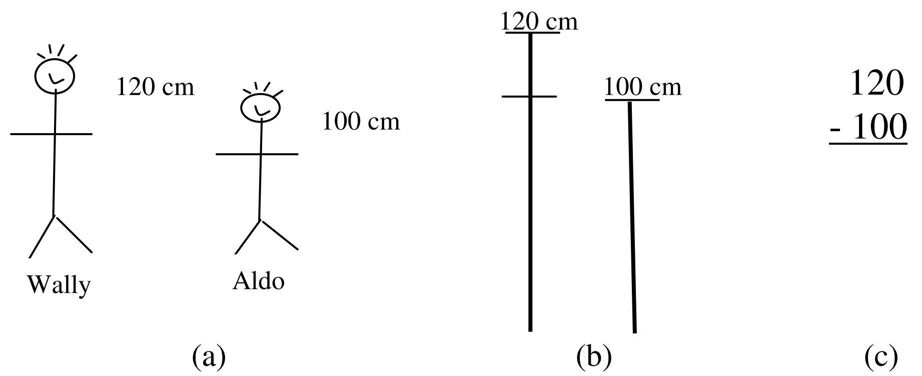


Figure 1. Diagrams drawn on the board at different stages during the discussion.

Excerpts from the Year 3/4/5 Classroom (Melissa's Classroom)

T:	Wally was 120 cm tall. (Both Wally and Ado are children in the class).
Children:	OOOHH.
T:	Here is Wally. Now Ado, he's a little bit younger so he is a little bit shorter. Ado was 100 cm tall. [Draws Figure 1 (a) on the Board] How much taller, listen carefully to the question. How much taller was Wally than Ado? How much taller was he than Ado? Think about it very very carefully. How much taller? [Paused]

T: We sometimes say what is the difference between them.
C1: 220cm.
T: That would be if he jumped up on his head.
[Gesturing the action of one jumping up on top of the other]
T: That how much they would be altogether. How much taller?
T: Here is 100cm which might be about here. [Marking off in the air 100cm with her left hand]
T: Wally is 120cm tall which might be about here. How much taller? [Gesturing 120 cm as a point above 100 cm and using both hands to focus their attention on the gap].
T: What is that difference between 100 and 120cm. What is that difference in there?
[Moving both hands backwards and forwards to emphasis the focus is on the gap between the two hands] Do you know?
C2: It could be 100 and something.
T: No that is an excellent go though. What is the difference between 100 and 120? What is that difference in there? How much is it Do you know Marley?
T: Lets look at this way. We have 120 cm is up here and 100cm is to here?
[Draws Figure 1 (b) on the board].
What is that difference in there? This is 100cm. What is the difference in between there?
[Pointing the difference between the two heights].
T: What is the difference in there? How many marks are between there?
C3: 50
T: No it's not 50.
C4: 100
T: No it is not 100. Think about it carefully. How many points go in between there and there. Very, very tricky. Think it about carefully.
C4: 10
C5: .8
C6: [shouted out] Miss 20
T: This is an easy way of doing this. We can do the difference between something by doing a take away. 120 take away 100
C7: 2
C8: 200
T: Lets think of it this way if you had 120 dollars and you took away 100 dollars how much is left. 120 dollars and you gave away 100. How much is left?
[Gestures with her closed fists the action of take away and then draws Figure 1 (c) on the board].
C9: 120

At this stage nearly all the class were whispering 20.
Children in unison: 20
Melissa then worked through the algorithm with them.

From a semiotic perspective the *object* is considered to be the beginning task, namely, "If Wally is 120 cm tall and Ado is 100 cm tall, how much taller is Wally than Ado" and the *signs* are the various representations that assisted in understanding the object. The *interpreters* were the students themselves. Melissa continually adjusted her representations as a response to students/ interpretations. The first representation (Figure 1(a)) did not seem to be interpreted by students as a difference representation, hence the introduction of the gesture, showing that the focus was on the difference between the children's two heights. This was further represented as a diagram with horizontal bars used to again focus attention on the difference (see Figure 1(b)). As Melissa proceeded along this trajectory she also changed the object itself from a comparison problem to a subtraction problem (by introducing the language of difference and then take away). Finally, she switched into the context of money thus the original object changed from *how much taller is Wally than Ado* to *if you had 120 dollars and gave away 100 dollars how much is left.* This process illustrates a common strategy used in many Indigenous classrooms, the context of money as a bridge to understanding mathematics. While the students successfully answered this problem, does this assist them in reaching an understanding of the original problems and do they see the analogy between each? This needs further research. Also another common characteristic of this conversation was the lack of ongoing dialogue about the problem itself. The students volunteered answers (which were often incorrect) but there was no ongoing conversation about their thinking. One concern that these teachers had was the "shame factor". Melissa was aware that Indigenous students do not like being asked questions in front of the whole class, and especially did not like their incorrect answers to be pursued, hence her continual positive reinforcing comments, such as, "good try" as the lesson proceeded. In some instances it appears that students are unable to go beyond the written mark; the literal interpretation.

The inherent triadic nature of sign relations (object, representations and interpretation) are exhibited in this research. The tasks presented in this research induce an interaction between these three dimensions but in this instance whether the interplay between different signs and their interpretations bring deeper meaning to the object itself is the key question. The use of gesturing was also explicit throughout the lesson. In fact the role of gesturing within a culture with a strong oral history, may in fact prove to be an important representation in the interpretation process. Recent research has evidenced that children are significantly more likely to reiterate the teacher's spoken strategy when it is produced in conjunction with gestures that conveyed the same strategy than when it is produced with no gestures at all (Goldin-Meadow, 2006).

From Duval's perspective, most of this lesson occurred within the mono-functional register, the use of language and diagrams to represent the problem at hand. This is considered to be an easier process than crossing across registers. While this framework indicated that the lesson was situated in a register which was considered to be "cognitively easier" the register gives little insight into how to work effectively within each or the role of gestures in creating meaning. This requires further research.

Excerpt from the Year 6/7 Classroom

The second expert was chosen for inclusion in this paper as it demonstrates students "code switching" as they interacted in the classroom context. The lesson began with a general discussion about what we mean by the term volume, how it differs from capacity, and the processes commonly used to calculate the volume of a three-dimensional cuboid. The students were then split into three rotational groups. The following except is from a conversation between an Australian Indigenous student and a Torres Strait Islander student.

C1: (Singing out loud in own language)
C2: You killed it
C1: You starting dissing each other
C2: You were going to start dissing, then they're going to start dissing and then your going to diss them
C1: Hello, Miss where are you from?
R: I am from Brisbane and where are you from?
C1: No, I'm from, I'm born in Rockhampton but I rear up in Yarrabah
C2: How many are there? [referring to the diagram of cubes]
C1: Twenty-four, yes that's right. 1, 2, 3, 4, 5, 6, 7, is that seven? Yep, it's seven. Twenty-four and I still need to do this one. [counting up the cubes in the diagram]

This short extract illustrates a typical conversation that occurred in the classroom. As the students worked and conversed with each other they continually switched between their own languages, but when it came to discussing mathematical concepts they expressed their ideas using the language of mathematics. It is conjectured that a possible reason for this is that their own language lacks the specific vocabulary needed to describe these mathematical situations.

Post Interview

The reflections at the end of the lessons between the teachers and the researcher focused on four broad themes, all of which impact on our main study. First, there are tensions between all the languages that exist in these situations and the need to pave the way to high levels of achievements in mathematics. There were at least two different languages in these classrooms, Australian Indigenous English and Creole. Both teachers, while they knew something about these languages felt that both languages lacked aspects that assisted them in working in a mathematical environment. For example, there appeared to be little attribute language in their home language. For length the predominant comparative words were "big", "bigger up", "small", and "boney". Hence, they felt a need to ensure that their lessons provided opportunities for Indigenous students to learn about and use the explicit language of mathematics. Second, there are culturally different styles in communication between home situations and school situations, especially when it came to direct questioning. Past research has evidenced that if an Indigenous student cannot answer the question then they experience a feeling of "shame", especially if they are singled out in front of others. Hence in both instances classroom discourse tended to avoid probing "incorrect thinking". Third, Indigenous students' engagement increases if the examples are related to their world and the approach is very hands on. Melissa commented that she always endeavoured to use the students themselves as the context she used when discussing mathematical ideas, hence the choice of Wally and Aldo for her comparative measurement problem. Fourth, given that their culture's communication is based on oral language there is a reluctance to "write" things down. All of these impacted on how both teachers conducted their lessons.

Summary and Implications

This pilot study begins to tease out particular issues that need to be taken into consideration as young Indigenous students move from a home environment to a school environment. The first implication for the main project is the need to explicitly link home environment to school environment, with the specific aim of allowing young Indigenous students access to white mathematics. The theoretical frameworks provided for this analysis give some insights into the classroom discourse. In the case of Melissa's class semiosis assisted in viewing the classroom interchange as consisting of three main dimensions, namely, object, representations, and interpretations. It also assisted in documenting how she changed the representations to assist the students reach some meaning about the object. But in this instance it was a backward mapping, starting with school and working back to home and the context of money. For the main project a more appropriate framework could be the notion of semiotic chaining, a means of building links between cultural practices and the teaching and learning of mathematics in school (Presmeg, 2005), an example of which was given by Walkerdine (1988) in her seminal work on mother – daughter relationships in the home environment. Semiotic chaining exemplifies the notion of layering to abstraction where the object and sign relationship build from the concrete to the abstract by the sign itself taking on the role of the "new object" for each subsequent layer (Presmeg, 2005). In this instance the initial object is situated in the home environment (e.g., guests coming to visit) and the final object is in the school environment (e.g., whole number). The impact of this framework on Indigenous learning needs further investigation.

The second implication is the recognition that Indigenous classrooms are bilingual and their home language, while sounding like English is in fact different from Australian Standard English. The two instances reported in this paper show that in their home language there is a lack of the vocabulary commonly used to describe mathematical situations (e.g., the lack of attribute language and the need to switch to mathematical code when describing mathematical situations). While this has been recognised as a problem in past research, there is a paucity of research focusing on the development of mathematical language with Indigenous students and its impact on mathematical achievement.

The third issue relates to the type of classroom discourse and choice of representations used to explore mathematical concepts. In particular, what style of discourse encourages students to engage in classroom discussions about mathematics concepts? How do we walk between the idea of justification and cultural notion of shame? What role do gestures have in supporting a culture based on an oral language tradition?

Although there is some recognition that many Indigenous students have English as a second language, their educational outcomes indicate there is still room for improvement. It is well recognised that oral communication is dominant in the lives of these students and that their experience with print and other literacies is often limited. By building on the oral language strengths of young Indigenous Australian students, the main study seeks to bridge the gap between home and school and assist students to enhance achievement in both literacy and numeracy. This pilot study reported in this paper begins to map the territory and provide indicators for the road ahead. As such, the research recognises the considerable capabilities of young Indigenous Australian students as they commence school and aims to assist them to engage in meaningful dialogue concerning literacy and numeracy in order to meet the challenge of improving long-term educational outcomes.

References

Bliss, J. (2004). Applying a cross-cultural framework to help Indigenous students learn to read and write. In B. Bartlett, F. Fryer & D. Roebuck (Eds.), *Conference Proceedings: Education: Weaving Research into Practice*, Volume 1, (pp.103-112). Retrieved May 9, 2006, from Informit database.

Bourke, P. C. J., & Rigby, K. (2000). *Better practice in school attendance: Improving the school attendance of indigenous students*. Canberra, ACT: DETYA.

Cairney, T. (2003). Literacy within family life. In N. Hall, J. Larson & J. Marsh (Eds.), *Handbook of early childhood literacy* (pp. 85-98). London: Sage Publications.

Cooper T., Baturo, A. Doig, S., & Warren E. (2004). Young white teachers' perceptions of mathematics learning of aboriginal and non-aboriginal students in remote communities. In M. Hoines & A. Fuglestad (Eds.), *Proceedings of the Twenty Eighth Conference of the International Group for the Psychology of Mathematics Education*, Volume 1, (pp. 239-246). Bergen: Bergen University College

Dickinson, D., McCabe, A., & Essex, M. J. (2006). A window of opportunity we must open to all: The case for preschool with high-quality support for language and literacy. In D. Dickinson & B. Neuman (Eds.), *Handbook of early literacy research*: (Volume 2, pp. 11-28). New York: The Guilford Press.

Dunn, M. (1999). Literacy development in an Aboriginal community summary of a research project: Literacy practices for diverse needs. *Australian Journal of Language and Literacy*, *22*(2), 103-119.

Duval, R. (1999). Representations, vision and visualization: Cognitive functions in mathematical thinking. Basic issues for learning. In F. Hitt & M. Santos (Eds.), *Proceeding of the Twenty first[t] Conference of the North American Chapter of the International Group for the Psychology of Mathematics Education.* Volume 1, (pp. 3-26). Morelos, Mexico.

Duval, R. (2002). The cognitive analysis of problems of comprehension in the learning of mathematics. Paper presented at *the Semiotics Discussion Group: 25th annual conference of the International Group for the Psychology of Mathematics Education*. Norwich, UK: PME.

Galloway, A. (2003). Responding to responses: Interaction between Indigenous Australian students and their non-Indigenous teachers. In P. L. Jefferies (Ed.) *Joint Australian Association for Research in Education and New Zealand Association for Research in Education 2003 International Education Research Conference*, Auckland, NZ. Retrieved April 25, 2006, from: http://www.aare.edu.au/03pap/gal03580.pdf

Goldin-Meadow, S. (2006). Talking and thinking with our hands. *Current Directions in Psychological Science, 15,* 34 - 39.

Haig, Y., Konisberg, P., & Collard, G. (2005). Teaching students who speak Aboriginal English. *PEN 150,* 1-12.

Holland, D., & Quinn, N. (Eds.) (1987). *Cultural models in language and thought.* Cambridge, England: Cambridge University Press.

Kirshner, D., & Whitson, J. A., (Eds.) (1997). *Situated cognition: Social, Semiotic and psychological perspectives.* Mahwah, NJ: Lawrence Erlbaum.

Lave, J., & Wenger, E. (1991). *Situated learning: Legitimate peripheral participation.* Cambridge, England: Cambridge University Press.

Matthews, S., Howard, P., & Perry, B. (2003). Working together to enhance Australian Aboriginal students' mathematics learning. In L. Bragg, C. Campbell, G. Herbert, & J. Mousley (Eds.). *Proceedings of the 26th annual conference of the Mathematics Education Research Group of Australasia*. Sydney: MERGA.

Ministerial Council on Education, Employment, Training, and Youth Affairs [MCEETYA] (2004). *Preliminary paper national benchmark results - Reading, writing and numeracy - Years 3, 5 and 7.* Retrieved May 1, 2005, from http://www.mceetya.edu.au/verve/_resources/ANR2004BmrksFinal.pdf

Peirce, C. S. (1960). *Collected papers*. Cambridge, Mass: Harvard University Press.

Presmeg, N. (2005). *The role of culture on the teaching and learning of mathematics.* Retrieved March 20 2007, from http://merg.umassd.edu/projects/symcog/bibliography/

Presmeg, N. C. (1997). A Semiotic Framework for Linking Cultural Practice and Classroom Mathematics. In J. Dossey, J. Swafford, M. Paramantie & A. Dossey (Eds.), *Proceedings of Nineteenth Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education,* Volume 1, (pp. 151-156). Columbus, Ohio: ERIC Clearinghouse for Science, Mathematics and Environmental Education.

Radford, L. (1997). On psychology, historical epistemology, and the teaching of mathematics: Towards a socio-cultural history of mathematics. *For the Learning of Mathematics, 17*(1), 26-33.

Saenz-Ludlow, A. (2001). Classroom mathematics discourse as an evolving interpreting game. In M. Anderson, A.Saenz-Ludlow, S.Zellweger & V.Cifarelli (Eds.), *Educational Perspectives on Mathematics as semiosis: From thinking to interpreting to knowing,* pp 253-284. Ottawa, Canada: Legas Press.

Sarra, C. (2003). Young and black and deadly: Strategies fro improving outcomes for Indigenous students. In Australian College of Educators, *Quality Teaching Series Practitioner Perspectives.* Retrieved May 12, 2006 from http://education.qld.gov.au/community/events/showcase/symposium/2005/pdf/chrissarra-conferencepaper.pdf

Simpson, L., & Clancy, S. (2005). Enhancing opportunities for Australian Aboriginal literacy learners in early childhood settings. *Childhood Education, 81*(6), 327-342.

Simpson, L., Munns, G., & Clancy, S. (1999). Language tracks: Aboriginal English and the classroom. *PEN 120*. Primary English Teaching Association.

Walkerdine, V. (1988). *The mastery of reason: Cognitive developments and the production of rationality.* New York: Routledge.

Warren, E. (2003). Language, arithmetic and young children's interpretations. *Focus on Learning Problems in Mathematics. 25*(4), 22-35.

Watson, A. (Ed.) (1998). *Situated cognition and learning of mathematics.* Oxford, England: University of Oxford Department of Educational Studies.

Zevenbergen, R. (2000). Language implications for numeracy: A study of language use of disadvantaged students. In *Improving numeracy learning*: What does the research tell us?. *Proceedings of the* Council for Educational Research conference 2000, Brisbane, October 15-17 (pp. 18-14). Retrieved June 1, 2007 from http://www.acer.edu.au/research_conferences/2000.html

Student Change Associated with Teachers' Professional Learning

Jane Watson
University of Tasmania
<Jane.Watson@utas.edu.au>

Kim Beswick
University of Tasmania
<Kim.Beswick@utas.edu.au>

Natalie Brown
University of Tasmania
<Natalie.Brown@utas.edu.au>

Rosemary Callingham
University of New England
<Rosemary.Callingham@une.edu.au>

Teachers and students in nine rural Tasmanian schools have been associated with a research project providing professional learning for teachers in mathematics in a reform-based learning environment. Students completed surveys to measure attitudes and mathematics skills and understanding late in 2005 and late in 2006. Teachers completed profiles late in 2005 and participated in professional learning activities from then throughout 2006. The professional learning program is described and change in student attitudes and performance reported.

The MARBLE project began in mid-2005, the acronym standing for "Mathematics in Australian Reform-Based Learning Environments." The aim of the project is to provide negotiated professional learning opportunities for a group of rural middle school teachers that will enhance the outcomes of their students in relation both to the quantitative literacy needs of today's society and to the opportunity to study further mathematics and contribute to innovation in Australia. The project reported on initial data collected from teachers and students in relation to beliefs and attitudes (Beswick, Watson, & Brown, 2006) and to performance on a mathematical task (Watson, Beswick, & Brown, 2006). Brown, Watson, Beswick, and Fitzallen (2006) also provided details of the overall teacher profile outcomes. The purpose of this paper is to report on professional learning program and the resulting student change following the first year of the project.

Professional learning program. All professional learning programs for teachers are limited to some extent by available resources and although this project was funded by the Australian Research Council, the Department of Education Tasmania (DoET), and the Catholic Education Office Hobart (CEO), care had to be taken to use resources carefully. Research elsewhere had suggested that important features of programs were:

> (a) ongoing (measured in years) collaboration of teachers for purposes of planning with (b) the explicit goal of improving students' achievement of clear learning goals, (c) anchored by attention to students' thinking, the curriculum, and pedagogy, with (d) access to alternative ideas and methods and opportunities to observe these in action and to reflect on the reasons for their effectiveness. (Hiebert, 1999, p. 15)

These features are related to Shulman's (1987a, b) seven types of teacher knowledge required for successful teaching – content knowledge, general pedagogical knowledge, curriculum knowledge, pedagogical content knowledge, knowledge of learners and their characteristics, knowledge of education contexts, and knowledge of education ends, purposes, and values – as well as to Hill, Rowan, and Ball's (2005) more recent focus on "Teachers' knowledge for teaching mathematics." Fitting all of these aspects into the time and resources was the challenge faced.

In particular in Tasmania, the Essential Learnings Framework (DoET, 2002; 2003) was the backdrop into which the professional learning was to fit in 2005. This curriculum framework, underpinned by a set of values and purposes, identified 18 Key elements within five Essential Learnings (Thinking, Communicating, Social Responsibility, World Futures and Personal Futures). Although the position of traditional Key Learning Areas (KLAs) was not specifically addressed in the framework, “Being Numerate” was identified as a key element in the Communicating Essential. This shift in emphasis recognised “Being Numerate” as a cross-curricular understanding and coincided with an increased focus on pedagogy and collaborative practice across the curriculum. Contemporaneously, a set of defined outcomes and standards (DoET, 2003) was produced for each key element. “Being Numerate” was one of the first against which teachers reported, in 2005.

Considerable professional learning to support teachers’ adoption of the reforms was provided through the Department of Education. This included appointment of curriculum and assessment leaders in schools/clusters, printed and on-line material (planning proformas, exemplar units, and work samples to guide assessment). Much of the professional learning was generic, with only three curriculum officers working with a “Being Numerate” focus across the state. Face-to-face professional learning in this element was therefore limited and dependent on individual schools or clusters adopting a numeracy focus. To assist in addressing this issue, the “Being Numerate” team developed an extensive on-line resource for teachers (DoET, 2007a).

In 2006, amid controversy over the implementation of the Essential Learnings, the incoming Minister for Education announced that there would be a new curriculum in Tasmanian schools. The Tasmanian Curriculum would be a refinement to “make it easier to understand, and more manageable for teachers and principals” (DoET, 2007b, para 1). An initial draft was circulated to stakeholders and following a consultation period the refined framework consisting of eight areas was announced. Mathematics/Numeracy became a defined area against which both primary and secondary teachers are required to report. Information and Communications Technology (ICT) was embedded in all curriculum areas (DoET, 2007b).

The MARBLE project provided an opportunity for two clusters of Tasmanian schools to have an intensive focus on numeracy in addition to the other professional learning that was taking place. Although this project was firmly grounded in the context of curriculum reform, specific content and pedagogical content knowledge in the area of numeracy were identified foci. Professional learning literature then informed the planning process. For example, Schifter (1998) found that engaging teachers with the content of the mathematics curriculum that they taught, in ways that challenged and deepened their own mathematical understandings, was effective in assisting them to make changes to their classroom practice. Hawley and Valli (1999) asserted that teachers should be involved in the identification of what they need to learn and the process to be used and that collaborative problem solving should be included.

In December 2003, the Australian Councils of the Deans of Education and the Deans of Science issued a draft report on professional learning in science, mathematics, and technology in Australia. The report lamented the lack of systematic evaluation of student outcomes and of improvements in teacher confidence and knowledge as a result of professional learning experiences (p. 43). Burkhardt and Schoenfeld (2003) further made a direct call for more extensive, evidence-based measures of outcomes to be developed to satisfy stake-holders, including politicians. These evaluations became among the aims of the MARBLE project with a specific focus of the research to evaluate whether the professional learning made an impact on teachers and students with respect to teaching and learning of Mathematics. This paper reports on the results of student surveys that included items to measure both attitude and mathematics performance, in terms of skills and understanding.

Attitudes to mathematics. The term attitude is used to describe an evaluative response to a psychological object (Ajzen & Fishbein, 1980) and hence individuals' attitudes to mathematics refer to their evaluation of mathematics. Hannula (2002) separated such evaluations of mathematics into four categories, namely: emotions experienced during mathematical activity; emotions triggered by the concept of mathematics; evaluations of the consequences of doing mathematics; and the perceived value of mathematics in terms of an individual's overall goals. Of course, these are dependent upon such things as the nature of the mathematical activity engaged in at the time, the aspects of mathematics being considered or what is believed to comprise mathematics, and expectations for the future in terms of mathematics. This means that an individuals' response to written items aimed at assessing their attitude to mathematics is likely to reflect rather transient states. Other authors have also described the multidimensionality of attitude in terms of dichotomous evaluations. These include: confidence or anxiety (Ernest, 1988); like or dislike; engagement or avoidance; high or low self efficacy; and beliefs that mathematics is important or not important, useful or useless, easy or difficult (Ma & Kishor, 1997), and interesting or not interesting (McLeod, 1992). There are connections between these eight dimensions and Hannula's (2002) categories but they tend to emphasise emotional reactions less.

The Program for International Student Assessment (PISA) (2003) incorporated measures of affect and their influence on mathematical literacy (Thomson, Creswell, &, De Bortoli, 2004). Thomson et al. (2004) found that for Australian 15-year-olds, mathematics self-efficacy and self-concept had the greatest impact on mathematical performance of all of the variables considered, and that anxiety about mathematics was negatively related to performance in the subject. In addition, students' inclination to engage in mathematics is likely to influence their decisions about pursuing the subject beyond the school years in which it is compulsory and hence is a likely contributor to the declining enrolments in tertiary mathematics in many countries (Boaler & Greeno, 2000). A decline in attitude to mathematics with increasing grade level was also been noted by Boaler and Greeno, (2000) and some evidence suggesting that this might apply particularly to students' inclination to engage with the subject, to like it, and to find it interesting was presented by Beswick et al. (2006).

Mathematical performance of students. Analysis of curriculum documents and previous research highlighted the mathematical concepts associated with the middle school that are the foundation for the quantitative literacy skills needed by all students and for the formal mathematical content of algebra, geometry, probability, and statistics needed by innovators in mathematics science and technology. The five concepts identified as forming a foundation to these understanding were Number Sense, Proportional Reasoning, Measurement, Uncertainty, and Relationships. These dual purposes, everyday numeracy and formal mathematics that pose a challenge for teachers and curriculum designers, are recognised in the Essential Learnings framework:

> Being numerate involves having those concepts and skills of mathematics that are required to meet the demands of everyday life. It includes having the capacity to select and use them appropriately in real life settings. Being truly numerate requires the knowledge and disposition to think and act mathematically and the confidence and intuition to apply particular principles to everyday problems.
> … Access to higher levels of abstract symbolic operation opens new ways of thinking and future academic and vocational pathways. (DoET, 2002, p. 21)

This extract echoes the work of Steen (2001), who sees quantitative literacy as an integral component of all mathematics curricula. Appreciating the purposes and applications of the mathematical thinking they have developed within the formal mathematics curriculum is seen as a critical need for elite students as well as those who will not go on to study higher levels of mathematics.

Methodology

The research was conducted in two rural clusters in different parts of Tasmania, comprising eight DoET schools and one CEO school. The professional learning program involved middle years (grades 5-8) teachers.

Sample. The survey was directed at students in Grades 5 to 8. Due to students entering and leaving schools, and progressing to higher grades in 2006, not all students had survey results for both years. Table 1 contains the number of students in each year and the number of repeating students. Although all schools were asked to administer surveys to all students whose teachers took part in the MARBLE project, there are some missing data from some schools.

Table 1
Number of Students in Each Grade Each Year (repeating student numbers in parenthesis)

Year	Grade 5	Grade 6	Grade 7	Grade 8
2005	182	220	181	128
2006	138	168 (141)	154 (144)	102 (94)

Survey items. The student surveys included items to measure both mathematical performance and attitude towards mathematics. In terms of mathematical performance, the survey was written to reflect the five foundation concepts identified in the literature. Of the 35 distinct items forming 17 questions on the initial student survey, there was overlap in terms of items reflecting these concepts. Fifteen items had links to two concepts with the coverage being 15 items on Number Sense, 6 items on Proportional Reasoning, 7 items on Measurement, 10 items involving Uncertainty, and 12 involving Relationships. The items had various sources including Watson and Callingham (2003), Callingham and Griffin (2000) and Department of Education, Community and Cultural Development (1997). Student outcomes for one of the problems based on fractional parts of a nebulous whole were discussed in Watson et al. (2006). Items were scored using scoring rubrics adapted from the original sources.

The subsequent student survey administered 12 months later contained eight items in common with the initial survey and 18 other items, providing a total of 13 items on Number Sense, 6 on Proportional Reasoning, 2 on Measurement, 7 on Uncertainty, and 5 on Relationships. This included three items that linked to three concepts and one item that linked to two. The change in emphasis reflected student outcome levels from the initial surveys and teacher intervention (through the professional learning program) in 2006.

Consistent with the study of Beswick et al. (2006) 16 items to measure attitude were included comprising two statements from each of the eight identified dimensions, to which respondents indicated the extent of their agreement on 5-point Likert scales ranging from Strongly agree to Strongly disagree.

Procedure. The outcomes from the 2005 student survey were reported to the teachers in the project at the beginning of 2006 and specific interventions were initiated by the teachers working in school-based groupings. The disappointing survey outcomes related to number sense and basic proportional reasoning in 2005 led to adopting more work with these concepts at the beginning of the year and less work with the other foundation concepts. Also relevant to these outcomes are the professional learning activities offered to the teachers during the final term of 2005 and throughout 2006. These are summarized briefly in Table 2. Professional learning was delivered in two ways. Whole of cluster sessions were combined with case studies, where each school was assigned a researcher to be involved in a project of its own choice. All schools except one completed a case study, which were reported to the Management Committee of the project at the end of 2006. These varied greatly in the degree of intervention by researchers and the quality of the outcomes. Brown, Rothwell, and Taylor (in press) reported on one case where teachers negotiated with researchers to develop a framework for the teaching of numeracy, drawing on curriculum support materials and teachers' understanding of the school context.

Table 2
Summary of Professional Learning Activities for Teachers

Focus of Professional Learning	*Mathematical content knowledge*	*Pedagogical content knowledge*	*Knowledge of students as learners*	*Curriculum knowledge*
Whole of Cluster Professional Learning	Fractions Measurement Ratio Problem solving Tinkerplots (Data collection, handling, representation, interpretation, evaluation) Mental computation Place value Accuracy Space Decimals Percentages Proportional reasoning Quantitative literacy (in media)	Fractions Pi Chance and Data; (Designing surveys, collecting data, representing data, interpreting data) Problem solving Numerate language Mental computation strategies	Division Fractions Applying rubrics to students' responses Progression statements	Coordinating the mathematics curriculum Assessment: Formative and summative including use design and use of rubrics Making inter-disciplinary connections with science; SOSE Planning units of work – connecting understanding goals with teaching, learning and assessment
School Case Studies	Tinkerplots Constructing a school scope and sequence Student produced resource kits	Mental computation strategies Tinkerplots Developing conceptual understanding of fractions	Mental computation and problem solving strategies	Implementing an Inquiry Whole-school numeracy audit

Analysis of attitudes. The 16 items related to attitude to mathematics were common to the student surveys administered on both occasions. Paired sample *t*-tests were used to compare the responses of those students who completed the survey on both occasions. Effect sizes were also calculated as described by Burns (2000). The eight pairs of items relating to the each of the identified aspects of attitude in the literature were also combined and the totals similarly compared. In all cases scoring was reversed for negatively worded items so that a higher score represented a more positive response.

Analysis of mathematical thinking. The data from the mathematics tasks were analysed using the Rasch Partial Credit Model (Masters, 1982) with Quest computer software (Adams & Khoo, 1996). A set of 8 link items common to both administrations was identified, and these items provided an anchor set that established the difficulties of the items at each test administration relative to each other (Griffin & Callingham, 2006). Estimates of person ability were identified for each student in both 2005 and 2006, anchored to the same set of link item difficulties so that genuine comparisons could be made. The performance of students in each grade was summarised for each year of the project. These measures provided a comparison of performance by grade. Also, summaries from students who completed both tests provided a measure of growth across time.

Results

Attitudes to mathematics. Table 3 shows changes in the mean responses of students who responded to the 16 attitude items included in the survey in both 2005 and 2006. Five of the changes were statistically significant and in each case the change was negative and the effect size was very small.

Table 3
Changes in Responses to Attitude Items from 2005 to 2006 (Negative statements in italics)

Attitude item	Mean 2005 (n=378)	Mean 2006 (n=378)	Diff. 2006-2005	Std Dev.	Sig. (2-tailed)	Effect size
1. I find maths an interesting subject.	3.56	3.39	-0.17	1.16	0.004**	0.15
2. Other subjects are more important than maths.	3.12	3.11	-0.01	1.32	0.866	0.00
3. I plan to do as little maths as possible when I get the choice.	3.45	3.54	0.09	1.37	0.202	0.07
4. I really do <u>not</u> enjoy maths lessons.	3.49	3.46	-0.04	1.36	0.571	0.03
5. I find most problems in maths fairly easy.	3.27	3.10	-0.17	1.20	0.005**	0.14
6. Maths helps to develop my mind and teaches me to think.	3.94	3.92	-0.02	1.16	0.689	0.02
7. Maths we learn at school is important in everyday life.	4.20	4.26	0.06	1.09	0.256	0.06
8. Maths makes me feel nervous and uncomfortable.	3.62	3.58	-0.03	1.31	0.609	0.02
9. Maths is a dull and uninteresting subject.	3.54	3.51	-0.04	1.35	0.594	0.03
10. I enjoy attempting to solve maths problems.	3.60	3.48	0.12	1.19	0.048*	0.10
11. The problems in maths are nearly always too difficult.	3.60	3.55	-0.04	1.03	0.395	0.04
12. I usually keep trying with a difficult problem until I have solved it.	3.79	3.67	0.11	1.11	0.052	0.10
13. I don't do very well at maths.	3.43	3.19	-0.24	1.14	0.000**	0.21
14. Having good maths skills will <u>not</u> help me get a job when I leave school.	4.34	4.33	-0.01	1.35	0.849	0.01
15. Most of the time I find maths problems too easy and unchallenging.	2.65	2.37	-0.28	1.17	0.000**	0.24
16. I don't get upset when trying to work out maths problems.	3.71	3.75	-0.03	1.54	0.665	0.02

*$p<0.05$. ** $p<0.01$.

Table 4 shows the changes in aggregated means for each of the eight aspects of attitude that underpinned the design of the items, and for total attitude. As expected on the basis of the individual items in Table 3, what statistically significant changes there were, were negative and effect sizes were again small.

Table 4
Changes in Responses to Attitude Dimensions and Total Attitude from 2005 to 2006

Attitude dimension (Item numbers in Table 1)	Mean 2005 (n=378)	Mean 2006 (n=378)	Diff. 2006-2005	Std Dev.	Sig. (2-tailed)	Effect size
Mathematics is interesting (1& 9)	7.10	6.89	-0.20	2.08	0.051	0.10
Mathematics is important (2 & 7)	7.31	7.37	0.05	1.74	0.555	0.03
Inclination to engage with mathematics (3 & 12)	7.24	7.21	-0.21	1.91	0.830	0.11
Liking for mathematics (4 & 10)	7.09	6.93	-0.16	2.11	0.137	0.08
Self-efficacy in relation to mathematics (5 & 13)	6.69	6.29	-0.41	1.80	0.000**	0.23
Mathematics is useful (6 & 14)	8.28	8.24	-0.37	1.90	0.705	0.19
Confidence in relation to mathematics (8 & 16)	7.33	7.33	0.00	2.12	1.000	0.00
Mathematics is easy (11 & 15)	6.24	5.92	-0.32	1.68	0.000**	0.19
Total Attitude (all items)	57.29	56.18	-1.13	8.63	0.013*	0.13

*p<0.05. ** p<0.01.

Mathematical thinking. Figure 1 shows the change in performance between like grades in each year of the project. The pattern of achievement across the grades is mixed. Although there is a general increase in performance as students move through school, within grades only Grade 7 shows a significant improvement from 2005 to 2006 (t = 2.01; df = 312; p = 0.045). It does seem that MARBLE has been somewhat more effective in addressing the primary/high school transition than at the other grade levels.

Figure 2 shows the growth over time of students who entered MARBLE in Grades 5, 6 and 7. When this growth was considered by comparing achievement in the lower grade with the same students' achievement in the higher grade, all improvements were significant. This is not unexpected due to the general cognitive development as students move through school. In terms of the rate of growth, those students who began the project in Grade 5 had a higher growth rate than students who started in either Grade 6 or Grade 7, who showed a very similar trajectory.

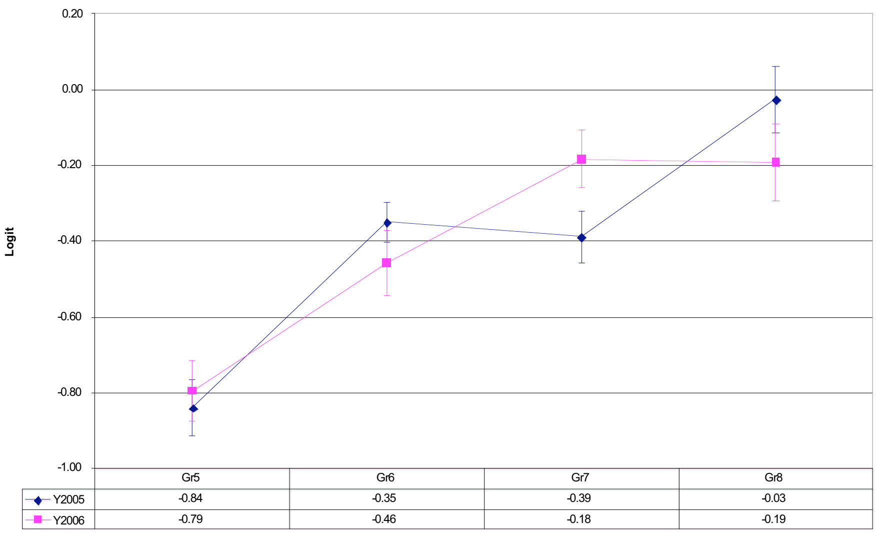


Figure 1. Change in performance by grade over time.

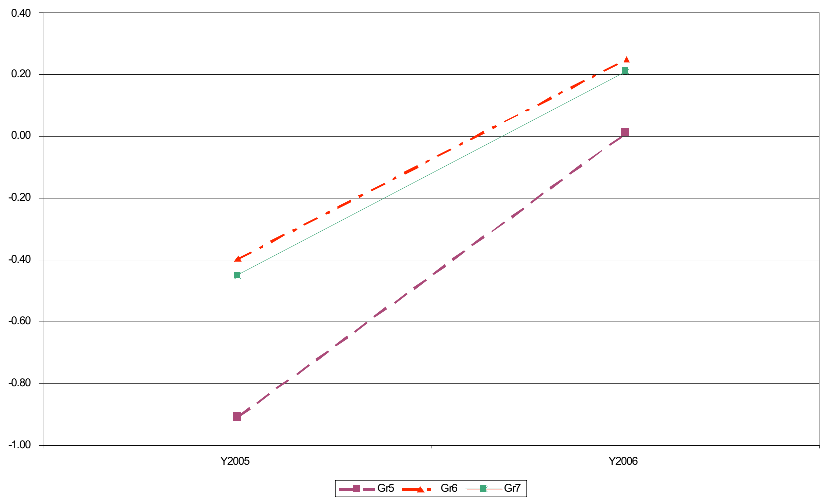


Figure 2. Growth over time by start grade in 2005.

Discussion

Although small, the direction of the changes in students' attitudes is disappointing. It seems likely that what we are observing is the previously noted deterioration of attitude to mathematics with year level (Beswick et al., 2006; Boaler & Greeno, 2000). Although linked (Thomson et al., 2004), the direction of causation between attitude towards, and achievement in mathematics is unclear with meta-analyses resulting in conflicting conclusions (compare Ma & Kishor, 1997 and Ma & Xu, 2004). In this study, the focus was very much upon improving teaching in the expectation that this would result in improved achievement and more positive attitudes to mathematics. A further possible explanation for these results lies in the transient and multifaceted nature of attitude to mathematics. In particular some aspects of attitude, particularly emotive responses (Hannula, 2002), are not readily accessible via written means.

The mathematical thinking outcomes were also disappointing across cohorts in the same grades, except for Grade 7. It is interesting, however, to note that to some extent the lack of improvement of performance at the high school transition, as noted for example by Callingham and McIntosh (2002) and Watson and Kelly (2004), was tempered, with improvement from Grade 6 to Grade 7. The stationary level of performance in 2006, of Grade 7 and Grade 8, was disappointing but it reflected a similar relationship of the Grade 6 and Grade 7 students in the previous year. This appears to reflect cohort differences in these grades.

Limitations. Several issues may have had an impact on the follow-up surveying of MARBLE project students after one year. The uncertainty associated with the curriculum and eventual change was distracting for many teachers and this was expressed at several of the professional learning sessions. Although the feedback from teachers following the professional learning sessions was positive, at times it was the impression of the authors that teachers were challenged by the topics covered (see Table 2) and may have been hesitant to implement them fully in their classrooms. There was also concern expressed by some teachers that the students were reluctant to try to the best of their ability in 2006 because the surveys did not count for their school assessment.

Implications. The outcomes from the 2006 student surveys were reported to teachers representing each of the nine schools at the beginning of 2007. At the meetings teachers were again, as in the previous year, asked to contribute to the planning in order to improve students' outcomes at the end of 2007. They were positive about the influence of the individual school case studies and wished to continue them as well as to work across schools within the clusters on topics of special interest at various grade levels. Taking into account the comments of Hiebert (1999) on the importance of sustained professional learning for teachers over time, it is hoped that another year will produce the desired outcomes.

Acknowledgement. The research was funded by Australian Research Council grant number LP0560543.

References

Adams, R. J., & Khoo, S. T. (1996). *Quest: Interactive item analysis system. Version 2.1* [Computer software]. Melbourne: Australian Council for Educational Research.

Ajzen, I., & Fishbein, M. (1980). *Understanding attitudes and predicting social behaviour*. Englewood Cliffs, NJ.: Prentice-Hall.

Australian Council of Deans of Education & Australian Council of Deans of Science. (2003). *Professional learning for enhancing teaching and learning within science, mathematics and technology in Australia.* [Draft report] Canberra: Author.

Boaler, J., & Greeno, J. G. (2000). Identity, agency and knowing in mathematics worlds. In J. Boaler (Ed.), *Multiple perspectives on mathematics teaching and learning* (pp. 171-200). Westport, CT: Ablex.

Beswick, K., Watson, J., & Brown, N. (2006). Teachers' confidence and beliefs and their students' attitudes to mathematics. In P. Grootenboer, R. Zevenbergen, & M. Chinnappan (Eds*.), Identities, cultures and learning spaces* (Proceedings of the 29 annual conference of the Mathematics Education Research Group of Australasia, Canberra, pp. 68-75). Adelaide, SA: MERGA.

