INTERVIEW DESIGN FOR RATIO COMPARISON TASKS

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In this article, which is part of an ongoing research, a classification is proposed for ratio comparison problems, according to their context, their quantity type, and their numerical structure. Deriving from this classification, an interview protocol was designed, and guidelines for the interpretation of answers into strategies were decided. A first case study was conducted and the results obtained are discussed, including comparisons of answers to problems of different contexts, quantity types, and number structures.

This paper reports part of an ongoing research on the strategies used by subjects of different ages and schoolings when faced to different kinds of ratio comparison tasks. A framework for the study of answers to ratio comparison probability tasks was presented at PME26 (Alatorre, 2002). The framework, which consists on two interrelated systems, one for interpreting answers as strategies and one for classifying the numerical structure of questions, was constructed within an investigation in which only adult subjects participated. One of the goals of this part of the research was to test its effectiveness with non-probability ratio comparison tasks and with younger subjects.

The partial investigations carried out so far show that the framework is indeed effective for a) the design of instruments aiming at studying the strategies used in different kinds of ratio comparison tasks, and b) the interpretation of the answers given by young subjects. This paper deals firstly with a classification of the kinds of problems; secondly, it presents a protocol design surging from this classification and in the frame of previous research; and thirdly, the actual results from a case study are discussed.

TYPES OF PROPORTIONALITY PROBLEMS

In the complex setting of proportional reasoning research, several ways have been put forward to classify the problems that can be proposed to subjects. They may be in turn grouped in classifications according to four issues that affect the subjects' responses: 1) the task, 2) the context, 3) the quantity type, and 4) the numerical structure.

The *task* that subjects have to accomplish was classified by Tourniaire and Pulos (1985) as "missing value problems" or "ratio comparison problems". To this basic classification other researchers, such as Lesh, Post and Behr (1988), later added more categories. In the research reported in this paper only ratio comparison problems are considered.

Among the classifications according to the *context*, Freudenthal (1983) distinguished couples of a) expositions, b) compositions, and c) □-constructs; Tourniaire and Pulos (1985) set apart d) physical, e) rate, f) mixture, and g) probability problems; and other authors, among which Lamon (1993), have distinguished h) well chunked measures, i) part-part-whole problems, j) associated sets, and k) stretchers and shrinkers. Although each of these classifications and categories corresponds to particular views and goals, the following considerations can be made. Categories a), d), e), h), and j) can all be considered as one and the same because they all involve two different quantities; the difference between d) and e) lies in the fact that the latter are word problems, and the difference between h) and j) lies in how familiar the subject finds them. Categories b) and i) can be considered as one, because they involve one quantity; f) and g) are in the same case and the difference among them may be considered important. Finally, categories c) and k), which are problems of a geometrical nature, can be considered as one. The left column of Table 1 displays the condensed classification of context resulting from these considerations. As examples, figure 1 shows a rate problem and figure 2 shows a part-part-whole mixture problem. This research does not deal with geometrical problems.

<table>
<tr><td colspan="2">Rate problems: couples of expositions</td><td>Intensive quantity surging from two quantities: both discrete, both continuous, or one of each type</td></tr>
<tr><td rowspan="2">Part-part-whole problems: couples of compositions</td><td>Mixture</td><td>One quantity, discrete or continuous</td></tr>
<tr><td>Probability</td><td>One quantity, discrete or continuous</td></tr>
<tr><td colspan="2">Geometrical problems: couples of □-constructs</td><td>Two continuous quantities</td></tr>
</table>

Table 1: Problem classification according to context and quantity types

The most generally accepted classification of *quantity types* separates discrete from continuous quantities (Tourniaire and Pulos, 1985). The right column of Table 1 shows the possibilities in each context category. It may be noted that in rate problems two quantities are at stake, and thus the intensive quantity formed (Schwartz, 1988) can have three different origins. In figure 1, one of the quantities at stake is discrete (amount of plants) and one is continuous (soil area), whereas in figure 2 the quantity at stake (liquid amount) is continuous, although discretely handled through amounts of glasses.

