LEVELS OF SOPHISTICATION IN ELEMENTARY STUDENTS' REASONING ABOUT LENGTH

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Because cognition is the core substance of understanding and sense making, cognition-based assessment is essential for understanding and monitoring students' development of powerful mathematical thinking. The Cognition-Based Assessment System (CBAS) project is applying the results, theories, and methods of modern research in mathematics education to create an assessment system that can be used to assess in detail the cognitive underpinnings of the progress students make in developing understanding and mastery of core mathematical ideas in elementary school mathematics. In this report, I briefly describe initial CBAS research on the development of students' reasoning about length.

THEORETICAL PERSPECTIVE

An important finding of modern research in the psychology of mathematics learning is that for particular mathematical topics and within particular age ranges, students' development of conceptualizations and reasoning can be characterized in terms of levels of sophistication. These levels start with the informal, pre-instructional reasoning typically possessed by students in the age range; the levels end with the formal mathematical concepts targeted by instruction. The levels describe not only what students can and cannot do, but their conceptualizations and reasoning, cognitive obstacles that obstruct learning progress, and mental processes needed both for functioning at a level and for progressing to higher levels. The pedagogical importance of these levels is that instruction that produces conceptual understanding and powerful reasoning for a mathematical topic must be firmly guided by detailed, research-based knowledge of the development of students' cognition for the topic (Carpenter & Fennema, 1991; Cobb, Wood, & Yackel, 1990).

METHOD OF ANALYSIS

To investigate the sophistication of students' thinking about length, the CBAS project examined how students reasoned about the lengths of a variety of straight and non-straight paths. Based on a review of the research literature on length, 19 assessment tasks were developed and administered to students in grades 2-5 (ages 7-11) in one-on-one, videotaped interviews. Levels of sophistication in students' reasoning about length were synthesized from analysis of videotapes, summaries, field notes, and transcriptions of students' work. I first summarize the levels, then the cognitive processes underlying these levels.

FINDINGS: LEVELS OF SOPHISTICATION FOR STUDENTS' REASONING ABOUT LENGTH

Measuring length involves determining how many unit lengths are contained in a given length; it therefore involves the use of number to make judgments about length. However, before students acquire the concept of length *measurement*, they often reason about length using non-measurement techniques. And, although non-measurement strategies appear before measurement strategies, non-measurement strategies continue to develop in sophistication even after measurement strategies develop. In fact, sophisticated non-measurement strategies are essential for understanding length-based geometric properties of shapes.

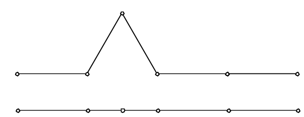
NON-MEASUREMENT (NM) REASONING

For non-measurement reasoning about length, students do not use numbers. Instead, they reason using vague judgments, direct or indirect comparisons, transformations, or geometric properties.

NM Level 1. Informal Holistic or Movement-Based Comparison

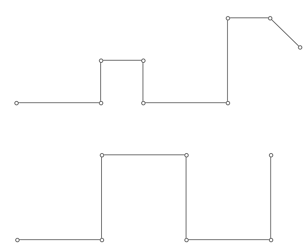
Students compare lengths holistically or using informal strategies that are imprecise and often vague. For instance, students might judge path lengths based on the amount of time or effort they imagine would be required to traverse the paths. Students have not yet separated length from the physical contexts in which they have experienced it. Elementary students may bring several perspectives to their reasoning, often with vestiges of the reasoning Piaget described for younger children. For instance, students might compare two non-straight paths by looking only at their endpoints, not what occurs between the endpoints.

Task L4. If an ant had to crawl along these paths, which path would be longer for the ant, or would they be the same? JAK drew segments joining the left endpoints and the right endpoints of the two paths and said, "I think they are pretty much the same."



Task L4. SAS said the top path was longer because "it wastes its time going up that way" [motioning to the two diagonal segments in the top path]. She said that the bottom path "just goes straight." When asked what she meant by "wastes its time," SAS said, "It is a lot easier if you just go straight and if you were carrying something it would be harder." When asked what makes it harder, she responded, "It is harder to hold something and when you turn it is harder."

Task L5. If an ant had to crawl along these paths, which path would be longer for the ant, or would they be the same? HT said that the second path would be longer, "cause [tracing the path with a finger]...like when it gets to these parts here [tracing along the square indentation in the bottom path] it has bigger squares, so it would take longer to get through."

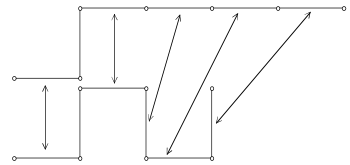


NM Level 2. Componential Comparison

Instead of reasoning about paths holistically, students systematically operate on components of paths. They compare paths segment-by-segment using visual, often transformation-based, strategies.

