MATHEMATICIANS ON CONCEPT IMAGE CONSTRUCTION: SINGLE '*LANDSCAPE*' VS '*YOUR OWN TAILOR-MADE BRAIN VERSION*'

Elena Nardi and Paola Iannone
School of Education
University of East Anglia, UK

Responding to an increasingly urgent need for collaboration between mathematicians and mathematics educators, the study reported in this paper engages mathematicians as educational co-researchers in a series of themed Focus Group interviews where a pre-distributed sample of mathematical problems, typical written student responses, observation protocols, interview transcripts and outlines of relevant bibliography is used as a trigger for reflection upon and exploration of pedagogical issues. In the extract exemplified here, a part of the sample that includes a question involving the concept of det(M), the determinant of an nxn matrix, triggers a conversation that reveals various images of the concept held by members of the group as well as their beliefs about personal and shared concept images.

Teaching mathematics at university and college level is rapidly changing as fewer and fewer students opt for exclusively mathematical studies (Holton 2001) and, at least in the UK, recruitment of good mathematics graduates to mathematics teaching is at an all-time low. Given the substantial gap between secondary and tertiary mathematics teaching approaches (LMS 1995), students feel increasingly alienated from the traditionalism of university-level teaching (HEFCE 1996). Moreover universities are more than ever accountable to society regarding the quality of their teaching. By the late 90s most responses to these changes were in terms of modifying the tertiary syllabus (Kahn & Hoyles 1997) and of meticulous topic-centred studies (e.g. calculus (Ganter 2000)). However, there has been a growing realisation that reform should be focusing on teaching, in terms of underlying principles and practices and, in particular, in terms of a consideration of students' experiences and needs (Jalling & Carlsson 1995). Moreover, research in mathematics education at this level, beyond offering assistance to university lecturers in this task (for example in the subtle, theory-informed yet accessible ways suggested in (Mason 2002)), needs to engage mathematicians themselves in pedagogically oriented self-reflective processes (e.g. Nardi & Jaworski 2002). The research we report here aims to contribute in this area.

AIMS AND METHODOLOGY

This project engages groups of mathematicians from seven institutions in the UK as educational co-researchers (the main body of data originates in the group of five mathematicians based in the University where the two authors work). It is a 15-month clinical partnership (Wagner 1997) between mathematicians and researchers in mathematics education that builds on the authors' (e.g. (Iannone & Nardi 2002) and others' (e.g. Jaworski, Nardi & Hegedus 1999) previous research. It is funded by the Learning and Teaching Support Network in the UK.

An appropriate forum for the type of collaboration intended in this project is the Focus Group (Wilson 1997): a meeting of a small number of individuals (our chosen 4-5 is seen in the literature as suitable) with one or more researchers to discuss participants' perceptions, attitudes, beliefs, emotions etc. on a selection of topics in a non-threatening, exploratory environment. Focus Groups encourage and utilise group interactions or in Madriz's (2001) words, allow observation of 'collective human interaction' (Madriz 2001). Madriz advocates the use of Focus Groups for the purpose of facilitating the expression of vulnerable groups of individuals and a group of highly educated, male, white, middle-class mathematicians does not yield an immediate association with vulnerability. However, given the historical fragility in the relationship between mathematicians and mathematics educators and the novelty of engaging with educational research for our participants, we see the method of Focus Group as one that addresses a type of vulnerability intrinsic to our study.

Furthermore, even though the initiation of the discussion (see set up of Focus Groups in the Data Collection section) originates in us, the researchers, we have deliberately chosen not to provide to the group our interpretation of the data discussed in the meetings and to minimise the unveiling of our interpretations in the course of the meetings. This resonates with the literature on Focus Groups, e.g. with Madriz (ibid.):

> …the interaction among group participants often decreases the amount of interaction between the facilitator and the individual members of the group. This gives more weight to the participants' opinions, decreasing the influence the researcher has over the interview process. (p836)
>
> …the researcher usually dominates the whole research process, from the selection of the topic to the choice of the method and the questions asked, to the imposition of her own framework on the research findings. Focus group minimises the control that the researcher has during the data gathering process by decreasing the power of the researcher over the research participants. The collective nature of the group interview empowers the participants and validates their voices and experiences. (p838)

Indeed it allows the emergence of diverse voices amongst the group as well (e.g. more and less experienced mathematicians, pure and applied ones (Hersh 1993) etc.).

