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*Inquiry Based Learning:
A Modified Moore Method Approach
To Encourage Student Research*

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ABSTRACT
INQUIRY BASED LEARNING:
A MODIFIED MOORE METHOD APPROACH
TO ENCOURAGE STUDENT RESEARCH

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The author of this paper submits that a mathematics student needs to learn to conjecture and prove or disprove said conjecture. Ergo, the purpose of the paper is to submit the thesis that learning requires doing; only through inquiry is learning achieved, and hence this paper proposes a programme of use of a modified Moore method (MMM) across the mathematics curriculum. The author of this paper has used the MMM in classes including an Introduction to Mathematics (general education liberal arts mathematics required as the minimum class that fulfils the mathematics requirement at Kutztown University of Pennsylvania (KUP)), Fundamentals of Mathematics I & II courses (mathematics for elementary education majors), Calculus I, II, & III, Set Theory, Linear Algebra, Bridge to Higher Mathematics, Probability and Statistics I & II, Real Analysis I & II, Topology, Senior Seminar, and Directed Reading.

The author of this paper has taught for approximately twenty-five years, much of it at Morehouse College (MC) an historically black liberal arts institution, but now teaches at a comprehensive university in the Pennsylvania State System of Higher Education (PASSHE) where use of the MMM has been met with mixed reception by students and faculty. This paper discusses the techniques used to facilitate learning and the successes or lack thereof of how the methods and materials in the courses taught established a meaningful inquiry-based learning environment, how the method assisted in forging some long-term undergraduate research, and encouraged some undergraduates to delve into research who might not have otherwise embarked on research.

So, this paper proposes an approach to mathematics education that centres on exploration, discovery, conjecture, hypothesis, thesis, and synthesis such that the experience of *doing* a mathematical argument, creating a mathematical model, or synthesising ideas is reason enough for the exercise - - and the joy of mathematics is something that needs to be instilled and encouraged in students by having them *do* proofs, counterexamples, examples, (informal) arguments, and counter-arguments in any mathematics course. Thus, the MMM used by the author is wholly a derivative of the Moore method and exists because of R. L. Moore, W. H. Mahavier, B. Fitzpatrick, M. Smith, C. Reed, D. Doyle, and other distinguished academics who instructed the author or the author's professors.

I. INTRODUCTION

It seems that it is generally accepted amongst those who subscribe to the Moore method, a modified Moore method (MMM), or another form of inquiry-based learning (IBL) that mathematics is built on a foundation which includes axiomatics, intuitionism, formalism, Aristotelian logic, application, and fundamental set theory.¹ The many branches of mathematics are not mutually exclusive. Oft times applied projects raise questions that form the basis for theory and result in a need for proof. Other times theory develops and later applications are formed or discovered for the theory. Hence, mathematical education should be centred on encouraging students *to think for themselves*: to conjecture, to analyze, to argue, to critique, to prove or disprove, and to know when an argument is valid or invalid. Perhaps the unique component of mathematics which sets it apart from other disciplines in the academy is proof – the demand for succinct argument from a logical foundation for the veracity of a claim.

Proof plays a central role in the inquiry-based learning environment that is created in the author's MMM classes. The author submits that students must be *active* – *not passive* in learning; thus, the student must learn to conjecture and prove or disprove said conjecture. Ergo, the author of this paper submits the thesis that learning requires *doing* – *not witnessing*; only through inquiry is learning achieved; and, hence this paper proposes that the experience of creating a mathematical argument is a core reason for any students' work in a mathematics class and should be advanced above the goal of generating a 'polished' proof, 'elegant' argument, etc..

Much mathematics education research suggests inquiry based learning is more effective, more engaging, and produces students who are enthusiastic learners. However, there are more practiconers of the 'I'll crack your head open and pour in the knowledge' in the academy who create a teacher-centered class than practiconers of inquiry-based learning. It may be the case that it is easier to lecture than to engage students by creating a student-centered class, because one might believe in student-centered learning but has succumbed to the educational establishment's dictates of doing more with less and packing students into classrooms like sardines, or because many people misunderstand what the Moore method, a modified Moore method, or inquiry-based learning is.

This paper outlines a programme of use of a modified Moore method (MMM) for use *across* the mathematics curriculum. The author of this paper has taught for approximately twenty-five years and has recently relocated to a comprehensive university in the Pennsylvania State System of Higher Education (PASSHE) where use of the MMM has been met with mixed reception by students and faculty. At Kutztown University of Pennsylvania (KUP) the author has been party to some successes and, unfortunately, some failures. He has written about such ideas for a number of years [27, 28, 29, 30, 31, 32] and is interested in mathematics curricular matters as well as methods of teaching mathematics.

This paper discusses the techniques used to facilitate learning and the successes or lack thereof of how the methods and materials in the courses taught established a meaningful inquiry-based learning environment, how the method assisted in forging

¹I could be wrong on this point; but, I am going to assume such for the sake of argument.

some long-term undergraduate research, and encouraged some undergraduates to delve into research who might not have otherwise embarked on research.

So, this paper proposes a pedagogical approach to mathematics education that centres on exploration, discovery, conjecture, hypothesis, thesis, and synthesis such that the experience of *doing* a mathematical argument, creating a mathematical model, or synthesising ideas is reason enough for the exercise - - and the joy of mathematics is something that needs to be instilled and encouraged in students by having them *do* proofs, counterexamples, examples, and counter-arguments in any mathematics course. Thus, the MMM used by the author is wholly a derivative of the Moore method and exists because of R. L. Moore, W. H. Mahavier, B. Fitzpatrick, M. Smith, D. Doyle, C. Reed, and other distinguished academics who instructed the author or the author's professors.

A course should be designed such that the instructor *guides* students through a carefully crafted set of notes that is pertinent to the syllabus (especially for a course that is multi-section so studentent in the MMM class do not 'fall behind' if the subsequent course is taught in a non-IBL manner) and allows the students build on material they create, discover, or construct. The material can then expand such that more and more complex ideas can be introduced or proposed. Further, the instructor *ought* constantly monitor the progress of individual students and adjust the notes or offer "hints," where appropriate so as to encourage inquiry and further study.

Use of the Moore method, a modified Moore method (MMM), or another form of inquiry based learning (IBL) cannot be undertaken or adopted as one changes shirts or ties dependent upon a whim, a mood of the day, or social convention - - one must agree with a philosophical position that humans have a natural inquisitiveness - - we must be *active* in order to learn and we must be *engaged* when learning. Adoption of said philosophy is not enough - - it must be *practiced* - - hence, the author submits that for an instructor to assist the student, the instructor himself must be an active learner.

In order to assist a student in building a belief in his ability to do basic mathematical research that might develop into something even greater than the project he might be engaged in at the time; there must already be in the instructor a belief in himself that he can learn through doing. Hence, having been a student in a class or classes that have been taught under the aegis of the Moore method or a modified Moore method *seems* to be pre-requisite to teaching in such a manner.

The MMM the author has used has at times been successful and other times (truthfully) not very successful but has resulted in producing more than a few seemingly successful students from the freshman to master's graduate level; but, it cannot be repeated enough that both the student and instructor must do research in the manner of Moore. The instructor doing research in the manner of Moore refines and enhances his abilities to do basic research which is critical for assisting a student doing research (how can one advise doing research lead when one cannot do research?) to the development of a student's understanding of mathematics.

Furthermore, for a student to truly understand a concept, he needs to contemplate the ideas within and about the concept; therefore, he needs to struggle with the problem and not be interfered with (or perhaps minimally interfered with [per the idea of 'hints']). Thus, the student struggles with concepts perhaps much as the concepts were originally wrestled with by the originators of ideas. The author

opines that it is more organic and naturalistic to leave the student free to work on creating a proof or counter-example and not be concerned with pace or how long a student takes to grasp a concept or produce a proof or counter-example; however, such is ideal but not practical. So, because of the constraints placed on instructor and student by academic 'standards' and curriculum at many (or most) universities a process (the MMM) is created to support the students in inquiry and assist in focusing or directing the inquiry.²

Therefore, those who subscribe to the Moore method, a modified Moore method (MMM), or another form of inquiry-based learning (IBL) should consider the identification of 'promising' students early in the mathematics programme to be a goal of his profession. By identify a 'promising' student or students, the instructor can then lead the student toward some interesting problems or areas of mathematics that are not a part of a 'standard' course which allows the student to be 'freed up' to *do real mathematics*.

²This is not to say that the instructor does not allow for students to follow to 'dead ends' or for students to make mistakes - - - the author has found that many a 'dead end' has made it easier for him to comprehend, discover, or grasp an idea better than if the mistake had not been made.

