PERSPECTIVE-TAKING IN MIDDLE-SCHOOL MATHEMATICAL MODELLING: A TEACHER CASE STUDY

Lyn D. English
Queensland University of Technology, Australia
Helen M. Doerr
Syracuse University, New York

Traditional word problems have not fulfilled the goal of mathematical sense-making for many students. Some studies have shown that authentic contexts, such as model-eliciting tasks, have the potential to engage students in making sense of realistic situations. However, there has been little research on the kinds of knowledge needed by teachers to support this type of student learning activity. In this paper, we report on the results of a case study that investigated the ways in which teachers respond to students' thinking while engaged in a model-eliciting task in data analysis. We describe how one teacher used perspective-taking to initially engage students with the task, to explain and justify their models, to assess the quality of their models, and to make connections to other mathematical ideas.

INTRODUCTION

Helping students understand the meaningfulness or significance of their mathematics learning is a major goal of education (Bransford et al., 2000). When students see the potential relevance of their mathematical experiences, they are more likely to engage in sense-making in their learning activities. The challenge for teachers lies in implementing authentic and mathematically rich learning experiences that are relevant and meaningful for students. Traditionally, one of the main ways in which teachers have attempted to bring meaning to students' mathematical learning is through word problems that comprise verbal descriptions of problem situations presented in a "real-world" context. The implicit idea behind the instructional goal of these problems is "to bring reality into the mathematics classroom, to create occasions for learning and practising the different aspects of applied problem solving, without the practical …. inconveniences of direct contact with the real world situation" (Verschaffel, 2002). At the elementary and middle school levels, solving such problems usually involves the application of one or more mathematical operations on the numbers contained in the problem (Verschaffel, Greer, & deCorte, 2000).

Numerous studies over the years, however, have indicated that traditional word problems have not fulfilled the goal of sense-making; that is, reality and school mathematics continue to remain as separate entities for many students. Students simply apply one or more operations without giving thought to possible constraints of the realities of the problem situation that may make such application inappropriate (e.g., Boaler, 2000; Lave, 1992; Schoenfeld, 1991; Verschaffel et al., 2000). Verschaffel (2002) reviewed several studies in which students were required to use judgement based on real-world knowledge and assumptions rather than the routine application of arithmetical operations (e.g., *John's best time to run 100 meters is 17 seconds. How long will it take him to run 1 kilometer*? p. 67). Not surprisingly, findings across several nations have revealed students' tendency to respond to such word problems in a stereotyped and non-realistic manner.

Related studies (e.g., De Franco & Curcio, 1997) have indicated that authentic problem contexts, where students participate directly in the problem situation, significantly improve students' inclination to apply their real-world knowledge to the solution process. One of the principles behind the model-eliciting problems that we used in the present study is the "Reality Principle" (Lesh, Hoover, Hole, Kelly, & Post, 2000). This principle, which governs the meaningfulness of the problems, emphasizes the importance of students making sense of the problem situation based on extensions of their personal knowledge and experiences. At the same time, the contexts of model-eliciting problems are designed to expand students' interests, rather than just catering to them.

A powerful, yet little researched, way in which teachers can support students' sense-making in a problem-solving situation is through perspective-taking. We define perspective-taking as understanding how the reality of a problem situation might be perceived from multiple points of view. This can be achieved by considering the problem situation from one's own perspective or from the perspective of others, such as the central characters within a problem context. We propose that when students are encouraged to adopt a particular perspective, they are being asked to create an imagistic system (Goldin, 1998). We take such systems to include not only the configurations of mathematical objects, but also how differing configurations might appear from different points of view. As such, imagistic systems play a significant role in interpreting, solving, and evaluating the solutions of modeling tasks.

We posit that students' development of significant mathematical models is dependent on the nature of the problem activities given and on the ways in which these problems are conceived and dealt with by the teacher. However, most research has focused on the nature of the problems, with considerably less research on the conceptions and strategies of the teacher. In this research, we have focused on the teachers' learning and reasoning as they implemented a sequence of modelling tasks. In particular, we were interested in understanding the ways in which teachers interpreted the tasks of teaching modelling problems in data analyses so that students engage in meaningful sense-making of the context and processes. In this paper we report on how a middle-school teacher, who participated in a teacher development program, initiated the use of perspective-taking as a way of promoting her students' mathematical modelling and sense-making activity.

DESCRIPTION OF THE STUDY

Participants

Seven middle-grade teachers and their classes participated in our study. They were from a co-educational private school situated within a middle class neighborhood in Brisbane, Australia. Neither the students nor their teachers had experienced modelling problems of the type implemented in this study. The teachers welcomed the opportunity to explore new ways to engage students in meaningful problem-solving activities. All the middle-grade mathematics teachers in the school, along with the head of the mathematics department and the school principal, were enthusiastic about participating in the project.

