LOGICO-MATHEMATICAL ACTIVITY VERSUS EMPIRICAL ACTIVITY: EXAMINING A PEDAGOGICAL DISTINCTION

Martin A. Simon

The Pennsylvania State University

I present a theoretical distinction that may prove useful in conceptualizing mathematics teacher education (and graduate education) and research on mathematics teacher education. Further, the distinction can contribute to developing frameworks on the design of mathematics curricula. The distinction between empirical activity and logico-mathematical activity focuses on the nature of a mathematical concept and how that concept develops, key issues in the quest to teach mathematics for understanding.

A primary goal of the mathematics education reform in North America during the last 15 years has been to promote students' learning of mathematics with understanding. This goal is in response to a widespread perception that too many mathematics students learn mathematics as a collection of disconnected and meaningless (to the learner) facts and procedures. This reform effort has been fueled by and has continued to require re-conceptualization of the nature of mathematics, what it means to do mathematics in school, how mathematical concepts are learned, and how mathematical concepts can be taught. In this article, I explicate a pedagogical distinction that could prove useful in conceptualizing the design of mathematics lessons and the education of mathematics educators. The theoretical distinction presented is grounded in a Piagetian empirical framework. Examples of data and author-generated lessons provide the basis for examining this distinction.

Over the last 6 years, my colleagues and I have been engaged in a research project, the Mathematics Teacher Development (MTD) Project[i]. The purpose of the project has been to understand the mathematical and pedagogical development of K-6 teachers (inservice and preservice) as they participated in a comprehensive reform-oriented teacher education program. This research has resulted in a set of distinctions about the pedagogical thinking that underlies the practice of teachers participating in the reform (cf., (Heinz, 2000; Simon, 2000; Tzur, Simon, Heinz, & Kinzel, 2001). In this article, I explore another distinction, deriving in part from the MTD research, that involves conceptualization of the nature of mathematical concepts, what it means to do mathematics in school and how mathematical concepts are learned.

One characteristic of classrooms and curricula guided by participation in the reform is an emphasis on students' active involvement in the development of new (to them) mathematical ideas. Different modes of active involvement have often been articulated (e.g., problem solving, looking for patterns, representing, explaining, justifying, finding counter examples). In the two lessons that follow, the first from MTD data and the second from one of the recent NSF-supported curricula, a similar lesson structure is used that makes use of pattern recognition. After describing these lessons, I will make a case for what I consider to be problematic aspects of the pedagogical conceptions underlying these lessons. I will then exemplify and briefly describe a contrasting framework for conceptualizing mathematics concept development and lesson design.

IVY'S LESSON ON AREA OF TRIANGLES

The MTD data that I describe in this section were included in a detailed analysis of Ivy's practice (Heinz, 2000). That analysis focused on the underlying structure of Ivy's practice. Subsequent observations, including situations that were not part of the MTD project, have led to a re-examination of these data and articulation of a new distinction.

Ivy, a sixth grade teacher (students age 11 years), was in her sixth year of teaching when she designed and taught this lesson on the area of triangles.

Ivy wanted her students to

> find the formula . . . I really believe that they forget what we just tell them and that they will remember what they figured out. And if they don't remember it, they can figure it out again and maybe faster the next time.
> . . . I want them to understand it.

Mathematical relationships that Ivy was aware of were the basis for her lesson design.

> We are building off those right triangle ideas because that is where the formula builds from, which is actually from rectangles. So I am trying to take them from rectangles to right triangles to non-right triangles to see how it is all related to the rectangle itself.

Following is an outline of Ivy's lesson:

1. Ivy led a review of how to find the area of a rectangle on a geoboard.
2. Students worked in small groups to find the area of a 2x3 right triangle.
3. The whole class discussed their strategies and results for step #2.
4. Students worked in small groups to find the areas of all of the right triangles they could make on their geoboards and recorded the measures of the base, height, and area for each triangle.
5. Students shared their data from step #4 with the whole class while Ivy recorded the information in a 3-column table
6. Students examined the table to come up with a formula.

