# STUDENTS' USE OF TECHNOLOGY IN MATHEMATICAL PROBLEM SOLVING: TRANSFORMING TECHNOLOGICAL ARTIFACTS INTO MATHEMATICAL TOOLS

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*Can textbook exercises be transformed into problem solving activities that encourage students to develop mathematical thinking? This study documents what high school students showed when they were explicitly asked to use technological tools to examine and solve a routine problem from different angles or perspectives. In this process, students dealt with several representations that were important to analyze and quantify concepts of variation or change attached to the problem, in terms of models of solutions.*

When does a technological artifact become a mathematical problem-solving tool for students? What process of appropriation do students go through in order to use such tool in mathematical practice? What type of mathematical resources and strategies do students need in order to transform the use technological artifacts into mathematical tools? What process do students take in order to meaningfully employ those tools in problem solving activities? What types of mathematical representations are enhanced through the use of technological tools? These are important questions that we use as a guide or reference to evaluate the potential of technological tools in students learning of mathematics. In particular, we are interested in documenting what high school students exhibit when are asked to systematically use dynamic software, excel, and symbolic calculators in problem solving activities. In this context, we recognize that there are multiple ways in which technology can be employed by students and we are interested in characterizing those ways in which technology becomes a powerful tool for students to identify and explore conjectures, to quantify and analyze (graphically or numerically) particular phenomena, and to identify patterns or relationships through the analysis of distinct representations. In this study a routine problem that often appears in first calculus course is used to identify and analyze different types of models that students construct to solve the problem. In this process, the use of technology becomes a powerful tool for students to represent and examine relationships through the use of resources and concepts that appear traditionally in domains such as algebra, geometry, functions, and trigonometry.

## CONCEPTUAL FRAMEWORK

There are different learning trajectories for students to take in order to achieve mathematical competence; however, a common ingredient is a need to develop a clear disposition toward the study of the discipline. Such a disposition includes a way of thinking in which students value: (a) the importance of searching for relationships among different elements or components of the tasks in study (expressed via mathematical resources), (b) the need to use diverse representations to examine patterns and conjectures, and (c) the importance of providing and communicating different arguments (Santos, 1998). Thus, it becomes important to encourage students to think of the discipline in terms of dilemmas or challenges to be met and resolved. This means that they need to conceptualize their learning experiences in terms of activities that involve posing questions, identifying and exploring relationships, and providing and supporting their answers or solutions (NCTM, 2000). It is necessary to value the students' participation and persuade them about the power of reflecting on what they do, in mathematical terms, during their interaction with tasks or mathematical content. "To be able to guide students' inquiry toward the learning of the mathematical content in the syllabus, teachers must first convince students that inquiry is a legitimate, safe, and productive way to learn in school" (Lampert, 1995, p. 215). Here, students' view of mathematics involves accepting that it is more than a fixed and static body of knowledge; it includes that they need to conceptualize the study of mathematics as an activity in which they participate actively in order to identify, explore, and communicate ideas attached to mathematical situations.

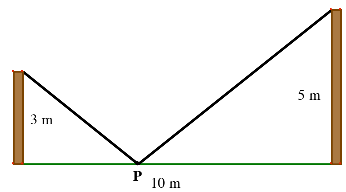
> …Students themselves become reflective about the activities they engage in while learning or solving problems. They develop relationships that may give meaning to a new idea, and they critically examine their existing knowledge by looking for new and more productive relationships. They come to view learning as problem solving in which the goal is to extend their knowledge (Carpenter & Lehrer, 1999, p. 23).

Students need to use different representational media to express their ways of thinking while dealing with tasks or problems. The students' constructions of powerful representational systems play an important role in developing distinct artifacts to understand and explain complex systems (Lesh, in press). The use of different tools offers students the possibility of examining situations from perspectives that involve the use of various concepts and resources. As a consequence, each representation might become a platform to identify and discuss mathematical qualities attached to the process of solution. Thus, during this process, a table might shed light on trends displayed by discrete data while an algebraic approach focuses on continuous behavior and general tendency (infinity). Geometric and dynamic approaches to the problem might provide a means for students to visualize and examine relationships that are part of the depth structure of the task. Specially, dynamic constructions help students focus their attention on common properties that appear while moving elements within the same configuration or representation. Lesh (in press) argues that "useful ways of thinking usually need to develop iterative and recursively, with input from people representing multiple perspectives". In this context, solving the task goes beyond reporting a particular solution, it is a process of constructing, investigating, representing, applying, interpreting, and evaluating several ways to solve the problem (Schoenfeld, 1998).