Program

The teacher development program comprised four teacher meetings, which were attended by all the teachers (except one who missed two meetings) and by the head of department. These meetings were intended to familiarize the teachers with the problem sequence by engaging them in a discussion of their own solutions to the problem as well as anticipated student solutions. The teachers' current practice included only limited use of group work, so pedagogical strategies for interacting with students in groups were discussed. The primary emphasis in terms of teaching strategies was to encourage and allow the students to develop, evaluate, revise and generalize their own solutions to the problems. The complete problem sequence was initiated over a period of 10 -12 lessons (depending on the teacher).

The student activities were a sequence of model development activities comprising five problem situations that require students to create usable rating systems in a range of contexts (cf. Doerr & English, in press). The core mathematical ideas were centered on notions of ranking, selecting and aggregating ranked quantities, and weighting ranks. For each problem, the students worked in small groups to analyze and transform entire data sets or meaningful portions thereof, for the purpose of decision making. The sequence of activities was designed so that the students could readily engage in meaningful ways with the problem situation and could create, use and modify quantities (e.g., ranks) in ways that would be meaningful to them and in ways that could be shared, generalized, and re-used in new situations. Our focus in this paper is on the first problem of the sequence, namely, the *Sneakers Problem.*

In the *Sneakers Problem*, the students encounter the notion of multiple factors that could be used in developing a rating system for purchasing sneakers and the notion that not all factors are equally important to all people. Students were asked "What factors are important to you in buying a pair of sneakers?" This generated a list of factors where not all factors were equally important to the students; the students then worked in small groups to determine how to order these factors in deciding which pair of sneakers to purchase. The students naturally produced different lists. The teacher then posed the problem that the sneaker manufacturer needed a single set of factors that represents the view of the whole class; in other words, the group rankings needed to be aggregated into a single class ranking. As we report below, the context of this problem, beginning with the point of view of the manufacturer, provided the teacher with the opportunity to use perspective-taking to support the sense-making efforts of her students.

Data Collection and Analysis

Of the seven teachers, we chose two for in-depth observation based on grade level, on prior observations of their lessons, on their mathematics background, and on their willingness to participate in this research. In this paper, we focus on the data collected from the first two lessons of one of these teachers, namely, a seventh-grade teacher whose primary teaching subject was biology not mathematics. This teacher frequently expressed to us her lack of confidence in teaching mathematics and how she felt that her mathematics teaching was more rote and less investigatory than her science teaching.

Each lesson was videotaped and audio-taped by the authors and detailed field notes were taken. The video-taping focused on the teacher and her interactions and exchanges with the students in her class The teacher meetings were audio-taped. The data analysis was completed in two stages. The first stage of analysis involved open-ended coding (Strauss & Corbin, 1998) of the transcripts of each lesson. Each author did this coding independently. This was followed by viewing the videotapes for each lesson, and adding annotations and clarifications to the transcript that were visible from the videotape. We then met to compare our coding and made revisions and refinements where needed.

The second stage of the analysis consisted of finding clusters of codes for each teacher that defined the critical features or characteristics for each lesson. These features describe the dominant events that governed the lessons, such as the ways in which the teacher encouraged student thinking, the ways in which she employed representations, the incidents in which she asked for meaning, explanations, and justifications. One feature that was prominent in the present teacher's interactions with the students' was the varied ways in which she encouraged perspective-taking. This feature was not emphasized in our teacher development program and was not evident in the other teacher's interactions with her students.

RESULTS: THE TEACHER'S USE OF PERSPECTIVE-TAKING

Our analysis of the data revealed a number of ways in which the teacher used perspective-taking to promote her students' mathematical modelling during the Sneakers Problem. These included using it as a means of introducing and focusing the problem for the students, as well as encouraging the students to adopt multiple perspectives for various purposes throughout the problem activity. These purposes included: to construct generalized models, to explain and justify models, to anticipate consequences, to assess, revise, and refine models, and to make connections.

Focusing the Problem

After the students had suggested a number of factors that they considered important in buying a pair of sneakers, the teacher used perspective-taking to draw the students' attention to the next component of the problem:

> A particular target group in the community are young teenagers like yourselves…Shoe companies actually want *your* input into the sorts of things you think are most important when you decide to buy a pair of shoes….. Now that doesn't necessarily mean whether Mum or Dad would agree but if you had the money and it was *your* decision, what things are most important for you to consider when you want to buy a pair of shoes?