Brown, N., Rothwell, P., & Taylor, S. (in press). Collaborative development of a Framework for Numeracy: A case study. In K. Milton & H. Reeves (Eds.), *Proceedings of the Biennial Conference of the Australian Association of Mathematics Teachers*. Adelaide: AAMT, Inc.

Brown, N., Watson, J., Beswick, K., & Fitzallen, N. (2006). *Numeracy in a reform-based learning environment*. Refereed paper presented at the annual conference of the Australian Association for Research in Education, Adelaide, December, 2006. Available at http://www.aaare.edu.au/06pap/bro06369.pdf

Burkhardt, H., & Schoenfeld, A.H. (2003). Improving educational research: Toward a more useful more influential, and better-funded enterprise. *Educational Researcher, 32*(9), 3-14. Retrieved 4 February, 2004 from http:// www. aera.net/pubs/er/eronline.htm

Burns, R.B. (2000). Introduction to research methods. French's Forest, NSW: Longman.

Callingham, R., & Griffin, P. (2000). Towards a framework for numeracy assessment. In J. Bana & A. Chapman (Eds.), *Mathematics Education beyond 2000* (Proceedings of the 23rd Annual Conference of the Mathematics Education Research Group of Australasia, pp. 134-141). Sydney: MERGA.

Callingham, R., & McIntosh, A. (2002). Mental computation competence across years 3 to 10. In B. Barton, K. C. Irwin, M. Pfannkuch & M. O. J. Thomas, (Eds.), *Mathematics education in the South Pacific: Proceedings of the 25th annual conference of the Mathematics Education Research Group of Australasia, Auckland* (pp. 155-163). Sydney: MERGA.

Department of Education, Community and Cultural Development. (1997*). Year 9 numeracy and assessment monitoring program*. Hobart: Office for Educational Review and Assessment Research Centre, University of Melbourne.

Department of Education Tasmania. (2002). *Essential learnings framework 1*. Hobart: Author.

Department of Education Tasmania. (2003). Department of Education Tasmania. (2003). *Essential learnings framework 2*. Hobart: Author.

Department of Education Tasmania. (2007a). *A focus on being numerate.* Retrieved 26 March from http://www.ltag.education.tas.gov.au/focus/beingnumerate/default.htm

Department of Education Tasmania. (2007b). *Tasmanian Curriculum Framework Parents Update*. Retrieved 26 March from http://www.education.tas.gov.au/dept/about/minister_for_education/curriculumupdateparents2

Ernest, P. (1988). The attitudes and practices of student teachers of primary school mathematics. In A. Babas (Ed.), Proceedings of the 12th international conference of the International Group for the Psychology of Mathematics Education (pp. 288-295). Veozprem, Hungary: PME.

Griffin, P., & Callingham, R. (2006). A twenty-year study of mathematics achievement. *Journal for Research in Mathematics Education, 37*(3), 167-186.

Hannula, M. (2002). Attitude towards mathematics: Emotions, expectations and values. *Educational Studies in Mathematics, 49*, 25-46.

Hawley, W.D., & Valli, L. (1999). The essentials of effective professional development: A new consensus. In L. Darling-Hammond & G. Sykes (Eds.), *Teaching as the learning profession: Handbook of policy and practice* (pp. 127-150). San Francisco: Jossey-Bass.

Hiebert, J. (1999). Relationships between research and the NCTM Standards. *Journal for Research in Mathematics Education, 30*, 3-19.

Hill, H. C., Rowan, R., & Ball, D. L. (2005). Effects of teachers' mathematical knowledge for teaching on student achievement. *American Educational Research Journal, 42*(2), 371-406.

Ma, X., & Kishor, N. (1997). Assessing the relationship between attitude toward mathematics and achievement in mathematics: A meta-analysis. *Journal for Research in Mathematics Education, 28*(1), 26-47.

Ma, X., & Xu, J. (2004). Determining the causal ordering between attitude toward mathematics and achievement in mathematics. *American Journal of Education*, *110*, 3, 256-280.

Masters, G.N. (1982). A Rasch model for partial credit scoring. *Psychometrika*, *47*, 149-174.

McLeod, D. B. (1992). Research on affect in mathematics education: a reconceptualisation. In D. A. Grouws (Ed.), *Handbook of Research on Mathematics Teaching and Learning* (pp. 575-596). New York: Macmillan Publishing Company.

Schifter, D. (1998). Learning mathematics for teaching: From a teachers' seminar to the classroom. *Journal of Mathematics Teacher Education, 1*, 55-87.

Shulman, L.S. (1987a). Assessment for teaching. An initiative for the profession. Phi Delta Kappan, *69*(1), 38-44.

Shulman, L.S. (1987b). Knowledge and teaching: Foundations of the new reform. Harvard Educational Review, *57*(1), 1-22.

Steen, L.A. (Ed.). (2001). *Mathematics and democracy: The case for quantitative literacy*. Washington, DC: Woodrow Wilson National Fellowship Foundation.

Thomson, S., Cresswell, J., & De Bortoli, L. (2004). *Facing the future: A focus on the mathematical literacy among Australian 15-year-old students in PISA 2003*. Camberwell, VIC: Australian Council for Educational Research.

Watson, J., Beswick, K., & Brown, N. (2006). Teachers' knowledge of their students as learners and how to intervene. In P. Grootenboer, R. Zevenbergen, & M. Chinnappan (Eds.), *Identities, cultures and learning spaces* (Proceedings of the 29 annual conference of the Mathematics Education Research Group of Australasia, Canberra, pp. 551-558). Adelaide, SA: MERGA.

Watson, J.M., & Callingham, R.A. (2003). Statistical literacy: A complex hierarchical construct. *Statistics Education Research Journal, 2*(2), 3-46.

Watson, J.M., & Kelly, B.A. (2004). A two-year study of students' appreciation of variation in the chance and data curriculum. In I. Putt, R. Faragher, & M. McLean (Eds.), *Mathematics education for the third millennium: Towards 2010* (Proceedings of the 27th Annual Conference of the Mathematics Education Research Group of Australasia, Townsville, Vol. 2, pp. 573-580). Sydney, NSW: MERGA.

Choosing to Teach in the "STEM" Disciplines: Characteristics and Motivations of Science, ICT, and Mathematics Teachers

Helen M. G. Watt
Monash University
<helen.watt@education.monash.edu.au>

Paul W. Richardson
Monash University
<paul.richardson@education.monash.edu.au>

James Pietsch
New College, University of New South Wales
<J.Pietsch@newcollege.unsw.edu.au>

This study examines prospective "STEM" [Science, Technology, Engineering, and Mathematics] teachers' motivations for undertaking a teaching career and their perceptions of the teaching profession, for undergraduate and graduate teacher education entrants from three major established urban teacher provider universities in the Australian States of New South Wales and Victoria (N=245). Motivations and perceptions were assessed using the recently developed and validated "FIT-Choice" [Factors Influencing Teaching Choice] Scale (Watt & Richardson, 2007). Differences are highlighted between males and females, and undergraduates and graduates, including switchers from previous careers. Demographic profiles for STEM teacher candidates are also provided. Findings provide important implications for enhancing the effectiveness of efforts to recruit mathematics, science, and ICT teachers.

It is now commonplace for governments around the globe to affirm that science, technology, engineering and mathematics ("STEM") disciplines are the drivers of technological advancement, innovation and provide the foundational infrastructure to secure a robust economic future (e.g., National Committee for the Mathematical Sciences of the Australian Academy of Science, 2006). The STEM disciplines are characterised as the engine-room of economic development in a world where the wealthiest nations secure their economic edge through increasingly knowledge-based economies. Advanced and developing economies alike seek to ensure that their education systems provide a sufficient number of tertiary educated people in STEM (Roeser, 2006). In some highly developed countries this avowed aim is not always easily achieved and is increasingly accompanied by tensions and problems when the education system is not able to fulfil the labour force demands for skilled and talented individuals (Jacobs, 2005). Other countries such as India and China are investing heavily to ensure that participation in these disciplines will result in sufficient numbers of people being prepared to pursue higher education and careers in STEM (Roeser, 2006).

The United States of America secured a leading edge in science, technological, and engineering innovation and development in the decades following World War II and through until the 1990s, by welcoming and educating top scientists from around the world. Now they are concerned that trends in educational attainment in secondary schools and universities have undermined that edge (e.g., Jacobs, 2005). Participation in the sciences and mathematics in secondary and tertiary education has exponentially declined in the USA over the last two decades, to the point where there is grave concern about the viability of those disciplines to sustain economic growth and development (Jacobs, 2005). A similar concern exists in Australia where there is an increasing decline in STEM participation and educational attainment (Dow, 2003b).

Not surprisingly, the Australian Government identifies the STEM disciplines as central to the critical infrastructure needed to secure economic success in an increasingly globally competitive and unpredictable world. Australia's future is seen to lie in its potential as a knowledge-based economy and society – one built on the knowledge, intellectual capabilities, and creativity of its people (National Committee for the Mathematical Sciences of the

Australian Academy of Science, 2006). To achieve this potential, it will be necessary to raise the scientific, mathematical and technological literacy and the innovative capacity of students; strengthen the education system that provides the platform from which world class scientists and innovators emerge; and support the development of a new generation of excellent teachers of science, technology and mathematics (Dow, 2003a).

Well educated university graduates in STEM are inexorably linked to the quality of education which children and adolescents receive at school. Clearly, well educated, specialist teachers of those disciplines are the critical link for the next STEM generation. Without proper planning and careful management to ensure the education system provides a sufficient flow of knowledge workers through the STEM "pipeline", Australia could find itself in a similar situation to Norway where secondary schools can no longer offer science (Lyng & Blichfeldt, 2003), creating a downward spiral of suitably qualified STEM professionals – including teachers. Even now in Australia, while there are acknowledged and increasingly insistent teacher shortages in rural and remote areas, there is also a specific shortage of STEM qualified teachers (Harris & Jansz, 2006; National Committee for the Mathematical Sciences of the Australian Academy of Science, 2006). Similarly pronounced lack of supply in STEM teachers is evident in a number of OECD countries (Lawrance & Palmer, 2003) a situation that is all the more concerning, given the rapid escalation in the need for STEM-related skills in the modern world, both in careers and everyday life.

Teacher Recruitment

In Australia, recruitment efforts for teachers have included a strong focus on graduate-level teacher preparation. Within this approach, individuals graduating from non-teaching university degrees as well as those working within other professions are eligible and encouraged to undertake a teaching qualification within a reduced timeframe. However, without well-educated teachers capable of drawing children and adolescents into a fascination with STEM fields, there will be little chance of sustaining the numbers who remain in the pipeline. The pipeline metaphor seems especially appropriate to STEM disciplines, in that later knowledge development is highly dependent on earlier knowledge frameworks. If children miss out earlier on, it will be all the more difficult for them to engage effectively with the higher levels of STEM study.

To make teaching more attractive, it has been argued that increasing the salary and improving the working conditions should attract school leavers, university graduates, and people from out of other careers into teaching (Harris & Jansz, 2006). Unfortunately, Australian university graduates from the STEM disciplines are not particularly attracted to teaching as a career; and STEM disciplines are not popular among those already enrolled in teacher education (Lawrance & Palmer, 2003). A national study published in 2001 and commissioned by the Deans of Science found that among science and technology graduates there was very little interest at all in a teaching career (McInnes, Hartley, & Anderson, 2001). The lack of enthusiasm by STEM graduates for a teaching career may be a direct function of the general shortage in STEM professionals, increasing the number and type of high-status and lucrative career options available to graduates in those fields, thereby exacerbating the difficulties of attracting new graduates and career switchers into a career teaching in STEM (Harris & Jansz, 2006). Parenthetically, few of the science education graduates in the national study held degrees in mathematics (2%), life and physical sciences (4 to 7%), or computer science (0%); (McInnes, Hartley, & Anderson, 2001), signalling a need to examine profiles across the different STEM domains rather than shortages and solutions at an aggregate level. The present study consequently disaggregates and contrasts findings for mathematics, science and ICT teacher graduands.

The Teacher Shortage

The teaching force is ageing in many of the OECD countries, with half the teaching force aged over 40 in some European countries (European Commission, 2000). In Australia the median age of teachers was 43 in 2001, with 44% older than age 45 (DEST, 2003). Australian mathematics teachers also appear older than the national average, signalling a particular imperative to encourage more people into mathematics teaching. Evidence from the Third International Mathematics and Science Study [TIMSS] further suggests that these teachers are not particularly happy with their jobs. Although the TIMSS study was designed to report on the learning of students aged 9, 13 and at the final year of secondary school from Africa, Asia, Europe, North America, South America, and Oceana (Australia and New Zealand), it also gathered fascinating data on the lives of teachers. Revealingly, it was the Australian and New Zealand teachers who represented the highest proportion who indicated they would "prefer to change to another career" (Lokan, Ford, & Greenwood, 1996, p.197). In mathematics in particular, 39% of teachers in a recent national study were undecided whether they would remain in teaching, and 16% actively planned to leave the profession (Harris & Jansz, 2006).

The retirement-fuelled exodus of teachers from the "baby boom" generation, who through their superannuation retirement packages receive financial inducements to leave work at 55, will quickly escalate shortages in the STEM disciplines, creating more difficulties in already hard-to-staff schools in rural and urban areas. Even if this generation of teachers could be persuaded to stay on until they reached the retirement age of 65, this would only alleviate problems in the shorter term. Faced with these dilemmas Education departments, teacher recruitment authorities and organizations are not able to solve their staffing problems by bringing in teachers from other countries as they did 30 years ago. On the contrary, recruiting companies from the UK, USA, and Asia are siphoning off new Australian teacher graduates into appealing positions overseas, making them unavailable to the Australian labour market until when and if they return.

A further deeply embedded problem is that males are heavily concentrated into the older age groups of teachers and that a "disproportionate number of male science, mathematics and technology teachers are aged over 45" (Dow, 2003b). Although teaching is increasingly a feminised profession in many OCED countries including Australia, fewer girls and women are retained in the STEM pipeline progressively through senior high school, university studies, and career choices; and women drop out of the STEM disciplines even when their achievement in those disciplines is equal to or higher than that of males (Jacobs, 2005). In Australia this has been well documented in the case of mathematics (see Watt 2005, 2006; Watt, Eccles, & Durik, 2006). In a highly competitive job market where Australia is facing a crisis in the availability of tertiary-trained workers (Birrell & Rapson, 2006), particularly in STEM, the women who *do* persist or excel in those domains can earn a higher salary and occupational status in careers other than teaching. The trend towards increasing numbers of women entering teaching, together with lower female participation in STEM disciplines, is likely to intensify the short-fall in STEM teachers.

The Present Study

We need first to be concerned about whether the shortage of STEM teachers can be met in the short and longer term; and secondly, whether those who are attracted into teaching in those disciplines have sufficient ability, personal interest in and enthusiasm for the sciences, mathematics and technology to enliven and sustain the interest of children and adolescents. Given the shortages of tertiary educated people across the labour market more generally, even those with low-level STEM skills may have attractive and lucrative career options. It is not desirable that 25% of mathematics and science teachers have no higher education in those domains (National Committee for the Mathematical Sciences of the Australian Academy of Science, 2006). To engage children and adolescents in STEM requires teachers with pedagogical as well as content expertise.

Given the potential for finding other more lucrative work, as well as the detractors we have outlined from teaching STEM, we ask the question why people still choose a teaching career in these domains. The purpose of our paper is to enquire into the profiles of characteristics, motivations, and perceptions of those who choose to pursue STEM qualifications with the intention of becoming teachers, including those who following a period of employment in another career have made the decision to become teachers. Our study makes two particularly important contributions to the existing literature. First, studies that have previously focused on teacher characteristics for specific discipline areas have tended to examine closely a particular group in isolation, with the consequence that it has not been possible to discover factors peculiar to those groups. A strength of our study is that the STEM teacher sample forms a subset of our larger sample of 1653 beginning secondary, primary, and early childhood teachers from across three major Australian universities. It is therefore possible to contrast characteristics and motivations for each of the mathematics, science and ICT subsamples, against the general profiles we have described previously (see Richardson & Watt, 2006). Second, although a recent influential national study focused on practising mathematics teachers (Harris & Jansz, 2006) has provided detailed statistics on their background characteristics and career intentions, we include additional information such as ethnic and socioeconomic backgrounds, and a stronger focus on motivations and perceptions. Teaching motivations were less rigorously investigated in the national study (via six "check-boxes" with an "other" option). Elsewhere we have argued the need for drawing upon established motivational frameworks and utilising rigorous measures in assessing motivations (Watt & Richardson, 2007). The present study meets both these needs, through implementing a comprehensive, validated, reliable measure for teaching motivations and perceptions, and exploring differences between mathematics, science, and ICT prospective teachers.

Method

Sample and Setting

Participants (N=245) were beginning teacher education candidates in STEM programs at three Australian universities, enrolled in either an undergraduate Bachelor of Education, or a graduate-entry 1- to 2-year teaching qualification. These participants comprise a subsample from our complete sample of teacher education candidates across those universities, for which demographic characteristics have been summarised by Richardson and Watt (2006). In the STEM subsample, both the proportion of women (53% vs. 67-84%), and of NESB [non-English speaking background] individuals (78% vs. 81-90%), were substantially lower than in the full sample (Table 1). Because teacher education candidates can undertake more than one specialisation, we identified the combinations of specialisations studied by prospective STEM teachers. Relatively low proportions of candidates undertook only one of mathematics (21%) or ICT (28%), while about half undertook science only (52%). The other profiles are presented in Table 2: most involved various combinations of STEM domains, although it was also interesting to observe combinations with the humanities, visual and performing arts, social studies, and languages. All participants were either undertaking (undergraduates) or had previously completed (graduates) a major in their area/s of specialisation.

Measures

Teacher education candidate characteristics. Participants stated their age in years, and checked boxes to indicate gender, undergraduate or graduate enrolment, and secondary teaching specialisation/s. Science specialisation was further disaggregated into general science, biology, chemistry, and physics at Monash university.

Table 1
STEM Representation Across University, Gender and ESB Groups

	Mathematics n's UG / grad	ICT n's UG / grad	Science n's UG / grad	Totals † UG / grad
USyd	12 / 13	2 / 2	23 / 20	29 / 26
Monash	13 / 30	6 / 20	16 / 54	24 / 78
UWS	11 / 33	3 / 17	14 / 38	20 / 68
Totals	36 / 76	11 / 39	53 / 112	73 / 172
% Female	42.9	44.0	55.2	52.7
% ESB	70.5	70.0	85.5	78.0

† *Note*. Totals for numbers of undergraduates and graduates within each university are not summed totals for mathematics, ICT, and science, because 82 individuals studied more than one STEM domain: 19 individuals are represented in each of mathematics and ICT, 62 in mathematics and science, and 1 in science and ICT.

Table 2
Teaching Specialisations

	Mathematics (N = 112)	ICT (N = 50)	Science (N = 165)
Mathematics	23†	19	62
ICT	19	14†	1
Science	62	1	86†
Humanity	3	5	5
Vis perf	1	2	0
SocStud	5	5	12
TESOL	0	4	0
LOTE	3	2	0

Note: † indicates number of students whose *only* method of study was mathematics, ICT or science.

Prior career background. Participants who indicated they had previously pursued another career were asked to provide details of that career. These were then classified in terms of STEM-relatedness or not.

Family background. Combined parental income from when participants were in high school was used as an indicative measure for background socioeconomic status (SES). Participants also nominated their parents' occupations, which were coded as STEM-related or not, and as teaching or not. Home language was coded as ESB [English-speaking background] vs. NESB [non-English speaking background].

Motivations for teaching. Motivations for choosing teaching as a career were assessed using the *FIT-Choice* [Factors Influencing Teaching Choice] scale (full details and good construct reliability and validity with this sample are reported in Watt & Richardson, 2007). Measured motivations include intrinsic values, personal utility values (job security, time for family, job transferability), social utility values (shape future of children/adolescents, enhance social equity, make social contribution, work with children/adolescents), self perceptions of individuals' own teaching abilities, the extent to which teaching had been a "fallback" career choice, social influences, and prior positive teaching and learning experiences. Each factor was measured by multiple item indicators with response options from 1 (not at all important) through 7 (extremely important). A preface to all motivation items was "I chose to become a teacher because …".

Perceptions about the profession. Participants rated the extent of their agreement with propositions about the teaching profession, with response options again from 1 (not at all) through 7 (extremely). Multiple propositions comprised factors concerning to the extent to which respondents perceived teaching as high in task demand (expert career, difficulty), and task return (social status, salary).

Career choice satisfaction. Participants' career choice satisfaction was measured by three items with response options from 1 (not at all) through 7 (extremely). As part of this section, participants also rated the extent to which they had experienced social dissuasion from teaching as a career.

Procedure

Surveys were conducted early in the academic year in 2002 at the University of Sydney, and 2003 at Monash University and the University of Western Sydney (UWS). They were administered in tutorial class groups to enhance data integrity and allow respondent queries. Administration was by the researchers and two trained assistants, with University ethics approval, consent of program coordinators, and informed consent of all participants. It took approximately 20 minutes to complete the survey.

Results

Who Chooses STEM Teaching?

Gender representation. Enrolments within each STEM strand were slightly more male dominated for mathematics and ICT, and conversely for science (Table 1). The mathematics statistics reflect the similar numbers of male and female practising teachers (Harris & Jansz, 2006).

Home language backgrounds. The majority of STEM teacher candidates were from ESB, and this was most pronounced for science (Table 1). Within disaggregated science strands at Monash, *all* teacher candidates studying biology, chemistry and general science were from ESB, compared with just under 85% studying physics. NESB concentrations among teacher candidates were higher in mathematics and ICT domains than across the full sample (Richardson & Watt, 2006). At the University of Sydney and UWS, NESB concentrations were higher than in the full sample (¼ NESB vs. 18% at USyd, 35% NESB vs. 19% at UWS), while the reverse was true at Monash (3% NESB vs. 10%).

Age profiles. Age profiles tended to be slightly higher for ICT, followed by mathematics and then by science (Figure 1). Summary statistics for science reflected typical ages of graduates in the full sample, whereas ICT and mathematics teacher candidates were an average 4-5 years older.

SES income backgrounds. Participant-reported combined parent income categories were somewhat lower on average for mathematics vs. science and ICT teacher candidates (Figure 2). For all three STEM domains, SES backgrounds were below those from the full sample, in which the median and modal category was $60,001-$90,000.

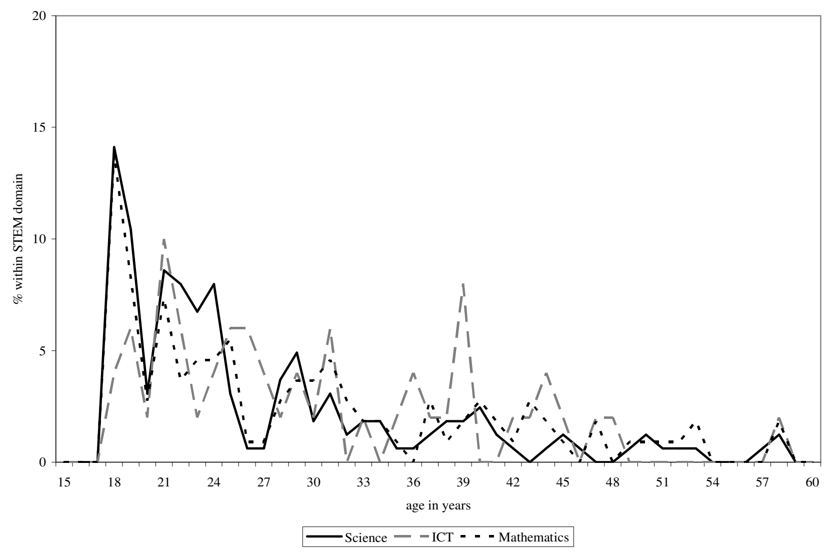


Figure 1. Age profiles for beginning teacher education candidates in STEM disciplines.[1]

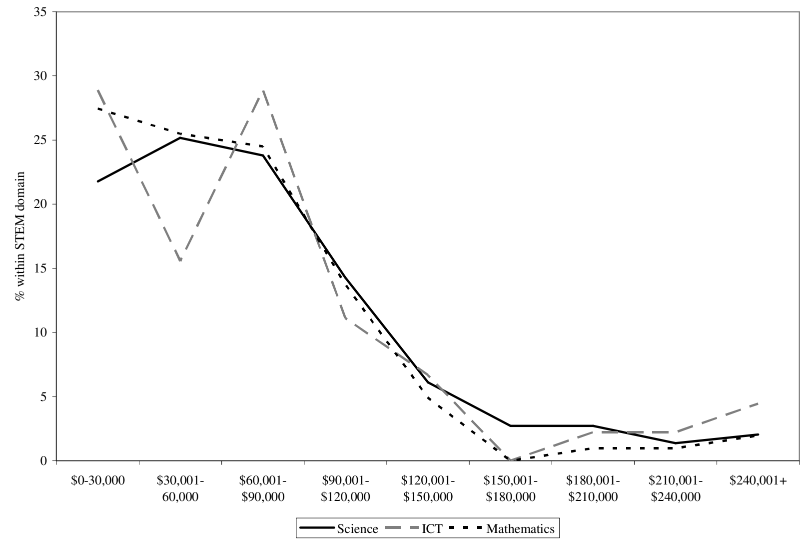


Figure 2. Combined parent income for beginning teacher education candidates in each STEM discipline (indicative SES).[2]

Parental careers. A considerable number of preservice STEM teachers (105, 43%) had parents who worked in STEM related areas (25–30% of fathers, ¼ of mothers): for science, 52 (31.5%) fathers and 43 (26.1%) mothers; for ICT, 11 (22%) fathers and 13 (26%) mothers; and for mathematics, 33 (29.5%) fathers and 27 (24.1%) mothers. Smaller proportions had teacher parents (25, 10%): for science, 25 (15%) had at least one parent who was a teacher (12% of mothers, 5% of fathers); for ICT, 6 (12%; 12% of mothers, 2% of fathers); and for mathematics, 10 (9%; 7% of mothers, 3% of fathers).

[1] . Summary statistics for science: *M*=26.92 *SD*=9.55, ICT: *M*=30.26 *SD*=9.57, mathematics: *M*=29.23 *SD*=10.62.

[2] . Summary statistics for science: *M*=2.96 *SD*=1.81, ICT: *M*=2.98 *SD*=2.07, mathematics: *M*=2.64 *SD*=1.64 (Income values: 1: $0-30,000, 2: $30,001-60,000, 3: $60,001-90,000, 4: $90,001-120,000, 5: $120,001-150,000, 6: $150,001-180,000, 7: $180,001-210,000, 8: $210,001-240,000, 9: $240,000 +)

"Career switcher" backgrounds. A large number of candidates in graduate programs in each of the STEM disciplines reported having pursued a prior career (46% in science, 55% in ICT, 47% in mathematics). Statistics for mathematics reflect those for early career teachers in the national study (Harris & Jansz, 2006). These proportions were considerably higher than the proportion of graduates in the full sample who had previously pursued other careers (Richardson & Watt, 2006). Of the STEM teacher candidates who had pursued a prior career, the proportion who had come from STEM-related occupations was very high. For mathematics and ICT teacher candidates who indicated they had pursued a prior career, over 90% had previously pursued careers in STEM, and 86% for science.

Why Choose Teaching?

Motivations for teaching. In each of mathematics, science, and ICT, the highest rated motivations for choosing a teaching career were perceived teaching abilities, the desire to make a social contribution, to shape the future of students, and the intrinsic value of teaching as a career. Positive prior teaching and learning experiences were also quite high, resonating with the importance of attracting quality teachers in mathematics emphasised in recent reports (Harris & Jansz, 2006; National Committee for the Mathematical Sciences of the Australian Academy of Science, 2006). The lowest rated motivation was consistently choosing teaching as a "fallback" career, followed by the social influences of others encouraging them to undertake teaching. These patterns of motivations are similar to those previously documented for teachers across different domains and areas of teaching (Richardson & Watt, 2006). Few systematic differences were evident between teaching motivations for undergraduates vs. graduates and males vs. females across the STEM domains (Figure 3).

- Male students studying to be *mathematics* teachers were more motivated than females by job transferability ($F(1,99)=5.4$, $p=0.02$; male $M=4.4$ SD 1.4, female $M=3.8$ SD 1.4), making a social contribution ($F(1,99)=5.2$, $p=0.03$; male $M=3.7$ SD 1.7, female $M=3.3$ SD 1.8), and choosing teaching as a fallback career ($F(1,99)=5.0$, $p=0.03$; male $M=2.6$ SD 1.4, female $M=2.1$ SD 1.4).
- Prior teaching and learning experiences were more important to undergraduates training to be *science* teachers compared with graduates ($F(1,142)=11.6$, $p=0.001$; undergraduate $M=5.4$ SD 1.1, graduate $M=4.6$ SD 1.6).
- Female students studying to be *science* teachers rated working with adolescents as a more important motivation than males ($F(1,140)=3.9$, $p=0.05$; male $M=4.7$ SD 1.4, female $M=5.0$ SD 1.6). However, there was also a significant interaction between gender and degree ($F(1,140)=5.2$, $p=0.02$), due to undergraduate males being more motivated by their desire to work with children than graduates, while graduate females were more motivated in this regard than undergraduates.

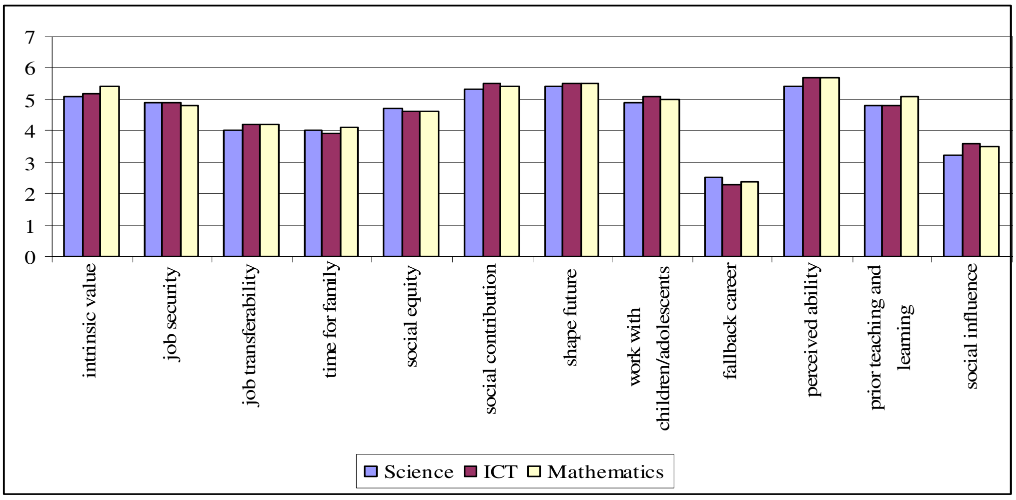


Figure3. Factors influencing teaching choice for teacher education candidates within STEM disciplines.

Perceptions about the profession. Participants generally perceived teaching as a career which is high in demand – and low in return. Participants rated teaching as a highly demanding career with a heavy workload that makes high emotional demands and requires considerable hard work; and as a highly expert career requiring specialised knowledge and abilities. At the same time, it was perceived to be relatively low in terms of salary and social status (Figure 4). Again, there were few differences by gender or undergraduate vs. graduate enrolment.

- For both *science* and *mathematics* candidates, graduates rated teaching significantly higher in demand than undergraduates (science: $F(1,140)=15.7$, $p=0.001$; undergraduate $M=5.6$ SD 1.1, graduate $M=6.2$ SD 0.8; mathematics: $F(1,99)=7.3$, $p=.008$; undergraduate $M=5.5$ SD 1.0, graduate $M=6.0$ SD 0.9).
- *Science* graduates also perceived teaching to require a higher level of expertise than undergraduates ($F(1,140)=4.1$, $p=0.05$; undergraduate $M=5.1$ SD 1.2, graduate $M=5.4$ SD 1.0). However this main effect was modified by a significant interaction of gender and degree, wherein graduate males rated expertise higher than undergraduates, and conversely for females ($F(1,140)=7.2$, $p=0.008$). Female *ICT* teacher candidates rated the demands of teaching to be higher than males ($F(1,45)=4.1$, $p=0.05$; male $M=5.9$ SD 0.9, female $M=6.5$ SD 0.6).
- Female *science* teacher candidates perceived teaching salaries as higher than males ($F(1,140)=5.0$, $p=0.03$; male $M=3.0$ SD 1.4, female $M=3.6$ SD 1.3).

Career choice satisfaction. Similar to the full sample, mathematics, science and ICT teacher candidates reported moderate experiences of social dissuasion from a teaching career. Despite this, and despite perceptions of teaching as a career high in demand and low in return, mean *satisfaction* ratings for teaching as a career choice were uniformly high (see Figure 5).

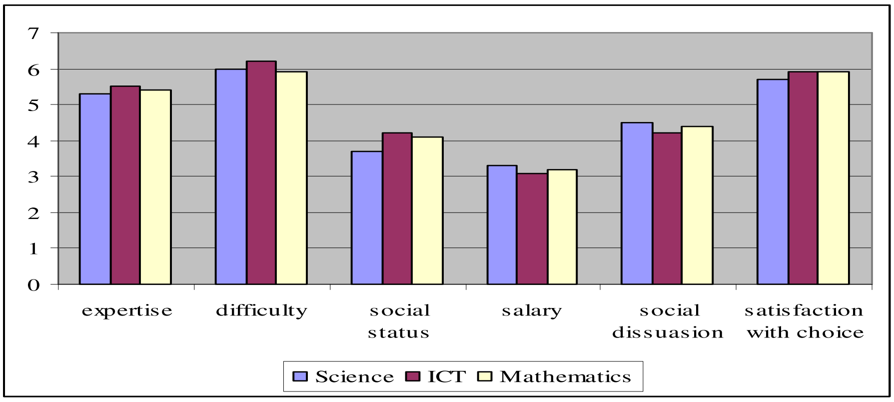


Figure 4. Perceptions about teaching for candidates within STEM disciplines.

Discussion

Our study has provided a detailed portrait of who chooses to undertake a teaching career in each of mathematics, science and ICT using a subsample drawn from a large-scale sample, which permits comparisons between these and other beginning teachers. We identified low proportions of women entering mathematics and ICT teaching, and despite women comprising approximately half of the science teacher candidates, they were very poorly represented in physics. Higher proportions of NESB individuals undertook mathematics and ICT teacher education compared with our full sample of teacher candidates, and they also tended to be older and from lower socioeconomic backgrounds. Roughly half the STEM teacher candidates had parents from STEM-related careers, and roughly half themselves came from prior STEM-related careers. Few had parents who were teachers. STEM teacher candidates mostly undertook specialisations within STEM domains, although it was also interesting to observe combinations with social studies and to a lesser extent humanities.

Teaching ability-related beliefs, personal (job security, time for family, job transferability) and social utility values (desire to shape the future, enhance social equity, make a social contribution, work with children / adolescents), and positive prior experiences of teaching and learning were all important motivations. Participants perceived teaching as a career that is highly demanding, and low in return in terms of salary and social status. They also reported relatively strong experiences of social dissuasion. At the same time, they had high levels of satisfaction with their choice of a teaching career. Importantly, these motivations and perceptions from the separate groups of STEM teacher candidates reflected those from our full sample (Richardson & Watt, 2006), and were generally similar for undergraduates vs. graduates, and males vs. females. The implications are that recruitment campaigns targeting these motivations should be effective for STEM teachers too, and suggest older graduates working in STEM-related careers as a fruitful group to aim to attract into teaching careers.

Acknowledgements. The authors contributed equally to the manuscript.

References

Birrell, B., & Rapson, V. (2006). Clearing the myths away: Higher education's place in meeting workforce demands. Retrieved 2 November 2006, from www.dsf.org.au

DEST. (2003). *Australia's teachers: Australia's future. Advancing innovation, science, technology and mathematics.* Canberra: Committee for the review of teaching and teacher education, Department of Education, Science and Training.

Dow, K. L. (2003a). *Australia's teachers: Australia's future: Advancing innovation, science, technology and mathematics. Agenda for action.* Canberra: Commonwealth of Australia.

Dow, K. L. (2003b). *Australia's teachers: Australia's future: Advancing innovation, science, technology and mathematics. Main report*. Canberra: Commonwealth of Australia.

European Commission. (2000). *Key data on education in Europe*. Brussels: Author.

Harris, K.-L., & Jensz, F. (2006). *The preparation of mathematics teachers in Australia: Meeting the demand for suitably qualified mathematics teachers in secondary schools*. Melbourne: Australian Council of Deans of Science, University of Melbourne.

Jacobs, J. E. (2005). 25 years of research on gender and ethnic differences in math and science career choices: What have we learned? *New Directions for Child and Adolescent Development, 110* (Winter), 85-94.

Lawrance, G. A., & Palmer, D. H. (2003). *Clever teachers, clever sciences: Preparing teachers for the challenge of teaching science, mathematics and technology in 21st Century Australia.* Canberra: Evaluations and Investigations Programme Research Analysis and Evaluation Group of the Department of Education, Science and Training .

Lokan, J., Ford, P., & Greenwood, L. (1996). *Maths & science on the line: Australian junior secondary students' performance in the Third International Mathematics and Science Study*. Melbourne: Australian Council for Educational Research.

Lyng, S. T., & Blichfeldt, J. F. (2003). *Attracting, developing and retaining effective teachers: Country background report* Norway: Work Research Institute.

McInnes, C., Hartley, R., & Anderson, M. (2001). *What did you do with your science degree? A national study of outcomes for science degree holders 1990-2000*. Melbourne: Australian Council of Deans of Science, University of Melbourne.

National Committee for the Mathematical Sciences of the Australian Academy of Science (2006). *Mathematics and statistics: Critical skills for Australia's future. The National Strategic Review of Mathematical Sciences research in Australia*. Canberra: Australian Academy of Science.

Richardson, P.W., & Watt, H.M.G. (2006). Who chooses teaching and why? Profiling characteristics and motivations across three Australian universities. *Asia-Pacific Journal of Teacher Education, 34*(1), 27-56.

Roeser, R. W. (2006). On the study of educational and occupational life-paths in psychology: Commentary on the special issue. *Educational Research and Evaluation, 12*, 409-421.

Watt, H.M.G. (2006). The role of motivation in gendered educational and occupational trajectories related to math. *Educational Research and Evaluation, 12*, 305-322.

Watt, H.M.G. (2005). Explaining gendered math enrollments for NSW Australian secondary school students. *New Directions for Child and Adolescent Development, 110* (Winter), 15-29.

Watt, H.M.G., Eccles, J.S., & Durik, A.M. (2006). The leaky mathematics pipeline for girls: A motivational analysis of high school enrolments in Australia and the USA. *Equal Opportunities International, 25*, 642-659.

Watt, H.M.G., & Richardson, P. W. (2007). Motivational factors influencing teaching as a career choice: Development and validation of the 'FIT-Choice' Scale. *Journal of Experimental Education*, *75*(3), 167-202.

Percentages as Part Whole Relationships

Paul White
ACU National
<paul.white@acu.edu.au>

Sue Wilson
ACU National
<sue.wilson@acu.edu.au>

Rhonda Faragher
ACU National
< rhonda.faragher@acu.edu.au>

Mike Mitchelmore
Macquarie University
<mike.mitchelmore@mq.edu.au>

Five practising teachers in regional NSW implemented Teaching for Abstraction for the Year 6 topic "Percentages". The authors constructed materials for a unit in which students explored familiar percentage contexts, searched for similarities in their mathematical structures and then applied their learnings to more abstract situations. Particular emphasis was given to additive versus multiplicative approaches in different percentage situations. After an introductory workshop, teachers taught the topic in eight 40 minute lessons. The results show that even though this approach is radically different from that to which students and teachers are accustomed, it has the potential to benefit student engagement, learning, and attitudes for both students and teachers. The overall conclusions have implications for how professional development for Teaching for Abstraction is addressed.

Mitchelmore and White (2004) outline an approach to teaching based on the fact that most elementary mathematical ideas are abstractions from experience. Emphasised is the importance of empirical abstraction in mathematics learning, focusing on an abstract concept as "the end-product of ... an activity by which we become aware of similarities ... among our experiences" (Skemp, 1986, p. 21). This view of abstraction leads to a theory for teaching early mathematical concepts called *Teaching for Abstraction* (Mitchelmore & White, 2000), where students engage in:

- *familiarising themselves* with the structure of a variety of relevant contexts;
- *recognising* the similarities between these different contexts;
- *reifying* the similarities to form a general concept, and then
- *applying* the concept in new situations.

Much of the theory has been developed from investigations into young children's understanding of the angle concept (Mitchelmore & White, 2000), but also from mathematical concepts involving rates of change (White & Mitchelmore, 1996), decimals (Mitchelmore, 2002), and percentages (White & Mitchelmore, 2005). Two further studies took place in 2006. The first was an extension of the earlier percentage study with Year 6, but in regional schools; the other was on rates and ratios with Year 8. The Year 6 study is reported here, the Year 8 study elsewhere.

Percentage as a Multiplicative Relation

Percentage is a multiplicative relationship that causes students particular difficulties—it forms a bridge between real-world situations and mathematical concepts of multiplicative structures (Parker & Leinhardt, 1995). The concise, abstract language of percentages often uses misleading additive terminology with a multiplicative meaning. Misailidou and Williams (2003) showed that inappropriate additive strategies were the dominant errors made by students aged 10-13 years. On the other hand, Van Dooren and De Bock (2005) claim that extensive attention to proportional reasoning in school mathematics results in the misapplication of

Proceedings of the 30th annual conference of the Mathematics Education Research Group of Australasia
J. Watson & K. Beswick (Eds), © MERGA Inc. 2007

proportional methods. Whatever the situation, a cursory look at the school mathematics curriculum shows that multiplicative relations underpin almost all number-related concepts studied in school (e.g., fractions, percentages, ratio, rates, similarity, trigonometry, rates of change). Hence, percentages and proportional reasoning in general are areas deserving research especially if a different methodology is adopted which goes beyond that in the research cited above.

