ONE LINE PROOF: WHAT CAN GO WRONG?

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Having an ability to appreciate, understand, and generate proofs is crucial in being able to evaluate students' mathematical arguments and reasoning. As such, the development of this ability in perspective teachers is imperative. This study examines the work of a group of preservice elementary school teachers in their efforts to generate one-line proofs on closure statements. We provide a framework that allows us to carry out a fine grain analysis of students' proofs and also provides a tool for diagnosis and remediation.

Mathematics occupies a privileged position among the sciences as the discipline that is most pure, most exact. The facet of mathematics that is most directly responsible for facilitating this honor is that of proof. The mathematical proof provides the certainty that is demanded in a field where precision and exactness is the currency of practice. A well-constructed deductive proof offers humans the purest form of reasoning to establish certainty. As such, proof is an important part of not only mathematical practice, but also of mathematical learning and teaching (Hanna, 1989).

However, research has repeatedly shown that proofs and the ability to understand and generate proofs is difficult for students in general (Hoyles, 1997) and for preservice elementary school teachers in particular (Barkai, Tsamir, Tirosh, & Dreyfus, 2002; Martin & Harel, 1989; Simon & Blume, 1996). This may be due, in part, to the fact that proficiency with proof requires the coordination of a number of competencies, identification of assumptions, and organization (or tracing) of logical arguments. The teacher, as the person who establishes the expectations and norms of a mathematics classroom, plays a crucial role in development of such competencies (Yackel & Cobb, 1996). Furthermore, the teacher's own aptitudes are requisite in the evaluations of students' arguments and mathematical reasoning (Barkai et. al., 2002). Therefore, it is imperative that, as difficult as it has proven to be, the ability to understand and generate proofs be instilled in perspective teachers.

Having said that, however, it may be beneficial to the achievement of such goals to find instances where the rigours and demands of exactness required in a proof are mediated by a reduction in the length of the proof. As such, this study examines preservice elementary school teachers' efforts to generate one-line proofs. We examine the abilities of the participants in generating such proofs as well as provide a framework for the analysis of their efforts. The framework, which allows for a fine grain analysis of the participants' work, also gives insights into the complex coordination of competencies that is required even for the writing a very short proof.

THE STUDY

Participants in this study were preservice elementary school teachers (n=116) enrolled in a course "Principles of Mathematics for Teachers", which is a core course in a teacher education program. One of the issues that wove itself through all the topics of the course was the need for support of mathematical claims. Extensive discussions and exercises were aimed at helping students understand when and where a general argument, or a proof, was needed and when an example was sufficient.

During the course the participants were exposed to the concept of closure as part of the discussion of number systems. The formal definition was provided – a set is said to be "closed" under operation if and only if for any two elements in the set, the result of the operation is in the set. Further, a variety of examples of sets closed or not closed with respect to certain operations were provided and a variety of problems in which students had to prove or disprove closure were posed. This included the invitation to prove or disprove claims such as *even numbers are closed under addition*, *multiples of 5 are closed under multiplication*, *rational numbers are closed under division*, *prime numbers are closed under addition*, among others.

In this study we analyze two questions that sought a written response from the participants:

> *(Q1) The set of perfect squares is closed under multiplication. Prove the statement or provide a counterexample.*
> *(Q2) The set of odd numbers is closed under multiplication. Prove the statement or provide a counterexample.*

MATHEMATICAL ANALYSIS AND FRAMEWORK

For our purposes, we considered the "ideal" solutions (that is, proofs) of these statements to be:

> *(Q1): Let a^2 and b^2 be any two square numbers.*
> *Then, $a^2 \cdot b^2 = (ab)^2$ which is itself a square number.*
> *(Q2): Let $(2m+1)$ and $(2n+1)$ be two odd numbers.*
> *Then $(2m+1)(2n+1) = 4mn+2m+2n+1 = 2(2mn+m+n)+1$, which is itself odd.*

However, the generation of such seemingly simple and short proofs is deceivingly intricate, requiring an appreciation of the need for, and the coordination of many skills (see Figure 1).

First and foremost is the recognition that a proof is indeed required for the purposes of establishing the truth of a statement. From a mathematical perspective, such a requirement is obvious. The establishment of the validity of a statement requires the treatment of the statement in general, as opposed to the examination of a few particular cases. Once a need for a proof has been established, the students then need to be sensitive to the fact that treatment of the general case requires the selection of some form of representation. Representations play a crucial role in mathematics; they are considered as tools for communication, as tools for symbolic manipulation, and as tools that promote and support thinking (e.g. Skemp, 1986, Kaput, 1991). Furthermore, the choice of representation is often linked to students' understanding of the content (Lamon, 2001).

However, the recognition that a representation is needed is not enough. The students must select one that is both correct and useful for the purposes of generating a proof. For Q1(above), for example, choosing to represent the two square numbers as *X* and *Y* is in itself not incorrect, but for the purposes of generating a proof, it is completely useless. A much more effective (and natural) representation of two square numbers is a^2 and b^2.