DATA COLLECTION

There are 12 Cycles of Data Collection, numbered 1-6 (data from this University's team) and 1X – 6X (for data from six other institutions in the UK). Six Data Sets will be produced one for each of the 1/1X – 6/6X Cycles. Provisionally each Cycle of Data Collection will focus on one of the following Themes (but this list is subject to change upon negotiation with the participants):

- Theme 1 Formal Mathematical Reasoning I: Students' Perceptions of Proof and Its Necessity
- Theme 2 Mathematical Objects I: the Concept of Limit Across Mathematical Contexts
- Theme 3 Mediating Mathematical Meaning I: Language and Notation
- Theme 4 Formal Mathematical Reasoning II: Students' Enactment of Proving Techniques and Construction of Mathematical Arguments
- Theme 5 Mathematical Objects II: the Concept of Function Across Mathematical Topics
- Theme 6 Mediating Mathematical Meaning II: Diagrams as Metaphors

For each Cycle a Data Set is produced. A Data Set consists of:

A short literature review on the theme supplemented by a bibliography for further consultation by the group.

Samples of data on the theme (e.g.: students' written work, interview transcripts, observation protocols) collected in the course of the authors' previous projects and doctoral work (e.g. (Iannone & Nardi 2002), (Nardi 1996)). There are usually five sets of examples in each Data Set, each on one mathematical question.

The group is asked to study the Data Set prior to a half-day meeting and be prepared to discuss their responses to the literature and the data sample in relation to their own experiences and views. They are also encouraged to support these views with brief samples of data that they have collected themselves. The Focus Group discussion at the meetings is audio-recorded on a digital sound recorder and the two researchers also bring along further examples to supplement and elaborate the issues raised in the Data Set. At the time of writing Cycles 1 and 2 have taken place and Cycles 3 and 4 have been set up. Volunteering institutions for Cycles 1X-6X have also been identified and the set up of these Cycles is currently in progress.

DATA ANALYSIS

Once a recording is complete, the digital sound file is transferred from the digital sound recorder to a computer and the Research Officer, the second author of this paper, produces a full transcript. Each recording, approximately 200 minutes long, gives a verbatim transcript of about 30,000 words. This text is roughly structured in parts according to the structure of the Data Set. Within each part the structure of the discussion may vary: sometimes the group starts from an epistemological analysis of the mathematical problem in question, including their own ways of responding to it, the question-setter's intentions, the prerequisite knowledge etc. and proceeds to an examination of the student examples and to an address of the general cognitive and pedagogical issues. Of course the conversation shifts backwards and forward from all of the above. As intended by the focus group methodology, the intervention by the two researchers is minimal and mostly of a co-ordinating and sometimes consolidating nature (see Example below).

The above structure, determined to a large extent by the participants but implicitly also dictated by the structure of the handout, has led to an almost natural emergence of Episodes from the text, namely self-contained pieces of conversation with a particular focus. In this paper we present one example of an Episode. In the spirit of data-grounded theory (Glaser and Strauss 1967) it is intended that Episodes will be the analytical units and, on the basis of the experience from Cycles 1 and 2, it is envisaged that from the Cycles of Data Collection 1-6 (plus the supportive data from Cycles 1X-6X) approximately 150 Episodes will emerge.

The two researchers are currently engaged with a first-level analytical triangulation (Jaworski, Nardi and Hegedus 1999) with regard to a consensus on their definition of the Episodes: working independently on a part of the transcript, they aim at achieving an agreement on a breakdown of each part in Episodes. A Story is then written up for each Episode, approximately 500 words long, namely a text which summarises the content as well as highlights the conceptual significance of the Episode for subsequent stages of analysis. A second-level analytical triangulation, regarding the content of the Stories, is also currently in progress.

