LOCATING FRACTION ON A NUMBER LINE

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Based on a survey of 3067 Finnish 5th and 7th graders and a task-based interview of 20 7th graders we examine student's understanding of fraction. Two tasks frame a specific fraction ($\frac{3}{4}$) in different contexts: as part of an eight-piece bar (area context) and as a location on a number line. The results suggest that students' understanding of fraction develops substantially from 5th to 7th grade. However, Part-to-Whole comparison is strongly dominating students' thinking, and students have difficulties in perceiving a fraction as a number on a number line even on 7th grade.

INTRODUCTION

Rational number is a difficult concept for students. One of the reasons is that rational numbers consist of several constructs, and one needs to gain an understanding of the confluence of these constructs. This idea was originally introduced by Kieren (1976, cited by Behr, Harel, Post & Lesh, 1992), and has since been developed by Behr, Lesh, Post & Silver (1983), who distinguish six separate subconstructs of rational number: a part-to-whole comparison, a decimal, a ratio, an indicated division (quotient), an operator, and a measure of continuous or discrete quantities. They consider the Part-to-Whole subconstruct to be "fundamental to all later interpretations" (p. 93). Toluk and Middleton (2001) regard division as another fundamental scheme that later becomes integrated into the rational number scheme. Based on a study of four case students they presented a schematic drawing of how students develop the connections between fractions and division. The highest developmental stage of their model is the confluence of Fraction-as-Division (a/b = a÷b, □ a/b) and Division-as-Fraction (a÷b = a/b, □ a/b < 1) into Division-as-Number (a÷b = a/b, □ a/b).

In mathematician's conception of (real)number, number line is an important element. (Merenluoto, 2001). If Part-to-Whole subconstruct is the fundament of the rational number construct, then ability to locate a fraction on a number-line could be regarded as an indication (although not a guarantee) of confluence of several subconstructs.

Novillis-Larson (1980, cited in Behr & al. 1983, p. 94) presented seventh-grade children with tasks involving the location of fractions on number lines. Novillis-Larson's findings suggest an apparent difficulty in perception of the unit of reference: when a number line of length of two units was involved, almost 25 % of the sample used the whole line as the unit. Behr & al. (1983) gave different representations of fractions for fourth graders, and their results show that number line is the most difficult one. For example, in case of the fraction $\frac{3}{4}$, the error rate with a rectangle divided in eight pieces was 21% and with a number line with similar visual cue, the error rate was 74%. When no visual cue was provided for the division, the error rate for rectangle was 1 % and for the number line 68%. For 3 years the students' text series had employed the number-line model for the whole-number interpretations of addition and subtraction. Considering that background, the results were surprisingly poor. Behr & al. (pp. 111-113) conclude that students "were generally incapable of conceptualizing a fraction as a point on a line. This is probably due to the fact that the majority of their experiences had been with the Part-to-Whole interpretation of fraction in a continuous (area) context."

The aim of this study is to deepen and broaden some results concerning students' understanding of fractions. We will explore the development of Finnish students' understanding of fraction both as Part-to-Whole comparison, and as a number on a number line. In addition, we intend to look at gender differences. In the qualitative part of the study we shall take an in-depth view of students' (mis)conceptions. The results will be compared with results from the two studies cited above. With respect to perceiving fraction as a number, there is an important difference between English and Finnish languages. While the English word "fraction" has no linguistic cue for the number aspect of the concept, the Finnish word for fraction ('broken-number') includes also the word 'number'. Hence, it will be interesting to see if Finnish students would more easily perceive fraction as a number.

METHODS

This paper is part of the research project 'Understanding and self-confidence in mathematics'. The project is directed by professor Pehkonen and funded by the Academy of Finland (project #51019). It is a two-year study for grades 5-6 and 7-8. The study includes a quantitative survey for approximately 150 randomly selected Finnish mathematics classes out of which 10 classes were selected to a longitudinal part of the study. Additionally, 40 students participate also a qualitative study.

