Crossing the Bridge to Higher Mathematics: Using a Modified Moore Approach to Assist Students Transitioning to Higher Mathematics

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Abstract

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The author of this paper submits that a mathematics student needs to learn to conjecture and prove or disprove said conjecture. Ergo, the purpose of the paper is to submit the thesis that learning requires doing; only through inquiry is learning achieved, and hence this paper proposes a programme of use of a modified Moore method in a Bridge to Higher Mathematics course to teach students how to do, critique, or analyse proofs, counterexamples, examples, or counter-arguments. Furthermore, the author of this paper opines that set theory should be the core of the course with logic and predicate calculus as antecedents to the set theory, and number theory, cardinal and ordinal theory, or beginning topology of the $\mathbb{R}$ as consequents of set theory.

The author of this paper has experienced teaching such a course for approximately fifteen years; mostly teaching the course at a historically black college. The paper is organised such that in the first part of the paper justification for use of a modified Moore approach - - both pedagogical and practical justification are submitted. In the second part of the paper the author submits the model for the Bridge course and focuses on what is effective for the students, what seems not useful to the students, and why; hence, explaining what practices were refined retained, modified, or deleted over the fifteen years. In the third part of the paper explanation is presented as to why the course was designed the way it was (content), how the course was revised or altered over the years and how it worked or did not for the faculty and students. The final part of the paper discusses the successes and lack thereof of how the methods and materials in the Bridge course established an atmosphere that created for some students an easier transition to advanced mathematics classes, assisted in forging a long-term undergraduate research component in the major, and encouraged some faculty to direct undergraduates in meaningful mathematics research. Qualitative and quantitative data are included to support what were or were not successes.

So, this paper proposes a pedagogical approach to mathematics education that centres on exploration, discovery, conjecture, hypothesis, thesis, and synthesis such that the experience of doing a mathematical argument, creating a mathematical model, or synthesising ideas is reason enough for the exercise - - and the joy of mathematics is something that needs to be instilled and encouraged in students by having them *do* proofs, counterexamples, examples, and counter-arguments in a Bridge course to prepare the student for work in advanced mathematics.

I. Introduction

Mathematics is built on a foundation which includes axiomatics, intuitionism, formalism, logic, application, and principles. Proof is pivotal to mathematics as reasoning whether it be applied, computational, statistical, or theoretical mathematics. The many branches of mathematics are not mutually exclusive. Oft times applied projects raise questions that form the basis for theory and result in a need for proof. Other times theory develops and later applications are formed or discovered for the theory. Hence, mathematical education should be centred on encouraging students to think for themselves: to conjecture, to analyze, to argue, to critique, to prove or disprove, and to know when an argument is valid or invalid. Perhaps the unique component of mathematics which sets it apart from other disciplines in the academy is proof - - the demand for succinct argument from a logical foundation for the veracity of a claim.

The author of this paper submits that in order for students to learn, students must be *active* in learning. Thus, the student must learn to conjecture and prove or disprove said conjecture. Ergo, the author of this paper submits the thesis that learning requires *doing*; only through inquiry is learning achieved; and, hence this paper proposes a philosophy such that the experience of creating a mathematical argument is a core reason for an exercise and should be advanced above the goal of generating a polished proof.

This paper outlines a programme of use of a modified Moore method (MMM) in a Bridge to Higher Mathematics (BHM) course [1] to teach students how to do, critique, or analyse proofs, counterexamples, examples, or counter-arguments. Furthermore, the author of this paper opines that Set Theory should be the core of the course with logic and predicate calculus as antecedents to the set theory, and number theory, cardinal or ordinal theory, or beginning topology of $\mathbb{R}$ as consequents of the set theory.

The course should be designed such that the instructor *guides* students through a carefully crafted set of notes that is a naïve introduction to dichotomous logic and predicate calculus, a basic axiomatic introduction to set theory, and which then builds introducing more and more complex ideas based on said set theory. Further, the instructor *ought* constantly monitor the progress of individual students and adjust the notes or offer "hints," where appropriate so as to encourage inquiry and further study.

Use of the Moore method or a modified Moore method cannot be undertaken or adopted as one changes shirts or ties dependent upon a whim, a mood of the day, or social convention - - one must 'buy into' a philosophical position that humans have a natural inquisitiveness - - we must be *active* in order to learn and we must be *engaged* when learning. Adoption of said philosophy is not enough - - it must be practiced - - hence, the author submits that the method of teaching that he suggests is a modified Moore method that he has used (to some success and producing seemingly successful students) in teaching mathematics from the freshman

[1] Also oft titled: Transition to Advanced Mathematics, Foundations of Higher Mathematics, Principles of Higher Mathematics, Introduction to Higher Mathematics, or Introduction to Set Theory. It is assumed that such a course is a course designed to transition the student from a computational or mechanical understanding of mathematics to a more abstract understanding of mathematics.

to senior level; but especially and most critically to the development of a student's understanding of mathematics in a BHM course.

II. Background

In this section, we will offer background material and offer justification for use of a modified Moore approach - - both pedagogical and practical justification are submitted.

It has become rather accepted in mathematics education to include in the canon a course or courses which transition students from the Calculus sequence to upper division course work (see [6] ,[9], [10], [11], [12], [13], [14]). R. L. Moore created or adapted a pseudo-Socratic method which bears his name (see [22], [23], [25], [37], [65], [66], [67]). He said, "that student is taught the best who is told the least."[2] It is the foundation of his philosophy and it sums up his philosophy of education simply, tersely, and succinctly. Moore believed that the individual teaches himself and the teacher is merely an informed guide who must not trample on the individual's natural curiosity and abilities.[3] The Moore method accentuates the individual and focuses on competition between students. Moore, himself, was highly competitive and felt that the competition among the students was a healthy motivator; the competition among students rarely depreciated into a negative motivator; and, most often it formed an *esprit d' corps* where the students vie for primacy in the class.[4]

However, the Moore method is, perhaps, best suited for graduate-level work where there is a rather homogenous set of students who are mature. Moore's philosophy of education is too often considered a mere method of teaching and, as such, can be taught, adopted, and practiced. The author opines that this is an error because it transcends the pedagogical and is a philosophy of education. Therefore, adoption of the methods that Moore created and practiced would be meaningless and could lead to harm for the students if the practitioner did not subscribe to Moore's philosophy and feel comfortable leaning in a Moore method context. Whyburn notes that Moore's beliefs "gives one the feeling that mathematics is more than just a way to make a living; it is a way of life, an orderly fashion in which you want to consider all things."[5]

Several authors have proposed a modified Moore method for teaching mathematical sciences courses ([2], [4],[19], [24], [43], [44], for example) which could be used in a transition course. If one agrees with the philosophical position conditional to the modified Moore method, then it is an entirely acceptable teaching methodology. However, most times it still requires that the individual learn without the aid of books, collaboration, subject lectures, and demands (uncompromisingly) talent

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[2] R. L. Moore, *Challenge in the Classroom* (Providence, RI: American Mathematical Society, 1966), videocassette. See, also, Miriam S. Davis, *Creative Mathematics Instruction* Ed.D. dissertation (Auburn, Al: Auburn University, 1970), 25; Benjamin Fitzpatrick, Jr., "The Teaching Methods of R. L. Moore." *Higher Mathematics* 1 (1985): 45; and, Lucille S. Whyburn, "Student Oriented Teaching - The Moore Method." *American Mathematical Monthly* 77, 4 (1970): 354.

[3] See Davis, *Creative Mathematics Instruction* and Paul R Halmos, How To Teach. In *I Want To Be A Mathematician* (New York: Springer-Verlag, 1985) for a more detailed discussion of Moore's tenets.

[4] See Davis, pages 21 and 119 and D. R. Forbes, *The Texas System: R. L. Moore's Original Edition* Ph.D. dissertation (Madison, WI: University of Wisconsin, 1971), pages 168, 169, 172, and 188 for a detailed psychological analysis of the formation of *esprit d' corps* from competition.

[5] Whyburn, page 354.

from the individual.[6] Most instructors who subscribe to the Moore method opine that the use of books causes the student to be a witness to mathematics, rather than a participant *in* mathematics.[7] Since the Moore method is based on the assumption that this talent is dormant or latent within the student,[8] the student is expected to do all that is necessary to tap into this dormant talent. There is not an expectation on the part of Moore's philosophy that this dormant talent will awaken easily or quickly. Thus, the pace of learning is set by the student or students.[9]

Several authors opine that a constructivist approach to teaching a mathematics course ([27], [51], [56], and [61]) is the proper method. The constructivist accentuates the community and focuses on cooperation amongst students. The constructivist approach includes alternate assessment, group projects, service learning, etc. and closely resembles pedagogically the National Council of Teachers of Mathematics [48] standards and (to some measure) Dewey's position ([16], [17], [18]). If one agrees with the philosophical position conditional to the constructivist method, then it may be an entirely acceptable teaching methodology. It seems that the constructivist method is best suited for elementary or secondary education where students have not completely matured and where the material is less sophisticated. The constructivist method is based on a philosophy that the individual learn with others and that reality is constructed. In its radical form it maintains "individuals construct their own reality through actions and reflections of actions."[10] So, under such a philosophy a compleat relativism antecedes such that objectivism is relegated to oblivion.

There does not seem to be a preponderance of authors who advocate for or prescribe a traditional German seminar approach to teaching mathematical sciences courses which could be used in a BHM course. Nonetheless, there seems to be a preponderance of books which follow such an instructional framework and many instructors who teach via said method. If one agrees with the philosophical position conditional to traditionalism, then one agrees that knowledge is obtained and the instructor has said knowledge, the student is interested in obtaining said knowledge and through an agreement between and betwixt students and instructor the instructor trains the student in the discipline. If one agrees with traditionalism, then it may be an entirely acceptable teaching methodology. However, most times it requires that the individual learn with books, the instructor lectures, and students learn *from* the book and the instructor.