Aims of the Study

The object of our research project was to build on the previous study (White & Mitchelmore, 2005) about how Year 5/6 classroom teachers adapt to using everyday situations and about how students abstract the multiplicative structure of percentages. That study developed a unit of work based on Teaching for Abstraction that emphasised underlying structure in percentage situations, including helping students to differentiate multiplicative from additive relations. The analysis showed that the approach was radically different to that which students and teachers are accustomed. Many students did learn to apply percentages even though the final level of achievement was not as high as had been expected. Two reasons for the lower than expected achievement were insufficient time to explore individual contexts in enough detail and inadequate attention to calculation skills. A new unit was developed which addressed fewer contexts and had a greater focus on calculating with percentages – using 10% as a base for calculations.

Method

Participants

Participants were students and teachers of five Year 6 classes in three regional primary schools. In each class, five students were selected as a representative “target group” for closer study.

Teaching Materials

The four phases of the theoretical framework for Teaching for Abstraction were used in planning the activities for the experimental unit as follows.

- *Familiarising*: Students explored individual, supposedly familiar contexts. Simple percentages were initially used (50%, 10%) but these increased in complexity to 25%, 75%, 20%, 30%, …, 90%, and 5%.
- *Recognising*: Activities required students to compare and contrast the use of percentages in different contexts. Calculations were based on first calculating10% and then multiplying by the appropriate factor.
- *Reifying*: Students were asked to make and explain generalisations based on the similarities found in the Recognising phase.
- *Application*: Students created their own problems.

The resulting lesson topics are shown in Table 1. The lesson titles used syllabus familiar terms, addressing the appropriate skills and outcomes. The lesson structure, however, followed the theory of abstraction: beginning with a context with embedded skills and concepts and leading on to discussion about the underlying abstract notions.

Table 1
Topics for Percentage Lessons

1. Thinking percent	Students interpret percentages in situations involving bar models. The focus is on percent as a part of 100.
2. Calculating percentages	Students extend their previous experience of percentages to simple percentages (multiples of 10%) of 200, 300 and 50 objects.
3. Calculating more percentages	Students further extend their previous experience of percentages to simple percentages (multiples of 10%) of any number of objects.
4. Discounts	Students investigate discounts and compare percentage discounts with fixed discounts.
5. How do I choose?	Students compare the appropriateness of additive versus multiplicative strategies.
6. Taxes	Students compare different ways the GST could have been charged and decide on fair ways of doing so.
7. What is the best way?	Students investigate problems involving different comparisons and decide the best way to solve these problems.
8. Summary	Students bring together the main ideas and skills learnt in this unit.

Procedure

The study took place in Term 4, 2006. A one-day orientation workshop was held, in which teachers were introduced to Teaching for Abstraction and the proposed teaching unit. They then taught the unit over a period of 2 to 3 weeks, and returned for a second workshop for an assessment of the effectiveness of the unit. The first three authors visited schools to assess students' understanding before and after the teaching, to observe lessons, and to interview teachers. Thus the following sources of data were generated.

- A written pre and post test assessment of all students on their ability to calculate with percentages.
- A 15-minute interview given before and after the teaching with the five targeted students in each class.
- Worksheets completed by the targeted students.
- Observations and subsequent interviews with the teachers. Each teacher was observed twice, once by the first author and once by the second or third author.
- Teachers' evaluations of each lesson and of the unit.

Results

Based on White and Mitchelmore (2005), the results are presented in two categories –calculating with percentages and interpretation of percentage contexts. The format centres on the pre and post quantitative data with support from qualitative data.

Calculating with Percentages

This section looks at the written pre/post test and the associated Lessons 1 – 3.

Written Test. The written test consisted of six questions requiring calculations with percentages. Question 1 asked "percent means out of ___" (this was not scored).

Question 2 involved calculating 10%, 20%, 25%, 50%, 75%, and 90% of 100 jelly beans in a jar. Question 3 asked for the same percentages of 200, and Question 4, the same percentages of 50. Question 5 required students to colour in 50% of a bar that was (a) 10 boxes, (b) 8 boxes long. Question 6 required colouring 25% of the same bars. The combined results for each question from the 5 classes are shown in Figure 1.

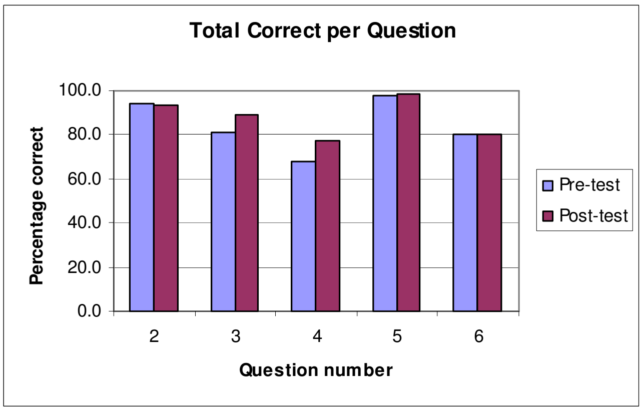


Figure 1. Aggregated percentage correct before and after teaching.

The results indicate no apparent change in Question 2 (percentage calculation out of 100) and Question 5 (colour in 50% of a bar). The scores were 94% and 98%, respectively. The consistently high scores can be attributed to the familiarity of students with calculating 50% and percentages out of 100. Question 6 (colour in 25% of a bar) also shows no change, with a pre and post result of about 80%. The lower score for Question 6 can be attributed to the less familiar 25% and the fact that in part (a) 25% of 10 required two and a half boxes to be coloured.

Questions 3 and 4 showed increases from 80% to 89% and 67% to 78% respectively. A closer look shows that the most common error in the pre test was calculating as if there were 100 jelly beans – that is, treating the percentage as always out of 100. This error did not occur in the post test. The overall lower facility of Question 4 arose because parts (c) and (e) involved fractional answers. In these calculations, only about 50% of students were able to respond correctly in the post test compared to about 43% in the post test. Also in Question 4, part (f) (find 90% of 50) correct responses rose from 60% to 81%.

In summary, the results indicate that 50%, 10%, and percentages out of 100 are familiar to students entering Year 6 and that the teaching here improved calculation facility for examples like 20%, 25%, 75% and 90% of numbers other than 100 except where fractional answers were involved.

Lesson Analysis. The first three lessons related to the written test as they focused on calculating percentages, beginning with 50% and 10% of 100 and moving on to more complex examples.

Teachers brought in food containers with percentages on them to introduce their early lessons. They discovered that students had an understanding of the difference between "percent fat" and "percent fat free" and that for any product, the two values added to 100%. They also discussed the use of these percentages as a marketing ploy. The contexts employed here clearly assisted students clarify their understanding of percentage. Similarly, little difficulty was found with Question 2 on the worksheet for Lesson 1 – colouring in 50% of a 14 cm bar with no box markings – because of the "half" connection. However, colouring in 10% and 90% proved more troublesome. A common mistake was to colour in 1 cm for 10% and 13 cm (14 cm – 1cm) for 90%.

Teachers' feedback indicated that these colouring activities and the discussion of errors like above helped students think beyond 50% and out of 100. This is consistent with the results of the quantitative analysis reported above. The observations suggest that using unmarked bars was an effective strategy and question the sense of only using marked bars in the written tests. The presence of markings goes some way to explaining why students had little difficulty with Question 5 of that test.

The 10% approach used in these lessons was given positive feedback by teachers and adopted by the majority of students. For a few students, however, the 10% approach conflicted with other rote learnt procedures. For example, although most students knew that you divide by ten to find 10% of something, one student wrote: *I take the first number if it's a 2 digit number or if it's a number greater than 100 I get rid of one zero*; *Move the decimal point forward once.* One author observed that in one class the "10% method" was effectively one recipe replacing another.

In conclusion, improvement in calculation facility where fractional answers were not involved is supported, but in some instances rote learning may have been the likely reason. Calculations resulting in fractional answers received no attention in the teaching.

Interpretation of Percentage Contexts

This section looks at the pre and post interviews and the associated Lessons 4 – 7.

Table 2
Interview Questions

Question
1. Two basketball players compare their shooting from the free throw line. The first player has scored 20 goals from 40 shots. The second player has scored 25 goals from 50 shots. Which player is the better shooter? Why?
2. Meg is 10 years old. Her little sister Lisa is 5 years old. How much older is Meg than Lisa? How old will Meg be when she is double her now? How old will Lisa be when Meg is double her age now? Explain your answer.
3. (a) Marcos purchases a new Mobile Phone. The original cost is \$100. Marcos is offered a choice of the cost being reduced by a 10% discount or having \$10 taken off the price. Which should Marcos choose? Explain your answer. (b) Pam purchases a TV. The original cost is \$200. Pam is offered a choice of the cost being reduced by a 10% discount or having \$10 taken of the price. Which should Pam choose? Explain your answer. (c) Does a 10% reduction or a \$10 reduction always give more off the price? (d) Give some examples to explain your answer.
4. (a) At one store, new joggers have a price of \$80, but because it is 'Cheap Tuesday', the price is reduced by 10%. How much do they cost on Cheap Tuesday? (b) At another store the same joggers have a price of \$100, but the store has a sale on and the price is reduced by 20%. What is the sale price? (c) Does a bigger percentage reduction always mean the price is cheaper? (d) Explain your answer.

Assessment Interviews. The interviews contained the four questions shown in Table 2. All questions were presented orally and in writing. They were administered to 21 of the 25 target students before and after the teaching. (The others were absent on one or both occasions.)

The overall performance on these questions went from 60% correct to 90% correct. Figure 2 shows the breakdown across the questions.

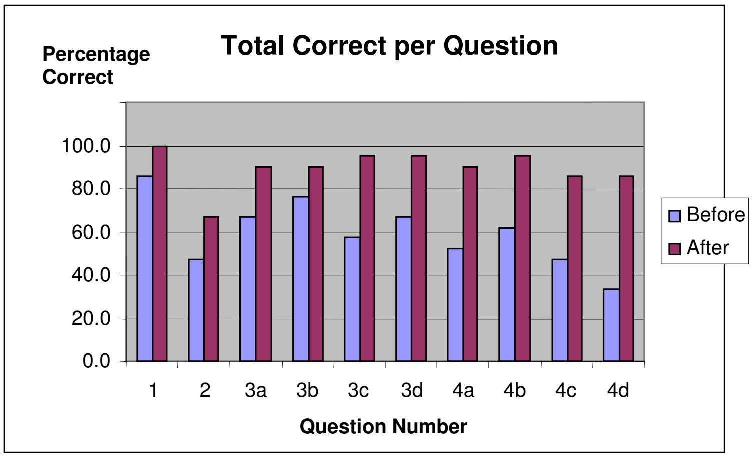


Figure 2. Aggregated percentage correct before and after teaching.

The correct responses to Question 1 included the expected assertions that each player scored 50%, but also scored as correct were arguments like: *The 25 was better because they were the same but kept it up longer.* The few errors fell into two categories: additive strategies (*First player because I took 10 away from the 50 and that equalled 40, so then I had to take 10 away from the 25 and that gave me the answer*) and incorrect multiplicative strategies (*First player because he got half. The other one got 25 out of 50 so he only got a quarter*). All errors disappeared after the teaching.

In Question 2, an additive response was required. For example: *15. She's 15 because it's only 5 years. They are 5 years apart. 20-5 would be 15.* Before the teaching, 48% used a correct strategy, rising to 67% afterwards. A few students attempted to use an additive strategy but made an arithmetical error. The major error, however, was the inappropriate use of a multiplicative strategy (43% before and 24% after) such as: *10. If I double Meg, I'll have to double Lisa because it will be the same time.*

In Question 3(a) and (b), improved arithmetic accounted for the improvement in facility. In 3(c), one third of students opted for the 10%—a response that virtually disappeared in the post test. For example: *10% gives more off the price because if the price was $100 and you take off $10 it would be $90, but if you take off 10% it would be $80; 10% gives more off the price. $10 reduction is just $10 but 10% depends on how much money you had.* In the pre interview, some reasons incorrectly relied on one example whereas others were basically sound but failed to come to the correct conclusion. In the post interview, the 95% facility for Question 3(d) shows that students' reasoning was clearer. For example: *It depends what the price is. If it is a higher number then $100 it is always a bit more than $10. If it is lower than $100, it is less.*

Like Question 3, improved facility in responses to Question 4 (a) and (b) was a result of improved arithmetic. In 4(c), the choice of the "bigger percentage reduction means a cheaper price" option fell from 52% to 14%. Like Question 3, pre interview conclusions were often based on a single example but also included some percentage misconceptions. For example: *Yes. Because the % usually means the same as the dollar amount so you take that off* and *Yes. The bigger the percent off, the less money* *you pay.* Of note is that in the pre interview only half of the 38% who said that the bigger percentage does not always give a cheaper price could give a valid reason, whereas in the post interview all could.

In summary, after the teaching, the number of students who could both calculate with percentages like 10% and 20% and use these percentages appropriately in context doubled. This included explaining why they came to the answers they did, in particular identifying percentage as a relative comparison and the need to identify "percentage of what". Question 2 seemed the most problematic. Of course, we would expect to see improvements after a teaching episode no matter what the approach was, especially when the same questions were used. But here the improvement in the students' explanation is striking and transcends what could be expected from either memory of questions or just currency of the concepts following teaching. The move from inappropriate additive strategies highlighted in the literature is particularly encouraging, whereas the issues with Question 2 could support the arguments of Van Doren and De Bock (2005) about over-use of proportional strategies or could also be attributed to the multiplicative language being misleading.

Lesson Analysis. Lessons 4 – 7 focused on using percentages in contexts like discounts, comparing discounts, and taxation, and investigations of when to use additive strategies and when to use multiplicative strategies. The overwhelming response here was that the extended discussion generated by the lesson materials was a great success and promoted student engagement and learning. For example, feedback from both teachers and students indicated that the time spent talking about what a discount is with examples from real life was particularly valuable. One teacher described Lesson 6 as "the epiphany lesson" where the students realised why they needed to be able to calculate percentages. The opportunity for students to elaborate their thinking was the main reason for the positive response to this aspect of the lessons. Students embraced the approach. Another teacher comment was: *The high point of the whole thing was that they did have to nut things out, discuss.*

In Lesson 4, students compared a fixed discount of $1 off meals deals for "math burgers" and whether it was better to buy two $5 deals (Nell) or one $10 deal (Grace). A typical answer was: *Nell, because she would get a $2 discount whereas Grace only gets a $1 discount*.

Using percentages to compare discounts was common in two classes at different schools but not mentioned in the other three classes. In one school a student came up with the idea (Nell gets 20% discount, Grace only 10%); it spread among more students and finally the teacher caught on and used it with other students.

When a comparison of a fixed tax of $10 over the 10% GST was discussed, students' reactions were mixed as to whether the GST was fairer. *Yes, because otherwise you could buy a $1 lollypop and the tax would come in and it would cost you $11 which is a rip off. No, it's not fair because if you get something that's expensive, you pay a lot of tax.*

In other questions where differing discounts of different amounts occurred, nearly everyone stated that a bigger percentage reduction does not always mean a cheaper buy because it depends on the original price—they observed that both the discount and "percent of what" were relevant. Some students gave a couple of examples to illustrate this point. However, the notion of "best" could still have different interpretations, with one student thinking the best deal was the lower cost not the bigger discount.

When asked which is a better result, 60 merit certificates in a school of 500 or 80 in a school of 800, most students compared the results of the two schools using percentages – one having 10% of students awarded certificates and the other school having a higher percentage. Only a couple of students tried to calculate what the other percentage was. One calculated it as being 15% and the other calculated it as 16% (the correct percentage being 12%), but their reasoning was correct.

Not all of the lessons were received positively. Lesson 5, which focused on the fact that additive comparisons are sometimes more appropriate (as with ages in interview Question 2), was seen as problematic. This lesson also involved answers which were value judgments (e.g., Which is worse: losing 50% of $1, $10, or $100?). Teachers reported being very uncomfortable with this lesson and, in fact, in one school the teacher handed over the teaching to one of the authors who was present.

Although the teachers agreed about the benefits of the open discussion, it was also a challenge because it went beyond what was their normal practice. Time was a factor especially when students got carried away with a digression like the size of a burger. With respect to tax, some thought GST is fair because the money comes back to you but one student was adamant that the government should not take 10% because they did not make the things.

Another aspect is that teachers differed in the way in which they marked worksheets.

- One teacher simply checked the worksheets for completeness and ticked once on the front page.
- Students marked each other's work. Every answer was ticked, even when the explanations showed that an answer was wrong.
- Students marked each other's work, but afterwards the worksheets were marked by the teacher. The teacher crossed out ticks and wrote specific comments such as "*50% of what?*" and "*It should be split into 10 parts!*"

Students' marking of each others work is a useful practice commonly followed in primary schools. However, marking an explanation is much more difficult than marking numerical answers and clearly requires a greater level of supervision by the teacher.

The other challenge brought forward by the teachers was the suggested order within the lessons. The materials began with contextual investigations without a great deal of scaffolding, and left discussion of the general principles to the end. Two teachers changed the sequence of this lesson by moving the final discussion (Step 4) to the beginning of the lesson. They then had little or no closing time in which students could discuss what they had learnt from the lesson. Another teacher agreed with these two, saying she had followed the prescribed order but in retrospect would choose to do it their way. In an observed "mathsburger" lesson, the teacher began by modeling a similar context where pets were sold for varying fixed discounts and talking extensively about what a fixed discount was. The teachers generally felt that the students needed more guidance before starting on the worksheet. There was the natural feeling, perhaps arising from traditional practice, that it is important for students to get worksheet answers correct. This is not likely to happen when worksheets are used to pose challenging problems for children to consider and learn from and to form the basis for later class discussion. Only one teacher said starting with the worksheet was a good way to proceed.

In conclusion, both qualitative and quantitative data support the claim that the extended discussion generated by the lesson materials was generally successful. The exception is Lesson 5, where the need for additive strategies and value judgements seemed too unusual for most teachers.

Conclusions

The results are consistent with White and Mitchelmore (2005) in showing that the approach taken has the potential to benefit student engagement, learning, and attitudes for both students and teachers. The regional setting did not seem to provide any different results to the previous study, except that the teachers indicated they did not normally have opportunities for such professional development. The decision to reduce the content of the unit does seem to have given sufficient time. Further development of the unit appears to be worth pursuing, with perhaps a further look at Lesson 5. What, however, do the results show about the theoretical model of *Teaching for Abstraction*?

We recall that Teaching for Abstraction consists of four stages:

- *familiarising* oneself with the structure of a variety of relevant contexts;
- *recognising* the similarities between these different contexts;
- *reifying* the similarities to form a general concept; and then
- *applying* the concept in new situations.

The *familiarizing* stage in the previous study showed a need to explore separate contexts in more detail. This aspect was successfully adopted here, with the choice of context exploration and discussion being strongly supported by teachers and students. A possibly negative aspect was that context discussion in areas like tax and discount was enriching but time consuming, and could provide different answers to the anticipated mathematical one. One need (expressed by teachers in the final workshop) was to learn more of the teaching approach adopted in the materials, especially the strategy of allowing the children to explore ideas and problems before the teacher telling them.

In the previous study, the *recognising* stage in calculation skills was identified as requiring attention. This extra attention was given here and the results indicate success apart from where fractional answers resulted. More contexts involving fractions are indicated as desirable.

The assessment of *reification* in the previous study indicated more emphasis needed to be put on explaining when and why percentages “work”. This unit actually showed explaining was a strength and the learning here is considered most valuable. The post interview analysis shows the students readily *applied* their knowledge to new situations and so, again, the discussion/investigation aspect of the unit was shown to be successful.

White and Mitchelmore (2005) emphasised that preparation for Teaching for Abstraction needs to be carefully thought out. It is again evident that this approach is radically different from that which students and teachers are accustomed to. In particular, the teachers’ inclination to reorder lessons to provide the general principle before immersion in the contexts shows a lack of comfort with or understanding of the Teaching for Abstraction approach. A possible conclusion is that the approach is too radical. It could be argued that the positive outcomes were simply the result of establishing interactive classrooms. However, we claim that the true cause was the context-based learning which is a feature of our theory. Our conclusion, therefore, is that the theoretical model (even if it was not followed rigorously) resulted in new directions for teachers and improved learning for students. The teachers involved were in fact extremely positive about the approach, and have asked for further professional development in this area. The challenges for them, though, are clear—addressing and assessing generalisations and when these are introduced in a lesson; accepting multiple answers and methods of doing calculations; and coping with a lack of confidence in working with new ideas. Teachers need more support in terms of both content and pedagogy. A project where teachers are assisted to develop their own materials following the Teaching for Abstraction model would seem an appropriate next step.

Acknowledgements. This research was funded by the ACT Hub of the National Centre for Science, Information and Communication Technology and Mathematics Education for Rural and Regional Australia (SiMERR). The projected is indebted to Heather McMaster for her contribution to the data analysis.

References

Misailidou, C., & Williams, J. (2003). Diagnostic assessment of children's proportional reasoning. *The Journal of Mathematical Behaviour*, *22*, 335-368.

Mitchelmore, M. C. (2002). The role of abstraction and generalisation in the development of mathematical knowledge. In D. Edge & Y. B. Har (Eds.). *Mathematics education for a knowledge-based era* (Proceedings of the Second East Asia Regional Conference on Mathematics Education and the Ninth Southeast Asian Conference on Mathematics Education, Vol. 1, pp. 157-167). Singapore: Association of Mathematics Educators.

Mitchelmore, M. C., & White, P. (2000). Development of angle concepts by progressive abstraction and generalisation. *Educational Studies in Mathematics*, *41*, 209–238.

Mitchelmore, M. C., & White, P. (2004). Abstraction in mathematics and mathematics learning. In M. J. Høines & A. B. Fuglestad (Eds.),, *Proceedings of the 28th annual conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 329-336). Bergen, Norway: PME.

Parker, M., & Leinhardt, G. (1995). Percent: A privileged proportion. *Review of Educational Research, 65*, 421-482.

Skemp, R. (1986). *The psychology of learning mathematics* (2nd ed.). Harmondsworth, England: Penguin.

White, P., & Mitchelmore, M. C. (1996). Conceptual knowledge in introductory calculus. *Journal for Research in Mathematics Education*, *27*, 79-95.

White, P., & Mitchelmore, M. C. (2005). Teaching percentage as a multiplicative relationship. In P. Clarkson, A. Dowton, D. Gronn, A. McDonough, R. Pierce, & A. Roche (Eds.), *Building connections: Theory, research and practice* (Proceedings of the 28th annual conference of the Mathematics Education Research Group of Australasia, Melbourne, pp. 783-790). Sydney: MERGA.

Van Dooren, W., & De Bock, D. (2005). Not everything is proportional: Effects of age and problem type on propensities for overgeneralization. *Cognition and Instruction, 23, 57-86.*

My Struggle with Maths May Not Have Been a Lonely One: Bibliotherapy in a Teacher Education Number Theory Unit

Sue Wilson
Australian Catholic University
<s.wilson@signadou.acu.edu.au>

Bibliotherapy provides a new approach to eliciting and understanding the affective responses of pre-service primary teachers. This paper further explores bibliotherapy as a reflective tool in teacher education by analysing affective responses of pre-service primary teachers studying an elective number theory unit. Pre-service teachers voluntarily wrote responses to readings about school students' learning, discussed their understanding of their own experiences in the light of the readings, and identified readings that impacted most on them. The paper describes the responses using the five stages of the bibliotherapy and identifies some factors which affect levels of engagement with the process.

We read to know
we are not alone
C S Lewis (in Attenborough, 1993)

Introduction

Bibliotherapy is a technique that was developed in psychology and library science. It aims to use guided reading and discussion to assist individuals to overcome negative emotions related to their real-life problems. Hendricks, Hendricks, and Cochran (1999) trace the development of the process and discusses its applications. Bibliotherapy has been used in preparing pre-service teachers to teach students with emotional and behavioural disorders (Marlowe & Maycock, 2000) and students with special needs (Morawski, 1997) by encouraging pre-service teachers to identify with the teachers in the readings. Bibliotherapy has also been used to help secondary students overcome mathematics anxiety (Furner & Duffy 2002; Hebert & Furner, 1997). Taken together, these studies suggest that bibliotherapy has potential as a technique to address mathematics anxiety in pre-service primary teachers.

Previous research of pre-service teachers in a unit focusing on mathematics and learning difficulties (Wilson & Thornton, 2005; 2006) suggested bibliotherapy as a promising new tool for eliciting and understanding pre-service teachers' affective responses and providing a framework and language for educators to understand and communicate about the reflective process. In that research, pre-service teachers reflected on their perception of themselves as learners of mathematics, identifying with students in case studies and re-evaluating their own experiences, developing a more positive self-image as learners of mathematics and gaining insight into how children's anxiety about mathematics can be minimised (Wilson & Thornton, 2005). These reflections had a dual nature, showing both affective and cognitive elements.

Not all pre-service teachers have the opportunity to participate in a unit focusing on students' learning difficulties in mathematics. This study extends the conversations about the use of bibliotherapy in mathematics teacher education by exploring its use in a unit that had a mathematics content focus, in this case an elective number theory unit. It describes a pilot study of the responses of students to the bibliotherapy process in the context of a unit

Proceedings of the 30th annual conference of the Mathematics Education Research Group of Australasia
J. Watson & K. Beswick (Eds), © MERGA Inc. 2007

where the readings were not part of the unit content nor set as an assessment task. A smaller selection of the readings from the previous study (Wilson & Thornton, 2006) was used, and a modified process of writing about a critical incident, followed by weekly reflections, was followed. This study will be used to inform further research investigating how bibliotherapy might be used during mathematics units for pre-service teachers to examine their attitudes towards themselves as learners and teachers of mathematics.

Theoretical Framework

The theoretical framework is based on research on three components: bibliotherapy; pre-service teachers' beliefs, attitudes and emotions; and mathematics anxiety.

Bibliotherapy

Bibliotherapy can be defined as "the guided reading of written materials in gaining understanding or solving problems relevant to a person's therapeutic needs" (Riordan & Wilson, 1989, p. 506, quoted in Myracle, 1995). It is a technique that aims to assist individuals to overcome negative emotions related to their real-life problem by guided reading about another person's problem. The readers identify with the protagonist in the story, but feel safe because they are not the one experiencing the crisis. Readers interpret through the lens of their own experiences. Reading is followed by discussion in a non-threatening environment (Aiex, 1996).

Clinical bibliotherapy involves a therapist working with individuals with serious emotional or behavioural problems. Developmental bibliotherapy, as in this study, is used to refer to the use of guided reading with students (Hebert & Furner, 1997, p. 170).

The stages of bibliotherapy can be summarised as:

identification - the reader identifies with and relates to the protagonist.
catharsis - the reader becomes emotionally involved and releases pent-up emotions.
insight - the reader learns through the experiences of the character and becomes aware that their problems might also be addressed or solved.
universalization – the recognition that we are not alone in having these problems, we "are in this together" (Slavson, 1950, quoted in Hebert & Furner, 1997, p. 170).
projection – the reader can envisage having a different concept of their professional identity.

Wilson and Thornton (in press) identified this fifth stage in their study of pre-service teachers, and describe it in terms of the literature on projective identity (2007). The process of bibliotherapy "requires a meaningful follow-up discussion" (Hebert & Furner, 1997, p. 169). Participants become involved in discussions and follow-up activities such as journal writing (Flores & Brittain, 2003).

Pre-service Teachers' Beliefs, Attitudes and Emotions

Thompson's (1992) review into affective elements of mathematics education concluded that teachers' beliefs limit their openness to change. In addition, Pajares (1992) noted that pre-service teacher beliefs about mathematics and mathematics teaching are established as a result of their own school experiences, and resist change. He used the metaphor for pre-service teachers as "insiders in a strange land". Unlike medical or law students, they enter a familiar environment and thus changing their conceptions of teaching can be particularly difficult. Buerk (1982, p.19), identified students who believed that "mathematics is only a collection of correct answers and proper methods", and whose views about mathematics knowledge conflicted with their general view of knowledge, and suggested that identifying and overcoming the disparity may address their negative feelings about mathematics. In a similar study, Seaman, Szydlik, Szydlik, and Beam (2005) identified contradictions in pre-service teachers' beliefs about the nature of mathematical behaviour that persisted throughout their program, and concluded that teacher education programs should encourage students to reflect on their existing beliefs. Borasi (1990, p. 179) emphasised the importance of students identifying their beliefs. Beliefs have both a cognitive and an affective aspect (Grootenboer, 2006). A significant number of primary school teachers identified their school experiences as a factor in their beliefs about mathematics (Carroll, 2005).

A number of studies have reported on the benefits of reflection in pre-service teacher education courses. Mathematical autobiographies have been used to encourage reflection by pre-service teachers (Ellsworth & Buss, 2000; Sliva & Roddick, 2001). Flores and Brittain (2003, p. 112) describe the use of writing "as a tool to help pre-service teachers reflect on their growth as they learn to teach mathematics". Ambrose (2004) states that reflection alone may not change pre-service teachers' belief systems and describes mechanisms that have potential for changing beliefs: providing emotion-packed, vivid experiences; becoming immersed in a community; reflecting on beliefs; and developing attitudes that help connect beliefs. Taken together, these studies provide a compelling case for focusing on pre-service teachers' perceptions of their own mathematics learning as an important strategy in addressing their attitudes about teaching mathematics.

Mathematics Anxiety

Mathematics anxiety has been identified as a learning difficulty for many children (Dossel, 1993). In addition, Hembree (1990) found that the level of mathematics anxiety of pre-service elementary teachers was the highest of any major on university campuses. Trujillo and Hadfield (1999) discussed the roots of mathematics anxiety in American pre-service primary teachers. Similarly, Haylock (2001) presented further evidence that many pre-service primary or early childhood teachers have anxiety about mathematics.

Research into primary teachers' effectiveness has emphasised deep and connected knowledge and a positive view of themselves as learners of mathematics (Askew, Brown, Rhodes, Johnson, & Wiliam, 1997; Ma, 1999), suggesting that pre-service teachers' mathematics anxiety is detrimental to their ability to teach mathematics effectively. As Wolodko, Willson, and Johnson (2003, p. 224) state:

> Our challenge is to help preservice teachers confront their past experiences and anxieties about teaching and learning of mathematics. If these are openly dealt with during their university education, fewer teachers may be content to teach just as they have been taught.

Recent studies of pre-service teachers with high levels of mathematics anxiety have shown low confidence levels to teach elementary mathematics (Bursal & Paznokas, 2006) and low mathematics teacher efficacy (Swars, Daane, & Giesen, 2006). The latter study concluded that "results of the interviews in this study seem to suggest that preservice teachers need experiences within mathematics methods courses which address their past experiences with mathematics" (p. 311).

Research investigating how university study might address this anxiety has focused on teaching mathematics to develop deeper knowledge (Chick, 2002) or on the impact of studying mathematics teaching strategies on pre-service teachers' beliefs and attitudes

(Frid, 2000). Wilson and Thornton (2005; 2006) concluded that enhancing pre-service self-image as learners and practitioners of mathematics using the bibliotherapy process may help them see mathematics as making connections and to encourage the view that all students can learn mathematics (Australian Association of Mathematics Teachers, 2002) as well as help them address their own mathematics anxiety.

Methodology

Research Context

The setting for this study was an elective number theory unit, at an Australian urban university in 2006. The unit explored aspects of number theory such as the historical development of the idea of number and number patterns. In addition, pre-service teachers wrote reflections on and discussed research papers that reported how school children feel about mathematics and about themselves as they learn mathematics and gave a broad overview of the difficulties that primary school students have in learning mathematics. The research papers included readings about mathematics anxiety (Dossel, 1993), understanding in mathematics (Skemp, 1976), how children learn mathematics, multiple approaches to learning mathematics, and children's beliefs about mathematics. The readings considered psychological and sociocultural aspects of learning mathematics, addressing both the affective and the cognitive domain. Readings were chosen for their potential to invoke an emotional response in the reader.

Data Sources and Collection Methods

In the first workshop, pre-service teachers were asked to describe a critical incident in their school mathematics education that impacted on their image of themselves as learners of mathematics. During the semester pre-service teachers wrote guided reflections on eight readings and wrote two in-class reflections, discussing these and their personal observations from schools. Suggested prompts such as: "What did you learn that was new?", "Something I disagreed with", "Something that surprised me", and "Something that confirmed what I thought", were used by some students while others wrote open-ended reflections. Pre-service teachers voluntarily agreed to participate in the study and chose which of the reflections they would complete. The students were aware that reflections submitted for the pilot study were not part of the content of the unit or its assessment, but had discussed the rationale for completing the readings as a valuable contribution to their professional learning.

Research Sample

The research sample for this study was a class of eleven (seven female and four male) pre-service primary teachers. The students were either in the second year of a four year education degree or the first year of a two year graduate entry education degree and hence differed in the amount of professional experience that they had completed. All had studied or were currently completing a unit focused on mathematical content. All 11 pre-service teachers agreed to participate in the study.

Data Analysis Methods

When the unit was completed, the critical incidents and journals were analysed for evidence of the stages of bibliotherapy. The quotations in this paper have been selected to provide an insight into the thinking of those who identified strongly with the readings, rather than as a representative sample from all pre-service teachers. This paper focuses on the extent that the bibliotherapy process was taken up by pre-service teachers in this context. Fictitious female names were assigned to all students to preserve anonymity.

Results and Discussion

Critical Incidents

The critical incidents indicated the pre-service teachers' initial feelings. As might be expected from a group of pre-service teachers who had chosen a mathematics elective, most (seven of the eleven) reported positive experiences of mathematics. Faith expressed it thus, "I am a huge maths lover". In the description of the critical incident several mentioned the positive and lasting influence that an individual teacher had on their attitude towards mathematics.

Hilary professed positive attitudes, "At a basic level, I love maths. I love that there is an absolute right or wrong answer" but then described her reactions to her year 11 experiences, "I didn't understand and everything began to move away too quickly. I questioned and questioned but still couldn't come to an understanding, so I quit." This avoidance exemplified the coping mechanisms that some pre-service teachers use in situations that they find stressful (Sliva & Roddick, 2001) and is similar to the pre-service teachers whose written critical incidents reflections highlighted a cycle of fear, failure and avoidance reported in previous research (Wilson & Thornton, 2005).

Four pre-service teachers who expressed disquiet about their mathematical experiences at school reported struggling with a lack of understanding. "We never understood what the formulas were or why they worked" (Joyce). "If I did finally work out how, as soon as the question changed slightly, I wouldn't be able to do them" (Christine).

Journal Reflections

All participants submitted the critical incident and at least one of the in-class reflections. All except one person submitted reflections on at least one of the eight articles, with almost half the class submitting reflections on four or more articles. The two pre-service teachers who identified themselves as having more issues with anxiety submitted the most reflections. In the first half of the semester more than half the class submitted reflections, with numbers diminishing towards the end of the semester. The researcher attempted to gauge which readings had the most impact by asking students to select three of the readings that had resonated most with them for the final in-class reflection. All participants except one chose the Dossel (1993) article about mathematics anxiety, even though it was some ten weeks since they had written the reflection on this article.

Using readings to clarify pre-service teachers' understanding of their own learning was central to the bibliotherapy technique. An important part of the pre-service teachers' reflections revolved around the view of mathematics that they had developed during their schooling. I "was able to retain the formula, and put the correct variable in it but I did not really understand the concept" (Debbie). These views are consistent with those reported in the research literature. Taylor (2003, p. 333) investigated the common misconception among United States students "about the nature of mathematics as being built on remembered procedures". The study presented an alternate conception of the nature of mathematics as making connections.

Although the commitment to doing the weekly writing seemed to vary inversely with the pre-service teachers' perceptions of themselves as mathematics learners, most described the experience as useful, although one student expressed some disquiet about taking time from the unit to discuss the readings in the anonymous student evaluations. Two students responded to the readings by undertaking further research on mathematics anxiety for assessment tasks in other units that did not have a curriculum focus. One chose it as the topic for an assignment for another unit and the second convinced the two team members in her group to use mathematics anxiety as the focus of their group presentation. One of the members of the class for this presentation described the panic she felt when suddenly presented with questions about mathematics in a context where she was not expecting them.

The journal entries provided evidence that some students had shown an emotional response to the readings, had reflected on their own experiences and had engaged in the stages of bibliotherapy.

Identification. The pre-service teachers' reflections showed that they identified with the character (in this case the students in the articles) and the situation in which they found themselves. "I have struggled with maths anxiety without being aware that I had it" (Debbie). The use of bibliotherapy encouraged pre-service teachers to reflect on themselves as learners of mathematics: "I have connected with the articles as a learner of maths too" (Bev).

Catharsis. Through their reading of the articles the pre-service teachers became emotionally involved and released pent-up emotion. "As soon as new maths concepts were presented I would get very panicky" (Debbie). Joyce felt the article (Dossel, 1993) confirmed a lot of her own experiences of high school, "Can anyone blame a girl for wanting to stick to what they feel they can cope with – rather than risking the humiliation of tackling the unknown connections between big ideas" and included a quotation attributed to Edward E. David Jr "mathematics courses are chiefly designed to winnow out the weak and grind down the ungifted". These students responded emotionally and connected with the readings.

Insight. Through their readings and discussion the pre-service teachers gained a different perspective from the experiences of others and became aware that their problems might also be addressed. "I had never heard of maths anxiety prior to this. It pieced many pieces together in this puzzle of mine" (Faith). "I have taken in as a learner that it is ok to get an answer that is different from everyone else" (Bev). Difficulties from school were because "the teacher hadn't explained in the class in a way that I understood, or was relevant to me" (Christine). Realising this was a valuable part of the process.

Universalisation. Reflecting on the readings and sharing of their experiences pre-service teachers were able to connect with each other and find that they were not alone in their feelings and experiences. Stories show that others have the same issues and one is not alone (Rizza, 1997). Joyce wrote: "I can see evidence of 'maths anxiety' every time I tell someone I am doing a subject called number theory". Debbie saw the process as incomplete, "I still feel that I have maths anxiety and it would take a while before I can overcome these feelings."

Projection. Their reflection on their own circumstances was followed by a consideration of what it could mean for the future and the implications of their insights for their teaching. "Reading about maths anxiety made me reflect on my own experiences as a child. It also made me think towards the future" (Alison). These pre-service teachers questioned not only the views that they had developed of themselves as learners of mathematics, but also the image that they had previously held of themselves as teachers of mathematics.

Bibliotherapy addresses Ambrose's (2004) criteria for mechanisms that have potential for changing beliefs, as it provides emotion-packed, vivid experiences, encourages pre-service teachers to become immersed in a reflective community, and connects beliefs and emotions. Pre-service teachers are thus able to modify their self-concept as "insiders" as identified by Pajares (1992) and re-image themselves as teachers who do not only teach "just as they have been taught" (Wolodko et al., 2003). This has important implications for developing pre-service teachers' ability to write reflectively. Askew, Brown, Rhodes, Johnson, and Wiliam (1997) found evidence that teachers' perceptions of mathematics and how it is learned were more important in promoting positive outcomes for students than different teaching methods or ways of organising classrooms.

Conclusion and Implications

The juxtaposition of bibliotherapy with mathematics teacher education units has proved to be a powerful strategy to address mathematics anxiety in pre-service teachers. Although teaching mathematics units well to pre-service primary teachers is important in their teacher education, a focus on learning (or learned) difficulties is necessary to address some of the anxiety felt. The strength of the bibliotherapy technique is that the identification, catharsis, insight, universalisation, and projection allow the pre-service teachers to reflect more coherently on their beliefs about mathematics learning and teaching. The special feature of the bibliotherapy approach of eliciting pre-service teacher reflections stems from its ability to call forth cognitive responses paralleled by emotional responses. In comparison to other reflective practices, the potential of bibliotherapy lies in opportunity to change the way pre-service teachers feel. The unique feature of using bibliotherapy to address mathematics anxiety is that, unlike other studies where pre-service teachers identify with teachers in the readings, the pre-service teachers in this study identify with the students.

This study investigated the extent to which the bibliotherapy process was taken up in a unit where readings were presented to the students as a valuable contribution to their professional learning rather than the content of the unit, and the reflections did not form part of an assessment item. These results and observations have implications for the way the bibliotherapy process could be incorporated into other teacher education courses. It might take more time to go through the final stages of the process in units such as these, although it is important to realise that everyone is unique and there is no schedule for recovery. From the responses of the pre-service teachers, it is apparent that the stages of bibliotherapy are not linear and do not only happen once. Each reading has the potential to stimulate a new cycle of responses which can be described as identification, catharsis, and universalisation. With each cycle pre-service teachers develop greater insight eventually leading to a robust projection into their future as teachers.

It would be valuable in future research to identify useful articles or readings that impact on the majority of pre-service teachers and to investigate successful ways of integrating bibliotherapy into a range mathematics teacher education courses in ways that benefit all students, not only those who suffer from mathematics anxiety.

The pre-service teachers' comments give voice to the concern that negative learning experiences will not reinforce negative beliefs and feelings about mathematics in the students they will teach and echo the concerns of teacher educators who identify this as an issue. "It is definitely worth the effort to free our students" (Debbie).

References

Aiex, N. (1996). Bibliotherapy. *Learning Disabilities OnLine*. ERIC Digest 82 Retrieved August 12, 2004, from http://www.ldonline.org/ld_store/bibliotherapy/eric_digest82.html .

Ambrose, R. (2004). Initiating change in prospective elementary school teachers' orientations to mathematics teaching by building on beliefs. *Journal of Mathematics Teacher Education, 7*, 91-119.

Askew, M., Brown, M., Rhodes, V., Johnson, D., & Wiliam, D. (1997). *Effective Teachers of Numeracy (Final Report).* London: King's College.

Attenborough, R. (director). (1993). *Shadowlands* [Motion Picture] Great Britain: Pinewood Shepperton Studios.

Australian Association of Mathematics Teachers, Inc. (2002) *Standards for excellence in teaching mathematics in Australian schools*. Adelaide: AAMT, Inc

Borasi, R. (1990) The invisible hand operating in mathematics instruction: Students' conceptions and expectations. In Cooney, T. & Hirsch, C. (Eds) *Teaching and learning mathematics in the 1990's.* (1990 Yearbook). (pp. 174-182) Reston, VA: National Council of Teachers of Mathematics.