## GENERAL PROCEDURES AND THE INITIAL PROBLEM

Twenty-four students (grade 12) worked on a series of tasks that included textbook problems (in addition to assignments of the course). Students met twice a week during 2.5 hours. In this report we document features of mathematical thinking that students exhibited while interacting with one problem. In general, each student had access to a computer (excel and dynamic software) and a calculator. In each session, students had opportunity to work individually, in small groups, and as a part of the whole group discussions. Some students had some experience in using the tools and often those students taught other students during and out of the regular sessions. The example used to illustrate the students' ways of thinking (models) involves a routine problem that regularly appears in calculus textbooks. Thus, models that students exhibit during their interaction with this task illustrate mathematical processes and content that appear when students use distinct representations to explore mathematical qualities attached to various methods of solutions. Ideas from arithmetic, algebra, geometry, and calculus emerge naturally as a means to analyze relationships that appear in each student's approach. It is important to mention that there is no intention to provide a detailed analysis of students' work, rather each students' approaches to the task is shown to highlights the type of representation used to solve the problem. In particular, attention is paid to the variety of ideas and strategies that emerged when students are encouraged to use different technological tools to represent and approach even routine exercises. Thus, the initial nature of the task can be transformed into sequences of students' mathematical explorations.

**The Initial Problem**[1]. The distance between two telephone poles is of 10 m as shown in the figure. The length of each pole is 3 and 5 meters respectively. To support the poles, a cable from the top of each pole will be tied to a point on the ground between the two poles. Where should that point be located in order to use the minimum length of cable?



Two poles and a cable

Students worked on this problem first individually and later as a part of a group of three students. When some small groups presented their approaches to the entire class, it was common that new ways to solve the task emerged from the class discussion and students had opportunity to rewrite their initial approaches. At the end of each session, the teacher directed the class to discuss advantages and limitations of what students had presented. So, in general, students became aware not only of the power of their own methods but also of the strengths of other students' approaches.

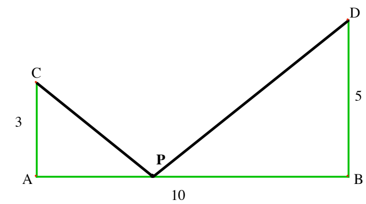
**Students' construction of models to solve the problem**

The term model is used to characterize ways in which students identify and employ ideas, concepts, representations, operations, and relationships to solve problems. So the construction of models is a process that involves constant exchange and refinements of students' ideas.

Students' initial interaction with the problem focused on identifying key ideas to detect particular relationships. Thus, understanding the task involved the introduction of a representation and notation that led students to discuss a set of questions.

[1] The general statement of this problem is known as Heron's Problem. It is often stated as "given two points on the same side of a line, find a point on that line such that the sum of its distances to the given points is minimal".

**(i)** An important mathematical idea embedded in this task is to recognize that the length of the cable varies when P is moved along the segment between the two poles. Here, it is also important to quantify that change. Students, in general, introduced particular notation that helped them identify key elements of the task.



Representation and a notation

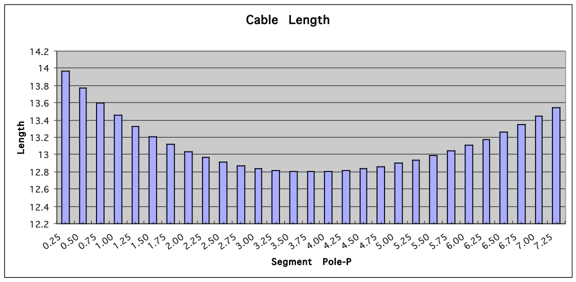
(ii) How can we know that the length of the cable change when point P is moved along the line between the two poles? How can we determine the distance between one pole and point P? What data do we have? Can we use the Pythagorean's Theorem? These were initial questions that helped students identify relevant information and explore relationships between the length of the cable and location of point P.