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[6] D. Reginald Traylor, *Creative Teaching: Heritage of R. L. Moore* (Houston, TX: University of Houston Press, 1972): page 171.

[7] Indeed this was the case in all the classes that the author took where a Moore adherent was instructing. Interestingly, to this day the author can not quote well known theorem like the 'Dedekind Cut Theorem,' but can sit down with a paper and pencil and reading the statement of the proof, prove it. Indeed, if memory serves the author correctly, a proof of said is contained in his master's thesis.

[8] Davis, pages 74, 75, 79, 81, and 185. Also, Traylor, page 169.

[9] See Davis, page 94; Forbes, page 156; F. Burton Jones, "The Moore Method." *American Mathematical Monthly* 84, 4 (1977): page 275; Traylor, page 131; and, R. L. Wilder, Axiomatics and the Development of Creative Talents. In *The Axiomatic Method with Special Reference to Geometry and Physics* (Amsterdam: North - Holland, 1959), page 479.

[10] Steffe and Kieren, "Radical Constructivism and Mathematics Education," *Journal for Research in Mathematics Education* 25, no. 6 (1994): 721.

If there was a way to teach mathematics, perhaps this paper would not exist. However, it seems commonly accepted that (1) different individuals learn in different ways and (2) there is a basic knowledge base that is necessary for the average student to obtain so that he has a higher likelihood to succeed in upper division courses. It is not as commonly accepted, perhaps, but is argued that (3) proving claims true or false is a skill that can be mastered through a programme of individual inquiry, practice, exposition, practice, discussion, practice, and revision.[11] Hence, dogmatically approaching pedagogy be it either through the prism of the Moore method, constructivism, traditionalism, or some other philosophical position deprives both the instructor and the student of learning opportunities. A pedagogy that is dogmatic in its approach deprives the instructor by creating as 'one way and only one way correct' mentality which is constricting, pedestrian, and seemingly unproductive. It deprives the student because his learning style might not be geared toward the dogmatic approach taken by the instructor.

Much of the recent general educational research centred on point (1), thus we shall not bother wasting paper addressing in detail this point. Much work of professional associations (in particular the Mathematical Association of America (MAA)) recent research and policy statements centred on defining point (2) and revising, enhancing, and reviewing point (2) [4, 5, 6, 7, 8, 9, 10, 27]. Regarding point (2) as it applies to a BHM course, the author submits that the course should teach students how to do, critique, or analyse proofs, counterexamples, examples, or counter-arguments. Moreover, the author of this paper opines that the majority of the course should be focused on aspects of set theory.

As to point (3), proving claims true or false the author opines is a skill is grounded in the philosophy of William James and the practice of George Pólya. Just as art schools teach composition techniques, architecture schools teach drafting, etc. schools of mathematics teach theorem proving as a skill that is grounded in logic. There are a finite number of techniques and students are encouraged to learn each one so as to have a basic competency when approaching mathematical claims.

So, herein is proposed a methodology which seeks not to dogmatise teaching. It is proposed that if (1) is true, then strict, uncompromising, and rigid adherence to a pedagogical method should *not* be employed. This is because it is highly unlikely that a class would be composed so homogeneously as to allow for one inflexible method of teaching to be employed and the goal of any course should include maximisation of the likelihood of success for students in the course *and in and beyond the courses subsequent to the course in which the students are enrolled.*

It is proposed that if (2) is true, then a Moore method being employed by an instructor causes a likelihood that the pace of the course will be slow (perhaps too slow). Further, it is proposed that if (2) is true, then the constructivist method being employed by an instructor might cause the pace of the course to be slow (perhaps too slow) and cause a likelihood that a student might not understand the material but only parts, elements, or pieces of the material. This is because group work does not necessarily imply that all in the group equally worked on a project, that all in a group learn each and every part of the project, and that much

[11]For if it is not something that can be learnt then the need for and the justification underpinning a course such as a Bridge to Higher Mathematics is vacuous and makes for an absurd curriculum. The author purposely repeats the word practice for the same reason that repetition is a part of the learning process for many (if not most).

difficulty arises between the work in a group translating to an individual being able to do the work without help. In some areas of academia this may be an acceptable outcome, but in mathematics - - especially in theoretical mathematics - - it can be lethal to a student's mastery of material in subsequent courses if mastery of present material is not obtained (or at least there is a maximisation of the probability that the student mastered the material).

It is proposed that if (2) is true, then a traditional German seminar method being employed by an instructor maximises the amount of material that can be 'covered;' but, 'coverage' does not necessarily imply mastery. Indeed, it can be argued that the traditional German seminar method (recitation) creates a likelihood that the pace of the course will be fast (perhaps too fast). Maximisation of expository material does not imply maximisation of the probability that the student mastered the material. In some areas of academia it may be an acceptable outcome that the student is *aware of much of the material but has not mastered the material*, but in mathematics - - especially in theoretical mathematics - - it can compromise or retard a student's mathematical progress.

Thus, use of a modified Moore method insofar as it employs the classic Moore method (students *doing* the proofs, counter-examples, etc.) allows for pace of the course to not be too fast; but, use of the *modified* Moore method with a book as a guide or reference for fundamental points which would probably be best learnt through discovery (but by using discovery would cause the pace to be too slow) allows for the pace of the course to not be too slow. Expository material (especially definitions) are contained within a book or instructor notes such that they are available to the student *before* material is discussed in the classroom but the expository material is not assumed to have been accessed *before* discussion of said within the classroom. Therefore, the pace of the class is largely determined by the students' abilities, schedules, interest, and needs; but, somewhat determined by a traditional idea of a syllabus and basic knowledge base that is necessary for the average student to obtain so that he has a higher likelihood to succeed in upper division courses. The use of a modified Moore method for instruction allows for the potential that material scheduled for the end of a course can be discussed or studied; but, does not guarantee it will be discussed or studied.

It is proposed that if (3) is true, then a BHM course is properly included in the mathematics canon. We shall take point (3) for granted (perhaps errantly, but we shall assume it). Thus, we shall assume (1), (2), and (3) are true for the subsequent discussion.

III. The Use of a Modified Moore Method in a Bridge Course

In this part of the paper the author submits the model for the Bridge to Higher Mathematics (BHM) course, what seems useful or not useful to the students, and why; hence, explaining what practices were refined retained, modified, or deleted over the fifteen years.

A basic tenet of the modified Moore method (MMM) employed by the author is 'if it works, then use it,' to paraphrase William James. The instructor must enter into the classroom without much 'baggage' - - that is to say he should be pragmatic, realistic, open to changes, revisions, and constantly assess whether or not the students are learning.

The MMM employed by the author is fundamentally derived from the Moore method: the author agrees with most of Moore's philosophy of education but relaxes several aspects of the Moore method. Moore's philosophy of education stated that a person learns best *alone* - without help or interference from others. The author's modified Moore philosophy of education states that a person learns best and most completely alone; *but*, sometimes needs a bit of help, encouragement, or reinforcement.

The Moore method assumes the student has a natural inquisitiveness, he must be active in learning, and as a consequent self-confidence and self-directedness is established and builds within the individual.[12]

However, the student is not always going to perform at peak efficiency given the constraints of human nature and the diversions of modern society. Therefore, the MMM employed by the author assumes there is a natural inquisitiveness in all humans; but, it ebbs and flows or intermittently turns on or off much as a distributor cap distributes a charge in an engine. Therefore, a student sometimes needs a bit of help, encouragement, or reinforcement. The help, encouragement, or reinforcement should not be actualised by giving solutions to a student; but, by asking a sequence of directed questions that the instructor 'knows' is perhaps one path toward an argument for or against a proposition. It is best if the instructor tries to put himself in the place of the student and imagine that he does not know the solution [13]

The Moore method demands that the student not reference any texts, articles, or other materials pertaining to the course save the notes distributed by the instructor and the notes the individual takes during class. Not every student is as mature and dedicated as to be able to follow such a regulation especially in an undergraduate setting and most especially in a BHM course. Thus, books are not banished in the classroom of our MMM. The class has a 'required' text that the author opines is fine for definitions and trite examples but is less than compleat or rigourous in its exposition or examples. The author opines that such a text is best so that it

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[12] See Davis, pages 17, 78, and 173; D. R. Forbes*, The Texas System: R. L. Moore's Original Edition* Ph.D. dissertation (Madison, WI: University of Wisconsin, 1971), page 181; Traylor, page 13; and, Whyburn, page 354.

[13] This is easy for a person such as the author who readily forgets much and oft remembers little.

does not give to or impose upon the student too much.[14] This philosophy of education does not seek maximal coverage of a set amount of material, but standard competency with some depth and some breathe of understanding of material under consideration. This requires time, flexibility, and precise use of language.

The Moore method *demands* that the students not collaborate. Moore stated this position clearly:

> I don't want any teamwork. Suppose some student goes to the board. Some other student starts to make suggestions. Suppose some how or another a discussion begins to start. One person suggests something, then another suggests something else... after all this discussion suppose somebody finally gets a theorem... who's is it? He'd [the presenter] want a theorem to be his - he'd want a theorem, not a joint product![15]

The modified Moore method employed by the author tempers the position Moore proposed and demands *no collaboration on material before* student presentations and *no collaboration on any graded assignment* and requests minimal collaboration on material after student presentations. After student presentations, if a student does not understand a part of an argument or nuance of said argument, the students are permitted to discuss the argument as well as devise other arguments.

