MATHEMATICAL PROOF AS FORMAL PROCEPT IN ADVANCED MATHEMATICAL THINKING

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In this paper the notion of "procept" (in the sense of Gray & Tall, 1994) is extended to advanced mathematics by considering mathematical proof as "formal procept". The statement of a theorem as a symbol may theoretically evoke the proof deduction as a process that may contain sequential procedures and require the synthesis of distinct cognitive units or the general notion of the theorem as an object like a manipulable entity to be used as inputs to other theorems. Therefore, a theorem could act as a pivot between a process (method of proof) and the concept (general notion of the theorem). I hypothesise that mature theorem-based understanding (in the sense of Chin & Tall, 2000) should possess the ability to consider a theorem as a "formal procept", and it takes time to develop this ability. Some empirical evidence reveals that only a minority of the first year mathematics students at Warwick could recognise a relevant theorem as a "concept" (having a brief notion of a theorem) and did not have the theorem with the notion of its proof as a "formal procept". A year later some more successful students showed a concept of the theorem as a "formal procept" and their capability of manipulating the theorem flexibly.

INTRODUCTION

Mathematical proof is one of the most important aspects of formal mathematics. From most mathematics textbooks we can simply see the process of a mathematical proof as the development of a sequence of statements using only definitions and preceding results, such as deductions, axioms, or theorems. Theoretically the *process* of a mathematical proof occurs when the proof is built up and looked at subsequently as a process of deducing the statement of the theorem from definitions and the specified assumptions. A proof becomes a *concept* when it can be used as an established result in future theorems without the need to unpack it down to its individual steps. I choose to focus on this sequence of proof as a process of deduction becoming encapsulated as a concept of proof in a manner that would seem natural to most mathematicians. It is noted that there are alternative theories, for example, Dubinsky and his colleagues (Dubinsky, Elterman & Gong, 1988) focus on the use of quantified statements as processes becoming turned into mental objects by applying the quantifiers. Pinto and Tall (2002), in contrast, show how some students are capable of building formal proofs by reconstruction of prototypical imagery used in thought experiments.

ORIGINAL NOTION OF PROCEPT

Gray and Tall (1994) suggested the notion of "*procept*", which was taken to be characteristic of symbolism in arithmetic, algebra and calculus, defined in the following terms:

> An *elementary procept* is the amalgam of three components: a *process* which produces a mathematical *object*, and a *symbol* which is used to represent either process or object.
>
> A *procept* consists of a collection of elementary procepts which have the same object.
>
> (Gray & Tall, 1994, p.120)

The original definition was made in the context where the authors were aware of a wide range of examples and the definition was framed to situate the examples within the definition. In this primary consideration it is a "descriptive definition", in the sense of a definition in a dictionary, rather than a "prescriptive definition", in the sense of an axiomatic theory. However, if we consider the definition of "procept" in a prescriptive view, it seems applicable to extend the original notion of "procept" to the notion of formal proof, which can be called "*formal procept*", by adding the following analysis.

EXTENDED NOTION OF FORMAL PROCEPT

It should be noticed that there are three components of an elementary procept: *process*, *object*, and *symbol*. Now we can put the frame of Gray & Tall's "procept", particularly in the form of an "elementary procept", on the notion of formal mathematical proof. The *symbol* is the statement of what is going to be proved (which can be a theorem). The *process* is the deduction of the whole proof. And the *object* is the concept of the general notion of proof. Therefore, a theorem, for example, which is considered as a formal procept could act as a pivot between a process (method of proof) and the concept (general notion of the theorem). It should be stressed that the individual is not considered to conceive the real meaning of a theorem until the theorem has become a formal procept. With the above interpretation we could see the role of a symbol as being pivotal not only in elementary mathematical thinking but also in advanced mathematical thinking to allow us to change the channel between using a symbol as a concept to reflect on and link to other concepts and as a process to offer the detailed steps to deduce a proof. However, an immediate argument arises. It seems that the above corollary does not always follow because even mathematicians sometimes use certain theorems without fully understanding their proofs. However, I find this viewpoint an *advantage* to our analysis, for it simply shows that such individuals are *not* using theorems as formal procepts, they only have *part* of the structure, usually the statement of the theorem which they then use as an ingredient in another proof without fully understanding the *totality* of the structure. I consider the whole notion of a theorem to be grasped when the notion of proof of the theorem is also assimilated in the individual's understanding. Some evidence here shows that only a few students understand the notion of proof as a formal procept, but the empirical research also shows that, over time, more students grasp the subtlety of the idea.