II. AN INSTRUCTOR WHO EMPLOYS A MODIFIED MOORE METHOD

It is assumed that justification for use of a modified Moore approach is not necessary for this paper given the audience to which it is presented. Nonetheless, there seems to be some sort of 'basic profile' that an adherent of the Moore method, of a modified Moore method (MMM), or of another type of inquiry-based learning (IBL) method fulfils. There are variations and deviation from the basic profile; but, the author opines that there are some constants.

It seems to the author that many if not most instructors who employ a teaching methodology that is inquiry-based are they themselves extremely curious and enjoy learning. Further, it has been the experience of the author that instructors using a MMM are by-and-large willing to point out mistakes (made by one's self most often). Thus, there seems to be more than a bit of self-critiquing as well as critiquing of others. There also seems to be an internal drive within the individual to succeed, little patience for failure within one's self, perseverance, and a passion to 'get it right.' It may not be as common, but amongst the instructors that the author had who used the Moore method, there was a healthy dose of humour employed by the instructors and a willingness to accept that the students did not always perform 'perfectly' (meaning there were days when students had little to nothing to present).

All of the aforementioned aspects are incorporated within the author's modified Moore method: an internal drive within the individual to succeed, little patience for failure within one's self, perseverance, a passion to 'get it right,' self-deprecating humour, and a willingness to accept that students don't always work 'optimally.' Further, a basic tenet of the modified Moore method (MMM) employed by the author is 'if it works, then use it,' to paraphrase William James. The author opines that as an instructor, he must enter into the classroom without much 'baggage' - - that is to say he should be pragmatic, realistic, open to changes, revisions, and constantly assess whether or not the students are learning. If they are not, then some material might need to be revisited or questions asked that focus the students' attention on concepts that might not be understood as well as need be for later material. Likewise, if students are 'having a bad couple of days' then the author attempts to allow for such and tries to encourage the students by noting his educational vicissitudes; moving off subject and chatting about how to learn, why we study, etc.; or, provoking discussion about some of the concepts that the students are struggling with and providing encouragement to the students.

III. THE AUTHOR'S MODIFIED MOORE METHOD

In this section, a description of the MMM employed by the author is outlined; then, it is compared & contrasted with what he understands is the Moore method. The author's MMM is necessarily derived from the Moore method: the author agrees with most of Moore's philosophy of education but relaxes several aspects of the Moore method. Moore's philosophy of education stated that a person learns best *alone* - without help or interference from others. The author's modified Moore philosophy of education states that a person learns best and most completely alone; *but*, sometimes needs a bit of help, encouragement, or reinforcement.

The author has found that what fundamentally drove him toward use of his MMM was that he learnt best under the Moore method (out of all the methods he was privy to be exposed to whilst an student (both undergraduate and graduate) and that he could learn under other methods but with diminished results. Two of the primary reasons for the diminishment of results were his laziness and ability to memorise. The Moore method or modified Moore method does not seemingly reward superficiality or non-contextual rote memorisation.

At Emory University, his Algebra instructor used the Moore method and his Analysis instructor a traditional method. In the Algebra class the author was enraptured by the material and found himself driven to try to do *every* exercise, example, proof, counter-argument, or counterexample. The author memorised proofs in Analysis and regurgitated them back on tests (one recalled with great ease is the proof that $\sqrt{2}$ is irrational). At Auburn University, the exact opposite was true: his Analysis instructor used the Moore method and his Algebra instructor a traditional method. The author memorised proofs in Algebra and regurgitated them back on tests; whilst thriving in the Analysis class. It is not a contention forwarded by the author that traditionalism caused his lack of understanding of material but that because it was easy *for the author* to memorise arguments presented by the instructor, cram before tests, and take 'short cuts;' but, that the Moore method does not reward such study habits (e.g.: it is harder to fall into the 'the long memory & short on understanding trap' in a class organised under the aegis of the Moore method or a modified Moore method).

The Moore method assumes the student has a natural inquisitiveness, he must be active in learning, and as a consequent self-confidence and self-directedness is established and builds within the individual.³

Nonetheless, the student is not always going to perform at peak efficiency and will, on occasion, 'having a bad day' given the constraints of human nature and the diversions of modern society. Therefore, the MMM employed by the author assumes there is a natural inquisitiveness in all humans; but, it is vicissitudinal so therefore intermittently reaches an apex or a nadir much like a sine wave where $f : [0, \infty) \rightarrow \mathbb{R} \ni f(t_M) = a \cdot \sin(b \cdot t_M) \ni t_M \geq 0, \quad a, b, \wedge t_M \in \mathbb{R}$ and t_M is some measure of time. Therefore, a student sometimes needs a bit of help, encouragement, or reinforcement (an intellectual 'push'). The help, encouragement, or reinforcement should not give solutions to a student; but, should be operationalised by asking a sequence of directed questions that the instructor 'knows' is (perhaps)

³ See Davis, pages 17, 78, and 173; D. R. Forbes, *The Texas System: R. L. Moore's Original Edition* Ph.D. dissertation (Madison, WI: University of Wisconsin, 1971), page 181; Traylor, page 13; and, Whyburn, page 354.

a path toward an argument for or against a proposition. It is best if the instructor tries to put himself in the place of the student and imagine that he does not know the solution ⁴

Class meetings commence with student presentations and those presentation take usually between one-half and two-thirds of the period whilst the rest of the time the instructor introduces new definitions, methods to prove or disprove claims, and new terminology, notation, etc. Students are encouraged to take more responsibility for their education and regard the instructor less as a teacher and more as a conductor. Nonetheless, it must be noted that some days there are no student presentations; so, the instructor must be prepared to lead a class in a discussion over some aspects of the material or be prepared to ask a series of questions that motivates the students to conjecture, hypothesise, and outline arguments that can later be rendered rigorous. In lower-level courses, if presentations are not forthcoming or time is not exhausted before presentations are, then sometimes students are presented with concepts or ideas they critique (faulty arguments, somewhat correct arguments (maybe missing a justification or two), faulty 'counterexamples,' correct counterexamples, etc.) In post-Calculus classes if presentations are not forthcoming or time is not exhausted before presentations are, then sometimes students are presented with claims and proposed proofs and counterexamples which they critique (faulty 'proofs,' correct proofs, faulty 'counterexamples,' correct counterexamples, etc.) The author tries to keep in the back of his mind at least a few such claims to 'run up the flag-pole and see who salutes it.'

The Moore method demands that the student not reference any texts, articles, or other materials pertaining to the course save the notes distributed by the instructor and the notes the individual takes during class. Not every student is as mature and dedicated as to be able to follow such a regulation especially in an undergraduate setting. Thus, books are not banished in the classroom of our MMM. The class has a 'required' text that the author opines is fine for definitions and trite examples but is less than complete or rigorous in its exposition or examples.⁵ The author opines that such a text is best so that it does not give to or impose upon the student too much.⁶ This philosophy of education does not seek maximal coverage of a set amount of material, but standard competency with some depth and some breathe of understanding of material under consideration.

The Moore method *demands* that the students not collaborate. Moore stated this position clearly:

I don't want any teamwork. Suppose some student goes to the board. Some other student starts to make suggestions. Suppose some how or another a discussion begins to start. One person suggests something, then another suggests something else. . . after all this discussion suppose somebody finally gets a

⁴This is easy for a person such as the author who readily forgets much and oft remembers little (like me).

⁵Indeed, I purposely try to adopt a book that is not 'great' so that students are not presented with an opportunity to just copy from a text. It should be noted that over the past decade or so, I have noticed students referencing the book less and less (I have wondered if they use it at all, save for exercises in certain classes such as Calculus).

⁶Indeed, the student is allowed to use as many books as he opines is necessary to understand the material. This has not been a problem, frankly, because the mathematics collection in the library at Morehouse College was and at Kutztown University of Pennsylvania is [fortunately] lacking.

theorem... who's is it? He'd [the presenter] want a theorem to be his - he'd want a theorem, not a joint product!⁷

The modified Moore method employed by the author tempers the position Moore proposed and demands *no collaboration on material before student presentations and no collaboration on any graded assignment* and requests minimal collaboration on material after student presentations. After student presentations, if a student does not understand a part of an argument or nuance of said argument, the students are permitted to discuss the argument as well as devise other arguments. Indeed, rather than calling on students like the Moore method [16, 17, 19, 26, 49] most often volunteers are requested like Cohen's modified Moore method [3]; but, there are times when calling on students can be useful and helpful.