The Moore method does not include subject lectures. The MMM employed by the author includes minimal lectures before student presentations over definitions, notation, and terminology, an occasional exemplar argument, counter-argument, example, or counterexample (especially early in the course), as well as subsequent discussions (facilitated, directed, or led by the instructor) after the students discuss the work(s) presented when the instructor finds there is confusion or misunderstanding about the material amongst the students. However, the MMM employed by the author is not as 'lecture heavy' as a traditional class - - the instructor does not enter the class begin lecturing and only end recitation at the end of the period.

In the BHM course where the author's modified Moore method is employed, everything *should* be defined, axiomatised, or proven based on the definitions and axioms whether in class or referenced. In this regard the MMM employed by the author is reminiscent of Wilder's axiomatic methods [31, 32, 33]. Everything cannot be defined, discussed, etc. within class; hence, the allowance for reference material. Indeed, the MMM employed by the author avails itself of technology; thus, additional class materials are available for students to download from an instructor created web-site. The materials on the web-site have several purposes including delving deeper into a subject; clarifying material in a text; correcting a text used in the class; reaction papers to student work; alternate solution(s) by student(s) other than the student who presented a solution to a claim in class, an alternate solution by the instructor to a claim which was presented in class, or posing several additional problems and question in the form of additional exercises.

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[14] Indeed, the student is allowed to use as many books as he opines is necessary to understand the material.

[15] Moore, *Challenge in the Classroom.*

Moreover, instructor created handouts on the web-site present students with material previously discussed, claims which were made during the class (by students or the instructor), exercises beyond the scope of exercises in the text, and conjectures that were not presented by students in the class along with proposed arguments as to the veracity of the claims. The students critically read the proposed arguments and note whether or not the proposed solution is correct. Thus, the modified Moore method employed by the author includes more reading of mathematics materials than the Moore method, though less than a traditional or constructivist method.

A superficial understanding of many subjects is an anathema to a Moore adherent; a Moore adherent craves a deep, full, and compleat (as compleat as possible) understanding of a subject (or subjects)[16] so, the pace in the author's BHM class is set by the instructor tempered by the instructor's understanding of what the students grasp. 'Coverage' of material is not a hallmark of the Moore method. On the other hand, traditional methodology includes the pace of the class set by the instructor (usually prior to the semester). 'Coverage' of material is a trademark of traditional methods. Maximal treatment of material is typical in a traditional classroom. However, the undergraduate experience is repleat with time constraints. Thus, pace is not determined by the students but is regulated and adjusted by the instructor. Hence, the BHM class using the MMM employed by the author attempts to balance the student-set pace (Moore method or constructivism) with the instructor-set pace (traditionalism). The author's MMM acknowledges that not all questions can be answered and that each time a question is answered a plethora of new questions arise that may not be not answerable at the moment. Therefore, the MMM employed by the author is designed to balance the question of 'how to' with the question of 'why.' The author opines that a subject that is founded upon axioms and is developed from those axioms concurrently can be studied through answering (or at least attempting to answer) the questions 'why' and 'how to.'

The author's modified Moore method includes the concept of minimal competency, that a student needs some skills before attempting more complex material. So, aspects of 'coverage' are included in the author's BHM class; that is to say that there is a set of objectives that the instructor attempts to meet when administering a class, that he is duty-bound to try to meet said objectives. However, the author's modified Moore method does not attempt to maximise 'coverage' of a syllabus. A syllabus designed by an instructor who adheres to the author's MMM would include 'optional' material and would have a built-in 'cushion' so that the set of objectives can be discussed (more than just mentioned), the students have a reasonable amount of time to work with the material, and more than that set of objectives is met each semester. The goal of education is not, under this methodology, 'vertical' knowledge (knowing one subject extremely well) nor 'horizontal' knowledge (knowing many subjects superfluously), but this philosophy attempts to strike a balance between the two.

Traditional methods include regularly administered quizzes, tests, and a final. The author's MMM also includes said assessments. However, a part of each quiz

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[16] See Davis, page 70; Fitzpatrick, "The Teaching Methods of R. L. Moore." *Higher Mathematics* 1 (1985): 44; Fitzpatrick, Some Aspects of the Work and Influence of R. L. Moore, *A Handbook of the History of Topology* 1996), page 9; Forbes, page 194; Paul R. Halmos, How To Teach. In *I Want To Be A Mathematician* (New York: Springer-Verlag, 1985), page 262; and, Edwin E. Moise, "Activity and Motivation in Mathematics." *American Mathematical Monthly* 72, 4 (1965): page 409.

or test (for a test no less than ten percent nor more than thirty-five percent) is assigned as ‘take home’ so that the student may autonomously compleat the ‘take home’ portion with notes, ancillary materials, etc. ’Take home’ work is more natural (reflecting the work mathematicians *do* after college); allows students to follow an honour code; allows students time to work on their arguments or examples; and, allows students to tackle more challenging problems than could be included on an ’in class’ quiz or test.

The author has found that what fundamentally drove him toward use of his MMM was that he learnt best under the Moore method (out of all the methods he was privy to be exposed to whilst an undergraduate) and that he could learn under other methods but with diminished results. Two of the primary reasons for the diminishment of results were his laziness and ability to memorise. The Moore method or modified Moore method does not seemingly reward superficiality or non-contextual rote memorisation.

It is the author’s understanding that constructivist or reform methods include class discussions, use of technology, applications, modelling, and group assignments. A BHM course directed under the author’s MMM includes class discussion and allows for the discussion to flow from the students (but be directed by the instructor). It should be expected that, on average, at least one-half of each class period be dedicated presentation of work, at least one-fourth of each class period be dedicated discussion of work presented or ideas about the definitions, axioms, or arguments. The author’s MMM class allows students to use machines[17] for applications (minimal discussion of applications exists in the BHM course since the emphasis is on the foundations of theoretical mathematics) and modelling (with regard to the fact that students present their arguments before the class and there exist exemplars for the students as well as critical reading exercises). The author’s modified Moore method does *not include any kind of group assignments nor any kind of group work.* In this regard it is much more similar to the Moore or traditional methods than constructivism.

At Emory University, his Algebra instructor used the Moore method and his Analysis instructor a traditional method. In the Algebra class the author was enraptured by the material and found himself driven to try to do *every* exercise, example, proof, counter-argument, or counterexample. The author memorised proofs in Analysis and regurgitated them back on tests (one recalled with great ease is the proof that $\sqrt{2}$ is irrational). At Auburn University, the exact opposite was true: his Analysis instructor used the Moore method and his Algebra instructor a traditional method. The author memorised proofs in Algebra and regurgitated them back on tests; whilst thriving in the Analysis class. It is not a contention forwarded by the author that traditionalism caused his lack of understanding of material but that because it was easy *for the author* to memorise arguments presented by the instructor, cram before tests, and take ’short cuts;’ but, that the Moore method does not reward such study habits (e.g.: it is harder to fall into the ’the long memory & short on understanding trap’ in a class organised under the aegis of the Moore method or a modified Moore method).

At Georgia State University, his Meta Analysis and Evaluation Theory, and

[17]Maple, Mathematica, Derive, etc. are acceptable (with restrictions - - they may be used outside of class, they may be used to explore, but they can not be used for graded assignments and the use of said is not encouraged).

Sampling Theory instructors used constructivist methods and his Mathematical Statistics and Linear Statistical Analysis instructors traditional methods, and his Edumetrics and Multivariate Statistics instructor used modified Moore methods. Frankly, it is surprising the amount of material that is recalled and not all together shocking that much (if not most) is forgotten from the courses. However, given that it was graduate school (and the second PhD programme for the author who was in his 30s when at Georgia State) it is not noteworthy that the author did learn some things. However, he found that many of the constructivist 'activities' were mind-numbing or less than instructive (in his humble opinion) and it is entirely possible that the methods were not executed well rather than the fault of the method, per se.

Another strength of the modified Moore method is that the focus of a MMM class is on the construction of arguments, examples, counter-arguments, or counterexamples. It is not the case that a traditionalist or constructivist teaching mathematics would do any less; but the MMM is very much suited for construction of arguments, examples, counter-arguments, or counterexamples and less suited for more applied pursuits or material.

A third strength of the modified Moore method is that there can be and oft is a detailed discussion and the instructor can focus student attention on the difference between and betwixt contradiction, contraposition, and contrarianism. The author has found that many student have great difficulty discerning the difference between and betwixt the three and oft confuse them. The author opines that contradiction, contraposition, and contrarianism are not usually properly contrasted and compared in a traditional class because the class is often instructor-focused rather than student-focused. The author opines that contradiction, contraposition, and contrarianism are not usually properly contrasted and compared in a constructivist class because the class often does not delve into material as deeply as with a MMM nor does it seem that contrasts and comparisons are done as much in a constructivist setting. [18]

A fourth strength of the modified Moore method is that many ideas naturally percolate from members of the class and that many ideas can be provoked by the instructor by asking a sequence of question so that the end result is ideas arise from individuals in the class. Such percolation of ideas is a hallmark of any Moore class; and such is a primary objective for the instructor in the author's MMM schemata. It very much seems to be the case that in a traditional scheme there is not such a movement of ideas from individuals to the group (the class) but from one individual (the instructor) to the group (the class) or from an outside source (the book) to the group (the class). Much of the literature about or advocating for constructivist instruction in mathematics contains a similar focus on natural percolation of ideas from members of the class and as such are share a commonality to a MMM class.