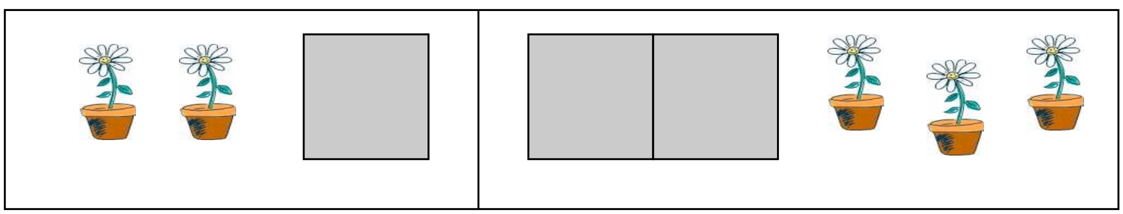

Figure 1. Garden problem (G). In which garden will the flowers be more cramped? (The squares stand for soil area where the flowers will be planted)

Among the classifications proposed for the *numerical structure*, two are considered here: Karplus, Pulos and Stage's (1983) and Alatorre's (2002). Before describing them, the notation used in this paper will be presented; it differs slightly from that of Alatorre (2002). In a ratio comparison there are always four numbers and two "objects" (1 and 2) involved. In each object there is an antecedent "a" and a consequent "c", and thus the four numbers may be written in an array, which is an expression of the form $(a_1,c_1)(a_2,c_2)$.

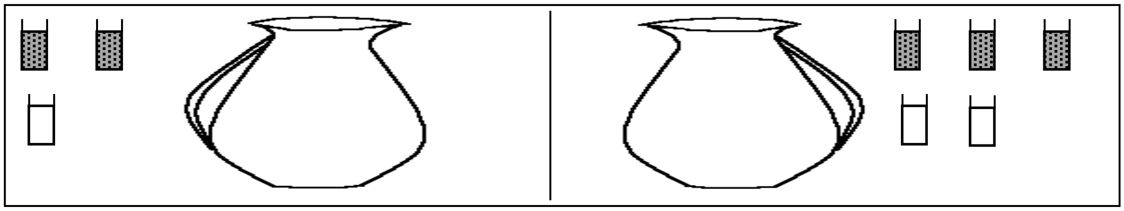

Figure 2. Juice problem (J). In which jar does the mixture taste stronger?
(The grey glasses contain concentrate and the clear ones contain water)

For instance, the numbers involved both in figures 1 and 2 are (2,1)(3,2). Also of interest may be the totals t=a+c (t_1=3 and t_2=5), the differences d=a–c (d_1=1=d_2), and the part-whole quotients p=a/c (p_1=2/3 and p_2=3/5). If in figure 2 there were 5 concentrate and 1 water glasses at the left side and 2 concentrate and 7 water glasses at the right side, the array would be (5,1)(2,7) and one would have a_1=5, c_1=1, t_1=6, d_1=4, p_1=5/6, a_2=2, c_2=7, t_2=9, d_2=□5, and p_2=2/9.

Karplus et al (1983) propose the categories W (integer ratio at least within one object), B (integer ratio between objects, in antecedents or in consequents), 1 (unit amount), and X (unequal ratios); they can be combined to form 5□2 categories: W, B, WB, WB1 and NIR (non integer ratio), all of which can occur in proportionality or non-proportionality (X) situations. For instance, the array (2,1)(3,2) of figures 1 and 2 is WB1X because the ratio 2:1 is integer both within (in garden 1 of figure 1, there are two plants for a square) and between (in figure 1 the area of garden 2 doubles that of garden 1), because one of the four numbers is 1, and because it is a non-proportionality situation.

Alatorre's (2002) proposition is a classification of all arrays in 86 different situations according to 17 different "combinations" –successions of results when an order relationship is established in the array between the pairs of numbers t, a, c, d, and p–, and 17 different "locations" –non-ordered pairs of the following alternatives for both quotients of the array: *u:* unit (p=1); *w:* win (1>p>_); *d:* draw (p=_); *l:* lose (_>p>0); and *n:*□nothing (p=0)–. The array (2,1)(3,2) of figures 1 and 2 belongs to combination K11 (<<<=>) because t_1=3<t_2=5, a_1=2<a_2=3, c_1=1<c_2=2, d_1=1=d_2, and p_1=2/3>p_2=3/5; and to location *ww*, because both p_1 and p_2 are 1>p>_. The 86 situations can be grouped in six difficulty levels, labelled I to VI.