HIERARCHY OF THE DEVELOPMENT OF SYSTEMATIC PROOF

Chin and Tall (Chin & Tall, 2000) postulated a hierarchy running through the development of systematic proof, in stages consisting of *concept image-based*, *definition-based*, *theorem-based*, and *compressed concept-based.* These stages show successive compressions of knowledge in the sense suggested by Thurston (1990). The first stage, which is concept image-based sees the student having a concept image of a particular concept built from experience, but very much at an intuitive stage of development. The transition to definition-based involves the first compression. From amongst the many properties of the concept-image, a number of generative ideas are selected and refined down to give the concept-definition. During the definition-based stage, the definitions are used to make deductions, all of which are intended to be based explicitly on the definitions. Many students, however, remain in the concept-image based stage, basing their arguments not on definitions and deductions, but on thought experiments using concept images (Tall & Vinner, 1981; Vinner, 1991). Bills and Tall (1998) introduced the term 'formally operable' definition (or theorem), proposing that:

> A (mathematical) definition or theorem is said to be *formally operable* for a given individual if that individual is able to use it in creating or (meaningfully) reproducing a formal argument. (Bills & Tall, 1998, p.104)

Tracing the development of five individuals over two terms in an analysis course, focusing on the definition of "least upper bound", they found that many students never have operable definitions, relying only on earlier experiences and inoperable concept images. Furthermore, it was also possible for a student to use a concept without an operable definition in a proof using imagery that happens to give the necessary information required. Thus, we already know that the development from the concept-image based stage to the compressed notion of operable definition is a difficult one for many students. Even so, they are then expected to move on to the next, theorem-based stage, when theorems that have been proved by the process of proof are now regarded as being compressed into *concepts* of proof, to be used as entities in the process of proving new theorems. For this to be fully successful, I hypothesise that students who have developed mature theorem-based understanding should possess the ability to consider a theorem as a "formal procept". I further hypothesise that individuals with this capacity to use theorems flexibly as processes or concepts are developing a compressed concept level of mathematical thinking that enables them to think with great flexibility and conceptual power.

EMPIRICAL STUDY

In the cross-sectional probe, 277 first year mathematics students, following a course in one of the top five ranked mathematics departments in the UK, responded to a questionnaire on equivalence relations & partitions" when just having learned the topic for several weeks. Thirty-six out of these 277 students were interviewed. In the longitudinal probe, fifteen selected students answered the same questionnaire and were interviewed during the first term in their second year. Their marks for the first year study are widely distributed — three are over 80, four between 70 to 79, four between 60 to 69, one between 50 to 59, and three between 40 to 49. This presentation is focused on two questions in the questionnaire which are generally designed to examine how the students manage to apply a relevant theorem to make their deductions. The plan of the study is to obtain a global perspective of the first year mathematics students' general understanding of some relevant theorems, then to investigate whether and how the students' understanding improves.

"EQUIVALENCE RELATION" AT THE THEOREM-BASED LEVEL

The following question is designed to examine if the students improve their understanding from the definition-based level to theorem-based level:

> A relation on a set of sets is obtained by saying that a set X is related to a set Y if there is a bijection $f: X \to Y$. Is this relation an ***equivalence relation***?

It is necessary to note specially that the following theorem, which can be directly applicable to this question, had been taught before the topic of "relations" was introduced in the lecture:

(1) The *identity* map is a bijection.

(2) The *composition* of bijections is a bijection.

(3) The *inverse* of a bijection is a bijection.

This involves compression of the proofs of (1), (2), (3) (as processes) into useable concepts (theorems).