A fifth strength of the modified Moore method as practiced by the author is that it encourages students to opine, conjecture and hypothesise by naming principles, lemmas, theorems, corollaries, etc. for individuals who proved the result or proposed the result. Such a technique, the author has found, advances the proposition of trying to opine and think about the ideas discussed in class - - hopefully giving

[18] It is all together possible that I could be wrong on this point; but, I have not been privy to much depth in courses instructed by a constructivist where I was a student nor have I been privy to much depth in courses instructed by a constructivist where I was a peer witness.

a modest 'push' to the students to try to stretch beyond that which is before them and try to induce ideas new to them (but not necessarily or most often new). The author's MMM is predicated on the proposition that we do not really care who first developed an idea in mathematics - - we are interested in the idea itself and whomever it was that thought of it or did it first does not matter for it is in the *doing of mathematics* we learn not through a discussion of history. But, does the history of a problem really matter in the greater scheme of things? Perhaps it did for such individual, but most often the ideas that the students propose are ideas first proposed and most often were solved by people who are dead and therefore are not complaining or seeking credit. [19]

A sixth strength of the modified Moore method as practiced by the author is that in such a scheme there is a pronounced, overt, and clear celebration of effort and an attempt at trying to solve a problem, create a proof, argue a point, forge an example, produce a counter-arguments, or construct a counterexample. Such was not the case in many classes in which the author was a student. In many a class instructed by a person using a Moore method or modified Moore method there was not such a celebration for an attempt and is some cases there was revulsion for or a taunting of an individual who tried but failed to produce a valid result.[20] It seems to be quite the case that in a traditional scheme there is not such a celebration for the vast majority of the work is refined through recitation by the instructor or exposition in the text. On this point from his review of the literature about or advocating for constructivist instruction, the author's MMM shares in a similarity with constructivism. A fortunate or fortuitous example exists in recent popular culture, the Disney film *Meet the Robinsons*, captured the sense of excitement the author attempts to create in his class and amongst his students for trying, trying again, celebrating the attempt, accepting that we are not always correct, and realising that we learn from mistakes (if we pay attention to the mistakes and analyse them), and always trying to "keep moving forward." Therefore, though it may seem trite, every student in an author's MMM class receives credit in the form earning a 'board point' for attempting to solve a problem, present a solution, do a proof, etc. Said points add into the student's total points at the end of the semester and there are a plethora of example where said points produced the 'rewarding' result of a student's grade being positively impacted.

Much of the points that highlight the strengths of the modified Moore method may be summed up as the MMM accents, celebrates, encourages, and attempts to hone an internal locus of control. The traditional scheme there does not appear to be as prevalent a focus on the internal; indeed, rather there is a clear focus on ideas from the external (the instructor or the book). The constructivist scheme there does not appear to be as prevalent a focus on the internal; indeed, rather there seems to be a focus on ideas from the external (the group or the book) and

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[19] Let it not be misunderstood that in the author's MMM class there is some sort of nihilistic or narcissistic atmosphere. The credit for principles, lemmas, theorems, corollaries, etc. and study of the history of mathematics is all well and good but is not a part of a BHM course nor is a focus of any course taught by the author.

[20] It should be noted that at least three individual instructors who used Moore methods or modified Moore methods were definitely NOT in this category: Michel Smith of Auburn University, Coke Reed of Auburn University, and John Neel of Georgia State University. Those three individual instructors were very encouraging, inspirational, and were part of the basis for the author's teaching style.

the internal and individual are not primary.

Nonetheless, there are weaknesses to the MMM as practiced by the author, not the least of which the pace of the course is slow and almost always when a section of a course is taught by the author and someone else using traditionalistic methods, the author has found 'coverage' lacks in his section.[21] Sometimes it is the case that an outside observer might opine that there is no pace seemingly at all in the course or that there is 'backward progression.' It is safe to say that sometime there is indeed 'backward progression' in a course taught by the author for if the author find there seems to be a prevalent misunderstanding, confusion, or downright erroneous concept being embraced by members of the class; such is usually discussed, confronted, or debated.

One example stands out in the author's mind. He had a fellow faculty member visit his class and there was a student who volunteered to present proof to a rather difficult theorem about the mean of a particular continuous random variable that day. The student did a wonderful job and she laid out the argument beautifully. Well, the claim was proven; but, there was another proof that the instructor thought of and wished for the class to consider. He called on another student in the class and had him go to the board and the instructor quizzed the student on some material to a point at which the student felt he had an idea how to prove the result differently than his classmate had. He preceded to do so, 'winging it,' and not producing by any means a polished result, but the essence of the argument was there, he had presented the class with the rough sketch of a fine argument, and it seemed to be quite a productive class. However, the author found that his colleague was not impressed and bemoaned that he had come on a day when the author had not taught anything at all and was disappointed that he had not had the opportunity to witness this 'modified Moore method' that others had mentioned!

Another weakness to the MMM as practiced by the author has (that it shares with the Moore method) is there is a heavy burden placed upon the student. Quite frankly, it seems there is a much greater expectation placed upon the student in the MMM class as described herein than under a traditional or constructivist rubric.[22] The expectation is that the students are adults, they are responsible for their education, they are *not required* to attend class, they are responsible to do the work, they are not forced to do any work or hand in any work (other than quizzes or tests), they are expected to 'try & try again,' they are placed in a position in the class to usually suffer through a barrage of questions from the author and be interrupted often whilst presenting, they are oft questioned whilst someone else is presenting (meaning that during one student's presentation the other students have to 'be on their toes' to expect that they might be asked why something is so or whether or not it is or is not [which is a back-handed way to get students to 'attack' another student's work on the board in a kind way], and they are asked to do all of this whilst attempting to take notes, etc. (which most do). Judging from some of the

[21]However, it is definitely NOT the case that most often a standardised syllabus in a course has not been 'covered' in a section the author has taught.

[22]Such was made very clear when the author moved to a new institution where there is a mathematics education programme, there seem to be as many traditionalists as at his previous institution, there are constructivists, and where there are no other Moore method faculty in the department.

comment made by students on the Student Ratings on Instruction (SRI) at Kutztown University the added expectation is not popular nor seemingly appreciated.[23] Additionally, the author found that it was easier for him in a class taught by a traditionalist or constructivist; it simply seemed that one was able to 'slack off' more (or more easily) and that in a class taught with the Moore or modified Moore method the accent was definitely focused on those who went to the board, it was difficult (if not impossible) to remain 'anonymous,' and though attendance was not mandatory for him it was absolutely necessary!

It seems to the author, that a third weakness to the author's MMM and the Moore method in general is it is not 'big on,' many of the educational psychological trends of the past quarter-century. It does not seem to be a method which attracts applause from students who seem drastically below the norm (if such exists). It is not a 'feel good' approach where no one fails, where everyone passes (no matter their work), and 'everyone gets a gold star.'[24] Indeed, there is very little from the last 25 years of educational trends that fits well with the author's MMM. Thus, there is a danger that the author and his MMM may become or is stagnant or rigid. The author constantly (almost obsessively) asks whether or not such is occurring, but like the forest for the trees might not be aware that such is the case. However, at least at the moment it seems through interviews with colleagues, peers, former students, and friends that such stagnation is not in existence presently. Nonetheless, nothing in regular life is clear like a good proof, and the Moore method (nor my modified version of the Moore method) is not inerrant, it is not flawless, and since it is a creation of man is not infallible.

[23]Such was the case at Morehouse College in some classes but the author received more positive feed-back there. Also, he received much positive feed-back from students after they graduated which included comments such as,"... at the time I did not care for it, but now I appreciate ..." The author has been in his present situation for two years, so, it may be such will also occur with Kutztown University alumni. However, it may be a case of the cart before the horse since the author was at Morehouse College for 17 years so that students may have acclimated to the MMM used by the author and students not inclined to such avoided his section of a class. There is a tad of anecdotal evidence to suggest that may have been the case due to the following: During first semester of the 2007-2008 year, enrolment in the author's Probability & Statistics I course at Kutztown was lower than another section taught by another instructor; but, the previous 2 years the author was the only person who taught the Probability & Statistics I course. The author heard that there was much jockeying by several students to get into the other section and not be in his section. Moreover, the author heard from more than one student that one or two mathematics education majors in particular were "desperate" to get into the other section and celebrated when they achieved their objective.

[24]Basically, the author is pointing out that the communality aspect of the constructivist philosophical, educational, and psychological method is not a part of the Moore method and is not a part of his modified Moore method. Fraternal, good-natured competition is a part of the Moore method and is definitely a part of the author's MMM. Such is encouraged and every attempt is made to create an atmosphere in the class of a community of individuals, of colleagues, rather than a community that is one an amorphous blob.

IV. The Content in and Form of the Bridge Course

In this part of the paper the author submits the content he opines should be in the Bridge to Higher Mathematics course, an explanation is presented as to why the course was designed the way it was (content), how the course was revised or altered over the years, and how it worked or did not for the faculty and students.

The author has studied under professors who have taught in each of the three ways that have been noted: the Moore, traditional, and constructivist methods during his formal educational experience which spans from the 1960s to the 1990s.[25]The author's MMM has been developed over the past twenty-five years of his college-level teaching experience (1982 – present). It is constantly being analysed, refined, and evaluated so it is a dynamic rather than static programme of thinking about mathematics and teaching mathematics. As such the development, revision, and evaluation of the MMM used by the author is an example of an action research model forged empirically.

The author taught the BHM course whilst at Morehouse College and the BHM course was designed to transition students to upper division courses. The BHM course was designed to be taken by the students concurrent with Calculus II or Calculus III (in a three four-credit hour Calculus sequence). In the model Bachelor of Science (BS) programme, the course is placed as a first semester sophomore course. The course is intended to prepare students for upper division courses with special emphasis on the Real Analysis and Abstract Algebra sequences which are normally started after the BHM (in the BS programme Real Analysis I, Abstract Algebra I, and either Real Analysis II or Abstract Algebra II are minimally required as part of the BS programme [of the traditional year of both Analysis and Algebra]).[26] The author will be teaching the BHM course at Kutztown University in the Spring semester of the 2007-2008 school year. The BHM course is designed to transition students to upper division courses. The BHM course seems to be designed to be taken by the students concurrent with Calculus II or Calculus III (in a four three-credit hour Calculus sequence). In the model Bachelor of Science in Mathematics (BSM) programme and in the Bachelor of Science in Mathematics Education (BSE) programme, the course is placed as a second semester freshman course. The course is intended to prepare students for upper division courses without special emphasis on any particular upper division course that the author is aware of since in the BSM and BSE programmes only Abstract Algebra I is minimally required as part of either programme [of the traditional year of both Analysis and Algebra]).[27]

At Morehouse College, for a short time in the late 1990s the transition to advanced mathematics was modelled on a 2-course sequence[28] the first course centred

[25] See http://faculty.kutztown.edu/mcloughl/curriculumvitae.html for a compleat curriculum vitae.