**I. A Discrete Model**. Some students focused on calculating particular cases that emerge when point P is moved along segment AB. Although they initially divided the segment of length 10 into two arbitrary segments, later they organized the lengths systematically into a table arrangement. AP represents the distance from the length of the shorter pole to point P; PB the length between point P and the other pole. P1 and P2 represent the lengths of the poles, D1 and D2 the corresponding hypotenuses and D1+D2 the length of the cable. A table that includes a refined partition of segment AB and the hypotenuses of the two triangles are shown next.

| AP | PB | P1 | P2 | D1 | D2 | D1+D2 |
|---|---|---|---|---|---|---|
| 1 | 9 | 3 | 5 | 3.16227766 | 10.2956301 | 13.4579078 |
| 1.25 | 8.75 | 3 | 5 | 3.25 | 10.0778222 | 13.3278222 |
| 1.5 | 8.5 | 3 | 5 | 3.35410197 | 9.86154146 | 13.2156434 |
| 1.75 | 8.25 | 3 | 5 | 3.473111 | 9.64689069 | 13.1200017 |
| 2 | 8 | 3 | 5 | 3.60555128 | 9.43398113 | 13.0395324 |
| 2.25 | 7.75 | 3 | 5 | 3.75 | 9.22293337 | 12.9729334 |
| 2.5 | 7.5 | 3 | 5 | 3.90512484 | 9.01387819 | 12.919003 |
| 2.75 | 7.25 | 3 | 5 | 4.06970515 | 8.80695748 | 12.8766626 |
| 3 | 7 | 3 | 5 | 4.24264069 | 8.60232527 | 12.844966 |
| 3.25 | 6.75 | 3 | 5 | 4.4229515 | 8.40014881 | 12.8231003 |
| 3.5 | 6.5 | 3 | 5 | 4.60977223 | 8.20060973 | 12.810382 |
| **3.75** | **6.25** | **3** | **5** | **4.80234318** | **8.0039053** | **12.8062485** |
| 4 | 6 | 3 | 5 | 5 | 7.81024968 | 12.8102497 |
| 4.25 | 5.75 | 3 | 5 | 5.20216301 | 7.61987533 | 12.8220383 |

Students observed that the length of the cable decrease up to some value and then increases again. Here, they identified that when the point on is 3.75 from the point A, then the cable gets the minimum length.

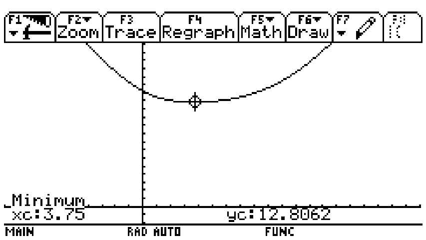
**A Visual Representation**. After students generated a table, they presented the corresponding graph. Here, they observed that when the point gets closer to one of the poles the length of the cable gets larger. Indeed, they observed that when the point was 3.75 the length of the cable reaches the minimum value. Here, students asked: When or under which conditions does the midpoint of the segment determine the minimum length of the cable? Students reported that when the lengths of the poles were the same then the midpoint of AB was the point at which the cable reaches its minimum length. Otherwise, the point will be located on the segment and close to the point with shorter length.



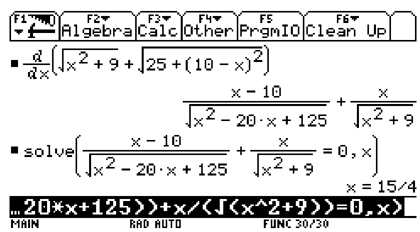

**VISUAL APPROACH**

The key ingredients of this model include the idea of analyzing particular cases, the process of refining the segment partition, the use of the Pythagorean theorem, the use of the tool (excel) and the visual representation of the data.