The Moore method does not include subject lectures. The MMM employed by the author includes very minimal lectures before student presentations over definitions, notation, and terminology, an occasional exemplar argument, counter-argument, example, or counterexample (especially early in a course), as well as subsequent discussions (facilitated, directed, or led by the instructor) after the students discuss the work(s) presented when the instructor finds there is confusion or misunderstanding about the material amongst the students. However, the MMM employed by the author is not as 'lecture heavy' as a traditional class - - the instructor does not enter the class begin lecturing and only end recitation at the end of the period. The author attempts to follow Moore's principle, "that student is taught the best who is told the least."⁸

In a course where the author's modified Moore method is employed, everything *should* be defined, axiomatised, or proven based on the definitions and axioms whether in class or referenced. In this regard, the MMM employed by the author is reminiscent of Wilder's axiomatic methods [44, 45, 46, 47, 48]. Everything cannot be defined, discussed, etc. within class; hence, the allowance for reference material. Indeed, the MMM employed by the author avails itself of technology; thus, additional class materials are available for students to download from an instructor created web-site. The materials on the web-site have several purposes including delving deeper into a subject; clarifying material in a text; correcting a text used in the class; reaction papers to student work; alternate solution(s) by student(s) other than the student who presented a solution to a claim in class, an alternate solution by the instructor to a claim which was presented in class, or posing several additional problems and question in the form of additional exercises. The author creates 'work-sheets' for many of his classes which change from semester to semester depending on the students in a class. The 'work-sheets' include questions posed by students in class as well as question that may have percolated naturally from a discussion in the class. Moreover, instructor created handouts on the web-site present students with material previously discussed, claims which were made during the class (by students or the instructor), exercises beyond the scope of exercises in the text, and conjectures that were not presented by students in the class along with proposed arguments as to the veracity of the claims. The students critically read

⁷R. L. Moore, *Challenge in the Classroom* (Providence, RI: American Mathematical Society, 1966)

⁸R. L. Moore, *Challenge in the Classroom* (Providence, RI: American Mathematical Society, 1966)

the proposed arguments and note whether or not the proposed solution is correct. Thus, the modified Moore method employed by the author seems to include a tad more reading of mathematics materials than the Moore method.

A shallow understanding of many subjects (or a subject) is a repugnant idea and is not a part of the author's MMM. Therefore, the pace in the author's classes usually is not as fast as the pace in a traditional lecture class; but, the pace is not as slow as some Moore method classes. The pace is set by the instructor's understanding of what the students grasp; by the academic 'standards' and curriculum of the university; and, by the content outlined in the traditional Mathematical Association of America (MAA) Committee for the Undergraduate Programme in Mathematics (CUPM) guides [4, 5]. A class using the MMM employed by the author attempts to balance the student-set pace (full Moore method) with the instructor-set pace (traditionalism). The author's MMM acknowledges that not all questions can be answered and that each time a question is answered a plethora of new questions arise that may not be answerable at the moment. Therefore, the MMM employed by the author is designed to balance the question of 'how to' with the question of 'why.' The author opines that a subject that is founded upon axioms and is developed from those axioms concurrently can be studied through answering (or at least attempting to answer) the questions 'why' and 'how to.'

The author's modified Moore method includes the concept of minimal competency, that a student needs some skills before attempting more complex material. So, aspects of 'coverage' are included in the author's classes (once again as defined by the CUPM guides [4, 5] from the 1960s)⁹; that is to say that there is a set of objectives that the instructor attempts to meet when administering a class, that he is duty-bound to try to meet said objectives. However, the author's modified Moore method does not attempt to maximise 'coverage' of a syllabus. The goal of education is not, under this methodology, 'vertical' knowledge (knowing one subject extremely well) nor 'horizontal' knowledge (knowing many subjects superfluously), but this philosophy attempts to strike a balance between the two. Truthfully, minimum competency and expecting pre-requisites to have been completed are quite problematic when considering students entering college from high school. It was the case in Georgia and is the case in Pennsylvania that few students come to college with a basic understanding of basic Arithmetic, simple Algebra, fundamental Geometry, simple Trigonometry, basic Conic Sections, or elementary Functions.¹⁰

Traditional methods include regularly administered quizzes, tests, and a final. The author's MMM also includes said assessments. However, a part of each quiz or test (for a test no less than ten percent nor more than thirty-five percent) is assigned as 'take home' so that the student may autonomously complete the 'take

⁹The author opines that the 1963 - 1965 Committee on the Undergraduate Program in Mathematics (CUPM) materials delineated that which is fundamental to a strong undergraduate programme and preparation for graduate school and that much of the work produced by CUPM post-1985 centres on 'service' courses, 'mathematics appreciation,' and computational mathematics applications with computers. Post-1985 CUPM guidelines [6, 7, 8, 9, 10, 11] are not 'bad' or 'wrong' but do not seem to accentuate the kind of or strength of preparation for advanced work in mathematics that the author opines should be a part of any course contained within a mathematics major.

¹⁰What (if any) use the NCTM Guidelines (see [35]) are or have proven to be is a subject that would be of great import. It would be nice if a dispassionate, objective, and quantitative study be designed and executed that see if the NCTM Guidelines are of use or a help. Further, some have questioned the need for pre-requisites (see [36] for example)

home' portion with notes, ancillary materials, etc. 'Take home' work is more natural (reflecting the work mathematicians *do* after university); allows students to follow an honour code; allows students time to work on their arguments or examples; and, allows students to tackle more challenging problems than could be included on an 'in class' quiz or test.

A course directed under the author's MMM includes class discussion and expects the discussion to flow 'organically' from the students (but be directed by the instructor). It should be expected that, on average, at least one-half of each class period be dedicated presentation of work, at least one-fourth of each class period be dedicated discussion of work presented or ideas about the definitions, axioms, or arguments. The author's modified Moore method does *not include any kind of group assignments nor any kind of group work*.

A focal point of the discussion of the methods of proof or how to properly argue under the author's MMM is the uncompromising demand for justification. An instructor who employs the author's MMM must insist that his students (and he himself) justify every claim, every step of a proof (at least during the early stages of a course or course sequence), and explain to the students the rationalé for such a policy.

If one happens upon a fact but really does not know why the fact is indeed so, does he really know the thing he claims to know? In classical philosophy, epistemologically in order for person A to know X: (a) X must exist; (b) A must believe X; and, (c) A must justify why X is. An instructor who employs the author's MMM allows for (a), does not request the students adopt (b), but must insist on (c). This is because there are enough examples of truths in mathematical systems such that (a) and (c) are the case but (b) certainly is not for the majority. One can over time come to accept (b) because of the irrefutability of the argument that establishes the certainty of the claim.

Further, the MMM requires the instructor adopt an approach such that inquiry is ongoing. A demand for understanding what is and why it is, what is not known and an understanding of why it is not known, the difference between the two, and a confidence that if enough effort is exerted, then a solution can be reasoned. In this way, the MMM is simply a derivative of the Moore method; it is perhaps a 'kinder, gentler' Moore method than the original. Consider:

Suppose someone were in a forest and he noticed some interesting things in that forest. In looking around, he sees some animals over here, some birds over there, and so forth. Suppose someone takes his hand and says, 'Let me show you the way,' and leads him through the forest. Don't you think he has the feeling that someone took his hand and led him through there? I would rather take my time and find my own way.¹¹

Within the context of a 'kinder, gentler' Moore method, the author's MMM includes volunteering as mentioned previously and the act of volunteering results in students earning points that are added to the final total points at the end of the semester. The volunteer earns a point whether right or wrong and earns another point if the work is correct. No points are earned for 'elegance,' brevity, etc. The

¹¹R. L. Moore, *Challenge in the Classroom* (Providence, RI: American Mathematical Society, 1966)

author has found that the points system assists in getting student who may normally not go to the board to go to the board and attempt to demonstrate their work. It seems to encourage without going 'overboard' and praising that which is not praiseworthy.¹²

However, the notion of confidence that if enough effort is exerted, then a solution can be reasoned must be tempered with humility and realism. Not everything can be known. Hence, one must be selective. The instructor and students must realise that they are not the most intelligent creatures in the universe. Hence, one must accept his limitations.¹³

A class taught with the modified Moore method discussed herein has as a hallmark the idea that ideas naturally percolate from members of the class and that many ideas can be provoked by the instructor by asking a sequence of question so that the end result is ideas arise from individuals in the class. Such percolation of ideas is a characteristic of any Moore class; and such is a primary objective for the instructor in the author's MMM schemata. The author opines that his MMM is designed to encourage students to opine, conjecture and hypothesise by naming principles, lemmas, theorems, corollaries, etc. for individuals who proved the result or proposed the result.¹⁴ Such a technique, the author has found, advances the proposition of trying to opine and think about the ideas discussed in class - - hopefully giving a modest 'push' to the students to try to stretch beyond that which is before them and try to induce ideas new to them (but not necessarily or most often new). The author's MMM is predicated on the proposition that we do not really care who first developed an idea in mathematics - - we are interested in the idea itself and whomever it was that thought of it or did it first does not matter for it is in the *doing of mathematics* we learn not through a discussion of history.