To exemplify the above and also demonstrate the theoretical perspectives used in the analysis, below we offer an example of an Episode (see introduction below and a compressed version of the transcript in Fig. 1). A preliminary analytical account of the Episode, an expansion of its Story, follows.

AN EXAMPLE FROM CYCLE 1

In the following the participants are four of the five mathematicians of this university's team (renamed as A, B, C and D) and one of the mathematics educators and first author of this paper (renamed as ME).

In this part of the recording, Part III, the group have been discussing a Linear Algebra question which involves proving certain properties of the adjoint (or adjugate) of an nxn matrix. The expressions to be proved and the proofs themselves involve an extensive use of det(M), the determinant of a matrix. Almost a quarter of an hour into the conversation, the focus has shifted (the issues thus far have included: the question setter's intentions being about enacting a certain handling of algebraic definitions and properties; typical tendencies in the student responses, such as providing arguments for the 2x2 case and assuming in their proofs properties of determinants hitherto unproved in the course) towards speculating into how students perceive of determinants. Or as A puts it **'what the students actually feel when they do these things. And when you see a determinant how do you… how is one supposed to relate to it?'**

Extract: Cycle 1, 13:54 – 22:06 from Part III (35:08)

A: Is it just a bit of garbage that is sort of coming your way where you have to apply certain rules or are you… do you have a mental image of what it might mean? Do you think of a determinant as a volume or … (*B offers* 'Something to be worked out'). And then how do you relate? You see when you see … the adjoint is a cute way of getting the inverse of a matrix. So that could be my way of it. If I see adjoint I think of inverse and I then I would work from there. But that is justified by my requirement that I cannot handle a thing that is very complicated. So… inverse I can just about understand and then I work with that….

D, referring to the case in which the matrix is singular, suggests then that this image 'removes the true power of the adjoint'. *A agrees and asks about* 'the guidance given to the student' *with regard to the ways in which the student will be* 'picturing what this is'.

B: This is another example of trying to understand and appreciate … the student's landscape. Everyone has their own personal landscape… I agree that determinant is a tricky one… even if they have seen it before at school. What I think this is… and I suggested earlier that the determinant might be thought of as a number to be worked out and I am sure that a lot of people think about this when they see an integral. A thing that you have to work out for all pieces involved in it …

A wonders whether this is 'what you want to instill' *in the students, that this is* 'something that you have to work out' *and B suggests that the students* 'can be given some structure'. *Given the diversity of images hitherto discussed, asks ME, is there any* 'sharing of *landscapes' between the mathematicians, who have been* 'working with these things for a long time', *and their students? D then offers the examples of sharing his images (of the Intermediate Value Theorem and of a 3x3 matrix as a transformation of a three space* – 'they find it difficult to imagine that a pure mathematics lecturer can think of a 3x3 matrix as a transformation of a three space, they somehow… for them that is *incongruous') with the students. ME suggests a conjecture: the students' attitude originates in the absence of links between the various courses. D agrees, suggests more examples where students were surprised at and at difficulty with how various* *concepts* 'could be connected' *and returns to the concept of determinants* 'for cross products and volumes' *which he did not pursue in his lectures (but they will see later, adds C). B suggests that the Linear Algebra problem in question* 'is a good example of an opportunity to make that link between determinants as something which just saves you writing down a large number of elements'. *So because, for example, the adjoint can be written out through the assistance of determinants, the students are given* 'an idea of the determinant as an object which just saves writing', *an image that his experience from the lectures suggests that students find helpful. A then returns to the issue of* 'sharing landscapes'.

