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IMPROVING DECIMAL NUMBER CONCEPTION BY TRANSFER FROM FRACTIONS TO DECIMALS

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Seventh and eighth grade students identified as holding an incomplete fraction conception of decimals were tested on related fraction knowledge. Most of these students (78%) had a problem in coordinating the size of the parts and the number of parts in comparing fractions. These students underwent several instructional sessions. Half of them worked on fraction coordination and on mapping it to decimals, while the other half had more instruction in decimals. Treating the source of the problem in fractions was found to be more effective in improving decimal conception. The remaining students (22%), who had a problem in decimals but not in fractions, improved in decimal conception following mapping instruction that promoted transfer of fraction knowledge to decimals.

This study focuses on treating a specific difficulty in decimal number conception (through the article we use the short name 'decimals' to stand for decimal numbers or decimal fractions, e.g. 0.23, 2.072, and the term 'fractions' to stand for rational numbers written as a/b, e.g. 2/3, 7/5). Several researchers in different countries (Sackure-Grisvard & Leonard,1985; Resnick et al., 1989; Nesher & Peled, 1986; Stacey & Steinle, 1999; Stacy et al., 2001) identified children's implicit models of decimals. They observed the different conceptions at a given grade and the changes over the years. The main task used to identify decimal conception was a number comparison task. Given this task, two main implicit number models were observed: treating decimals as if they are natural numbers (this conception leads to two rule variations) and treating decimals as fractions using an incomplete fraction conception (fraction rule). Some other, more technical conceptions, involve the use of rules that do not connect directly to a specific conception (e.g. "it's the opposite of fractions").

The term "fraction rule" was used in the research literature to describe the rule used by children that compared decimals by (only) using their "parts". For example, in comparing 0.2 with 0.34 these children would say that 0.2 is bigger because it has tenths, which are bigger than the hundredths that 0.34 consists of. Similarly, they would say that 0.45 is bigger than 0.457 because hundredths are bigger than thousandths.

According to an international research (Resnick et al., 1989) carried out in the US, France and Israel, about a third of the children show evidence of using the fraction rule in the first year of learning decimals (usually sixth grade). With further instruction some of these children become experts, and yet the ratio of children using this rule does not change much in the next two years, since some of the children holding a more primitive conception (whole number rule) shift to the fraction conception.

In higher grades there is some decline in the ratio of children using this rule. In a series of studies researchers in Australia followed children's conceptions of decimals. In one study Moloney and Stacey (1997) tested children in 4^{th} to 10^{th} grade and found that the fraction misconception persisted in higher grades, and was used by 20% of year 10 students.

Stacey and Steinle (1998) observed that some children behave similarly to children that have the fraction misconception (as defined above) and yet have other reasons for this "overt" behavior. That is, rather than choose 0.3 as bigger than 0.47 because tenths are bigger than hundreds, some of them use, what the researchers term "reciprocal thinking", and choose 0.3 because in fractions 1/3 is bigger than 1/47. Some others use "negative thinking" and choose 0.3 as bigger because they conceive of these numbers as negative numbers, and in negative numbers -3>-47.

In this study we focus on children that hold the fraction conception with the "denominator focused thinking", as Stacey and Steinle (ibid) term the "tenths are bigger than hundredths" explanation. These children have a relatively good decimal conception.

Our research hypotheses were that a large number of students use the fraction rule in 7th and 8th grades, and that most of these children would also have a similar problem with common fractions. We hypothesized that those who have a problem in fractions would benefit from instruction in fractions, and that their new knowledge would transfer to decimals. We also hypothesized that students that use the fraction rule but have no problem in common fractions, would benefit from help in making connections between their fraction knowledge and their decimal number knowledge.

METHOD

Three similar number comparison tests were used in the study. Each of them tested performance and understanding in comparing pairs of fractions and in comparing pairs of decimals. In each item the student was asked to circle the bigger number (or mark that the numbers are equal), and explain her answer.

A pretest was given to 261 seventh and eights grade students, and 59 students were identified as using the fraction rule (FR) in decimals. Out of the 59 students, 46 (also) used a similar rule in common fractions while 13 had no problem in comparing fractions. Following this diagnosis, the students were divided into three groups presented in Table 1: The 13 students, who had no fraction problem, were assigned to the mapping group. The 46 students, who used the fraction rule in both fractions and decimals, were randomly assigned to an experimental group and a control group.

Group: Study plan	Experimental n=23	Control n=23	Mapping n=13
Pretest results:	FR in decimals	FR in decimals	FR in decimals
	FR in fractions	FR in fractions	Ok in fractions
Instruction sessions 1 & 2:	Coordination in fractions	More in decimals	More in decimals
Posttest 1	Posttest1 given to	all groups	
Instruction session 3:	Mapping to decimals	More in decimals	Mapping to decimals
Posttest 2	Posttest2 given to	all groups	

Table 1: Group allocation and study plan following pretest results. (FR=Fraction Rule)

Each group had 3 sessions of instruction. During two of these sessions the experimental group worked on coordinating the number of fraction parts with the size of the part in perceiving the fraction magnitude. The control group and the mapping group had "more of the same" instruction on decimals (i.e. continued doing regular activities in decimals). During a third session, the control group continued with decimal instruction, while the experimental group and the mapping group received mapping instruction. This instruction involved a guided discussion mapping fractions and decimals (more details on this session are given in the results section).

Two posttests were given following instruction. The first was given after two sessions, and the other after the third session. The purpose of giving two tests was to differentiate between the effect of coordination instruction and the effect of mapping instruction.

RESULTS

As expected, a large number of students, 23% of the 261 seventh and eighths grade students, used the fraction rule in the pretest when they compared decimals. Most of them, 73% of the 59 students using the fraction rule in decimals, had problems in comparing fractions. It should be noted that in checking students' explanations we did not find students that used the fraction rule with any other explanation besides the "denominator focused" explanation. That is, all the students referred to the parts ("tenths are bigger than hundredths") and no one used the "reciprocal thinking" or "negative thinking" explanations that were observed by Stacey and Steinle (1998).

Table 2 presents an example of comparison items together with representative answers of students that use the fraction rule in decimals, but differ in their fraction knowledge. Duha compared fractions incorrectly, focusing on their parts, fifths and sixths, and disregarding the number of parts. Faddy compared the two fractions correctly by finding a common denominator. As a result, Faddy was assigned to the mapping group, while Duha was placed among the 46 that were split into the experimental and control groups.

Test item	Duha	Faddy
2.8 2.85	2.8 is bigger. 2.8 has tenths while 2.85 has hundredths. Tenths are bigger than hundredths because they are less parts.	2.8 is bigger. Tenths are bigger than hundredths.
5/6 4/5	4/5 is bigger. In 4/5 we divide into fifths, while in 5/6 we divide into sixths, and fifths are bigger than sixths.	5/6 is bigger. 5/6 is the same as 25/30, and 4/5 is 24/30. So now it's easy to compare, 25>24 so 5/6 is bigger.

Table 2: An example of children behaving similarly in decimals and differently in fractions.

Following instruction that focused on parts and number of parts in fractions, the experimental group showed significantly improved (paired t test, $p<.05$) decimal performance. Group performance improved even more following an additional session in which children were encouraged to make connections between fractions and decimals. Similar shifts were observed in the mapping group that participated in decimal sessions and 1 mapping session (Table 3).

	Average (scale	Decimal 0-9)	scores
	experimental	control	mapping
Pretest	2.56	2.73	2.30
Posttest1	4.26	3.82	4.53
Posttest2	7.26	5.39	7.30

Table 3: Average decimal scores in all groups.

The control group, that received more decimal instruction without further fraction instruction, showed some improvement (Table 3). The change following the first two sessions was not significant, and yet following a third session the change (between posttest1 and posttest2 scores) was significant (paired t test, $p<.05$).

In addition to looking at the change within the different groups, a comparison of decimal scores was done between groups. The differences in scores between the groups following the first 2 sessions (tested in posttest1) were not significant. Following the third session (tested in posttest2) a significant difference in decimal scores was found between the experimental group and the control group.

The groups were also compared on their fraction knowledge (Table 4). It was found that the experimental group that had 2 sessions of fraction instruction improved to the extent that it approached the knowledge level of the mapping group (that had no problem in fraction comparison to begin with).

	Average	fraction	scores
	experimental	control	mapping
Pretest	0.91	1.13	4.00
Posttest1	3.69	1.52	4.00
Posttest2	4.00	1.95	4.00

Table 4: Average fraction scores in all groups.

As a result of these shifts in decimal and fraction knowledge, when the three groups were compared on a total score (combining decimals and fractions) the state of the experimental group relative to the mapping group shifted. In the first test the differences in the total score were mainly attributed to the mapping group that was better in fractions than the two other groups. In the second posttest, the differences in scores were attributed to the control group. Although the control group improved to some extent, it still lagged behind the other groups in both decimal knowledge and fraction knowledge. The experimental group and the mapping group improved in decimals in a similar manner, and the experimental group closed the gap in fractions, making the two groups equally good.

As mentioned earlier, the third session in the experimental group and in the mapping group involved the same mapping instruction that was conducted through guided discussion. The purpose of this session was to help the students improve their knowledge in decimals by making connections to their fraction knowledge. As seen in Table 3 and Table 4, the postest1 fraction scores of these two groups were high: 3.69 for the experimental group and 4.0 for the mapping group (on a scale of 0-4). The corresponding decimal scores were 4.26 and 4.53 (on a scale of 0-9).

During mapping instruction the students faced a conflict situation when they got one (incorrect) answer by comparing two decimals, 0.24 and 0.253 and yet a different (correct) answer by comparing the corresponding pair of fractions, 24/100 and 253/1000. In the course of discussion the students immediately realized that the answers should have been the same. They all agreed that the answer in fractions was correct, but then wondered how 0.253 could be bigger than 0.24. One of the students suggested that 0.253 has more parts, and the idea was further elaborated and accepted by others. When one student (S1) wondered about the original rule, his colleague (S2) answered and others (S3, S4) added and summarized:

S1: So the rule about hundredths being bigger than thousandths and tenths bigger than hundredths is incorrect?

S2: I believe that one part out of a hundred is [still] bigger than one part out of a thousand.

S3: But that doesn't mean that the more we [continue to] take parts of [that] hundred it would stay smaller than the number of parts we took of ten.

S4: It's not enough to look at the size of the part, but also at the number of parts that we color.

The mapping session effect can be observed by comparing decimal scores in posttest1 and posttest2 (Table 2 and Table 4). The shift for the experimental group was from an average of 4.26 to 7.26 (on a scale of 0-9), and for the mapping group from 4.53 to 7.30. In both cases the differences were significant (paired-t test, $p<.001$ for the experimental group, $p<.05$ for the mapping group).