Ivy's instructions for step #6 were:

> Look at how these numbers are in this chart with our areas . . . and see if you can figure out a pattern that you can use every time using the numbers [measures of base and height] to come up with the area. . . . There is something that you can do to these [measures of] the bases and the heights to get the area.

A PUBLISHED LESSON ON EQUIVALENT FRACTIONS

In the United States, mathematics educators often consider the state of the art in reform-based mathematics education curricula to be represented by recent National Science Foundation supported curricula.[ii] It is my experience that the lessons within each curriculum, although generally superior to those found in preexisting curricula, are uneven in quality. One explanation for this phenomenon might be the multiple authors involved in writing each of the curricula. However, I would argue that a more important reason is the lack of or inadequacy of explicit frameworks for guiding lesson design. This latter point suggests work to be done in mathematics education. The pedagogical distinction that I explicate in the next section may prove useful in curricular design efforts.

I include the first 7 steps of the "At a Glance" (*Math trailblazers: A mathematical journey using science and language arts (K-5)*, 1999) that summarizes the lesson on equivalent fractions.

1. Ask students to use their fraction chart from Lesson 3 to find all of the fractions that are equivalent to 1/2. List these on the board or overhead.
2. Ask students to compare the numerators and the denominators of the equivalent fractions in order to look for patterns.
3. Ask students to suggest other fractions that are equivalent to 1/2.
4. Write number sentences on the board or overhead showing the equivalencies.
5. Students look for patterns in the number sentences.
6. Students use the patterns (multiplying or dividing the numerator and the denominator by the same number) to find fractions equivalent to 3/4, 1/3, and 2/5.
7. Students use the patterns to complete number sentences involving equivalent fractions.

ANALYSIS OF THE TWO LESSONS AND DEVELOPMENT OF DISTINCTIONS

The two lessons, just described, have similar goals and structure. The goals involve the generation of a computational strategy (generalization) or formula with "understanding."[iii] The structure involves generating a set of examples, finding the numerical pattern (relationship) among the parts of the examples, and establishing that pattern as a generalization for computing the missing number in further examples.

Lessons of this type, if criticized, are generally criticized on the basis of issues of justification. That is, although examining a set of examples to find a pattern is appropriate for generating a conjecture, it does not constitute mathematical proof that the relationships involved are true for all cases of the type being considered. There remains a need for deductive justification. This is an important mathematical issue, but not the one that I focus on here.

Let us consider what students might learn from these lessons. In Ivy's lesson, students are likely to learn that there is a fixed relationship among the base, height, and area of a triangle and that it can be represented as $A=bh/2$. Similarly, in the lesson on equivalent fractions, students might learn that there is a numerical relationship among equivalent fractions. To produce an equivalent fraction, one can multiply the numerator and denominator by the same number (not zero and not necessarily an integer). Is this what we mean by "understanding" in mathematics? I argue that it is not.

Understanding is a broad term, and a single definition is unlikely to capture all significant meanings (cf., Piaget, 2001; Sierpinska, 1994; Simon, 2002). However, for the purpose of analysis and contrast with the lessons described above, I offer the following characterization of understanding. *Mathematical understanding is a learned anticipation of the logical necessity of a particular pattern or relationship(s)*[iv].

In the lesson on area of triangles, the reader might see how the work with the geoboard could result for *some* students in an understanding of the logical necessity of the relationship among the base, height, and area of a triangle. However, the derivation of the formula from the numerical pattern structures the lesson towards learning *that* the formula is appropriate as opposed to *why*. Likewise, the lesson on equivalent fractions does not foster an anticipation of the logical necessity of the patterns found. In the next section, I will discuss how a lesson that begins as Ivy's lesson did could be designed to foster anticipation of the logical necessity.

PROMOTING ANTICIPATION OF LOGICAL NECESSITY

Contrasting Lessons

In this section, I present lessons on the same two topics. These lessons are meant to provide a useful juxtaposition, allowing examination of the underlying pedagogical constructs and how these constructs are related to the development of mathematical understanding (as I defined it above). The lessons that follow are not necessarily appropriate for any particular group of students.