Does the history of a problem really matter in the greater scheme of things? Perhaps it did for an individual who invented, discovered, or created something; but, most often the ideas that the students propose are ideas first proposed and most often were solved by people who are dead and therefore are not complaining or seeking credit.¹⁵ In some of the course taught by the author, he has found that some students have developed ideas or arguments in a way similar to how they were originally created but often (if the author understands how some principles were

¹²The 'everyone is a winner,' 'anti-competition,' or 'pro-cooperation' fad presently in vogue in education.

¹³This point seems in contrast to the internal confidence point previously mentioned or the idea of belief in one's abilities. However, I claim it is not; for to believe one can do something does not mean he can - - it means he is willing to try. Failure exists (in me and in the world) and I cannot succeed at everything; but, I can certainly give it a try (as long as my interest is aroused). The notion of 'quiet arrogance' was proposed by my sainted mother. My sister and I have tried to pass this idea down to the next generation in our family. My mother, may she rest in peace, meant that one's work should speak for itself and one should not 'shove it' in another person's face if we figure something out and they do not. I suppose it is a variation on the idea that actions speak louder than words.

¹⁴I believe I stole this idea from Dr. Coke Reed or Dr. Michel Smith, although I might be mistaken,

¹⁵Let it not be misunderstood that in the author's MMM class there is some sort of nihilistic or narcissistic atmosphere. The credit for principles, lemmas, theorems, corollaries, etc. and the study of the history of mathematics is all well and good, I suppose, but is not a focus of any course taught by the author. Someone whose passion includes the history of mathematics may differ greatly with the point I am making in this section of the paper and that is fine. 'Different strokes' for 'different folks,' comes to mind.

created or founded) not precisely as was originally created or discovered.

Much of the points that highlight aspects of the author's modified Moore method may be summed up as it accents, celebrates, encourages, and attempts to hone an internal locus of control. Present trends or fads in mathematics education does not seem to focus on the internal; indeed, there seems to be a focus on ideas from the external: the instructor, a calculator, a computer (not programming a computer but just using it), a book, or a group 'cooperative learning,' and the internal and individual are deemphasised or ignored completely.

Nonetheless, everything is not 'a bed of roses' for an instructor who uses the MMM as practiced by the author. First and foremost, as previously mentioned, the pace of a course is often slow and almost always when a section of a course is taught by the author and someone else using traditionalistic methods, the author has found 'coverage' lacks in his section.¹⁶ Sometimes it is the case that an outside observer might opine that there is no pace seemingly at all in the course or that there is 'backward progression.' It is safe to say that sometime there is indeed 'backward progression' in a course taught by the author for if the author find there seems to be a prevalent misunderstanding, confusion, or downright erroneous concept being embraced by members of the class; such is usually discussed, confronted, or debated.

One example stands out in the author's mind. He had a fellow faculty member visit his class and there was a student who volunteered to present proof to a rather difficult theorem about the mean of a particular continuous random variable that day. The student did a wonderful job and she laid out the argument beautifully. Well, the claim was proven; but, there was another claim that was true that the instructor thought of and wished for the class to consider. He called on another student in the class and had him go to the board. He asked the student about the claim (whether he thought it true or not) and the instructor quizzed the student on some material to a point at which the student felt he had an idea how to prove the result. He preceded to do so, 'winging it,' and not producing by any means a polished result, but the essence of the argument was there, he had presented the class with the rough sketch of a fine argument, and it seemed to be quite a productive class. However, the author found that his colleague was not impressed and bemoaned, "since Dr. McLoughlin knew that he will be observed, I wished he planned to present some new material to demonstrate his teaching."

The aforementioned incident is an exemplar of what seems to be the case: that is, the Moore method or a MMM such as the author employs is so different from a traditional classroom that a traditionalist can misunderstand the method and opine that nothing is being accomplished. It is the case that in an inquiry-based learning classroom nothing might be accomplished on a given day, but that was not the case in the example previously mentioned. Hence, one can easily misconstrue that which occurs in an inquiry-based learning environment which could lead to professional difficulties (lack of an award of tenure, poor class assignments, etc.). There are many examples of problems between mathematicians who use the Moore method and those who don't from the twentieth century to fill several volumes. Thus, for practical reasons an instructor who creates an IBL class needs to have the support of his colleagues or at least the support of those in charge of the department.

¹⁶However, it is definitely NOT the case that most often a standardised syllabus in a course has not been 'covered' in a section the author has taught.

Another 'thorn' of the MMM as practiced by the author (that it shares with the Moore method) is there is a heavy burden placed upon the student. Quite frankly, it seems there is a much greater expectation placed upon the student in the MMM class as described herein than under a traditional or a constructivist rubric.¹⁷ The expectation is that the students are adults, they are responsible for their education, they are *not required* to attend class, they are responsible to do the work, they are not forced to do any work or hand in any work (other than quizzes or tests), they are expected to 'try & try again,' they are placed in a position in the class to usually suffer through a barrage of questions from the author and be interrupted often whilst presenting, they are oft questioned whilst someone else is presenting (meaning that during one student's presentation the other students have to 'be on their toes' to expect that they might be asked why something is so or whether or not it is or is not [which is a back-handed way to get students to 'attack' another student's work on the board in a kind way], and they are asked to do all of this whilst attempting to take notes, etc. (which most do).

Judging from some of the comments made by students in class, to the chairman of the department or colleagues, or on the Student Ratings on Instruction (SRI) at Kutztown University, the added expectation is not popular nor seemingly appreciated.¹⁸

The aforementioned 'problems' for or with students also could lead to professional difficulties (lack of an award of tenure, poor class assignments, etc.). At the university where the author teaches, there seems to be a heavy accent on student evaluations and some colleagues have advised him that faculty need to award many A's and few F's whilst maintaining a large enrolment (few withdrawals) to avoid employment problems. It would be interesting to note if such occurs at other universities or if such has any correlation to the existence of a College of Education at the university where the author teaches for there was not a College of Education

¹⁷Such was made very clear when the author moved to a new institution where there is a mathematics education programme, there seem to be as many traditionalists as at his previous institution, there are constructivists, and where there are no other Moore method faculty in the department.

¹⁸Such was the case at Morehouse College in some classes but the author received more positive feed-back there. The amount of positive feed-back was higher for higher-level courses at Morehouse. The exact opposite is the case at Kutztown University. Also, he received much positive feed-back from students after they graduated which included comments such as,"... at the time I did not care for it, but now I appreciate ...". The author has been in his present situation for three years, so, it may be such in the future with Kutztown University alumni. However, it may be a case of the cart before the horse since the author was at Morehouse College for 17 years so that students may have acclimated to the MMM used by the author and students not inclined to such avoided his section of a class. There is a tad of anecdotal evidence to suggest that may have been the case due to the following: During first semester of the 2007-2008 year, enrolment in the author's Probability & Statistics I course at Kutztown was lower than another section taught by another instructor; but, the previous 2 years the author was the only person who taught the Probability & Statistics I course. The author heard that there was much jockeying by several students to get into the other section and not be in his section. Moreover, the author heard from more than one student that one or two mathematics education majors in particular were "desperate" to get into the other section and celebrated when they achieved their objective. The other instructor taught in a traditional German seminar recitation style and expected no proofs to be done in or out of class opining that the course was an 'applied' course and mentioning the author was a 'pure' person whilst he (the other instructor) was an 'applied' person. The same held for the second semester of the 2007-2008 year when comparing enrolment in the author's Foundation of Higher Mathematics course versus another section taught by another instructor.

at the college where the author taught and there seemed far less antipathy toward the Moore method or a modified Moore method. The author could be wrong and there may not be antipathy amongst the faculty at the university, but there is no doubt there is antipathy amongst the students at the university toward the MMM and it exists most often amongst the student pursuing a Bachelor of Science in Mathematics Education.