Both classifications may be used as complementary. Considering that six of the 17 combinations (K0 and K12 through K16) bear proportionality and the rest does not, Alatorre's 86 situations may be crossed with Karplus et al's basic categories W, B, WB, WB1 and NIR.

PROTOCOL DESIGN AND INTERPRETATION OF ANSWERS

One of the purposes of the research is to compare the strategies used by subjects when faced to different kinds of ratio comparison settings. As stated above, these may differ in their context, their quantity type(s), and their numerical structure. A protocol was designed, with ten different problems and fifteen different numerical questions.

Figures 1 to 8 show the eight main problems. The gardens, lemonade, notebooks, and blocks problems are rate problems. They vary in the types of the quantities at stake – one discrete and one continuous in the first two, two discrete ones in the third, two continuous ones in the fourth. The juice and the exams problems are part-part-whole mixture problems – the first one is continuous and the second one is discrete. The spinners and marbles problems are part-part-whole probability problems – respectively with a continuous and a discrete quantity. Two other problems were used as controls, one for the fractions algorithmisation and one for the concept of randomness.

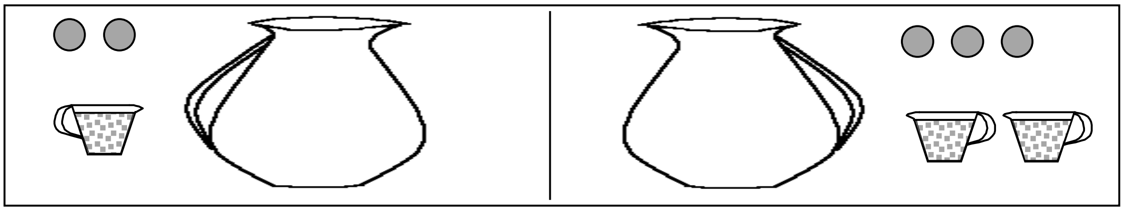

Figure 3. Lemonade problem (L). In which jar is the lemonade's taste stronger?
(The round figures stand for lemons and the cups contain sugared water)

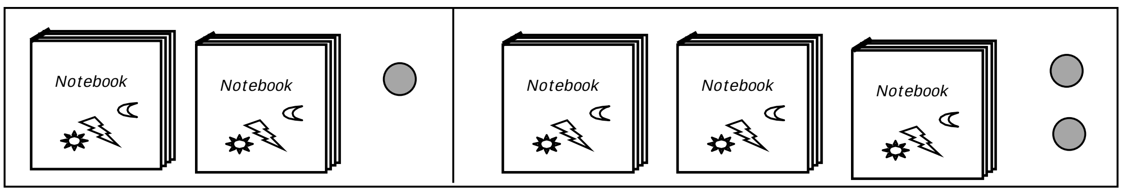

Figure 4. Notebooks problem (N). In which store are the notebooks cheaper?
(The round figures stand for coins)

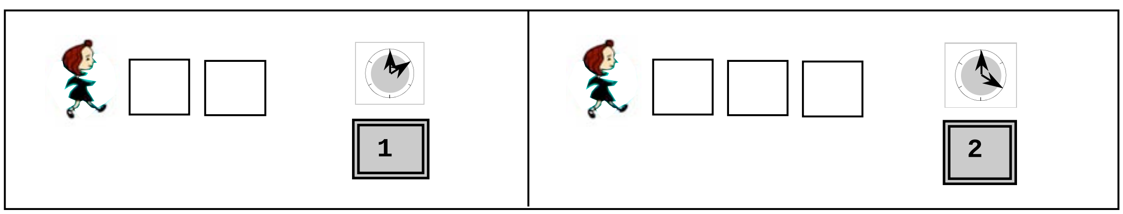

Figure 5. Blocks problem (B). Which of the girls walks faster?
(The squares stand for blocks, and the numbers represent minutes for walking them)