DISCUSSION

Following research that identified decimal number conceptions, this study focuses on the source of a specific conception, the fraction rule. Two instructional treatments were used in the study to improve decimal conception of seventh and eighth grade students that were identified as using this rule: coordination of the size of the fractional part and number of parts in comparing common fractions, and mapping fraction knowledge to decimal knowledge. Students who used the fraction rule and had a similar problem in fractions underwent coordination instruction. This instruction improved their understanding and performance in fraction comparison. It also improved their understanding of decimals by transfer of relevant (size of part & number of parts coordination) fraction knowledge.

The second type of instruction, mapping instruction, promoted connections from fractions to decimals, followed by knowledge accommodation. This instruction created conflict for students that got different results in fractions, where they performed well at that point, and decimals, where they still had problems. Group discussion caused pressure to adjust decimal conception accommodating it to allow for transfer of fraction knowledge.

Mapping instruction was used with the experimental group following coordination instruction, achieving further (significant) improvement in decimals. It was also used successfully with students who, to begin with, had no trouble with fractions, and yet needed some help in transferring their fraction knowledge and reorganizing their decimal number knowledge to allow for taking both size of parts and number of parts into account.

The effect of these two types of instruction was compared to the effect of "more of the same", i.e. more decimal instruction for the control group. Following two sessions of this instruction the control group improved, not significantly, in decimals and fractions. An additional session resulted in significant improvement in decimals but not in fractions, with the average decimal score still significantly below the average final score for the experimental and mapping groups. The improvement in decimals apparently resulted from improving knowledge within decimals without making connections to fraction knowledge.

This study used remedial teaching, i.e. treated the assumed source of the problem after learning decimals and after the fraction rule was observed by some of the students. The effect of the treatments implies that working with students on coordinating the number of parts with the size of parts and helping them transfer this knowledge to decimals, can help students construct better decimal knowledge. Instruction of this kind can be used either prior to teaching decimals or after teaching decimals in order to prevent or to remediate fraction rule conception.

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THE DEVELOPMENT OF STUDENT TEACHERS' EFFICACY BELIEFS IN MATHEMATICS DURING PRACTICUM

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In this study we examine the development of prospective primary teachers' efficacy beliefs (TEB) with respect to teaching mathematics during practicum. The analysis of the responses of 89 student teachers in three repeated scale distributions indicated a marked positive change in all groups formed by cluster analysis. Eight students were interviewed at the commencement, in the middle, and at the end of the course to investigate which factors contributed to this change and how. We found that the effect of broad sources informing efficacy beliefs identified by Bandura (1997) supplemented by the influence of individuals involved in the course played a major role in the cyclical process of efficacy change. Implications for further research and for developing practicum courses are drawn.

INTRODUCTION

Nowadays research on mathematics teaching focuses on multiple components of learning, including variables of the affective and social domain. The affective domain can be conceived as a complex structural system consisting of four main components: emotions, attitudes, beliefs, and values. In this study, teachers' self-efficacy beliefs with respect to teaching mathematics are examined. Bandura (1997) defines self-efficacy as one's beliefs about his or her ability to organize and execute tasks to achieve specific goals. In this context, the teacher's efficacy beliefs (TEB) refer to a teacher's sense of ability to organize and orchestrate teaching that promotes learning. The importance of TEB derives from its role in determining both the teachers' professional behavior and pupils' motivation and performance (Ghaith & Shaaban, 1999; Henson, 2001; Brouwers & Tomic, 2001).

Though researchers seem to agree on the significance of TEB, issues such as the measurement and development of these beliefs remain open to discussion. Specifically, researchers are now more sensitive to problems related to the reliability and validity of the measuring instruments; in this respect the appropriateness of widely used scales, such as the Gibson & Dembo scale has been under criticism (Deemer & Minke, 1999). Recently, Tschannen-Moran & Hoy (2001) developed and tested a scale of 24 items (Teacher Sense of Efficacy Scale-TSES), which consisted of three subscales: pupils' involvement in the learning process, adoption of teaching strategies and classroom management. It was further found that the three factors could be grouped into one second-order factor, meaning that all of them measure a wider construct, namely TEB. The authors proposed further studies to examine the validity and reliability of the TSES in different cultures and specific domains. The need to examine the development of TEB in specific areas was also pointed out by other researches (Henson, 2001).

Tschannen-Moran, Hoy & Hoy (1998) proposed a comprehensive cyclical model representing the evolution and development of TEB.

This model includes the four broad sources of efficacy information proposed by Bandura (1997): mastery experience, vicarious experience, social persuasion, and physiological and emotional arousal. Mastery or enactive experience is considered as the most powerful source of efficacy information, vicarious experiences may alter TEB through comparison with peers attainments, social persuasion refers to feedback provided by significant others, and finally, the feelings of relaxation and positive emotions signal self-assurance and anticipation of future success. The model assumes that the information derived from these sources is cognitively processed and weighted vis-à-vis existing beliefs structure and subsequently influence the development of TEB. According to the model, teachers assess what will be required of them in the anticipated teaching situation (*analysis of teaching task*) and take into consideration their capabilities in a certain domain (*assessment of personal teaching competencies*). However, the proposed model has not been yet empirically verified.

The aim of this study was to shed some light on how the aforementioned factors work analyzing the stages of development of TEB, and especially the growth of student teachers' beliefs during practicum. Practicum is one of the most important parts of teachers' education; it functions as the bridge of students' theoretical understanding and real classroom practice, and evidently provides students with real hands on experiences (Ebby, 2000). Furthermore, the teaching training period offers students the opportunity to interact with others and specifically with their mentor (Tillema, 2000).

Based on the above analysis, the aims of this study were to: (1) Examine the development of preservice teachers' efficacy beliefs in teaching mathematics during the course of their final teaching practice program (TPP), using the TSES, and (2) Verify the cyclical model of development of TEB using empirical data.

METHODS

A questionnaire based on the TSES (with 24 statements on a 9 point Likert scale), reworded to reflect TEB in mathematics was administered to the 89 four-year students who attended the TPP from January to April 2002[1]. The students' beliefs were measured at the commencement of the program, after the 1st part, and at the end of it. The internal reliability of the scale was extremely high in each administration (Cronbach's alphas: a1= 0.96, a2=0.97 and a3=0.98, respectively). After analyzing the data from the first measurement, purposive sampling procedure was used to select the eight students who were interviewed. More specifically, the students who participated in the interviews were, in terms of the clusters formed by initial TEB, one from G1 (S11), two from G2 (S21, S22), four from G3 (S31, S32, S33, and S34), and one from G4 (S41). Their scores in courses in mathematics were below average (S11 and S32), average (S22, S31, S33 and S41) or higher than average (S22 and S34). Students were interviewed three times, one at

[1] The course lasts for 13 weeks and it is divided into two parts. The students are assigned to lower (1st to 3rd grades) and higher school cycle (4th to 6th grade) in each part, with a week break in the middle for group discussion and reflection on practice.

the beginning of the TPP, one at the middle and one at the end of it. The interviews were semi-structured and lasted for 45 minutes; they were based on questions which asked students to provide a detailed description of their beliefs in each of the three thematic factors of the scale (i.e., teaching strategies, student involvement, and classroom management). The constant comparative method (Denzin & Lincoln, 1998) was used to analyze the interview data.

FINDINGS

Exploratory factor analysis applied separately for each of the three scale administrations (see Charalambous & Philippou, 2003) showed a significance level of .001 for the Barlett's test of sphericity and high KMO values (.907, .933 and .948, for each measurement, respectively). The varimax rotation to the data from the first administration resulted in two factors that explained 60.53% of the variance. The first factor consisted of 16 statements (same as those in the first two factors of the TSES) and explained 34.02% of the variance, and the second factor consisted of 8 statements (identical to those in the third factor of TSES) and explained 26.51% of the variance. Applied to the data of the second and third measurement, the two-factor model was found to explain 69.00% and 76.13% of the variance, respectively. Two items of the first factor were discarded because in the analysis of the data from the third measurement they loaded on the second factor rather than on the first. Thus the two-factor solution remained with 22 items; the first factor (F1) consisted of 14 items that referred to TEB in teaching strategy use and activating students during mathematics classes (teaching mathematics, hereafter), and the second (F2) of 8 items related to TEB in managing the mathematics classroom.

The mean TEB in each measurement were found as 5.61, 6.50, and 7.05 for F1, and 5.73, 6.56, and 7.01 for F2, indicating that the students started with rather positive beliefs in both factors; these beliefs were improved in the course of the program. The repeated measures technique and the Bonferroni test (to avoid carry over effects) revealed that improvement was statistically significant; there was no interaction between the two factors ($F_{(2,84)}$= 1.79, p=.173, pillai's =.041), and the observed differences were due to participation in TPP (due to the factor "administration") ($F_{(2,84)}$= 87.65, p<.001, pillai's =.674). Significant differences were found between the first and the second administration ($\bar{x}_{\text{2nd adm.}} - \bar{x}_{\text{1st adm.}}$ = .85, p< .001), and between the second and the third administration ($\bar{x}_{\text{3rd adm.}} - \bar{x}_{\text{2nd adm.}}$ = .55, p< .001).

To search for patterns of development of TEB, cluster analysis was applied to the data emerged from the first administration. The Ward's method of hierarchical cluster analysis identified four homogenous groups. The four-cluster solution was justified since the Agglomeration schedule showed a fairly large increase in the value of the distance measure from a three-cluster (15.84) to a four -cluster solution (25.41). Table 1 shows the mean and variances of each of these groups for both scale factors. Clearly, G1 students entered the program with somewhat higher beliefs than the sample mean; their beliefs were improved mainly during the first part of program. G2 students started with slightly lower TEB but they got the most out of the program, compared to the other students, particularly during the 1st part of the program. The majority of students (G3) entered the program with higher TEB and these beliefs continued to be above the sample mean level. Finally, the last "group" (G4) consisted of only two students with extremely low TEB;

despite some positive change, their beliefs failed to surpass the scale mean (i.e., remained negative).

Factors	F1 (TEB in teaching strategies)						F2 (TEB in class management)					
	1st Admin.		*2nd Admn.*		*3rd Admin.*		*1st Admin.*		*2nd Admin.*		*3rd Admin.*	
Groups *	$\bar{x}$	**SD**	$\bar{x}$	**SD**	$\bar{x}$	**SD**	$\bar{x}$	**SD**	$\bar{x}$	**SD**	$\bar{x}$	**SD**
G1 (N=25)	5.20	.80	5.67	.78	6.07	.84	5.25	.85	5.68	.82	6.04	.89
G2 (N=13)	4.84	.78	6.74	.61	7.43	.48	4.44	.61	6.43	.54	7.08	.56
G3 (N=45)	6.18	.58	7.02	.63	7.60	.56	6.53	.73	7.26	.72	7.71	.68
G 4 (N=2)	2.46	.45	2.67	1.16	3.96	.15	1.81	.62	1.88	.88	2.43	1.86

* Complete data were selected of only 85 students (nine point scale: 1=not at all good and 9=very good)

Table 1: Means and variances of the four groups formed by cluster analysis

The analysis of the interview data justified the differences in the development of students' TEB concerning both factors (teaching mathematics and managing the mathematics class). The following excerpts are indicative of the patterns witnesses in the above analysis.