As the teacher observed each group making their lists, she again reminded the students of the perspective they were to adopt ("Now is this a list based on what you believe or what your parents might believe?"). In this way, she was encouraging them to order the factors from their point of view, which was important to the shoe company.

Once the students had generated their various lists of ordered factors, the teacher again highlighted the perspective of the shoe manufacturer to direct the students' attention to the differences in the lists and thereby introduce the next component of the problem:

Teacher: Ok girls. You've just made a really important decision but I don't know how I'm going to go back to this manufacturing company with these lists because what do you notice about them?"

Students: They're different.

Teacher: They're all different. But you're all 12 year-old girls and surely you all think the same when it comes to fashion and shoes?

Students: No.

Teacher: Well, what's the dilemma here? Do we have a problem?

In this instance, the teacher takes on the perspective of the shoe manufacture to pose the essential difficulty in the task, namely, that the different lists of each group will need to be combined into one list for the manufacturer. This, then, is used to engage the students in the mathematics of combining ranked lists of data.

Encouraging Explanation, Justification, and Generalized Models

The teacher displayed a strong focus on the students' construction of generalized models and on their ability to explain and justify their models. In doing so, she frequently posed different perspectives for the students to consider, including that of a marketing researcher, a shoe manufacturer, and a mathematician. For example, one group gave a subjective explanation of how they arrived at their model:

Student: Most people thought that size was more important than comfort… you should get a size that suits you and then see if it's comfortable.

The teacher prompted the students to consider the perspective of the manufacturer:

Teacher: OK. But if a shoe company had that information to work with, would they use that same logic to decide on a final list?

In this way, the teacher encouraged the students to re-consider the logic of their justification that was based on their opinions rather than the information in the lists. In response to a student's suggestion that more lists were needed a "bigger data" set, the teacher continued to challenge the students' justification by posing the perspective of a mathematician engaged in sense-making. One student, however, could not see the purpose of adopting such a perspective:

Teacher: Ok, how do you think the mathematicians who are trying to make some sort of sense out of those lists, how do you think they would go about trying to figure out which came fist, which came second? Do you think they would use this method?

Student 1: Probably not.

Student 2: Anyway, we're not looking at what mathematicians would do.

Teacher: Well, ultimately, we're after a list that is best for the shoe company to market the shoe, so they need to know whether they've got to focus on fashion, or whether they've got to consider price as the most important.

In other interactions with the students, the teacher emphasized the importance of a mathematical justification. For example, after observing one group of students use a subjective approach to working the problem, the teacher asked, "Are you sure that this isn't just what you think the order should be? Are you sure that this has been justified mathematically?" In questioning another group, the teacher asked, "Are you confident you're going to be able to get up there [before the class] and explain this [their model] as marketing researchers with degrees in mathematics?"

Assessing Models

After the groups had generated their models, the students presented their work to the class. Once the students had described and displayed their solutions, the teacher asked them to compare the final lists of factors they had produced. In doing so, the teacher adopted the perspective of a market researcher as she posed the question: "Which lists do you think the market research people would go with?" The students chose a list that had been created by three of the groups, namely, 1. price (most important), 2. size, 3. fashion, 4. style, 5. comfort, 6. quality, 7. color, 8. purpose, 9. brand (least important). One student explained why this list is preferred:

> Because they actually make sense, like, compared to the others. Some people might not be able to understand the bottom ones [of the displayed lists] whereas they can understand the ones at the top.

In response, the teacher asked the students what they would have anticipated as a solution, given their original lists of factors:

> So you think the market research people would go with that list because three groups came up with that list? Now what would we have expected, considering that our original six lists were very different from each other? What would we have expected to get from working on the ultimate list—this new list—what should we have seen?

By asking the students the above, the teacher was helping them appreciate the importance of developing a generalized model. In the class discussion the students were able to explain that each of the three lists was generated by the same model, namely, ranking, summing the ranks, and then re-ranking (some students also averaged the ranks).

Making Connections

On several occasions, the teacher focused on connections between the students' existing mathematical knowledge and their new modelling experiences, as well as on connections between the modelling experiences and the real world. One example of this occurred on conclusion of the modelling program. The teacher asked the students to reflect on their modelling activities and to indicate whether they had applied some of their prior mathematics learning to these new experiences. This led to a discussion about their recent work on data and chance and the use of surveys. The teacher then posed the perspective of a market researcher: "How do you think market researchers use mathematics to work out what people like in a community?" Since one of the students was to take part in a weekend survey conducted by market researchers, a unique opportunity arose for further use of perspective-taking. A class discussion followed in which the students considered the mathematical nature of survey work from the perspective of marketing companies.