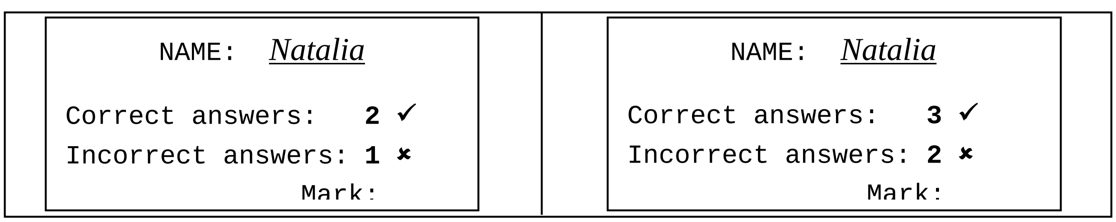

Figure 6. Exams problem (E). In which exam did Natalia do better?
(Additional question: what were her marks in both exams?)

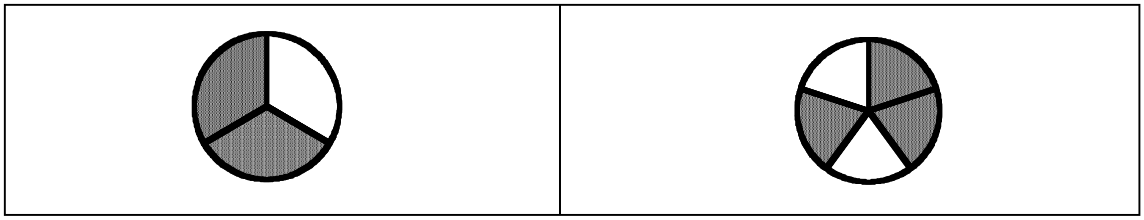

Figure 7. Spinner problem (S). In which spinner is a dark sector more likely to be marked?

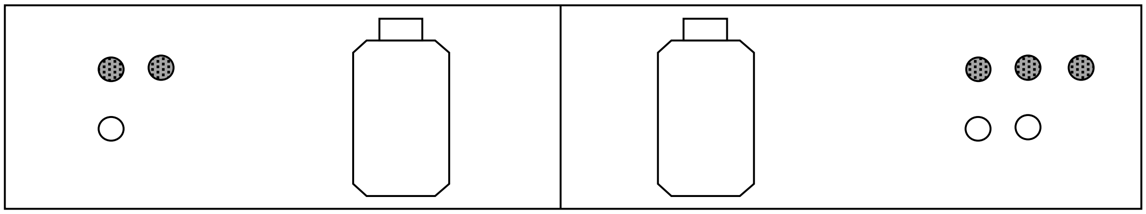

Figure 8. Marbles problem (M). In which bottle is a dark marble more likely to come out at the first try? (Bottles will be shaken with marbles inside)

Table 2 displays the 15 questions that were designed according to numerical structure. All the problems may be presented to subjects in each of 15 questions, except for question number 7, which does not have any sense in most rate problems. In figures 1 to 8 all the problems are shown in question number 4, which is the array (2,1)(3,2).

Question number	Left		Right		Karplus et al's	Alatorre's		
	a	c	a	c		Combination	Location	Difficulty level
1	2	3	2	3	B	K0	*l=l*	II
2	1	4	3	2	WB1X	K3	*wl*	III
3	2	3	2	2	WBX	K5	*wd*	III
4	2	1	3	2	WB1X	K11	*ww*	VI
5	3	3	1	1	WB1	K16	*d=d*	IV
6	2	2	3	2	WBX	K4	*wd*	III
7	3	3	2	0	WX	K7	*ud*	II
8	2	1	4	2	WB1	K15	*w=w*	IV
9	2	5	1	3	WB1X	K8	*ll*	VI
10	3	6	1	2	WB1	K14	*l=l*	IV
11	5	2	7	3	NIR	K9	*ww*	VI
12	4	6	2	3	B	K14	*l=l*	IV
13	3	2	5	3	NIR	K6	*ww*	V
14	2	4	3	5	W	K10	*ll*	VI
15	8	4	4	2	WB	K15	*w=w*	IV

Table 2. Antecedents (a), consequents (c), and classifications of the 15 questions

Since one of the purposes of the research is to observe the occurrence of different strategies, it is important that the protocol contains a variety of situations distributed among the 17 combinations and the 17 locations. Questions 11 to 15 are designed to be posed only to subjects capable of proportional reasoning, because of their higher difficulty (Karplus et al's categories, difficulty levels, and larger number sizes).