[26] For a compleat overview of the mathematics programme at Morehouse see the mathematics web-site, http://math.morehouse.edu/

[27] For a compleat overview of the mathematics programme at Kutztown see the mathematics web-site, http://math.kutztown.edu/

[28]For a more thorough description of the courses see [39] or [40]

on logic and the second course centred on naïve set theory.[29]

In each course students were required to understand definitions and axioms and create proofs, examples, and counter-examples. It was an extension of the material that was part of the Math 255 (3 credit hour), An Introduction to Set Theory, that was adopted under the revised curriculum at Morehouse College in 2002 and enacted in 2003. The author found anecdotally that the material being spread over two semesters made discussion of the material easier, the transition for students seemingly better, and it was easier to used a MMM to teach the course. However, due to philosophical objection by some faculty over having a five semester-hour transition to advance mathematics and a practical problem of the number of hours of the major, the precursor course, Math 157, was deleted from the final adopted curriculum at Morehouse College in 2002.[30]

The BHM course should at a minimum introduce first order logic, predicate calculus, syllogistic arguments, existentials, universals, basic set theory, mathematical notation used in upper division course work (as such as needed and arises), more advanced set theory and the axioms,[31] generalised collections of sets, the field axioms of the reals, the order axioms of the reals, natural numbers, integers, rationals, irrationals, Cartesian product sets, relations, equivalence relations, partial orders, functions, cardinality, and ordinality.

The author recommends that the material in the course include and accent material that from [7] and [8] more than [9, 10, 11, 12, 13, 14]. The author opines that the 1963 - 1965 Committee on the Undergraduate Program in Mathematics (CUPM) materials delineated that which is fundamental to a strong undergraduate programme and preparation for graduate school and that much the work produced by CUPM post-1985 centres on 'service' courses, 'mathematics appreciation,' and computational mathematics applications with computers.[32] When one analyses the parts of the course in more detail, one can see that there is much that is of import as introductory material that does not need to be delved in deeply. For example equivalence relations, partial orders, maximal or minimal elements in an order, etc. all are frequently discussed in more detail in an (subsequent to the BHM course) Algebra course (and rightly so). Indeed that which should be introduced but should not be over-emphasised included in the logic portion of the course truth tables,[33] existentials & universals,[34] and syllogistic arguments. That which should be introduced but should not be over-emphasised included in the post-logic portion of the course include, of course, relations, but also generalised unions and intersections,[35] and ordinals.

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[29] The course were numbered and titled: Math 157 (2 credit hour), The Principles of Mathematics (designed for second semester freshmen) and Math 255 (3 credit hour), An Introduction to Set Theory (designed for first semester sophomores). Archival syllabi, grading policies, etc. can be requested from the author.

[30]Math 157 was included in all the versions but the last of the curriculum during the revision process from 1998 through 2002.

[31]The axioms of Set Theory in the BHM are noted and the axiom of null in particular used but the rest of the set axioms are not formally used.

[32]Post-1985 CUPM guidelines are not 'bad' or 'wrong' but do not seem to accentuate the kind of or strength of preparation for advanced work in mathematics that is the focus of this paper.

[33]They are nice but simply getting the logic principles established is all that matters.

[34]These are best accentuated in the context of Set Theory.

[35]Such are frequently discussed in more detail and used in an (subsequent to the BHM course) Analysis course.

That which needs to be accented, emphasised, repeated, stressed, and even (perhaps) drilled include in the logic portion of the course *modus ponens, modus tollens, reducto ad absurdum*, hypothetical syllogism, the law of the excluded middle, cases, contrarianism, the fallacy of assuming the conclusion, the fallacy of denying the hypothesis, and the fallacy of appealing to authority. That which needs to be accented, emphasised, and stressed in the post-logic portion of the course are sets, sets, and more sets! To wit, all the basic elementary properties of sets, the algebra of sets, and basic topological properties of $\mathbb{R}$ investigated with the axioms of $\mathbb{R}$ should be emphasised. Moreover, all aspects of functions (domain, codomain, range, corange of a function, image sets, inverse image set, the union or composition of functions, the creation of or proofs or refutations about claim on well-defined injective, surjective, or bijective functions between sets). Experience has shown the author that a plethora of claims about aspects of injectiveness or surjectiveness (and as a result bijective functions) and work the students can 'sink their teeth into' can be set in the context of cardinality. By doing so, the establishment of some semblance of rudimentary understanding of cardinality oft results for students which aids them in subsequent course-work.

It is essential that the course centre on foundational logic and on naïve set theory. There should be a clear focus on proof: direct proof (three different methods), indirect proof (two methods), *reducto ad absurdum*, mathematical induction, proof by example for existentials, generalised point method of proof for sets, as well as counterexample for universals and counter-arguments for existentials. The author has used Lin & Lin's *Introduction to Set Theory* (no longer in print) as well as Barnier & Feldman's *Introduction to Advanced Mathematics* (2^{nd} Ed., Prentice-Hall). Currently, the author uses the 3^{rd}edition of the Barnier & Feldman text. In addition his sequence of notes and hand-outs are liberally used.[36] There are more thorough books, for example [20], [21], [29], [32], [38,] [41], [57], and [59]; or less through like [15], [26], [30], [53], [54], and [60].

The first meeting day of the BHM class, students are given a syllabus, given a grading policy, told of the expectation of student responsibility, told of the website, etc. Definitions are presented for dichotomous logic, the concept of a domain of definition and prime statements presented, and letting$P$ and Q be statements, truth tables are developed for connectives $\neg P, P \wedge Q, P \vee Q, P \Rightarrow Q, P \iff Q$, order of operations, and they are sent home with some basic drill exercises. The second meeting day student presentations begin (but do not dominate the class as they do once proofs work on proofs begin). Rather than calling on students like the Moore method [36], volunteers are requested like Cohen's modified Moore method [2]. Truth tables are dispensed with in the first week and methods of proof in logic commenced by the third meeting of the class. Throughout the first few weeks the class proceeds in this fashion with short talks about new definitions, methods to prove or disprove claims, and introduction to new terminology, notation, etc. By the end of the first third of the semestre, the amount of time the instructor talks decreases from perhaps half of the class period (at the end of the class session) to perhaps a fourth or not at all. In this manner, the students are encouraged to take more responsibility for their education and regard the instructor less as a teacher and more as a conductor. Nonetheless, it must be noted that some days there are no student presentations; so, the instructor must be prepared to lead a class in

[36] Handouts, worksheets, ancillary materials, etc. are available at the author's web site.

a discussion over some aspects of the material or be prepared to ask a series of questions that motivates the students to conjecture, hypothesise, and outline arguments that can later be rendered rigorous. If presentations are not forthcoming or time is not exhausted before presentations are, then sometimes students are presented with claims and proposed proofs and counterexamples which they critique (faulty 'proofs,' correct proofs, faulty 'counterexamples,' correct counterexamples, etc.) The author tries to keep in the back of his mind at least a few such claims to 'run up the flag-pole and see who salutes it.'

Moreover, by the end of the first third of the semestre, the discussion of the class focuses on basic matters in set theory: the algebra of sets, connectives, and claims about said (unions, complements, relative complements, symmetric differences, etc.) Throughout these weeks the class proceeds in this fashion with presentations for most of at least the first half of the class, discussions on methods to prove or disprove claims (maybe anther way to do the same claim), and late in the period some discussion over new definitions, terminology, notation, etc. In this manner, the students are continually encouraged to take even more responsibility for their education and the instructor reminds the students that he is not a teacher but a guide. Also still if presentations are not forthcoming or time is not exhausted before presentations are, then the author has material to discuss with the class, questions to ask the class, or sometimes a story or two and some encouraging words (if it seems members of the class are suffering from 'burn out.'

So, when the last third of the semestre is upon the class, typically discussions of relations and their aspects are winding down and attention turns to functions. Throughout these weeks the class usually is filled with presentations and there is little else done (perhaps a bit of talk still conversing about methods to prove or disprove claims, and perhaps some discussion over new definitions, terminology, notation, etc. Toward the closing of a typical semester the class focuses its compleat attention on applications of functions by discussing, considering, and doing proofs and counterexamples over the cardinality of sets. Characteristically, many more claims are considered than are proven or disproven during this time; but, oft it is the most satisfying part of the course for all since many students find they have a 'handle' on the material and the instructor has a wonderful time watching the students struggle, opine, revise, and often do very well with the concepts of cardinality. Nonetheless, it is also the time when many students are startled with delights such as $\aleph_0 + 1 = \aleph_0$, $\aleph_0 + \aleph_0 = \aleph_0$, etc. At least twice in the 15 years the author has taught BHM, classes have begun to work on ordinals also so that the wonderful contrast to cardinals has been considered by the class such as $1 + \omega_0 = \omega_0$, but $\omega_0 + 1 \neq \omega_0$

Though the BHM course contains a hodgepodge of introductory material the students will use or study in upper division courses, *all of it* truly centres on logic and sets. For each subject, definitions, terminology, and notation are established and a series of facile claims are proposed for the students to prove or disprove. In the first part of the BHM course, the objective is to introduce the student to the subjects and to allow them to work on elementary claims in the areas so they may connect the logic and methods of proof earlier discussed with the particular aspects of set theoretic mathematics under discussion at the time. For example, if there is a combinatorial claim or a claim about the sum of two natural numbers students are encouraged to note the Peano axioms, the field axioms of the reals, the order axioms of the reals, and then given parochial claims on factorals, permutations, combinations, etc. Throughout the first few weeks the class proceeds in this fashion with short talks about new definitions, brief reviews of methods to prove or disprove claims, comments on additional methods to prove or disprove claims (such as the second principle of mathematical induction, two place quantification, etc.) and introduction to new terminology and notation (where applicable). The intensity of the discussion intensifies as the semester proceeds, the sophistication of the discussion increases, the claims included in the course are of a more challenging form (for the most part) than earlier in the course.[37]

The BHM course continues to frame more of the foundation for preparation intended for advanced course work by including the axioms of the reals and the axioms of sets. In particular, the discussion of the axioms of the reals should be revisited in a first Real Analysis course, so that the claims considered and the discussion of the axioms of the reals in the BHM course is very much so a basic axiomatic introduction to the real line. In almost every semestre the author taught the course one of the first student proposed claims is that x·0 = 0. Most times, one or more students object to the claim stating that it is obvious! The author normally replies, "if it is obvious, then the proof should be facile and if it must be assumed, then where in the axioms is it listed (it is not)?" Hence, it usually becomes a part of the handout exercises that are codified after the meeting that it is proposed.