Buerk, D. (1982). An experience with some able women who avoid mathematics. *For the Learning of Mathematics, 3*(2), 19-24.

Bursal, M., & Paznokas, L. (2006). Mathematics anxiety and preservice elementary teachers' confidence to teach mathematics and science. *School Science and Mathematics, 106*(4), 173-80.

Carroll, J. (2005). Developing effective teachers of mathematics: Factors contributing to development in mathematics education for primary school teachers. In P. Clarkson, A. Downton, D. Gronn, M. Horne, A. McDonough, R. Pierce, & A. Roche (Eds.), *Building connections: Research theory and practice.* (Proceedings of the 28th annual conference of the Mathematics Education Research Group of Australasia, pp. 202-209). Sydney: MERGA.

Chick, H. (2002). Evaluating pre-service teachers' understanding of middle school mathematics. In B. Barton, K. Irwin, M. Pfannkuch, & M. Thomas (Eds.), *Mathematics Education in the South Pacific* (Proceedings of the 25th annual conference of the Mathematics Education Research Group of Australasia, pp.179-186). Sydney: MERGA.

Dossel, S. (1993). Maths anxiety. *Australian Mathematics Teacher*, *49*(1), 4-8.

Ellsworth, J., & Buss, A. (2000). Autobiographical stories from preservice elementary mathematics and science students: Implications for K-16 teaching. *School Science and Mathematics, 100*(7), 355.

Flores, A., & Brittain, C. (2003). Writing to reflect in a mathematical methods course. *Teaching Children Mathematics*, *10*(2), 112-118.

Frid, S. (2000). Constructivism and reflective practice in practice: challenges and dilemmas of a mathematics educator. *Mathematics Teacher Education and Development, 2*, 17-34.

Furner, J., & Duffy, M. (2002). Equity for all students in the new millennium: Disabling math anxiety. *Intervention in School and Clinic, 38*(2), 67-75.

Haylock, D. (2001). *Mathematics explained for primary teachers*. London: Paul Chapman.

Hebert, T., & Furner, J. (1997). Helping high ability students overcome maths anxiety through bibliotherapy. *Journal of Secondary Gifted Education, 8*(4), 164-179.

Hendricks, C., Hendricks, J., & Cochran, L (1999). *Using literacy conversations for healing: The significant conversationalists*. American Reading Forum Online Yearbook Vol XIX Retrieved 8 March 2007 from http://www.americanreadingforum.org/99_yearbook/pdf/8_hendricks_99.pdf

Hembree, R. (1990). The nature, effects and relief of mathematics anxiety. *Journal for Research in Mathematics Education, 21*, 33-46.

Grootenboer, P. (2006). Mathematics educators: Identity, beliefs, roles and ethical dilemmas. In P. Grootenboer, R. Zvenbergen, & M. Chinnappan (Eds.), *Identities, Cultures and Learning Spaces* (Proceedings of the 29th annual conference of the Mathematics Education Research Group of Australasia, 270-277). Sydney:MERGA.

Ma, L. (1999). *Knowing and teaching elementary mathematics*. Mahwah, NJ: Lawrence Erlbaum.

Marlowe, M. & Maycock, G. (2000). Phenomenology of bibliotherapy in modifying teacher punitiveness. *Journal of Genetic Psychology. 161*(3), 325-336

Myracle, L. (1995). Molding the minds of the young: The history of bibliotherapy as applied to children and adolescents. Th*e Allan Review* 22(2). Retrieved August 12, 2004, from http://scholar.lib.vt.edu/ejournals/ALAN/winter95/Myracle.html .

Morawski, C. (1997). A role for bibliotherapy in teacher education. *Reading Horizons, 37*(3), 243-259.

Pajares, M. (1992). Teachers' beliefs and education research: Cleaning up a messy construct. *Review of Educational Research,* 62(3), 301-332.

Riordan, R., & Wilson, L. (1989). Bibliotherapy: does it work? *Journal of Counselling and Development*, *67*, 506-507.

Rizza, M. (1997). *A Parent's Guide to Helping Children: Using Bibliotherapy at Home.* Retrieved January 2, 2005, from http://www.sp.uconn.edu/~nrcgt/news/winter97/wintr972.html .

Seaman, C., Szydlik, J., Szydlik, S., & Beam, J. (2005). A comparison of preservice elementary teachers' beliefs about mathematics and teaching mathematics: 1968 and 1998. *School Science and Mathematics*, *150*(4), 197-210.

Skemp, R. (1976). Relational understanding and instrumental understanding. *Mathematics Teaching*, *77*, 20-26.

Slavson, S. (1950). *Analytical group psychotherapy with children, adolescents, and adults*. New York: Columbia University Press.

Sliva, J., & Roddick, C. (2001). Mathematics autobiographies: a window into beliefs, values, and past mathematics experiences of preservice teachers. *Academic Exchange Quarterly,* Summer 2001 *5*(2) 101-107.

Swars, S., Daane, C & J. (2006). Mathematics anxiety and mathematics teacher efficacy: What is the relationship in elementary preservice teachers? *School Science and Mathematics, 106*(7), 306-315.

Taylor, A. (2003). Transforming pre-service teachers' understanding of mathematics: Dialogue, Bakhtin and open-mindedness. *Teaching in Higher Education, 8*(3), 333-344.

Thompson, A. (1992). Teachers' beliefs and conceptions: A synthesis of the research. In D. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 127-146). New York: Macmillan.

Trujillo, K. & Hadfield, O. (1999). Tracing the roots of mathematics anxiety through in-depth interviews with pre-service elementary teachers. *College Student Journal*, 33(2), 219-232.

Wolodko, B., Willson, K., & Johnson, R. (2003). Preservice teachers' perceptions of mathematics: Metaphors as a vehicle for exploring. *Teaching Children Mathematics, 10*(4), 224-230.

Wilson, S. & Thornton, S. (2005). I am really not alone in this anxiety: Bibliotherapy and pre-service primary teachers' self-image as mathematicians. In P. Clarkson, A. Downton, D. Gronn, M. Horne, A. McDonough, R. Pierce, & A. Roche (Eds.), *Building connections: Research theory and practice.* (Proceedings of the 28th annual conference of the Mathematics Education Research Group of Australasia, pp. 791-798). Sydney: MERGA.

Wilson, S., & Thornton, S (2006). To heal and enthuse: Developmental bibliotherapy and pre-service primary teachers' reflections on learning and teaching mathematics. In P. Grootenboer, R. Zevenbergen & M. Chinnappan (Eds.), *Identities, Cultures and Learning Spaces* (Proceedings of the 29th annual conference of the Mathematics Education Research Group of Australasia, Canberra (pp. 36-44).

Wilson, S., & Thornton, S. (2007). *Bibliotherapy: Describing and understanding affective change in pre-service teachers.* Manuscript submitted for publication.

Wilson, S., & Thornton, S. (in press) "The factor that makes us more effective teachers": Two pre-service teachers' experience of bibliotherapy. *Mathematics Teacher Education and Development.*

Students' Conceptual Understanding of Equivalent Fractions

Monica Wong
University of Sydney
<monica.wong@optusnet.com.au>

David Evans
University of Sydney
<d.evans@edfac.usyd.edu.au>

This paper investigates students' conceptual understanding of equivalent fractions by examining their responses to questions using symbolic and pictorial representations. Two hundred and thirteen students in Years 3 to 5 from three Sydney primary schools were administered a general mathematics achievement test and a fraction assessment. Five questions from this fraction assessment instrument were analysed. The different types of knowledge used to answer each question were examined and common misconceptions identified. The responses of students with limited general mathematics achievement were compared to those of their more competent peers. The differences that emerged between the two groups in their conceptual understanding of equivalent fractions, were highlighted.

The development of conceptual understanding involves seeing the connections between concepts and procedures, and being able to apply mathematical principles in a variety of contexts. It is a central focus of the NSW Mathematics curriculum (Board of Studies NSW (BOS NSW), 2002). Considering the difficulties experienced by students in mastering equivalent fractions and the many misconceptions they hold (e.g., Gould, 2005a, 2005b; National Research Council (NRC), 2001; Pearn, Stephens, & Lewis, 2003), identifying the nature of the differences in conceptual understanding between students of varying levels of general mathematical proficiency provides a mechanism to inform the teaching of this particular concept (NRC, 2001).

As part of a larger study which examined students' conceptual understanding of equivalent fractions, an Assessment of Fraction Understanding (AFU) instrument was developed. The pencil and paper test contained 34 questions that were used to measure students' conceptual understanding, their ability to solve routine problems and to adapt their understanding to non-routine problems (NRC, 2001; Shannon, 1999). Three schools participated in this phase of the study. All students were administered the AFU instrument and some students also participated in semi-structured interviews.

This paper focuses specifically on five fraction questions from the AFU and their diagnostic potential in identifying students' misconceptions. Comparisons between the responses of students with naïve and with more advanced mathematical understanding assist in defining the progressive learning sequences followed by students to master and understand equivalent fractions.

Theoretical Perspective

Systems of Knowledge

Mathematics is a reasoning activity that involves observing, representing and investigating relationships in the social and physical world, or between mathematical concepts themselves (BOS NSW, 2002). A mathematical concept is not a single isolated idea but one idea in a structured system of knowledge or schemata (Anderson, 2000; Lesh, Landau, & Hamilton, 1983). Information-processing models of cognitive development suggest that within these structured systems of knowledge, information stored in memory can be categorised into declarative and procedural knowledge (Anderson, 2000).

Declarative knowledge is knowledge of specific facts and ideas (Anderson, 2000). Mathematical definitions of procedural knowledge assume a foundation of declarative knowledge: "a familiarity with the individual symbols of the system and with the syntactic conventions for acceptable configurations of symbols" (Hiebert & Lefevre, 1986, p. 7). Procedural knowledge also incorporates the awareness of how to approach a task and its related steps or algorithms (Anderson, 2000).

Conceptual understanding in mathematics develops when students "see the connections among concepts and procedures and can give arguments to explain why some facts are consequences of others" (NRC, 2001, p. 119). Facts are no longer isolated but become organised in coherent structures based on relationships, generalisations and patterns, Conceptual understanding has also been described as "conceptual knowledge" (Anderson, 2000; Rittle-Johnson, Siegler, & Alibali, 2001) and "relational understanding" (Skemp, 1986). Rittle-Johnson et al. (2001) found that developing students' procedural knowledge had positive effects on their conceptual understanding, and conceptual understanding was a prerequisite for the students' ability to generate and select appropriate procedures.

Thus, conceptual understanding is intertwined with procedural knowledge. This makes the isolated study of either difficult, requiring more than the determination of the correctness/incorrectness of a student's answer. It requires further investigation into the response, which can provide valuable insight into the thinking (Gould, 2005a; 2005b).

Fraction Knowledge

A *common fraction* (fraction) is often described as the ratio or quotient of two whole numbers, *a* and *b*, expressed in symbolic form $\frac{a}{b}$, where *b* is not zero (BOS NSW, 2002). It is a symbol that has meaning and can be interpreted and manipulated. The fraction schemata includes five interconnected, yet distinct interpretations (Lamon, 2001), as shown in Table 1. Using these interpretations, one can explore the various characteristics and manipulations of fractions (such as proper and improper fractions, mixed numerals, fraction equivalence, comparison, addition, multiplication and division). The concept of fractions is also linked to other mathematical concepts such as geometry, number-lines, and whole number multiplication and division.

Table 1

Different Fraction Interpretations for the fraction $\frac{3}{4}$

Interpretations	Example
Part/whole	3 out of 4 equal parts of a whole or set of objects or collection
Measure	$\frac{3}{4}$ means a distance of 3 ($\frac{1}{4}$ units) from 0 on the number line
Operator	$\frac{3}{4}$ of something, stretching or shrinking
Quotient	3 divided by 4, $\frac{3}{4}$ is the amount each person receives
Ratio	3 parts cement to 4 parts sand

Fraction concepts can be explained by teachers and students using a combination of external representations such as written symbols, spoken language, concrete materials, pictures, and real world examples (Lesh et al., 1983).

Conceptual Understanding of Fraction Equivalence

Fraction equivalence is one concept within the extensive fraction schemata. Equivalence implies similar worth. Thus two common fractions are considered equivalent when they have the same value (BOS NSW, 2002; Skemp, 1986). A fraction represents a number with an infinite number of names. Listing some of these names makes it apparent that each individual fraction is part of an "equivalence set". For example, the equivalence set for the fraction $\frac{1}{2}$ can be represented as $[\frac{1}{2}, \frac{2}{4}, \frac{3}{6}, \frac{4}{8}, ...]$. Implicit in the concept of equivalence is the knowledge that each fraction in the set is interchangeable with the others.

Conceptual understanding of equivalent fractions involves more than remembering a fact or applying a procedure. It is based on an intricate relationship between declarative and procedural knowledge; between fraction interpretation and representation. Students should able to: (a) make connections between fraction models by understanding the sameness and distinctness within these interpretations (Lesh et al., 1983; NRC, 2001); (b) make connections between the different representations (Lesh et al., 1983), and (c) show that a fraction represents a number with many names. The present study examines a small portion of the large body of knowledge associated with fractions.

Figure 1 depicts the scope of the questions used to identify students' conceptual understanding of equivalent fractions. At the lowest level, knowledge is declarative and procedural, loosely linked to specific examples of equivalent fractions (NRC, 2001) and not generalised across representations or interpretations. As students develop understanding, their knowledge becomes generalised and applied more broadly.

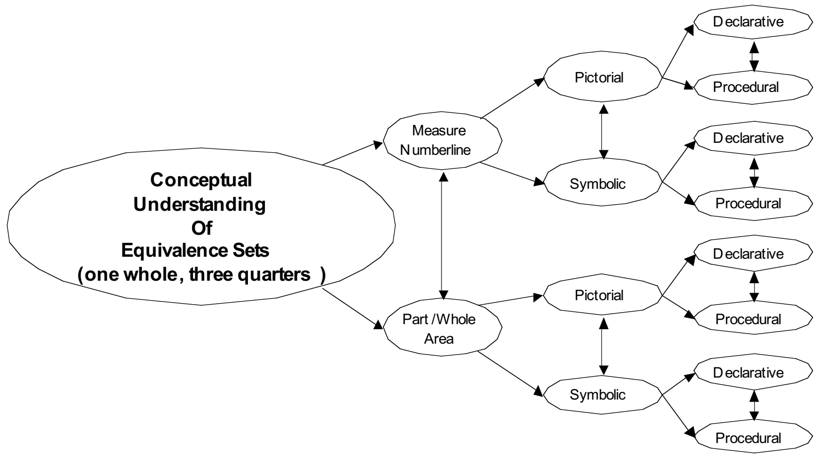
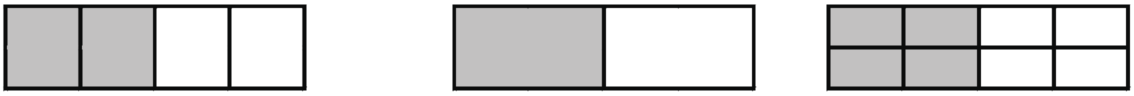


Figure 1. Model used to develop equivalent fraction questions (Adapted from Lamon (2001, p.151)).

In this study, students were presented with tasks that aimed to elucidate their level of thinking. The demands of the tasks were restricted to identifying symbolic and pictorial representations and representing fractions using part/whole area and measure models. They incorporated "skill" questions that required the recall of a practised routine or procedure, and "conceptual" questions that required students to apply their knowledge and explain their actions (Shannon, 1999).

Tasks that incorporate pictorial representations with visual distractors provide one method of measuring students' conceptual understanding of equivalent fractions. Such tasks have been found to highlight the unstable nature of a student's fraction knowledge

(Ni, 2001; Niemi, 1996). Pictorial representations of part/whole area and measure models can be described as "simple representations" when the total number of equal parts in the shape matches the fraction denominator. They allow students to count the parts (see Figure 2a). The shaded part is associated with the numerator and the entire shape is associated with the denominator. Equivalent pictorial representations are visually challenging. They occur when the number of equal parts of the whole is a multiplicative factor less or greater than the denominator (Niemi, 1996), as shown in Figures 2b and 2c. The areas of the whole and shaded part never change, but the number of equal parts into which the whole is divided can alter dramatically. Thus different fraction names can be offered for the shaded area and an equivalence set identified. Simple and equivalent representations for a measure model appear in Figure 3.



(a) simple (b) equivalent - 2 equal parts (c) 8 equal parts

Figure 2. Part/whole area model simple and equivalent representations for two quarters.

(a) simple (b) equivalent - 8 equal parts

Figure 3. Measure model simple and equivalent representations for two quarters.

Equivalent fraction tasks using symbolic notation (see Figure 4) are more cognitively demanding as up to four dimensions need to be simultaneously co-ordinated: the original two-dimensional fraction, $\frac{3}{8}$ and its equivalent, $\frac{12}{32}$ (English & Halford, 1995). Questions that incorporate the interpretation and manipulation of symbolic notation are ideal for identifying the levels of students' conceptual understanding of equivalent fractions. Evaluation of their responses provides an insight into the students' thought patterns, conceptual understanding and procedural knowledge. Teachers who understand how students develop this knowledge, and are able to help them to see the links between various representations are providing the most effective fraction programs for students.

(a) $\frac{3}{8} = \frac{\quad}{32}$ (b) $\frac{3}{8} = \frac{12}{\quad}$ (c) $\frac{3}{8} = \frac{\quad}{\quad}$ Answer for a and b: $\frac{3}{8} = \frac{12}{32}$

Figure 4. Typical equivalent fraction question and answer employing symbolic representations only.

The purpose of this study was to evaluate students' understanding of equivalent fractions through their responses to questions that incorporated symbolic and pictorial representations, and required them to identify measure and part/whole interpretations. Firstly, the types of knowledge used by students to answer these questions were investigated. Secondly, responses by students of varying general mathematical achievement were compared to examine the differences evident in their developing mastery of equivalent fractions.

Methodology

Participants

Two hundred and thirteen students from Years 3 to 5 from three Sydney primary schools participated in the study. Their details appear in Table 2.

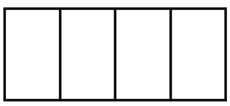
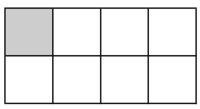
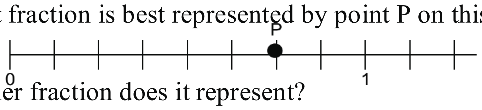
Table 2
Participant Details

Grade level	Sample size (n)	Age (years)		Gender	
		Range	Avg.	% Boys	% Girls
3	64	8.15-10.21	8.84	48.4	51.6
4	80	7.97-11.08	9.84	50.0	50.0
5	69	10.02-12.75	10.81	37.7	62.3

Instruments

The *Progressive Achievement Tests in Mathematics* (PATMaths) was used to measure students' general mathematics achievement (Australian Council for Educational Research [ACER], 2005). As recommended by ACER, different tests were used for grades 3 to 5. All tests were norm referenced and scores calibrated on a common scale. The questions for the Assessment of Fraction Understanding (AFU) were derived and adapted from various assessment instruments including the Trends in Mathematics and Science Study, the North Carolina Testing Program, the California Standards Test and the Success in Numeracy Education program (Catholic Education Office, 2005). The questions analysed in this paper related to the fraction "one whole" and "three quarters" and appear in Table 3, along with the representation mapping used for each question.

Table 3
Questions Analysed

Fraction	Symbolic to Pictorial	Symbolic to Symbolic
One whole	14. Shade in $\frac{2}{2}$ of the shape below?  Can you think of another name for the fraction shaded?	29. Circle the fractions that are equal to 1? $\frac{8}{8}$ $1\frac{1}{100}$ $\frac{1}{1}$ $\frac{9}{10}$ $\frac{4}{4}$ $1\frac{1}{8}$ $\frac{7}{8}$ $\frac{10}{9}$ $\frac{9}{8}$ How did you work this out?
Three quarters	13. In the figure, how many small squares need to be shaded so that $\frac{3}{4}$ of the small squares are shaded? 	28 (b). $\frac{6}{8} = \frac{\square}{\square}$
	Pictorial to Symbolic	
Three quarters	18. What fraction is best represented by point P on this number line? _____  What other fraction does it represent?	

The questions were linked to the *Mathematics K-6 Syllabus*, as shown in Table 4. Stage 2 (NS2.4) knowledge and skills are generally taught in years 3 to 4, whereas Stage 3 (NS3.4) skills are taught in years 5 to 6. All questions were open-ended, which allowed for students' understanding to be examined more effectively. Part/whole area questions used an equivalent area representation, and the measure question used a simple number-line representation. Questions 29 and 14 examined the concept of one whole, whereas questions 13, 28b, and 18 examined three quarters. Question 18 further illuminated the sophistication of the students' connections between measure and part/whole interpretations.

Table 4
Mathematics K-6 Syllabus Reference (BOS NSW, 2002)

Question	Syllabus Reference (knowledge and skills)	
14, 29	NS2.4 (1)	Renaming $\frac{2}{2}, \frac{4}{4}, \frac{8}{8}$ as 1
13	NS2.4 (2)	Finding equivalence between halves, quarters and eighths using concrete materials and diagrams, by re-dividing the unit
28(b)	NS3.4 (2)	Developing a mental strategy for finding equivalent fractions, e.g., multiply/divide the numerator and the denominator by the same number
18	NS2.4 (2)	Placing halves, quarters and eighths on a number line between 0 and 1 to further develop equivalence

Procedure

All participants were tested during term three, 2006, over two consecutive days. The PATMaths test was administered on the first day and the AFU the following day. Both tests were administered following standardised protocols. Each pencil and paper test was of 45 minutes duration. Calculators were not permitted. Participants were asked to show all working for the AFU in their test booklet.

Results

Most Australian states and territories identify students "at risk" as the lowest achieving 20 percent of students (Doig, McCrae, & Rowe, 2003). Participants with limited *general mathematics achievement* (GMA) are identified as those students who score below the 20th percentile on their particular PATMaths test, when compared with the norming data (ACER, 2006). Students scoring in the middle 60% are considered to be developing mathematical knowledge at an appropriate level, whilst the upper most 20% are identified as more competent. Participants were categorised into achievement levels (see Table 5).

Table 5
Student Achievement

	General Mathematics Achievement (GMA)							
	Limited (N = 28)		Avg. (N = 152)		High (N = 33)		Total (N = 213)	
Grade	*n*	% of Grade	*n*	% of Grade	*n*	% of Grade	*n*	% of Total
3	6	9.4	45	70.3	13	20.3	64	30.0
4	8	10.0	61	76.3	11	13.8	80	37.6
5	14	20.3	46	66.7	9	13.0	69	32.4
Total	28	13.1	152	71.4	33	15.5	213	100.0

A Rasch analysis was conducted to determine the difficulty of each question used in the AFU. The relative difficulty of each item and other associated Rasch statistics are shown in Table 6. The 'fit residual' statistic confirms whether the item is over or under-discriminating in comparison to the theoretical dichotomous Rasch model (which has an acceptable fit statistic between -2 and 2) (Bond & Fox, 2001). The chi-square probability statistic verifies whether there is a statistically significant difference between the theoretical and observed item discrimination for each question (RUMM Laboratory, 2004). There were no significant deviations from the theoretical Rasch model for any of the five questions analysed in this study.

Table 6
Item Difficulty

Question	Difficulty	SE	Fit Residual	Chi-square	*p*
29. Circle fractions equal to 1	-0.691	0.150	-1.851	9.359	0.052
14. Shade in 2/2 of the shape	0.061	0.141	0.533	5.484	0.241
13. Shading 3/4 of small squares	0.211	0.141	-1.145	6.917	0.140
28 (b) 6/8 =	0.611	0.145	-0.564	2.729	0.604
18. Fraction represented on a number-line	1.821	0.157	-0.291	4.875	0.300

The easiest questions (i.e., 29 and 14) required students to identify one whole. Students were more able to identify three quarters of an equivalent area model than 1) to determine an equivalent fraction for three quarters using only symbolic representation or 2) to identify a fraction using a measure model.

Further question analysis identified the knowledge structures participants employed to solve these equivalent fraction problems. Commencing with the easiest question (29), Table 7 shows the percentages of students who answered the question correctly and incorrectly. Eighty percent of students who answered the question justified their response by stating that the top number and bottom number were the same. Participants explained their thinking by using procedural knowledge, which does not exclude conceptual understanding. The participants who provided an incorrect response provided no observable pattern of reasoning. From the incorrect responses given, many students selected fractions that contained the number 1 as part of the fraction (either 1/1 or mixed numerals containing the whole number 1).

Table 7
Responses to Question 29: Circle the Fractions that are Equal to 1 (n=61)

	Limited GMA (N = 28)		High GMA (N = 33)	
Answer selected	*n*	%	*n*	%
CORRECT:1/1, 4/4 and 8/8 selected	10	35.7	25	75.8
1/1 only	5	17.9	3	9.1
Two or more of the following selected: 1/1, 1 1/8, 1 1/00	7	25.0	1	3.0
Other	4	14.3	2	6.0
No response	2	7.1	2	6.0

The application of students' knowledge in linking symbolic to pictorial representations of "one whole" was tested in question 14 using an equivalent pictorial representation. Responses are tabulated in Table 8. Nearly all the participants who were able to answer the question correctly were also able to give another name for the fraction shaded. Only 10.7% (n = 3) of participants with limited GMA and 54.5% (n = 18) of participants with high

GMA were able to answer questions 29 and 14 correctly. Thus, these participants were able to show greater conceptual understanding as they applied their symbolic understanding of one whole to an equivalent pictorial representation. For those participants who answered the question incorrectly (shading 2 small squares), approximately half wrote “1/2” for the fraction shaded.

Table 8
Responses to Question 14: Shade in 2/2 of the Shape (n=61)

	Limited GMA (N = 28)		High GMA (N = 33)	
Number of small squares shaded	*n*	%	*n*	%
CORRECT: 4	5	17.9	22	66.7
Participants are able think of another name for the fraction shaded	4	14.3	20	60.1
2	22	78.6	10	30.3
Participants gave response 1/2 for fraction shaded	9	32.1	5	15.2
Other or Missing	1	3.6	1	3.0

Although 3/4 is a commonly presented fraction, low GMA participants had difficulty representing the fraction using an equivalent part/whole area diagram. Responses from all participants for question 13 are shown in Table 9. Their most common incorrect response was to shade three small squares. Six of these participants also shaded 2 squares in question 14, suggesting they used the value of the numerator in both questions to determine the number of squares to shade.

Table 9
Responses to Question 13: How Many Small Squares Need to be Shaded (n=61)

	Limited GMA (N = 28)		High GMA (N = 33)	
Number of small squared shaded	*n*	%	*n*	%
CORRECT: 6	4	14.3	23	69.7
2	5	17.9	1	3.0
3	12	42.9	8	24.2
1, 4, 5	4	14.3	0	0.0
Missing	2	7.1	1	3.0

Question 28b presented a symbolic to symbolic equivalent fraction question and participant responses are shown in Table 10. This question can be solved procedurally by multiplying the top and bottom by the same number. Some participants gave either the response 4/6 or 8/10, indicating that they may have separated the fraction into two components, with the bottom number being two greater than the top one. An equivalent fraction was then constructed with a similar pattern. Three limited GMA students answered questions 13 and question 28b correctly. For the high GMA group, 14 participants answered questions 13 and 28b correctly. Only 2 limited GMA students answered all four questions 29, 14, 13, and 28 correctly compared to 11 from the high GMA group.

The number of participants who were able to identify point P on the number line is shown in Table 11. Only 33.3% (n = 11) of the high GMA group answered the question correctly. Only seven of these participants were able to list another name for the fraction. These seven participants answered all three “3/4” questions and question 29 (identifying symbolic representations of one whole) correctly. Only four of these seven participants answered all five questions correctly. It was these four participants who showed the greatest conceptual understanding of equivalent fractions, as they were not only able to link symbolic and pictorial representations but also offer another name for a specific fraction and apply their knowledge across different fraction representations consistently. No participants in the low GMA group were able to answer all questions correctly. They did not apply their knowledge consistently across representations and were unable to transfer their knowledge to the measure interpretation.

Table 10
Responses to Question 28b: 6/8 = __/__ (n=61)

	Limited GMA (N = 28)		High GMA (N = 33)	
	n	%	*n*	%
CORRECT equivalent fraction given	7	25.0	18	54.5
4/6 or 8/10	2	7.1	4	12.1
Other	8	26.6	7	21.2
Missing	11	39.3	4	12.1

Table 11
Responses to Question 18: Identify point P on the number line (n=61)

	Limited GMA (N = 28)		High GMA (N = 33)	
	n	%	*n*	%
CORRECT: 6/8	1	3.6	8	24.2
CORRECT: 3/4	1	3.6	3	9.1
6/10	0	0.0	3	9.1
6	3	10.7	2	6.1
Other	15	53.6	9	27.2
Missing	8	26.8	8	24.2

Discussion

Students' conceptual understanding of equivalent fractions was examined in this study through their responses to mathematical problems that required them to make connections between equivalent pictorial and symbolic representations incorporating measure and part/whole area interpretations.

Students demonstrated the use of procedural knowledge when answering equivalent fraction problems presented in symbolic form. In some instances, whole number reasoning was exhibited in the procedures they used. Many students were unable to represent a symbolic fraction using an equivalent area diagram. Students who successfully linked symbolic and pictorial part/whole area interpretations for one whole and three quarters showed their knowledge was more generalised and were more able to apply their understanding to pictorial representations using a number-line (measure interpretation). However, the difference between the students in the limited and the high general mathematics achievement groups seems to lie not in the errors they made as similar types of errors were observed. Rather, the depth of their procedural and declarative knowledge and the strength of their connections between procedures and concepts varied as shown in the percentage of questions answered correctly and the types of questions answered correctly.

Conceptual understanding and procedural knowledge are delicately intertwined. The analysis of additional questions or the interview data may assist in clarifying students' level of conceptual understanding. It may also corroborate the findings of Siemon, Izard, Breed, and Virgona (2006) who demonstrated that students with developing fraction knowledge were able to perform simple fraction tasks, but were unable to explain or justify their thinking in writing.

References

Anderson, J. R. (2000). *Cognitive psychology and its implications* (5th ed.). New York: Worth Publishers.

Australian Council for Educational Research. (2005). *Progressive achievement tests in mathematics* (3rd ed.). Melbourne: ACER Press.

Board of Studies NSW. (2002). *Mathematics K-6 syllabus*. Sydney: Author.

Bond, T. G., & Fox, C. M. (2001). *Applying the rasch model: Fundamental measurement in the human sciences*. Mahwah, NJ: Erlbaum.

Catholic Education Office. (2005) *Success in numeracy education [SINE] YEARS 5-8: Fraction screening tests and interviews.* Melbourne: Author.

Doig, B., McCrae, B., & Rowe, K. (2003). *A good start to numeracy*. Canberra: Commonwealth Department of Education, Science and Training.

English, L. D., & Halford, G. S. (1995). *Mathematics education: Models and processes*. Mahwah, NJ: Lawrence Erlbaum Associates.

Gould, P. (2005a). Drawing sense out of fractions. In M. Coupland, J. Anderson & T. Spencer (Eds.), *Making mathematics vital* (Proceedings of the 20th Biennial Conference of the Australian Association of Mathematics Teachers, pp. 133-138). Adelaide: AAMT.

Gould, P. (2005b). Year 6 student's methods of comparing the size of fractions. In P. Clarkson, A. Downton, D. Gronn, M. Horne, A. McDonough, R. Pierce, & A. Roche (Eds.), *Building connections: research, theory and practice* (Proceedings of the Annual conference of the Mathematics Education Research Group of Australasia, pp. 393-400). Sydney: MERGA.

Hiebert, J., & Lefevre, P. (1986). Conceptual and procedural knowledge in mathematics: An introductory analysis. In J. Hiebert (Ed.), *Conceptual and procedural knowledge: The case of mathematics* (pp. 1-27). Hillsdale, NJ: Erlbaum.

Lamon, S. J. (2001). Presenting and representing: From fractions to rational numbers. In A. Cuoco & F. R. Curcio (Eds.), *The roles of representation in school mathematics* (pp. 146-165). Reston, VA: The National Council of Teachers of Mathematics.

Lesh, R. A., Landau, M., & Hamilton, E. (1983). Conceptual models and applied mathematical problem-solving research. In R. A. Lesh & M. Landau (Eds.), *Acquisition of mathematics concepts and processes* (pp. 263-341). Orlando, FL: Academic Press, Inc.

National Research Council. (2001). *Adding it up: Helping children learn mathematics*. Washington, DC: National Academy Press.

Ni, Y. (2001). Semantic domains of rational numbers and the acquisition of fraction equivalence. *Contemporary Education*, *26*, 400-417.

Niemi, D. (1996). *Instuctional influences on content area explanations and representational knowledge: Evidence for the construct validity of measures of principled understanding. CSE technical report 403.* Los Angeles: National Center for Research on Evaluation, Standards, and Student Testing, University of California.

Pearn, C., Stephens, M., & Lewis, G. (2003). Assessing rational number knowledge in the middle years of schooling. In M. Goos & T. Spencer (Eds.), *Mathematics: Making waves* (Proceedings of the 19th Biennial Conference of the Australian Association of Mathematics Teachers, pp. 170-178). Adelaide: AAMT.

Rittle-Johnson, B., Siegler, R. S., & Alibali, M. W. (2001). Developing conceptual understanding and procedural skill in mathematics: An iterative process. *Journal of Educational Psychology*, *93*, 346-362.

RUMM Laboratory. (2004). Interpreting RUMM2020: Part I: Dichotomous data: RUMM Laboratory Pty Ltd.

Shannon, A. (1999). *Keeping score*. Washington D.C.: National Academies of Science.

Siemon, D., Izard, J., Breed, M., & Virgona, J. (2006). The derivation of a learning assessment framework for multiplicative thinking. In J. Norotna, H. Moraova, M. Kratka, & N. Stehlikova (Eds.), *Proceedings of the 30thannual conference of the International Group for the Psychology of Mathematics Education* (Vol. 5, pp.113-120). Prague: PME.

Skemp, R. (1986). *The psychology of learning mathematics* (2nd ed.). London: Penguin Books.

Statistics Teachers as Scientific Lawyers

Joanne Woodward
The University of Auckland
<jwoo088@auckland.ac.nz>

Maxine Pfannkuch
The University of Auckland
<m.pfannkuch@auckland.ac.nz>

When Year 10 students are introduced to reasoning from box plots the type of classroom discourse that will lead them to understand statistical inferential argumentation is unknown. In this paper the discourse of one teacher and her class is analysed. Although the teacher required evidence for claims and introduced statistical vocabulary, she argued with the medians, lacked uncertainty, did not answer the original question or make sense of the conclusion. The implications for teaching are discussed.

According to Tukey (1977) statisticians are like detectives since they need to unlock stories in data. But they also need to be narrators of the stories they discover and excellent lawyers presenting reasoned arguments (Abelson, 1995). Little research has been done on the way teachers communicate inferences from data and yet this is critical for student learning (Rubin, Hammerman, & Konold, 2006). The teacher, as the master of statistical discourse, provides the accepted vocabulary, language structure, and behaviour, guiding and scaffolding the students to attend to the correct features of graphical representations and to build meanings recognised by the statistics community. This paper considers the language of one Year 10 teacher as she formulates inferences from box plots in the conclusion step of one investigation to discover how closely her argument models that of a statistician. Particular attention is given to whether the teacher displays her statistical thinking as reasoned arguments. In other words, is this teacher enculturating her students into a community of statistical practice with argumentative skills equal to a lawyer, or a community where statistical thinking is not present?

Background

Tukey's (1977) focus on discovery of stories in data using innovative visual representations was revolutionary. His quick pencil and paper methods of graph construction, such as the box plot, summarised the data in a more succinct way. The strength of the box plot, however, is also one of its weaknesses since students tend to reason solely using the five-number summary values, the cut-off points, rather than seeing the box plot as representing a distribution (Biehler, 2004). Such deficient inferential reasoning may result in *shaky* conclusions and give students the idea that statistics is deterministic (Ben-Zvi, 2006). In Year 10 students are expected to make informal inferences about populations by comparing samples displayed as box plots, that is, informally draw inferences by mainly looking at, comparing, and reasoning with box plots (Pfannkuch, 2006). Information represented in a box plot is dense, which makes it conceptually demanding (Bakker, 2004). There is limited research, however, on how students and teachers reason with and draw inferences from box plots and on describing how teachers guide and model to students the process of informal inference.

In the conclusion step of the investigative process, the need for inference is most obvious as in this step all the evidence must be presented and weighed. This involves interpretation of the results of analysis, bringing in new ideas, and communicating them in an appropriate format for the audience. Interrogation of the results is also present in the drawing of conclusions – possibilities for explanations are generated within the context, more information is sought, interpretations are critiqued, and unrealistic explanations are discarded whereas support is given for plausible ones (Wild & Pfannkuch, 1999). This interrogation process is very similar to a lawyer developing and presenting an argument.

Krummheuer (1995) considers classroom interaction as collective argumentation to develop working interims, which eventually become accepted knowledge. He describes argumentation as consisting of four elements: claims, grounds, warrants, and backings. In his scheme claims are usually the postulated solution to the problem, grounds are the facts to support the claim, warrants are the information joining the claims and grounds, whereas the backings are the global contexts, which give the warrants authority. Lampert (1990) describes a zig-zag process for the formation of conclusions that begins with conjecture, examines premises, and proposes counter arguments before agreement is reached, whereas Bakker, Derry, and Konold (2006) describe an inferential view of data as being a social exchange of questions, explanations, and justifications. Social interaction is also an important part of learning. Sfard (2000) likens discourse to playing a game. If the teacher is doing all the thinking then effectively the teacher is playing the game in her head. By verbalising her thoughts the teacher is inviting the students to play the game, which is important as in order to develop shared statistical meanings both the teacher and the students need to participate. Learning how to argue with data is the result of wanting to play the game, or in other words to communicate more effectively and recognise the superior discourse of the master, the teacher (Ben-Yehuda, Lavy, Linchevski, & Sfard, 2005).

Informal inference is a complex process, one in which researchers are still defining the rules of the game about how to talk about box plots (Pfannkuch, 2006). Even if all the rules of the game were understood, it would be impossible to make them all explicit. Instead the students need to experience thinking about data displayed in box plots and presenting inferences as reasoned arguments through interacting first with the teacher's thoughts.

Method

Two teachers and their Year 10 classes of students participated in a case study, which considered the language used during six classroom lessons on informal inference. Both teachers taught at the same urban girls' school and both classes were in the average ability stream. The majority of the girls were of Pacific Island ethnicity, many of whom speak English as their second or third language.

The first researcher wrote three class activity outlines, which included student worksheets, overhead transparencies, and teachers' notes. The teachers participating in the study and those teaching at their school were consulted in the development of the resources. The activities encouraged students to act as statisticians unlocking the story in the data and learning in the spirit of Tukey (1977) through exploratory data analysis. Wild and Pfannkuch (1999) found practicing statisticians use an investigative cycle of defining the problem, planning, data management, analysis, and formulating conclusions. To emphasise this cycle the steps were used as section headings on the teacher's notes. The conclusion step was written by completing two statements, *I notice…* and *I wonder…,* as these were found to provide a useful structure to overcome the initial inertia students experience in writing conclusions (Pfannkuch & Horring, 2005).

The lessons were videoed and transcribed. To illustrate key findings about the language used by both teachers when formulating conclusions in investigations, this paper uses one of the teachers, a female with 6 years teaching experience, and one lesson, the sixth and last lesson on an activity called *Big Foot*.

Analysis

Informal inference has only recently been recognised as an important step to develop more formal inference concepts. The definition and concepts of informal inference are still being developed and so the tools to analyse the language of inference are also being created. A micro-analysis of the language used was based on an adaptation of Cadzen's (2001) initiation, response, and evaluation model for discourse analysis. Krummheuer's (1995) argumentation categories and the question categories of explanation, justification, noticing, wondering, and closed were added to her model to reflect the data captured better.

The fictional context for the *Big Foot* activity was provided as a story in the teacher's notes. The teacher read the story to the class about Alice and her twin brother going to their cousin's farm for a holiday. Normally Alice fits into bigger gumboots than her twin brother, but this year she notices that he has bigger gumboots. She wonders whether he has thick socks on or whether his feet are actually bigger than her feet. The problem to investigate is: Who have bigger feet, girls or boys? To answer this question a sample of real data from New Zealand CensusAtSchool was provided on the right foot length of 9, 11, and 13 year old male and female students. The sample size for each of the six groups was 24. Each group of students received data on one of the three age groups. Figure 1 shows box plots of the data but note that the students' graphs did not show outliers.

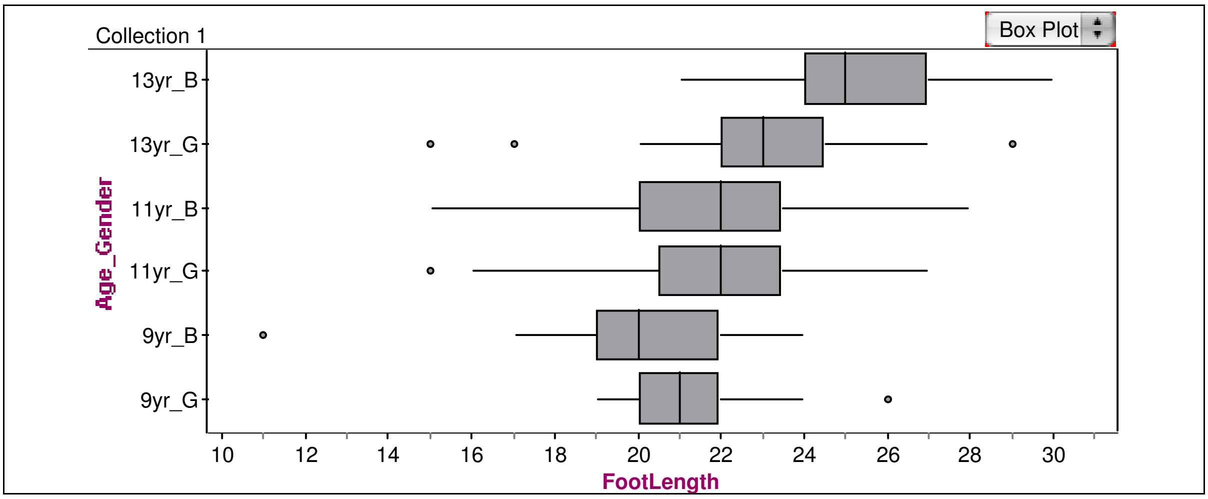


Figure 1. Box plots of the data provided for the *Big Foot* activity.