In the cross-sectional probe, only a small percentage of the students (13%, 36 out of 277) tried to apply the above theorem to make their deductions. Nearly a half of the students (132 out of 277) still went back to examine the definitions step by step to answer this question (they were categorised as "definition-based"). More than a half of these thirty-six students (14 out of 36) only briefly referred to the theorem without giving more detailed interpretation. That is these fourteen students could only state the theorem but seemed not able to unpack its meaning. For these fourteen students, the notion of proof cannot be considered as a formal procept yet because they did not seem to know the process (method of proof) but only the brief concept (statement of theorem). In addition, it should be noticed that, within the thirty-six interviewees (six out of these thirty-six interviewees were categorised as "theorem-based"), thirty-three expressed that they had impression of the relevant theorem. It seems to suggest that most of the students should know or, at least, have some kind of impression of this relevant theorem, even though the majority did not manage to apply the theorem to the practical question.

In the longitudinal probe, twelve out of fifteen were able to apply the theorem in the second year, whilst only three were categorised as "theorem-based" in the first year. As was found in the cross-sectional probe, the students' concept images of this topic were not solid at that time. Although most of the students seemed to know the relevant theorem, they did not really have a clear idea how to apply it to this practical problem. JULSON (68% for his first year study) was an example offering a definition-based response (as follows) but he vividly expressed in the first year interview — "I remember I learned it [the theorem] in the lecture a couple weeks ago, but I'm sorry I haven't put it in my head yet."

Always bij. $f: X \to X$, ∴ set $X \sim$ set X
If bij $f: X \to Y$, then bij $f': Y \to X$
If bij $f: X \to Y$ and bij $f': Y \to Z$, then bij $f'': X \to Z$ (JULSON 68%, 1st year)

Compared with their former responses, the quality of these fifteen students' deductions seems to indicate that the notion of the theorem had become more workable in their concept images. JULSON's recent response (classified as "theorem-based") could offer us some evidence.

– id = bij : $X \to X$
– if f = bij : $X \to Y$ then f^{-1} = bij : $Y \to X$
– if f = bij : $X \to Y$ and g = bij : $Y \to Z$
then gf = bij : $X \to Z$

(JULSON 68%, 2nd year)

In the second year, JULSON not only stated the theorem but also explained how the theorem can be proved (in the interview). Thus he clearly showed that the notion of proof of this theorem had become a “formal procept” in his concept image as he knew both the statement of theorem (as general concept) and the method of proof (as process).

The following quoted conversations recorded in the interview with DIAHUM might offer us some more delicate insight into how the successive moves — from informal to definition-based, then on to theorem-based conceptions — happened with the individual.

DIAHUM (48% for his first year study) gave the following response (classified as "informal definition-based") in the first year:

$a \to a$
If $a \to b$ $b \to a$.
If $a \to b$ $b \to c$ have same no. of elements
$a \to c$.

(DIAHUM 48%, 1st year)

He cleared up what he meant in his response as follows:

> I was trying to apply the definition of equivalence relation to make the answer more formal. But I don’t think my answer was formal enough because I didn’t really know how to apply the definition even though I can remember it. And another problem is I can’t recall the definition of bijection. What I can remember is a bijection is one-to-one and onto. That means the two sets have the same number of elements (he explained later that this idea was from what he learned at A-level).

He also expressed that he knew the theorem which is directly relevant to this question in the interviews. But the theorem seemed to be something only in his understanding in a theoretical manner rather than in his intuition which can be freely referred to at any time.

In the second year, he responded in terms of the relevant theorems as follows:

reflexive: Yes since $\exists$ a bijection $X \to X$.
symmetric: Yes since if $\theta : X \to Y$ then $\theta^{-1} : Y \to X$.
transitive: Yes since if $\theta : X \to Y$ $\phi : Y \to Z$
then $(\phi\theta)(X \to Z)$.