Another amusing discussion usually develops when the focus is on sets and the discussion is centred upon the two claims that given any set A defined after a well defined universe U, it is the case that $\emptyset \cap A = \emptyset$ and that given any set A defined after a well defined universe U, it is the case that $A \cap U = A$. The similarity to $x \cdot 0 = 0$ and $x \cdot 1 = x$ where $x \in \mathbb{R}$ is noted. The students oft find it rather objectionable that one was proven ($x \cdot 0 = 0$ where $x \in \mathbb{R}$) and one axiomatised ($x \cdot 1 = x$ where $x \in \mathbb{R}$) in the reals but both claims ($\emptyset \cap A = \emptyset$ and $A \cap U = A$) are proven in sets. It is in the discussions amongst the students and between the students and the instructor that the best elements of the Moore method and make for a wonderful educational experience (hopefully) for the students and a meaningful experience for the instructor. Management of the discussion centres on the instructor, but control of the discussion is left to the students. Students are free to debate the subject, discuss the subject (in the class - - not outside the class), opine, hypothesise, conjecture, and attempt to resolve the seemingly contradictory evidence before them. The instructor is responsible to explain the significance of the axioms and expose the students to the beauty of mathematics and proof. That is to say, that the axioms provide a framework or set of rules of a game or a puzzle, that logic provides the structure for deducing answers to questions or ways to solve the game or puzzle, that the students have the ability to solve the game or puzzle, and then encourage them to do so. In this manner, the MMM creates a student-centred experience as does the Moore or constructivist methods.

A third discussion that is of great significance to the students and usually develops occurs when the focus is on the cardinality of sets; specifically the nature of transfinite cardinals and the rather bothersome (in the minds of many students)

[37] The claims about sets are not *necessarily* more challenging. Oft times, students propose claims in the first class which are quite challenging. Indeed, some students offer claims in the BHM class which are not answered during the semester. The instructor becomes more a conductor as the class progresses but, to reiterate, must constantly be prepared to lead discussions in case there are no student presentations.

fact that $\aleph_0 < c$. The fact triggers emotive responses from some and usually creates a rather long series of conjectures which do not induce answers during the semestre. The students are left to wonder about many of the conjectures and are encouraged to opt for a directed reading course in their subsequent programme so that they might more fully investigate the nature of cardinals (and ordinals). In this manner, the BHM course taught with the MMM fulfils the promise of academe - - opening new areas of inquiry for the student and leaving him with wanting more. Indeed, by the very nature of the manner in which the elucidation of the conjectures occurred: percolating up from the students causes more than a few to act upon their curiosity and study the conjectures in a directed reading course or when they take Senior Seminar.[38]

A focal point of the discussion of the methods of proof under the MMM is the uncompromising demand for justification. An instructor who employs the author's MMM must insist that his students (and he himself) justify every claim, every step of a proof (at least during the first third of the BHM course), and explain to the students the rationalé for such a policy. If one happens upon a fact but really does not know why the fact is indeed so, does he really know the thing he claims to know? In classical philosophy, epistemologically in order for person A to know X: (a) X must exist; (b) A must believe X; and, (c) A must justify why X is. An instructor who employs the author's MMM allows for (a), does not request the students adopt (b), but must insist on (c). This is because there are enough examples of truths in mathematical systems such that (a) and (c) are the case but (b) certainly is not for the majority. One can over time come to accept (b) because of the irrefutability of the argument that establishes the certainty of the claim.

The MMM requires the instructor adopt an approach such that inquiry is ongoing. A demand for understanding what is and why it is, what is not know and an understanding of why it is not known, the difference between the two, and a confidence that if enough effort is exerted, then a solution can be reasoned. In this way, the MMM is simply a derivative of the Moore method; it is perhaps a 'kinder, gentler' Moore method than the original. Consider:

> Suppose someone were in a forest and he noticed some interesting things in that forest. In looking around, he sees some animals over here, some birds over there, and so forth. Suppose someone takes his hand and says, 'Let me show you the way,' and leads him through the forest. Don't you think he has the feeling that someone took his hand and led him through there? I would rather take my time and find my own way.[39]

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[38] Senior Seminar is the 'capstone' course in the mathematics programme at Morehouse College. Students (in different traditions depending on instructor) choose an advisor and research a problem set; then do a formal paper (AMS style, research paper) and presentation at the end of the semestre. Senior Seminar is also the 'capstone' course in the mathematics programme at Kutztown University. The course seems to be organised in a combination of constructivist and German seminar traditions: Students choose a topic, research a problem set, then do a formal paper (not AMS style, more of a report or synthesis paper), and presentation at the end of the semestre.

[39] Moore, *Challenge in the Classroom.*

However, the confidence must be tempered with humility and realism. Not everything can be known. Hence, one must be selective. The instructor and students must realise that they are not the most intelligent creatures in the universe. Hence, one must accept his limitations.

At least one quiz is administered each week or week-and-a-half, part in class part take home, or all take home (on quiz work a majority of the work is 'take home')in which the students are asked to prove or disprove conjectures. They are required (of course) to work alone. The quizzes are graded and commentary included so that feedback is more than just a grade. Also, there are three or four major tests during the semestre and a comprehensive final; thus, the MMM is grading intensive for the instructor. The frequency of the quizzes creates a benchmark for the students so they do not fall behind. The final and the tests (no less than three tests nor more than four, depending on the length of the semestre) gives the students the ability to demonstrate competency over a part of the course and an opportunity and responsibility to digest and synthesise the material. The testing schedule differs from the Moore and reform method and shares a commonality with the traditional method. It may be a tad more 'quiz intensive' than traditional methods, but the author has found that many of his colleagues who employ traditional methods grade homework (which is not a part of the MMM) so it might be similar to the traditional methods in that regard.

Experience with many different course sizes over the past twenty-five years has led the author to conclude that optimal course size is between approximately fifteen and twenty. When there are less than about fifteen students, then the class discussions often suffer for a lack of interaction. When the class size is more than about twenty students, then class discussions are often difficult to facilitate and can be problematic because so many students wish to be heard simultaneously. Also, if the class size exceeds approximately twenty, then the burden of grading so many papers becomes quite heavy and the turn around time lengthens which is detrimental. It seems that it is best to provide feedback in a timely manner so that the students have time to reflect on their work and discuss the work in follow-up session during office hours. If too much time has elapsed between the times students hand the papers in and they get the papers back, their memory of *why* they thought what they thought dwindles and the educational experience for the student suffers.

V. Course-work and Students' Further Study Subsequent to the Bridge Course

In this part of the paper, a discussion of the successes (or lack thereof) of the methods and materials in the Bridge to Higher Mathematics course that the author has taught using his modified Moore method. A primary goal of the course and the method is to establish an atmosphere that created for some students an easier transition to advanced mathematics classes, assisted in forging a long-term undergraduate research component in the major, encouraged some students to further their study of mathematics (that is to say go beyond the BS in Mathematics), and encouraged some faculty to direct undergraduates in meaningful mathematics research (by finding out from individuals what they found most interesting and attempting to suggest to the student further study and then approaching faculty members with an interest in said area and attempting to team the two up.).

First and foremost, one should consider what attributes, behaviours, skills, or understanding a student should take with him after successful completion of the BHM course. The author submits that, minimally:

a student be able to explain what a proof is and discern between a valid proof and claim that a proof has been performed, but in reality has not;
a student be able to read a proof of a statement.
a student be able to construct a valid proof using different methods which include: direct, proof by cases, indirect, contradiction, induction (weak and strong forms, and contraposition;
a student be able to construct valid counterexamples to propositions which are false;
a student be able to recognise and avoid common fallacies in arguments including begging the question, circular reasoning, assuming the conclusion, and denying the hypothesis;
a student be able to prove statements about sets using the generalised point method *both* directly and indirectly;
a student be able to use Venn (Euler) diagrammes to assist in the construction of a proof or counterexample of a claim in set-theory (and realises a diagramme does not prove nor disprove a claim);
a student be able to use Haas diagrammes to assist in visualising partial orders or equivalence relations;
a student be able to find a domain, codomain, range, 'corange' of a relation;
a student be able to define a binary relation between sets, prove claims about product sets, define an equivalence relation, prove claims about a relation (is or is not a relation, is or is not an equivalence relation, is or is not a partial order, is or is not a linear order, etc.;
a student is able to find a domain, codomain, range, 'corange' of a function;
a student be able to find an image set, inverse image set, etc. of a function between sets;
a student be able to find the union and composition of functions;
a student be able to create well-defined injective, surjective, or

bijective functions between sets;
a student be able to prove or disprove rudimentary claims about said;
a student be able to define a function between sets, prove claims about functions (is or is not a well-defined function, is or is not injective, surjective, bijective, etc.;
a student be able to prove or disprove claims about the cardinality of sets: denumerability, countability, infinite, finite, and uncountability.