After the students drew their box plots and formulated their conclusions the teacher had a class discussion. From a detailed analysis of her argumentation language four main themes emerged, explaining the evidence, justifying the evidence, drawing conclusions from the evidence, and making sense of the conclusion.

Explaining the Evidence

During the discussion of the conclusion, the teacher required the students to explain the claims they made and she emphasised selected evidence to support her arguments. Features of the teacher's language were requesting explanations, using statistical terms, using gestures, and using quantitative measures as she supported and guided the students to develop more complete arguments in their conclusions. The following excerpt, which illustrates some of these language features, occurred during the formulation of the conclusion from the foot length data for 11-year-olds.

T: These two sets of data here now are 11 year olds. The groups have done their own “I notice” “I wonder” but what do the rest of you think about 11 year old boys’ and girls’ foot size?
S: They're very similar.
T: Thank you, okay, they're very similar. What tells you that? Because. They're very similar because what?
S: They're both like.
S: The range.
T: The spread’s similar, similar range.
S: The boxes is similar.
T: The boxes are similar, the interquartile range is similar, real similar (measures them with her fingers on the overheads) the interquartile range.
S: The medians.
T: The medians are the same now. Do you know, did you notice that before they were one centimetre apart. Now at 11 years old what's happened?
S: They're the same.
T: They're the same. Great. Okay. At 11 year old, at 11 years old the girls from these data values have the same foot lengths.

The teacher often asked students to explain their observations either by asking directly “what are you looking at?” or by rewording the student’s answer as a clarification type question. A feature of her language that can be noted in the excerpt above is her use of closed questioning, which was often used to ask the students to explain their claims. She supplies another word such as “because” using a raised intonation and by revoicing the student’s response with the additional words, “What tells you that? Because. They're very similar because what?” Another feature of the teacher’s language was to revoice the students’ responses using statistical vocabulary so that rather than using the term *boxes*, the teacher uses the term *interquartile range* and then reinforces this substitute term, thereby implying that these are right words with which to argue. Further reinforcement is through hand gestures, pointing to or measuring the differences between the interquartile ranges. Another strong feature of her language for explaining the evidence, which is not illustrated in the excerpt, was her requirement for quantitative measures. A student would say, “the box is bigger” to which she would reply, “by how much?” To the student’s response “bigger by 5” she would revoice and typically add the measure of centimetres to the student’s response, thereby referencing and reinforcing the context. These quantitative measures, however, were not used as evidence for her argument but rather were observations.

The excerpt above also demonstrates how she typically guided the students to see the median as the most important feature. The students offered a variety of views to support the claim that the foot sizes for 11-year-old boys and girls are similar. They suggest the range, boxes, and the medians are the same. Although the teacher provided a visual explanation by drawing along the length of the boxes to highlight the range and the boxes, the median received the most attention from the teacher and appeared to be the answer she was requiring. In an earlier lesson she explicitly stated that the median was the most important feature of a box plot. So although the teacher did require explanations, these tended to focus on only one feature of the box plot, the median. The only time she used numbers to support the argument was when she reasoned with the median.

Justifying the Evidence

Although the teacher provided the backing for the claim that "girls have a larger foot length than the boys" using the values of the medians, the teacher did not provide any warrants for using the median. For example: "so what we're saying girls is the median value is 21 and here it's 20. So for year 9 [9-year-old] girls the average, typical value of their right foot is one centimetre bigger than for the boys". The concept of using the median as a representation of the data set was not used to support the teacher's arguments in the conclusion, instead the use of the medians was presented in a way that suggested representativeness was a ground, an understood and accepted concept in the classroom.

Drawing Conclusions from the Evidence

Several features of the teacher's language may have conveyed a sense of certainty about the conclusion that was drawn from the evidence. One feature was the way the conclusion was reached. The teacher focused on a single statistic, the median to compare the data sets. This may have communicated to the students that only the median should be attended to and the rest of the information in the graph could be ignored. The teacher described the boxes as being at the median, as if the whole box was just a single line: "See this little box for boys is at 25 and this one's at 23". Reasoning with only one feature of the data also suggested there was a single procedure to follow to formulate a conclusion, rather than a weighing of the evidence. If the boys' median foot length was numerically larger than the girls' median foot length then the boys had a larger foot length, and vice versa. A sense of finality may have been communicated to the students, which could have prevented them from exploring the data any further.

Definite language was often used rather than expressing uncertainty by using phrases such as *tends to* or *could show* and so students may have assumed there was a single right answer. The teacher did occasionally use informal variable language; for example, she described the 13-year-old boys' foot length as being "on average, two centimetres bigger than girls' foot length". Usually, however, the teacher used exact language such as, "at 11 years old the girls from these data values *have* the same foot lengths".

The *Big Foot* activity was introduced by a story about Alice and her twin brother. Although the teacher did answer the statistical question, about whether boys or girls had bigger feet, the purpose for the investigation, to tell Alice whether her brother's feet had grown or whether he was just wearing thicker socks was not part of the conclusion drawn. Therefore the hallmarks of the teacher's drawing of conclusions from data were certainty and a lack of reference to the original problem under consideration.

Making Sense of the Conclusion

For the conclusion the teacher did not discuss the significance of the differences in the box plots in terms of the context, that is, age and foot length. The difference between the boys' and girls' foot length was simply stated as "1 centimetre", "the same", and "2 centimetres".

Often the *I wonders* came from students' personal opinions or experience of the context rather than the data. This is evident when the students suggested growth spurts occurred when they were 14 years old, for which they did not have data, or for years the data showed the opposite.

T: Alright girls, very quickly, people over at this table were talking about growth spurts, lovely, I wonder if boys undergo a growth spurt between the ages of …
S: 9.
S: 13 and…
S: 14 (several students).
T: 11 and 13.
S: Between 9 and 13.
S: I reckon 10 and 14.
T: Shh, S10's saying she did the year 9s but she said she found out the boys had smaller feet, year 9 boys had smaller feet. She said actually that's probably not true because she knows that her Dad's got bigger feet than her Mum, probably.

As shown in the excerpt, the teacher engaged with these personally-based wonders but did nor challenge them and hence lost an opportunity to explore the difference between evidence-based statistical reasoning and personal experience. Although her words reflected the data she did not explicitly redirect students' attention back to the box plots and data.

Discussion and Conclusion

Informal inference is a recent introduction into the curriculum. In this study the teacher was learning a new way of teaching statistics but more importantly she had not experienced or been enculturated into the discourse of informal inference. To expect to see a perfect statistical discourse modelled in a real classroom is unrealistic. However, there are some issues that arise from the analysis that need to be considered.

Abelson (1995) identified two facets of argumentation: inference, which is the process of deriving logical conclusions from data, and providing persuasive arguments based on the analysis. Students enter a classroom expecting the teacher knows and will provide the correct answers. They also expect there is a single correct answer. Yet this is not the case in statistical investigations. Analysis of data usually provides a multiplicity of results rather than one clear answer and some are contradictory (Biehler, 1997). The teacher in this study presented only one interpretation of the data and did not request alternative interpretations from the students. The argument was one sided, with the teacher developing her stance only. Her conclusion was certain, resulting in a deterministic rather than a probabilistic stance. Perhaps the teacher focused on simplifying her process of reasoning with the data and so she removed the arguments she was having in her head and only verbalised the *winning* argument.

The teacher also tended to reduce the complex relationships in the box plots by only using the medians as evidence for the arguments and by not linking her observations back to the context or the problem under investigation. The teacher did not verbalise her thinking or reasoning process, nor did she justify the use of the median by providing a warrant, instead she used a series of questions to funnel the students to focus on the median. Although the students could answer the questions, they were not learning about how to think about the box plot or the reasoning process. If this is not occurring while the teacher is present then the students are unlikely to think for themselves when the teacher is gone (Mason, 2000). Wild and Pfannkuch (1999) call for statistical thinking to be articulated: in a classroom this call surely should be louder. The verbalisation of the inner dialogue alerts the student to its existence; noticing features on a graph becomes a process rather than a plucking of ideas from the air. Thinking is learnt in the same way knowledge is learnt, through interaction with a knowledgeable other (Perkins, Jay, & Tshman, 1993) or as Mason (2000, p. 97) states “a student learns to think mathematically by being in the presence of a relative expert who makes their thinking processes explicit”. When students interact with the thinking of teachers they have a model for thinking and the experience of thinking. Modelling also provides a way for students to hear how the language and discourse is used in the context, and how it is structured.

Increasing the fluency of students’ discourse will mirror an increased understanding of graphs (Ainley, Nardi, & Pratt, 2000). If understanding emerges in use (Bakker et al., 2006) then teachers need to invite the students to participate in a learning dialogue. The teacher in this study did ask the students to offer their opinions but rather than exploring the students’ stances by asking them for the basis of their claims, as Bakker et al., (2006) suggest, the teacher evaluated them. The teacher did invite the students to support their views by explaining them, but the judgement still rested with the teacher rather than inviting the other students to agree or disagree.

Formulating thoughts into words helps clarify students’ thinking. Words can act as a pump for statistical ideas that do not yet exist for students (Sfard, 2000). When the teacher introduced the terms spread, range, and interquartile range the students then had the words to argue with and new referents for exploration and elaboration. Technical knowledge, however, is not sufficient to interpret graphs to provide a meaningful answer in terms of the problem being investigated. To synthesise an answer the evidence needs to be weighed (Pfannkuch, 2006). The teacher in this study did not explicitly model this process although there were periods of silence where she may have been thinking through the evidence. In particular she did not challenge the students when they were attempting to make sense of the conclusion. Critical thinking is required during the evaluation process, and includes weighing the quantitative evidence and contextual knowledge. In real investigations *correct* solutions do not occur, instead statisticians must present their best conclusion fully supported. Abelson (1995) describes statisticians as requiring the narrative and argumentative skills equal to lawyers. Statisticians may require these skills but unlike lawyers their arguments seek to find the truth from the story in the data, not to present one side of a story or a winning argument. Also statisticians’ language is tempered by uncertainty whereas lawyers argue with certainty. Hence statisticians’ argumentative skills are those of scientific lawyers.

If teachers want to encourage students to engage in argumentation then it is their responsibility to initiate and guide students towards a shared understanding of how the discourse is structured. By presenting and allowing only a single interpretation of the data, which is evaluated only by the teacher, as was the case in this study, the statistical process of inference is not being modelled (Figure 2). Teachers instead need to use questioning and revoicing to support the development and structuring of alternative views and to model critical thinking when evaluating the stances and presenting the argument just as a scientific lawyer would (Figure 2).

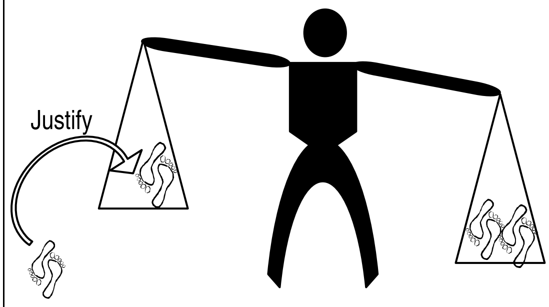
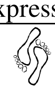
Argumentation of teacher studied	**Proposed argumentation for teacher**
	
While the teacher did notice some features of the box plot and did require an explanation for claims made [shown by presence of feature on scale], she only used the median as evidence in the conclusion. No justification was given for using the median. The conclusion was expressed with certainty and not discussed in terms of the problem.	Features of the box plot are weighed as evidence in the conclusion. Inclusion of features is justified [shown by presence of contexualised features on both sides of scale]. Features are contextualised in order to determine the meaning of the findings. The evidence is discussed and weighed in terms of the problem, its context, and statistical issues such as sampling variability and quality of data. The conclusion is expressed in terms of uncertainty.
Legend:  Represents that feature such as the median is discussed quantitatively.	Legend:  Represents that feature such as the median is discussed within the context.

Figure 2. Summary of argumentation used and proposed.

The teacher was beginning to enculturate her students into a statistical community of practice but her focus on the production of box plots and the formulation of the *correct* conclusion obscured the investigative process and statistical thought, a facet of pedagogic purpose that Mason (2000) has also found. Several researchers (e.g., Biehler, 1997) have found teachers are unsure about how to talk about graphs and so the findings of this study are not unique but contribute to the growing call to discover ways of developing teachers' talk. Statistical thinking is complex and involves searching for the story in the data. Statistics teachers as scientific lawyers need to narrate their thinking, providing an account of how they are reasoning, arguing, and weighing the evidence for the story that they have unlocked, in order to answer the problem posed at the beginning of the investigation.

References

Abelson, R. (1995). *Statistics as principled argument*. Hillsdale, NJ: Lawrence Erlbaum Associates.

Ainley, J., Nardi, E., & Pratt, D. (2000). The construction of meanings for trend in active graphing. *International Journal of Computers for Mathematical Learning*, *5*(2), 85-114.

Bakker, A. (2004). *Design research in statistics education: On symbolizing and computer tools*. Utrecht, The Netherlands: CD-ß Press, Center for Science and Mathematics Education.

Bakker, A., Derry, J., & Konold, C. (2006). Using technology to support diagrammatic reasoning about center and variation. In A. Rossman & B. Chance (Eds.), *Proceedings of the Seventh International Conference on Teaching Statistics, Salvador, Brazil, July, 2006: Working cooperatively in statistics education.* [CD-ROM]. Voorburg, The Netherlands: International Statistical Institute.

Ben-Yehuda, M., Lavy, L., Linchevski, L., & Sfard, A. (2005). Doing wrong with words. *Journal for Research in Mathematics Education*, *36*(3), 176-247.

Ben-Zvi, D. (2006). Scaffolding students' informal inference and argumentation. In A. Rossman & B. Chance (Eds.), *Proceedings of the Seventh International Conference on Teaching Statistics, Salvador, Brazil, July, 2006: Working cooperatively in statistics education*. [CD-ROM]. Voorburg, The Netherlands: International Statistical Institute.

Biehler, R. (1997). Students' difficulties in practicing computer-supported data analysis: Some hypothetical generalizations from results of two exploratory studies. In J. Garfield & G. Burrill (Eds.), *Research on the role of technology in teaching and learning statistics* (pp. 169-190). Voorburg, The Netherlands: International Statistical Institute.

Biehler, R. (2004, July). *Variation, co-variation, and statistical group comparison: Some results from epistemological and empirical research on technology supported statistics education*. Paper presented at the 10th International Congress on Mathematics Education, Copenhagen

Cazden, C. B. (2001). *Classroom discourse: The language of teaching and learning* (2nd ed.). Portsmouth, NH: Heinemann.

Krummheuer, G. (1995). The ethnography of argumentation. In P. Cobb & H. Bauersfeld (Eds.), *The emergence of mathematical meaning: Interaction in classroom cultures* (pp. 229-270). Hillsdale, NJ: Erlbaum.

Lampert, M. (1990). When the problem is not the question and the solution is not the answer: Mathematical knowing and teaching, *American Educational Research Journal, 27*(1), 29-63.

Mason, J. (2000). Asking mathematical questions mathematically. *International Journal of Mathematical Education in Science and Technology*, *31*(1), 97-111.

Perkins, D., Jay, E., & Tshman, S. (1993). Beyond Abilities: A dispositional theory of thinking. *Merrill-Palmer Quarterly*, *39*(1), 1-21.

Pfannkuch, M. (2006). Comparing box plot distributions: A teacher's reasoning. *Statistics Education Research Journal*, 5(2), 27-45. [Online: http://www.stats.auckland.ac.nz/serj]

Pfannkuch, M., & Horring, J. (2005). Developing statistical thinking in a secondary school: A collaborative curriculum development. In G. Burrill & M. Camden (Eds.), *Curricular development in statistics education: International Association for Statistical Education (IASE) Roundtable, Lund, Sweden 28 June-3 July 2004,* (pp. 204-218). Voorburg, The Netherlands: International Statistical Institute.

Rubin, A., Hammerman, K., & Konold, C. (2006). Exploring Informal inference with interactive visualization software. In A. Rossman & B. Chance (Eds.), *Proceedings of the Seventh International Conference on Teaching Statistics, Salvador, Brazil, July, 2006: Working cooperatively in statistics education.* [CD-ROM]. Voorburg, The Netherlands: International Statistical Institute.

Sfard, A. (2000). On reform movement and the limits of mathematical discourse. *Mathematical Thinking and Learning*, *2*(3), 157-189.

Tukey, J. (1977). *Exploratory data analysis*. Reading, MA: Addison-Wesley.

Wild, C.J., & Pfannkuch, M. (1999). Statistical thinking in empirical enquiry (with discussion). *International Statistical Review*, *67*(3), 223-265.

Developing Pedagogical Tools for Intervention: Approach, Methodology, and an Experimental Framework

Robert (Bob) Wright
Southern Cross University
<bob.wright@scu.edu.au>

David Ellemor-Collins
Southern Cross University
<david.collins@scu.edu.au>

Gerard Lewis
Catholic Education Office, Melbourne
<GLewis@ceo.melb.catholic.edu.au>

This paper reports on a project aimed at developing pedagogical tools for intervention in the number learning of low-attaining 3rd- and 4th-graders. Approaches to instructional design and intervention are described, and the use of the design research methodology is outlined. A major outcome of the project, an experimental framework for instruction, is described. The framework consists of five aspects: number words and numerals, structuring numbers to 20, conceptual place value, addition and subtraction to 100, and early multiplication and division. The descriptions of aspects include a discussion of low-attaining students' knowledge and difficulties, and details of instructional approaches developed in the project.

The Numeracy Intervention Research Project (NIRP) has the goal of developing pedagogical tools for intervention in the number learning of low-attaining third- and fourth-graders (8- to 10-year-olds). This paper reports on the NIRP by providing overviews of (a) the approach to intervention, (b) the use of a design research methodology, and (c) an experimental instructional framework consisting of five key aspects.

Approach to Intervention

A significant proportion of students have difficulties learning basic arithmetic (Louden et al., 2000). This limits their development of numeracy (McIntosh, Reys, & Reys, 1992; Yackel, 2001). Low-attainment is of particular concern in the context of the emphasis on numeracy, both nationally and internationally (e.g. *The national numeracy project: An HMI evaluation*, 1998; *Numeracy, a priority for all*, 2000). Furthermore, there are very few instructional programs to address numeracy difficulties and very few Australian schools systematically address this problem (Louden et al., 2000). Hence there are calls "to identify effective remedial approaches for the various identified weaknesses" (Bryant, Bryant, & Hammill, 2000, p. 174) and to develop approaches from the research-based reforms recommended for general mathematics education (Rivera, 1998). Researchers have developed programs of intervention in early number learning (Dowker, 2004; Gervasoni, 2005; Pearn & Hunting, 1995; Wright, Martland, Stafford, & Stanger, 2006; Young-Loveridge, 1991) focusing on topics such as counting and early addition and subtraction. The NIRP aims to extend this work with a focus on basic whole number arithmetic including reasoning with numbers in the hundreds and thousands, multidigit addition and subtraction, and early multiplication and division.

Organising by key aspects. Recent intervention programs have described early number knowledge in terms of components (Dowker, 2004) and domains (Clarke, McDonough, & Sullivan, 2002). These descriptions highlight the idiosyncratic nature of students' number knowledge (Gervasoni, 2005) and learning paths (Denvir & Brown, 1986). In this paper we use a framework approach (Wright et al., 2006) to set out five key aspects of number learning that we regard as important for intervention with 3rd- and 4th-graders. Our approach involves developing instructional activities relevant to each of these key aspects. In this approach, constructing a framework of key aspects is important in developing a domain-specific theory for intervention instruction. Further, this framework can be applied to all students and can inform classroom instruction.

Instructional design. Progressive mathematisation refers to the development from informal, context-bound thinking to more formal thinking (Beishuizen & Anghileri, 1998; Gravemeijer, 1997; Treffers, 1991). As in the emergent modelling heuristic (Gravemeijer, Cobb, Bowers, & Whitenack, 2000), instructional design involves anticipating a potential learning trajectory, and devising an instructional sequence of tasks which foster students' progressive mathematisation along the trajectory, through levels of thinking from informal to formal. Particular settings, such as manipulative equipment or notation systems, can have an important role in an instructional sequence. A setting can be established as a context for students' initial context-dependent thinking, and then become a *model for* more independent numerical reasoning, thus mediating the crucial development from concrete toward more abstract thinking (Gravemeijer, 1997). An instructional sequence consists of *instructional procedures*, each of which serves to incrementally distance the student from the materials, advance the complexity of the task, and potentially raise the sophistication of the student's thinking. Detailed assessment of the student's knowledge informs the teacher's selection of instructional procedures. Instruction focuses on engaging the student in independent, sustained thinking, and observational assessment enables tuning instruction to the cutting edge of the student's knowledge (Wright et al., 2006).

Approach to number instruction. Our approach to instruction emphasises flexible, efficient computation, and strong numerical reasoning (Beishuizen & Anghileri, 1998; Heirdsfield, 2001; Yackel, 2001). Mental computation, in particular, is foundational for efficient computation, numerical reasoning, and number sense (McIntosh et al., 1992; Treffers, 1991). Learning builds from students' own informal mental strategies (Beishuizen & Anghileri, 1998; Gravemeijer, 1997). However, students need to develop flexible, efficient, mathematically sophisticated strategies. Low-attaining students often use inefficient count-by-ones strategies, and error-prone rote procedures, and depend on supporting materials or fingers (Gray, Pitta, & Tall, 2000; Wright, 2001). Hence, intervention instruction needs to develop students' number knowledge to support non-count-by-ones strategies, and to move students to independence from materials.

Methodology

The NIRP adopted a methodology based on design research (Cobb, 2003; Gravemeijer, 1994), consisting of three one-year design cycles. The NIRP aimed to develop pedagogical tools for intervention, consisting of a framework, assessment tasks, and instructional sequences. Each design cycle consisted of (a) initial development of the pedagogical tools, (b) use of the tools in an intervention program with teachers and students, (c) analysis of the learning and teaching in the program, and (d) refinement of the tools based on the analysis. Within each cycle, analysis and development were on-going, in meetings of the researchers and project teachers, in analysis of assessments, and in teachers' daily lesson planning. The analysis of the learning and teaching in the intervention program is informed by a teaching experiment methodology (Steffe & Thompson, 2000). Interview assessments and instructional sessions were videotaped, providing an extensive empirical base for analysis. The approach to the development of intervention programs described in this paper is an appropriate response to Ginsburg's (1998) call for teaching experiments focusing on students with learning difficulties.

The Study

The intervention program for each year involved eight or nine teachers, each from a different school, across the state of Victoria. In each school, 12 students were identified as low-attaining in arithmetic, based on screening tests administered to all third- and fourth-graders. In each school (a) in term 2, these 12 students were assessed in individual interviews; (b) in term 3, eight of the low-attaining students participated in intervention teaching cycles; and (c) in term 4, the 12 students were again assessed in individual interviews. The teaching cycles involved teaching sessions of 30 minutes duration, for four days per week, for 10 weeks. Two students were taught as singletons and six as trios, and all of the instructional sessions with singletons were videotaped. Across the three years of the project, in each of 25 schools, the project teacher assessed 300 low-attaining students, each on two occasions, taught 50 students individually and 150 students in trios.

Development of the Instructional Framework

Through the cycles of design research, the framework of key aspects of knowledge developed from four considerations. Firstly, areas of significance were identified in our analysis of low-attaining students' number knowledge and difficulties, areas that seem to be characteristic of what successful students can do and what low-attaining students cannot do. Secondly, these areas were clarified in making a coherent framework for teachers to use for analysing assessments and profiling students' learning needs. Thirdly, the key aspects became further defined as the key instructional sequences and their associated settings emerged. Fourthly, the key aspects were refined in articulating a coherent framework for instruction. The framework is experimental in the design research sense – it is intended to be further trialled, analysed, and developed.

The resulting instructional framework consists of the following five aspects: (A) Number Words and Numerals; (B) Structuring Numbers 1 to 20; (C) Conceptual Place Value; (D) Addition and Subtraction 1 to 100; and (E) Early Multiplication and Division.

Experimental Instructional Framework

For each of aspects A to D, we describe (a) the significance of the aspect; (b) low-attaining students' knowledge and difficulties; and (c) instructional sequences. Due to space limitations, aspect E is not described in this paper.

Aspect A: Number Words and Numerals

Low-attaining students' knowledge and difficulties. Early number curricula focus on number word sequences (NWS) to 20, and to 100, and learning to read and write 2-digit numerals. Students' early difficulties are well-documented (e.g., Fuson, Richards, & Briars, 1982). Classroom instruction on NWSs and numerals tends to decrease as students progress through school. However, low-attaining third- and fourth-graders have significant difficulties with these areas (Hewitt & Brown, 1998). Errors with NWSs in the range 1 to 100 include: (a) “52, 51, 40, 49, 48…” and (b) “52, 51, 49, 48…”. Students are aware of the chains of number words in each decade – 41 to 49 and 51 to 59, and link these chains incorrectly when going backwards (Skwarchuk & Anglin, 2002). Errors with NWSs in the range 100 to 1000 occur at decade and hundred numbers, for example: (a) “108, 109, 200, 201, 202…”; (b) “198, 199, 1000, 1001…”; and (c) “202, 201, 199, 198…” (Ellemor-Collins & Wright, in press). When students respond correctly on these tasks, in many cases they lack certitude. Knowledge of sequences of tens off the decade is an important part of knowledge of base-ten structures (Ellemor-Collins & Wright, in press), and is a prerequisite for mental jump strategies (Fuson et al., 1997; Menne, 2001). Some low-attainers cannot skip count by tens off the decade. Given the task “Count by tens from 24”, responses included: (a) “24, 25, 20”; (b) “24, 30, 34, 40”; (c) “24…34…44” with each ten counted by ones subvocally; and (d) “I can’t do that”. As well, there is a range of significant errors with sequences of tens beyond 100 (Ellemor-Collins & Wright, in press). Some 3rd- and 4th-graders make errors with 3-digit and 4-digit numerals involving zeros (Hewitt & Brown, 1998): 306 is identified as “360”; 6032 is identified as “6 hundred and 32”, or "60 thousand and 32”; and 1005 is written “10 005”.

Instruction in number words and numerals. Facility with number word sequences and numerals is important, and requires explicit attention for low-attainers (Menne, 2001; Wright et al., 2006). This aspect includes identifying and writing numerals to 1000 and beyond. Instruction focuses on reciting and reasoning with number word sequences and numeral sequences, without structured settings such as number lines or base-ten materials. Students develop knowledge of the auditory and visual patterns, somewhat separate from numerical reasoning about quantity and position (Hewitt & Brown, 1998; Skwarchuk & Anglin, 2002). We have found that explicit instruction focusing on bridging 10s, 100s and 1000s, forwards and backwards, is productive. Saying sequences by tens and hundreds, on and off the decade, supports development of place value knowledge. Saying sequences by 2s, 3s, and 5s, on and off the multiple, supports development of multiplicative knowledge. Students can and should learn number word sequences and numerals in a number range well in advance of learning to add and subtract in that range because familiarity with a range of numbers establishes a basis for meaningful arithmetic (Wigley, 1997).

Exemplar instructional sequence: the numeral track. The numeral track is an instructional device consisting of a sequence of ten numerals, each of which is adjacent to a lid which can be used to conceal the numeral (Wright et al., 2006). In the instructional sequence, first the lids are opened sequentially, and the student names each numeral in turn, after seeing the numeral. Second, when the sequence is familiar, the student’s task is to name each numeral in turn, before seeing the numeral. In this case, the opening lids enable self-verification. Third, the number sequence can be worked backwards. Finally, more advanced tasks can be used. For example, one lid is opened and the teacher points to other lids for the student to name: the number before, the number two after, and so on. In this setting, learning about NWSs supports and is supported by learning about sequences of numerals. The teacher selects the sequence: bridging 110, a tens sequence off the decade, a 2s sequence, and so on. The teacher can observe a student’s specific difficulty, and finely adjust the instructional tasks. The lids allow incremental distancing from the material and internalisation of the sequence.

Aspect B: Structuring Numbers 1 to 20

Facile calculation in the range 1 to 20. Learning arithmetic begins with learning to add and subtract in the range 1 to 20. Students' initial strategies involve counting-by-ones (Fuson, 1988; Steffe & Cobb, 1988) and developing this facility is an important aspect of early number learning. Students can then develop strategies more sophisticated than counting by ones, such as adding through ten (eg., $6 + 8 = 8 + 2 + 4$), using fives ($6 + 7 = 5 + 5 + 1 + 2$), and near-doubles (eg., $6 + 7 = 6 + 6 + 1$). Developing these strategies builds on knowledge of combining and partitioning numbers (Bobis, 1996; Gravemeijer et al., 2000; Treffers, 1991). Efficient calculation also involves knowledge of additive number relations, such as commutativity ($8 + 9 = 9 + 8$), and inversion ($15 + 2 = 17$ implies that $17 – 15 = 2$). The development of efficient, non-count-by-ones calculation in the range 1 to 20 is important. Counting-by-ones can be slow, and error-prone. Further, facile calculation promotes number sense and numerical reasoning (Treffers, 1991), and develops a part-whole conception of numbers (Steffe & Cobb, 1988), providing a basis for further learning such as the construction of units of 10 and multiplicative units (Cobb & Wheatley, 1988).

Low-attaining students' knowledge and difficulties. Low-attaining 3rd- and 4th-graders typically will solve addition and subtraction tasks in the range 1 to 20 by counting on and counting back (Gray et al., 2000; Wright, 2001). As well, they will not necessarily use the more efficient counting strategy, solving $17 – 15$ for example, by making 15 counts back from 17 and keeping track on their fingers. They do not seem to partition numbers spontaneously when attempting to add or subtract. These students typically have difficulty with tasks such as stating two numbers that add up to 19. They might know all or most doubles in the range 1 to 20, but will not use a double to work out a near-double addition ($6 + 7$). As well, they might solve without counting, addition tasks with 10 as the first addend ($10 + 5$) but will not apply the ten structure of teen numbers (14 is $10 + 4$) to solve addition ($14 + 4$) or subtraction ($15 – 4$), and will not use adding through 10 to solve tasks such as $9 + 5$. This preference for counting-by-ones has been explained as a preference to think procedurally (Gray et al., 2000).

Instruction in structuring numbers 1 to 20. The arithmetic rack (Treffers, 1991) is an important instructional device, enabling flexible patterning of the numbers 1 to 20 in terms of doubles, five, and ten. Instruction proceeds in three phases: (a) making and reading numbers; (b) addition involving two numbers; and (c) subtraction involving two numbers (Wright et al., 2006). In each phase, the teacher advances the complexity, from tasks with smaller numbers and more familiar structures, to tasks with larger numbers and less familiar structures. In each phase, the teacher also uses screening and flashing to progressively distance the student from the setting. The student is actively reasoning, in the context of the structured patterns. The intention is that activity with the rack is increasingly internalised and the student shifts from reasoning with numbers as referents-to-the-beads, to numbers as independent entities (Gravemeijer et al., 2000). Instruction with the arithmetic rack can overcome low-attainers' reticence to relinquish counting-by-one strategies.

Aspect C: Conceptual Place Value

Multidigit knowledge. Research evidence supports building multidigit arithmetic on students' informal understandings of number, and emphasizing mental strategies with 2-digit numbers (Beishuizen & Anghileri, 1998; Fuson et al., 1997; Yackel, 2001). Efficient mental strategies require sound knowledge of structures in multidigit numbers such as: (a) additive place value (25 is 20 and 5); (b) jumping by ten, on and off the decade (40 + 20 = 60, 48 + 20 = 68); (c) jumping within and across decades (68 + 5 = 68 + 2 + 3 = 73); and (d) locating neighbouring decuples (linking 48 + 25 to 50 + 25) (Ellemor-Collins & Wright, in press; Heirdsfield, 2001; Menne, 2001; Yackel, 2001). These structures are based on the decade patterns and units of ten. Other important structures include doubles and halves: double 25 is 50, double 50 is 100. Together these structures form a rich network of number relations, the basis of flexible and efficient computation (Foxman & Beishuizen, 2002; Heirdsfield, 2001; Threlfall, 2002). Instruction on these multidigit structures can be distinguished from formal place value instruction. Thompson and Bramald (2002), for example, observe that students' intuitive strategies depend on quantity value, the informal additive aspect of place value, which they distinguish from column value, the formal written aspect of place value. Place value tasks involving manipulation of numerals and knowing column value are problematic for many students, especially low-attainers (Beishuizen & Anghileri, 1998; Thompson & Bramald, 2002). Younger students reason about numbers first in terms of verbal sequences and quantities, rather than written symbols, so addition by formal manipulations of symbols is not necessarily meaningful for these students (Cobb & Wheatley, 1988; Treffers, 1991). For example, a student might understand the result of jumping by ten, but not of adding one in the tens column. Where regular place value instruction is intended to support the development of standard, written algorithms, we propose conceptual place value as an approach to support the development of students' intuitive arithmetical strategies.

Low-attaining students' knowledge and difficulties. Low-attaining third- and fourth-graders typically will not increment or decrement by ten off the decade when solving 2-digit addition and subtraction tasks. In a task presenting, with base-ten materials, 48 + 2 tens and 5 ones, some low-attainers find the total by counting by ones from 48. Other students will attempt to use a split strategy (40 + 20 and 8 + 5) to solve these tasks but will have difficulty recombining tens and ones (Cobb & Wheatley, 1988). These students either lack place value knowledge or are unable to use place value knowledge in dynamic situations, that is, situations that involve increasing or decreasing numbers by ones, tens or hundreds. We regard these difficulties as symptomatic of a lack of important knowledge about multidigit numbers (Ellemor-Collins & Wright, in press).

Instruction in conceptual place value. Conceptual place value encompasses instructional sequences that develop knowledge of the structure of multidigit numbers, as a foundation for mental computation. The main instructional sequence involves flexibly incrementing and decrementing by ones and tens, and later hundreds and thousands, in the context of base-ten materials. Two important settings are: (a) bundling sticks and (b) dots on laminated card organised into ten strips and hundred squares. These seem to be more authentic and hence more useful than MAB blocks. Instructional tasks include firstly, building 2-digit numbers, and then incrementing and decrementing by one ten, two tens, one ten and two ones, and so on. The teacher incrementally distances the student from the setting. Initially, the material is visible. The student answers, and then might reorganise the tens and ones to verify their answer. As the instructional sequence develops, the material is screened and the screens are removed to enable verification. This instruction elicits reasoning about quantities in the range 1 to 100, thus providing a basis for 2-digit addition and subtraction using jump strategies (aspect D). As well, this instruction is extended to flexibly incrementing and decrementing 3- and 4-digit numbers, by ones, tens and hundreds. In this way, students' first learning of place value is strongly verbal and occurs in an additive sense. We have also found that Arrow Cards (Hewitt & Brown, 1998; Wigley, 1997) can be very useful in further supporting this learning.

Aspect D: Addition and Subtraction to 100

Flexible, efficient multidigit computation. Developing facile mental strategies for addition and subtraction involving two 2-digit numbers is a critically important goal of arithmetic learning in the first three or four years of school. This lays a strong foundation for all further learning of arithmetic, including multiplication and division, fractions and decimals, and so on. As well, strong mental strategies will support learning of the standard written algorithms and efficient use of calculators in mathematical problem solving (Beishuizen & Anghileri, 1998). Two main categories of efficient strategies are jump strategies and split strategies. Variations and alternatives abound. (Foxman & Beishuizen, 2002; Fuson et al., 1997; Klein, Beishuizen, & Treffers, 1998; Thompson & Bramald, 2002). All these strategies involve jumping in tens and jumping through ten. Jumping through ten can be used for example, to solve 68 + 7 as 68 + 2 + 5, and more generally involves adding and subtracting to and from a decuple ($60 + 8$, $47 + x = 50$, $74 - x = 70$, $60 - 4$).

Low-attaining students' knowledge and difficulties. As with tasks involving base-ten materials described in aspect C above, some low-attaining third- and fourth-graders seem to interpret written tasks such as 38 + 24 and 63 – 24 as an instruction to make 24 counts forwards or backwards respectively (Wright, 2001). Also, low-attainers frequently try to use a split strategy for written tasks, but have difficulty recombining tens and ones (Beishuizen, Van Putten, & Van Mulken, 1997; Fuson et al., 1997). As well, when solving a task such as 46 + 53, by adding 40 and 50 and 6 and 3 (split strategy), they will typically count-on to work out each of the two sums (40 + 50 and 6 + 3). These students do not know about jumping in tens and jumping through ten to add or subtract in the range 1 to 100 (Menne, 2001). Research suggests that most successful students use jump strategies, whereas most low-attainers use split strategies; further, low-attainers who do use jump have more success and flexibility than those who use split (Beishuizen et al., 1997; Foxman & Beishuizen, 2002; Klein et al., 1998). As well, students who have been taught place value in the traditional way, are likely to have a preference for split strategies.

Instruction in addition and subtraction to 100. Incrementing and decrementing by ten is one important prerequisite for learning to use jump strategies in the range 1 to 100. A second is having facile strategies for addition and subtraction in the range 1 to 20 (Menne, 2001). Our experience is that low-attainers who are facile in the range 1 to 20 require explicit instruction in applying this knowledge when adding and subtracting two 2-digit numbers. For this instruction we have found it useful to use ten frame cards in two forms – a ten frame card for each of the numbers 1 to 9, and full ten frame cards for the decuples. In this setting, 38 can be shown using 3 ten-cards and one eight-card. The ten frame cards used in this way, can supports students' reasoning about adding and subtracting to and from a decuple. This approach can be extended firstly to addition and subtraction involving a 1-digit and a 2-digit number (64 + 3, 78 + 6, 47 – 4, 82 – 7) and finally to addition and subtraction involving two 2-digit numbers. We use a notation system in conjunction with mental strategies. The notation is used to record the mental strategy rather than providing a means of solving the task. Notation supports reflection and communication, and is important for increasing robustness, curtailment and flexibility (Gravemeijer et al., 2000; Klein et al., 1998). We have found four notation systems useful. The empty number line notation (Klein et al., 1998) is used for jump and related strategies. Also used for jump strategies is the simple arrow notation (48 + 25, 48→50, 50→70, 70→73). The so-called drop-down notation is used for split strategies and notation involving a progression of number sentences (arithmetical equations) can be used for either jump or split strategies.

Conclusion

An important intention of the framework is to bring together aspects of number variously identified as areas where low-attaining students do not progress. A second important intention is to bring together powerful instructional sequences specific to each of those aspects. The consistent approach to instructional design in terms of progressive mathematisation promotes coherence across the framework. Further, by and large, instruction in the aspects proceeds concurrently, and the teacher makes connections between the aspects (Treffers, 1991). The goal, overall, is the coherent development of students' facility with whole number arithmetic.

The experimental framework initiates further lines of inquiry at four levels: (a) analyse further, low-attaining students' learning within each aspect; (b) refine the instructional sequences and their connections; (c) assess students' and teachers' responses to intervention programs based on the framework; (d) Clarify the design research approach to developing pedagogical materials for intervention.

Acknowledgements. The authors gratefully acknowledge the support for this project from the Australian Research Council under grant LP0348932 and from the Catholic Education Commission of Victoria. As well, the authors acknowledge the contribution of Cath Pearn (partner investigator) to this project. Finally, the authors express their sincere thanks to the teachers, students, and schools participating in the project.

References

Beishuizen, M., & Anghileri, J. (1998). Which mental strategies in the early number curriculum? A comparison of British ideas and Dutch views. *British Educational Research Journal, 24*(3), 519-538.

Beishuizen, M., Van Putten, C. M., & Van Mulken, F. (1997). Mental arithmetic and strategy use with indirect number problems up to one hundred. *Learning and Instruction, 7*(1), 87-106.

Bobis, J. (1996). Visualisation and the development of number sense with kindergarten children. In J. T. Mulligan & M. Mitchelmore (Eds.), *Children's number learning* (pp. 17-34). Adelaide: AAMT.

Bryant, D. P., Bryant, B. R., & Hammill, D. D. (2000). Characteristic behaviours of students with learning disabilities who have teacher-identified math weaknesses. *Journal of Learning Disabilities, 33*, 168-177,199.

Clarke, B., McDonough, A., & Sullivan, P. (2002). Measuring and describing learning: The early numeracy research project. In A. Cockburn & E. Nardi (Eds.), *Proceedings of the 26th annual conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 181-185). Norwich, UK: PME.

Cobb, P. (2003). Investigating students' reasoning about linear measurement as a paradigm case of design research. In M. Stephan, J. Bowers, P. Cobb, & K. Gravemeijer (Eds.), *Supporting students' development of measuring conceptions: Analyzing students' learning in social context* (pp. 1-16). Reston, VA: National Council of Teachers of Mathematics.

Cobb, P., & Wheatley, G. (1988). Children's Initial Understandings of Ten. *Focus on Learning Problems in Mathematics, 10*(3), 1-26.

Denvir, B., & Brown, M. (1986). Understanding of number concepts in low attaining 7-9 year olds: Part II. The teaching studies. *Educational Studies in Mathematics, 17*, 143-164.

Dowker, A.D. (2004) *Children with difficulties in mathematics: What works?* London: DfES.

Ellemor-Collins, D., & Wright, R. J. (in press). Assessing pupil knowledge of the sequential structure of number. *Educational and Child Psychology, 24*(2).

Foxman, D., & Beishuizen, M. (2002). Mental calculation methods used by 11-year-olds in different attainment bands: A reanalysis of data from the 1987 APU survey in the UK. *Educational Studies in Mathematics, 51*(1-2), 41-69.

Fuson, K. C. (1988). *Children's counting and concepts of number*. New York: Springer-Verlag.

