UNDERGRADUATE STUDENTS' MENTAL OPERATIONS IN SYSTEMS OF DIFFERENTIAL EQUATIONS

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This paper reports on research conducted to understand undergraduate students' ways of reasoning about systems of differential equations (SDEs). As part of a semester long classroom teaching experiment in a first course in differential equations, we conducted task-based interviews with six students after their study of first order differential equations and prior to instruction on SDEs in order to obtain baseline data on their conceptual resources for SDEs. Interpretative analysis of the interview data generated the following three themes pertaining to student reasoning in situations dealing with SDEs. First, students used their conception of rate as a reasoning tool. Second, students used quantification as a mental operation, and third, students enacted what we call a function-variable scheme in their efforts.

INTRODUCTION

Undergraduate mathematics education research is a growing area of interest and current areas of concern include, but are not limited to, calculus, proof, abstract algebra, number theory and differential equations. Research in differential equations has primarily focused on student understandings of and difficulties with single differential equations (e.g., Rasmussen, 2000; Artigue, 1992; Habre, 2000; Zandieh & McDonald, 1999) while much less research has focused on students' understandings of SDEs (for some brief reports in this area see Rasmussen, 2000 & Trigueros, 2000). The purpose of this paper is to begin to fill this gap by offering a theoretical account of students' mental resources and ways of reasoning when thinking about and solving SDEs.

THEORETICAL BACKGROUND

Since differential equations are expressions of rate, we build upon previous work regarding student conceptions of rate in our analysis of student thinking about SDEs. Piaget (1970) documented that young children's concept of speed exists before their concept of time. According to Piaget, children initially understand speed in a relational sense. That is, if one person "passes" another then the person passing has a greater speed. Children first develop a concept of speed as a "quantified motion" and later conceptualize speed as the coordination of distance and time in proportion to each other. Thompson (1994a) extends Piaget's seminal research to older children's understanding of rate and ratio, finding that students initially understand rate (speed) as a single quantity that is a value attached to motion and not as a ratio of distance and time. Developing a conceptual understanding of rate involves integrating distance and time together as a ratio. Complementarily, Confrey and Smith (1994) suggested that rate is a unit per unit comparison and focus on the idea of "per" and the mental construction of units for both distance and time.

Thompson & Thompson (1994) presented levels that they suggest are needed to construct a sophisticated understanding of speed as rate. To summarize this framework, students initially understand speed as a unit of motion. For example, going 25 mph means going

25 miles in one hour (a unit they call speed length). For students, this conception evolves into an understanding of the relationship of distance, speed, and time. This relationship can be briefly explained in the following way: as time changes, the distance changes proportionally, the ratio □d/□t remains constant, and that ratio is the speed (rate).

More recently, Carlson, Jacobs, Coe, Larsen, and Hsu (2002) suggested that we might understand students' progression of rate and covariational reasoning in terms of progressively more complex mental actions. These researchers suggested five levels of mental actions as students mature in their understanding of covariational reasoning. To give an example of this framework, the first level is identified when a student understands there is a relationship between two variables. Level 5 is identified when students understand instantaneous rate in terms of instantaneous values for a function.

In their work, Rasmussen & Whitehead (2002) posit that an interconnected but somewhat hierarchical set of stages may also frame students' understanding and use of rate in differential equations. These stages depict students' reasoning as it becomes increasingly more sophisticated. The stages range from using rate as a ratio of two discrete values to dynamically using rate as a function to infer changes to the structure of the space of solutions for single differential equations. All three of the above frameworks offer ways to think about students' understanding that we can build on in SDEs.

Adapting Piaget's notions of image, mental operations, and schema to rate, Thompson (1994a) aimed at "capturing the multiple reconstitutions that take place in individuals as they progress toward the construction of mathematical objects such as ratio and rate" (p. 180). In this report we aim to capture and describe analogous progress students make as they construct increasingly sophisticated understandings in SDEs and in our analysis we make use of these same constructs. According to Thompson (1994b), an image is "the kind of knowledge that enables one to walk into a room full of old friends and expect to know how events will unfold" (p. 125). It is more than a mental picture; it is a dynamic construct originating in a student's experience, both mathematically and otherwise. A mental operation, on the other hand, "is a system of coordinated actions that can be implemented symbolically, independent of images in which the operation's actions originated" (Thompson, 1994a, p. 182). For example, when a student puts two values in order, she uses a mental operation. A scheme is an organization of actions that is repeatable and generalizable (Piaget, 1970). Finally, images develop through students' intentions to meet goals at any given time with the mental resources, operations, and schema they enact in that situation, but mental operations and schema exist within images as well. The images of rate and solution that students develop when learning SDEs can be framed using these constructs.