The research team Markku S. Hannula, Hanna Maijala, Erkki Pehkonen, and Riitta Soro designed the survey questionnaire. It consisted of five parts: student background, 19 mathematics tasks, success expectation for each task, solution confidence for each task, and a mathematical belief scale. The survey was mailed to schools and administered by teachers during a normal 45-minute lesson in the fall 2001. The mathematics tasks in the test were designed to measure understanding of number concept and it included items concerning fractions, decimals, negative numbers, and infinity. Task types included bar-representation of fractions, locating numbers on a number line, comparing sizes of numbers, and doing computations. In this study, we examine student responses to certain items on fractions. There are three levels of analyses to this task: a large survey (N = 3067), a more detailed analysis of different types of answers (N =97), and an analysis of task-based interviews with 20 students.

The bar task required the students to shade fractional proportions of a rectangle divided into eight pieces (an eight-piece bar). This topic is usually covered in Finland during third and fourth grade. We will look at student responses to the task in which the proportion was $\frac{3}{4}$ (Figure 1). The second task required the students to locate three numbers on a number line, where only zero and one were marked (Figure 2). We shall focus on how students located the number $\frac{3}{4}$ on the number line. In Finland the number line is in some schools introduced during second grade, while other schools may not introduce until with diagrams during fourth grade. Likewise, not all schools choose to use number line with fractions. However, in forthcoming new curriculum the students ought to learn fraction, decimal number and percentage and the connection between these - and also the number line representation for all. There were yet another three items in the test that measured more computational skills with fractions: to compare $\frac{5}{8}$ to $\frac{5}{6}$, to compare $\frac{1}{5}$ to 0.2, and to calculate $3 \bullet \frac{1}{5}$

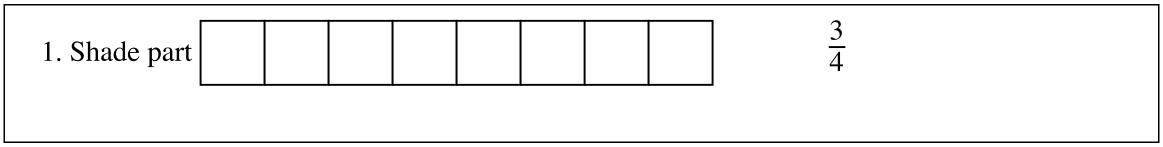

Figure 1. The bar task of the test.

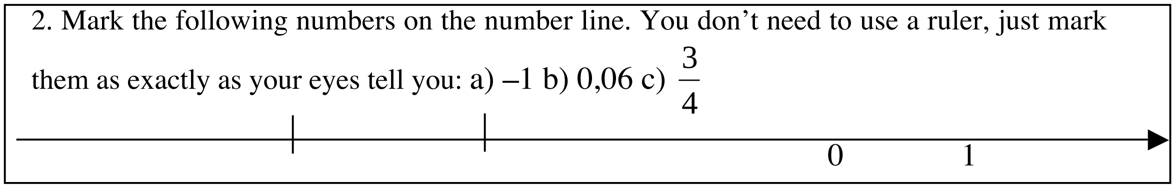

Figure 2. The number line task of the test.

From the full sample of 159 classes, five 7th grade classes were selected for a qualitative longitudinal study[1]. We shall analyze the different types of incorrect answers to the 'fraction on a number line' -task given by these 97 7th grade students.

Based on student responses in the survey and teacher evaluations, four students from each of these classes were chosen to represent different student types. The qualitative study is still ongoing, but during the first year, three lessons of each class were observed and video-recorded. The focus students of each class were interviewed in groups in May 2002, more than six months after the test[2]. The video- recorded interviews consisted of a semi-structured interview on mathematics-related beliefs and a clinical interview with students who were solving some mathematical tasks.

In one of the tasks the group had a number line on a paper (magnified from the one in the task), and they were asked to put numbers 3, -1, 0.06, $\frac{3}{4}$, 1.5, and $2\frac{1}{5}$ on the number line. The numbers were written on cards that were given one by one. The students were first asked to think where they would locate the number, and after they indicated that they had decided, they were asked to put their notes on the number line at the same time. They were also asked to explain how they solved the task.