<u>Area of a triangle.</u> For brevity and because it is sufficient for my purpose, I will describe a lesson that develops only a generalization for the area of a right triangle. The lesson begins in a similar way and is based on the assumption that students understand the relationship between the area of a rectangle and the measures of its sides.

1. Students are asked to find the area of particular right triangles using geoboards and to justify their approach. (It is anticipated that students add rubber bands to make the right triangle into a rectangle.)
2. Students are given a ruler, asked to find the area of right triangles that have been drawn on plain paper, and asked to justify their approach. The drawings involve right triangles whose legs are not parallel to the sides of the paper. This is meant to preempt overgeneralization that could result from work with the geoboard. (It is anticipated that students will draw two sides to complete a rectangle. Some students may have already abstracted the relationship from step 1.)
3. Students are asked to anticipate (without drawing) what they would do with a triangle of side measures 3. 4, and 5 units to find the area and what the area would be? Likewise for a triangle of side measures 5. 12, and 13 units. (It is anticipated that students will think about drawing a rectangle and then consider how the side measures of the triangle would give them information about the size of the rectangle).
4. Students are asked to write a generalization for how to calculate the area of a right triangle given the measures of the sides.

<u>Equivalent fractions.</u>

Again for brevity and because it is sufficient for my purpose, I will describe a lesson that develops only a part of the concept involved. This lesson promotes a generalization for making equivalent fractions when the new fraction is expressed in terms of smaller fractional parts (e.g. making 1/2 into 4/8) and for which the new numerator is the unknown. Students are assumed to have an understanding of whole number multiplication and division and knowledge of multiplication/division number facts through 10x10. Further, they are assumed to have a basic understanding of fractions, including representation using area diagrams and the meaning of the numerator and denominator.

1. Students are asked to draw a rectangle with 1/2 shaded. They then are instructed to draw lines on the figure so that the figure is divided into sixths and to determine 1/2 =?/6.
2. Students are asked to draw a rectangle with 2/3 shaded. They then are instructed to draw lines on the figure so that the figure is divided into twelfths and to determine 2/3 =?/12.
3. Students are asked to draw diagrams to determine the following:
    a. 3/4=?/8
    b. 4/5=?/15
    c. 1/4=?/20
4. Drawing diagrams to solve equivalent fractions problems is not much fun when the numbers get large. For the following do not draw a diagram. Rather think about what would happen at each step if you were to draw a diagram. Use that thinking to answer the following:
    a. 2/9=?/90
    b. 7/9=?/72
5. Use a calculator to calculate the following. Write down each step that you do and the result you get. Justify each step in terms of how it is related to cutting up a rectangle.
    a. 16/49=?/147
    b. 13/36=?/324
6. Write a calculator protocol for calculating a problem of the form a/b=?/c.

UNDERLYING PEDAGOGICAL PRINCIPLES

In Ivy's lesson and the published lesson, the students engage in an *empirical* process. Students are involved in collecting a set of results and identifying a pattern in those results. The process does not require any insight into why that pattern is produced (the logical necessity). What is it about the latter set of lessons that has the potential to foster understanding as the anticipation of logical necessity?

Let us look more closely at the lesson on equivalent fractions. Students begin by using an activity sequence that they already have available (further subdividing a rectangle) in service of a goal that they have established[v] (to determine the numerator of the equivalent fraction). The students determine how many subdivisions must be made in each of the original fractional parts to change their initial diagram to one that will portray the equivalent fraction (e.g., to convert thirds into twelfths, each fractional part must be subdivided into 4 parts). The student then is able to examine the diagram to determine the number of subdivisions in the shaded region (e.g., total = 8), the new numerator. If it does not happen spontaneously, Problem 4 is designed to focus the students on the relationship between the subdivisions of all the original parts and the resulting subdivisions of the shaded region.