METHODOLOGY

We conducted videotaped, task-based, semi-structured individual interviews with six students as part of a semester long classroom teaching experiment (Cobb, 2000) in differential equations for engineers at a midsize university in the United States. The theoretical purposes of the teaching experiment were to characterize both individual and collective reasoning about differential equations (from a dynamical systems point of view) and to contribute to the theoretical body of knowledge about undergraduate students' mathematical thinking as it evolves in the complexity of classroom life.

Instruction sought to develop a participation structure (Erickson, 1986) where students routinely explained their thinking, provided reasons for their conclusions and interpretations, and attempted to make sense of other students' ideas and approaches (Yackel, Rasmussen, & King, 2000). As a result, students approached the interview sessions with an appreciation that the research team was genuinely interested in understanding their thinking and reasoning.

In both the day to day and retrospective analysis of classroom events, we used the interpretive framework developed by Cobb and Yackel (1996) that strives to coordinate psychological and sociological perspectives. In the analysis reported here, we foreground a psychological perspective while a sociological point of view remains in the background. Analysis was informed by a grounded theory approach as described by Glaser and Strauss (1967) and proceeded in the following manner: Transcriptions of the interviews were summarized per student per task. Analyses of these summaries were discussed with both authors in order to develop a shared sense of interpretation and to minimize inappropriate interpretations. We often returned to the original data as we compared interpretations and sought confirming and/or disconfirming evidence for conjectures. Systematic review of these summaries then led to the identification of three themes that cut across most all of the problems dealing with SDEs.

RESULTS

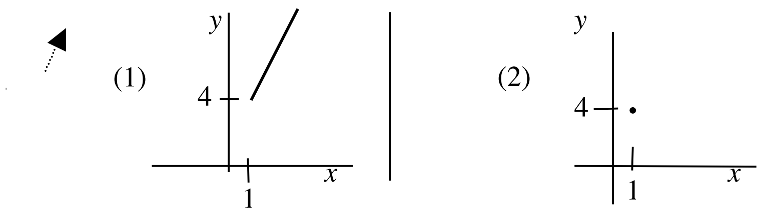
The three interview tasks dealing with SDEs were designed to afford us insight into ways of reasoning that were available to students before instruction on SDEs. Students had not solved any problems like these in the past, but they had completed approximately seven weeks of instruction dealing with single differential equations. We next review the three tasks, provide telegraphic results of student responses, and then discuss our analysis and interpretation in terms of three cross cutting themes.

TASK 1: In this task, we look at systems of rate of change equations designed to inform us about the future populations for two species that are either competitive (that is, both species are harmed by interaction), or cooperative (that is, both species benefit from interaction, for example bees and flowers). Which system of rate of change equations describes competing species and which system describes cooperative species? Explain your reasoning.

$$\text{(A)}\quad \begin{array}{l} dx/dt = -5x + 2xy \\ dy/dt = -4y + 3xy \end{array} \qquad \text{(B)}\quad \begin{array}{l} dx/dt = 3x - 2xy \\ dy/dt = y - 4xy \end{array}$$

TELEGRAPHIC RESULTS: All students were able to correctly predict that system A was cooperating and B competitive. A typical strategy used the fact that the second term in each DE in (A) would cause the rate of change of one variable to increase when the other variable increased and the second term in each DE in (B) would cause the rate of change of one variable to decrease when the other variable increased. Another strategy used involved considering what would happen if x or y was 0. Students also used their understanding of the situation to reason about the rate quantities, using specific values to reason about the behavior of the size of the populations. For two students, however, we could not tell if they were thinking about a rate of change quantity or about an x or y quantity.

TASK 2: Imagine the following: You are up in a hot air balloon looking down at a skateboarder. He has a piece of chalk on the bottom of his board that is drawing a line on the ground as he moves. Here is what you see on two different rides.

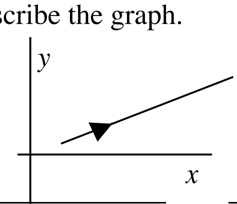


Sketch possible x(t) and y(t) graphs for each ride.

Follow-up A: Students were given concave up graphs for x-t and y-t and asked to trace the line in ride (1) that would produce the given x-t and y-t graphs.
Follow-up B: Repeat for a concave up graph for x-t and a concave down graph for y-t.

TELEGRAPHIC RESULTS: All students in the interviews sketched appropriate x-t and y-t graphs for both of the skateboard rides. For (1), they sketched linear graphs; their approach involved imagining the skateboarder's trace as it developed over time and then sketching the horizontal and vertical displacement of the line on the x-t and y-t axes. Two students also suggested concave up and increasing x-t and y-t graphs if the skateboarder speed was not constant. For (2), all students sketched horizontal lines for the x-t and y-t graphs. The typical reasoning students used to justify their graphs was that time "goes on" and the skateboard does not move, so a horizontal lines are the best representation. In Follow-up A, four students discussed the increasing speed of the skateboarder, using the changing slope as justification. Two also reasoned that two concave down graphs would also be acceptable and explained that the skateboarder would be slowing down. In Follow-up B, four students claimed it would not be possible to produce such graphs while two said that it would be possible.