RESULTS

Survey results

As a first, rough picture we can see that 70 percent of students answered correctly to the bar task (Table 1), while 60 percent gave no answer, or a robustly incorrect location for the fraction $\frac{3}{4}$ on the number line (Table 2). We see that in both tasks 7th graders perform notably better than 5th graders. The majority of the students seem to learn the bar task

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[1] Another researcher of the team is doing similar study with five 5th grade classes.

[2] Two of the students were absent on the day of the interview. They were later interviewed individually.

during 5th or 6th grade. However, only half of the students learn to locate the fraction $\frac{3}{4}$ as a positive number smaller than one. Most likely, others do not perceive the fraction as a number at all. We see also a significant gender difference favoring boys in task 2 b ($p < 0.001$), and among 5th graders also in task 1c ($p < 0.01$) (using Mann-Whitney U-test).

	5th graders (N=1154)	7th graders (N=1903)	All (N=3067)
Girls (N=1522)	43 %	85 %	69 %
Boys (N=1525)	50 %	86 %	73 %
All (N=3067)	46 %	86 %	71 %

Table 1. Percentage of correct answers in shading $\frac{3}{4}$ of an eight-piece bar.

	5th graders (N=1154)	7th graders (N=1903)	All N=3067
Girls (N=1522)	15 %	41 %	31 %
Boys (N=1525)	25 %	58 %	46 %
All N=3067	20 %	50 %	38 %

Table 2: Percentage of answers locating $\frac{3}{4}$ within the interval 0-1.

There was a clear relation between the bar task and the number line task, mastering one being a requisite for being able to solve the other. If the student was unable to solve the bar task correctly, the likelihood of him/her solving the number line task correctly was only 8%. Moreover, of those who were able to solve the number line task, 93% had solved the other task correctly. Even the computational tasks were difficult for the 5th graders. The low success rates (23 - 43%) are easily explained by the fact that these topics had not been yet taught in most schools. Most 7th graders (83%) answered correctly that $\frac{5}{8} < \frac{5}{6}$ and 66 percent gave correct answers to the two other computational tasks. Especially interesting here is that 30% of those 7th graders, who knew (with high certainty) that $\frac{1}{5} = 0.2$ located $\frac{3}{4}$ outside the interval 0 - 1. Thus, it seems, that even if a student is able to transform a fraction into a decimal, s/he may be unable to perceive it as a number.

Error analyses

Analyzing the answers of the 97 students in the five chosen 7th grade classes we found out that the correct answer was most common one (49 %) in the number line task (Figure 3). Another 5% had located $\frac{3}{4}$ incorrectly but somewhere between zero and one. Furthermore, a quarter of students had located it between 2.5 and 3.5, and 1 % of the answers were between one and 2.5. One student had marked the fraction on the right side of 3.5, and 6% had not given any answer.

Interview data

One thing that became clear in the interviews of the 20 7th graders was that an improper fraction $2\frac{1}{5}$ was much easier to put on the number line than $\frac{3}{4}$, and no one made a mistake with that task. Furthermore, it was possible to identify two different ways to solve the number line task correctly, and five different misconceptions behind students' incorrect answers in the number line task.

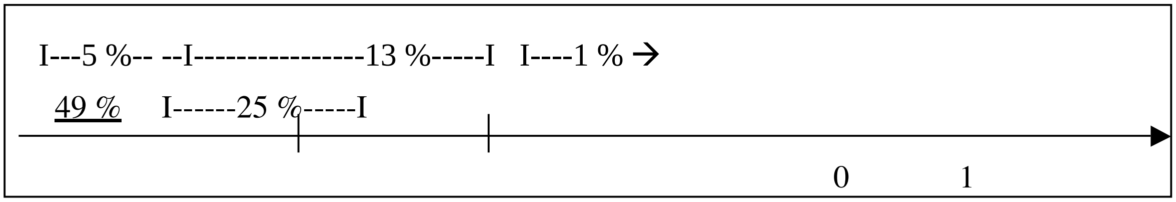

Figure 3. Amounts of seventh-grade students' locating _ within different intervals on a number line (N=97).