Many students who were in the author's BHM course (and usually at least two or more courses beyond the BHM course with the author) went on the graduate school. In the period of 1999 - 2005 (beginning with the entering class of 1995), 17 students who were in the author's BHM pursued post-baccalaureate work in the mathematical sciences. The author does not claim it was him but the *the modified Moore method* which was key in encouraging the students to pursue post-baccalaureate work. In fact, there is a possible explanation for the number of students who were in the author's BHM who pursued post-baccalaureate work - - it may have been due to student self-selection. If the student in the back of his mind thought of the possibility of graduate school or subliminally had the self-confidence necessary to do such work, then he may have selected the author's BHM class (and other classes) because they were reputed to be 'hard.'

Indeed, there may be another a possible explanation for the number of students who were in the author's BHM who pursued post-baccalaureate work - - the author's own bias toward 'smart' students![40] Again, there may be *yet* another possible explanation for the number of students who were in the author's BHM who pursued post-baccalaureate work - - the programme at Morehouse College was redesign between 1998 and 2002 and was revised beginning in 2003. Such 'success' as the author had in teaching students who ultimately went to graduate school might not be as great in a department that is more focused on the 'applied' or mathematics education. Hence, there is a strong caveat in inducing any 'success' at all the author seems to have had from the number of students who pursued further study in mathematics. If such trends are found after 10 or more years at Kutztown University, then perhaps, a more credible case could be made for 'success' for the author.

When one considers some quantitative data about the grades earned in the BHM course and grades earned in Abstract Algebra I; and, grades earned in Real Analysis I for a sample of students who took the author's BHM class in either the Fall of 1999 or the Spring of 2000. We considered grades earned in Abstract Algebra I or Real Analysis I for the period of Spring 200 through Spring 2001.

Let g be grade earned in Abstract Algebra I, a be grade earned in Real Analysis I, b be grade earned in Bridge to Higher Mathematics (BHM). Of the 61 students sampled, 34 had earned grades in BHM and Abstract Algebra I; with correlation estimate and regression equation estimates of $\hat{r} \doteq 0.554, p < 0.01$ and $\hat{g} \doteq 0.989 + 0.519 \cdot b$. Of the 61 students sampled, 31 had earned grades in BHM and Real Analysis I; with correlation estimate and regression equation estimates of $\hat{r} \doteq 0.461, p < 0.01$ and $\hat{a} \doteq 1.233 + 0.415 \cdot b$.

[40]There has always been claims by some that the Moore method favours the 'already mathematically inclined.' Such a view seems to assume there is a latent mathematical ability, not everyone possesses it or possesses as strong an ability, and that adherents to the Moore method subliminally favour 'better' students.

Furthermore, it was interesting to note that of the 61 students enrolled in either the Fall of 1999 or the Spring of 2000; 30 earned a grade which was less than a 'C' and therefore had to repeat the BHM course before moving on to either Abstract Algebra I or Real Analysis I. Moreover, 22 had not completed Abstract Algebra I by the next academic year and 22 had not completed Real Analysis I by the next academic year (which were not the same 8 students [6 completed both successfully]).[41]

It should be emphasised quite strongly and emphatically that no inferences should be derived from these statistics; they are only descriptive and illuminating insofar as it seems to suggest that for this particular group of students in the particular time-frame the BHM was a positive predictor of the grade earned in either Abstract Algebra I or Real Analysis I.

Clearly, there is a strong case for and a need for a dispassionate, objective, and quantitative study to be designed and executed that could delve into the question of whether or not a particular teaching method results in more students pursuing advanced degrees or more students having success in subsequent course-work in a mathematics programme. Such a study might prove impossible to create and might be controversial since there are more than a few faculty in mathematics departments in the U. S. A. who opine that 'subsequent course-work' is a misnomer, that pre-requisites should be minimised or done away with, and that much (if not most) of the positions forwarded in this paper are contrary to what said faculty believe should constitute a mathematics programme.

[41]The correlation between the grade in Abstract Algebra I and Real Analysis I was $\hat{r} \doteq 0.796, n = 31$.

VI. Summary and Conclusion

In sum, the author described using a modified Moore method (MMM) to teach a Bridge to Higher Mathematics (BHM) course and described the material in the course to assist students transitioning from Calculus to the upper-division mathematics course-work, outlined some of the strategies employed, discussed the syllabi, policies, and ancillary materials used. Perhaps the most important part of this modified Moore method is the caution that one should remain flexible, attempt to be moderate in tone and attitude, be willing to adjust dependent upon the conditions of the class, and not be doctrinaire about methods of teaching. It is my belief that this method maximises educational opportunity for the most students by attempting to teach to as heterogeneous a group as possible. For each individual instructor, the method employed should be that which is most comfortable for him and connects with the students.

I opine that this pseudo-Socratic method should be considered by more instructors of mathematics. I deem this because many of the students taught in this method have gone on to graduate school or entered the work-force and have communicated with me that they felt that the BHM course taught in this manner (or other courses taught in the manner) was the most educationally meaningful for them. Whilst a student myself over the course of many years, I was exposed to each of the methods discussed in this paper (traditional, Moore, and constructivist) and aspects of those methods are a part of my modified Moore method because I found that each had its strengths and weaknesses. Thus, I attempted to create a method that, hopefully, included the best of each and discarded to worst of each. I can honestly say that I moderately succeeded in almost every class taught with the Moore, traditional, or constructivist methods. That which I learnt the best was that which I *did* myself, rather than be told about, lectured to, or even read about. I must *do* in order to *understand*. That I can not explain something does not mean it does not exist, it simply means that I do not know it (at this point or perhaps it is never knowable).

The MMM seeks to minimise the amount of lectures, but allows for students to read from multiple sources and converse (after presentations). It acknowledges that learning is a never-ending process rather than a commodity or entity that can be given like the metaphor of an instructor cracking open the head of a student then pouring the knowledge into said head. In that regard it is very much reminiscent of reform methods and the philosophy of John Dewey. Dewey stated, "the traditional scheme is, in essence, one of imposition from above and from outside,"[42] and "understanding, like apprehension, is never final."[43]

The queries contain open questions from the perspective of the students (and perhaps the instructor) without indication as to whether they are true or false under the axioms assumed. But, unlike the Moore method, necessary lemmas or sufficient corollaries *are* oft included; thus, affording the students a path to construct their arguments.

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[42] John Dewey, *Experience and Education* (New York: Macmillan, 1938), page 18.

[43] John Dewey, *Logic: The Theory of Inquiry* (New York: Holt and Company, 1938), page 154.

P. J. Halmos recalled a conversation with R. L. Moore where Moore quoted a Chinese proverb. That proverb provides a summation of the justification of the MMM employed in teaching the transition sequence. It states, "I see, I forget; I hear, I remember; I *do*, I *understand*."

It is in that spirit that a core point of the argument presented in the paper is that a BHM mathematics course is useful, it is worthy of being a part of the curriculum, it should be centred on *set theory*, the method of teaching the course should be carefully considered (I recommend a modified Moore method, of course) for the nature of the course and material in the course seems to lend itself to a particular pedagogy. Also, the method of teaching the course should be carefully considered because in order to have an educationally meaningful experience for the students and in order to properly transition the student from an elementary understanding to a more refined understanding of mathematics, every effort should be made to see that the students reach beyond a mundane, pedestrian understanding of mathematics (in every course, not just this course). An innovation in the pedagogy proposed is that not all questions posed in the courses are answered. Many of the questions posed in the courses are left for the student to ponder during his matriculation and answer at a later date. Examples of proofs, counterexamples, etc. are given but *most* of the actual work is done by the students.

So, this paper proposes a philosophy such that the experience of doing a mathematical argument is reason enough for the exercise; but, the author recognises the practical need for task completion so student completed proofs, counterexamples, examples, counter-arguments, etc. form the framework type of mathematical education proposed herein.

Hence, this paper proposes a pedagogical approach to mathematics education that centres on exploration, discovery, conjecture, hypothesis, thesis, and synthesis such that the experience of doing a mathematical argument, creating a mathematical model, or synthesising ideas is reason enough for the exercise - - - and the joy of mathematics is something that needs to be instilled and encouraged in students by having them *do* proofs, counterexamples, examples, and counter-arguments in a Bridge course to prepare the student for work in advanced mathematics.

Nonetheless, it is not argued that this is <u>the</u> way to teach, for as Halmos asked in [33], "what is teaching?" I do not know; yet I try to do it!

References

[1] Barnier, William & Feldman, Norman. Introduction to Advanced Mathematics. 2^{nd} Ed. Upper Saddle River, NJ: Prentice Hall, 2000.

[2] Chalice, Donald R., "'How to Teach a Class by the Modified Moore Method." *American Mathematical Monthly, 102*, no. 4 (1995) 317 - 321.

[3] Chartrand, Gary, Polimeni, Albert, & Zhang, Ping. *Mathematical Proofs: A Transition to Advanced Mathematics*, Boston, MA: Addison-Wesley, 2003.

[4] Cohen, David W., "A Modified Moore Method for Teaching Undergraduate Mathematics." *American Mathematical Monthly* 89, no. 7 (1982): 473 - 474, 487 - 490.

[5] Conference Board of the Mathematical Sciences, *Statistical Abstract of Undergraduate Programs in the Mathematical Sciences in the United States,* Washington, DC, Mathematical Association of America and the American Mathematical Society, 1995.