Task L1. If these were wires and I straightened them, which would be longer, or would they be the same? Each segment between dots is the same. *AR claimed that the two wires were the same.*

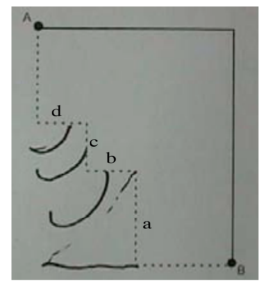
She matched segments in the bottom wire with segments in the top wire (as shown in the figure), saying, for each match, "This [segment] is the same as this [segment]."



Task L4. On the top path, MB drew one segment at the right end and another under the triangular indentation, saying, "One of them will make this [pointing to the

added segment on the right] and one of them will be at the bottom [pointing under the indentation]. And it [the top path] will be a little bit bigger by one line [pointing to the added segment on the right]."

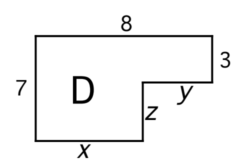
Task L12. Which path from A to B is shortest? JOK concluded that both paths are the same length: "If you put this here [dotted segment *a* onto the horizontal base] and this here [dotted segments *b, c, d* onto the left side]…they match up and they make a straight line." [Although this reasoning is not far off in this case, it is a strictly visual componential comparison that is logically flawed.]



NM Level 3. Property-Based Comparison

Students compare lengths using concepts such as perpendicularity, parallelism, and geometric properties of shapes. Students' use of these properties, however, may be informal and not stated explicitly.

In determining the perimeter for Shape D, AK said, "Side X plus side Y equals 8 because these sides are across from the top which is 8. Side Z is 4 because it and the side of length 3 are across from the side of length 7."



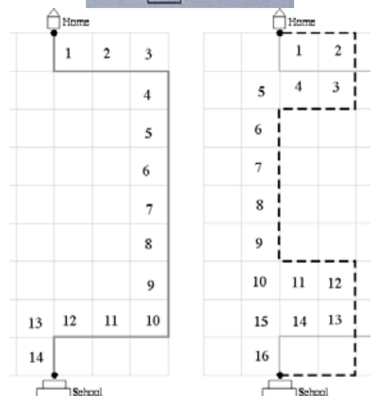
MEASUREMENT (M) REASONING

M Level 0: Pseudo-Measurement

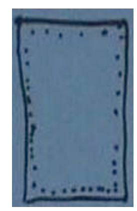
The numbers students use to reason about length do not properly represent the iteration and enumeration of unit lengths. Students fail to properly connect counting acts to unit lengths. For instance, they might recite numbers as they continuously move their fingers along a segment. Or they might count dots or squares in ways that are inappropriate for length measurement.

Task L4. CG counted the number of dots on both shapes and said the paths would be the same length because they each have 6 dots. He said that if you straightened the top shape, it would be "perfectly in line" with the bottom shape.

Task L3. Which path from home to school is shortest, the gray path, or the dotted path? SA said, "You can count the squares and whoever [sic] has the less is the shortest." As shown in the following figure, SA counted squares along the gray path 1-14, then along the dotted path 1-16 [but mistakenly switched back to the gray path at the very end, 13-16]. SA said, "So the gray line is shorter because it has less squares."



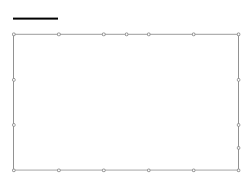
Task L16 Supplemental. The interviewer asked JAK if she could draw a rectangle with a distance around of 40. When asked how she knew the rectangle she drew was 40, JAK drew dots along its inside edge, stopping when she counted 40. (There are actually 38 dots on her paper.)

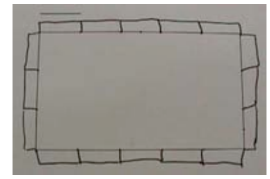


M Level 1: Enactive/Figurative Unit Length Iteration

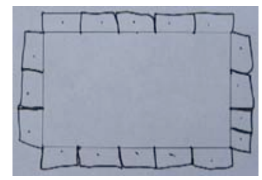
Students use 2d or 3d shapes to represent unit lengths, but unlike in M Level 0, these 2d/3d shapes have a 1-1 correspondence to properly located unit lengths (though there may be gaps, overlaps, and variations in length). Although students have not disembedded unit lengths from physical manifestations such as rods or squares, they have abstracted the linear extent of 2d/3d units sufficiently to use them as representations of length units. Indeed, because students are attending to linear extent—not, for example, to squares per se—there is no consistent double counting around corners.

Task L9. How many black rods does it take to cover around the gray rectangle? SAS drew rectangles around the outside perimeter of the rectangle, then answered 16. Each rectangle she drew had endpoints that matched the given dots on the rectangle, and so were equal in length to the given rod. When asked how she got 16, she responded, "I thought of this one, [motioning across the given rod, then the rod she drew], and I tried to measure it as much as that one was."