**II. A Symbolic Model and the Use of a Calculator**. A small group of students expressed the hypotenuse in each right triangle as $\sqrt{x^2+9}$ and $\sqrt{25+(10-x)^2}$ respectively and the length of the cable as the sum of these two expressions. That is, $l(x)=\sqrt{x^2+9}+\sqrt{25+(10-x)^2}$. How can we find the minimum value of $l(x)$ in this expression? Can we graph this function? With the help of a calculator, some students graphed the function $l(x)$ and identified the value in which the minimum value is reached. Other students, who decided to follow algebraic procedures to find the minimum value of $l(x)$, realized that they could contrast their results with those obtained through the use of the symbolic calculator. So the calculator functioned as a monitor of the students' work.




**Graphic representation**

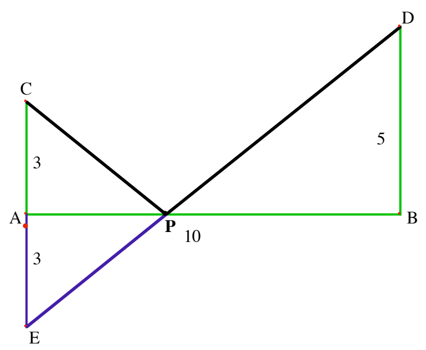



**Symbolic approach**

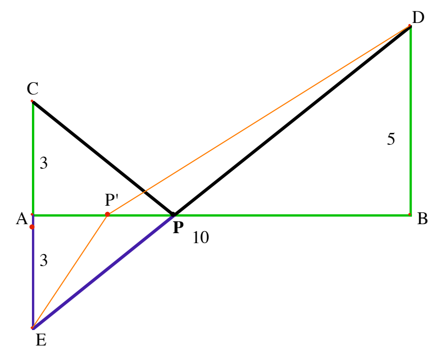
In this model, students focused on representing the function that related the position of the point to the length of the cable. Here, the fundamental components of this model include the use of particular notation, algebraic procedures, and establishing connections between the graph representation and the problem. The calculator had a dual purpose: to graph the function and identify the solution and to verify algebraic operations (derivative, roots of equation).

Iii.a geometric model. Another method suggested by the instructor was to examine the case in which one pole was reflected on its vertical line with respect to segment that joins the two poles (figure below). They observed that angles APC and APE are congruent.

Students recognized that in this case the segment ed that intersects AB at P is the minimum length of the cable. They argued that any other point different p' will generate a triangle EP'D in which the sum of EP' and P'D will be always longer than ED (figure below). The argument was based on using the triangle inequality. That is, they showed that in ΔEP'D, EP' P'D ED.




**Reflection method**




**Justification of the solution**

To find the distance AP, Some students recognized that triangles APE and BPD were similar. Therefore, the corresponding sides held proportionality, that is, $\frac{x}{3} = \frac{10 - x}{5}$ which led to $x = \frac{15}{4}$.

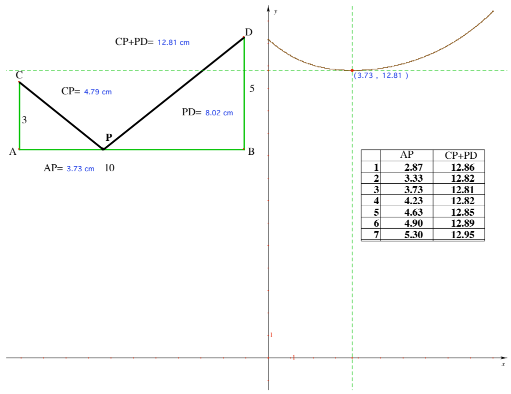
**Slope Approach**. Some students also realized that the minimum distance of the cable is obtained when the slopes of the two lines CP and PD are the same but with opposite sign. That is, when the angles APC and BPD are congruent. Here, they introduced a coordinate system with A as its origin point. Thus, they calculated the slopes of the line that passes by (0, 3) and (x, 0) and the line that passes by (x, 0) and point (10, 5).