The author has found that the some colleagues do not appreciate the MMM; one incident at Morehouse College helps illuminate this. In Calculus I, as student was perturbed with the generalised power rule integration as outlined in the text since it did not include when the exponent is negative one. The student opined as to the nature of

$$\int (x^{-1}) dx$$

and it's possible solution. After some dead ends the student was encourage to consider

$$g : \mathbb{R} \longrightarrow (0, \infty) \ni g(x) = e^x$$

and it's derivative; then, to consider the inverse function of g . The student deduced

$$f : (0, \infty) \longrightarrow \mathbb{R} \ni f(x) = \ln(x) \Rightarrow$$

$$f'(x) = \frac{1}{x} \Rightarrow$$

$$\int (x^{-1}) dx = \frac{1}{x} + C \ni C \in \mathbb{R}$$

The next semester the student was chastised publicly in class by a colleague who stated (to paraphrase as best as possible), "you do not know that because I haven't taught you that yet!" The colleague spoke to the author and was perturbed because (to paraphrase as best as possible), "that was not on the Calculus I syllabus; what do you think you are doing in that class? (why exponentials were and logarithms were not included is beyond the author's ability to comprehend mathematics or education, but that really does not matter)"

IV. THE USE OF A MODIFIED MOORE METHOD IN DIFFERENT MATHEMATICS COURSES

As previously stated, the author's MMM has been developed over the past twenty-five years of his teaching experience (1982 – present). It is constantly being analysed, refined, and evaluated so it is a dynamic rather than static programme of thinking about mathematics and teaching mathematics. In this part of the paper the author submits how the modified Moore method he uses is implemented in different courses, how it worked or did not for the students, and how it (ideally) sets a stage for encouraging student to do research. The author's MMM is used in all of his classes: at Kutztown University that means from Introduction to Mathematics (Math 017)¹⁹ through Senior Seminar (Math 380) and Directed Reading (Math 370 or 372).²⁰ This section's discussion will centre upon examples of claims that are produced by students in courses previous to Calculus (that are non-major courses). A less comprehensive discussion of major courses will conclude this section.²¹

The Introduction to Mathematics (ITM) is a 'mathematics appreciation' course. As such, there are few to no students interested in mathematics that take the course; but, there are some very talented students who take the course and some quite wonderful ideas oft arise from the class. These talented students are simply interested in subjects other than mathematics.²² Nonetheless, that does not nullify creating an inquiry-based learning environment. The material considered in the course include (but is not limited to) Aristotelian logic, *basic* set theory, *basic* number theory, *basic* probability theory, and descriptive statistics.

Let us consider the logic portion of the course. First definitions are presented for dichotomous logic, the concept of a domain of definition and prime statements presented, and letting P and Q be statements, truth tables are developed for connectives $\neg P, P \wedge Q, P \vee Q, P \rightarrow Q, P \iff Q$, order of operations, and they are sent home with some basic drill exercises. Then the fun begins! Some of the most enjoyable IBL examples come from the logic portion of the course: asking students to explain the law of the excluded middle and what if such were not the case; why the law of addition is for 'or' whilst the law of simplification is for 'and;' requesting student consider $P \rightarrow P$ as oppose the $P \vee \neg P$, etc.

Some students begin to question the need for dichotomy (giving rise to trichotomous logic or more generalised logic), some suggest other connectives (usually at least one suggests the exclusive or), etc. When the subject turns to *basic* set theory, some students immediately note the similarity to the logic (some try to justify everything based on the similarity of notation - - and get shot down), whilst the claims that are proposed by students oft involve the universe, \emptyset , and one of my favourites: the claim that $\forall x \in A \Rightarrow \exists x \in A$.²³ Examples abound for what

¹⁹A course for non-science majors which fulfils the student's general education requirement for a 3-hour mathematics course

²⁰The author uses his MMM to teach graduate courses also (Math 512, 540, and 545); but, that is not the focus of this paper. It is worth noting; however, that at Kutztown University there is not a Masters of Science in Mathematics programme but there is Masters of Science in Mathematics Education programme.

²¹For a more comprehensive discussion of other courses see [28], [31], or [32]

²²Misguided people that they are (a joke).

²³Which provokes the wonderful conversation about the axiom of null.

students can study or what they induce should or might be when the subject turns to number theory; many centre on the nature of 0 and 1 (for example: Let $x \in \mathbb{R}$ and consider such concepts as $\frac{1}{0}, \frac{0}{0}, \frac{1}{1}, \frac{0}{1}, \frac{x}{0}, \frac{x}{x}, \frac{1}{x}, \frac{x}{1}$ to name but a few generate a plethora of claims, questions, ideas, etc.). Something as rudimentary as, "Let $x \in \mathbb{R}, x \cdot 0 = 0,$ " prompts calls that it is obvious! The author normally replies, "if it is obvious, then the proof should be facile and if it must be assumed, then where in the axioms is it listed (it is not)?" For many ITM students believe that $x \cdot 1 = x$ where $x \in \mathbb{R}$ is an axiom and object that one can (should) prove $x \cdot 0 = 0$ where $x \in \mathbb{R}$.

The simplicity and beauty of the claim $x \cdot 0 = 0$ where $x \in \mathbb{R}$ is one that has been the cause of many a diatribe of students toward the author. One student 'showed' the author that of course it was true - - producing a sequence of keystroke on his calculator! The understanding of what constitutes meaningful evidence ensued and a demonstration of how to err with a calculator shown (which was as putting a match to a forest fire - - - it did not 'help'). Since formal proof is not a part of the course, the author relaxes the requisites for arguing a point in the class and allows for one to 'show' rather than 'prove' a claim.²⁴ Nonetheless, in the ITM course there exist more than a few students that welcome the concepts and find that they could (or might) use them in areas such as Criminal Justice, Law, or Sociology. Indeed, such inquiry-based learning discussions and exercises continue in the probability theory and descriptive statistics portion of the class.

In the Fundamentals of Mathematics I and II sequence (mathematics for elementary education majors or mathematics for special education majors) the material overlaps the ITM math course but includes more number theory (properties of whole numbers, integers, rational numbers, real numbers, and decimals), "problem-solving;" mathematical systems; systems of numeration; informal geometry; informal rudimentary functions; and, measurement. Many of the same or similar claims, discussions, debates, etc. arise within the class. Given that the material is more comprehensive but not more advanced, the opportunity to discuss fundamentals (note the title of the course) of arithmetic and arithmetic processes dominates the first course. Two of the most wonderful claims that are presented to each Fundamentals of Mathematics I class is that letting $U = \mathbb{R}, 0 < 1$ and $0.\bar{9} = 1$.²⁵ It could be that such are classic claims and are a part of most IBL number theory courses; however the author has found that they are *not* a part of most non-IBL taught courses. It might not be the case but from what the author has witnessed many non-IBL course begin and end with whatever text is being used and deviation from the text is rare or non-occurring.

In the Fundamentals of Mathematics I and II sequence, some of the interesting student claims that the author has been privy to include:

1. Claim: $U = \mathbb{R}, \nexists x \ni 1 < x < 2,$
2. Claim: $U = \mathbb{R}, \frac{1}{0} = 1$
3. Claim: $U = \mathbb{R}, \frac{1}{0} = 0$

²⁴The author uses the word 'show' to mean that an outline of an argument is presented such that whilst not a formal proof; the outline could serve and a sketch that could later become a proof if a student were to ever be asked or were interested in proving a mathematical claim.

²⁵The claim $U = \mathbb{R}, 0 < 1$ is a part of most classes the author instructs - - he inevitably attempts to provoke a discussion about it in each class he teaches.

and a host of other fun claims that are incorrect and produce some interesting discussions and allow for students to truly inquire as to the nature of \mathbb{R} , \mathbb{Q} , \mathbb{I} , \mathbb{N} , \mathbb{N}^* , etc.²⁶

Truth be knownst, of the topics contained within the sequence (Aristotelian logic, *basic* set theory, *basic* number theory, *basic* probability theory, descriptive statistics, and a discussion of elementary functions, "problem-solving;" mathematical systems; systems of numeration; informal geometry; and, measurement) the author deemphasises "problem-solving;" systems of numeration, and measurement and emphasises more than other faculty the Aristotelian logic. The aforementioned deemphasised topics may contain a wealth of opportunity for IBL; but, as of this writing the author has not found nor been found that such exists with regard to said topics. In the Calculus sequence, a plethora of opportunities arise for IBL and a cornucopia of discussion topics exist. We shan't dwell on the Calculus sequence for it seems that such topics are so familiar to the reader.

Perhaps the most pleasant (and clearly a favourite area for the author) is what is commonly called a Bridge to Higher Mathematics (BHM), Foundations of Mathematics (FOM), Introduction to Advanced Mathematics (IAM), or Introduction to Set Theory course that has become part of the canon over the last twenty years or so and is designed to transition students to upper division courses. Typically, the course is placed as a sophomore course but at some schools it is a freshman or junior level course. At Kutztown University it is titled Foundations of Mathematics (FOM), so we shall refer herein to it as such.

The FOM course includes first order logic, predicate calculus, syllogistic arguments, existentials, universals, basic set theory, mathematical notation used in upper division course work (as such as needed and arises), more advanced set theory and the axioms,²⁷ generalised collections of sets, the field axioms of the reals, the order axioms of the reals, natural numbers, integers, rationals, irrationals, Cartesian product sets, relations, equivalence relations, partial orders, functions, cardinality, and ordinality.