During the 1st interview S_{11} expressed concerns indicating somehow low confidence. After the 1st part she felt much better about teaching mathematics: She stated in the 2nd interview: "*I can teach mathematics provided I know the content and I have a detailed lesson plan,…, to know what to do every minute, irrespective of whether I would follow it*". By the end of the program, the student could teach mathematics effectively "*provided that she works hard*". A similar pattern of improvement was witnessed in the case of S_{22}. At the opening, the student valued her mathematical knowledge quite sufficient, but yet she was concerned about her teaching competence. In the 2nd interview she admitted: "*I think I am getting over my fears. I realized that I could flexibly teach mathematics. My teaching trials succeed!*" At the end of the program the student claimed that she was convinced that she could teach mathematics effectively. The G3 students indicated that their initial positive feelings about teaching mathematics improved as a result of the program. For example, S_{31} pointed out: "*I came to believe that I could teach mathematics. That feeling was improved during the TPP. I had the chance to teach younger and elder children, obedient and disobedient. Thus I had an inclusive teaching experience, which made me believe that I can even do better in the future*". At the commencement of TPP, S_{41} felt totally incapable of getting through; she could hardly understand mathematics, especially the concepts taught at upper grades. In the 2^{nd} interview she was still far from confident, though some successful lessons encouraged her to say, *"I am not that bad in teaching mathematics"*. The final part of the TPP improved her feelings, though the student failed to get over some of her initial concerns: "*I think that the second part eliminated my fears about teaching mathematics… Now I feel more comfortable in teaching that subject. But perhaps I have been lucky to teach easy concepts. I do not know if I would do the same well in the future*". S_{41} expressed analogous concerns about managing class during a mathematics lesson even at the end of the TPP: "*I learnt some useful things, but discipline continues to be a difficult issue… I cannot do much. The idea that I cannot control the class really terrifies me"*.

The qualitative data were also informative of the developmental process of TEB, explaining at the same time the aforementioned different patterns of improvement. First of all, the analysis revealed that students entered the program with some TEB that were formed on the basis of their overall experiences in mathematics. For example S_{11} admitted: "*Mathematics was my weak point. Until the age of 13 years old, math used to be among my favorite subjects, but then it changed...it started to become more complicated to push me away, too hard to understand*". Similarly, S_{32} indicated, "*I failed and had to retake the first math course. In general my mathematics grades were below average, and that influenced me*". On the contrary, S_{22} stressed, "*Well, I had begun loving mathematics in the primary school through the junior school. In the high school a math teacher influenced me, and I wanted to become a mathematician. He was superb in terms of knowledge and approach*".

Hands on experience influenced students' TEB mainly through regular involvement and a sensed feeling of accomplishment. For instance, S_{11} emphasized the catalytic role of these experiences: "*My attitude towards mathematics was negative… I knew* that *my knowledge in that domain was deficient. But the TPP made me realize that I could overcome these deficiencies… I had to teach subjects that I was totally unfamiliar with. I prepared a lot, and eventually, my lessons were very good"*. S_{33} pointed out the almost daily teaching of mathematics, which let her believe that she could be efficient in this task. S_{32} characterized the TPP as "*a first class experience… a baptism in the job*", which helped him to get rid of his worries and insecurity feelings. The second part of the program strengthened more students' TEB, since students had the opportunity to test their ideas and strategies tried during the first part of the program, in another environment and with students of a different level. The contribution of teaching experiences to improving students' TEB is illustrated by S_{21}'s assertion at the conclusion of the program: "*TPP was the most important part of my studies. Since the beginning of the TPP I had no idea about the way a school works. We have learnt a lot of theories, but I felt insecure to teach mathematics. Now, at the end of the program I realized that teaching is not so hard, as I thought before"*. Even S_{41} indicated: "*I realized that my teaching was getting better. I was not lost, as I felt in the first part of TPP. I was more effective"*.

Interaction with individuals involved in the TPP (mentors, headmasters, tutors, fellow students) seemed to influence the development of students' beliefs. More specifically, mentors operated either as teaching models or as feedback providers. Yet, the interaction with mentors did not work equally well for all students. S_{32} felt that his mentor was completely different from him: *"He was much older than me, and he used to teach mathematics in a rather traditional way. I tried to teach mathematics in a different way. I was competing my mentor, and that motivated me to try hard in teaching mathematics…I became more confident when the mentor failed to recognize pupils' difficulties in decimals, and I was able to do that!"* On the other hand, S_{33}'s TEB initially declined, as a result of the interaction with her mentor: "*She used to teach in a rather mechanical way. She helped pupils in solving all the textbook exercises. She hardly left pupils work on their own. I tried to do something different, and I asked pupils to try harder in order to solve the exercises on their own. The mentor nodded her head, showing her dissatisfaction. Even if she avoided telling me anything, I felt that she was thinking: "you failed to teach mathematics in the proper way, and I have to teach that concept again"*. Mentors' feedback also influenced students' TEB. For instance, S_{22}'s mentor during the 2[nd] part was very supportive, "*She shared her initial teaching experiences with me. She tried to persuade me that we all do mistakes in the beginning. Thus, she helped me a lot*".

Despite luck of students and headmasters' interaction, one student (S_{31}) referred to a very positive incident with the school headmaster: "*I was preparing lessons very well, but since pupils were disobedient, I failed to reach the expected outcomes. I was very disappointed...But the headmaster persuaded me that it was not my fault and that, in a different school I could definitely do better"*. University tutors seemed to affect students' TEB, but not equally well for all students. For instance, S_{31} claimed that she weighted most her mentors' opinion than the tutors' opinion, since the mentor attended all her lessons. On the other hand, S_{21} had a totally different approach asserting, "tutors *are experts in the domain of teaching mathematics but mentors are not"*. Finally, some of the students seemed to be influenced by fellow students' comments and achievement as indicated in the following excerpt by S_{34}: "*I had the feeling that some of my ideas were not successful, while other students were going fine. Listening to others supposed successes was deteriorating myself image as teacher."*

The students' individual style and characteristics functioned as a filter differentiating the effect of practically similar information on their efficacy beliefs. For instance, S_{11} pointed out: "*I weighted more the interaction with pupils, and the learning outcomes of my lessons. I avoided being influenced by fellow students, my mentor or the university tutors. Whatever their reaction to my lessons was, I stopped analyzing it. I told myself: "Leave it at the backside of your mind. Do not allow yourself be encouraged or discouraged"*. Cognitive processing of the incoming information was also apparent in the following extract by S_{31}: *"I used to take into consideration the mentors' feedback. But I valued more pupils' reaction. On the other hand, if I was convinced that I taught a concept properly, and pupils did not seem to grasp it, I used to look at my mentor. If he was nodding with satisfaction, I could realize that it wasn't my fault. Pupils were probably tired"*. Attributing failures to other factors than the personal performance also helped in keeping students' TEB to the same level. As S_{41} mentioned, "*Pupils did not respond to my lessons as I expected, since they were thinking of me as a student-teacher and not as a teacher"*. Similarly, S_{22} indicated: "*At first I attributed the responsibility solely to myself, but after trying several methods and means, I came to believe that it was not always my fault. Now I do not assume full responsibility; I realize that some pupils are not willing to participate, simply because they do not care*". Finally, analysis of task and assessment of personal competence seemed to interact with the cognitive processing of the efficacy information, as it is evident from the following extract: "*I think I am able to teach mathematics. During the first part of the TPP I was efficient in that domain* [mastery experience]. *I believe that it is all a matter of choosing the correct activities* [analysis of teaching task]. *And I had no problem in that domain* [assessment of personal competence]. *Thus, I foresee that I can also do well during the second part of the TPP, provided that the students are obedient* [analysis of teaching task] (S_{34}, interview in the middle of TPP).

DISCUSSION

The TSES was proved helpful in describing the development of preservice teachers' TEB in the domain of mathematics. Based on the two-factor solution, students' TEB were found to gradually improve while participating in the TPP. However, the four emerging clusters of improvement suggest that the development of TEB was not the same of all students, as a result of the factors involved in the process of forming TEB. More specifically, the findings of the interviews further verify the opinion that the main source of the development of efficacy beliefs is "mastery experience" i.e, actual experiences in a certain domain. Yet, the role of other sources and namely vicarious experience and verbal persuasion was also prevalent. Students' interactions with mentors, tutors, and fellow student-teachers seemed to be an important part of the their experiences, which modified their beliefs. The sources of efficacy information did not operate in the same way for all students, but through a cognitive processing of the incoming information, as Bandura (1997) suggested. Furthermore, the results indicated the presence of task analysis and appraisal of competencies, verifying the model proposed by Tschannen Moran et al. (1998). However, cognitive processing, analysis of task and assessment of competencies seemed to function simultaneously and the whole process of forming these beliefs appears as a reciprocating process, rather than a linear one provided that students were at the same time referring to their experiences, their abilities, and the teaching task. Finally, interviews revealed that the causal attribution was present during the cognitive processing of the incoming information. This result indicates that attribution theory and efficacy theory need not be considered as two distinct or rival theories, as they were viewed before, but rather as two complementary theories (Lyden, Chaney, Danehower & Houston, 2002).

Overall, the present study indicated that the practicum should provide preservice teachers with many "hands on" experiences with instructing and managing opportunities in a variety of contexts. Special attention should also be paid to the individuals that are engaged in teaching practice programs. Future research should expand the attempt to study TEB in certain domains. Furthermore, the claim that the development of TEB should not be conceived as a linear process, as well as the need for intergrading social cognitive theory with attribution theory offer a wide spectrum for further research. Finally, research into the relations between teachers' efficacy beliefs in Mathematics and their effectiveness, measured through the progress made by their pupils, is needed. Implications of such research for authentic professional development of teachers and for the improvement of teaching practice could be drawn.

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RECOGNISING EQUIVALENT ALGEBRAIC EXPRESSIONS: AN IMPORTANT COMPONENT OF ALGEBRAIC EXPECTATION FOR WORKING WITH CAS

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The ability to recognize equivalent algebraic expressions quickly and confidently is important for doing mathematics in an intelligent partnership with computer algebra. It is also a key aspect of algebraic expectation, the algebraic skill that parallels numeric estimation. The progress in this ability of 50 students learning mathematics with CAS over two school years was monitored using a novel instrument called the Algebraic Expectation Quiz. Students began with very low facility and confidence and made moderate progress, confirming the importance of addressing this obstacle to using CAS explicitly in the curriculum. We offer conclusions about test items, students' strengths and weaknesses and suggest possible teaching actions. The results demonstrate that recognizing equivalence, even in simple cases, is a significant obstacle for students.