$m_1 = \frac{-3}{x}$ and $m_2 = \frac{5}{10 - x}$; to hold the condition students observed that:

$$\frac{-3}{x} = -\frac{5}{10 - x} \Rightarrow -30 + 3x = -5x \Rightarrow x = \frac{15}{4}$$

The components attached to this model involve the use of properties of triangles (congruence and similarity) to identify the solution. Supporting the solution was also a key part of this model. In addition, the use of a coordinate system played an important role to introduce basic ideas of analytical geometry.

**IV. A Dynamic Model**. Yet another approach that students showed while dealing with this problem was to represent the problem through the use of dynamic software. This software allowed students to determine graphically the relationship between the distance AP and the length of the cable (CP + PD). Here, by moving point P along segment AB a graph of the behavior of the length is generated. It is also important to mention that a table including some values of distance CP and the corresponding value of CP+PD can also be obtained. In this case what students reported was an approximation of the minimum length of the cable, that is, 12.81. In addition, students could drag basic parameters and generalize their results (for example, varying distance between poles or poles lengths).




| | AP | CP+PD |
|---|---|---|
| 1 | 2.87 | 12.86 |
| 2 | 3.33 | 12.82 |
| 3 | 3.73 | 12.81 |
| 4 | 4.23 | 12.82 |
| 5 | 4.63 | 12.85 |
| 6 | 4.90 | 12.89 |
| 7 | 5.30 | 12.95 |

**A dynamic representation**

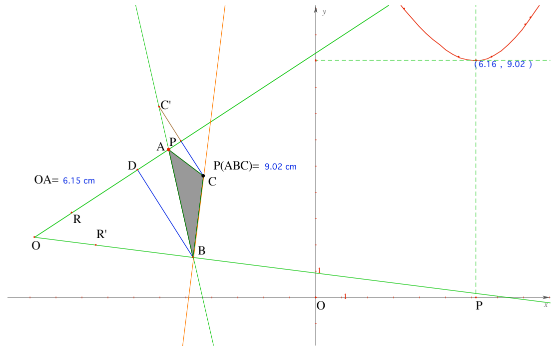
A key component of this model was to represent dynamically the relationship between the point of the segment and the length of the cable. The use of a coordinate system to show the graphic representation of that relationship was also an important ingredient. In addition, students in this model could explore easily other cases in which they change parts of the initial representation (lengths of poles and segment).

Students had the opportunity to discuss advantages and disadvantages attached to each model. In particular, they noticed that geometric and dynamic models did not involve algebraic procedures to find the solution.  To close the session, the teacher posed the following related problem:

> Let C be a given point in the interior of a given angle. Find points A and B on the sides of the angle such that the perimeter of the triangle ABC is a minimum.

Students' first approach was to represent the problem with the use of dynamic software. In using the software it was also important to introduce a particular notation (figure below).  Let OR and OR' rays with a common point O and C a point on the interior of angle ROR'. Thus, their first goal was to find the minimum length from point C to any of the angle side. They drew segment CB from point C and perpendicular to ray OR'. Now, from C they drew a perpendicular line to OR and from B a perpendicular segment BD perpendicular to OR. They recognized that segment DQ and segments QC and DB represented an *analogous case* to the original problem. That is, the objective was to identify a point on DQ such that BA + AC was minimum. Using that the line joining BC' (CQ = QC') intersects DB at P. and this point determines the minimum distance. Therefore, the triangle with the minimum perimeter is triangle CPB.

An extension: An angle, an interior point and a triangle with minimum perimeter, and graph showing an approximate solution using dynamic software.




## REMARKS

When students openly search for various approaches while working on mathematical tasks it is common to identify different types of representations that help them examine and use different problem solving resources and strategies. Some tasks or problems that often appear in regular textbooks can be taken as platform to engage students in mathematical practice. In particular, the use of technology became a powerful tool to explore properties and relationships that did not appear in paper and pencil approaches. It was evident, that students' ideas about solving routine problems get enhanced when explicitly they search for various ways to represent and solve the tasks. That its, routine problems are seen as a means to encourage students to extend and reflect on their mathematical thinking. Thus, teachers might use initially some of their textbook problems as a way to engage students in the search of powerful representations to elicit and refine their previous mathematical ideas. These ideas eventually are transformed in models that are useful to solve problems.

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