Alatorre's (2002) framework is to be used in the interpretation of answers. Because of space limitations, only a succinct summary will be offered here, with slight modifications in the notation. Simple strategies can be centrations or relations. Centrations can be on the totals CT, on the antecedents CA, or on the consequents CC. Relations, either "within" or "between", can be order relations RO (when an order relationship is established among a and c elements of each object and the results are compared), or subtractive relations RS (additive strategies), or proportionality relations RP. Composed strategies can be conjunctions *X&Y* (*X* and *Y* dominate), exclusions *X*¬*Y* (*X* dominates), compensations *X***Y* (*X* dominates), or counterweights *X*□*Y* (neither dominates). Strategies may be labelled as correct, sometimes depending on the situation (combination and location) in which they are used. Correct strategies are RP; RO in *wl*, *wd*, or *dl* locations; CA in locations with *n*; CC in locations with *u*; and some composed strategies.

Alatorre's (2002) framework was constructed for double urn probability problems (similar to problem M), where the antecedents are the favourable and the consequents are the unfavourable ones. Before any experimental evidence was obtained, it was anticipated that: 1) the strategies system would be applicable not only in probability and mixture part-part-whole problems (see Alatorre, 2000) but also in rate problems; 2) there would be no comparisons, additions or subtractions among a and c within objects in rate problems, because a and c belong to different quantities, and therefore no CT, RO, or "within" RS strategies were expected in rate problems; and 3) part-whole "within" proportionality strategies RP would only be observed in part-part-whole problems, and part-part "within" or "between" RP would be observable in all problems.

DISCUSSION OF THE RESULTS OBTAINED IN A CASE STUDY

A first interview has been conducted with Sofía, a ten-year-old fifth grader from a Mexico City private school. All the 8×15 questions were previously colour printed in 5"×8" cardboard cards. The interview, which lasted 57 minutes, was videotaped. Sofía was presented one problem at a time, for which question number 1 served in each case as an introduction to the context, and then followed one by one questions 2 to 10 (questions 11 to 15 were not posed to Sofía). The problems were posed in this order: marbles, juice, spinners, exams, notebooks, blocks, gardens, and lemonade. The question asked could always be answered by "left side" or "right side" or "it is the same"; Sofía was requested in each case to justify her answer. She was allowed to handle the cards at will.

The interview was analysed and an attempt was made to interpret Sofías's answers using Alatorre's (2002) framework. This was possible in all but two answers that will be commented later on. The results are shown in table 3.

These results can be analysed in several ways, and some conclusions may be drawn. The first and very important one is that Alatorre's (2002) framework's strategies system can be successfully used with ratio comparison problems different from the double urn task for which it was originally constructed. The two only answers for which the system does not have any category were given in problem S. Both of them display the same kind of reasoning, which can best be exemplified in the answer to question S-5 (see figure 9):

Sofía: The left side, because if you join the three parts they are bigger and besides they are more spread apart.

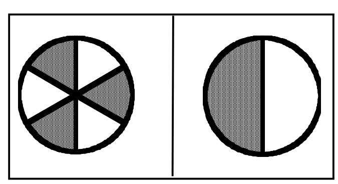

Figure 9. Question S-5

This reasoning seems to be based on the size and the distribution of the spinners' sectors. Both factors are not related to the ratio comparison task but to the concept of randomness and to the visual perception of the sectors.

An overall analysis of the strategies used by Sofía was performed. Centrations were slightly more used than relations and than composed strategies, which is also the case with adult subjects in double urn tasks (Alatorre, 2002). Among centrations, CA and CC were by far the most frequently used, and it is noticeable how they were often competing against each other, alternatively excluding each other or in counterweight. Sofía seemed to be equally attracted by the side with the largest antecedent or with the smallest consequent. All relations were "within", except for a "between" RS strategy in question N-4 ("it's fair that for one notebook more they charge one peso more", see figure 4). RS and RP relations were only used when they led to the answer "it is the same" (question 4 for RS and questions 5, 8, and 10 for RP). Also noticeable is that once a correct strategy is found, it may be thrown down by an incorrect one, as in questions J-5 and N-5.