INTRODUCTION

The purpose of this paper is to report on students' ability to recognize the equivalence of algebraic expressions and their confidence in their judgments, obtained from a novel instrument called the Algebraic Expectation Quiz. We report the results of using this instrument with 50 students over 21 months, during their final two years of secondary schooling learning mathematics with CAS, drawing conclusions about test items, students' strengths and weaknesses and finally suggesting possible teaching actions. The results demonstrate that recognizing equivalence is a more significant obstacle for students than we and their teachers had expected and that students lack confidence.

The work in this paper is motivated by the need to develop and monitor students' algebraic thinking when computer algebra systems (CAS) are available for doing, teaching and learning mathematics. It has been carried out as part of a project (see CAS-CAT Research Project website) where students at three secondary schools studied a new tertiary-preparation mathematics subject for which CAS calculators (graphics calculators that additionally have a symbolic manipulation facility) were available at all times, including for examinations. Students learning and doing mathematics using CAS will not need the same mastery of algebraic routines as do other students. They will be able to use CAS where other students need a by-hand technique. However to use CAS effectively and to attain other educational goals, students will still require conceptual understanding and technical facility with algebra, the exact nature of which is a topic of debate and concern in the literature (see, for example, Drijvers (2000); Herget, Heugl, Kutzler & Lehmann (2000); Pierce (2002)). Artigue (2001) suggests that there is a complex dialectic between conceptual and technical work in algebra. Passing technical work over to CAS means that it still needs attention, although possibly different attention, in the curriculum. Controlled use of technology (such as CAS) almost certainly requires specific competencies that are not covered by the standard curriculum.

Equivalent forms and Algebraic Expectation

The ability to recognise equivalent forms of algebraic expressions, which is the focus of this paper, has been identified as a central part of working with CAS, and one that is likely to take on new importance in future curricula. Many researchers, for example Heid (1989), Lagrange (1999) and Drijvers (2000), comment on the importance of this ability for a CAS user. Firstly, when students are entering expressions with complex syntax into a CAS they need to be able to recognise whether the expression entered is equivalent to the one they are working from on paper. For example, in order to enter into a CAS the expression sin $2x$, which requires no brackets in by-hand mathematics, students need to appreciate that brackets are required in (most) CAS and then that sin ($2x$) is not the same as (sin 2)x. To enter $(a + p)/q$ (possibly written with a vinculum instead of brackets), students must appreciate that this is not the same as $a + p/q$. Secondly, students need to be able to convert output into a standard form since CAS does not always present results in the manner which is conventional in a given educational setting. The roots of the quadratic equation x^2-10x-16 = 0, for example may be given as $\sqrt{41} + 5$ and $-(\sqrt{41} - 5)$, whereas teachers in our local schools would almost invariably give these as $5 + \sqrt{41}$ and $5 - \sqrt{41}$. Thirdly, students need knowledge of equivalent expressions to be able to deal with the automatic simplification feature of CAS. This feature means that an input is often processed to an equivalent form immediately on entry. When an expression such as $2f - g + 3f - g$ is entered into a CAS such as the TI-89, the input is automatically simplified and appears as $5f - 2g$, the input $12x/6x$ appears as 2, $(b + a)^2$ is reordered alphabetically and appears as $(a + b)^2$ and $6 + (4a + 2b)/2$ appears as $2a + b + 6$. Lagrange (1999) reported that students reflected deeply on events like these, when teachers wanted them to consider deeper phenomena. Effective CAS users need to be able to quickly recognize that the inputs have been made correctly, even though the form appearing on the screen is different. Fourthly, on-going monitoring of the progress of a calculation requires a general alertness to the appropriateness of intermediate results and quick, confident decisions. At all stages of their work, CAS users need to recognise simple equivalences quickly and confidently, so that they are not derailed by the trivial but are alerted to the significant unexpected. Guin and Trouche (1999) noted that a surprising CAS result did not necessarily induce questioning in students who could not compare expressions.

We see the ability to recognize equivalent expressions as one indicator of a more general ability of *algebraic expectation* (Pierce, 2002; Pierce & Stacey, 2002). An important component of the algebra needed when CAS is used is ‘symbol sense’, rather analogous to the ‘number sense’ that is needed for doing arithmetic with a four function calculator. Arcarvi (1994) outlines what ‘symbol sense’ might be and how it is involved in all aspects of solving problems with algebra, including formulating problems algebraically, technical work within the mathematical world and interpreting algebraic answers in a real context. This describes the understanding of algebra required for working in partnership with technology. Within this overarching symbol sense, Pierce and Stacey (2002) have identified *algebraic expectation* as the algebraic parallel to numerical estimation. It is the thinking process that takes place when a mathematician considers the nature of the symbolic result expected as the outcome of some algebraic process. Its essence is making a quick assessment of the expected characteristics of algebraic results. Pierce and Stacey

(2002) define and analyse algebraic expectation in terms of several different elements: knowing conventions and basic properties of operations, and being able to identify the structure and key features of algebraic expressions. Algebraic expectation is the general ability needed to monitor the succession of expressions appearing on a CAS screen, making on-going rough checks for mathematical sense.

Because of the centrality of the ability to quickly recognize algebraic equivalence to the more general ability of algebraic expectation, we call the instrument described below the Algebraic Expectation Quiz. However, adequate monitoring of algebraic expectation requires other assessments, so in the CAS-CAT project (CAS-CAT Research Project) the Algebraic Expectation Quiz was supplemented by two tests that are not reported here: a 6-item Constructed Response Test and a separate test of both multiple choice and constructed response items assessing the ability to link representations.

THE ALGEBRAIC EXPECTATION QUIZ

Design of instrument

Outside the CAS experience, one of the daily experiences students have of needing to recognise equivalence of algebraic expressions is in checking their work from the back of a textbook. Very often, a student's answer is not in the same form as the answer in the book and so the student must decide whether their answer is correct or not. This situation was used to supply a realistic context for the Algebraic Expectation Quiz. Students were told that they would be presented with a series of slides, each showing two expressions. One expression was the answer given by a mythical student (Julie) to a problem and the other was the textbook answer. For example, the first item (item 1 of Table 1) presents on one slide $x \div y$ (Julie's answer) and $y \div x$ (the textbook answer). The students' task was to decide if Julie was definitely wrong, probably wrong, probably right or definitely right. Students unable to decide could choose the option 'no idea'. This mechanism enables a measure of confidence in the decision to be obtained simultaneously with accuracy. To reduce confusion of right/wrong and correct/incorrect in reporting results, we will refer to the items where Julie's answer matched the textbook answer as true items and the other items as false.

The Algebraic Expectation Quiz tries to capture the essence of making quick, real-time decisions, mirroring the way in which algebraic expectation provides constant and almost unconscious on-going monitoring of algebraic results. Hence it was important to present items with a restricted time to respond. To do this, the test was designed to be administered using Microsoft PowerPoint with a fixed time of 10 seconds for each item (time based on prior trialing by Pierce (2002)). Because of the need for unbroken concentration, the test time cannot exceed about 5 minutes, so that approximately 25 items can be used.

Items were selected based on previous research on learning algebra or using CAS (see for example Drijvers, 2000; Kieran,1992; MacGregor and Stacey,1997)), Pierce's quiz (2002) and the researchers' experience of working with students at this level. To confirm appropriateness and validity, items were reviewed by experienced mathematics teachers, who were asked whether they expected that students in Years 11 and 12 could quickly establish equivalence (or otherwise) of the expressions without any written steps. Whether or not a student can be said to have "good algebraic expectation" depends on their age and stage, so this heavily influences the creation of items. With few exceptions, we chose items from curriculum topics in Year 10 and below, using the four basic operations and square root. Informative items will canvas topics that a teacher assumes students already confidently know as well as what they need to know for their coming studies. Links between the items and skills required for using CAS can be seen by comparing the items with the examples used in the Introduction to this paper.

Table 1 lists 22 items, ordered according to item facility at the first test time. All items involve knowledge of conventions and basic properties of operations (for example that addition is commutative, the square root of a sum is not the sum of square roots), but this is not always the source of challenge for an item. Analysis of the first results of administering the Algebraic Expectation Quiz (Ball, Pierce & Stacey, 2001) showed that the most effective items required at most two intermediate steps which were simple enough to be carried out mentally. For example, Item 12 requires canceling the negatives and splitting the fraction. More complicated items were discarded.

Data collection

Along with other assessments, the Algebraic Expectation Quiz was administered to students in three project schools four times during the 2001 and 2002 school years to monitor their developing algebraic knowledge and skills. The complete cohort contained students learning with CAS and without it. However, the results in this paper are from the fifty students who completed the test in both February 2001 (beginning of Year 11) and September 2002 (just before the end of Year 12 and their final external examinations) and who undertook the new mathematics subject that permitted CAS. The results presented in this paper are for the 22 items common to the February 2001 (Feb 01) and September 2002 (Sep 02) administrations of the quiz. Information about the three project schools indicates that the 50 students may have academic ability better than the state average.

The quiz was administered by two of the present authors and one other researcher. They began by outlining the purpose of the quiz, explaining that each slide would only be shown for ten seconds and that there would not be enough time to do any written working but that students were expected to make a quick judgment based on their mathematical experience. Two sample items were demonstrated before the timed sequence began and they were used to stress that the students needed to judge the algebraic equivalence of the expressions rather than nicety of form. It was also explained that the intention was not to look for special cases, but to consider general expressions. For example, $x \div y$ is indeed equal to $y \div x$ when $x = \pm y$, but not in general, so Julie's answer in Figure 1 is wrong (i.e. Item 1 is a false item). The Algebraic Expectation Quiz took less than 5 minutes followed by other assessments not reported here. Students completed the Algebraic Expectation Quiz without use of pen and paper or technology except to record answers.

RESULTS

For each item Table 1 reports (i) the *facility* (% of students answering correctly i.e. *definitely right* or *probably right* on true items, and *definitely wrong* or *probably wrong* on false items) and (ii) percentage of students with well-placed certainty (both certain and correct i.e. answering *definitely wrong* for a false item or *definitely right* for a true item) and (iii) misplaced certainty (certain and incorrect i.e. *definitely wrong* for true or *definitely right* for false) in Feb 01 and Sep 02.