DISCUSSION

In these lessons, we observed how the teacher initiated the use of perspective-taking as a means of promoting her students' mathematical modelling. The context of the modeling problem facilitated multiple perspective-taking, which encouraged students in their realistic sense-making efforts. The teacher's use of perspective-taking served a number of purposes.

The teacher used the perspective of the shoe manufacturer and the students' own perspective on desirable features in a pair of sneakers as a way of focusing the students' attention on the problem. In the initial part of the problem, the teacher directed the students to consider desirable features from their own perspective, rather than that of others. The teacher then switched the perspective-taking focus from the students to the shoe manufacturer. The purpose here was to highlight the difficulty faced by the manufacturer when there are multiple lists of different features to consider. This naturally led to the next component of the problem, namely, to develop a model that would enable the lists to be aggregated.

In her efforts to encourage the students to construct generalized models, the teacher again used the manufacturer's perspective, along with that of market researchers and mathematicians. When students used a subjective approach to aggregating the lists, the teacher encouraged them to consider the perspective of the manufacturer. That is, she was prompting the students to reconsider the logic of their justification, given the needs of the manufacturing company. When students continued to justify their subjective methods, the teacher challenged their thinking by posing the perspective of a mathematician engaged in sense-making.

The perspective of a market researcher was used on several occasions, including to prompt the students to assess the models generated by the different groups, as well as to draw connections between the students' modeling experiences and real life. The ways in which market researchers use mathematics formed the basis of a concluding class discussion.

The teacher's decision to employ perspective-taking for multiple purposes reflected her awareness of the importance of students' sense-making as they worked the modeling problems. This decision also highlighted the knowledge and understanding the teacher had gained about implementing model-eliciting problems. For example, she stressed the need for students to construct generalized models, to explain and justify their models, and to assess, revise, and refine their models. Finally, the teacher's efforts in helping the students connect their new modeling experiences both to their prior learning and to the outside world were enhanced through perspective-taking.

References

Boaler, J. (2000). Introduction: Intricacies of knowledge, practice, and theory. In Boaler, J. (Ed.), *Multiple perspectives on mathematics teaching and learning (*pp. 1-18). Mahwah, NJ: Lawrence Erlbaum Associates.

Bransford, J., Zech, L., Schwartz, D., Barron, B., Vye, N., & The Cognition and Technology Group at Vanderbilt. (2000). Designs for environments that invite and sustain mathematical thinking. In P. Cobb, E. Yackel, & K. McClain (Eds.), *Symbolizing and communicating in mathematics classrooms: Perspectives on discourse, tools, and instructional design* (pp. 325-360). Mahwah, NJ: Lawrence Erlbaum Associates.

DeFranco, T. C., & Curcio, F. R. (1997). A division problem with a remainder embedded across two contexts: Children's solutions in restrictive versus real-world settings. *Focus on Learning Problems in Mathematics, 19(*2), 58-72.

Doerr, H. M., & English, L. D. (2003). A modeling perspective on students' mathematical reasoning about data. *Journal for Research in Mathematics Education*, March issue.

Goldin, G. A. (1998). Representational systems, learning, and problem solving in mathematics. *Journal of Mathematical Behavior, 17*(2), 137-166.

Lave, J. (1992). Word problems: A microcosm of theories of learning. In P. Light & G. Butterworth (Eds.), *Context and cognition: Ways of learning and knowing* (pp. 74-92). New York: Harvester Wheatsheaf.

Lesh, R. A., Hoover, M., Hole, B., Kelly, A., & Post, T. (2000). Principles for developing thought-revealing activities for students and teachers. In A. E. Kelly & R. A. Lesh (Eds.), *Handbook of research design in mathematics and science education* (pp. 591-646). Mahwah, NJ: Lawrence Erlbaum Associates.

Schoenfeld, A. H. (1991). On mathematics as sense-making. An informal attack on the unfortunate divorce of formal and informal mathematics. In J. F. Voss, D. N. Perkins, & J. W. Segal (Eds.*), Informal reasoning and education* (pp. 311-343). Hillsdale, NJ: Lawrence Erlbaum Associates.

Strauss, A., & Corbin, J. (1998). *Basics of qualitative research: Techniques and procedures for developing grounded theory* (2nd ed.). Thousand Oaks, CA: Sage Publications.

Verschaffel, L. (2002). Taking the modeling perspective seriously at the elementary level: Promises and pitfalls. In A. D. Cockburn & E. Nardi, (Ed.), *Proceedings of the 26th Annual conference of the International Group for the Psychology of Mathematics Education* (vol. 1, pp. 64-80). Norwich: University of East Anglia.

Verschaffel, L., Greer, B., & De Corte, E. (2000). *Making sense of word problems*. Lisse, The Netherlands: Swets & Zeitlinger.