As expected, CT was only observed in a part-part-whole problem (S-4). However, RO and RS did not respond as expected. Although RO appeared more in part-part-whole problems, it did come out four times in rate problems: G-3, L-2, G-6 and L-6 ("right side, because there are more lemons than cups"). Also, RS did appear twice in a "within" form in rate problems: L-4 and G-4 ("it is the same, because there's one more flower than squares in each" – see figure 1), contrary to what was expected. This apparently shows that Sofía faced no dilemma in putting together different quantities for comparisons.

	Marbles (M)	Juice (J)	Spin-ners (S)	Exams (E)	Notebooks (N)	Blocks (B)	Gardens (G)	Lemo nade (L)
2	CA	CA	***RO***	***CA & CC***	***CA & CC***	***CA & CC***	***CA & CC***	***RO***
3	CC	CC	CA	CC	***CC * CA***	***CC * CA***	***RO***	CC
4	CA RS	CA □ CC CA	CT CT	CA CA □ CC	CA ¬ CC RS	CC ¬ CA CA □ CC	RS	RS
5	CA * RO	***RP*** CA	Other	***RP***	***RP*** CA	CC ***RP***	***RP***	***RP***
6	CA	RO	CA		***CA * CC***	CA	RO	RO
7	***CC***	***CC***	***CC***	***CC***	***CC***			***CC***
8	RO	CA * RO	Other	RO	CA ¬ CC	CC ¬ CA	***RP***	***RP***
9	CA ¬ CC CC ¬ CA	CC ¬ CA		CC RO		CA ¬ CC CA □ CC		CC
10	RO	CC ¬ CA		RO			***RP***	CC

Table 3. Strategies used by Sofía. Correct strategies appear in bold italics.
(Two successive answers were given to some of the questions 4, 5, and 9).

Among the RP strategies, only one was a part-whole "within" strategy: E-5 ("it is the same, because in both exams Natalia had one half correct and one half incorrect answers"), and all other RP's were part-part "within" strategies. This behaviour follows the predictions. Most RP's appeared in problems with at least one continuous quantity.

The results in table 3 were also analysed column-wise and row-wise. A column-wise analysis allows comparing the eight different contexts and the part-part-whole (first four) vs. the rate (last four) problems. Rate problems were apparently easier for Sofía to answer correctly than part-part-whole ones, which goes in agreement with Tourniaire and Pulos' (1985) opinion in the sense that it is easier to handle two different quantities than only one. Among the rate problems, G was the one with more correct solutions and B was the one with the least. Among the part-part-whole problems, E was the easiest and M the most difficult. These results could also be due to the order in which the problems were administered; further interviews should vary this order.

A row-wise analysis of table 3 allows comparing the results obtained in questions with different numerical structure. The results were consistent with the levels proposed by Alatorre (2002): The easiest question was number 7 (level II, see table 2), which was always answered with a correct strategy, followed by level III questions 2 and 3, and the most difficult questions were level VI questions 4 and 9, in which incorrect strategies were always used. Among the level IV proportionality questions, the correct RP strategy was more frequently used in question 5, where the ratio was 1:1, than in questions 8 or 10, where it was 1:2 or 2:1. Proportionality was easier for Sofía to distinguish in 1:1 ratios than in other ratios.

CONCLUSION

Alatorre's (2002) framework has proved effective for the design of interview protocols as well as for the interpretation of answers to ratio comparison tasks of different kinds, that is different contexts (rate and part-part-whole, mixture and probability) and different quantity types. It also proved effective with a young subject. The only answers that were not interpretable in terms of the framework could be attributed to flaws in the concept of randomness or related to the visualisation.

The framework also allowed a comparison of the subject's answers among problems of different contexts and quantity types, and among questions of different number structure. It is to be expected that it also permits to compare the behaviour of different subjects towards ratio comparison tasks.

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