[6] Conference Board of the Mathematical Sciences, *The Mathematical Education of Teachers,* Washington, DC: Mathematical Association of America, 2001.

[7] Committee on the Undergraduate Program in Mathematics, *Pre-graduate Preparation of Research Mathematicians*, Washington, DC, Mathematical Association of America,1963.

[8] Committee on the Undergraduate Program in Mathematics, *A General Curriculum in Mathematics for College*, Washington, DC, Mathematical Association of America,1965.

[9] Committee on the Undergraduate Program in Mathematics, *Reshaping College Mathematics*, Washington, DC, Mathematical Association of America, 1989.

[10] Committee on the Undergraduate Program in Mathematics, *The Undergraduate Major in the Mathematical Sciences*, Washington, DC, Mathematical Association of America,1991.

[11] Committee on the Undergraduate Program in Mathematics, *CUPM Discussion Papers about the Mathematical Sciences in 2010: What Should Students Know?)*, Washington, DC, Mathematical Association of America, 2001.

[12] Committee on the Undergraduate Program in Mathematics, *Guidelines for Programs and Departments in Undergraduate Mathematical Sciences (Working Paper)*,Washington, DC, Mathematical Association of America, 2001.

[13] Committee on the Undergraduate Program in Mathematics, *CUPM Interim Reports: Toward a Working Draft Curriculum Guide (Working Paper)*, Washington, DC,Mathematical Association of America, 2001.

[14] Committee on the Undergraduate Program in Mathematics, *Undergraduate Programs and Courses in the Mathematical Sciences: CUPM Curriculum Guide 2004*, Washington, DC,Mathematical Association of America, 2004.

[15] Devlin, Keith, *Sets, Functions, and Logic*, 3^{rd} Ed. Boca Raton, FL: Chapman & Hall, 2004.

[16] Dewey, John, *Democracy and Education*. New York: Macmillan, 1916.

[17] Dewey, John, *Experience and Education*. New York: Macmillan, 1938.

[18] Dewey, John, *Logic: The Theory of Inquiry.* New York: Holt, 1938.

[19] Duren, LowellR., "An Adaptation of the Moore Method to the Teaching of Undergraduate Real Analysis - - A Case Study Report." Ph.D. dissertation, The Ohio State University, 1970.

[20] Enderton, Herbert, *Elements of Set Theory*. London, New York: Academic Press, 1977.

[21] Fine, Nathan. *An Introduction to Modern Mathematics*. Chicago, IL: Rand McNally, 1962.

[22] Fitzpatrick, Benjamin, Jr., "The Teaching Methods of R. L. Moore." *Higher Mathematics* 1 (1985): 41 - 45.

[23] Forbes, D. R., "The Texas System: R. L. Moore's Original Edition." Ph.D. dissertation, University of Wisconsin, Madison, 1971.

[24] Foster, James A., Barnett, Michael, & Van Houten, Kare, "(In)formal methods: Teaching Program Derivation via the Moore Method." *Computer Science Education*,6, no. 1 (1995): 67 - 91.

[25] Frantz, J. B., The Moore Method. In *Forty Acre Follies*, 111 - 122. Dallas, TX:Texas Monthly Press, 1983.

[26] Garnier, Rowan & Taylor, John. 100% Mathamtical Proof. Chinchester, England, UK: Wiley, 1996.

[27] Gersting, Judith L. and Kuczkowski, Joseph E., "Why and How to Use Small Groups in the Mathematics Classroom." *College Mathematics Journal* 8, no. 2 (1977): 270 - 274.

[28] Gold B., Marion W. & S. Keith. *Assessment Practices in Undergraduate Mathematics.* MAA Notes, 49. Washington, DC: Mathematical Association of America, 1999.

[29] Goldrei, Derek, *Classic Set Theory*. London, England, UK: Chapman & Hall, 1996.
[30] Hale, Margie. Essentials of Mathematics. Washington, DC: MAA, 2003.
[31] Halmos, Paul R., How To Teach. In *I Want To Be A Mathematician.* New York:Springer-Verlag, 1985.
[32] Halmos, Paul R., *Naïve Set Theory,* Princeton, NJ: D. Van Nostrand, 1960.
[33] Halmos, Paul R., "What Is Teaching?" *American Mathematical Monthly* 101, no. 9 (1994): 848 - 854.
[34] Halmos, Paul R., Moise, Edwin E. and Piranian, George, "The Problem of Learning to Teach." *American Mathematical Monthly* 82, no. 5 (1975) 466 - 476.
[35] Hu, Sze-Tsen, *Introduction to Contemporary Mathematics*. San Francisco: Holden-Day, 1966.
[36] Hummel, Kenneth, *Introductory Concepts for Abstract Mathematics*. Boca Raton, FL: Chapman & Hall, 2000.
[37] Jones, F. B. "The Moore Method." *American Mathematical Monthly* 84, no. 4 (1977): 273 - 278.
[38] Kaplansky, Irving. *Set Theory and Metric Spaces*,2^{nd} Ed. New York: Chelsea Publishing, 1977.
[39] Kline, William, Oesterle, Robert, & Willson, Leroy.*Foundations of Advanced Mathematics*, Lexington, MA: D. C. Heath, 1975.
[40] Krantz, Steven, *The Elements of Advanced Mathematics*, 2^{nd} Ed. Boca Raton, FL: Chapman & Hall, 2002.
[41] Lay, Steven R. Analysis with an Introduction to Proof. 3^{rd} Ed. New Jersey: Prentice Hall, 2000.
[42] Lakatos, Irme, *Proofs and Refutations*. New York: Cambridge University Press, 1976.
[43] McLoughlin, M. P. M. M. "Initiating and continuing undergraduate research in mathematics: Using the fusion method of traditional, Moore, and constructivism to encourage, enhance, and establish undergraduate research in mathematics." Paper presented at the annual meeting of the Mathematical Association of America, Baltimore, Maryland, 2003.
[44] McLoughlin, M. P. M. M. "The fusion method of traditional, Moore, and constructivism and the incorporation of undergraduate research through the mathematics curriculum." Paper presented at the annual meeting of the Mathematical Association of America, Baltimore, Maryland, 2003.
[45] McLoughlin, M. P. M. M. "On the nature of mathematical thought and inquiry: A prelusive suggestion." Paper presented at the annual meeting of the Mathematical Association of America, Phoenix, Arizona, 2004.
[46] Moise, Edwin E. "Activity and Motivation in Mathematics." *American Mathematical Monthly* 72, no. 4 (1965): 407 - 412.
[47] Murtha, J. A. "The Apprentice System for Math Methods." *American Mathematical Monthly* 84, no. 6 (1977): 473 - 476.
[48] National Council of Teachers of Mathematics. *Curriculum and Evaluation Standards for School Mathematics* (1989). Reston, VA: NCTM.
[49] National Science Board Committee on Undergraduate Science and Engineering Education, *Undergraduate Science, Mathematics, and Engineering Education*, Washington, DC, National Science Board, 1986.
[50] O'Shea, D. & H. Pollatsek, *Do We Need Prerequisites*? Notices of the American Mathematical Society, 1997.
[51] Page, Warren "A Small Group Strategy for Enhancing Learning." *American Mathematical Monthly* 86, no. 9 (1979): 856 - 858.
[52] Polya, George *How to Solve It.* New York: Doubleday, 1957.
[53] Schumacher, Carol, *Chapter Zero* Reading, MA: Addison-Wesley, 1996.
[54] Smith, Douglas, Eggen, Maurice, & St. Andre, Richard. *A Transition to Advanced Mathematics*, 4^{th} Ed. Pacific Grove, CA: Brooks-Cole, 1997.
[55] Solow, Daniel, *How to Read and Do Proofs* 3^{rd} Ed. New York: Wiley, 2002.
[56] Steffe, Leslie and Kieren, Thomas, "Radical Constructivism and Mathematics Education," *Journal for Research in Mathematics Education* 25, no. 6 (1994): 711 – 733.
[57] Suppes, Patrick, *Axiomatic Set Theory*. New York: Dover, 1972.
[58] Tucker, Adam, *Models That Work: Case Studies in Effective Undergraduate Mathematics Programs.* MAA Notes, 38. Washington, DC: Mathematical Association of America, 1995.
[59] Vaught, Robert, *Set Theory: An Introduction*, 2^{nd} Ed. Boston, MA: Birkhäuser, 1995.

[60] Velleman, Daniel, *How to Prove It: A Structured Approach* New York: Camridge University Press, 1994.

[61] Weissglagg, Julian “Small Groups: An Alternative to the Lecture Method.” *College Mathematics Journal* 7, no. 1 (1976): 15 - 20.

[62] Wilder, R. L. “The Nature of Mathematical Proof.” *American Mathematical Monthly* 51, no. 6 (1944): 309 - 325.

[63] Wilder, R. L. “The Role of The Axiomatic Method.” *American Mathematical Monthly* 74, no. 2 (1967): 115 - 127.

[64] Wilder, R. L. Axiomatics and the Development of Creative Talents. In *The Axiomatic Method with Special Reference to Geometry and Physics*, edited by L.Henkin, P. Suppes, and A. Tarski. Amsterdam: North - Holland, 1976.

[65] Wilder, R. L. “Robert Lee Moore, 1882 - 1974.” *Bulletin of the American Mathematical Society* 82, no. 3 (1976): 417 - 427.

[66] Wilder, R. L. The Mathematical Work of R. L. Moore: Its Background, Nature, and Influence. In *A Century of Mathematics in America: Part III*, edited by Peter Duren. Providence, RI: American Mathematical Society, 1989.

[67] Whyburn, Lucille S. “Student Oriented Teaching - The Moore Method.” *American Mathematical Monthly* 77, no. 4 (1970): 351 - 359.