A: I think that's the reason why we are still in business, as lecturers. I think that that is what it is about. You are not just communicating facts, you are saying that this is one way you can view it and that is another way you can view it, let's put these together somehow. And we are sharing such picture somehow…

ME and A insist on the importance of such link construction and D ponders on the difficulty of making such links 'good'*:*

D: … because on the opposite extreme if you think of determinants as cross products and volumes and so on… What the heck is a 4x4 determinant? Yes, of course it is a volume but mathematically you want them to think of an nxn determinant…

A: You see, making the connection I think becomes a personal issue because you need to link your pictures, not somebody else's pictures and this is quite important. I… I don't like this forced, you know, networking of all mathematics … everything relates to everything…(*that* 'assumes there's just one network' *suggests ME*). Yes .., it is currently destructive, it is currently destructive. You need to have your own tailor-made brain version of what the thing is. It will have… it will be a tree, all fits… ordered according to your view. And you need to have … and you need to do this in your own time, under the instruction or guidance and teaching and not, you know, some ... (*It is no good to impose any one* 'landscape' *on students, closes ME*).

Figure 1.

A PRELIMINARY ANALYTICAL ACCOUNT

Here we focus our examination mostly on the evidence on concept image construction (Vinner & Tall 1981) : the various images of the concept of det(M), the determinant of an nxn matrix, held by members of the group as well as their beliefs about the nature of these images.

The discussion on perceptions of the concept of determinant is initiated by A's 'a bit of garbage that is sort of coming your way?', his evocation of an initial image of the concept as vacant of meaning, as devoid of a *raison-d'-être* possibly held by students (the obstacles set by such images have been explored e.g. in (Nardi 1999)). Before A launches into an exposition on the type of image that he personally finds helpful, another attempt at exploring student-held images of det(M) makes its first appearance (and will subsequently become a pivotal one in the course of the conversation): B's 'something to be worked out'. We return to this a bit later. A in the meantime describes a *raison-d'-être*, a powerful instrumental image (Skemp 1978) for the concept of the adjoint of an nxn matrix: through the property $\mathrm{Adj}(M)M = \det(M)I_n$ of the adjoint one, under certain conditions, can derive an expression for the inverse of matrix M. The image emerges from his desire ('justified by my requirement') for simplicity, itself dictated by his learning need ('I cannot handle a thing that is very complicated'). D is more hesitant about the potency of this image: 'it removes the true power of the adjoint' he proposes. In doing so he introduces a voice of caution with regard to the likely pitfall of fostering potentially limiting images of the concept. He returns to the difficulty of avoiding this later. His intervention seems to prompt A's question about the role of the 'guidance given to the student' with regard to building images of the concept.

B's subsequent proposition takes this role further: beyond instilling concept images, one needs to try to 'understand and appreciate the student's landscape', a point with a flavour of constructivism in it (von Glasersfeld 1995). He advocates building up images of det(*M*) from the students' palpable understanding of it as 'a number to be worked out'. A's response reflects a certain suspicion towards algorithmic images as potential impediments to conceptual understanding (Skemp ibid.). B defends the approach as providing 'some structure' for the students.

ME then wonders whether 'landscapes' of images with clearly differing degrees of complexity are 'shared' with the students. D responds with an array of examples and a poignant observation: students find the idea that 'a pure mathematics lecturer can think of a 3x3 matrix as a transformation of a three dimensional space' 'difficult', even 'incongruous'. By reporting the students' resistance to his suggestion of a cross-topical image (one that blends elements of Algebra and Geometry), he initiates a discussion on the limitations of a 'compartmentalised view of mathematics' (used by another mathematician in (Nardi, Jaworski & Hegedus, in preparation). Returning to determinants, D continues his advocation for multi-context introduction of new concepts. He proposes viewing determinants in the context of cross-products and volumes – a view which he has not yet put forward in his lectures (but does come later in their studies, as C confirms).

B then returns to his earlier proposition to build subtler yet sturdy instrumental images of the concept on students' initial understandings and proposes det(*M*) as 'something which just saves you writing down a large number of elements'. He supports this with uses of this image for writing out the adjoint of a matrix and grounds his belief in his observation of student positive reactions to it in the lectures. A commends the student-centredness of B's proposition for the role of the lecturer as it contributes to building collectively shared landscapes. D's subsequent skepticism ('it is not easy to make it good') highlights a difficulty of the task: if we see determinants in the context of cross products and volumes, what is then a 4x4, and even further, an nxn one, the ultimate abstract image that one needs to aspire that the students will eventually come to possess?