The author has found Barnier & Feldman's *Introduction to Advanced Mathematics* (3rd Ed., Prentice-Hall is a 'good' book to use (since the use of a book is required at Kutztown presently). Currently, the author uses the Barnier & Feldman text for there is little (if any) detailed exemplars in it which allows for his MMM to be used to facilitate the students' learning. In addition, his sequence of notes and hand-outs are liberally used.^{28, 29}

Many of the claims that we have previously discussed do arise within the context of the FOM class, and some more advanced claims also arise or new and interesting ideas proposed. In almost every semester the author taught the course one of the first student proposed claims is that $x \cdot 0 = 0$ just as it arises oft in courses before

²⁶ $\mathbb{N} = \{1, 2, 3, \dots\}$ and $\mathbb{N}^* = \{0, 1, 2, 3, \dots\}$

²⁷The axioms of Set Theory in the FOM are noted and the axiom of null in particular used but the rest of the set axioms are not formally used.

²⁸Handouts, worksheets, ancillary materials, etc. are available at the author's web site: <http://faculty.kutztown.edu/mcloughl/index.asp>.

²⁹Hale's book [21] seems to be an attempt to have a book that uses or encourages a MMM; but, such seems fraught with problems the least of which is that ideas do not percolate up from the student but seem to be imposed upon them (I could be wrong on this). Other texts (for example, Solow's [39] or Velleman [42]) attempt to turn mathematics into a completely algorithmic exercise which, I opine, is not only an inaccurate portrayal of math but probably harmful.

the Calculus sequence as previously mentioned. Prototypical claims that are a part of the FOM class include:

what if for the axioms of \mathbb{R} , one were to not claim $1 \neq 0$?

what if for the axioms of Set Theory, one were to not claim \emptyset exists?

must every branch of mathematics have an axiom system?

what is more important, pure mathematics or applied mathematics?

and, how do I drop this major (a small joke)?

An amusing discussion usually develops when the focus is on sets and the discussion is centred upon the two claims that given any set A defined after a well defined universe U , it is the case that $\emptyset \cap A = \emptyset$ and that given any set A defined after a well defined universe U , it is the case that $A \cap U = A$. The similarity to $x \cdot 0 = 0$ and $x \cdot 1 = x$ where $x \in \mathbb{R}$ is noted. The students oft find it rather objectionable that one was proven ($x \cdot 0 = 0$ where $x \in \mathbb{R}$) and one axiomatised ($x \cdot 1 = x$ where $x \in \mathbb{R}$) in the reals but both claims ($\emptyset \cap A = \emptyset$ and $A \cap U = A$) are proven in sets. It is in the discussions amongst the students and between the students and the instructor that the best elements of the Moore method and make for a wonderful educational experience (hopefully) for the students and a meaningful experience for the instructor. Students debate the subject, discuss the subject (in the class - - not outside the class), opine, hypothesise, conjecture, and attempt to resolve the seemingly contradictory evidence before them. The instructor is responsible to explain the significance of the axioms and expose the students to the beauty of mathematics and proof. That is to say, that the axioms provide a framework or set of rules of a game or a puzzle, that logic provides the structure for deducing answers to questions or ways to solve the game or puzzle, that the students have the ability to solve the game or puzzle, and then encourage them to do so. In this manner, the MMM creates a student-centred experience as does the Moore or perhaps some other IBL method.

Many of the ideas that arise in the FOM class that student confront (meaning that the ideas they were presented with or came to believe) are concerned with product sets, relations, and functions - - especially functions.³⁰ This author tries to focus students' attention on the concepts of relations and functions by accentuating cardinality of sets. Characteristically, many more claims are considered than are proven or disproven during this time; but, oft it is the most satisfying part of the course for all since many students find they have a 'handle' on the material and the instructor has a wonderful time watching the students struggle, opine, revise, and often do very well with the concepts of cardinality. Nonetheless, it is also the time when many students are startled with delights such as $\aleph_0 + 1 = \aleph_0$, $\aleph_0 + \aleph_0 = \aleph_0$, etc. At least twice in the 15 years the author has taught FOM, classes have begun to work on ordinals also so that the wonderful contrast to cardinals has been considered by the class such as $1 + \omega_0 = \omega_0$, but $\omega_0 + 1 \neq \omega_0$

A discussion that is of great significance to many students and usually develops occurs when the focus is still on the cardinality of sets; specifically the nature of transfinite cardinals and the rather bothersome (in the minds of many students) fact that $\aleph_0 < c$. The fact triggers emotive responses from some and usually creates

³⁰An aside: many texts and some faculty speak of invertible functions of one-to-one functions. Have I missed something? Is it not the case that when we let U be a well defined universe, $D \subseteq U, \wedge C \subseteq U$ that $f : D \rightarrow C$ is invertible if and only if f is a bijection? Allowing for restriction or extension functions doe not alter this, no?

a rather long series of conjectures which do not induce answers during the semester. The students are left to wonder about many of the conjectures and are encouraged to opt for a directed reading course in their subsequent programme so that they might more fully investigate the nature of cardinals (and ordinals). In this manner, the FOM course taught with the MMM fulfils the promise of inquiry based learning (IBL) which should be the hallmark of academe - - opening new areas of inquiry for the student and leaving said person wanting more! Indeed, by the very nature of the manner in which the elucidation of the conjectures occurred: percolating up from the students causes more than a few to act upon their curiosity and study the conjectures in a directed reading course or when they take Senior Seminar.³¹

Other course where many a pleasant idea is proposed by a student, in which some wonderful claims are made, discussions provoked, or arguments hatched are Probability & Statistics I, Probability & Statistics II, Real Analysis I, Real Analysis II, and Introduction to Topology. Within the Probability & Statistics sequence basic claims about outcomes, events, and the sample space that are logical equivalents to basic set theoretic claims about elements, sets, and the universe or to basic Aristotelian logic about atoms, statements, or arguments usually are the first student produced claims which merit investigation. Throughout the sequence, many more arise including claims about transformations of univariate or multivariate discrete or continuous random variables, moments of univariate or multivariate discrete or continuous random variables, etc.³² Within the Real Analysis sequence, many of the claims following from the axioms of the reals are student produced claims and it simply cascades from there to include claims about sequences and series, limits, continuity, etc. .³³ For Topology (Point-Set Topology), the student produced claims oft are consequences of the axioms, definitions, lemmas, theorems, corollaries, metrics, and examples that are based upon the notes that Moore created and were handed down to Mahavier and from him to Smith whence to McLoughlin. .³⁴

Throughout the courses the author teaches that are post-Calculus, many ideas are forwarded by students or suggested by the author then produce an idea or a kernel of an idea for a topic for a student to do research about or on. Exemplars are too numerous to include but a comprehensive list exists on the author's website (<http://faculty.kutztown.edu/studentresearch.asp>). Many focus on concepts or ideas that the author is fascinated by or are ideas that a student presented to the author of which the author had interest in. If a student is interested in an area of mathematics or an idea to which the author has little or no interest, he makes

³¹ Senior Seminar is the 'capstone' course in the mathematics programme at Morehouse College. Students (in different traditions depending on instructor) choose an advisor and research a problem set; then do a formal paper (AMS style, research paper) and presentation at the end of the semester. Senior Seminar is also the 'capstone' course in the mathematics programme at Kutztown University. However, the course seems to be organised in a combination of constructivist and German seminar traditions: Students choose a topic, research a problem set (using books and citing much of the history of the problem, then do a formal paper (not AMS style, more of a report or synthesis paper), and do a presentation at the end of the semester.

³²My favourite basic claims are when we let S be a well-defined sample space and $E \wedge F$ events claiming that $E \subset F \Rightarrow Pr(E) < Pr(F)$ versus $E \subseteq F \Rightarrow Pr(E) \leq Pr(F)$ -- to get (finally) to the correct $E \subset F \Rightarrow Pr(E) \leq Pr(F)$.

³³My favourite claims usually centre on the classic Cantor set or a derivative of it and on metrics, dimensionality, etc.

³⁴My favourite claims usually centre on compactness and connectedness which lead to continua, metrics, dimensionality, etc.

every effort to engage the student in a discussion with a colleague who's interests are commiserate with the concept, idea, or topic.³⁵

³⁵One may perhaps rightly claim that this is rather selfish of the author, to wit he accepts guilt.