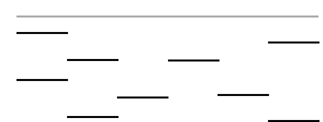
Task L9. SA drew 2 vertical segments from the endpoints of the black rod down to the gray rectangle. She then created 4 additional, same-size rectangular figures on the top side of the gray rectangle. On the right side of the gray rectangle, she drew 2 rectangular shapes corresponding to the given dots, then, using the given dots to guide her work, she created2 square-like shapes.



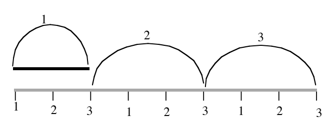
SA continued along the bottom of the gray rectangle, making 5 rectangular shapes, and on the left side, creating 3 more rectangular shapes. SA then counted the rectangular units she made, getting 17.

M Level 2: Measurement by Iterating Unit LENGTHS: Unit Length NOT Properly Maintained or Located During Iteration

Because students have abstracted unit lengths to the level necessary to disembed them from physical manifestations, they specifically count length units. However, because students do not properly coordinate unit lengths with either each other or the whole object being measured, gaps and overlaps occur. There is an inability to maintain the unit. Also, because there is a lack of proper structuring of the *set* of iterated unit lengths, students often lose track of their counting.



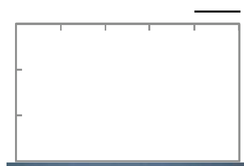
Task L7. How many black rods does it take to cover the gray rod? JAK drew a rectangle on the far right black rod, copied that length above the rod and under the gray rod, then continued to draw black rods under the gray rod. She gave an answer of 5 rods, explaining, "I measured this one [pointing to the far right black rod] with 2 little short lines and then a long line and it gave me a clue for how long it was and then you just draw how long the lines are and that gives you how many."

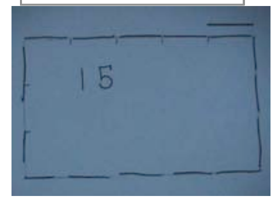


Task L8. How many black rods does it take to cover the gray rod? SA said that she knew that the black rod takes 3 hash marks on the gray rod. She drew a vertical segment from the right end of the black rod to the third hash mark on the gray rod. SA then counted the fourth, fifth, and sixth hash marks, "1, 2, 3" and marked the sixth hash mark and said, "have one."

She counted, "1, 2, 3" on the seventh, eighth, and ninth hash marks and said "have one." She returned to the beginning of the gray rod, pointed to each section she created, and counted "1, 2, 3."

Task L10. How many black rods does it take to cover around the gray rectangle? JAK drew segments all the way around the rectangle's perimeter, rotating the paper so that she could draw horizontally. She made and counted 15 segments. When asked to explain, JAK said, "These little squiggly lines [given hash marks on top and left sides] helped me measure….This side [pointing to right side of the rectangle] and on the bottom…I was picturing in my mind that there were squiggles."



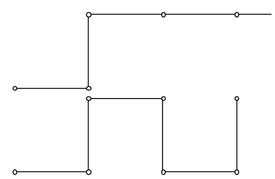


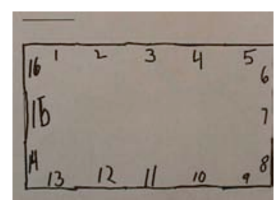
Task L15. [Given a 5-by-7 rectangle and a "broken" ruler that starts just before 1 and ends just after 9.] Use the broken ruler to measure the distance around this rectangle. JOK lined up the 1 on the ruler with the left endpoint of the base, then moved the ruler to the height, again lining up the 1 on the ruler. He said the distance around the rectangle was 28. When asked for the dimensions of the rectangle, JOK replied, "6 by 8."

M Level 3: Measurement by Iterating Unit LENGTHS: Unit Length Properly Maintained

Students are able to operate on their abstractions of unit lengths. They can use the coordination operation to properly relate the position of each unit with the position of the unit that precedes it so that gaps and overlaps are eliminated. Some students at this level can also coordinate and integrate unit lengths with the whole—so the whole is clearly seen as iterations of the unit. Some students can also iterate composites of unit lengths.

Task L1. If these were wires and I straightened them, which would be longer, or would they be the same? SS said, "Just by looking at it I can tell that these are like the same size between two dots, and so I would count by twos." SS counted the top wire by twos as she pointed to the segments to get 6, then counted the bottom wire by twos to also get 6. SS said, "And so I would know that they were the same."





