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The meaning constructed beyond symbols

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Abstract Algebra and calculus students have difficulties to express themselves in a statement of mathematical symbols and to comment on written mathematical statements to end with equivalent mathematical symbols statement. In this study, the reasons behind the meaning students load to the mathematical symbols and written mathematical statements was investigated. Moreover the relation between students' understanding of algebra/calculus and written/mathematical symbol statements were also examined. This study is conducted with descriptive and exploratory purposes to the enquiry. Because of the nature of the data, this research uses qualitative data collection tools which are written examination to students, interviews with students and lecturers and classroom observations. Concurrent verbal protocols were also conducted by students to gain insight into their thinking. Results showed that teaching for testing, which seems to be one of the reasons for Skemp's instrumental understanding, before the university have influences on students' performance on dealing with mathematical symbols at algebra and calculus. It is found that students mostly take symbols as object with some meaning rather than thinking of process object duality.
Keywords : meaning of symbols, process-object, understanding

1 Introduction

1.1 Literature

Symbolic expressions, formulas, diagrams, drawings and other forms of representations are used extensively in mathematics, both to support thinking and to communicate ideas in terms of understanding and constructing the knowledge. Such mathematical representations have traditionally been generated using pen and paper technique which still plays an important role even though the progress in the technology recently. In the classroom, during the teaching and learning process mathematics becomes a language to communicate which highlights mathematical conversations, mathematical argumentation, mathematical vocabulary and mathematical writing (Pimm, 1987, p. 198). Written mathematics in the exam papers or on the blackboard in the classroom differs from other disciplines with the property of having vast amount of symbols. Hence, mathematical writing process has many features; it is a stage that learner shows what s/he can perform by the knowledge acquired throughout problem solving on the paper or in the classroom, so that it is a means to express thoughts in social context of the classroom. In the studies of the writing process it is claimed that new knowledge is generated in the writing process. The writing activity can be seen as a process that attempts to explain some messages in a way that could be understood by potential reader.

Representations are used throughout the writing activity and so the development of new notation is critical to breakthrough in the history of mathematics (Sfard, 1991). Manipulating and using different forms of representations of mathematical object in the necessary processes is an essential ability in mathematics. Choosing the right representation for the right task is very important, and often difficult (Duval, 2000a). Symbols are special feature of the representations in mathematics. Hayes-Roth et al. (1983, p. 61) describes symbols as strings of characters and of symbol structures as a type of data structure such as Apple, Transistor-13, Running, Five, 3.14159. The symbols are the mathematical marks, that do not constitute ordinary language, and that are manipulated according to certain well-defined rules. Even though symbols would have their meaning in mathematics individuals might have their own constructed meaning which might be shaped socio-cultural factors, experiences, knowledge, cognitive abilities. Mathematical objects are typically pointed as a semiotic representation. Raymond Duval (2001) stresses that all mathematical objects has more than one semiotic representation and that it is a crucial and typical error to mistake one of these representation difficulties in mathematics that can stem from the complicated nature of mathematical signification.

Symbols play an important role at the understanding of mathematics to construct meanings and knowledge and concepts. In the undergraduate mathematics, it has been observed that, the processes of manipulating and using of symbols with reference to meaningful concepts, procedures and representations changed student to student. Algebra and calculus students have difficulties to express themselves in a statement of mathematical symbols and to comment on written mathematical statements to end with equivalent mathematical symbols statement. Although students seemed to have "something to say" they were not able to transform symbolic mathematical statement and written mathematical statements to each other and then use them effectively during the lecture and problem solving process. In this study, the reasons behind the meaning students load to the mathematical symbols and written mathematical statements were investigated. Moreover the relation between students' understanding of algebra/calculus and written/mathematical symbol statements were also examined.