V. ENCOURAGING STUDENT RESEARCH

In this part of the paper, a discussion of the successes (or lack thereof) of the MMM the author uses in courses is outlined which leads to students doing mathematical research (with the author or not). The author opines that a primary goal of any course is to establish an atmosphere that creates for students an interesting and challenging intellectual environment which ideally encourages students to further their study of mathematics. When teaching a previous to Calculus (that is a non-major course) course, the encouragement of further study in mathematics oft is done by pointing out the student's ability to grasp the material and produce ideas rather than just read about ideas. Such is done easily when the class is taught within the context of the Moore method or a modified Moore method since the class is 'student-centred.' The encouragement of further study in mathematics is actualised by offering a suggestion of a course or course a student could take, perhaps suggesting a change of major (to mathematics or a related subject), or perhaps minoring in mathematics. For Elementary Education majors, the author sometimes tries to suggest the student specialise in mathematics (such exists at Kutztown University) or change to Secondary Education. For Secondary Education majors, the author attempts to suggest the student change his major to mathematics rather than mathematics education.

Clearly, much of the encouragement of further study in mathematics and of research in a topic of interest to the student and the author occurs from Calculus onward. Such has arisen, for example, in Calculus II with a student or two studying aspects of the Gamma Function and in Calculus III with a student studying aspects of Césaro summability or LaPlace transforms to name but two. From the Foundations of Mathematics (FOM) forward, the opportunity to encourage further study and in more depth arises within every course the author has taught and has assisted in forging a long-term undergraduate research component for some majors (by identifying the student in FOM typically). The existence of Senior Seminar at Morehouse College allowed for such since students were aware they were required to do a Senior Seminar thesis which is not the case at Kutztown University. However, the author has been successful in finding students who are simply curious, who are in the Honours Programme at Kutztown, who are interested in a career in a field where mathematics is used, or who are interested in graduate school (so far).^{36, 37}

Many students who were in the author's Calculus, FOM, Real Analysis, Probability & Statistics, or Senior Seminar courses went on to graduate school. In the period of 1999 through 2005 (beginning with the entering class of 1995), 17 students who were in the author's classes pursued post-baccalaureate work in the mathematical sciences. The author does not claim it was him but that by teaching in a manner that inquiry-based learning can be achieved by the students *the modified Moore method* was key in encouraging or directing the students to pursue

³⁶I opine that there are *many* students who probably would do undergraduate mathematics research if the opportunity arose and a faculty member simply offered such.

³⁷I opine that undergraduate research is a *great* experience for students – most of all because it helps the student focus on doing 'real' math slowly, purposefully. Such can assist the student in illuminating whether a love of math exists within him, in clarifying his objectives and goals (graduate school, etc.). IBL, the Moore method, or the MMM relay helps this point, I believe.

post-baccalaureate work. In fact, there is a possible explanation for the number of students who were in the author's classes who pursued post-baccalaureate work - - it may have been due to student self-selection. If the student in the back of his mind thought of the possibility of graduate school or subliminally had the self-confidence necessary to do such work, then he may have selected the author's classes because they were reputed to be 'hard' but 'fair, and 'challenging.'

Indeed, there may be another a possible explanation for the number of students who were in the author's classes who pursued post-baccalaureate work - - the author's own bias toward 'smart' students!³⁸ Again, there may be *yet* another possible explanation for the number of students who were in the author's classes who pursued post-baccalaureate work - - the programme at Morehouse College was redesign between 1998 and 2002 and was revised beginning in 2003. Such 'success' as the author had in teaching students who ultimately went to graduate school might not be as great in a department that is more focused on the 'applied' or mathematics education. Hence, there is a strong caveat in inducing any 'success' at all the author seems to have had from the number of students who pursued further study in mathematics. If such trends are found after 10 or more years at Kutztown University, then perhaps, a more credible case could be made for 'success' for the author.

Descriptive papers, such as this, assist in creating anecdotal evidence to suggest a teaching method derived from the Moore method does seem to be successful. Furthermore, common sense seems to suggest that an inquiry-based learning (IBL) environment would seem likely to result in more students pursuing advanced degrees or more students having success in subsequent course-work in a mathematics programme by the very nature of IBL and human curiosity.³⁹ It seems rather clear that a strong case for and a need for a dispassionate, objective, and quantitative study to be designed and executed that could delve into the question of whether or not a particular teaching method results in more students pursuing advanced degrees or more students having success in subsequent course-work in a mathematics programme. Such a study might prove impossible to create and might be controversial; but, the author opines it would be very interesting to do such and the results would be fascinating (no matter which teaching method showed promise or even if no difference (resulted in more students pursuing advanced degrees or more students having success in subsequent course-work) existed between and betwixt methods).

³⁸There has always been claims by some that the Moore method favours the 'already mathematically inclined.' Such a view seems to assume there is a latent mathematical ability, not everyone possesses it or possesses as strong an ability, and that adherents to the Moore method subliminally favour 'better' students.

³⁹However, such may be a circular argument.

VI. SUMMARY AND CONCLUSION

In sum, the author described using a modified Moore method (MMM) to teach across the curriculum and described the material in several courses to assist students in inquiring about the nature, structure, or foundations of mathematics, outlined some of the strategies employed, and discussed how such led to successes (or lack thereof) of creating an interest in students to do mathematical research (with the author or not). Perhaps the *most* important part of this modified Moore method is the caution that one should remain flexible, attempt to be moderate in tone and attitude, be willing to adjust dependent upon the conditions of the class, and not be doctrinaire about methods of teaching. It is the belief of the author that this method maximises educational opportunity for the most students by attempting to teach to as heterogeneous a group as possible. For each individual instructor, the method employed should be that which is most comfortable for him and connects with the students.

I opine that this pseudo-Socratic method should be considered by more instructors of mathematics. I deem this because many of the students taught in this method have gone on to graduate school or entered the work-force and have communicated with me that they felt that a course or courses taught in this manner was *the most educationally meaningful* for them (as it was for me). Whilst a student myself over the course of many years, I was exposed many different methods instructors used to teach mathematics (traditional, German Seminar method, recitation, mimicry, Moore, constructivist, and radical constructivism 'critical theory') and aspects of each of those methods are a part of my modified Moore method because I found that each had its strengths and weaknesses. Thus, I attempted to create a method that, hopefully, included the best of each and discarded to worst of each. I can honestly say that I moderately succeeded in almost every class taught with the Moore or modified Moore method. That which I learnt the best was that which I *did myself*, rather than be told about, lectured to, or even read about. I must *do* in order to *understand*. That I can not explain something does not mean it does not exist, it simply means that I do not know it (at this point or perhaps it is never knowable).

The MMM seeks to minimise the amount of lectures, but allows for students to read from multiple sources (which are used for definitions, examples, etc.) and converse (after presentations). It acknowledges that learning is a never-ending process rather than a commodity or entity that can be given like the metaphor of an instructor cracking open the head of a student then pouring the knowledge into said head. In that regard it is very much reminiscent of reform methods and the philosophy of John Dewey. Dewey stated, "the traditional scheme is, in essence, one of imposition from above and from outside,"⁴⁰ and "understanding, like apprehension, is never final."⁴¹

The queries contain open questions from the perspective of the students (and perhaps the instructor) without indication as to whether they are true or false under the axioms assumed. But, unlike the Moore method, necessary lemmas or sufficient corollaries *are* sometimes or oft included; thus, affording the students a

⁴⁰John Dewey, *Experience and Education*, (New York: Macmillan, 1938), page 18.

⁴¹John Dewey, *Logic: The Theory of Inquiry* (New York: Holt and Company, 1938), page 154.

path to construct their arguments.

I always end a paper that discusses the Moore method or a modified Moore method with a retelling of story of R. L. Moore quoting a Chinese proverb. I cannot put into words how inspiring or meaningful the proverb was for me, is for me, and will continue to be for me. P. J. Halmos recalled a conversation with R. L. Moore where Moore quoted a Chinese proverb. That proverb provides a summation of the justification of the MMM employed in teaching the transition sequence. It states, "I see, I forget; I hear, I remember; I *do*, I *understand*."

A core point of the argument presented in the paper is that a method of teaching a course should be carefully considered (I recommend a modified Moore method, of course) and mathematics lends itself nicely to learning through inquiry. An inquiry-based learning environment seems to create an educationally meaningful experience for students, helps transition students from an elementary to a more refined understanding of mathematics, and encourages students to reach beyond a mundane, pedestrian understanding of mathematics. An innovation in the pedagogy proposed is that not all questions posed in the courses are answered. Many of the questions posed in the courses are left for the student to ponder during his matriculation and answer at a later date. Examples of proofs, counterexamples, etc. are given but *most of the actual work is done by the students*.

So, this paper is one in a sequence of papers the author has written which proposes a pedagogical approach to mathematics education that centres on exploration, discovery, conjecture, hypothesis, thesis, and synthesis such that the experience of doing a mathematical argument, creating a mathematical model, or synthesising ideas is reason enough for the exercise - - - and the joy of mathematics is something that needs to be instilled and encouraged in students by having them *do* proofs, counterexamples, examples, and counter-arguments rather than hear about or witness someone else do proofs, counterexamples, examples, and counter-arguments. Nonetheless, it is not argued that this is the only way to teach, for as Halmos asked in [24], "what is teaching?" I do not know; yet I try to do it!