Item no	Item	Item Facility Feb 01	Item Facility Sep 02	Well-placed certainty Feb 01	Well-placed certainty Sep 02	Misplaced certainty Feb 01	Misplaced certainty Sep 02
21	$\frac{2a^3+5}{a^2} \quad 2a+\frac{5}{a^2}$	10	40	4	36	38	16
12	$\frac{s}{t}+\frac{p}{t} \quad \frac{\square(s+p)}{\square t}$	16	36	6	22	30	12
16	$\frac{4+b}{4} \quad 1+\frac{b}{4}$	16	56	8	32	32	12
20	$\sqrt{2}+\sqrt{y} \quad \sqrt{2+y}$	18	34	10	22	42	44
11	$a+p\div q \quad \frac{a+p}{q}$	24	52	18	36	58	34
4	$\frac{12x}{6x} \quad 2x$	24	58	20	42	54	30
7	$\sqrt{16x-4y} \quad 2\sqrt{4x-y}$	26	48	14	24	26	16
3	$\frac{2+x}{y+2} \quad \frac{x}{y}$	28	40	20	20	32	42
15	$2x^2-y^2 \quad (2x-y)(2x+y)$	28	82	18	42	38	10
17	$6+(4a+2b)\div 2 \quad 6+2a+b$	34	28	16	14	12	8
10	$\frac{1}{3}+\frac{1}{y} \quad \frac{2}{3+y}$	34	62	14	36	28	10
9	$\frac{a-b}{d-c} \quad \frac{a}{d}-\frac{b}{c}$	38	38	34	26	16	26
14	$sin(2x) \quad (sin\,2)x$	54	64	24	36	8	6
8	$h=x^2 \quad x=\pm\sqrt{h}$	54	90	28	76	12	2
5	$\frac{12x}{6x} \quad 2$	56	70	36	58	34	20
23	$(b+a)^2 \quad a^2+b^2+2ab$	58	94	42	78	28	2
2	$n^2(n^3+1) \quad n^5+n^2$	60	88	32	78	32	4
1	$x\div y \quad y\div x$	72	96	52	92	22	0
24	$\sqrt{xy} \quad \sqrt{x+y}$	82	96	66	92	6	2
13	$2f-g+3f-g \quad 5f-2g$	88	94	64	72	4	2
22	$-2y+6 \quad -2(y-3)$	92	94	78	82	6	0
6	$5m \quad m^5$	92	100	78	94	6	0
	Average over items	46	66	31	51	26	14

Key	<40%	40-75%	>75%

Table 1: Algebraic Expectation Quiz results Feb 2001 and Sept 2002 (N=50)

The facilities in Table 1 show that the items covered a wide range of difficulty. For example, Item 6 (the comparison of $5m$ with m^5) is a false item, and in Feb 01, 92% of students answered correctly *definitely wrong* or *probably wrong*. By Sep 02, this had increased to 100% suggesting that students had good algebraic expectation for items of this type for the duration of the study. The table also indicates that most of the students were certain of their correct result (78% in Feb 01, 94% in Sep 02). If an erroneous answer of this type was obtained from CAS work, we would expect students to identify it. Notational items (items 6 and 24) were well done as was linear factoring (item 22) and collecting like terms (item 13). Simple reorganizations such as the slightly unusual presentation of $(a+b)^2$ (item 23) were unexpectedly difficult although there was a significant improvement in both facility (Wilcoxon signed ranks test comparing median item scores for Feb 01 and Sep 02: W=228.5, p=0.000) and well-placed certainty (W=226.5, p=0.000) in Sep 02. Almost all fraction items (for example, items 12 and 21) had very low facility on both occasions. The relatively high percentage of students with misplaced certainty shows that many students remain unaware of their errors. On average, however, the facility of items improved by 20% over the two years, as did the well-placed certainty, whilst happily the misplaced certainty decreased (W=21, p=0.001).

Figure 1 shows that certainty (total of well-placed and misplaced certainty) and item facility are moderately related (correlation Feb 01=0.60, Sep 02 =0.76). The scattergrams show that items with highest facility in Sep 02 generally had high certainty, whereas in Feb 01 there was lower correlation. There are many interesting individual items. For example, in Sep 02 confidence for item 15 remains low, despite high facility. On this item, in Feb 01 students displayed misplaced certainty; in Sep 02 more are correct but they are unsure of their response. Items 4, 11 and 20 have low success and high certainty. In Feb 01, about half of the students showed misplaced certainty, and still about one third in Sep 02. These items correspond to persistent misconceptions (see MacGregor et al, 1997, for example). We conclude that many students are unlikely to identify errors of this nature in their work. In contrast, the three most difficult items (12, 16 and 21) have low facility, but moderate certainty. By Sep 02, only 13% showed misplaced certainty on these items. Although they were not sure of the correct answers, most students had at least become aware that these items represented areas where care is needed.

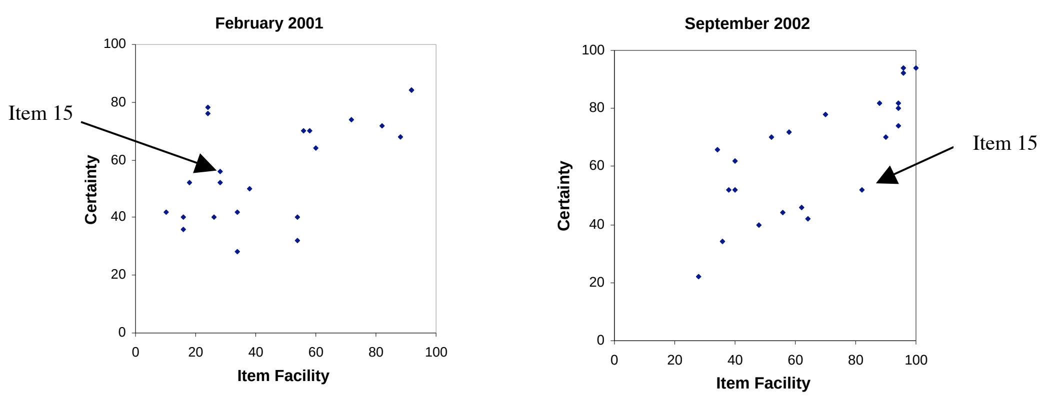

Figure 1: Scattergrams of Certainty versus Item Facility for Feb 01 and Sep 02

DISCUSSION

This discussion of the usefulness of the Algebraic Expectation Quiz as an instrument, information about students' algebraic expectation, and implications for teaching, is based on the results, together with feedback from the researchers who administered the quiz.

The Algebraic Expectation Quiz format was very successful. Students understood the scenario of comparing student and textbook answers and enjoyed the fun, fast Powerpoint presentation. The opportunity to indicate certainty reduced pressure on students and it provided deeper insight into their algebraic thinking. Since the quiz is quick to administer and easy to assess, the format is likely to be useful. A disadvantage is that the quiz format requires continuous concentration and therefore the number of items is limited.

The quiz results provide a partial but useful indicator of students' algebraic expectation, specifically their ability to recognize equivalent algebraic form. Overall, the results demonstrated both low facility and low levels of confidence. Improvement over two years was moderate, given that algebra becomes more important in the curriculum during years 11 and 12. Improvement in well-placed certainty was also moderate, although accompanied by a pleasing reduction in misplaced certainty. It is likely that students were increasingly aware of areas where they needed to be careful. The identification of areas of misplaced certainty (such as items 21, 12, 20, 3) provides information to teachers about firmly entrenched student misconceptions. Lack of certainty may also lead to excessive reliance on the machine: we have sometimes observed students who insist on copying machine output absolutely precisely, including factors of 1 or $ln(e)$, for example.

Overall, the low facility and low confidence demonstrate that students may benefit from the support of CAS in more situations than teachers expect. At the same time, it is likely that they will find CAS harder to use than might be expected, because of the difficulties outlined above related to identifying equivalent algebraic forms. This is especially so, given that participating schools were of above average academic attainment.

As noted, to develop controlled use of technology, it is likely that the standard curriculum will need to change. The results show that giving more emphasis to algebraic expectation and considering a wider range of algebraic forms is a vital direction for change. Targeting items to known CAS peculiarities may be useful e.g. changing the notation of the hand-written vinculum for division to the one line calculator format using a division sign or negative power. Pierce, who studied tertiary students learning with CAS extensively and documented their progress (Pierce, 2002) suggests that teachers discussing expressions may routinely ask about structure and key features. This can take little time, and become a routine for teacher and students, which develops algebraic expectation. Lagrange (1999) and Guin and Trouche (1999) have also been concerned to develop strategies.

Managing the tensions created by differences in algebraic form between by-hand and by-CAS algebra was always a concern for teachers working in the CAS-CAT project. In an interview at the end of the second year of teaching with CAS Lucy, one teacher in the CAS-CAT project, commented that her students were increasingly better able to deal with differences in form and she identified a teaching strategy developed to help them:

> "I'm finding increasingly that when the kids have done it by-hand and then they use the CAS, they're actually better prepared for the differences in form. And I notice one other thing.

> When I'm doing symbolic procedures more often at the end now, I'll write three or four answers and then I'll ask the kids, 'Which one did the calculator give you?' . . . Sometimes the way that it will express something that has surds or has fractions is not consistent [with by-hand conventions] so … you have to be able to be confident to say that's just the same thing."

As teachers like Lucy experiment with teaching with CAS, experience of how to develop better algebraic expectation will grow. Algebraic knowledge is required to identify equivalent forms (or non-equivalent forms if errors are made) quickly, with well placed certainty, for on-going monitoring. The data presented in this paper has demonstrated that students may have greater needs in this area than have been appreciated.

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HEURISTICS OF TWELFTH GRADERS BUILDING ISOMORPHISMS*

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This report analyzes the discursive interactions of four students to understand what heuristic methods they develop as well as how and why they build isomorphisms to resolve a combinatorial problem set in a non-Euclidian context. The findings suggest that results of their heuristic actions lead them to build isomorphisms that in turn allow them to justify a conjecture of theirs, using transitivity.

This research report focuses on four twelfth graders who, in an extended, self-structured problem-solving session, build heuristics and isomorphisms. The study arises from our general research program into the development of mathematical ideas by individual students as they work collaboratively in a small group. Specifically, this investigation connects to our inquiry into students' discursive practices (Powell & Maher, 2002) and how through their discursive practices they structure their own investigation and build mathematical practices and ideas appropriate for their problem task. The data is part of an ongoing, fifteen-year longitudinal, cross-sectional research project of the Robert B. Davis Institute for Learning, directed by Maher, that has been conducted in public elementary and secondary schools in a suburban, working-class, and immigrant town of New Jersey. Overall, our longitudinal study aims to contribute basic scientific understanding of cognitive behaviors as well as pedagogical conditions for which mathematics learning occurs as a process of sense making.

The participants in the present study are four students, Brian, Jeff, Michael, and Romina, in their senior year of high school who, from their entry into first grade have participated in mathematical activities of our longitudinal study. Over the years, these students have engaged tasks from several strands of mathematics, including algebra, combinatorics, probability, and calculus both in the context of classroom investigations as well as in after school settings (Maher, 2002). In this study, during an after-school problem-solving session, the students collaborate on a culminating task—The Taxicab Problem—of the research strand on combinatorics:

> A taxi driver is given a specific territory of a town, shown below. All trips originate at the taxi stand. One very slow night, the driver is dispatched only three times; each time, she picks up passengers at one of the intersections indicated on the map. To pass the time, she considers all the possible routes she could have taken to each pick-up point and wonders if she could have chosen a shorter route.
>
> What is the shortest route from a taxi stand to each of three different destination points? How do you know it is the shortest? Is there more than one shortest route to each point? If not, why not? If so, how many? Justify your answer.