TASK 3: The following is a sketch of x vs. y that is produced over time. Look at each of the following suggested systems of differential equations and decide if any of the choices could describe the graph.



(1) $\frac{dx}{dt} = x + y$, $\frac{dy}{dt} = \square x + y$ (2) $\frac{dx}{dt} = x + y$, $\frac{dy}{dt} = x + y$ (3) $\frac{dx}{dt} = 2x \square 2y$, $\frac{dy}{dt} = x \square y$

TELEGRAPHIC RESULTS: Four of the students correctly identified (3) as a possibility by imagining a parameterization of x and y as functions of t and using their image of the graph to predict what the rate of change equations could be. A typical strategy involved the fact that the line has a slope less than one and so x would be greater than y, and since the line has positive slope, both x and y would be increasing. They coordinated x and y as dynamic quantities (as variables and functions). One student showed surprise when he realized that the one x-y graph could represent both rate of change equations and then went on to correctly solve the problem. The other two students made no progress.

DISCUSSION AND ANALYSIS

We developed the following three themes to characterize students' mathematical understandings of systems of differential equations: rate use, quantification as a mental operation, and function-variable scheme. These themes are intended to be complementary rather than disjoint. For example, we often saw students using the mental operation of quantification to construct a rate quantity, which they then used to reason about the values of the variables/functions x and y. This overlapping or simultaneity of themes occurred frequently, but for discussion purposes we illustrate and clarify each theme separately.

Rate Use. Other researchers have offered ways to describe how students' understanding of rate develops. We build on and extend this work to detail students' use of rate as a tool for building other mathematical images such as population predictions (Task 1) and function (Tasks 1, 2 & 3). Several of the interviewed students used their understanding of rate to reason and reach conclusions. For example, they typically used the idea of positive and negative rates as being increasing and decreasing, which connects with Carlson et al.'s (2002) framework for mental action (level 4/5). They also used rate as a ratio of displacement and time (Thompson & Thompson, 1994), discussing changes in x and y compared to a change in t by coordinating the values discretely (Task 2). Finally, they used rate to make predictions about how functions change over time in Task 1,2, and 3.

As an illustration, consider the following transcript with one of the students, Arthur, who used his understanding of rate to reason that the given graphs in Follow–up B to problem 2 were not possible.

> Arthur: I can't do this. Um. Um. As time elapses, it goes faster in the y direction. And it slows down in the x direction. On this one [indicates problem 2 Follow-up B] I don't know how to move my pen. Because, on that one, [indicates problem 2 Follow-up A], they were both, they both increased. On this one, they both looked about the same. On this one, um, [Pause] I'm lost.
>
> Karen: OK. Could you say what's bothering you about it?
>
> Arthur: Well. This one's increasing and this one's [Refers problem 2 Follow-up B]. This one's this one's increasing, too. But the rate of the rate of this one is increasing. The rate of increase on this one is decreasing. So, I've got a problem ...

Arthur used rate as a tool to reason about the possible x-t and y-t graphs in this problem. In the last statement, he says that the rate is increasing for the x-t graph and decreasing for the y-t graph. This is a conflict for him because he looks at the original skateboarder's graph in the x-y plane and reasons that the rate (slope) of it requires that the x-t and y-t graphs need to have similar rates (note his first statement) for the constant slope in the x-y plane to be maintained. Rate is then a tool for him, and slope, rate, and speed all integrate together to be used in reasoning about the skateboarder's rate in the parameterized curves representing movement in the horizontal and vertical direction.

For us, rate use means that the students have an image of rate that provides them powerful ways to reason in tasks involving motion and/or change, as exemplified in Arthur's excerpt. Situated within a problem setting, students' images involve a conception of rate as a quantity that determines the dynamic behavior of a function, such as if it is increasing and decreasing, and rate as a comparison of quantities as they change dynamically.

Quantification as a mental operation. Thompson (1994a) posits that quantification is an important mental operation and our analysis finds support for this position. He defines a quantitative operation as "a mental operation by which one conceives a new quantity in relation to one or more conceived quantities" (p. 185). An example of a quantitative operation is combining two quantities additively. Quantification begins with a mental action and creates a new quantity (not a specific number, but an image of a value) by operating on original quantities. In each of the tasks, quantification was an important part of students' sense making and reasoning.