Once such a representation is established, the students must then be able to work with it. That is, they must be able to perform correctly any manipulations necessary to transform the expression into the form that clearly represents the nature of the number. In the example of Q1 such a manipulation is not onerous. Q2, however, requires much greater adeptness with algebraic manipulation in order to mould the expression into one that clearly expresses its inherent ‘oddness’. There is an assumption in this last sentence, though.

STUDENTS’ RESPONSES

A complete and correct proof was provided by 19% of the participants for Q1 and by 37% of the participants for Q2. However, it is not our purpose to quantify student’s responses. Instead, we have organized students’ incorrect responses according to the framework provided above. What follows are exemplars of this organization.

<u>(Not) Recognizing the need for a proof</u>

Among participants who did not attempt a proof we recognize two kinds of arguments. One is a narrative style that reiterates the statement, at times explaining what is to be proven. For example:

> (Q1): *The set of perfect squares is closed under multiplication because no matter what 2 perfect squares you multiply together, your answer will always be another perfect square.*

Another is justification with a single numerical example. For instance:

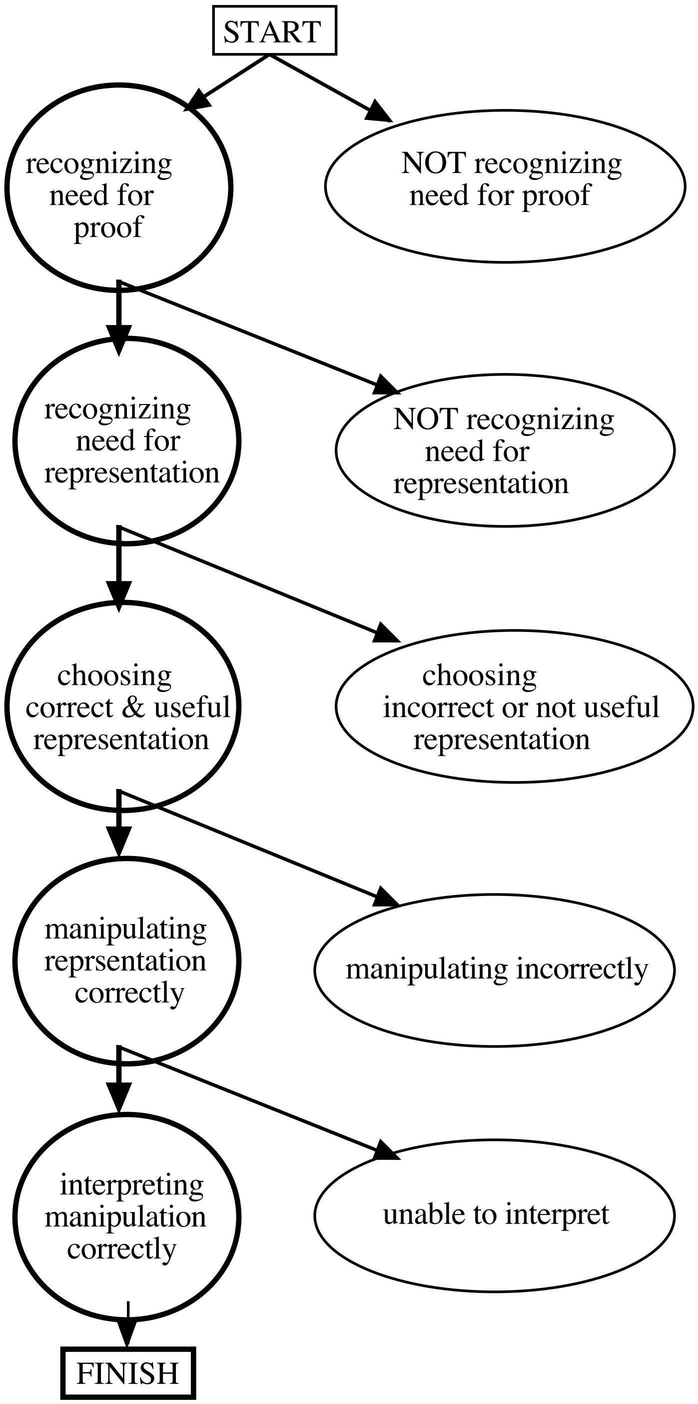
> (Q2): *{1,3,5,7,9, …}*
> *3×5 =15*
> *The product of 2 odd numbers is an odd number.*

<u>(Not) Recognizing the need for representation</u>

We believe that a general argument in this case requires a representation of the objects in question. However, some of the participants justified their decision of closure with an inductive argument, as exemplified below.

> (Q1): *The set of perfect squares is closed under multiplication because if you multiply 2 of the numbers inside the set you get another perfect square as a result.*
> *4×25=100* *4×36=144* *25×36=900*
> $\sqrt{100} = 10$ $\sqrt{900} = 30$ $\sqrt{144} = 12$
> *etc.*

> (Q2): *O={1,3,5,7,9,11,13…}*
> *5×7=35* *11×9=99* *15×13=195*
> *True, the set of odd numbers is closed under multiplication. Any two odd numbers multiplied together will result in an odd number.*



The phrase *clearly expresses* assumes that the students are able to interpret the result of their manipulation as representative of what they are aiming to show. This is the last step in the proof process. The students must be able to constantly interpret their manipulations in order to know what they have found, and when they have found it.