In his final comment A makes another, related point of caution (one that also resonates with constructivist views of learning): this image construction is a *personal* venture. A 'forced networking of all mathematics' where 'everything relates to everything' is a futile aspiration that detracts from the 'need to have your own, tailor-made brain version of what the thing is'. The futility lies in the assumption that this 'just one network' can in fact exist, concurs ME perhaps in an attempt to deter the possibility of the skepticism prevailing around the table about the *overall* value of the venture: in any case having *a singular* image cannot be in itself good. In fact it is 'destructive' concurs A. In her final comment ME suggests that imposing any one 'landscape' may not be a commendable aspiration but fostering multiple ones is (Janvier 1987): especially when coupled with the awareness that it is impossible for any one 'landscape' to be comprehensive.

Another central observation we wish to suggest in this account is how the discussion fluctuates creatively between an epistemological analysis of a particular concept (E), to a psychological one ((Psy) - an exploration of individually held images and beliefs) and, eventually, a pedagogical one ((Ped) - the role of teaching as facilitating the students' concept image construction). It is this spiralling development that our methodology of Episodes/Stories etc. aspires to preserve and highlight benefits of. The way in which the consensus on the role of teaching was achieved by the group at the end of the Extract (E□ Psy□ Ped) can be instructive: through a creative and complementary juxtaposition of personally held views (images of the determinant of an nxn matrix), a pedagogical strategy is distilled, commonly agreed and firmly owned by the group. In this study, building on recently developed ideas on 'non-deficit' models of teaching (e.g. (Brown et al 1993)) we conjecture that the impact on practice of this sense of ownership has the potential to exceed the impact of externally imposed pedagogical prescriptions.

Furthermore we conjecture that the group's analyses are potent in a more theoretical way too: what is on offer in the data above and, significantly, grounded in the views of practicing mathematicians, is a description of concept image spaces as *dynamic* loci of human cognition. This is a description that enriches the *snapshot-static* ways in which the concept image / concept definition theory, one of the most defining tools of research in the area since its first appearance in the 1980s (e.g. (Vinner & Tall 1981)), is sometimes used.

Finally, ***in place of a conclusion*** and as an indication of how the analysis of each Episode is re-embedded in the original aims of the study (ultimately we aim at identifying cross-Episode patterns in attitudes, beliefs and practices), we wish to offer some brief observations on certain elements of the interaction between mathematicians and mathematics educators occurring in the Focus Group interviews.

In the extract at least two discreet but distinct roles of the mathematics educator are exemplified. In one occasion she poses a question (about a 'sharing of landscapes' between the mathematicians and their students) which appears to shift the conversation towards a more overt and focused consideration of student needs and teaching practices. In other occasions she repeats some of the words used by members of the group as if to consolidate the views discussed (e.g. when she brings together the comments on the importance of 'link construction'). These two roles, when adopted judiciously, resonate with the relevant descriptions in the literature (e.g. (Madriz 2001) and with the collaborative aims of the study.

It is possible to argue that, despite a minimal participation of the mathematics educators in the conversations themselves, their presence is conspicuous in the sense that they hold an almost exclusive responsibility for offering the triggers for discussion through the pre-distributed Data Set. As evident however in the data collected so far in Cycles 1 and 2, the vibrancy of the group's views and the enthusiasm with which the members of the group engage in the conversation is helping the content of the conversations escalate beyond the remit of the pre-determined themes. In this sense the Data Set, while offering a concrete, solid basis for discussion, does not appear to be a straightjacket imposed by the researchers on the participants. Indeed the participants, by constantly re-shaping the focus of the discussion, are determining the actual content of the data and eventual focus of the research. They are thus *becoming co-researchers* – which is at the heart of what we believe to be a topical and much needed pedagogical enterprise.