REFERENCES

- [1] Barnier, William & Feldman, Norman. Introduction to Advanced Mathematics. 2nd Ed. Upper Saddle River, NJ: Prentice Hall, 2000.
- [2] Chalice, Donald R., "How to Teach a Class by the Modified Moore Method." *American Mathematical Monthly*, 102, no. 4 (1995) 317 - 321.
- [3] Cohen, David W., "A Modified Moore Method for Teaching Undergraduate Mathematics." *American Mathematical Monthly* 89, no. 7 (1982): 473 - 474, 487 - 490.
- [4] Committee on the Undergraduate Program in Mathematics, *Pre-graduate Preparation of Research Mathematicians*, Washington, DC, Mathematical Association of America, 1963.
- [5] Committee on the Undergraduate Program in Mathematics, *A General Curriculum in Mathematics for College*, Washington, DC, Mathematical Association of America, 1965.
- [6] Committee on the Undergraduate Program in Mathematics, *Reshaping College Mathematics*, Washington, DC, Mathematical Association of America, 1989.
- [7] Committee on the Undergraduate Program in Mathematics, *The Undergraduate Major in the Mathematical Sciences*, Washington, DC, Mathematical Association of America, 1991.
- [8] Committee on the Undergraduate Program in Mathematics, *CUPM Discussion Papers about the Mathematical Sciences in 2010: What Should Students Know?*, Washington, DC, Mathematical Association of America, 2001.
- [9] Committee on the Undergraduate Program in Mathematics, *Guidelines for Programs and Departments in Undergraduate Mathematical Sciences (Working Paper)*, Washington, DC, Mathematical Association of America, 2001.
- [10] Committee on the Undergraduate Program in Mathematics, *CUPM Interim Reports: Toward a Working Draft Curriculum Guide (Working Paper)*, Washington, DC, Mathematical Association of America, 2001.
- [11] Committee on the Undergraduate Program in Mathematics, *Undergraduate Programs and Courses in the Mathematical Sciences: CUPM Curriculum Guide 2004*, Washington, DC, Mathematical Association of America, 2004.
- [12] Dewey, John, *Democracy and Education*. New York: Macmillan, 1916.
- [13] Dewey, John, *Experience and Education*. New York: Macmillan, 1938.
- [14] Dewey, John, *Logic: The Theory of Inquiry*. New York: Holt, 1938.
- [15] Duren, Lowell R., "An Adaptation of the Moore Method to the Teaching of Undergraduate Real Analysis - A Case Study Report." Ph.D. dissertation, The Ohio State University, 1970.
- [16] Fitzpatrick, Benjamin, Jr., "The Teaching Methods of R. L. Moore." *Higher Mathematics* 1 (1985): 41 - 45.
- [17] Forbes, D. R., "The Texas System: R. L. Moore's Original Edition." Ph.D. dissertation, University of Wisconsin, Madison, 1971.
- [18] Foster, James A., Barnett, Michael, & Van Houten, Kare, "(In)formal methods: Teaching Program Derivation via the Moore Method." *Computer Science Education*, 6, no. 1 (1995): 67 - 91.
- [19] Frantz, J. B., The Moore Method. In *Forty Acre Follies*, 111 - 122. Dallas, TX: Texas Monthly Press, 1983.
- [20] Gersting, Judith L. and Kuczkowski, Joseph E., "Why and How to Use Small Groups in the Mathematics Classroom." *College Mathematics Journal* 8, no. 2 (1977): 270 - 274.
- [21] Hale, Margie. Essentials of Mathematics. Washington, DC: MAA, 2003.
- [22] Halmos, Paul R., How To Teach. In *I Want To Be A Mathematician*. New York: Springer-Verlag, 1985.
- [23] Halmos, Paul R., *Naïve Set Theory*, Princeton, NJ: D. Van Nostrand, 1960.
- [24] Halmos, Paul R., "What Is Teaching?" *American Mathematical Monthly* 101, no. 9 (1994): 848 - 854.
- [25] Halmos, Paul R., Moise, Edwin E. and Piranian, George, "The Problem of Learning to Teach." *American Mathematical Monthly* 82, no. 5 (1975) 466 - 476.
- [26] Jones, F. B. "The Moore Method." *American Mathematical Monthly* 84, no. 4 (1977): 273 - 278.
- [27] McLoughlin, M. P. M. M. "The central role of proof in the mathematics canon: The efficacy of teaching students to create proofs using a fusion of modified Moore, traditional, and reform methods." Paper presented at the annual summer meeting of the Mathematical Association of America, Burlington, Vermont, 2002.

- [28] McLoughlin, M. P. M. M. "Initiating and continuing undergraduate research in mathematics: Using the fusion method of traditional, Moore, and constructivism to encourage, enhance, and establish undergraduate research in mathematics." Paper presented at the annual meeting of the Mathematical Association of America, Baltimore, Maryland, 2003.
- [29] McLoughlin, M. P. M. M. "The fusion method of traditional, Moore, and constructivism and the incorporation of undergraduate research through the mathematics curriculum." Paper presented at the annual meeting of the Mathematical Association of America, Baltimore, Maryland, 2003.
- [30] McLoughlin, M. P. M. M. "On the nature of mathematical thought and inquiry: A preliminary suggestion." Paper presented at the annual meeting of the Mathematical Association of America, Phoenix, Arizona, 2004.
- [31] McLoughlin, M. P. M. M. "Crossing the bridge to higher mathematics: Using a modified Moore approach to assist students transitioning to higher mathematics." Paper presented at the annual meeting of the Mathematical Association of America, San Diego, California, 2008.
- [32] McLoughlin, M. P. M. M. "A Modified Moore Approach to Teaching Mathematical Statistics." Paper to be presented at the annual meeting of the American Statistical Association, Denver, Colorado, 2008.
- [33] Moise, Edwin E. "Activity and Motivation in Mathematics." *American Mathematical Monthly* 72, no. 4 (1965): 407 - 412.
- [34] Murtha, J. A. "The Apprentice System for Math Methods." *American Mathematical Monthly* 84, no. 6 (1977): 473 - 476.
- [35] National Council of Teachers of Mathematics. *Curriculum and Evaluation Standards for School Mathematics* (1989). Reston, VA: NCTM.
- [36] O'Shea, D. & H. Pollatsek, *Do We Need Prerequisites?* Notices of the American Mathematical Society, 1997.
- [37] Page, Warren "A Small Group Strategy for Enhancing Learning." *American Mathematical Monthly* 86, no. 9 (1979): 856 - 858.
- [38] Pólya, George *How to Solve It*. New York: Doubleday, 1957.
- [39] Solow, Daniel, *How to Read and Do Proofs 3rd Ed*. New York: Wiley, 2002.
- [40] Steffe, Leslie and Kieren, Thomas, "Radical Constructivism and Mathematics Education," *Journal for Research in Mathematics Education* 25, no. 6 (1994): 711 - 733.
- [41] Tucker, Adam, *Models That Work: Case Studies in Effective Undergraduate Mathematics Programs*. MAA Notes, 38. Washington, DC: Mathematical Association of America, 1995.
- [42] Velleman, Daniel, *How to Prove It: A Structured Approach* New York: Cambridge University Press, 1994.
- [43] Weissglagg, Julian "Small Groups: An Alternative to the Lecture Method." *College Mathematics Journal* 7, no. 1 (1976): 15 - 20.
- [44] Wilder, R. L. "The Nature of Mathematical Proof." *American Mathematical Monthly* 51, no. 6 (1944): 309 - 325.
- [45] Wilder, R. L. "The Role of The Axiomatic Method." *American Mathematical Monthly* 74, no. 2 (1967): 115 - 127.
- [46] Wilder, R. L. Axiomatics and the Development of Creative Talents. In *The Axiomatic Method with Special Reference to Geometry and Physics*, edited by L.Henkin, P. Suppes, and A. Tarski. Amsterdam: North - Holland, 1976.
- [47] Wilder, R. L. "Robert Lee Moore, 1882 - 1974." *Bulletin of the American Mathematical Society* 82, no. 3 (1976): 417 - 427.
- [48] Wilder, R. L. The Mathematical Work of R. L. Moore: Its Background, Nature, and Influence. In *A Century of Mathematics in America: Part III*, edited by Peter Duren. Providence, RI: American Mathematical Society, 1989.
- [49] Whyburn, Lucille S. "Student Oriented Teaching - The Moore Method." *American Mathematical Monthly* 77, no. 4 (1970): 351 - 359.