The implied claim that students can pay attention to (perceive) this relationship is worth examining. Piaget's(1977) central construct of *assimilation* maintains that a learner can only attend to that for which s/he already has the assimilatory schemes to structure the experience. In the example of using the drawing to solve 2/3=?/12, the student intentionally subdivides each of the thirds (including those that are shaded) into 4 parts. It is therefore well within her/his capacities to come to anticipate that, as a result of subdividing, there are 4 times as many small parts as there were larger parts in the shaded region.

To describe this process more generally, the students' activity (subdividing by a particular number) produces particular effects (an augmentation of the shaded parts by a factor of that number). Through repeated use of the activity to accomplish a goal, the students are able to pay attention to the effects of their activity and eventually to see a pattern in the relationship of the activity and its effects (reflective abstraction). This mechanism for explaining the learning of mathematical concepts is developed in greater detail in Simon, Tzur et al. (2000; 1999).

Note, that this mechanism is further applied in the design of Problem 5. Here the task is designed to encourage abstraction based on the activity of determining the factor relating the original denominator and the new denominator.

I leave it to the reader to go through a similar analysis of the lesson on the area of a right triangle.

POTENTIAL SIGNIFICANCE OF THIS PEDAGOGICAL DISTINCTION

I have used examples from data to articulate a pedagogical distinction between lessons that engage students in *empirical activity* and lessons that involve *logico-mathematical* activity. I use these terms[vivii] because, although the distinction is not equivalent to Piaget's (2001)) distinction between empirical and reflective abstraction, the distinction can be thought of as analogous to it. I emphasize that the distinction is not simply one of the need for deductive justification. Rather this is a distinction that is fundamental to what is meant by a mathematical concept and the process by which concepts in mathematics are learned. The distinction highlights the difference between a mathematical generalization (e.g., theorem) and a mathematical concept. The former can be arrived at and proved without development of an anticipation of its logical necessity. A concept involves understanding and thus anticipation of logical necessity. The distinction between these two types of activity is potentially useful in conceptualizing the design of effective mathematics lessons and the education of mathematics educators (teachers, researchers, curriculum developers). I expand on each of these points.

In recent years, mathematics students have benefited from curricular efforts based on mathematics education research conducted in the last 30 years. As I mentioned above, although overall curricula are improving, there is still considerable unevenness between and within curricula. The distinction offered in this article is intended to contribute to the development of useful frameworks for guiding lesson design. It also provides a lens for viewing existing curricula. One established curricular effort that consistently builds on students' activity in a logico-mathematical process is the Dutch Realistic Mathematics Education (Gravemeijer, 1994). Using the distinction I have presented, their notion of *model of* becoming a *model for* can be understood as a technology for representing students' activity as a basis for students' reflection on the relationship between their activity and its effects.

Data from our research (e.g., Ivy's lesson) and analysis of recent curricula (e.g., lesson on equivalent fractions) suggest that some educators who intend to teach mathematics for understanding are generating lessons that engage students in only empirical activity. This distinction is one through which teacher educators and graduate educators can look at the prospective and practicing educators with whom they work. A useful (and ambitious) goal for the education of mathematics educators would be to promote their understanding of mathematics conceptual learning as built on a logico-mathematical process.

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[i] This research was supported by the National Science Foundation under grant REC-9600023. The opinions expressed do not necessarily reflect the views of the Foundation.

[ii] There are approximately a half dozen such curricula at each of the elementary, middle school, and high school levels.

[iii] "Understanding" is in quotes because of the lack of shared meaning for the term. Discussion of this point is up coming.

[iv] Although this articulation of "understanding" is my own, it is consistent with the ideas of others, most notably Piaget (2001).

[v] Although the teacher poses the problem, each student's activity is based on the goal that s/he sets. It is anticipated that the students' goals will be compatible with the intention of the teacher.

[vi]

[vii] I request that readers who find this choice of terminology to be problematic and/or who have ideas for other terminology, to communicate with me. If you find the terminology to be appropriate, I would be interested in knowing that as well.