Fuson, K. C., Richards, J., & Briars, D. (1982). The acquisition and elaboration of the number word sequence. In C. J. Brainerd (Ed.), *Progress in cognitive development: Vol. 1 Children's Logical and Mathematical Cognition* (pp. 33-92). New York: Springer-Verlag.

Fuson, K. C., Wearne, D., Hiebert, J., Murray, H., Human, P., Olivier, A., Carpenter, T., & Fennema, E. (1997). Children's conceptual structures for multidigit numbers and methods of multidigit addition and subtraction. *Journal for Research in Mathematics Education, 28*, 130-162.

Gervasoni, A. (2005). The diverse learning needs of young children who were selected for an intervention program. In H. L. Chick & J. L. Vincent (Eds.), *Proceedings of the 29th annual conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 33-40). Melbourne: PME.

Ginsburg, H. (1998). Mathematics learning disabilities: A view from developmental psychology. In D. P. Rivera (Ed.), *Mathematics education for students with learning disabilities* (pp. 33-58). Austin, TX: Pro-ed.

Gravemeijer, K. P. E. (1994). Educational development and developmental research in mathematics education. *Journal for Research in Mathematics Education, 25*(5), 443-471.

Gravemeijer, K. P. E. (1997). Mediating between concrete and abstract. In T. Nunes & P. Bryant (Eds.), *Learning and teaching mathematics: An international perspective*. East Sussex: Psychology Press.

Gravemeijer, K. P. E., Cobb, P., Bowers, J. S., & Whitenack, J. W. (2000). Symbolizing, modeling and instructional design. In P. Cobb, E. Yackel, & K. J. McClain (Eds.), *Symbolizing and communicating in mathematics classrooms: Perspectives on discourse, tools, and instructional design* (pp. 225-273). Hillsdale, NJ: Lawrence Erlbaum Associates.

Gray, E., Pitta, D., & Tall, D. (2000). Objects, actions, and images: A perspective on early number development. *Journal of Mathematical Behaviour, 18*(4), 401-413.

Heirdsfield, A. (2001). Integration, compensation and memory in mental addition and subtraction. In M. Van den Heuvel-Panhuizen (Ed.), *Proceedings of the 25th annual conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 129-136). Utrecht, Netherlands: PME.

Hewitt, D., & Brown, E. (1998). On teaching early number through language. In A. Olivier & K. Newstead (Eds.), *Proceedings of the 22nd annual conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 41-48). Stellenbosh, South Africa: PME.

Klein, A. S., Beishuizen, M., & Treffers, A. (1998). The empty number line in Dutch second grades: Realistic versus gradual program design. *Journal for Research in Mathematics Education, 29*(4), 443-464.

Louden, W., Chan, L., Elkins, J., Greaves, D., et al. (2000). *Mapping the territory: Primary students with learning difficulties*. Canberra: DETYA.

McIntosh, A., Reys, B. J., & Reys, R. E. (1992). A proposed framework for examining basic number sense. *For the Learning of Mathematics, 12*, 2-44.

Menne, J. (2001). Jumping ahead: An innovative teaching program. In J. Anghileri (Ed.), *Principles and practices in arithmetic teaching - innovative approaches for the primary classroom* (pp. 95-106). Buckingham: Open University Press.

The national numeracy project: An HMI evaluation. (1998). London, UK: OfSE.

Numeracy, a priority for all: Challenges for Australian schools. (2000). Canberra: DETYA.

Pearn, C., & Hunting, R. P. (1995). Mathematics intervention: An overview of the first two years. In B. Atweh & S. Flavel (Eds.), *Galtha* (Proceedings of the 18th annual conference of the Mathematics Education Research Group of Australasia, Darwin, pp. 446-452). Darwin: MERGA.

Rivera, D. P. (1998). Mathematics education and students with learning disabilities. In D. P. Rivera (Ed.), *Mathematics education for students with learning disabilities* (pp. 1-31). Austin, TX: Pro-ed.

Skwarchuk, S., & Anglin, J. M. (2002). Children's acquisition of the English cardinal number words: A special case of vocabulary development. *Journal of Educational Psychology, 94*(1), 107-125.

Steffe, L. P., & Cobb, P. (1988). *Construction of arithmetic meanings and strategies*. New York: Springer-Verlag.

Steffe, L. P., & Thompson, P. W. (2000). Teaching experiment methodology: Underlying principles and essential elements. In A. Kelly & R. Lesh (Eds.), *Handbook of research design in mathematics and science education* (pp. 267-306). Mahwah, NJ: Lawrence Erlbaum Associates.

Thompson, I., & Bramald, R. (2002). *An investigation of the relationship between young children's understanding of the concept of place value and their competence at mental addition (Report for the Nuffield Foundation)*. Newcastle upon Tyne: University of Newcastle upon Tyne.

Threlfall, J. (2002). Flexible mental calculation. *Educational Studies in Mathematics, 50*(1), 29-47.

Treffers, A. (1991). Didactical background of a mathematics program for primary education. In L. Streefland (Ed.), *Realistic mathematics education in primary school* (pp. 21-56). Utrecht: Freudenthal Institute.

Wigley, A. (1997). Approaching number through language. In I. Thompson (Ed.), *Teaching and learning early number* (pp. 113-122). Buckingham, UK: Open University Press.

Wright, R. J. (2001). The arithmetical strategies of four 3rd-graders. In J. Bobis, B. Perry, & M. Mitchelmore (Eds.), *Numeracy and beyond* (Proceedings of the 25th annual conference of the Mathematics Education Research Group of Australasia, Vol. 2, pp. 547-554). Sydney: MERGA.

Wright, R. J., Martland, J., Stafford, A. K., & Stanger, G. (2006). *Teaching number: Advancing children's skills and strategies* (2nd ed.). London: Paul Chapman Publishing.

Yackel, E. (2001). Perspectives on arithmetic from classroom-based research in the United States of America. In J. Anghileri (Ed.), *Principles and practices in arithmetic teaching – Innovative approaches for the primary classroom* (pp. 15-31). Buckingham: Open University Press.

Young-Loveridge, J.M. (1991). *The effects of early mathematics intervention: The EMI-5s study, Volumes 1 & 2*. Hamilton, NZ: University of Waikato.

Pedagogy and Interactive Whiteboards: Using an Activity Theory Approach to Understand Tensions in Practice

Robyn Zevenbergen
Griffith University
<r.zevenbergen@griffith.edu.au>

Steve Lerman
London South Bank University, UK
<lermans@lsbu.ac.uk>

In studying the use of Interactive Whiteboards (IWBs) we have observed that there are concerns in relation to measures of pedagogy. Using a productive pedagogies framework to analyse the use of IWBs in middle school classrooms, we found very low rating on aspects of pedagogy related to intellectual quality. Using an activity theory framework, and drawing on observations and interview data, we theorise the tensions in the uptake and use of IWBs to support mathematics learning.

Promoters of IWBs have been very strategic in the use of case studies to illustrate the novelty and support that can be achieved through the clever use of the tool (Edwards, Hartnell, & Martin, 2002). However, as reported elsewhere (Zevenbergen & Lerman, 2006), there are notable concerns in terms of how the IWBs are used in Australian classrooms. In this paper, we draw on these contradictions with the use of IWBs to theorise the use of IWBs. Drawing on the principles of activity theory to frame the analysis, we draw particularly on the notion of tools, in this case IWBs, which mediate pedagogic relationships. Within activity theory, tools can refer to both concrete and semiotic tools. As such, we draw on a range of tools that can be used to explain the complex milieu of classrooms and the uptake of IWBs. The values and beliefs that teachers hold about pedagogy and/or technology mediate the ways in which they will use such technologies. The beliefs and values may relate to the pedagogical approaches that are adopted or to the technological tools themselves. Where teachers hold particular views about how children best learn mathematics, then they are most likely to employ strategies that align with those beliefs. Similarly, if they see technology as a tool that can undertake particular functions (such as a calculator can be used for working out arithmetic tasks), then the technology will be used in that fashion. In exploring computer-mediated learning using activity theory, Waycott, Jones, and Scanlon (2005, p. 107) reported that there is a reciprocity between the tools and the learner where "the user adapts the tools they use according to their everyday practice and preferences in order to carry out their activities; and how, in turn, the tools themselves also modify the activities that the user is engaged in." Drawing on activity theory, we explore the ways in which IWBs were used in a number of classrooms, provide an evaluation of the approaches being used by teachers, and then seek to explain the observations that were made in these classrooms.

Interactive Whiteboards as Mediating Tools: A Background

The implementation of interactive whiteboards in schools in the UK has been strongly supported by the government (Beauchamp, 2005) with over £50m being spent on their implementation in primary and secondary schools (Armstrong et al., 2005). However, it has not received the same fiscal support in Australian schools. Many schools are supporting the implementation of these devices through various means but without systematic support. In most cases, the implementation of IWBs is a school-based decision and as such is supported by funds raised by the schools. How the IWBs are implemented within a given school is dependent upon the resources of the school to provide the equipment and the beliefs of the teaching staff as to the value of the tool. As such, there is considerable variation across Australia as to their uptake and implementation. This can range from how IWBs are placed in classrooms (who has them and where they are physically located), how teachers use them, and access to professional development.

In taking up new forms of technology Glover and Miller (2002) reported that their experienced teachers were skeptical of these new forms of pedagogy whereas, in contrast, preservice teachers saw these new technologies as an integral and valued component of their future practice. In the process of moving from the novice user to one who integrates the IWB into their repertoire of pedagogic skills, Beauchamp (2005) contends that there needs to be a considerable investment for teachers to learn to develop their technical competence alongside their pedagogical skills. In terms of how the IWB is used in the classroom, Glover and Millar (2002) contend that teachers need to recognize that there is considerable interactivity associated with the use of IWBs. They argue that the IWB can engender an approach that fails to radicalize pedagogy and where the IWB is used to enhance students' motivation rather than become a catalyst for changing pedagogy. To be competent with the use of IWBs, it was recommended that teachers need daily access to such tools (Armstrong et al., 2005) so that teachers are able to develop their repertoire of skills and to integrate it into practice (Glover & Miller, 2001). Greiffenhagen (2000) argued that the availability of IWBs as a teaching aid is only of value where it becomes part of the regular pattern of classroom life. Others argue that teachers also need to have access to a wide range of software and applications that are subject specific (Armstrong et al., 2005) and that on-going training with the use of IWBs helps teachers develop their skills and knowledges with regard to the affordances of these tools.

Changing Technology, Changing Pedagogy?

In considering the impact of IWBs on classroom practice, Smith, Hardman, and Higgins (2006) reported that there is a faster pace in lessons using IWBs than non-IWB lessons, that answers took up considerably more of the overall duration of a lesson, and that pauses in lessons were briefer in IWB lessons compared with non-IWB lessons. They also reported a faster pace in numeracy lessons than in literacy lessons. Although they reported some support for the potential of IWBs, they concluded that overall the use of IWBs was not significantly changing teachers' underlying pedagogy. The majority of teacher time was still spent on explanation and that recitation-type scripts were even more evident in IWB lessons. They found that although the pace of the lessons increased, there had been a decline in protracted answers from students and that there were fewer episodes of teachers making connections or extensions to students' responses.

Although there is a suggestion that IWBs have considerable potential to change interactions and modes of teaching, this has not been found to the case in practice (Smith, Hardman, & Higgins, 2006). These authors claim that there is a faster pace in lessons but less time is being spent in group work. There is a tendency for teachers to assume a position at the front of the class when using IWBs (Maor, 2003). Similarly Latane (2002) suggests that there needs to be a move from teacher-pupil interaction to one of pupil-pupil interaction. In studying mathematics classrooms, Jones and Tanner (2002) reported that interactivity can be enhanced through quality questioning where the quality of the questions posed and the breadth of questioning need to be developed to ensure interactivity in mathematics teaching when using IWBs.

IWBs and Activity Theory

The literature alerts us to the affordances and constraints of this new technology. In considering this within the context of activity theory, we are particularly drawn to third generation activity theory Engeström's third generation framework (e.g., 2000, p. 31), where the mediating tools were extended and elaborated substantially to identify the participants and resources present in an activity, and their different roles and responsibilities. His elaborate representation of these elements and their connections enables an identification of tensions and contradictions in activity systems and hence the potential for development. His model of activity is represented in Figure 1.

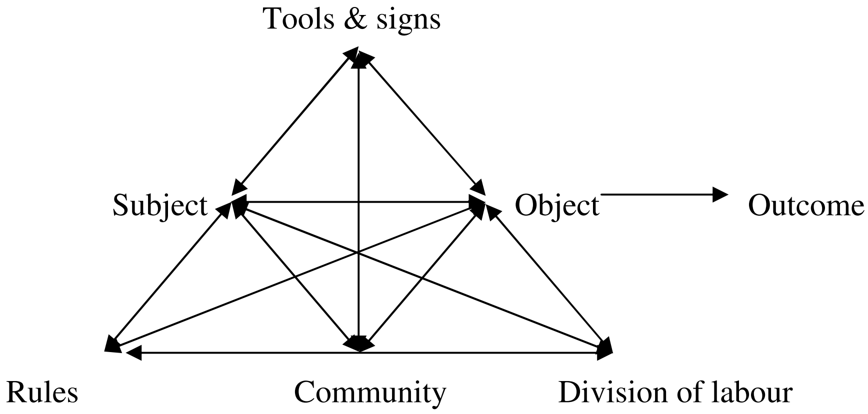


Figure 1: Engeström's third generation activity theory.

The model proposed by Engeström extends the work of Leont'ev so as to consider not only the tension and contradiction between points in the framework but also the context within which learning occurs. For us, the theory allows us consider the results we have observed as being related to these tensions. We draw on this model to understand better the outcomes of this research. It allows us to theorise more constructively the analysis made possible through the analytic lens which we applied to the classroom videos. Rather than explain our outcomes in some deficit framing, Engeström's proposition allows the tensions within the activity system – in this case, classrooms – to be understood more holistically.

Data Collection

The research reported here is drawn from a much larger study where we were concerned with the ways in which technology (ICTs) were being used to support mathematical learning in the middle years of school. As this larger project unfolded over the four years of data collection, we were fortunate to see the introduction of IWBs into some of our participating schools. This provided an intended aspect to the project. The process for data collection involved teachers or someone from the research team taking video of lesson where teachers used ICTs or, more specifically for this paper, IWBs. These tapes were subsequently analysed using a productive pedagogies framework.

When using this well documented framework on the IWB lessons, there were many worrisome scores when teachers used IWBs in mathematics lessons. To better understand this outcome, we returned to the schools to interview teachers, and returned to the tapes to undertake observations of those lessons. For the IWB aspect of the project, we had two schools using the tools – one in Queensland and one in Victoria. Across these schools, five classrooms were using IWBs.

Descriptive Overview of Pedagogy

In viewing the tapes, a number of commonalities were evident in the observed lessons. Our data confirmed the research of Smith, Hardman, and Higgins (2006) where we observed the level of questioning being used by teachers in these lessons. There were more recall questions than those requiring deeper levels of understanding. This type of questioning also allowed for a quicker pacing of the lesson since teachers were able to ask quick fire questions where there was little depth in the responses required. The predominant approach used by teachers when using the IWBs was that of whole class teaching. In these settings, the teacher controlled the lesson, inviting students to participate in manipulating the objects. In all cases, the teachers used the IWBs as the introduction to the lesson. Once the students had been involved in the introductory component of the lesson, they returned to their desks to work on activities related to the topic being introduced. Depending on the resources used by the teacher, there were instances where the IWB made possible a rich introduction to aspects of mathematical language.

Productive Pedagogies Analysis

Although the observations provided us with some indicators of how the IWBs were being used in the classroom, we also employed a quantitative measure to document the use of IWBs. This measure allows us to analyse the lessons more rigorously. We have used this approach in analyzing the use of ICTs in classrooms so were able to compare those data against the use of IWBs. The process involves three observers observing the lessons that had been videotaped. Each observer rates the lesson against nominated criteria on a scale of 0-5 where 0 indicates that there was no evidence of that criterion in the lesson and 5 indicates that it was a strong feature that was consistent throughout the lesson. The ratings are made at the completion of the lesson and the score is for the overall lesson. If there is some evidence of a criterion in the opening phase of the lesson but does not appear again, then this means that it was not a strong feature of the overall lesson. The three observers rate their observations independently and then come together to come up with a common score. This involves a process of negotiation to arrive at the common outcome. In most cases, there was usually a difference of 1 between the ratings and the ensuing discussion meant that the observers needed to negotiate their ratings with the other two. The framework we have used come from the work of the Queensland Schools Longitudinal Reform Study (Education Queensland, 2001) in which the researchers analysed one thousand lessons in terms of the pedagogies being used by teachers. The method was that described above and where the criterion for each rating was based on the Productive Pedagogies. There are four dimensions within the framework – Intellectual Quality, Relevance, Supportive School Environment, and Recognition of Difference – in which there are a number of pedagogies that are evident of that theme. The Productive Pedagogies are outlined in Table 1.

Table 1
Productive Pedagogy Dimensions, Items and Key Questions (from Gore, Griffiths, & Ladwig, 2006)

	Productive Pedagogy	Key question
Intellectual quality	Higher order thinking	Are higher order thinking and critical analysis occurring?
	Deep knowledge	Does the lesson cover operational fields in any depth detail or level of specificity?
	Deep understanding	Do the work and response of the students provide evident of understanding concepts and ideas?
	Substantive conversation	Does the classroom talk break out of the initiation/response/evaluation pattern and lead to sustained dialogue between students, and between students and teachers?
	Knowledge as problematic	Are students critiquing and second guessing texts, ideas, and knowledge?
	Metalanguage	Are aspects of language, grammar and technical vocabulary being foregrounded?
Relevance	Knowledge integration	Does the lesson range across diverse fields, disciplines and paradigms?
	Background knowledge	Is there an attempt to connect with students' background knowledge?
	Connectedness to the world	Do lessons and assigned work have any resemblance or connection to real life contexts?
	Problem based curriculum	Is there a focus on identifying and solving intellectual and/or real world problems?
Supportive School Environment	Student control	Do students have any say in the pace, direction or outcome of the lesson?
	Social support	Is the classroom a socially supportive, positive environment?
	Engagement	Are students engaged and on-task
	Explicit Criteria	Are criteria for student performance made explicit?
	Self regulation	Is the direction of students' behaviour implicit and self-regulatory?
Recognition of difference	Cultural knowledges	Are diverse knowledges brought into play?
	Inclusivity	Are deliberate attempts made to increase participation of all students from different backgrounds?
	Narrative	Is the teaching principally narrative or expository?
	Group Identity	Does teaching build a sense of community and identity?
	Citizenship	Are attempts made to foster active citizenship?

Gore et al. (2006) argue that the productive pedagogies framework is most useful as a tool for reflecting on practice. In analysing the classroom video, two or three researchers observed the lesson using the categories to rate the overall lesson. A scale of 0[1] (not a feature of this lesson) through to 5 (an integral part of the lesson) were scored for each lesson. These were undertaken independently by the members of the research team. Once the lesson had been completed, the team met to view their ratings and to come to a

[1] This model has been validated by the QSLRS team and where each score is more clearly articulated than is possible within this paper.

consensus on the score. In most cases, the scores were very similar so there was little negotiation. However, there were a number of instances where there was considerable debate but this was often centred on clarification of the definitions and the perceptions around whether the score could be applied to the full lesson.

Within the Productive Pedagogy approach, there is a strong emphasis on raising the quality of teaching in terms of the intellectual experiences and the social learning. The outcomes of the Queensland study (Education Queensland, 2001) indicated that teachers were very good at providing a supportive learning environment but that the intellectual quality was quite poor. When the analysis was undertaken across key learning areas, it was reported that the learning environments in mathematics scored the least favourably suggesting that the intellectual quality in mathematics (across all years of schooling) was poor.

Scoring IWBs – New Pedagogy or Problematic Pedagogy?

In seeking to explore the use of IWBs in mathematics classroom, we undertook the same analysis of the classroom videos. As can be seen in Table 2, the scores are low in most areas. We have included the analysis of classroom data where ICTs were used in mathematics classrooms as a comparison.

Table 2
Productive Pedagogy Analysis of IWB use in Upper Primary Classrooms.

	ICTs		IWBs	
Dimension of Productive Pedagogy	Mean	SD	Mean	SD
Depth of knowledge	1.64	1.36	1.5	1.46
Problem based curriculum	2.19	1.38	*0.92*	0.83
Meta language	1.69	1.07	1.25	1.87
Background knowledge	1.76	1.16	1.67	1.63
Knowledge integration	1.48	1.27	*0.42*	0.45
Connectedness to the world	1.38	1.44	*0.42*	0.45
Exposition	1.19	1.64	0.83	0.82
Narrative	0.31	0.78	0.17	0.18
Description	2.24	1.02	*1.42*	1.25
Deep understanding	1.43	1.47	1.25	1.19
Knowledge as Problematic	1.14	1.47	1.33	1.36
Substantive conversation	1.26	1.40	*0.5*	0.46
Higher order thinking	1.31	1.55	1.33	1.36
Academic engagement	2.23	1.38	*1.5*	1.46
Student direction	0.79	0.92	0.33	0.28
Self regulation	3.24	1.12	*2.5*	2.45
Active citizenship	0.30	0.78	0	0
Explicit criteria	2.83	1.17	*1.33*	1.28
Inclusivity	0.33	0.75	0	0
Social support	2.51	0.25	*1.25*	0.62

These data indicate that when using the IWBs as a pedagogical device, their effectiveness may be somewhat limited. We have reported the data for when teachers used ICTs to support numeracy learning elsewhere (Lerman & Zevenbergen, 2006) and this showed very low levels of quality learning potential. However, when using the same framework to analyse the use of IWBs, the results are even lower. Nine out of the twenty pedagogies (those in italics) scored substantially lower when using IWBs. Most of the lower scores were in those two dimensions that relate to the intellectual aspects of mathematics learning. From these data we can conclude that the use of IWBs actually reduces the quality of mathematical learning opportunities, provides fewer opportunities for connecting to the world beyond schools, and offers little autonomous/independent learning opportunities for students. Because these data are alarming in terms of their low scores, we sought to understand the phenomenon noted earlier in this paper. Whereas the low scores would suggest that there was potential for low levels of mathematical learning, our observations of the lessons indicated that despite these perceived low scores, there were few behaviour problems with students.

Activity Theory: Coming to Understand the Use of IWBs

In this final section, we analyse, using Activity Theory, the outcomes in the productive pedagogies table alongside interview data and classroom observations. We focus on the notion of the artifact mediating learning. Within activity theory, signs and tools mediate learning so, in our case, the IWBs were seen as artifacts that shape the ways in which learning can occur. The teachers found the resources that were available through the IWB – such as pre-planned lessons and digital tools (protractors, rulers, etc.) – offered different ways of working with the students. Not only were the resources shaping the ways in which teachers taught and planned, but also they impacted on other aspects of their work.

> Shane: I find that there are a whole lot of really good lessons that I can just use. If I am doing something on area for example, there are lessons already made up. Some other teachers have developed them so they have to be good ones. I am sure that the company only puts up the best examples. I have found these to be very handy and they save me doing the preparation work. I guess I change them a bit to suit me and the kids but they are pretty much there.

Most of the teachers had some comment about the time factor in the use of IWBs. It was seen to save preparation time in two different ways. As evident in the comment by Shane he drew on the resources that had been made by other teachers as these were "tried and proven" examples of lessons that worked. In observing his lessons, he would select from the databank and then implement the lesson. Another teacher commented on how, when using the IWB, the toolkit meant that the resources were all in the one place so she did not have to hunt around for them. Knowing that the protractor, ruler, clock, calculator were all on the screen and at the touch of the board, was seen to be a considerable timesaver. Other teachers made similar comments about the tools that were available on the IWB.

> Sarah: I think that the tools on the whiteboard are just great. They are done in a way that the children like them. When I pull up the calculator, for example, it looks exciting. It is much more interesting than the overhead projector type. I think that these kids expect a bit more from their computers and this is possible with the interactive whiteboard.

These built-in tools were seen to help teaching by reducing time spent not only on preparation of lessons but also within the lesson. This helped to make for a quicker pacing of lessons. The quicker pace was seen to enhance learning opportunities by engaging students. When using the IWBs, it would appear that the teachers were aware of the faster pace of the lessons. Having the ready-made resources available meant that little time was "wasted" moving from one site to another or drawing representations on the traditional boards or papers. They articulated that they posed a lot more questions and the students had greater opportunities for participating in the lessons due to the increased questioning.

> Maxine: One of the things that I like about the whiteboards is that I can ask a lot more questions. You just have to click on the menu and there is the lesson or the things you need so you are not wasting a lot of time putting up overheads or drawing things on the board. I can ask more questions to the kids to see what they know and to get them to think about things. Like when we did the lesson with the clocks. You just click on the clock and there it is. You can just move the time around as quick as they kids respond. I think they like the quicker speed. They seem to enjoy the race of the lesson. If they answer quickly, then we can do another one or something a bit different

The IWB offers other potentials that were not possible in previous media. In the following observation of a lesson, we were able to see how the accuracy of the IWB makes the teaching of fractions possible in new and novel ways.

> While the teacher poses the questions, these are teacher-initiated questions and tend to be of a low level- that is, recall-type questions. Observing a lesson on fractions, the teacher had used the fraction creator. In this, the teacher used the circle and made various numbers of segments. With each new model of fractions, she posed questions including “How many pieces are there?” "What fraction is that?" The pacing of questions was faster than would be possible if the teacher were to draw the objects on the board and then create sections. What was possible in this format was that the accuracy of the sections made for less confusion as to the size (and hence equality in those sizes) but also made possible the more difficult representations (such as sevenths or fifths). (Lesson Observation)

However, although the accuracy of representation was a strength of the IWB, it is noted that the overall pedagogy remained similar to most lessons we have observed in the more traditional modes of teaching. The depth of questioning remained at a relatively superficial level where low levels of questions were posed. Thus there remained considerable tension in what was offered and what could have been asked. While some aspects of pedagogy had changed, others had remained in place.

One of the observations in the use of IWBs was that it seemed to be used for the introduction to the lessons. In following this observation, teachers were asked if this were the case and if so, why. In the interviews, it was confirmed that the teachers tended to use the IWB to orientate the lesson and to motivate the students.

> Heidi: I use it to get the lesson started. The kids are all together, there are all on the one task, they know what we are doing. That is a good way to start the lesson. It is also good as the kids are very motivated by the boards so they are keen to get into the lesson.

In examining the role of the artifact one must also ask what it is replacing, both physically and in how it used during teaching. The IWB largely replaces the standard whiteboard in that whilst it is also available for pupils to be called to the board to present their ideas, proposals, and outcomes of their problem solving, it can also be used to present content previously prepared and it enables the teacher to choose high quality accurate representations as they are called for during the progress of the lesson. The IWB enables the same variety of font formats and other visual effects as word processing packages too. In most classrooms the whiteboard remains on the wall alongside the IWB. There is some sense that the students in classrooms expect a higher level of digital media in their lives. Following one lesson where the teacher had been working with some number work and using the calculator, we discussed the approach and what was offered through the IWB environment that would not be possible with the non-digital environments. In the case being observed, it was posed that the same learning could have occurred had the teacher used the traditional whiteboard and an overhead projector, which would have been a substantially cheaper option. The teacher commented as follows.

> Marcie: What I think is the key to this is that the calculator is already there. I click it on and there it is. I don’t have to walk to the OHP and use that medium. There is no time being wasted. The

> calculator (on the IWB) is a neat one and the kids like it. I think that they are so savvy with technology that they come to expect that, you know, the instant appearance of things – like the calculator. They get turned off by wasting time moving around, they like things to come up at the touch of a key. They just expect it, they have grown up with computers and they just expect that that is the way the world is.

The overhead projector (OHP) can be used to project pre-prepared transparencies onto the whiteboard but in our experience in these schools the OHP is rarely used. Our observations indicate the predominance of the latter two uses of the IWB, pre-prepared materials and impressive formats, causing some frustration amongst pupils as they want to "have a go" at using the IWB themselves. Writing on the whiteboard is a slow process, calling for the teacher to be turned away from the pupils. Projecting PowerPoint work or other resources sets up fast paced lessons and greater control of pupils' behaviour.

Thus the key tension here appears to be between the artefact and the division of labour. Although it is clear that the IWB offers great potential for higher level interactions between teacher and pupils, the need to be in control of the class and in this case the artefact militates against any pedagogic shift towards greater intellectual challenge. The identification of this tension also opens up the possibility for development with teachers; the specific focus offers a way in to engagement with what is blocking a positive move.

Conclusion

There is little doubt that IWBs have the potential to enhance learners' opportunities to experience mathematical representations and develop their mathematical thinking. As with all resources, mathematical or other, internalising a tool, be it the number line or a calculator, LOGO, dynamic geometry or Graphic Calculus, or presentation tools such as overhead projectors or IWBs, transforms the world, in this case of mathematical pedagogy for the teacher. That transformation is always mediated by other experiences; however by themselves they will not transform pedagogy, no matter what their potential. Indeed, as we have reported in this paper, the technologically impressive features of the IWB can lead to it being used to close down further the possibility of rich communications and interactions in the classroom as teachers are seduced by the IWB's ability to capture pupils' attention. We suspect, also, that teachers' advance preparation for using the IWB, often via the ubiquitous PowerPoint package or pre-prepared lessons for the IWB, are leading to a decreased likelihood that teachers will deviate in response to pupils' needs and indeed might notice pupils' needs less frequently through the possibility to increase the pacing of mathematics lessons.

References

Armstrong, V., Barnes, S., Sutherland, R., Curran, S., Mills, S., & Thompson, I. (2005). Collaborative research methodology for investigating teaching and learning: The use of interactive whiteboard. *Educational Review, 57*(4), 457-469.

Beauchamp, G. (2004). Teacher use of the interactive whiteboard in primary schools: Towards an effective transition framework. *Technology, Pedagogy and Education, 13*(3), 337-348.

Education Queensland. (2001). *The Queensland school longitudinal reform study*. Brisbane: GoPrint.

Edwards, J.-A., Hartnell, M., & Martin, R. (2002). Interactive whiteboards: Some lessons from the classroom. *Micromath, Summer*, 30-33.

Engestrom, Y. (2000). Activity theory as a framework for analyzing and redesigning work. *Ergonomics, 43*(7), 960 - 974.

Glover, D., & Miller, D. (2001). Running with Technology: the pedagogic impact of the large-scale introduction of interactive whiteboards in one secondary school. *Journal of Information Technology for Teacher Education, 10*(3), 257.

Glover, D., & Miller, D. (2002). The Interactive Whiteboard as a Force for Pedagogic Change: the experience of five elementary schools in an English education authority. In *Information Technology in Childhood Education* (Vol. 1, pp. 5-19). Norfolk: Association for the Advancement of Computing in Education.

Gore, J., M., Griffiths, T., & Ladwig, J., G. . (2004). Towards better teaching: productive pedagogy as a framework for teacher education. *Teaching and Teacher Education, 20*, 375–387.

Griffenhagen, C. (2000). *A report into whiteboard technologies: A published report*. Oxford: Oxford University Laboratory.

Jones, S., & Tanner, H. (2002). Teachers' interpretations of effective whole-class interactive teaching in secondary mathematics classrooms. *Educational Studies, 28*, 265-274.

Latane, B. (2002). Focused interactive learning: A tool for active class discussion. *Teaching of Psychology, 28*(1), 10-16.

Maor, D. (2003). The teacher's role in developing interaction and reflection in an on-line learning community. *Educational Media International, 401*(1/2), 127-138.

Lerman, S., & Zevenbergen, R. (2006). Maths, ICT & Pedagogy: An examination of equitable practice across diverse settings. In J. Novotna, H. Morova, M. Kratka, & N. Stehlikova (Eds.) *Mathematics in the Centre: Proceedings of the 30th annual conference of the international group for the Psychology of Mathematics Education.* (Vol 4, pp. 49-56) Prague: PME..

Smith, F., Hardman, F., & Higgins, S. (2006). The impact of interactive whiteboards on teacher–pupil interaction in the National Literacy and Numeracy Strategies. *British Educational Research Journal, 32*(3), 443 - 457.

Waycott, J., Jones, A., & Scanlon, E. (2005). PDAs as lifelong learning tools: An activity theory based analysis. *Learning, Media & Technology, 30*(2), 107 - 130.

Zevenbergen, R., & Lerman, S. (2006). *Numeracy, equity and ICTs: Final Report*. Brisbane: Griffith University.

International Perspectives on Early Years Mathematics

Jillian Fox
Queensland University of Technology
<j.fox@qut.edu.au>

In recent decades the development of mathematical proficiency has been recognised as a key issue for children and their education. The purpose of this paper is to identify key international perspectives that influence Australian mathematics education in the early years especially those that are in a similar state of technological development to Australia. There are four key trends deserving of discussion: (1) development in the early years, (2) mathematical proficiency in the early years, (3) mathematics policy and curriculum designed for young children, and (4) the existing research evidence-base.

In recent decades there has been universal interest in the benefits surrounding early childhood education and the crucial role early childhood development plays in societies economic and social growth has been internationally recognized and acknowledged (Dodge, 2004). This commitment to early childhood education and care is evident in the report *Starting strong: Early childhood education and care* (OECD, 2001, 2006). Policy-makers have "recognised that equitable access to quality early childhood education and care can strengthen the foundations of lifelong learning for all children" (OECD, 2001, p. 7). Correspondingly, the importance of mathematics in the lives of young children has been recognised (Ginsburg, Cannon, Eisenband, & Pappas, in press).

The past few decades have seen substantial changes in thinking about young children's ability to reason mathematically and their propensity to learn mathematical concepts and acquire associated skills (e.g., Baroody, 2000; Clements & Sarama, in press; Ginsburg, Balfanz, & Greenes, 2000). The development of mathematical proficiency has been recognised as a key issue for children and their education. In recent years it has also been acknowledged that the advances in technology have influenced the need for increased and enhanced numeracy practices. Steen (2001) credits the rise in the use of quantitative data, numbers, and information to the universal increase in the usage of technology, computer, and the internet. Our very young children are born into a world that is built on digital technology and a world where having competence and dispositions to use mathematics in context is essential. The importance of a numerate society in a technological age is recognized globally (Her Majesty's Inspectorate, 1998; National Council of Teachers of Mathematics (NCTM), 2000).

The coalescence of contemporary understandings (i.e., early learning, mathematical proficiency, technology) has created a juncture in the field of early childhood mathematics. The purpose of this paper is to identify key international influences on Australian mathematics education in the early years, especially those from countries that are in a similar state of technological development to Australia. There are currently four international trends guiding early childhood mathematics – the nature of early years development, the mathematical capabilities of young children, early childhood policy and curricula to inform practices, and the call for evidence-based research.

Proceedings of the 30th annual conference of the Mathematics Education Research Group of Australasia
J. Watson & K. Beswick (Eds), © MERGA Inc. 2007

The Early Years of Development

The internationally defined period of early childhood spans the years from birth to eight years (Bredekamp & Copple, 1997). These first eight years of life constitute two distinct learning periods: first, the development that occurs prior-to-school in informal learning situations, and second, the development that occurs in the first three years of schooling, which is often regarded as formal learning. High quality educational programs in the prior-to-school years facilitate the development of the child in all its dimensions and have considerable long-lasting effects on the child's life (OECD, 2001, 2006). Perry (2000) argues that during this period the growth of fine and gross motor skills, understanding and expression of emotional and social competence, cognitive changes, and the development of language are extensive. Contributing to the current perceptions of young children and their learning capabilities are the findings of neuroscience research.

Neuroscience research findings confirm the connection between young children's experiences and achievements later in life (Bruer, 1999). These studies have suggested that brain growth is highly dependent upon children's early experiences. Original research by Chugani (1998) provides evidence that "environmental enrichment" stimulates brain development. Worldwide, there has been excitement about the potential of the studies from neuroscience to inform early childhood education (e.g., Meade, 2000). Consequently, it has been widely accepted that early childhood development prior to school helps prepare young children to succeed in school (Bowman, 1999) and that long-term success in learning requires quality experiences during the "early years of promise" (Carnegie Corporation, 1998).

Mathematical Proficiency in the Early Years

International research provides evidence advocating the salient nature of early childhood development and mathematical growth. An increased recognition of the importance of mathematics (Kilpatrick, Swafford, & Findell, 2001) coupled with research findings has confirmed that mathematical development occurs in the early years and is critical to success and achievement in both school and life pursuits. Studies have shed light on the many general and specific mathematical skills, abilities, knowledge, and dispositions acquired by young children. For example, Hughes' (1986) research clearly showed that children exhibit knowledge of subtraction before attending school, while Carpenter and Moser (1984) noted children use informal knowledge to solve simple addition problems. Feeney and Stiles (1996) produced research findings that describe children's spatial ideas. Measurement, data, and probability have also been identified as features in early childhood learning (Perry & Docket, 2002). Other studies have reported on problem solving (Cobb et al., 1991), data sense (Jones, Langrall, Thornton, & Nisbet, 2002), numerical competence (Wynn, 1998), and counting (Sophian, 2004). Research findings on children's early mathematical growth together with the growing number of children who spend time in early childhood programs has created an impetus for the creation of policies and curricula that support the development of early years care and education. In the digital age with the vast majority of jobs requiring more sophisticated skills than in the past, mathematical proficiency has become as important a gatekeeper as literacy (Baroody, Lai, & Mix, in press).

Mathematics Position Statements and Curriculum in the Early Years

Position statements and curricula advocating mathematics education in the early years recognises young children's mathematical potential. A position statement developed by the National Association for the Education of Young Children (NAEYC, 2002), and NCTM (2002) have affirmed that high-quality, challenging, and accessible mathematics education for 3- 6-year-old children is a vital foundation for future mathematics learning. Besides advocating a robust mathematical base for these young members of society, the position statement acknowledges that higher levels of mathematical proficiency are required in the 21st century (NAEYC, 2002; NCTM, 2002). However, none of the ten research-based recommendations designed to guide classroom practice discusses the role of technology on young children's mathematical development.

Curriculum documents create the foundation for much that happens in compulsory and pre-compulsory settings. The NCTM (2000) proposes that mathematics is a way of thinking about relationships, quantity, and pattern via the processes of modelling, inference, analysis, symbolism, and abstraction (NCTM, 2000). They stipulate that, "the foundation for children's mathematical development is established in the earliest years" (p. 73) and have created principles and standards that promote this viewpoint. American researchers involved in early childhood mathematics have made a concerted effort to foster curriculum reform informed by the mathematics standards. Clements, Sarama, and Di Biase (2003) have compiled a list of assumptions, themes, and recommendations that evolved from the conference on *Standards for Pre-Kindergarten and Kindergarten Mathematics Education* (NCTM, 2000). Clements et al. (2003) emphasise that the guidelines for developing standards and curricula should be based on available research and inform practice.

Evidence-based Practice

The belief that mathematical capabilities and competencies of young children are extensive and impressive (Clements & Sarama, in press) is validated by contemporary research findings. Yet a dearth of research on early childhood mathematics especially in the years prior-to-school has been reported (Perry, 2000). Hiebert (1999) cautions that an adequate evidence base is essential to inform teachers who are trying to improve children's achievement in mathematics. In a recent study, I reviewed 208 articles on early childhood mathematics education sourced from the ERIC database that were published between 2000 and 2005 in order to determine the adequacy of the literature. Overall, this study revealed: (1) a lack of peer-reviewed articles that discuss, investigate, examine, or debate early childhood mathematics; (2) a limited emphasis in the prior-to-school years; and (3) a paucity of literature on technology and problem solving. Traditional mathematical topics are represented in the research, such as mathematical concepts and instruction, but the literature was limited in other significant ways. For example, scant research on technology use by young children was reported. When considering the mathematical proficiency to be developed by young children to function effectively in everyday life, technology plays an important role.

A further consideration in the adequacy of the literature base is the rigour of the research. The requirement for evidence-based policy and curricula in the USA has drawn attention to the quality of the research base from which policy and curricula are developed in other countries. In the United States landmark legislation titled, *No Child Left Behind* (NCLB) (U.S. Department of Education, 2001) aims to improve the performance of school students by increasing the standards of accountability as well as providing parents more flexibility in choosing which schools their children will attend. This act specifically calls for scientifically-based research to inform education policy and practice and that decisions regarding policy and practice be evidence-based. Thus, not only is there a need for more early childhood research on contemporary topics such as technology, but there is a corresponding need for this research to be rigorous.

Conclusion

Four international trends influence contemporary early childhood mathematics: (1) world wide beliefs about the salient nature of early years' development, (2) the mathematical capabilities of young children, (3) the development of position statements and curricula to inform early childhood practices, and (4) the evidence base. However, some essential areas remain under-researched. For example, the influence of technology on human life in the new millennium has created a world characterized by diverse and energetic communication, vast amounts of information, rapid change, and high levels of numeracy. Technology affects the daily lives of every person, directly or indirectly (Williams, 2002) yet currently it is featuring little in the early childhood mathematics research literature. The research base providing evidence of the salient nature of prior-to-school years and mathematical development is also lacking. These first years of life should not be overlooked due to their important role in brain development, lifelong learning, and life chances.

This paper has identified some of the international perspectives influencing early childhood mathematics education and some of the areas in need of research. These international perspectives contribute to the research agendas and directions in Australian early childhood.

References

Baroody, A. J. (2000). Does mathematics instruction for three-to-five year olds really make sense? *Young Children, 4*, 61-67.

Baroody, A. J., Lai, M., & Mix, K. S. (in press). The development of young children's early number and operation sense and its implications for early childhood education. To appear in B. Spodek & O. Saracho (Eds.), *Handbook of research on the education of young children* (2nd Ed.). Mahwah, NJ: Lawrence Erlbaum.

Bowman, B. (1999). Policy implications for math, science, and technology in early childhood. In *Dialogue on early childhood science, mathematics, and technology education* (pp. 40 - 49). Washington, DC: American Association for the Advancement of Science.

Bredekamp, S., & Copple, C. (1997). *Developmentally appropriate practice in early childhood programs.* Washington, DC: National Association for the Education of Young Children.

Bruer, J.T. (1999). In search of brain-based education. *Phi Delta Kappan, 80,* 648-654, 656-657.