Accompanying this problem statement, the participants have a map, actually, a 6 x 6 rectangular grid on which the left, uppermost intersection point represents the taxi stand. The three passengers are positioned at different intersections as blue, red, and green dots,

* We are grateful to Hanna N. Haydar for his discerning comments and suggestions.

respectively, while their respective distances from the taxi stand are one unit east and four units south, four units east and three units south, and five units east and five units south.

Besides the new, non-Euclidean geometric setting, this task has an underlying mathematical structure and encompasses concepts that resonate with those of other problems the participants have worked on in the longitudinal study. They, therefore, potentially revisit and deepen mathematical ideas they have already built as well build new ideas. Their implicit task was to formulate and test conjectures. We explicitly announced that they were to explain and justify conclusions. After they worked on the problem for about an hour and a half, we listened as they presented their resolution and asked questions to follow movements in their discourse toward further justification of their solution. Their resolution goes beyond the problem task to generalize it and to propose isomorphic propositions. A research objective is to inquire into and track how the participants develop their resolution of the problem task. Expressly for this report, we explore the following two questions: What heuristics do the participants employ? How and why do they build an isomorphism between the problem task and other problems on which they have worked?

THEORETICAL FRAMEWORK

Our theoretical perspective involves notions concerning the development of representations, models of the growth of understanding, and ideas about the generation of meaning references for which are detailed in Maher (2002). Here we build on this perspective and incorporate into it specific criteria for noticing, within the fine details of discourse, propositions that lead toward building isomorphisms. We explore a conceptual category about the contents of mathematical experience as proposed by Gattegno (1987). He theorizes how the human capacity to be aware of something and attach importance to can beget different sciences. For him, the instrument of knowing that allows scientists to be cognizant of the content of their awareness is "a dialogue of one's mind with one's self" (p. 6). Different sciences develop from the repeatable findings that stem from dialogues of minds with themselves specializing, for instance, on different human senses and on specific ways of knowing. He discusses a special "conquest of the mind at work on itself":

> mathematics…is the clearest of the dialogues of the mind with itself. [It] is created by mathematicians conversing first with themselves and with one another.…Based on the awareness that relations can be perceived as easily as objects, the dynamics linking different kinds of relationships were extracted by the minds of mathematicians and considered per se. (pp. 13-14).

From Gattegno's view on the psychological and dialogic development of mathematics, three notions of the contents of human experiences upon which the discipline is built can be identified: objects, relations among objects, and dynamics linking different relations. As the data from this study show, an additional category concerns heuristics. It pertains to methods of responding to questions raised in dialogues of the mind with the self. Extending Gattegno's categories, in mathematics, the content of experiences, whether internal or external to the self, can be *objects, relations among objects, and dynamics linking different relations*, and *heuristics*.

The notion of dynamics linking different relations provides guidance for identifying isomorphic propositions. Powell (1995) gives an example of this notion. It concerns the correspondence one can perceive between the two processes of (a) raising 2 to consecutively increasing integral powers and (b) the multiplicative process of doubling. In each process, we have objects (2 and the implicit objects … $\frac{1}{8}$, $\frac{1}{4}$, $\frac{1}{2}$, 1, 2, 4, 8…) and a relation (raising to a power and doubling). In the study, we employ this notion to identify and then analyze the propositions of participants that contribute to building of isomorphisms.

METHOD

Our data sources consist of the problem task; a video record of about 100 minutes of the activity of the four participants from the perspective of two video cameras; a transcript of the videotapes combined to produce a fuller, more accurate verbatim record of the research session; the participants' inscriptions; and researcher field notes. The participants' inscriptions are scanned and saved as picture documents. The video recordings are digitized, compressed, and stored on five compact disks as MPEG1 files. The transcript is a textual rendering of verbal interactions, specifically, turn exchanges among the participants and between them and researchers and in all consists of 1,619 turns at talk. Our analytic method employs a sequence of phases, informed by grounded theory (Charmaz & Mitchell, 2001), ethnography and microanalysis (Erickson, 1992), and approaches for analyzing video data (Pirie, 1996). Specifically, our method of data analysis involves the following nine non-linear, interacting phases: (1) attentively viewing the videotapes several times without intentionally imposing a specific analytic lens; (2) describing consecutive time intervals; (3) identifying critical events; (4) transcribing the video record; (5) inductive and deductive synchronous coding of transcript, videotape, and inscriptions; (6) writing analytical memoranda; (7) categorizing codes, identifying properties, and dimensionalizing properties within categories; (8) constructing a storyline; and (9) composing a narrative. (For an elaboration and examples of these phases, see Powell, Francisco, & Maher, 2001).

RESULTS

The problem-solving session lasted for approximately 1 hour and 40 minutes. Analysis of the video data reveals that, without assistance from the researchers, the participants through their conversational exchanges structure their own investigation. Further analyses of their discourse and inscriptions reveal that they use their time to understand and plan how to resolve the problem task; develop problem-solving strategies and overcome heuristic hurdles; hypothesize and create combinatorial algorithms; build explanations and justifications of their ideas; challenge each other to clarify their explanations and justifications as well as accept challenges of the same from researchers; and formulate isomorphisms, focusing on the one between the Taxicab Problem and the Towers Problem.[1]

[1] The Towers Problem is to build towers (for example, with Unifix cubes) of particular heights when selecting from a certain number of colors. From grades 3 to 10, the participants have worked on versions of this problem with varied conditions.

During the session, the participants develop and employ sixteen heuristics. The following are their different heuristics with indication of when ([hours:minutes:seconds]) from the start of the session they initially implement each one: (1) counting routes from the taxi stand to a pick-up point while outlining without drawing the routes [0:02:30]; (2) traveling on the grid lines only east and south [0:04:41]; (3) parceling out different mini-tasks among group members as well as collecting and recording the data they generate [0:05:59]; (4) counting routes to a pick-up point while drawing the routes on the same sub-grid [0:06:15]; (5) attending to dynamical links among objects and relations between two systems [0:07:31]; (6) attending to numeric patterns in generated data [0:12:10]; (7) doing easier sub-problems; counting routes from the taxi stand to nearby intersection points while outlining without drawing the routes [0:14:53]; (8) counting routes from the taxi stand to nearby intersection points while drawing the routes on the same sub-grid [0:16:12]; (9) parceling out the same mini-task, each counting routes to intersection points nearby the taxi stand, drawing them on the same sub-grid, as well as recording and comparing resulting data [0:16:50]; (10) planning to count systematically points nearby the taxi stand and anticipating that a numerical pattern will emerge [0:22:12]; (11) talking aloud how one is finding all shortest routes an intersection point [0:24:23]; (12) drawing each route between the taxi stand and an intersection point on a separate sub-grid [0:24:23]; (13) finding opposite routes to each drawn route to ensure that all possible routes are found [0:25:00]; (14) parceling out mini-tasks to compare the data generated from different combinatorial algorithms [0:39:08]; (15) building isomorphisms among the Taxicab, Tower, and Pizza Problems, using Pascal's triangle as an iconic representation upon which to build the isomorphisms [1:02:37]; and (16) processing their findings with researchers to see *where they lead themselves* through their presentation of their ideas [1:04:34]. Some participants supplant some heuristics with others, and some heuristics once initiated remain active strategies for some participants.

With these heuristics, the participants generalize the problem task and propose isomorphic propositions. They notice relational connections between this problem task and others on which they have worked. They develop combinatorial algorithms with which they generate reliable data from which to perceive numerical patterns. Based on these patterns, they conjecture that the underlying mathematical structure is Pascal's triangle of binomial coefficients. To convince themselves of the veracity of their conjecture, they build an isomorphism between the problem task and the Towers Problem. They know from previous work on block-tower tasks that Pascal's triangle underlies their mathematical structure. In what follows, we further focus our analysis on the isomorphism they build by identifying their discursive propositions about dynamical links that establish one-to-one correspondences between, on the one hand, objects and relations or actions in one system and, on the other hand, objects and relations of another system in such a way that an action on objects of one system maps to an analogous action on the corresponding objects in the other system.

A prerequisite to articulating a proposition that indicates an isomorphism is to attend to particular features of objects and relations among the objects within each system to determine whether dynamical links can be formulated between the systems. Early in the session, the participants manifest embryonic thinking about an isomorphism.

Romina wonders aloud: "can't we do towers on this" (turn 159).[2] Her public query catalyzes a negotiatory interlocution among Michael, Jeff, and her. Jeff, responding immediately to Romina, says, "that's what I'm saying," (turn 160) and invites her to think with him about the dyadic choice ("there or there" turn 162) that one has at intersections of the taxicab grid. Furthermore, he wonders whether one can find the number of shortest routes to a pick-up point by adding up the different choices one encounters in route to the point (turn 162). Romina proposes that since the length of a shortest route to the red pick-up point is 10, then "ten could be like the number of blocks we have in the tower" (turn 169). Romina's query concerning the application of towers to the present problem task prompts Michael's engagement with the idea, as well. As if advising his colleagues and himself, he reacts in part by saying, "think of the possibilities of doing this and then doing that" (turn 180). While uttering these words, he points at an intersection; from that intersection gestures first downward ("doing this"), returns the to point, and then motions rightward ("doing that"). Similar to Jeff's words and gestures, Michael's actions also acknowledge cognitively and corporally the binomial aspect of the problem task. He, Jeff, and Romina have put into circulation the prospect of as well as insights for building an isomorphism between the Taxicab and Towers Problems.

The prospect and work of building such an isomorphism reemerges several more times in the participants' interlocution. With each reemergence, the participants further elaborate their insights and advance more isomorphic propositions. Eventually the building of isomorphisms dominates their conversational exchanges. Approximately thirty-five minutes after Romina first broached the possibility of relating attributes of the Towers Problem to the problem at hand, the participants reengage with the idea. Romina speculates that between the two problems one can relate "like lines over" to "like the color" and then "the lines down" to the "number of blocks"(turn 738). What is essential here is Romina's apparent awareness that each of the two different directions of travel in the Taxicab Problem needs to be associated with different objects in the Towers Problem.