$\frac{3}{4}$ = 3.4 The first kind of misconception is a simple wrong interpretation of the mathematical symbolism. The only clear example of that comes from the student S10 who had written 3.4 under the tick he had drawn on a number line. This was a systematic error by the student who also in the interview explained why he put the note on the right side of 3: "I thought that this is 3.4." Such interpretation of $\frac{3}{4}$ was appealing for another student S8 in the same interview group. He had originally located the note correctly after a long hesitation, but later moved it to where S10 had put his note, and explained that he was thinking it as "a decimal thing".

$\frac{3}{4}$ is Not Really a Number. A fundamental conceptual misunderstanding became evident in an interview with student S11. She could not perceive $\frac{3}{4}$ at all as a number on a number line. When asked to put the fraction on the number line, she could not do it.

S11: I don't know. (I don't have ---) {Lets the note fall from her hand. Pulls her arms into her lap.}[3]

I: If I required you to put it (on the number line, where would you put it?)
S11: I don't know
I: Is that a number?
S11: No.
I: What is it then?
S11: A number {laughs}. I dunno.

She could not locate $\frac{3}{4}$ anywhere. However, in the following tasks she was able to locate 1.5 and $2\frac{1}{5}$ correctly on the number line. Hence, I returned to the fraction $\frac{3}{4}$

[3] Text in brackets represents the plausible words of unclear speech, non-verbal communication is written in curly brackets.

I: Would you like to try that $\frac{3}{4}$ again.
S11: Nope. Because it has no forenumber.
I: What forenumber there (--)
S11: That (two or one {points to 2 1(5 and 1.5} --)
I: If we put zero as the forenumber? (Zero whole and _?)
S11: {Takes $\frac{3}{4}$ in her hand} Umm. So then it would be somewhere {thinks, puts approximately to $\frac{3}{4}$} (somewhere) {slides the note to right place} must be there, I don't know. Somewhere so, that it's before one.
I: Is this {points to $\frac{3}{4}$} (same as zero whole $\frac{3}{4}$?)
S11: No
I: What's the difference?
S11: There's zero in there.

Hence, at the end part of the interview we can see, that in her understanding _ is not the same as 0 _. The latter has a unique location on a number line, while _ is something else.

Three out of four. The next error type interprets $\frac{3}{4}$ as "three out of four" which equals 3. In the test the student S2 had drawn a following figure as her answer in the test, which illustrates such line of reasoning (Figure 4). However, in the interview she put the note to the correct place and was able to give a clear explanation.

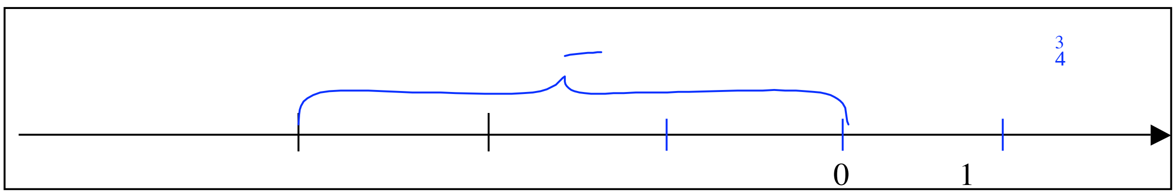

Figure 4. A drawing by S2 in the test.

$\frac{3}{4}$ of what? The next family of errors is based on an understanding of $\frac{3}{4}$ as three parts of a whole divided into four. However, these students incorrectly think of the drawn number line as the whole, or they think of the end segment of the number line from zero or one to the arrowhead. Student S1 stands out as a clear example of such thinking. He put his note to the number three on the number line and explained his thinking.