(DIAHUM 48%, 2nd year)

Although he did not use the term “identity” to mention the bijection mapping from the set *X* to itself, he could precisely write down the composition of two bijections whilst some others mentioned it in the wrong order. In addition, he could explain the idea to prove the theorem in the interview. When being asked why he answered in this way this time, he gave the following explanation:

> Well, I think it’s fairly natural for me to make the deduction like this. When I faced the question, the theorems burst upon my head and I just wrote down the proof.

DIAHUM's case seems to suggest that he cannot freely apply a formal conception until it is assimilated in his concept image as an embodiment. When DIAHUM could only recite the formal definition of equivalence relation but was still struggling with the implication of it, it is natural for him to consult the relevant ideas he learned at school to make his first deduction because they were more embodied and secure in his concept image. Having a year of time to digest all these notions, the theorem, which he only knew about before, had been assimilated into his concept image as a formal procept that he could recall intuitively in the second test.

SUB-SUMMARY

In the students' (written or oral) responses, we can see that most students seemed to apply the relevant theorem directly to this practical question in the second year whilst most of them only gave a definition-based response in the previous year. This kind of result is consistent with the successive move from definition-based conceptions to theorem-based conceptions over time during which the ideas are being used formally (Chin & Tall, 2000). From the improved quality of the students' deductions, I consider, at least for some students, the notion of proof of the theorem seemed to have become a "formal procept" in their concept images. They only appeared to know the general concept (statement of the theorem) but not the process (method of proof) of the notion of proof of the theorem before. But, a year later, some students seemed to be able to unpack the notion of the theorem to the proof process and to apply the theorem to the question more flexibly.

LINKAGE BETWEEN "EQUIVALENCE RELATIONS" AND "PARTITIONS" (AT THE COMPRESSED CONCEPT-BASED LEVEL)

Theoretically the notion of "equivalence relations" is linked to the notion of "partitions" as there is a theorem stating that "an equivalence relation can produce a partition of a set and vice versa" which is always formulated as the conclusion of the topic. The following question is asked in order to examine whether the students appreciate the idea practically.

> Write down two different ***partitions*** of the set with four elements, $X=\{a,b,c,d\}$. For the first of these, please write down the ***equivalence relation*** that it determines.

In the cross-sectional probe, the students' reponses to this rather easy question with only four elements in the set reveal that only few students (sixty-one out of 277) show there is a workable linkage between the two notions in their concept images. The others gave two correct partitions with incorrect or without corresponding equivalence relations, or incorrect partitions with incorrect or without corresponding equivalence relations, or totally wrong answers. However, all the thirty-six interviewees said that they knew there is a theorem linking the two notions "equivalence relations" and "partitions" together, whether they appreciated it or not. It seems fairly clear that being aware of the statement of a theorem does not mean that the theorem is operable in one's concept image. I consider that the notion of proof has not become a "formal procept" yet, since the students could only remember the statement of the theorem as general concept but did not have the access to proof as process, the method of proof. Thus they could still not apply the theorem to this practical problem in the first year.

In the longitudinal probe, there were only five out of the fifteen subjects being able to apply the idea of the relevant theorem by successfully giving two correct partitions with a correct corresponding equivalence relation in the first year, and the number increased to eleven in the second test. As to the other four students, three gave two correct partitions without corresponding equivalence relation and one even failed to offer two correct partitions without giving any corresponding equivalence relation. Please note that all the fifteen expressed that they remembered they had seen, in the lecture, the theorem which links the two notions together.

HELTON, getting 61% for his first year study, can be a representative of those who failed to offer a correct response before but solved the question successfully in the second test. In the interviews, he expressed that he could just remember the theorem without really understanding the meaning of it. But when preparing the examination, he studied how the theorem is proved and then grasped the idea of the theorem. Thus he could simply solve the problem in the second year. However, MAUHAM (71% for her first year study), offering two correct partitions without giving the corresponding equivalence relation twice, is someone who confessed that he only recited the statement of the theorem and had no idea how the theorem can be proved.