Carnegie Corporation of New York. (1998). *Years of Promise: A comprehensive learning strategy for America's children*. Retrieved December 26, 2006, from www.carnegie .org/sub/pubs/execsum.html.

Carpenter, T., & Moser, J. (1984). The acquisition of addition and subtraction concepts in grades one through three. *Journal for Research in Mathematics Education*, *15*, 179-202.

Chugani, H. (1998). A critical period of brain development: Studies of cerebral glucose utilisation with PET. *Preventative Medicine*, *27*, 184-188.

Clements, D. H., & Sarama, J. (in press). Early childhood mathematics learning. In F. Lester (Ed.), *Second handbook of research on mathematics teaching and learning*. Greenwich, CT: Information Age Publishers.

Clements, D. H., Sarama, J., & DiBiase, A. M. (Eds.). (2003). *Engaging young children in mathematics – Standards for early childhood mathematics education*. Mahwah, NJ: Lawrence Erlbaum.

Cobb, P., Wood, T., Yackel, E., Nicholls, J., Wheatley, G., Trigatti, B. & Perlwitz, M. (1991). Assessment of a problem-centred second grade mathematics project. *Journal for Research in Mathematics Education*, *22*, 3-29.

Dodge, D. (2004). Human capital, early childhood development and economic growth. In R.E. Tremblay, R. G. Barr, & RDEeV Peters (Eds.), *Encyclopedia on Early Childhood Development.* (online), Montreal, Quebec: Centre of Excellence for Early Childhood Development. Retrieved September 2006. from www.exellence-earlychildood.ca/documents/DodgeANG.pdf.

Feeney, S. M., & Stiles, J., (1996). Spatial analysis: An examination of preschooler's perception and construction of geometric patterns. *Developmental Psychology*, *32*, 933-941.

Ginsburg, H. P., Balfanz, R., & Greenes, C. (2000). Challenging mathematics for all young children. In A. L. Costa, (Ed.), *Teaching for intelligence: A collection of articles* (pp.245-258). Arlington Heights, IL: Skylight Professional Development.

Ginsburg, H. P., Cannon, J., Eisenband, J., & Pappas, S. (in press). Mathematical thinking and learning. In K. McCartney & D. Phillips (Eds.), *The handbook of early child development*. Malden, MA: Blackwell Publishing.

Hiebert, J. (1999). Relationships between research and the NCTM Standards. *Journal for Research in Mathematics Education, 30*(1), 3-19.

Her Majesty's Inspectorate. (1998). *The National Numeracy project: An HMI evaluation*. London: Office for Standards in Education.

Hughes, M. (1986). *Children and number: Difficulties in learning mathematics.* New York: Basil Blackwell.

Jones, G. A., Langrall, C. W., Thornton, C. A., & Nisbet, S. (2002). Elementary students access to powerful mathematical ideas. In English, L. D. (Ed.), *Handbook of international research in mathematics education*. (pp. 113-141). Mahwah, NJ: Lawrence Erlbaum.

Kilpatrick, J., Swafford, J., & Findell, B. (Eds.). (2001*). Adding it up: Helping children learn mathematics*. Washington, DC: National Academy Press.

Meade, A. (2000). The brain and early childhood development. *Pitopito Korero*, (Ministry of Education, New Zealand), *23*, 7-12.

National Association for the Education of Young Children. (2002). *Early childhood mathematics: Promoting good beginnings. Position statement*. Retrieved 14 December 2006, from http://146.145.134.3/resources/position_statements/psmath.htm.

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics*. Reston, VA: Author.

Organisation for Economic Co-operation and Development. (2001). *Starting strong: Early childhood education and care.* Paris: OECD.

Organisation for Economic Co-operation and Development. (2006). *Starting strong 2: Early childhood education and care.* Paris: OECD.

Perry, B. (2000). *Early childhood numeracy*. Australian Association of Mathematics Teachers. Retrieved September 2006 from http://www.aamt.edu.au/home.html.

Perry, B., & Dockett, S. (2002). Young children's access to powerful mathematical ideas. In L. D. English (Ed.), *Handbook of international research in mathematics education* (pp. 81-112). Mahwah, NJ: Lawrence Erlbaum.

Sophian, C. (2004). Mathematics for the future: Developing a Head Start curriculum to support mathematics learning. *Early Childhood Research Quarterly*, *19*(1), 59-81.

Steen, L. A. (2001). Mathematics and numeracy: Two literacies, one language. *The Mathematics Educator, 6*(1), 10-16.

U.S. Department of Education .(2002). *No Child Left Behind: A desktop reference*, Washington, D.C: Author.

Williams, R. (2002). *Retooling: A historian confronts technological change*. Cambridge, MA: MIT Press.

Wynn, K. (1998). Numerical competence in infants. In C. Dolan (Ed.), *Development of mathematical skills* (pp. 1-25). Hove, England: Psychology Press.

Early Childhood Mathematics Education Research: What is Needed Now?

Bob Perry
Charles Sturt University
<bperry@csu.edu.au>

Sue Dockett
Charles Sturt University
<sdockett@csu.edu.au>

In the last four years there have been a number of calls for research into many aspects of early childhood mathematics education. As well, there has been an unprecedented increase in Australasian research in this field. How have these two factors matched? That is, are mathematics education researchers studying the aspects of the field that have been identified for further research? This paper provides the beginnings of a discussion around this question by highlighting particular Australasian early childhood mathematics education research endeavours and linking them to recent statements calling for further research in the field.

In our chapter for the recent MERGA review of research in mathematics education research in Australasia (Perry & Dockett, 2004), we concluded with the following statement on future research in early childhood mathematics education.

> From this critique of early childhood mathematics education research in Australasia in the period 2000 – 2003, fruitful areas for future research would seem to include:
>
> - approaches to assessment and teaching / learning in numeracy and possible mismatches between these;
> - successful approaches to the mathematics education of young Indigenous students;
> - successful approaches to the mathematics education of young children from culturally and linguistically diverse backgrounds;
> - technology in the mathematics education of young children;
> - play in the mathematics education of young children;
> - development of mathematical concepts among children before they start school;
> - continuities and discontinuities of learning in children as they move from prior-to-school to school settings, and
> - recognition of young children as capable learners of mathematics and the results of such recognition in their mathematical outcomes in the first years of school.
>
> The field of early childhood mathematics education research beckons as an exciting forum in which committed researchers can make a difference. While a lot has already been done, there is still much to do in an area which has been neglected to some extent but which is now enjoying a resurgence of interest. (pp. 119-120)

Similar statements have been made in other contexts. For example, Ginsburg and Golbeck (2004, p. 190)

> argue that researchers and practitioners should examine carefully not only the possibility of unexpected competence in young children, but also its complexity and the limits on it; investigate the socio-emotional context of learning and teaching; attend closely to those children in need of extra help, including low-socio-economic status (SES) children, children with disabilities, and children who receive schooling in an unfamiliar language; create sensitive evaluation strategies that examine program quality, the effectiveness of teachers and administrators, and children's achievement; develop creative and enjoyable curricula that stress thinking as well as content and integrate an organized subject matter with projects and the thoughtful use of manipulatives; investigate the complex processes of teaching in various contexts; and investigate the possible benefits and disadvantages of parental involvement in early mathematics and science education.

Proceedings of the 30th annual conference of the Mathematics Education Research Group of Australasia
J. Watson & K. Beswick (Eds), © MERGA Inc. 2007

Clearly, there are many similarities in these two statements. Together, they can be taken to articulate an agenda for further early childhood mathematics education research.

The latest comprehensive review of this research in Australasia (Perry & Dockett, 2004, p. 119) suggests that

> there is a vibrant and important early childhood mathematics research agenda in Australasia. Growing worldwide recognition of the importance of the early childhood years – both in and of themselves and in preparation for future learning experiences – and of the valuable, innovative and critical research being undertaken in Australasia augurs well for growth and continued influence.

How are we Travelling?

It is particularly gratifying to be able to report that, over the last 4 years since this statement was made, the quantity and quality of early childhood mathematics education research in Australasia have both moved in very positive directions. Much of this research has been stimulated by large systemic numeracy programs. Bobis, Clarke, Clarke, Thomas, Young-Loveridge, & Gould (2005) provide a comprehensive comparison of these programs) such as *Count Me In Too* (Bobis & Gould, 2000), *Early Numeracy Research Project* (Clarke & Clarke, 2004; Clarke, Clarke, & Cheeseman, 2006), and *First Steps* (Willis, Devlin, Jacob, Treacy, Tomazos, & Powell, 2004) in Australia and the *Early Numeracy Project* in New Zealand (Thomas, Tagg, & Ward, 2003). Based on the pioneering work of Bob Wright (e.g., Wright, 1994; Wright, Martland, Stafford, & Stanger, 2002), these programs have revolutionised early numeracy teaching and learning in Australia and provided a great deal of stimulus for further research in early childhood mathematics education.

The lists of “needed” research compiled by Ginsburg and Golbeck (2004) and Perry and Dockett (2004) are extensive. It is well beyond the scope of this paper to report on achievements in each of the areas listed. Rather, as examples, we choose two areas in which a great deal of work has been done by Australasian mathematics education researchers. These promote the central tenet of this paper that much has been done but that there is still much to do.

Young Children are Capable Mathematics Learners

One area identified above in terms of further research that has been carefully considered by these systemic programs has been that of recognition of young children as capable learners of mathematics and how this recognition impacts on the curriculum and pedagogy of the first years of school. The notion that children come to school able to access powerful mathematical ideas is not new but has received renewed emphasis through several initiatives in Australasia and beyond. For example, the recently published *Position Paper on Early Childhood Mathematics* (Australian Association of Mathematics Teachers and Early Childhood Australia (AAMT/ECA), 2006, p. 2) states that:

> The Australian Association of Mathematics Teachers and Early Childhood Australia believe that all children in their early childhood years are capable of accessing powerful mathematical ideas that are both relevant to their current lives and form a critical foundation for their future mathematical and other learning. Children should be given the opportunity to access these ideas through high quality child-centred activities in their homes, communities, prior-to-school settings and schools.

Research in Australasia (Clarke et al., 2006; Perry, Dockett, Harley, & Hentschke, 2006; Thomson, Rowe, Underwood, & Peck, 2005; Young-Loveridge, 2004) and beyond (Aubrey, 1993; Aubrey, Dahl, & Godfrey, 2006; Sarama & Clements, 2004; Seo & Ginsburg, 2004) provides backing for this profoundly important statement. Many of the systemic numeracy programs mentioned earlier in this paper adhere to this position and reflect it in the ways that they assess their participants in order to ascertain the extent to which the powerful ideas are present.

Assessment in Early Childhood Mathematics Education

Prior to the publication of the Australian position statement on early childhood mathematics education (AAMT/ECA, 2006), the peak professional bodies in mathematics and early childhood education in the United States of America had published their own position statement (National Association for the Education of Young Children and National Council of Teachers of Mathematics (NAEYC/NCTM), 2002). Assessment of young children's mathematical learning features as one of the critical elements of high quality mathematics education. The following statement is included:

> Assessment is crucial to effective teaching. Early childhood mathematics assessment is most useful when it aims to help young children by identifying their unique strengths and needs so as to inform teacher planning. Beginning with careful observation, assessment uses multiple sources of information gathered systematically over time. … Mathematics assessment should follow widely accepted principles for varied and authentic early childhood assessment. For instance, the teacher needs to use multiple assessment approaches to find out what each child understands--and may misunderstand. Child observation, documentation of children's talk, interviews, collections of children's work over time, and the use of open-ended questions and appropriate performance assessments to illuminate children's thinking are positive approaches to assessing mathematical strengths and needs. (NAEYC/NCTM, 2002, pp. 12-13).

The Australian position statement suggests that

> Early childhood educators should adopt pedagogical practices that assess young children's mathematical development through means such as observations, learning stories, discussions, etc. that are sensitive to the general development of the child, their mathematical development, their cultural and linguistic backgrounds, and the nature of mathematics as an investigative, problem solving and sustained endeavour. (AAMT/ECA, 2006, p. 3)

Clearly, assessment of mathematics learning is an important part of early childhood mathematics education. There has been and continues to be a great deal of work in Australasia in this area. For example, the work of Doig and his colleagues (Doig, 2005; Thomson et al., 2005) has developed and used standardised approaches to assessment that are claimed to have highly valid and reliable statistical characteristics, making them very useful in large scale reporting. Mulligan and her colleagues (Mulligan, Prescott, Papic, & Mitchelmore, 2006) have developed a particular assessment approach, based on those used in *Count Me In Too* and other systemic numeracy projects, to assess the development of pattern and structure in young children. Fox (2006) has used extensive structured observations to study possible links between patterning activities and the development of algebraic reasoning in preschool children. Young-Loveridge and her colleagues (Young-Loveridge, 2004; Young-Loveridge & Peters, 2005) have used individual task-based interviews to assess the numeracy development of children across the early childhood years and to evaluate the effectiveness of many different teaching approaches. Perry and his colleagues (Perry et al., 2006; Perry, Dockett, & Harley, in press) have used the learning stories approach developed by Carr (2001) and linked it to an extensive numeracy matrix constructed jointly by researchers and practitioners to assess and plan for preschool children's mathematical learning within the context of a mandatory reporting regime. All of these approaches to assessment show great potential to further enhance young children's mathematical learning and the teaching that will facilitate this.

Conclusion

There are many further examples where Australasian early childhood mathematics education researchers have taken up the challenge to undertake research that has been identified through the literature as 'needed'. More needs to be done but much has been achieved. For example, there is a particular need for practice-based research on ways in which culturally and linguistically diverse learners might better engage with mathematics education in both prior-to-school and school settings. One possible approach that could be applied to mathematics education has been documented by Fleer and Kennedy-Williams (2002). Much has been done in the area of technology use in early childhood mathematics education (Kilderry & Yelland, 2005) and the importance of continuity in approaches to mathematics learning and teaching as children make the transition to school has been recognised, although there is still a long way to go before this recognition results in practical changes (Thomson et al., 2005). The advent of documents such as the Australian position statement on early childhood mathematics education (AAMT/ECA, 2006) shows that the professions relevant to early childhood mathematics education are taking notice of the advances being made and the avenues being opened by this research. This recognition provides the early childhood mathematics education research community with strong motivation to continue its work.

References

Australian Association of Mathematics Teachers and Early Childhood Australia. (2006). *Position paper on early childhood mathematics.* Available on-line from http://www.aamt.edu.au/about/policy/earlymaths_a3.pdf

Aubrey, C. (1993). An investigation of the mathematical knowledge and competencies which young children bring into school. *British Educational Research Journal, 19*(1), 27-41.

Aubrey, C., Dahl, S., & Godfrey, R. (2006). Early mathematics development and later achievement: Further evidence. *Mathematics Education Research Journal, 18*(1), 27-46.

Bobis, J., Clarke, B., Clarke, D., Thomas. G., Wright, R., Young-Loveridge, J., & Gould, P. (2005). Supporting teachers in the development of young children's mathematical thinking: Three large scale cases. *Mathematics Education Research Journal, 16*(3), 27-57.

Bobis, J., & Gould, P. (2000). Changing the professional knowledge of teachers. In J. Bana & A. Chapman (Eds.), *Mathematics education beyond 2000* (Proceedings of the 23rd annual conference of the Mathematics Education Research Group of Australasia, Fremantle, pp. 47-54). Sydney: MERGA.

Carr, M. (2001). *Assessment in early childhood settings: Learning stories*. London: Paul Chapman.

Clarke, D. M., & Clarke, B. A. (2004). Mathematics teaching in Grades K-2: Painting a picture of challenging, supportive, and effective classrooms. In R. N. Rubenstein & G. W. Bright (Eds.), *Perspectives on the teaching of mathematics* (66th Yearbook of the National Council of Teachers of Mathematics, pp. 67-81). Reston, VA: NCTM.

Clarke, B., Clarke, D., & Cheeseman, J. (2006). The mathematical knowledge and understanding young children bring to school. *Mathematics Education Research Journal, 18*(1), 78-103.

Doig, B. (2005). Developing formal mathematical assessment for 4- to 8-year-olds. *Mathematics Education Research Journal, 16*(3), 110-119.

Fleer, M., & Kennedy-Williams, D. (2002). *Building bridges: Literacy development in young Indigenous children.* Watson, ACT: Australian Early Childhood Association.

Fox, J. (2006). Connecting algebraic development to mathematical patterning in early childhood. In J. Novotná, H. Moraová, M. Krátká, & N. Stehliková (Eds.), *Mathematics in the centre* (Proceedings of the 30th annual conference of the International Group for the Psychology of Mathematics Education, Vol. 3, pp. 89-96). Prague: Charles University.

Ginsburg, H. P., & Golbeck, S. L. (2004). Thoughts on the future of research on mathematics and science learning and education. *Early Childhood Research Quarterly, 19*(1), 190-200.

Kilderry, A., & Yelland, N. (2005). A day in the life: Rethinking numeracy in the information age. *Journal of Australian Research in Early Childhood Education, 12*(1), 113-121.

Mulligan, J., Prescott. A., Papic, M., & Mitchelmore, M. (2006). Improving early numeracy through a pattern and structure mathematics awareness program (PASMAP). In P. Grootenboer, R. Zevenbergen, & M. Chinnappan (Eds.), *Identities cultures and learning spaces* (Proceedings of the 29th annual conference of the Mathematics Education Research Group of Australasia, Canberra, pp. 376-383). Sydney: MERGA.

National Association for the Education of Young Children and National Council of Teachers of Mathematics. (2002). *Early childhood mathematics: Promoting good beginnings.* Available on-line from: http://www.naeyc.org/about/positions/pdf/psmath.pdf

Perry, B., & Dockett, S. (2004). Mathematics in early childhood education. In B. Perry, G. Anthony, & C. Diezmann (Eds.), *Research in mathematics education in Australasia 2000 – 2003* (pp. 103-125). Flaxton, QLD: Post Pressed.

Perry, B., Dockett, S., Harley, E., & Hentschke, N. (2006). Linking powerful mathematical ideas and developmental learning outcomes in early childhood mathematics. In P. Grootenboer, R. Zevenbergen, & M. Chinnappan (Eds.), *Identities cultures and learning spaces* (Proceedings of the 29th annual conference of the Mathematics Education Research Group of Australasia, Canberra, pp. 408-415). Sydney: MERGA.

Perry, B., Dockett, S., & Harley, E. (in press). Learning stories and children's powerful mathematics. *Early Childhood Research and Practice.*

Sarama, J., & Clements, D. H. (2004). *Building Blocks* for early childhood mathematics. *Early Childhood Research Quarterly, 19*(1), 181-189.

Seo, K.-H., & Ginsburg, H. P. (2004). What is developmentally appropriate in early childhood mathematics education? Lessons from new research. In D. H. Clements, J. Sarama, & A.-M. DiBiase (Eds.), *Engaging young children in mathematics: Standards for early childhood mathematics education* (pp. 91-104). Mahwah, NJ: Lawrence Erlbaum.

Thomas, G., Tagg, A., & Ward, J. (2003). *An evaluation of the Early Numeracy Project 2002: Exploring issues in mathematics education.* Wellington: Ministry of Education.

Thomson, S., Rowe, K., Underwood, C., & Peck, R. (2005). *Numeracy in the early years*. Melbourne: Australian Council for Educational Research.

Willis, S., Devlin, W., Jacob, L., Treacy, K., Tomazos, D., & Powell, B. (2004). *First Steps in mathematics.* Sydney: Rigby Heinemann.

Wright, R. J. (1994). A study of the numerical development of 5-year-olds and 6-year-olds. *Educational Studies in Mathematics, 26,* 25-44.

Wright, R., Martland, J., Stafford, A., & Stanger, G. (2002). *Teaching number: Advancing skills and strategies*. London: Paul Chapman Publications / Sage.

Young-Loveridge, J. M. (2004). Effects on early numeracy of a program using number books and games. *Early Childhood Research Quarterly, 19*(1), 82-98.

Young-Loveridge, J., & Peters, S. (2005). Mathematics teaching and learning in the early years in Aotearoa / New Zealand. *Australian Journal of Early Childhood, 30*(4), 19-24.

Trimangles and Kittens: Mathematics Within Socio-dramatic Play in a New Zealand Early Childhood Setting

Shiree Lee (formerly Babbington)
The University of Auckland
<s.babbington@auckland.ac.nz>

In prior-to-school early childhood settings mathematical play can occur in a natural and unstructured manner. This paper describes the findings of a case study of children in an urban Auckland early childhood centre engaging in socio-dramatic play in the family corner. This data gives rise to the notion that foundational mathematical knowledge can, and does, develop in very young children.

The socio-constructivist approach and theoretical framework as espoused within the New Zealand early childhood curriculum framework *Te Whãriki* (Ministry of Education, (MoE) 1996) implies that children are gaining experiences, which relate to future academic learning, while they play, in natural and in institutional environments. Within *Te Whãriki* (MoE, 1996) curriculum is defined as "the sum total of the experiences, activities, and events … which occur within an environment designed to foster children's learning and development" (p. 10). It is these experiences, activities and events that can contain rich mathematical activities and in turn form the foundation of future mathematical skill (Babbington, 2003). This paper highlights some of the key findings of a recently conducted case study, focussing on children's natural mathematical play, which was observed in a prior-to-school, early childhood setting.

Method

This case study investigation included observation of a family play area (family corner) incorporating voice recording of children's conversations. The data collected included a combination of photographs of the physical layout, researcher journal, and voice recording in order to record the language children used while playing. Participants included all children who chose to enter the family play corner during the periods of observation. The age range of these participants was between 18 months and 4 years of age, although the main players throughout the observational period of two weeks were all over 2 years of age.

Results and Discussion

On the first morning of the case study data collection the children were rearranging the family play corner equipment within the setting. During this time several significant mathematical aspects occurred. Two children carried a child-sized bed into the new area and placed it along one wall in the room, however, approximately 30cm of the bed was jutting into a doorway. The children noticed this and one child stated that it did not fit and they would have to find another place for it. After several minutes of trying a variety of places they decided to move a set of drawers so that the bed could fit in, and it did. The practical measurement and geometrical knowledge (spatial rearrangement) evident in this anecdote is supported by the work of Giglio-Andrews (1996) where she states that actions such as these

Proceedings of the 30th annual conference of the Mathematics Education Research Group of Australasia
J. Watson & K. Beswick (Eds), © MERGA Inc. 2007

build the foundations upon which children learn about formal geometric concepts.

Once most of the furniture was in place an adult placed a basket of plastic cutlery into the centre of the table. A child (4 years) tipped it out and sorted the cutlery into categories by colour; "red ones here, white ones here" as he placed them into a cutlery tray.

Classification of this type is also seen in the work of Kirova and Bhargava (2002) where they researched mathematics within a play-based curriculum and found evidence that mathematical understanding can be observed in children's socio-dramatic play. Their findings described a variety of early concepts specifically those of one to one correspondence, classification and seriation. This anecdote shows evidence of classification by attribute (colour and shape) and occurred in a variety of play episodes recorded over subsequent observations.

Another example of this was when some plastic crockery and cutlery had just come out of the dishwasher and was placed onto the table in the family corner. A 4-year-old child immediately sat down and started to dry this equipment with a tea towel, as they were still wet. As she dried each piece she placed them carefully into discrete groups of plates, cups, knives, spoons, forks, and bowls. The actions of this child showed her knowledge of hygiene practices and routines in the home and at the centre and an understanding that objects can be categorised into groups showing further evidence of this young child's classification skills.

Intellectualising about number knowledge was observed. A group of children were sitting at a table, one 4-year-old child had a plastic "play" biscuit and was pretending to cut it down the centre with a knife. As she did this she stated "half for you and half for me".

The concept of halves was also discussed at other times. A 3-year-old child had placed a small amount of play dough onto a plate "Toast is on the plate but I still need more honey, not enough, I going to cut it in half", as she cut the play dough into two pieces. This demonstrated an understanding of the concept that one half is one of two pieces regardless of whether she understood the equivalent nature of fractions. These two anecdotes support the work of Smith (1998), "It is important that teachers and parents realise that when children play imaginatively they are not being frivolous but are practicing important intellectual and social skills, which will help them develop in many areas" (p. 27). The intellectual skills that children exhibited while engaged in the play described above were observed in further episodes.

Birthdays as an aspect of number were a recurring theme particularly when play dough was available in the area. For example, a 3-year-old child said "look at my cakes, 'tis for you (looking at researcher) you are four, gonna be four." She then pushed three small forks into her play dough "cake". At this point another child (4 years) at the table stated, "no that's three you need another candle." The cake was then cut into six small pieces as the 3-year-old counted, "one, two, one, two, one, two." The second child in this anecdote displayed the skill of subitising, as she immediately knew how many "candles" were on the play dough cake without needing to count them. She was also able to show her knowledge of simple addition when she recognised the first child's mistake. The patterns of counting were beginning to be explored here by the 3-year-old as she counted six pieces of "cake" in twos. Carr, Peters, and Young-Loveridge (1991) described this clearly in their work with 4-year-olds where children could count in twos and in fives when prompted. This early mathematical conceptual understanding is the foundation for future mathematical skill and understanding in a wide range of mathematical areas such as addition, subtraction, and multiplication (Maclellan, 2000).

Other aspects of number that were observed were those of simple addition and subtraction. In a recurring and very popular game of “mum and the kittens”, three girls (3 years, 3 years, and 4 years of age respectively) were approached by a fourth child to join their game. When he stated that he wanted to be a kitten too, the “mum”, a human mother character, responded that there were not three kittens only two but he could be a dog if he wanted to. When one of the “kittens” left the play area “mum” then shouted out “hey there’s only one kitten now!” The inherent knowledge that two plus one more makes three, and that two minus one equals one, was clearly part of this child’s experience. This simple addition and subtraction is one of the major aspects of mathematical relationships as it eventually leads to the child’s understanding of quantification (Geist, 2001).

Counting as a measure of time was observed alongside geometrical shape knowledge. A 3-year-old child at the play dough table had cut a small piece of dough into an equilateral triangle,” here’s your trimangle [sic] cake, it’s your favourite, vamilla [sic].” She took the play dough “cake” to the play oven and placed it inside saying, “1, 2, 3, 4, 5, 6, 7, 8, 9, 10, ready now.” Children may use counting as a way to measure time, length, weight, distance, speed, or volume (Maclellan, 1998) and this anecdote clearly supports this claim. Identification of common geometrical shapes is a natural experience and in the anecdote above the child describes a play dough creation as a triangle. This is supported by the work of Oberdorf and Taylor-Cox (1999) where they describe children’s geometry as the way in which they make sense of their world.

These findings give strong evidence to the tenet that very young children have complex mathematical knowledge. Unless children’s play is viewed with a mathematical lens the mathematics can go unnoticed and seem frivolous (Pound, 1999). Of course not all play is mathematical or has mathematical components but there are obvious examples as discussed in this paper. Within this study children exhibited clear knowledge and understanding of classification, geometrical shapes, counting as a measure of time, patterns in numbers and routines or rituals, passage of time in relation to age, spatial awareness, simple fractions, and addition and subtraction. These mathematical experiences and conceptual understandings will provide the basis upon which future mathematics can be built (Babbington, 2003; Dockett & Perry, 2002; Hedges, 2003; Geist, 2001).

Conclusion

The findings of this study have shown that the young children within this setting performed mathematical inquiry naturally and without adult interaction or intervention. Ginsberg (2006) further supports the key findings of this investigation and refers to children’s everyday mathematics as a natural and fundamental aspect of all children’s learning: “children have the capacity, opportunity, and motive to acquire basic mathematical knowledge” (p.148). Ginsberg goes on to claim that early mathematics is the foundation for learning in many other subject areas such as reading, scientific knowledge, and construction.

The notion of play as the catalyst for children’s learning (Dockett & Perry, 2002) will continue to be explored through gathering further empirical evidence of the ways in which mathematical exploration occurs in early childhood and will continue to inform research in, and about, early childhood education. This could include considering the importance of listening to children and carefully observing their play in order to identify mathematical knowledge. This is highlighted strongly within the examples in this paper but much more data could be gathered and analysed to find out what children know and can do. It is also important to remember that any learning and teaching should be enjoyable for all involved. As *Te Whãriki* states, it is expected that mathematical ideas will amuse, inform, delight, and excite (MoE, 1996) for all those who engage with early childhood education.

References

Babbington, S. (2003, September). *The magic of mathematics.* Paper presented at the 8th Early Childhood Convention, Palmerston North, N.Z.

Carr, M., Peters, S., & Young-Loveridge, J. (1998). *New Zealand Council for Education Research; Australian Council for Education Research. SET Research Information for Teachers, No.2.* Wellington: New Zealand Council for Educational Research.

Dockett, S. & Perry, B. (2002). Playing with maths. *Square One 12*(3), 14-17.

Geist, E. (2001). Children are born mathematicians. *Young Children, 3*(1), 12-19.

Giglio-Andrews, A. (1996). Developing spatial sense: A moving experience. *Teaching Children Mathematics, 2*(5), 290-293.

Ginsberg, H. P. (2006). Mathematical Play and Playful Mathematics: A guide for early education. In D. G. Singer, R. Michnick Golinkoff, & K. Hirsh-Paslek (Eds.), *Play = Learning. How play motivates and enhances children's social-emotional growth,* pp. 145-168. New York: Oxford University Press.

Hedges, H. (2003). A response to criticism and challenge: Early literacy and numeracy in Aotearoa/ New Zealand. *New Zealand Research in Early Childhood Education, (6)*1, 13-22.

Kirova, A., & Bhargava, A. (2002, Spring). Learning to guide preschool children's mathematical understanding: A teacher's professional growth. *Early Childhood Research and Practice*, Vol. 4, No. 1. Retrieved June 4, 2005 from http://ecrp.uiuc.edu/v4n1/kirova.html

Maclellan, E. (1998). The importance of counting. In I. Thompson (Ed.), *Teaching and learning early number.* Milton Keynes: Open University.

Ministry of Education. (1996). *Te whaariki he whaariki mo nga mokopuna o Aotearoa, Early childhood curriculum.* Wellington: Learning Media.

Oberdorf, C., & Taylor-Cox, J. (1999). Shape up. *Teaching Children Mathematics, 2*(5), 340-345.

Pound, L. (1999). *Supporting mathematical development in the early years*. Buckingham: Open University Press.

Smith, A.B. (1998). *Understanding young children's development.* Wellington, NZ: Bridget Williams Books.

Children's Number Knowledge in the Early Years of Schooling

Ann Gervasoni
Australian Catholic University
<Ann.Gervasoni@acu.edu.au>

This paper explores the number knowledge of 1015 children who began school in 2006 and of a further 3000 children in Grades 1-3. The data show that number knowledge varies considerably when children begin school, and that this variation extends as schooling proceeds. Teachers need to be aware of each child's current knowledge and ways to customise learning experiences if they are to meet each child's learning needs.

Introduction

Every child arrives at school on the first day with lots of number knowledge. Each child constructed this knowledge throughout their first 5 years of life as they interacted with their families, friends and environment. Because children's experiences and interests vary so much, then the number knowledge of children within a class is likely to vary, even when they first begin school. To examine this premise, this paper explores the number knowledge of children throughout the first 3 years of schooling.

Assessing Children's Number Knowledge

The data presented in this paper was collected in 2006 from over 4000 children attending 52 primary schools in the Ballarat Diocese of western Victoria, enabling a rich picture of children's number knowledge in this region to be formed. The practice in these schools is for teachers to assess each student in the first week of school using the *Early Years Interview* (Department of Education Employment and Training, 2001) for the purpose of gaining insight about each child's current mathematical knowledge.

Such assessment interviews are now widely used by teachers in Australia and New Zealand, due to the experience of three large-scale projects that informed policy formation (e.g., Gould, 2000; Clarke et al., 2002; Higgins, Parsons, & Hyland, 2003). A common feature of these projects was the use of a one-to-one interview and a research-based framework to describe progressions in mathematics learning (Bobis et al., 2005).

The development of the *Early Years Interview* and the associated framework of growth points are reported in detail elsewhere (e.g., Clarke, 2001; and Clarke, Sullivan, & McDonough, 2002), but it is important to note that the growth points describe major learning along a hypothesised learning trajectory (e.g., Cobb & McClain, 1999) and formed the basis for the development of assessment items. In the Ballarat Diocese, children's responses were analysed by the teacher to determine the growth points children reached in Counting, Place Value, Addition and Subtraction, and Multiplication and Division. To increase the validity and reliability of the data, teachers followed a detailed interview script, recorded answers and strategies on a detailed record sheet, and used clearly defined rules for assigning growth points. Children's growth points were entered into an excel spreadsheet and each school's data was aggregated to form the data set reported on here. The region's Numeracy Advisors and School Co-ordinators managed this process.

Children's Number Knowledge

The Place Value growth points associated with the *Early Numeracy Interview* describe children's knowledge of reading, writing, ordering, and interpreting numbers for one-digit to four-digit numbers and beyond. The assessment tasks provide insight about concepts of quantity, number partitioning, use of a mental number line, and application of place value conventions for reading and writing numerals. Figure 1 describes the highest Place Value growth points reached by Prep to Grade 3 children in the Ballarat Diocese in 2006.

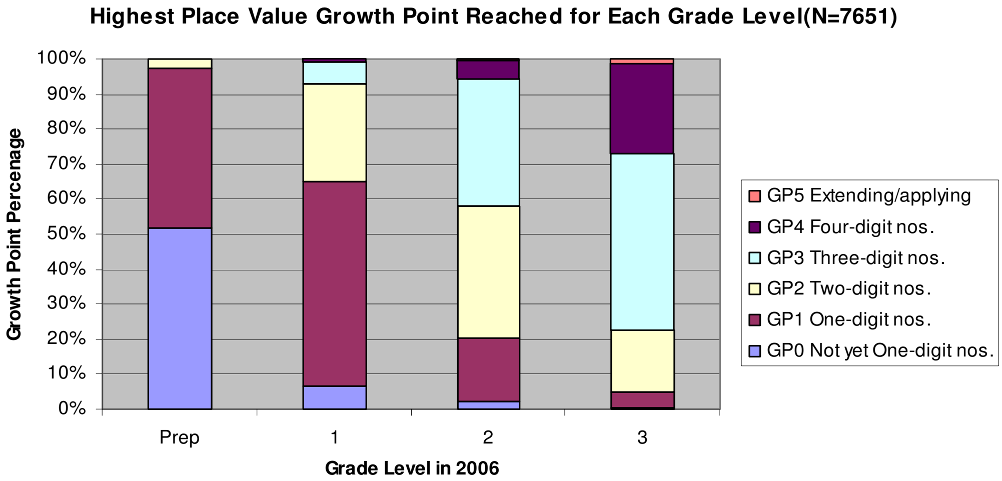


Figure 1. Percentage of Prep to Grade 3 children in 2006 reaching each of the place value growth points.

The data indicate that children's knowledge develops significantly during the first 3 years of school. Further, the complexity of the teaching process is highlighted by the spread of growth points within each grade. This spread of knowledge within one grade level has been noted in many previous students (e.g., Bobis et al., 2005).

Examination of the Prep data suggests that the children beginning school formed two distinct groups: those who knew how to read, write, and order all 1-digit numbers and who therefore required opportunities to explore 2-digit numbers, and those who did not. It is important to note that some children beginning school could already read, write, order, and interpret 2-digit numbers and thus required opportunities to explore at least 3-digit numbers. Similar to the findings of Wright (1992) this challenges a curriculum that typically focuses on numbers ranging from 1-20 when children begin school.

The data also suggest that for many children beginning Grade 1, a key issue was learning to interpret 2-digit and 3-digit numbers, although some needed opportunities to extend their number knowledge to at least 4-digit numbers. The issue for most children beginning Grade 2 was exploring 3-digit and 4-digit numbers. However, 20 percent of children beginning Grade 2 were not yet able to interpret 2-digit numbers, and it can be argued that these children would benefit from assistance to accelerate their learning. By the beginning of Grade 3, children's knowledge was spread from GP1-GP5. The extent of this range is highlighted by the fact that one-quarter of students were able to interpret 4-digit numbers, whereas another quarter were still learning to interpret 2-digit and 3-digit numbers.

The growth points reached by children in the Place Value domain provide an indication for teachers of the range of numbers that children may be expected to use for calculations and problem solving. The relevance of this becomes apparent when children are introduced to conventional written algorithms for calculations involving 2- and 3-digit numbers. Given that one quarter of Grade 3 children in this study were still learning to interpret 2- and 3-digit numbers, these children may be unlikely to understand the place value concepts underpinning formal algorithms, and this situation may impede their development of powerful mental reasoning strategies for calculating (Narode, Board, & Davenport, 1993). Indeed, data compiled for the Addition and Subtraction and Multiplication and Division Domains, but not shown here due to space constraints, show that 39% of Grade 3 children still used counting-based strategies for addition and subtraction calculations, and 47% needed to use models to solve multiplication and division problems. These children are unlikely to understand the abstract ideas associated with conventional algorithms and may focus only on the procedural knowledge associated with conventional algorithms.

Number Knowledge of Children Beginning School

When children first begin school, the data presented in Figure 1 show that that their ability to read, write, order, and interpret numbers varies considerably. It is useful to know if this finding extends to the other number domains also. For this purpose, Table 1 shows the percentage of Prep children who reached each Growth Point in the domains of Counting, Addition and Subtraction Strategies, and Multiplication and Division Strategies.

Table 1
Percentage of Prep Children in February 2006 Who Reached Each of the Counting, Addition and Subtraction and Multiplication and Division Growth Points

Counting Growth Points	Percent n=1015	Addition & Subtraction Strategies Growth Points	Percent n=925	Multiplication & Division Strategies Growth Points	Percent n=923
0. Knows some number names & sequences	46	0. Not Yet	59	0. Not Yet	67
1. Rote Counting (to at least 20)	19	1. Count all	34	1. Count all	28
2. Collections (at least 20 items)	32	2. Count on	6	2. Modelling when groups are perceived	5
3. Forward/backward (to at least 110)	2	3. Count down to	1	3. Modelling when groups not perceived	0
4. Skip counting (by 2s, 5s, 10s)	1	4. Basic strategies	0	4. Multiplication strategies	0
5. Skip counting from x (by 2s, 5s, 10s)	0	5. Derived Strategies	0	5. Division Strategies	0

The major issue to emerge from these data is the spread of growth points in every domain, right from the time children begin school, a finding noted in previous studies (e.g., Bobis et al., 2005). In Counting, just over half of the group knew the number word sequence to 20, and many of these children could count a collection of 20 items. The remaining children were still becoming familiar with number names and sequences to 20. However, some children counted beyond 110, and others skip counted by 10s, 5s, and 2s.

In Addition and Substraction, 7% of the children used the *count-on* strategy when asked to find the total of two collections (with nine items screened and another four items unscreened). In contrast, 34% used the *count-all* strategy, whereas the remaining children, on this occasion, were not able to solve the problem.

In the Multiplication and Division Domain, the data show that one-third of children beginning school solved the initial task that involved finding the total of 4 groups of two items. The others were not successful, and it is likely that the greatest factor in the task's difficulty was being able to interpret the demands of the task. In contrast, 5% of children on GP2 required learning opportunities focusing on developing mental images associated with groups and arrays in order to prompt the use of abstract multiplicative strategies.

In summary, the range of number knowledge in all domains for children beginning school is striking and highlights the importance of teachers identifying children's current knowledge and customising learning experiences to meet the individual learning needs.

Discussion and Conclusion

The data presented in this paper confirm the finding of previous studies that highlight the extent and diversity of children's number knowledge when they begin school and throughout the first three years of schooling. Teachers need to respond to this situation with ongoing monitoring and assessment to identify children's current number knowledge and customise learning experiences that cater for the range of learning needs.

Clearly, some children lag behind or stride ahead of their peers. These children may not always receive the opportunities needed to extend their knowledge further. For example, the Victorian Prep curriculum focuses on numbers ranging from 1-20, but many students have this knowledge when they first arrive at school. This highlights the fact that curriculum guidelines do not always match the learning needs of children and need to be refined by teachers if all children are to have the opportunity to thrive mathematically.

References

Bobis, J., Clarke, B., Clarke, D., Thomas, G., Wright, R., Young-Loveridge, J. & Gould, P. (2005). Supporting teachers in the development of young children's mathematical thinking: Three large scale cases. *Mathematics Education Research Journal*, *16*(3), 27-57.

Clarke, B. A., Sullivan, P., & McDonough, A. (2002). Measuring and describing learning: The Early Numeracy Research Project. In A. Cockburn & E. Nardi (Eds.), *Proceedings of the 26th annual conference of the International Group for the Psychology of Mathematics Education,* (Vol. 1, pp. 181-185). Norwich, UK: PME.

Clarke, D. (2001). Understanding, assessing and developing young children's mathematical thinking: Research as powerful tool for professional growth. In J. Bobis, B. Perry, & M. Mitchelmore (Eds.), *Numeracy and beyond.* (Proceedings of the 24th Annual Conference of the Mathematics Education Research Group of Australasia, Vol. 1, pp. 9-26). Sydney: MERGA.

Clarke, D., Cheeseman, J., Gervasoni, A., Gronn, D., Horne, M., McDonough, A., Montgomery, P., Roche, A., Sullivan, P., Clarke, B., & Rowley, G. (2002). *Early Numeracy Research Project Final Report*. Melbourne: Australian Catholic University.

Cobb, P., & McClain, K. (1999). *Supporting teachers' learning in social and institutional contexts*. Paper presented at the International Conference on Mathematics Teacher Education, Taipei, Taiwan.

Department of Education Employment and Training. (2001). *Early numeracy interview booklet*. Melbourne: Author.

Gould, P. (2000). Count Me In Too: Creating a choir in the swamp. In *Improving numerary learning: What does the research tell us? (Proceedings of the ACER Research Conference 2000, pp. 23-26).* Melbourne: Australian Council for Educational Research.

Higgins, J., Parsons, R., & Hyland, M. (2003). The Numeracy Development Project: Policy to practice. In J. Livingstone (Ed.), *New Zealand annual review of education*. (pp. 157-174). Wellington: Victoria University of Wellington.

Narode, R., Board, J., & Davenport, L. (1993). Algorithms supplant understanding: Case studies of primary students' strategies for double-digit addition and subtraction. In J. R. Becker & B. J. Preece (Eds.), *Proceedings of the 15th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 254-260). San Jose, CA: San Jose State University.