References

Brown, S. & McIntyre, D. 1993. *Making Sense of Teaching*. Buckingham, Philadelphia: Open University Press

Ganter, S. L. 2000. *Calculus Renewal: Issues for Undergraduate Mathematics Education in the New Decade*, Dordrecht / Boston / London: Kluwer Academic Publishers

Glaser, B. G. & Strauss, A. L. 1967. *The Discovery of Grounded Theory: Strategies for Qualitative Research,* New York: Aldine de Gruyter

Glasersfeld, E. Von. 1995. *Radical Constructivism: a Way of Knowing and Learning* London, Washington D.C.: The Falmer Press

HEFCE. 1996. *Mathematics Learning and Assessment: Sharing Innovative Practices*. London: Arnold

Hersh, R. 1993. Proving Is Convincing and Explaining. *Educational Studies in Mathematics*. v24 n4 p389-99 1993

Holton, D.A. 2001. *The Teaching and Learning of Mathematics at University Level: An ICMI Study*. Kluwer Academic Publishers

Iannone, P. & Nardi, E. 2002. A Group as a 'Special Set'? Implications of Ignoring the Role of the Binary Operation in the Definition of a Group, *Proceedings of the 26th Annual Conference of the International Group for Psychology in Mathematics Education,* 21-26 July 2002, Norwich, UK Volume 3, p121-128

Jalling, H. & Carlsson, M. 1995. 'An Attempt to Raise the Status of Undergraduate Teaching', *Studies of Higher Education and Research* No 1995: 2/3

Janvier, C. 1987. *Problems of Representation in the Teaching And Learning of Mathematics*, Hillsdale, NJ: Erlbaum

Jaworski, B., Nardi, E. & Hegedus, S. 1999. 'Characterising Undergraduate Mathematics Teaching'. p.121-128. In Zaslavsky, O.(ed.) *Proceedings of the 23rd Annual Conference of the International Group for Psychology in Mathematics Education*, Volume III. Technion University, Israel

Kahn, P. E. & Hoyles, C. 1997. 'The Changing Undergraduate Experience: a Case Study of Single Honours Mathematics in England and Wales'. *Higher Education* 22 (3), 349-362

London Mathematical Society. 1995. *Tackling the Mathematics Problem.* London, LMS

Madriz, E. 2001. Focus groups in feminist research, in *Handbook of Qualitative Research,* second edition, edited by N K Denzin and Y S Lincoln

Mason, J. 2002. *Mathematics Teaching Practice: A Guide for University and College Lecturers*. Horwood Publishing Series in Mathematics and Applications

Nardi, E. 1996. *The Novice Mathematician's Encounter With Mathematical Abstraction: Tensions in Concept-Image Construction and Formalisation*. Unpublished doctoral thesis. University of Oxford

Nardi, E. 1999. The Novice Mathematician's Inquiry About New Concepts: Bestowing Meaning Through Ambivalent Uses of Geometrical Metaphors. p57-68. In BROWN L (ed.) *Making Meanings in Mathematics*, a collection of extended and refereed papers from BSRLM - the British Society for the Research into Learning Mathematics. QED Publications: York

Nardi, E. & Jaworski B. 2002. Developing a Pedagogic Discourse in the Teaching of Undergraduate Mathematics: On Tutors' Uses of Generic Examples and Other Pedagogical Techniques, *Proceedings of the 2nd International Conference on the Teaching of Undergraduate Mathematics,* 1-6 July 2002, Crete, Greece. Available on CD ROM and at http://www.math.uoc.gr/~ictm2/

Nardi, E., Jaworski, B. & Hegedus, S. In preparation. From 'natural' but 'silly little tricks' to 'arsenal of techniques': A Spectrum of Pedagogical Development for the Teaching of Undergraduate Mathematics, revision to be submitted to *Journal for Research in Mathematics Education* by 30 June 2003

Skemp, R. 1979. *Intelligence, Learning and Action*, London: Wiley

Vinner, S. & Tall, D. 1981. 'Concept image and concept definition in mathematics with particular reference to limits and continuity.', *Educational Studies in Mathematics*, 12, 151-169

Wagner, J. 1997. 'The Unavoidable Intervention of Educational Research: A Framework of Reconsidering Researcher-Practitioner Cooperation.', *Educational Researcher,* 26 (7). 13-22

Wilson, V. 1997. Focus Group: a useful qualitative method for educational research?, *British Educational Research Journal,* vol. 2, n.2, 209-225