1.2 Methodology

This study is conducted with descriptive and exploratory purposes to the enquiry (Robson, 1993, p. 42). It uses an anti-positivist paradigm which is interpretivist with a naturalistic enquiry approach. Multi method approach is used in the study to get rich data for answering research questions (Cohen et al., 2000). Because of the nature of the data, this research uses qualitative data collection tools. Purposeful sampling is a dominant strategy in qualitative research which seeks information-rich cases that can be studied in depth (Patton, 1990, pp. 182-183). The most appropriate purposeful sampling strategy to this study is convenience sampling in which available individuals are taken or the cases are taken as they occur (Patton, 1990, pp. 169-183). The student sample consisted of 100 algebra students and 100 calculus students. The study involves administering written examination to students, interviews with students and lecturers and classroom observations. To obtain data on students' manner of 'expressing' written mathematical statements in terms of mathematical symbol statements and vice versa and using them in problem solving, however, concurrent verbal protocols were conducted by (10 algebra and 10 calculus) students to gain insight into their thinking. Students were selected for the protocol work to represent a range of attainments and for their ability to communicate well, based on their teachers' recommendations and tests scores. Categorizations and descriptive statistics were used to analyze the data.

1.3 Data collection and analysis

After designing the appropriate research instruments in terms of the paradigm of the research, purposes of the inquiry and defining the sample, data was collected by administering all research instruments to the designated sample in the university. In the middle of the spring term, first week, 100 algebra students and 100 calculus students were taken the written examinations. Second week, interviews and concurrent verbal protocols were conducted with (10 algebra and 10 calculus) students and interviews were conducted with 4 lecturers. Algebra and calculus classes were observed during data collection and spring term time.

The collected qualitative data is analyzed to make sense of this data in terms of the written accounts of teachers, students and documents about the situation, noting patterns, themes, categories and regularities (Cohen at al. 2000, p. 147). Qualitative data, which were categorized in terms of themes relevant to research questions and then these categorizations were coded. Coding qualitative data was helpful to comment on the overall picture in terms of the categories created in the light of the research questions and also it also gave a tidy and structured view of massive data. Coffey and Atkinson (1996) stated that coding is a procedure which tries to link all related fragments under a key idea or concept. There was ongoing analysis throughout the data collection, which seems to be a suggested and typical approach to qualitative data analysis (see Robson 1993, p. 384). In analyzing data, Miles and Huberman's (1984) first and second level coding notion and Driver and Erickson (1983) nomothetic and ideographic approaches were utilized to make categorizations. In a nomothetic approach, students' answers are analyzed against a group of predetermined accepted categories that might emerge from a view of what constitutes the incorrect answer to a question. In the ideographic approach, however, the students' answers are analyzed in their own terms rather than categorizing them into predetermined groups of categories as is the case of the nomothetic approach.

Data was gathered from students' answers to midterm examination questions. In both the algebra and calculus examinations there are two questions which differ in the form. The first form requires the translation of symbolic mathematical expressions into explanations involving different forms of representations (i.e. mathematical or non-mathematical written statements, diagrams, pictures, graphs etc. Second form involves the translation of concepts/terms into symbolic and purely mathematical expressions.

Examples of first form of questions include:
$A \Delta B$, $A \times B$, $infA$, , $A_n=\{k^n: k\in S\}$ and $A=\{A_i: i\in I\}\wedge k\in I \Rightarrow \bigcap_{i\in I} A_i \subseteq A_k \subseteq \bigcup_{i\in I} A_i$ (Algebra)

$\frac{1}{2}+\frac{1}{2.3}+\frac{1}{3.4}+...+\frac{1}{m(m+1)}$; $(3a-b)^n$;N={x; $|x-a|\langle\varepsilon, x\in R$ }; $\frac{-1}{5},\frac{3}{8},\frac{-5}{11},\frac{7}{14},\frac{-9}{17},....$; $(2x+y)^5$ (Calculus)

Examples of second form of questions include:
$\frac{1}{2}+\frac{1}{2.3}+\frac{1}{3.4}+...+\frac{1}{m(m+1)}$; $(3a-b)^n$;N={x; $|x-a|\langle\varepsilon, x\in R$ }; $\frac{-1}{5},\frac{3}{8},\frac{-5}{11},\frac{7}{14},\frac{-9}{17},....$; $(2x+y)^5$

Separate analyses were made for the two forms of questions. (Since similar categorizations were obtained for the two question forms only the analysis of the first question form is given in the paper). For the first form, analysis revealed two broad categories which were further subcategorized (Examples below are chosen from calculus and algebra exam responses).
First and the most frequent response pattern is the use of single form of representation which was subcategorized into the following:

- Symbolic expressions are explained in other forms of symbolic expressions without a single explanatory effort. For example in explaining the meaning of $A \Delta B$, considerably high proportion of responses only includes the expansion of the statement, namely $A \Delta B = (A \setminus B) U (A \setminus B)$.