The left side of the diagram in Figure 1 represents the steps towards a complete and correct proof. The right side represents the potential obstacles at every step.

Figure 1. Pathway towards (and digression from) a one line proof

We distinguish justification by a single example from justification by a series of examples. In the former case a student may believe that one example is sufficient. In the latter case we recognize an attempt to build an inductive argument, which identifies students' empirical proof schemes (Harel & Sowder, 1998). While empirical verification is very useful in clarifying the problem, it is only a preliminary stage towards a proof. However, it is very common for students to prefer an empirical argument over any sort of deductive reasoning (Hoyles, 1997).

(Not) Providing a useful representation

Once the need for representation has been recognized, it is essential to choose a correct and useful representation. Examples below demonstrate students' choice of representation that is inappropriate for the task at hand.

> (Q1): *Let X equal any whole number, then* X^2 *equal a perfect square.*
>
> X^2 □ X^2 *also a perfect square?*
>
> X^2 □ $X^2 = X^4$ *, yes* X^4 *is a perfect square so the set of perfect squares is closed under multiplication.*
>
> (Q2): *k*□*W,* □ *k+1*□*W, k+3*□*W*
>
> $(k+1)(k+3) = k^2+3k+k+3 = k^2+4k+3 = (k^2+4k)+3$
>
> *adding 3 makes the # odd, so the set of odd #s is closed under multiplication.*

The above response to Q1 does not satisfy the generality; while X^2 is an appropriate representation for a square number, using it for two different numbers compromises the argument. In the response to Q2, *k+1* and *k+3* represent odd numbers only if *k* itself is even, but this constraint was omitted. Furthermore, consideration of consecutive odd numbers compromises the intended generality of a proof.

(Not) Manipulating the representation correctly

The ability to chose a correct and useful representation is a necessary condition, but not sufficient. The next step is the ability to manipulate the chosen representation successfully. Unfortunately, as shown below, manipulation of algebraic symbols presented an obstacle for some participants.

> (Q1): *True. X,Y are whole numbers*
>
> *a=* X^2 *= perfect square, b =* Y^2 *= perfect square*
>
> *Prove: ab = number*^{2}
>
> $ab = X^2Y^2 = XY^2$ *= perfect square*
>
> (Q2): *a and b and c are whole #s,*
>
> *(2a+1)* □ *(2b+1) = 2ab+1, this is odd, since ab must be whole (set of whole #'s closed under multiplication), and any whole # times 2 is even, so plus one must be odd. So any odd # multiplied by any odd # equals an odd #.*

Note the perfect structure of the argument in the second example. Unfortunately, it is based on an incorrect symbolic manipulation.

(Not) Interpreting the manipulation

The ability to manipulate algebraic expressions pays off only if a learner is able to interpret the result of such manipulation. However, the data show, that several students were on the brink of completing the proof, but did not recognize it. That is to say, they were not able to interpret the result of their manipulation. Consider for example the following response to Q2:

Let (2n+1) and (2m+1) represent odd numbers.

$(2n+1)(2m+1) = 4mn+2n+2m+1$

$2\,|\,4mn+2n+2m+1; \quad 2\,|\,2mn+n+m+1$

This student has chosen a useful representation and also manipulated it correctly. However, this correct manipulation is followed by rather random symbol pushing, and no conclusion with respect to the closure of the set in question is presented. It appears that this student wasn't sure how to proceed in interpreting her manipulation.

CONCLUSION

The purpose of this study is twofold. One, it provides a framework for analyzing short proofs related to the notion of closure. Two, it shows viability of this framework by providing an analysis of students responses. For examining the work of students that did not complete a proof, the framework assists in identifying the obstacle that threw the student "off track". In such it provides an avenue for remedial instruction.

The study demonstrates that the concept of closure was generally well grasped. That is to say, the majority of students understood that they were expected to show that the product of two perfect squares is a perfect square, and the product or two odd numbers results in an odd number. What is considered as proper "showing" is a more general issue, extensively discussed in prior research (e.g. Harel & Sowder, 1998).

However, it is troublesome that what prevented some students from completing the proof was not their understanding of closure, or appreciation of the need for a proof, but a poor ability to choose an appropriate representation or inability to manipulate the chosen representation. The latter draws the focus from undergraduate teacher education and invites regression to skills of simple algebraic manipulation. Lack of competence in these skills presents an obstacle not only for correct manipulation, but also for interpreting the meaning of manipulation, that is, the ability to represent the manipulated expression in a desired form.

We believe that experience with one-line proofs is a valuable tool for sharpening the proof skills. The content of closure provides appropriate grounds not only for generating these proofs, but also for appreciating the role of useful representation. Furthermore, we suggest that the framework that we developed is appropriate for analyzing a variety of short proofs related to number properties. Future research will determine the scope of applicability of this framework.

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