SUB-SUMMARY

The result of this question appears to parallel the former question in many instances. All the students sensed the relevant theorem linking the two notions together but only a few could practically apply the theorem to the question in the first year. A year later, some students' understanding had progressed to reach a more mature theorem-based level. The theorem was no longer a "general concept" only but also a "process" which suggests the method of proof to make the whole notion of proof of the theorem as a "formal procept" in their concept images. However, only trying to recite the statement of a theorem without understanding the notion of proof of the theorem is not helpful for improving the student's understanding.

DISCUSSIONS AND CONCLUSIONS

The proceptual encapsulation in advanced mathematics seems to be slightly different from that in simple arithmetic (Gray & Tall, 1994), in which pupils appear to build up the notion of proceptual structure from encapsulating various processes, to obtaining the concept, then on to forming the procept of a symbol. The empirical data of this presentation reveal that most students, at the university level, seem to have the product (the statement of a theorem) first, then to develop the notion of proof if possible. There is evidence that being stuck in processes of calculation seemingly prevents pupils from obtaining the concept (e.g. Blackett, 1990, Gray & Pitta, 1997). However, the use of the computer to carry out the process, and so enable the learner to concentrate on the product, significantly improves the learning experience (Gray & Pitta, op. cit.; Gray & Tall, op. cit.). This kind of evidence suggests that concentrating on the product first, then to develop the notion of procept is possible and also helpful for improving student's learning.

The notion of formal procept is applicable to trigonometry and calculus. Many trigonometric formulae and theorems, for example, Mean Value Theorem and

Intermediate Value Theorem, in calculus can be seen as formal procepts. If the students only recite the product (the statement of the theorem) without understanding the idea of the proof, they could not be able to apply these formulae or theorems to solve practical problems flexibly. Besides, when more and more formulae and theorems are learned, the less able students will become trapped in reciting all these products which increase the burden upon an already stressed cognitive structure.

The empirical evidence presented in this paper gives us confidence to make a conclusion that the notion of procept of Gray & Tall can be extended to advanced mathematics. At the beginning, most students just have the product (the brief notion of the theorem) in their concept images only. But they cannot grasp the essence of the theorem and have more flexible thinking until they perceive the notion of proof of the theorem. Therefore, the ambiguity of process and product represented by the notion of formal procept also provides a more natural cognitive development at the university level which gives the students enormous power to develop more flexible mathematical thinking.

References

Bills, L. & Tall, D.O. (1998). Operable definitions and advanced mathematics: the case of the least upper bound. In A. Olivier & K. Newstead (Eds.), *Proceedings of the 22nd Conference of the International Group for the Psychology of Mathematics Education*, *2*, 104-111.

Blackett, N. (1990). *Developing understanding of trigonometry in boys and girls using a omputer to link numerial and visual representations*. Unpublished dotoral dissertation, University of Warwick, U.K.

Chin, E-T. & Tall, D.O. (2000). Making, having and compressing formal mathematical concepts. In T. Nakahara & M. Koyama (Eds.), *Proceedings of the 24th Conference of the International Group for the Psychology of Mathematics Education*, *2*, 177-184.

Dubinsky E., Elterman & Gong. (1988), The student’s construction of quantification. *For the Learning of Mathematics 8*(2), 44–51.

Gray, E.M. & Pitta, D. (1997). Changing Emily’s images. *Mathematics Teaching, 161*, 38–51.

Gray, E.M. & Tall, D.O. (1994). Duality, ambiguity and flexibility: a proceptual view of simple arithmetic. *Journal for Research in Mathematics Education*, *26*(2), 115-141.

Pinto, M.M.F. & Tall , D.O. (2002). Building formal mathematics on visual imagery: a theory and a case study. *For the Learning of Mathematics*, *22*(1), 2–10.

Tall, D.O. & Vinner S., (1981). Concept image and concept definition in mathematics, with special reference to limits and continuity. *Educational Studies in Mathematics*, *12*, 151–169.

Thurston, W.P. (1990). Mathematical Education. *Notices of the American Mathematical Society*, *37***,** 844-850.

Vinner, S. (1991). The role of definitions in teaching and learning mathematics. In D. O. Tall (Ed.), *Advanced Mathematical Thinking* (pp. 65–81). Kluwer: Dordrecht.