Task L9. HW said, "This [black rod] is about as long as between these two [dots] here." She then drew a black path around the rectangle, one segment between two dots at a time. For the third segment on the right side, she ignored the extra dot. HW counted each segment as she drew it, writing the corresponding numerals inside the rectangle. She got 16.

M Level 4: Abstract/Applied Measurement: Reasoning about Length without Iterating Units; Using Rulers Meaningfully

Students can operate on lengths numerically *without iterating unit lengths*. Iterable units have reached the symbol level.

Task L10. BW counted the spaces between the hash marks on the top side of the rectangle, getting 5 rods. He said that since the bottom was the same as the top, it would also be 5. He then counted 3 rods on the left side (which also has hash marks). He said that the left side was equal to the right side, so the right side also would be 3. BW then said, "3+3=6 and 5+5=10. So it takes 16 black rods."

Task L16. Give the lengths of the sides of three different rectangles that have a total distance around of 200 units. SS said, "If you want all the sides the same, it would be 50 for all sides." SS next said, "60+40 and that would be 100, since you need two." For a third rectangle, SS said, "It could be 70+30=100, since you have two of the 70+30's, so 200."

COGNITIVE PROCESSES

General Processes

Among the cognitive processes necessary for mathematical reasoning, abstraction is critical. *Abstraction*, which has several levels, is the process by which the mind selects, coordinates, unifies, and registers in memory some aspect of the attentional field (Battista, 1999). At its most basic or *perceptual level*, abstraction isolates something in the experiential flow and grasps it as an item. When material has been sufficiently abstracted so that it can be re-presented in the absence of perceptual input (visualized), it has reached the *internalized level*. Material has reached the *interiorized level* when it has been disembedded from its original perceptual context and it can be freely operated on in imagination, including "projecting" it into other perceptual material and utilizing it in novel situations. At the second level of interiorization, one can operate on symbols (in von Glasersfeld's "pointer" sense) as substitutes for abstractions.

Three additional processes that are fundamental to understanding students' reasoning about length are spatial structuring, coordination, and use of mental models. *Spatial structuring*, a type of abstraction, is the mental act of constructing an organization or form for an object or set of objects (Battista, 1999). It determines an object's nature or shape by identifying its spatial components, combining components into spatial composites, and establishing interrelationships between and among components and composites. *Coordination* arranges abstracted items in proper position relative to each other and relative to the wholes to which they belong. *Mental models* are nonverbal recall-of-experience-like mental versions of situations; they have structures isomorphic to the perceived structures of the situations they represent (Battista, 1999). Mental models consist of integrated sets of abstractions that are activated to interpret and reason about situations that one is dealing with in action or thought. In particular, they permit visual reasoning.

PROCESSES APPLIED TO LENGTH

The levels-of-sophistication example episodes described earlier illustrate that in reasoning about length, level of abstraction is important because it determines the sophistication of the abstractions and operations students can apply in reasoning about length. For instance, to use the non-measurement strategy of componential comparison, students must interiorize the paths so that they can decompose them into parts and establish a one-to-one correspondence between the parts. For measurement strategies, once students have interiorized a length unit, they can employ the *units-locating* process to locate unit lengths by coordinating their positions with each other and (later) with the whole. The sophistication of this coordination is a major factor in determining the level of students' reasoning about length measurement. At first, students exhibit no coordination of unit lengths. Then, as they attempt to iterate a unit length, they coordinate each successive unit with the unit that precedes it. Next, students see a particular unit length in relation to the sequence of unit lengths; for instance, they see a unit as the third unit from one end, enabling them to understand the location of this unit. Finally, students see the whole length as a composite of unit lengths. In the latter two instances, the units-locating process is integrated with the process of *structuring* all or a portion of the whole length. Indeed, interiorization enables students to integrate successive iterations of abstracted length units into an operationalized, structured system that allows students to (a) properly interrelate iterated units to avoid gaps and overlaps, (b) relate iterations to the whole object so that the whole can be conceptualized as a composite of units, and (c) maintain, via the generalized and systematized unit, the invariance of the unit length in multidimensional contexts. A second level of interiorization enables students to operate on measurements as symbols—that is, it enables students to meaningfully reason about measurement numbers without having to iterate unit lengths.

The example episodes also illustrate that the exact substance of what is abstracted is critical to reasoning properly about length. For instance, for holistic comparison of paths, do students abstract distances between endpoints or motions along the paths? Or, with measurement strategies, students often use two-dimensional shapes such as squares or rectangles as their length units. They have not yet abstracted length from these units, even though length is embedded in them. Indeed, most perceptual manifestations of length used by students in classrooms—rods, squares, cubes, and so on—possess multiple length dimensions (as well as area and volume), so students often have great difficulty properly abstracting linear extent. (This difficulty suggests that more thought and research should be directed to the types of concrete materials that are used instructionally for measuring length.)

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