Romina uses this insight later in the session. She transfers the data that she and her colleagues have generated from a transparency of a 1-centimeter grid to plain paper. Their data are equivalent to binomial coefficients. She identifies one unit of horizontal distance with one Unifix cube of color *A* and one unit of vertical distance with one Unifix cube of color *B*:

> Like doesn't the two- there's- that I mean, that's one- that means it's one of *A* color, one of *B* color [pointing to the 2 in Pascal's triangle]. Here's one- it's either one- either way you go. It's one of across and one down [pointing to a number on the transparency grid and motions with her pen to go across and down]. And for three that means there's two *A* color and one *B* color [pointing to a 3 in Pascal's triangle], so here it's two across, one down or the other way [tracing across and down on the transparency grid] you can get three is two down [pointing to the grid]. (turn 1210)

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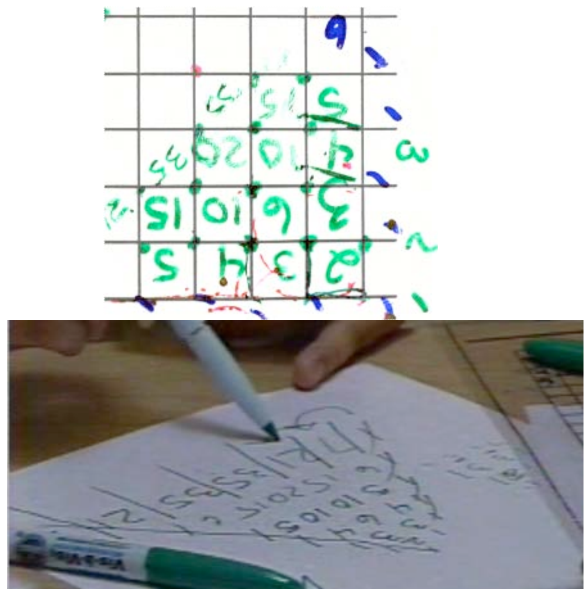
[2] For Romina and other participants in the longitudinal study, this comment is pregnant with mathematical and heuristic meaning derived from their constructed, shared experiences with tasks and inscriptions in the combinatorial and probability strands of the study (see, for instance, Kiczek, 2000; Martino, 1992; Muter, 1999).

Furthering the building of their isomorphism, Michael offers another propositional foundation. Pointing at their data on the transparency grid and referring to its diagonals as rows, he notes that each row of the data refers to the number of shortest routes to particular points of a particular length. For instance, pointing the array—1 4 6 4 1—of their transparency, he observes that each number refers to an intersection point whose "shortest route is four" (turn 1203). Moreover, he remarks that one could name a diagonal by, for example, "six" since "everything [each intersection point] in the row [diagonal] has shortest route of six"(turn 1205). In terms of an isomorphism, Michael's observation points in two different directions: (1) it relates diagonals of information in their data to rows of numbers in Pascal's triangle and (2) it notes that intersection points whose shortest routes have the same length can have different numbers of shortest routes.



A B

Figure 1. Participant's data arrays (from their perspective): (A) In green, empirical data of shortest routes between the taxi stand and nearby intersection points. Jeff wrote the ones in blue to augment the appearance of the numerical array as Pascal's triangle. From the participant perspective, to the left of Jeff's numbers, Romina wrote in green the numbers 1, 2, and 3 to indicate the row numbers of the triangular array. (B) The first five rows contain empirical data; the remaining two rows contain assumed data values based on the addition rule for Pascal's triangle.

Later in responding to a researcher's question, the participants develop a proposition that relates how they know that a particular intersection in the taxicab grid corresponds to a number in Pascal's triangle. They focus their attention on their inscriptions, A and B, in Figure 1. Michael and Romina discuss correspondences between the two inscriptions. Referring to a point on their grid that is five units east and two units south, Romina associates the length of its shortest route, which is seven, to a row of her Pascal's triangle by counting down seven rows and saying, "five of one thing and two of another thing"(turn 1313). Michael inquires about her meaning for "five and two" (turn 1314). Both Romina and Brian respond, "five across and two down"(turns 1317 and 1318). She then associates the combinatorial numbers in the seventh row of her Pascal's triangle to the idea of "five of one thing and two of another thing," specifying that, left to right from her perspective, the first 21 represents two of one color, while the second 21 "is five of one color" (turn 1320), presuming the same color. Using this special case, Romina hints at a general proposition for an isomorphism between the Taxicab and Towers Problems.

The above presents evidence that students work to build an isomorphism during the course of the problem-solving session. The content of the phases include the following with indication of when from the start of the session each occurs: (1) there exists a relationship between the Towers and Taxicab Problems, [0:07:37]; (2) Similar to the Towers Problem, the Taxicab Problem has a dyadic choice or binomial aspect, [0:07:39 and 0:08:55]; (3) The length of a shortest route to an intersection point corresponds to the height of a tower, [0:08:15]; (4) Each of the two different directions of travel in the Taxicab Problem needs to be associated with different objects in the Towers Problem, [0:44:26]; (5) Rebuild the meaning of 2 to the n in the environment of the Towers Problem, [0:08:26 and 0:44:51]; (6) Identify one unit of horizontal distance with one Unifix cube of color A and one unit of vertical distance with one Unifix cube of color B, [1:14:59]; (7) A row "diagonal" of their data contains the number of shortest routes for intersection points whose shortest distance from the taxi stand is n, [1:16:00]; (8) Intersection points whose shortest routes have the same length can have different numbers of shortest routes, [1:16:37]; (9) A tower 3-high with 2 of one color and 1 of another color, to routes to a point 2 down and 1 across, [1:18:40]; and (10) Intersection point five units east and two south from the taxi stand corresponds to five of one thing and two of another thing and, therefore, go the seventh row of Pascal's triangle and the second and fifth entries of the triangle to find the number of shortest routes from the taxi stand to the intersection point five units east and two south from the taxi stand, [1:22:40].

DISCUSSION

The forgoing has presented the mathematical processes and strategies that participants employ as they resolve the problem task. Through their various heuristic actions, among other consequences, the participants generate data that they consider reliable. Reflecting on numerical patterns in their data, they conjecture that Pascal's triangle is the underlying mathematical structure of the problem task. How do they justify this conjecture? The data suggest that to justify their conjecture is the reason why the participants build an isomorphism between the problem task and the Towers Problem. Furthermore, to understand how they build their isomorphism, we have focused analytic consideration on one of their heuristics: *attending to dynamical links among objects and relations between two systems*. By doing so, we have identified the locus of how they build an isomorphism. We observe that early in the problem-solving session by attending to dynamical links three participants—Romina, Jeff, and Michael—articulate awareness of object and relational connections between their current problem task and a former one, the Towers Problem. Later, upon noticing that their array of data resembles Pascal's triangle and conjecturing so, the participants embark on building an isomorphism between the Towers Problem and the Taxicab Problem as an approach to justifying their conjecture since from previous experience they know that Pascal's triangle underlies the mathematical structure of the Towers Problem. In this sense, their strategy can be interpreted as justifying their conjecture by transitivity: (a) Pascal's triangle is equivalent to Towers and (b) Towers is equivalent to Taxicab; therefore implying that (c) Pascal's triangle is equivalent to Taxicab. They know (a) is true and embark on demonstrating (b) to justify and conclude (c).

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INTERACTIVE WHITEBOARDS AND THE CONSTRUCTION OF DEFINITIONS FOR THE KITE

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This paper reports early work from a project examining the affordances offered by Interactive Whiteboards (IWBs). Here the focus is on the teaching of the definition of quadrilaterals through the use of Cabri Géomètre. We discuss the work of two 11-year-old children, who are exploring the kite. The protocol highlights the complexities inherent in understanding definitions of quadrilaterals, as reported widely by other researchers. We conjecture that a change of emphasis in the use of the IWB may encourage figural concepts (Fischbein, 1993) to be controlled by conceptual rather than visual components.

INTRODUCTION

The interactive whiteboard (IWB) is a combination of a projection system and a touch sensitive whiteboard, which allows the teacher (or children) to manipulate the computer screen through physical interaction with the whiteboard. Some IWBs require a special pen; others such as the one in this study can be operated by any device including the human finger. Unlike almost any other new technology introduced to education, IWBs have generally been welcomed by teachers, who seem to find their use enhances their normal conventional teaching style. Levy (2002), Glover and Miller (2001), and Smith (2001) all investigated the use of IWBs in one or a few British schools. They report generally on keenness to use the technology.

This study represents the second phase (first iteration) in a broader study in which we are teasing out the affordances of an interactive whiteboard in whole class teaching. In the first phase, we interviewed 14 teachers, most of who were identified by the ICT Coordinator in the school as being enthusiastic users of this technology. Two teachers in particular had used the technology but infrequently. These interviews generally supported the positive responses found in the previous research. Our analysis of their responses seemed to indicate two specific affordances, what we have come to call *visual* and *kinaesthetic*. The former relates to the size, clarity and colourful impact of the computer graphics, writ large on the whiteboard. The latter relates to the potential impact of dynamically manipulating the screen in such a way that the teacher's (or child's) agency in the process is far more impressive than merely following a small mouse arrow. The interviews suggested that the visual affordances of IWBs often impact through carefully prepared material by the teacher, but this material rarely exploited the kinaesthetic affordances.

It is nevertheless unclear how such affordances might shape children's learning. There is no guarantee that, when teachers talked to us about the technology, they were tuning into learning. It is entirely plausible that their responses were in fact guided by feeling in control of the class, by being the focus of attention at the IWB or by the excitement of using such impressive new technology. In order to focus on learning, we decided to examine teaching and learning in a context where it was reasonable to suppose that visual and kinaesthetic affordances would have particular potential.

The Royal Society and JMC (Royal Society, 2001) specifically advocated the teaching of geometry using a dynamic geometry software package (DGS) on an IWB. There is clearly a danger here of what Papert (1991) has referred to as technocentrism in focusing on a piece of hardware such as IWBs. It would be naïve in the extreme to claim special properties in a piece of hardware, or software for that matter, that mystically transforms the learning process. We aspire to relate the affordances of the IWB not only to the use of DGS but also to the teaching process itself with particular emphasis on the tasks and the teacher/class discourse. The focus of this paper is on work related to the construction and definition of quadrilaterals.

Tall & Vinner (1981) distinguish between what they term concept image and concept definition. Whereas the concept definition is seen as a form of words used to specify the concept, the concept image is used "… to describe the total cognitive structure that is associated with the concept" (p.152). The concept definition may be that offered by the teacher and used in rote fashion or it may be a personal construction of what has been offered and in fact departs quite significantly from the teacher's version. Classroom practice may have focussed on certain aspects of the concept definition, resulting in a concept image that scarcely incorporates the teacher's concept definition but instead emphasises certain particularities and a rather more limited personal concept definition.

Whereas Tall and Vinner's use of this terminology is intended to apply rather widely to a range of types of concepts, Fischbein (1993) has focused on a specific knowledge domain. Fischbein analyses geometrical reasoning and points out that such reasoning involves the manipulation of what he refers to as figural concepts. Such concepts fuse (at least ideally) sensorial spatial imaginings (visualisations) with conceptual attributes or properties. In fact, this ideal of fusion is modified in practice. According to Fischbein, "the figural component is usually influenced by figural-Gestalt forces and the conceptual components may be affected by logical fallacies. With age, and as an effect of instruction… the fusion between the figural and the conceptual facets improve." (p145).