Wright, R. (1992). Number topics in early childhood mathematics curricula: Historical background, dilemmas, and possible solutions. *Australian Journal of Education*, *36*(2), 125-142.

Listening to Students' Voices in Mathematics Education

Brian Doig
Deakin University
<brian.doig@deakin.edu.au>

Susie Groves
Deakin University
<susie.groves@deakin.edu.au>

Coral Campbell
Deakin University
<coral.campbell@deakin.edu.au>

Judith Mousley
Deakin University
<judith.mousley@deakin.edu.au>

Gaye Williams
Deakin University
<gaye.williams@deakin.edu.au>

> The essence of the demand for freedom is the need of conditions which will enable an individual to make his own special contribution to a group interest, and to partake of its activities in such ways that social guidance shall be a matter of his own mental attitude, and not a mere authoritative dictation of his acts. (John Dewey)

In many tertiary institutions, mathematics education staff teach courses from early childhood education through to professional development courses at Masters level. Similarly, research into teacher education processes spans these contexts. Common principles that underpin this work include staff willingness to be responsive to students' needs. This symposium focuses on the importance of listening to students' voices in mathematics teaching and research – no matter how old students are.

The "voices" (Belenky, Clinchy, Goldberger, & Tarule, 1986, p. 7) in the 3 following papers are those of primary, secondary, and tertiary students. "Listening", here, includes a range of research activities, including interviews, stimulated recall discussions, and written surveys. Doig and Groves interviewed a group of Year 5 and 6 students, seeking their thoughts on effective teaching practices in mathematics. Williams' interview with a Year 8 student threw light on the reasons for what seemed to be low-level engagement as well as barriers that he overcame. The students surveyed by Mousley and Campbell provided opinions about a new form of assessment. In each case, the student voices make available information about teaching and learning that would otherwise be inaccessible to mathematics educators.

The symposium as a whole draws on the fact that students have more experience of teaching practices than any other group of people. Listening to their voices is educative. Robinson and Taylor (2007) describe four values related to the notion of student voice: a conception of communication as dialogue, potential for participation and democratic inclusivity, recognition that power relations are unequal and problematic, and possibilities for transformation. In the following set of three papers, one sees evidence of how each of these components plays out in research that aims to support educational change by capturing students' ideas in primary, secondary, and tertiary mathematics education contexts.

Reference

Belenky, M., Clinchy, B., Goldberger, N., & Tarule, J. (1986). *Women's ways of knowing: The development of self, voice and mind.* New York: Basic Books.

Robinson, C., & Taylor, C (2007). Theorising student voice: Values and perspectives. *Improving Schools, 10*(1), 5-17.

Students' Pedagogical Knowledge: A Source of Pedagogical Content Knowledge

Brian Doig
Deakin University
<badoig@deakin.edu.au>

Susie Groves
Deakin University
<susie.groves@deakin.edu.au>

This paper presents the results of interviews with Year 5 and 6 students about their views of effective teaching practices in mathematics. The students interviewed were part of a large-scale study into improving middle years mathematics and science. Their views confirm findings from the literature and other data sources from the project, and provide valuable insights into student perceptions of effective teaching practice in middle years mathematics.

> In our efforts at "school improvement" we need to tune into what pupils can tell us about their experiences and what they think will make a difference to their commitment to learning, and, in turn, to their progress and achievement. (Rudduck & Flutter, 2000, p. 75)

A fact we often forget is that students have more experience of teaching practices than any other group. In other words, their *fund of knowledge* (Moll & Greenberg, 1990) of teaching practice is extensive. However, whether they are able to articulate their knowledge to assist teachers make teaching practices more effective is an open question.

This present research was stimulated, in part, by van den Heuvel-Panhuizen (2005). In the research reported at the *International Symposium Elementary Maths Teaching* (SEMT), van den Heuvel-Panhuizen interviewed two very articulate students. These girls had a clear idea of what they saw as good teaching practice, and what was not. This appeared to come from the professional background of their parents; they knew and used educational jargon. Examples of their insights included those related to explaining: that teachers should use visuals to aid their explanations, and that the "why" should be explained as well as the "how".

This paper presents the results of interviews with four Year 5 and 6 students about their views of effective teaching practices in mathematics.

Background

The student interviews were conducted as part of the *Improving Middle Years Mathematics and Science: The role of subject cultures in school and teacher change* (IMYMS) project, which investigated the role of subject knowledge and cultures in mediating change processes in the middle years of schooling. The project worked with 5 secondary and 27 primary schools located in urban, regional, and rural areas of Victoria. The project had its roots in the *Science in Schools* research project, which developed a successful strategy for improving teaching and learning science (Gough & Tytler, 2001).

IMYMS is based on an action planning process that involves auditing the practice of mathematics and science in schools. The major foci of the audit are teacher practice and beliefs, and student perceptions and learning preferences (e.g., Doig, Groves, Tytler, & Gough, 2005). Students also took part in written and performance assessments.

Methodology

At the end of the 2005 school year, four students who were part of the IMYMS project sample at one urban primary school were selected for a group interview. As Osborne and Collins (2001, p. 443) point out, (focus) group interviews offer "a means of exploring the principal issues of interest in a dynamic manner which utilizes the group interaction to challenge, and probe, the views and positions espoused by individual members in a non-threatening, relatively neutralized social setting". As in van den Heuvel-Panhuizen's (2005) *student consultancy study*, students were selected on the basis of their likelihood to be able to give informed advice on effective teaching. In this case, the students were chosen based on their results on the IMYMS written mathematics assessment, and their teachers' assessment that these four students were very capable in mathematics and likely to be articulate in an interview situation.

Two boys, Ian (Year 5) and Nick (Year 6), and two girls, Ursula and Eve (Year 6), were interviewed by the authors. A semi-structured interview protocol was used, with most subsidiary questions following up students' responses to the main questions. As with the IMYMS survey, the emphasis was on how students believe they best learn mathematics and hence how it should be taught. Students were told at the start of the interview that they were regarded "as consultants about good ways of teaching and learning and your thoughts about teaching and learning maths and science". The five "main questions" were:

1 The first question for us is just your thoughts on – well let's say do you enjoy maths … why do you like it?
2 So if you were going to be the maths teacher for next year, for Year 6 somewhere, how would you make things different? What would you do?
3 How do you think you actually learn maths best? What's the best way of learning maths?
4 What about the kids who are not doing so well or find it harder? … What's the best thing for the teacher to do to help those kids?
5 Any other suggestions for us? Things you would recommend, like to see, or think would help?

The interview was audio-taped and transcribed. The authors examined the transcript of the interview and identified phrases in the students' comments that were qualitatively and substantively different from one another. Phrases with like focus were then placed into categories that then formed the basis for this paper.

Results and Discussion

Only data relating to mathematics are reported here, with the three major themes emerging for mathematics being discussed below.

Challenging but Accessible Content

Due to the way students were chosen for the interview, it was not surprising that they all liked mathematics. However, a frequent complaint was that the way it was taught was boring. For example, Nick, the most articulate of the four students, commented:

Nick: Well I like maths just full stop, but I don't really like the way we do it. … [it's] boring, that's the word. … [they make] us do really easy things over and over and over again.

While the students were very aware that they were not typical in their mathematics classes and were "probably the wrong people to ask" about students who are weak at maths, they demonstrated a great deal of insight into the problems teachers face:

Nick: The teacher's trying to teach the whole class … It's a bit hard 'cos there's a massive range of abilities in maths and … people who understand it as soon as they saw it would get really bored.

When Australian teachers are faced with a wide range of mathematical abilities, their most likely strategy is to use groups – in most cases heterogeneous groups. However, all the students advocated the use of ability grouping, although they had some reservations:

Nick: The problem [is] the people who aren't so good at maths … don't like being put in a group by themselves … Even if they probably learn better that way, they're not happy doing it that way.

Eve: Yeah so it's hard to make them feel good.

Ursula: But some people like just accept the fact that they're not so good at maths and they want to learn more. Different people's personalities.

The strength of feeling was evident by the fact that when asked at the end of the interview what advice they would give to trainee teachers, three of the students responded:

Ursula: Put your students into groups. Different working groups.

Eve: And make sure that the people who are better at maths don't get put [into] like easier maths. They sort of need to be challenged more.

Ursula: And yeah other way round. The people who aren't so good at maths don't get really hard work that they can't even do or understand because that doesn't do anything.

Eve: We don't really get challenged. …

Ian: Instead of easy maths all the time for the people who aren't so good [need] different levels.

Teaching Strategies to Support Learning

Mirroring the IMYMS student survey, the students were asked how they best learn mathematics. There were divergent views on the role of teacher explanation:

Eve: I like the teachers sort of explaining how you do it and then just doing it. I think the worksheet helps you and then she'll come around and if you don't understand something she can tell you.

Ursula: I don't really like it when they explain 'cos they go on for hours and hours and you kind of lose concentration.

Eve: Well she just sort of explains it on the floor and then we go to our desks. But the people that still don't understand it, they go back on the mat to do it again. So maybe get them back on the mat and explain it more than what she's explaining.

However, except for the social aspects, the use of worksheets was, by and large, condemned by the students:

Nick: They make photocopies of text books.

Ursula: Yeah, and they give us a sheet and we stick it in the maths book. … [but mainly] she'll put up lots of sums on the board ... which is all right because I work with my friends … [that's] fun 'cos we sort of talk and do it as well instead of just sitting at your desk and not talking.

Ian: Yeah, we get lots of work sheets every time we have maths, yeah.

Nick's single piece of advice to trainee teachers was:

Nick: Don’t take the easy way out by just giving us just tons of worksheets!

Students also advocated teachers probing student understanding to avoid repetition:

Nick: I’d probably find out who needs to like revise something ’cos if the whole class knows it there’s no point going over it. But if only one or two people need to revise it, you could just work with those one or two people while the others can do something. So instead of going over it for the whole class, just go over it for a few people … Wouldn’t it be better to give them like a one page test at the start of the year … just to see where you’re at?

Linking Mathematics to Students’ Interests

While condemning worksheets, students were in favour of more “hands-on” work:

Nick: It would be better if we could do more hands-on things. Actually instead of just getting a worksheet and have to do sums … it would be better if we could actually like – this is just an example to teach you how to multiple and minus and stuff – pretend to run your own shop or something and people would come and buy things, just pretend. …

Ursula: Going outside measuring the basketball court. Like finding the area, perimeter, and so on.

The three Year 6 students, who were in extension classes for a while, particularly appreciated the project work they did:

Ursula: That was like going around the school plotting things in a project.

Nick: Well some of it was hands-on and the rest of it was boring like normal maths. … Oh yeah, the projects were good.

Ursula: Like water usage and we made a ramp, did actions for a ramp.

Nick: And you were allowed to do … choose what you wanted … and how much solar panels cost and things like that. … Actually doing things. Like you’re never just going … like once you go from high school, if you go to uni and stuff, you’re never just going to get a sheet, you won’t probably just get a sheet of sums for no reason whatsoever. It would be better if we actually used them in context of what we’re actually going to use them for.

Conclusion

The high-achieving students interviewed felt frustrated by the repetition of mathematical content they felt that they had already mastered. They did not believe that the mathematics they were doing provided them with sufficient challenge, nor that lower-achieving student were being well served by the strategies their teachers were employing. Moreover, students maintained that applying mathematics to real situations was better than completing many, similar, context-free, arithmetic exercises. This, of course, resonates with the literature on effective mathematics teaching (e.g., Doig, 2005; Department of Education, Science and Training, et al., 2004).

These results also confirm those from the IMYMS student surveys, where the four items (of the 24) that most primary students found “very helpful” for learning mathematics were: “Being able to choose how I present things”; “Doing hands-on activities”; “Doing investigations or projects of my own choice”; and “Doing activities that challenge me to think”. “Doing worksheets” only rated fifteenth. The results align with the IMYMS *Components of Effective Teaching and Learning* that focus on conceptual challenge, supporting meaningful understanding, and linking with students' lives and interests (Doig, et al., 2005).

The reactions of pupils to what occurs in the classroom has been identified, by teachers themselves, as one of the most important determinants of their practice in several studies, with the influence that pupils exert on teachers being seen by Bishop and Nickson (1983) as stemming from "the part they play in the social arena of the classrooms" (p. 15). Although the results from the interviews may be seen as only confirming other data, these students' comments provide richer insights for teachers and researchers than would otherwise be available. As in McIntyre, Pedder, and Rudduck's (2005) study, students provided constructive advice on what helps their own and other students' learning. However, unlike that study, the timing of our interview meant that there was no opportunity for teachers to act on these views. Although teachers in the IMYMS project were provided with summaries of the student responses to the written surveys, based on our experience with the interview we would recommend that projects focusing on improving classroom practice consider the incorporation of student interviews at an early stage.

Acknowledgments. IMYMS was funded by an Australian Research Council Linkage Grant, with Industry Partner the Victorian Department of Education and Training. The Chief Investigators were Russell Tytler, Susie Groves, and Annette Gough.

References

Bishop, A., & Nickson, M. (1983). *Review of research in mathematics education*. Windsor, Berks: The NFER-Nelson Publishing Co. Ltd.

Department of Education, Science and Training; New South Wales Department of Education and Training; Catholic Education Commission, New South Wales; The Association of Independent Schools of New South Wales Ltd. (2004). *What's making the difference? Achieving outstanding numeracy outcomes in NSW primary schools – Main report.* Retrieved 29 March 2007 from: http://www.dest.gov.au/sectors/school_education/publications_resources/profiles/making_the_difference_main_report.htm

Doig, B. (2005). *Research pointers to practice: A review of research to inform middle years mathematics teaching and learning*. Adelaide: Australian Association of Mathematics Teachers.

Doig, B., Groves, S., Tytler, R., & Gough, A. (2005). Primary and secondary mathematics practice: How different is it? In P. Clarkson, A. Downton, D. Gronn, M. Horne, A. McDonough, R. Pierce & A. Roche (Eds.), *Building connections: Theory, research and practice* (Proceedings of the 28th annual conference of the Mathematics Education Research Group of Australasia, Vol. 1, pp. 305-312). Sydney: MERGA.

Gough, A., & Tytler, R. (2001). Researching effective teaching and learning in science: Victoria's Science in Schools research project, *Proceedings of the 2001 annual conference of the Australian Association for Research in Education, Fremantle.* Retrieved 30 March 2005 from the World Wide Web: http://www.aare.edu.au/01pap/gou01515.htm

McIntyre, D., Pedder, D., & Rudduck, J. (2005). Pupil voice: Comfortable and uncomfortable learnings for teachers. *Research Papers in Education, 20*(2), 149-168.

Moll, L. C., & Greenberg, J. B. (1990). Creating zones of possibilities: Combining social contexts for instruction. In L. Moll (Ed.), *Vygotsky and education: Instructional implications and applications of sociohistorical psychology* (pp. 319-348). Cambridge: Cambridge University Press.

Osborne, J. & Collins, S. (2001). Pupils' views of the role and value of the science curriculum: A focus-group study. *International Journal of Science Education, 23*(5), 441-467.

Rudduck, J. & Flutter, J. (2000). Pupil participation and pupil perspective: "Carving a new order of experience". *Cambridge Journal of Education, 30*(1), 75-89.

van den Heuvel-Panhuizen, M. (2005). Children's perspectives of the mathematics classroom. In J. Novotna (Ed.), *Proceedings of the International Symposium Elementary Maths Teaching* (pp. 23-33). Prague: Charles University, Education Faculty.

Research Enriched by the Student Voice

Gaye Williams
Deakin University
<gaye.williams@deakin.edu.au>

This paper illustrates the enriched understanding of classroom activity that can occur when the student voice is included in the research design and valid data is generated. It presents interpretations of classroom activity based solely on lesson video, and the same activity interpreted though video stimulated post lesson student interviews. It draws attention to the need to synthesize data from complementary sources.

This paper discusses data collected through student interviews, and identifies the problematic nature of research designs that do not include the student voice. A Year 8 student's observed classroom behaviour as visible on video was initially examined. This was compared with interpretations based on student reconstruction of lesson activity in post lesson video-stimulated interviews. Differences in interpretations are considered.

Literature Review

Clarke's (Clarke, Keitel, & Shimizu, 2006) use of video-stimulated recall enabled students to view their classroom activity and to become involved in listening to, discussing, and writing about their mathematical experiences. Clarke (2001) had previously found that observational data alone had led to misinterpretation of the activity of a particular student. The teacher's characterisation of this student as inattentive was consistent with the video record and the student herself, on viewing the videotape, commented that it did not look as though she was paying attention. During her post-lesson video-stimulated interview, this student provided convincing evidence that she had been engaged with the lesson content. Clarke has found that considering the data from all three sources (lesson video, teacher interview, and student interview) reduced the potential for such misinterpretation. This example illustrated that student behaviour needs to be well grounded in the individual's documented statements or actions, and, where possible, corroborated by other data sources such as post-lesson interviews (Clarke, Keitel, & Shimizu, 2006). This paper demonstrates that valuable data can be lost without attention to the student voice, and illustrates inconsistencies between complementary interpretations and synthesis of these interpretations.

Limitations to using students interviews as a data source have been identified (e.g., Krutetskii, 1976; Fine & Sandstrom, 1988; Barnes, Williams, & Clarke, 2001) and strategies to overcome these limitations have been developed (e.g., Ericsson & Simons, 1980; Fine & Sandstrom, 1988; Williams, 2005). Krutetskii found students did not always share unproductive pathways and Ericsson & Simons suggest the use of salient stimuli (like video) in post-task interviews to reduce this occurrence. Where a subject can spontaneously "describe one or more specific sub-goals, … [that] were both relevant to the problem and consistent with other evidence of the solution process, …" (Ericsson & Simons, 1980, p. 217) the validity of the student reconstruction increases. On the other hand, if the interviewer asked questions that included constructs the subject had not identified in the interview, the subject could "generate answers without consulting memory traces" (Ericsson & Simons, 1980, p. 217). Descriptions of grounded theory approaches to interviews illustrate how to generate data by letting the subject focus the content and select the language, and the researcher probe to identify the meaning the subject intended for the language used (e.g., Bowers, 1989).

In addition to the ethical issues associated with student unease during interviews (Fine and Sandstrom, 1988; Barnes, Williams, & Clarke, 2001) the validity of data can also be affected by such unease if the student responds, as they perceive the interviewer expects rather than through reconstructing their experiences.

> We believe that ethical behaviour when interviewing children includes a requirement for the researcher to help child informants to feel comfortable and at ease, and as far as possible to avoid placing them under stress. To achieve this, every effort must be made to minimise the negative effects of the inevitable power imbalance. … [this also] helps to ensure that the resulting data are meaningful and valid. (Barnes, Williams, & Clarke, 2001)

The interviews undertaken in this study took into account limitations identified herein. The intention of this paper is to draw attention to the problematic nature of interpreting student activity from video observation alone, and draw attention to how valid data can be collected through the student voice when complementary accounts are synthesised.

Research Design and Mathematical Setting

The Year 8 lesson from which the data were drawn was the twelfth lesson in a sequence of 14 lessons in one set of the Australian data within the broader *Learners' Perspective Study,* designed to explore the teaching and learning of mathematics as viewed from the perspective of the learner. The methodology included videotaping a sequence of lessons, post-lesson video-stimulated student and teacher interviews, collection of student work and lesson tasks, and teacher questionnaires (Clarke, Keitel, & Shimizu, 2006). The primary data were collected by three video cameras that operated simultaneously in the classroom to display the actions of (a) the whole class; (b) the teacher; and (c) a pair of focus students. Following the lesson, the focus students took part in individual audio-taped interviews stimulated by a mixed image of the video of themselves (large image) and the teacher (small insert). During the interview after Lesson 12, the student (Leon) controlled the video remote and fast-forwarded to the parts of the lesson that were important to him and talked about what was happening, what he was thinking, and what he was feeling. By asking Leon to find and discuss what was important to him, he had opportunity to discuss the lesson without the interviewer imposing language and ideas. The following type of statement at the start of the interview was intended to put students at ease:

> I [interviewer] just wanted to say is there is nothing right or wrong about anything you say 'cause what we are interested in is how the maths classroom looks to you when you're in it and the sort of thinking that you are doing in the class. We are not really concerned about whether the things that you are saying about the maths are exactly right or- or [pause] wrong or [pause] in the middle or whatever, … [the interviewer then explained the purpose of the research, and added we wanted to know what the student was thinking and feeling] and you know and we don't.

Like other students, Leon smiled, and his body language indicated he was more relaxed. By undertaking non-judgemental interviews, some ethical considerations raised by Fine and Sandstrom (1988) and Barnes, Williams, and Clarke (2001) were addressed. In addition, attention was paid to the 'equality of interaction' (Alro & Skosmove, 2004) to reduce power differentials and increase the richness of the data. This type of interaction, identified in classroom interactions, was adapted to the interview process by Williams (2005).

> A dialogue maintains equality including a respect for diversity. This does not mean that a dialogue presupposes similarity or symmetry. We are speaking of interpersonal equality and human respect. In dialogue there should be no use of power or force, no persuasion of the other and no winning. ... to be productive, a dialogue develops as a dynamic process between equal communicating partners. … Even when the teacher is a more knowing or competent party to the dialogue, classroom conversations can be dialogic. The roles can be different and so can the competencies. (Alro & Skovsmose, 2004, p. 41)

Mutual respect was built by: reinforcing the value of student responses to the research team, the absence of coercion to respond in certain ways, and the demonstrated fallibility of the interviewer as a *person*: "Now I wonder have I remembered to turn on the tape?" and "Yes- I sometimes make that mistake too" – [pointing remote control at monitor not video recorder]. Leon's interview resulted in a collaborative interaction. As interviewer, I was unable to decipher a comment made by Leon early in the interview as we watched the video. I commented as a matter-of-fact "out loud" reflection: "well sometime I am going to have to transcribe that and what you were saying there so eventually I'll know, I'll go back over it a hundred times [laugh] till I find it [laugh]". This changed the interview dynamics. Leon responded: "I can rewin- can I rewind it? … I can tell you what I was saying if I rewind it". My casual sharing of the work involved in deciphering almost inaudible statements led to Leon volunteering assistance and continuing to do so throughout the interview. He even deciphered the talk of students laughing at a question he had asked.

Lesson Context

Students worked in pairs to find the area of one triangle (see Figure 1). Leon and his partner Pepe worked with Triangle 1. The measurements of side lengths of each triangle were written on the diagrams on the board. Pepe began trying to represent Triangle 1 on the A3 sheet provided. He did not know how to construct triangles when three sides are given and did not ask anyone for assistance.

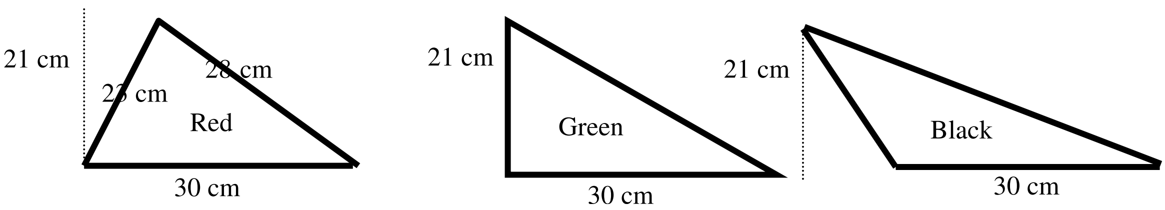


Figure 1. The three triangles on the board in Lesson 12

Pepe remained focused on finding his own way to construct this triangle over the next five and a half minutes. He no longer engaged in off-task activity, he requested various implements from students seated around him, and leant over his page using pencil, compass, and ruler. When Leon began to ask Pepe questions about what he was doing, he did not explain. If he responded to Leon's queries at all, these responses were generally short and/or abrupt. Pepe demanded that Leon watch and work it out himself, or explained the problem but not how it was overcome. On occasions, Pepe enlisted Leon's assistance. For example, when he found the compass span insufficient for the length required, he directed Leon to hold the end of the ruler as a pivot so the ruler could act as a compass.

Video Record of Leon's Practice in the Classroom

From the video record of the lesson, the following observations about Leon's activity were considered indicators of his inattentiveness (Williams & Clarke, 2002):

- Whispered prompt from Pepe, prompting an answer about homework (~ 4 min)
- Whispered prompt from Earl, prompting answer to a teacher question (~ 12 min)
- Question to Elena regarding the triangle the pair should be working on (~24 min)
- Question to Pepe regarding which triangle they were working on (~32 min)
- Pepe's attempts to get Leon on task, which included slapping his face (~35 min)
- Question to Pepe, prompted frustrated "If you've been listening, Leon" (~37 min)

Observations of Leon's activity in class (without access to the interview data) suggested a student whose level of engagement with the mathematics fluctuated frequently during the course of the lesson and a student not inclined to precise work. For example, when Pepe was attempting to construct Triangle 1, Leon stated, "Go twenty-one centimetres straight up, go in a little bit." Pepe demanded to be left to do it his own way. Pepe appeared committed to a careful construction, and was annoyed with Leon's lack of care. Pepe's insistent repeated comment to Leon "Leon. Make sure that, make sure that twenty-eight always stays on … look it idiot (hits Leon's face)" appeared to indicate Leon's lack of care in holding the ruler as a pivot while Pepe made an arc.

The Post-lesson Video-stimulated Reconstruction of Leon's Classroom Practice

These interpretations arise from Leon's video-stimulated post-lesson interview. Table 1 includes excerpts of lesson transcript and Leon's interview reconstruction.

Table 1
Leon's Reconstructions Lesson Activity

Lesson Transcript	Interview Reconstruction
Leon [to Pepe]: What are you doing?	Leon: He's got the compass and I didn't know what he was doing [laugh in voice] with the compass because he was supposed to be ruling straight lines.
Pepe: Ah, watch yourself. [Pepe stretched the compass along the ruler and Leon watched. Pepe kept his eyes on the equipment and the page]	Leon: Sometimes Pepe is a better teacher than the teacher.
Pepe [to self]: It doesn't make like twenty-eight centimetres [returns ruler, hesitates, extends hand]. Wait wait wait wait wait, I still need it.	
Leon: What the hell are you doing Pepe? Are we actually going to do any work or not? [Pepe has drawn a line and started using the ruler for something else.]	He was doing the circle thing and I didn't know why he was doing the circle thing … that sort of threw me off course.

Leon's interview (see Table 1) showed he was unaware of the process for constructing triangles and did not recognise the compass' possible purpose when he saw Pepe using it. After the interaction in Table 1, Leon continued to challenge what Pepe was doing. Pepe became frustrated and used expletives and a sharp, voice to clarify the problem:

> We don't know how … bloody … ooh, twenty eight centimetres is. We don't know where the fucking measurement is [slapped hands down on the page indicating both ends of the line]. Think about it. [Leon looked at the triangle on the board for a short period of time remaining motionless].

Leon then suggested his approximate way to replicate the triangle. His interview comments (Table 1) show he was not aware of Pepe's more precise way. Pepe responded:

"Let me do it my way" and Leon returned to teasing the girls beside him. When Pepe hit Leon, it was to regain his attention. Pepe's intense repeating of how Leon needed to hold the ruler occurred when he was trying to regain Leon's attention not because Leon was holding the ruler inappropriately. As Leon held the pivot, his new realisation about Pepe's activity occurred:

> and he's [Pepe] gone all the way around it [compass arc] "What are you doing that for?" and he's gone "Just watch" and I've gone "Oh, so you can get the angle that is sloping down?" and Pepe has gone "Yes exactly". That was where I understood it.

Analysis, Discussion and Conclusions

Pepe's frustrated outburst triggered a short burst of intense focus by Leon that resulted in his realising there was a problem. Leon's suggested way of overcoming this problem was initial thinking about a newly discovered complexity. His rudimentary method is consistent with him being unaware of other methods at that stage. Pepe's request to proceed alone led to Leon's subsequent inattention. Later, Pepe repeated his instructions about how to hold the pivot to regain Leon's attention rather than because Leon was not holding the pivot appropriately. Once Pepe had gained his attention, Leon leant over and concentrated intently on holding the pivot. This was when Leon reported realising the relationship between the position of the side of the triangle and the angle formed. This illustration emphasises the need to include the student voice in classroom research to add to the validity of interpretations. What on the surface appeared to be lack of care and an inattention to detail was at least partly due to a lack of understanding of the mathematics involved. Williams and Clarke (2002) provide another example of synthesis of complementary interpretations in other activity of these students. Including the student voice increases the need to develop strategies to synthesise complementary interpretations.

References

Alro, H., & Skovsmose, O. (2004). Dialogic learning in collaborative investigation. *Nordic Studies in Mathematics Education*, 9(2), 39-59.

Barnes, M., Williams, G., & Clarke, D. J. (2001, July). *Methodological and ethical Issues in Interviewing Schoolchildren.* Paper presented at the Australian Qualitative Research Conference, Critical Issues in Qualitative Research, Duxton Hotel, Melbourne.

Bowers, B. (1989). Grounded theory. *Paths to knowledge: Innovative research methods for Nursing*, *15*(Serial No. 2233), 33-57.

Clarke, D. J. (Ed.). (2001). *Perspectives on practice and meaning in mathematics and science classrooms*. Dordrecht, The Netherlands: Kluwer.

Clarke, D., Keitel, C., & Shimizu, Y. (Eds.). (2006). *Mathematics classrooms in twelve countries: The insider's perspective*. Rotterdam, The Netherlands: Sense Publications.

Ericsson, K., & Simons, H. (1980). Verbal reports of data. *Psychological Review*, *87*(3), 215–251.

Fine, G., & Sandstrom, K. (1988). *Knowing children: Participant observation with minors.* Newbury Park, CA: Sage.

Krutetskii, V. (1976). *Psychology of mathematical abilities in schoolchildren*. In J. Kilpatrick, & I. Wirzup (Eds.), J. Teller (Trans.). Chicago: University of Chicago Press. (Original work published in 1968).

Williams, G. (2005). *Improving intellectual and affective quality in mathematics lessons: How autonomy and spontaneity enable creative and insightful thinking.* Unpublished doctoral dissertation, University of Melbourne, Melbourne, Australia. Accessed at http://eprints.infodiv.unimelb.edu.au/archive/00002533/

Williams, G., & Clarke, D. J. (2002). The contribution of student voice in classroom research. In C. Malcom & C. Lubisi (Eds.), *Proceedings for the South African Association for Research in Mathematics, Science and Technology Education 2002* (pp. 398–404). Durban, South Africa: SAARMSTE.

Listening to Student Opinions about Group Assessment

Judith Mousley
Deakin University
<judith.mousley@deakin.edu.au>

Coral Campbell
Deakin University
<coral.campbell@deakin.edu.au>

This paper illustrates the way two teacher-researchers are listening to mathematics education students' voices in a Masters course. Group assignments have their advantages but it is difficult to ensure strong collaboration, high-level analysis and discussion, a good spread of work between group members, and positive social interactions. This research set out to explore one way of attending to these problems in a mathematics education Masters unit. Students submitted (unmarked) individual essays before combining them to create (graded) group assignments. They completed surveys about group work before and after this activity, and some were interviewed. Expecting individual work before group work led to increased levels of engagement, very high quality work, use of skills in analysis and critique, and good levels of student satisfaction.

Introduction

Assessment is an important part of any educational process. Accomplishment is not signalled only by the completion of a unit of work, but also by the extent to which students have engaged with the topic at hand. Further, generic skills including effective communication, information seeking, analysis, and critique are being valued increasingly. Therefore, assessment must fulfil a number of different purposes.

In teachers' professional development courses, the most common form of assessment is formative, using feedback that identifies potential for improvement (Falchikov, 2005). Formative tasks are also used as the basis for teaching and learning when assessment tasks emphasise the student's role in coming to understand content. It is also a common part of mathematics teacher and teacher education rhetoric that group learning situations provide educational advantages for students and develop learning that is a mix of knowledge acquisition and collaborative skills. Through participating in group learning, students (whether children or teachers) should develop improved communication and negotiation skills, and better understanding as well as critical thought and deeper learning through debate. Supposedly, students working in groups reflect on their learning and are more able to externalise their thought processes as well as helping others to understand the content.

However, research on group assessment has shown that many group tasks fail to deliver on these skills, in particular the opportunity for the development of active debate and critical reflection. Too often, both mathematics education and teacher education tasks allow for inequitable contributions and for dominance by one or two individuals to occur (Nightingale et al., 1996; Ramsden, 2003).

The research reported in this paper aimed to investigate and respond to students' voices about group work. We sought to find out what Masters mathematics education students saw as advantages and disadvantages of group assignments, and to change our practices accordingly. A new approach was developed, and again student voices were captured in the evaluation of this change.

Proceedings of the 30th annual conference of the Mathematics Education Research Group of Australasia
J. Watson & K. Beswick (Eds), © MERGA Inc. 2007

Methodology

Students enrolled in the wholly-online 2006 Masters unit entitled *Teaching Mathematics Successfully* were sent an initial survey on-line, together with ethics forms. The survey question requested their opinions of the advantages and disadvantages of group work in mathematics classrooms as well as group assignments in their own studies. (This paper focuses on the teachers' opinions about a change made in their group assignment.) The survey data were grouped into three categories related to group assignments: perceived advantages, perceived problems, and advice to lecturers. Each of these categories was divided into subgroups of types of advantage, problem, or advice.

After completing the initial survey, the students completed the same group assignment that had been set in 2005. However, given negative 2005 student feedback on the group assignment, and consequent discussion amongst the lecturers about this, it had been decided that the group assignment would be completed in two stages. Thus the students completed and submitted the assessment task individually before working together as a group of 3 or 4 to produce and submit a group assignment, drawing on the best features of their efforts. Only the group assignments were graded. The initial, individual assignments were filed as a record of students' efforts, but not marked.

We had been satisfied with the previous year's group assignment. It involved writing a literature review, analysis and critique of relevant, common practices, and action research in a mathematics classroom leading to the writing of a report. Student feedback on the assessment task itself, an inquiry into one of the "6 components of quality mathematics teaching" identified by Sullivan and Mousley (1996), had been very positive. In addition, while allowing for some student choice it was inquiry-based and had the potential to seek evidence of extensive research and reflection. It also had seemed to meet many characteristics of good higher education assessment practice. For example, our review of research by Biggs (2003), James, McKinnis, and Devlin (2002), Nightingale et al. (1996), Ramsden (2003), and Nulty and Kift (2003) suggested that effective assessment:

- is closely aligned with course content and expected outcomes;
- is valid, reliable, and ethical in nature and free of cultural bias;
- requires completion of authentic activities with emphasis on promoting learning;
- focuses on eliciting student understandings and demonstration of higher order skills;
- provides constructive, diagnostic feedback;
- utilises a variety of methods across assessment tasks;
- allows for some student choice and caters for different learning styles; and
- is cognisant of staff and student workloads.

However, the feedback indicated that the 2005 students (practising mathematics teachers in primary and secondary schools) did not like the fact that it was a group assignment. It seemed ironic that mathematics teachers who generally praise and use group work with their own students objected to group assignments, and this contradiction stimulated this research. Our primary aim was to find a way to listen to students' voices in a way that respected their views about group work but responded to their criticisms of it.

After submitting the group assignment then receiving their grades and feedback, the students who had responded to the first survey were sent a second one. Those who responded and indicated a willingness to be interviewed were later telephoned. Again they were asked for their opinions and advice about the use of group assignments.

Forty-four students responded to the first survey, 38 to the second, and 26 were interviewed by telephone. To meet ethics guidelines, the surveys and interviews were implemented by Coral Campbell, who did not teach the unit.

Results

Data from the initial survey showed that the teachers used group work in their classrooms and they described many reasons for this. Overall, the idea of group work in classrooms was received positively, with no respondent citing problems other than classroom management issues such as children's off-task chatter. The most common advantage cited was the development of mathematical understanding resulting from children sharing of ideas and procedures, peer explanations, and combinations of individual expertise (40 responses, 90%). The most common responses are shown in Table 1.

Table 1
Teachers' Views of Group Tasks in Mathematics Classrooms

Advantage/*Disadvantage*	**n=44**	**%**
Sharing of ideas, explanations, and expertise	40	90
Development of social skills	33	75
Peer support and tutoring, modelling	29	65
Children's off-task chatter	*28*	*63*

It is recognised that assessment of group products is different from merely working in groups in class, but the teachers' responses in the initial survey were quite different from the above when it came to their own assessment. Lesser numbers of students listed advantages, and they listed many disadvantages, as exemplified in Table 2, the primary one being "Others always share the marks I alone should have earned". Advantages still included sharing of knowledge where "the sum is more than the parts", and peer support was mentioned as "helpful in distance education".

Table 2
Teachers' Views of Group Assignments in Masters Courses (pre group assignment)

Advantage/*Disadvantage*	**n=44**	**%**
Sharing of knowledge	9	20
Peer support	5	11
"Free riders" sharing equal marks	*32*	*70*
Poor distribution of responsibilities (18, 41%)	*18*	*41*
Time management (getting others to complete their sections on time)	*12*	*27*

After completion and submission of the individual assignment, then working together to write and submit a group assignment that drew on the best elements of each, and after all students' work had been returned, the second survey was administered. The results were quite different. Students listed many more advantages for group work this time, especially in relation to the teacher education context (Table 3).

Table 3
Teachers' Views of Group Assignments in Masters Courses (post group assignment)

Advantage/*Disadvantage*	**n=38**	**%**
More ideas/insights	28	73
Each person bringing their own strengths	22	57
Wider range of resources	12	31
Sharing or reading/workload	5	13
Rich discussion / alternative views	5	13

Students contacted by telephone mainly talked about social and academic support available to them working on line as off-campus students (50%), learning about different ways of responding to the assessment criteria (42%), and learning academic skills such as structuring and presentation of essays (19%). It was clear that some students had been stretched by the group's analysis of their individual work.

> It takes you out of your comfort area. Having your ideas challenged, engaging with others' ideas.
> It forces you to look outside your line of thought.
> Primary, secondary, adult, and special ed. all talking about the same issue. It was SO enlightening.

Collis (1998) and Reeves (2000) write about the importance of collaborative group work to set the context for on-line students' support. There were a few complaints during the interviews about the logistics of on-line group work including multiple versions (15%), about "different ideas and ideologies" (1%), and the "need to agree on a common structure" and "melding of writing styles" (1%).

However, in both the interview and the students' later formal evaluations of the unit and its teaching, there were no complaints about the primary disadvantages cited before undertaking the new version of his assignment: uneven input or difficulties getting group members to communicate and contribute. On the contrary, it was surprising how many groups commented that they "must have been lucky being in a group where everyone contributed heaps". Two students "felt pressure to continue" despite family problems but received "much-needed support". One student in Hong Kong wrote, "I met people!" However, there were also hints of the usual group conflicts when a student mentioned that a member of her group had "dominated all maths discussions", and another felt that her "assignment contributions were not included well".

One student felt that 10 to 20% of the marks should be given to the individual drafts, but other students did not mention spontaneously the fact that it was not marked; and when asked, made comments like "It worked well that way and we knew you had it there if there were any arguments". When asked what advice they had for lecturers, students advised that less time was needed for individual essays and more for group improvements. Three friends suggested that "geography could be considered to allow face-to-face and telephone exchanges", echoing the advice of Herrington, Oliver, and Reeves (2003) who encouraged lecturers involved in distance to consider how students can arrange occasional face-to-face meetings.

In summary, it is clear that the students – teachers who generally valued group work in their own mathematics classroom – felt more positive about group assessment after undertaking the two-step process. While the "think, pair, share" process is commonly used in primary classrooms, it would be useful to try this process for group work and assessed tasks in secondary mathematics classrooms. We will continue to use it in teacher education.

Conclusion

Despite these suggestions for minor improvements, the idea of unassessed individual essays becoming the basis for an assessed group assignment was very well received. A few of the teachers commented that they were going to try this idea in mathematics classes; and indeed, this would be a worthy topic for further research.

When the research was introduced to students, they were told that the lecturers were interested in their ideas and perceptions of the assignment experiences. Many of them commented in the interviews about their interest in the research process and expressed their appreciation of "an opportunity to try new learning methods".

> This is the first time that anyone has asked me what I think about assignments. It is surprising that it's a maths unit, but it has me thinking about ways of listening to my own maths students. I wonder if they like group work and see advantages. I am going to ask them. We are going to keep studying the same units and critiquing each others' work even if we can't do group assignments.

It has been exciting to find a way of organising group work that requires but values the contributions of individuals. We will continue to research other aspects our teaching in both pre-service and postgraduate mathematics education, and to listen to students' voices.

References

Biggs, J. (2003). *Teaching for quality learning at university: What the student does*. Berkshire: Open University Press.

Collis, B. (1998). WWW-based environments for collaborative group work, *Education and Information Technologies, 3*(3), 231-245.

Falchikov, N. (2005). *Improving assessment through student involvement.* London: Routledge Falmer.

Herrington, J., Oliver, R., and Reeves, T., (2003), Patterns of engagement in authentic learning environments. *Australasian Journal of Educational Technology*, *19*(1), 59-71.

James, R., McKinnis, C., & Devlin, M. (2002). *Assessing learning in Australian universities*. Melbourne: Centre for the Study of Higher Education, The University of Melbourne.

Mousley, J., & Sullivan, P. (1996). *Learning about teaching: An interactive tutorial program to facilitate learning about teaching*. Adelaide: Australian Association of Mathematics Teachers.

Nightingale, P., Te Wiata, I., Toohey, S., Ryan, G., Hughes, C., & Magin, D. (1996). *Assessing learning in universities*. Sydney: Professional Development Centre, University of New South Wales.

Nulty, D., & Kift, S. (2003). *A framework for ensuring the quality of student assessment*. Paper presented at the Australian Technology Network Evaluations and Assessment conference, QUT, Brisbane, Qld.

Norris, S. P., & Ennis, R. H. (1990). *Evaluating critical thinking*. Melbourne: Hawker Brownlow.

Ramsden, P. (2003). *Learning to teach in higher education*. London: Routledge Falmer.

Reeves, T. C. (2000). Alternative assessment approaches for online learning environments in higher education, *Journal of Educational Computing Research, 23*(1), 101-111.