- In another response pattern, students attempt to provide what may be called a textbook explanation for the symbolic expression in question. . For example, to the question asking the meaning of infimum A, a very frequent response is just saying that it is “the *biggest lower boundary of the set A*”
- Some students are satisfied with writing only the name of the symbolic expression again without further explanation. For example some students write ‘A cartesian product B’ for the ‘*A x B*’ statement.
- Concepts/terms are explained with examples, e.g. writing, “$a_n = 5^n$ *is a sequence*” for describing the concept of sequence.
- Symbolic expressions are explained with non-mathematical statements of involving the use of some mathematical terms, e.g. “Sequence is a function whose range only contains positive integers” or providing superficial description of the term/concept, e.g. calling ‘*the final boundary value*’ for the concept of limit.

  Second and less frequent response pattern is the use of multiple representations for the answers.
- Symbolic expressions are explained with non-mathematical language with the help of symbolic mathematical expressions. “Given that A,B are non empty sets, A ∆ B is the union of the difference of A from B and the difference of B from A,
  i.e. $A \Delta B = (A \setminus B) \cup (A \setminus B)$”
- Symbolic expressions are explained statements with the help of drawings and diagrams. In describing the concept of ‘neighborhood some students used the real number line with the statement involving epsilon-delta definition of limit. In the explanation of the symmetrical difference some students used the Venn diagram.
- Only a small proportion of students attempt to use multiple representations in their explanations which includes the use of the multiple forms of explanations. These responses usually include a mixture of mathematical and non-mathematical statements, (mostly Venn) diagrams and drawings (e.g. number lines).

Findings of the analysis of interviews and the verbal protocols is not presented here because space limitation but will be partially discussed in the arguments section.

2 Argument

Results showed that teaching for testing, which seems to be one of the reasons for Skemp’s (1977) instrumental understanding, before the university, has influences on students’ performance on dealing with mathematical symbols at algebra and calculus. Algebra and calculus classes have mixed-ability students from different regions of the Turkey. The only reason they were together was the university entrance examination and their choice. In the interviews with students and lecturers it is revealed that students were taught for test so that they were memorizing and merely applying the rules in the questions. That may explain why students mostly see symbols as objects with some superficial meaning rather than thinking of process or both process and object. The nature of mathematical thinking might be influenced by the symbols that one uses to represent the mathematical concepts. This seemed to limit students at problem solving process and the meaning they give to written or mathematical symbol statements.

The cultural code/rules and school background for expressing thoughts in mathematical writing that the writer subscribes to is important parameter for the knowledge generation. However, the comment and organized knowledge used at transforming symbol and symbol expression description grammar to each other is relevant to a direct interaction between the representations on the paper and the author’s internal cognitive system. Individuals may ‘see’ connections and associations in a different way when they are in the process of writing in two ways symbolic and in sentence form. Symbols gain their meaning constructed in mind, but the reflection would be appear with pen and paper technique which provides the flexibility for mathematical writing and the strong dependence that mathematical writing continues to owe this medium. Mathematical symbolic expression was almost exclusively supported by pen and paper or blackboards concretely. In their study Toyota et al. (2006) proposed a tree representation for each mathematical formula by considering positional relations between component characters/symbols. By affective use of Toyota et al.’s suggestion teaching strategy may help to see process and object as one. In tree representation each characters/symbols can be taken as object and then the relation in terms of mathematics and position between them can be related to each other and that may provide relational understand with rich concept image and cognitive units (Barnard and Tall, 1977). In tree representation students need to write symbolic expression in description statement. Practically, expressing symbols and “symbolic expression description grammar” in terms of each other might be seen as one of the ways to understand mathematics.

3 Conclusions

Mathematical objects do not have only one representation, it has at least a written description statement. Having a rich concept image of mathematical objects provides understanding. Teaching for testing has influences on students’ performance on dealing with mathematical symbols at algebra and calculus. Since students memorize expressions by heart they mostly take symbols as object with some meaning rather than thinking of process object duality. Being not able to see a mathematical concept from two perspectives, which are symbolic and its description form, seemed to limit students at problem solving process and the meaning they give to written or mathematical symbol statements.

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