For example, on Task 1, x and y represent the populations of two different species. Students mentally operated on the two different population quantities to conceive a new quantity: the rate of change of each of the populations. By mentally acting on the notion of "how big is the population," students conceived of rate as related to quantified values for population. Most of them did this by assigning values to x and y and reasoning with the results. The following is an example from Jere's response on Task 1.

Jere: This one here would be cooperating because they're both helping to the, the general equation, they're both adding to it. This one, they're both, I guess, pulling away from the, from the function. From the differential equation. And, only one is putting in.

Karen: I'm just trying to understand your, your thinking. When you say both are putting in.

Jere: Yeah. Both species. If you have bees and flowers, then, you know, your bees are helping your flowers.

Karen: Oh. So. When you say add to the equation can you say a bit more about?

Jere: They're increasing. Here, you only one species that is negative. So. If you would add. Oh. For a cooperative species, you'd want a positive rate of change. Right? You'd wantthem to help each other… and a competing species, one of them, one of the, one of these equations over time is going to drop, and one is going to go up.

Jere was mentally operating on the x and y quantities to create new quantities (rates of change of x and y). For example, he said, "they're both adding to it…the differential equation." Rate use then affords him a means to mentally act on these DEs to expand and quantify the two systems in relationship to each other and to the actual situation.

In Tasks 2 and 3, students also used quantification as a mental operation. They first quantified the rate of change in x and y that is present in (A) (in the x-y plane) and coordinated the variation of x, y and t (time). They operated on the graph to conceptualize tri-variation of the three quantities and produced two new quantities, the co-variation relationship of x vs. t and y vs. t. Then they moved to representing these as graphs of x-t and y-t that displayed similar co-variation with respect to time.

The mental operation of quantification in these students' mathematical reasoning is complex. The quantification of rate from the variables x and y is intertwined with the quantification of the functions x(t) and y(t) using rate and both are grounded in the students' understanding of the situations portrayed in the tasks.

Function-variable scheme. In earlier studies of single differential equations, Rasmussen (2000) found that one of the most difficult images students need for schemes of mental operation is that x and y need to simultaneously be both variable and function for a robust conception of differential equations and solutions to differential equations. Students who have developed ways to think about these "things" as both variables and functions then create a mental scheme that puts together images and understandings from both function and variable to create new ways to reason. Students who were able to reason appropriately about the third question showed that they might be at different depths of development for this scheme. In fact, three students showed a particularly strong understanding of this function-variable relationship. We illustrate this third theme with a portion of transcript from Jeff's reasoning on Task 3.

Jeff: The change in y is positive. The change in x is positive. Both increasing. For all t that we see, it's moving slightly faster in the x direction that it is in the y direction. They look almost equal but not quite. [Pause] So. OK. dx/dt is greater than dy/dt. And dx/dt and dy/dt are both greater than zero. They're both positive. That's positive x. That's a positive change in x. This says x is greater than y. Which means this is a negative value. So, I would say that this [system (1)] is invalid... [pause]

Karen: Tell me what you're thinking.

Jeff: Although initially they could be equal, my value of x is greater than my value of y. This is greater than this y. Which means that this is a negative slope for y. Which it is not.

Jeff: This [system (2)] is telling me that change in x is equal to the change in y. Seems close but that's not quite right. That would be a little bit steeper. [About system (3)] So 2x minus 2y is going to be positive because x is greater than y. So this is positive. And x still greater than y. So, this is positive. But this is still greater than this. So, I would have to go with choice three.

We were actually quite surprised at the depth of reasoning that Jeff used before having ever studied SDEs. Jeff was able to reason about this because, in our view, he conceptualized x and y as both function and variable operating simultaneously in this situation. When he needed to see x and y as variables to substitute numerical values for and use to determine something about dx/dt and dy/dt, he did so. When he needed to see them as functions of time that are changing continuously and operating "underneath" the situation, he did that. He seemed to move effortlessly between the two mathematical objects enacting a scheme that was easy for him to use and us to observe. His mental operations coordinated into a robust scheme to reason in this situation. Other students also showed development of this mental scheme, although with different depths of understanding. We posit that this function-variable scheme developed for our students in their study of first order differential equations and served them well in reasoning about mathematical ideas in SDEs.

CONCLUDING REMARKS

This research offers a snapshot of students' reasoning and sense making about SDEs prior to formal instruction. Its significance lies in the fact that it builds on theoretical constructs developed for younger children, laying a foundation for understanding student thinking about SDEs. In addition, although not discussed here, this theoretical analysis is proving to be pragmatically useful in our efforts to refine and revise instructional materials. We are currently analyzing data from the classroom teaching experiment (including classroom videorecordings and end of the semester interviews) as students' progress in their study of SDEs. We anticipate that the theoretical ideas developed here will be useful in this retrospective analysis of student thinking and instructional design and may extend to other areas as well.

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