S1: (I was thinking of) three fourths of the whole that number line.
I: (-- Where from did you start counting the whole number line?)
S1: {Points to the zero} (From there approximately --)

$\frac{3}{4}$ Of Which Unit? Yet another family of mistakes was based on an understanding of $\frac{3}{4}$ as three quarters of a unit, but of a wrong unit. Thus, the number could be put before one, two, three or four. In interview, the student S6 had a hard time deciding where to put $\frac{3}{4}$ on the number line, and her utterances reveal this problem of specifying the unit.

Students S4 and S5 put their notes to right place, S6 becomes confused.
S6: Heyy! {sounds desperate} {Begins to giggle confusedly}

I: Tell now, where you would have< (Where was you thinking to put it.)

S6: {Becomes serious} but how can it be a fourth, if it is there? {smile}

I: Well, you tell how YOU thought it?

S6: No, but. Sort of< I thought that, it is somewhere there after three, you know. Heheh (--)

I: So you would have put it somewhere here? {points to number line on the right side of three}

S6: Yes. But, how can it then be, like before one? Because if, you know, in principle it could be like tw< before two? Or something.

Flexible fraction concept. The student S3 had located the fraction correctly in the test, but written also a comment "(out of one (?))" next to her answer. Such comment is related to aforementioned misconception. In the interview she also located the note correctly, but when she was explaining her thinking, she accepted also the interpretation made by the student S1, that $\frac{3}{4}$ could be measured out of the whole number line (see an earlier transcript: "$\frac{3}{4}$ of what?").

S3: Yeah, me too, (I chose out of one) three fourths. (So) one could have put it also here {points to number three} where it would have been out of four, or out of the number four three fourths

Taken together, this student showed flexibility in her conception of $\frac{3}{4}$. She chose to locate it to 0.75, but she realized that one could choose a different whole and end up with a different answer.

Correct answers. Most of the students who solved the task correctly halved the segment 0 - 1, and then halved the segment $\frac{1}{2}$ - 1 to find $\frac{3}{4}$ However, two students transformed the fraction into a decimal. They explained that they had thought of $\frac{3}{4}$ as 0.75, which they knew to be a little less than one.

CONCLUSIONS

With respect to learning fractions, there is considerable development from 5th to 7th grade. Robust gender differences were found when task was difficult for the age group. When a task had became routine (e.g. the bar task and computing $3 \bullet \frac{1}{5}$ for 7th graders), the gender differences diminished. Such pattern of gender differences can be understood in the light of a general conclusion made by Fennema and Hart (1994). According to them, gender differences in mathematics remain within the most difficult topics, although the differences in general seem to be getting smaller.

Although most 7th graders had learned to compute with fractions their conceptual understanding was weak. Similarly to previous studies, we found that Part-to-Whole-Comparison was the dominating scheme also for Finnish 7th graders. In case of simple fractions, many students could not locate it correctly on the number line. The main difficulty for students was to determine what was the 'whole' wherefrom to calculate the fraction. However, in case of improper fractions 7th graders had no such difficulty.

Comparing the findings of this survey with the results by Behr & al. (1983), we see that Finnish 5th graders perform drastically worse and 7th graders notably better than the 4th graders in that study. Furthermore, 7th graders in the study by Novillis-Larson (1980) performed considerably better than Finnish 7th graders in this study. However, we should remember that the number line that was used in this study was different than in the other ones, and the nature of visual cues seems to be important.

A hypothesis was made that because of a linguistic clue Finnish students might be inclined to perceive fraction as a number with a unique value. However, there was no clear evidence for it. One of the interviewed students simply refused to locate $\frac{3}{4}$ on number line and she was ambivalent on whether it really is a number or not. Several others could not locate the fraction $\frac{3}{4}$ within the right interval between zero and one. Error rate with number line task was greater in this study than in the cited studies with English-speaking subjects. However, these differences may also be due to different curricula or differences in the test items.

Students' understanding of rational number concept develops considerably from 5th to 7th grade. However, half of the 7th graders are still unable to locate a simple fraction even roughly to a right place on a number line. Their problem seems mainly to be in sticking to a Part-to-Whole schema while being unable to identify the whole correctly.

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