Ideally the conceptual aspect should control the figural concept. In practice, children typically allow the visual component a controlling role even when they know the definition of the figure. Thus, a student who knows the definition of a parallelogram may nevertheless find difficulty in recognising the various shapes that correspond to that definition. An oblique parallelogram, a square and a rectangle are so figurally different that the unifying effect of the common concept simply vanishes.

Mariotti (1994) used Fischbein's theory of figural concepts to examine the interplay between the figural level ("observing the object as it appears", p234) and the conceptual level ("relating to the properties which characterise the geometrical figure, embodied by the object", p234) in a teaching experiment. She concludes from this episode that there is often a conflict between the figural, which stresses the differences in perception, and the conceptual, which attempts to emphasis the similarities within a classification. Mariotti places some emphasis on the role of the teacher to raise this conflict, arguing that it is only by making the tension between the figural and the conceptual explicit that it can disappear.

The importance of the visual aspect of figural concepts is made quite apparent in a review by Clement and Battista (1992). They highlight the findings by Herschkowitz (1989) who showed how visualisation was a crucial element in the process by which a student constructed geometric ideas. Furthermore, she connected that process to Van Hiele levels 1 and 2 (1986). Thus, the student in an initial stage of development will typically refer to a protypical visual image. At level 1, any figures offered to the student will be compared to the student's prototype as a critical part of the identification of the shape. According to Herschkowitz, as the student moves towards level 2, she will begin to discriminate critical aspects of the prototype and use these as part of the judgement. When the student has reached level 2, she will be able to use those attributes independently of the prototype and the figures will be seen as mere instances of the shape.

Herschkowitz pointed out that the use of prototypes at levels 1 and 2 could often be limiting as the student may be tied too closely to that single image, preventing the discrimination of the critical attributes. The use of the concept might be quite limited because the student relies too heavily on the prototypical image.

We can conclude from this literature that children narrowly using prototypical thinking are using the visual rather than the conceptual aspects of the figural concept, and that this unsophisticated concept image may at least be in part due to pedagogical circumstances.

Some researchers have found evidence that the use of software in carefully designed teaching experiments might support a move from Van Hiele level 1 to level 2. Markopoulos & Potari (1996) discuss their work with 11-year-old children using specially designed software to investigate children's construction of the concept of geometric shapes. The authors claim that, through the teaching experiment, the children began to develop connections between the figures and their properties and formed hierarchical relationships between classes of shapes. A study by Jones (2000) suggested that a teaching programme based on a DGS package enabled some children to move from Van Hiele level 1 to level 2.

De Villiers (1998) is interested in how we might encourage children to appreciate the process of defining. In a teaching experiment, De Villiers focussed on what he calls *descriptive* defining in which the current definition is improved by the selection of appropriate subsets of properties from which the other properties can be deduced. The experiment aimed to support children's appreciation of the goal of achieving "economic" definitions. De Villiers speculates that dynamic geometry software may have a special role to play in supporting the construction of hierarchical classifying by children.

There is at face value a possible connection between the two aspects of figural concepts with the two affordances of IWBs, namely the visual and the kinaesthetic. It seemed plausible that the IWB could support the enhancement of children's concept image for quadrilaterals by supporting the figural component through the visual impact of the IWB whilst encouraging the conceptual component through appropriate use of the kinaesthetic affordance. The latter type of support was the more problematic, since, although we could envisage the teacher exploiting the dynamic nature of the whiteboard to examine many non-prototypical configurations of a quadrilateral, it was far from clear to us what sort of activity might support the fusion between these two components as described by Fischbein.

METHODOLOGY

The first iteration of the second phase of this study involved a class of 11/12-year-old children at a secondary school. Neither they, nor the teacher, who was an experienced head of mathematics department, had any previous experience of DGS. The teacher's experience of using an IWB was also limited.

The researchers constructed a set of Cabri-based tasks, which we saw as possible resources for lessons. In this first iteration, our intention was not to guide the lesson planning too strongly as we were interested in how the teacher herself would employ (or not) these resources in planning a sequence of 8 Cabri lessons, which focussed on quadrilaterals, triangles, reflection and translation. It is also true to say that we did not feel ourselves to be experts who could offer ready-made lessons, but rather that we hoped the research would enable us to be better informed to plan the use of the IWB more effectively in the second iteration. The intention was that each lesson would have an introduction and a final plenary involving the use of the IWB, and that the children, working in pairs, would use PCs for personal exploration in the middle of each lesson.

3 pairs of students were chosen by the class teacher on the basis that they generally worked well together and had good attendance. These pairs were interviewed for about 30 minutes before the programme started to gain some insight into their attitudes to the school, computers and mathematics. In addition, some of their relevant mathematical ideas were briefly explored. During most of the lessons, the IWB work and the computer work of the 3 pairs of children were recorded[1]. At the end of the program, the same pairs of pupils were interviewed again. They were briefly questioned about their attitudes to the lessons.

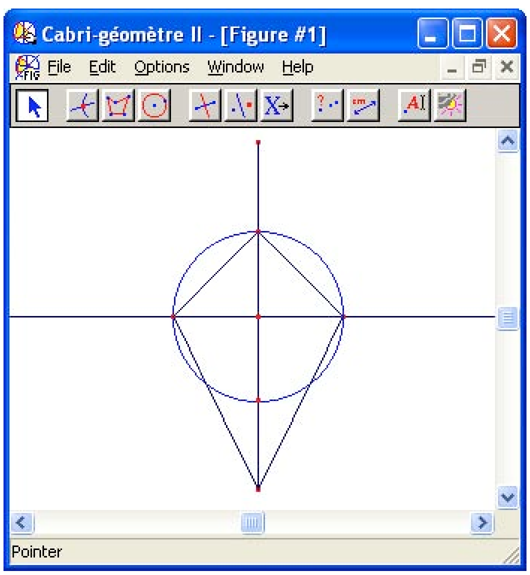

Fig. 1: The girls construct their kite.

Their geometrical reasoning was explored by asking them to work on tasks using Cabri Géomètre. In this paper, we focus on part of that final interview where two girls, whom we shall call Christine and Michelle, consider which shapes can be formed by dragging a prepared construction.

From this range of lessons, perhaps the most relevant to the episode below were lessons 3 and 4. Lesson 3 began with an introduction by the teacher in which a rhombus was made by folding a piece of paper. In this whole class work, the idea that a square is a special type of rhombus was raised in the final plenary on the IWB, as was the idea that a rhombus was a special type of kite. There was however no attempt to reinforce these ideas through further examples, probably because of time constraints. In lesson 4, there was initial discussion on the IWB about the properties of a parallelogram. In the main part of the lesson, the children attempted to construct quadrilaterals from properties based on the

[1] This was achieved by use of Camtasia software, which captured the screen images and recorded mouse 'clicks' as well as capturing speech via attached microphones. Camtasia is produced by TechSmith Corporation, 1780 E. Grand River Ave. East Lansing, MI 48823 USA http://www/techsmith.com/products/camtasia/camtasia.asp

diagonals. Christine and Michelle managed to construct a square with some help from the teacher. After one or two false starts, they also constructed a kite (Figure 1) by drawing a vertical line segment and a perpendicular bisector to the segment. They placed a point on the bisector and constructed a circle through this point, centre at the intersection of the two lines. The kite was formed by constructing a polygon through the three points of intersection between the circle and the lines, and the end of the line segment. It is worth pointing out that this kite is not general since the angle between the horizontal diagonal and either of the upper sides is fixed at 45^{o}. Interestingly, the construction disappears when the lowest point is dragged upwards beyond the point on the circle where the figure becomes a square.

AN ILLUSTRATIVE VIGNETTE

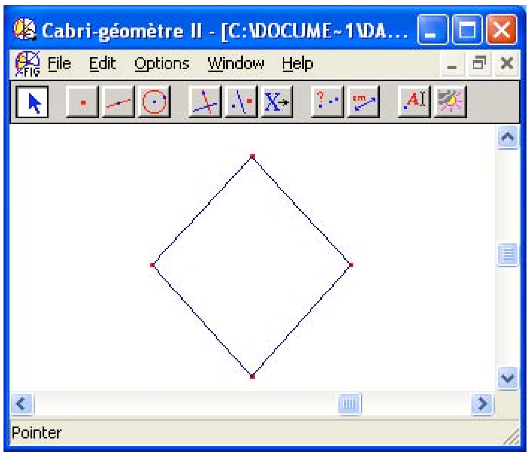

Fig. 2: The initial "rhombus" kite

During the interview, which took place after the sequence of lessons, we presented Christine and Michelle with the construction in Figure 2. This figure is constructed more generally than the one that Christine and Michelle constructed in Lesson 4. All 4 vertices can be dragged , allowing the figure to appear as a rhombus, square or kite. This figure can also be dragged into a deltoid shape.

The following discussion took place before the girls attempted to drag the figure.

M: So you can look at it like a kite. It's like a rhombus, but turned.
C: Mmm. Yes it is. Aren't kites meant to be longer at the bottom than at the top. It's like a square, but it's changed shape.
M: Yeah. If it's like that, then it's like a rhombus.
C: Yes
M: If you look at it that way, it's like a rhombus. I don't know if it is this way. Sort of like a rhombus, but upside down, to the side.
C: Yes, it's slanted.
M: So you can look at it like a kite. It's like a rhombus, but turned.
C: Yes, it's a special kind of a rhombus.

Both girls are at this stage attending to the visual aspects. The task has directed them towards the visual as they have not been asked to interact with the figure other than through their perceptions. There is evidence that both girls seem to refer to prototypical thinking as in Van Hiele level 1. Michelle seems to have in mind a prototype rhombus with its base horizontal and is able to imagine turning the rhombus from its current heading to that of her prototype (lines 1, 5, & 7). It is unclear what she means when she refers to a kite (lines 1 & 7) but Christine clearly imagines a prototypical kite in which two adjacent equal sides are longer than the other two adjacent but equal sides (line 2). It seems that this figure does not conform to her concept definition of a kite.

In the next stage, we encouraged the girls to interact with the shape by dragging it to see whether it could be "messed up" (Noss & Hoyles, 1994). They began by rotating the figure into a position that was close to the prototypical rhombus described above:

M: That's a rhombus.
I: Did you say it was a rhombus before?
M: Yes, but it didn't look so much a rhombus. [laugh] It was sort of turned.

To begin with they only dragged points, which changed the orientation. Then they picked up a point that altered the figure so it was clearly no longer a rhombus. They recognized the kite but were confused because they felt the shape had been constructed to be a rhombus and dragging certain points was consistent with the notion that the rhombus could not be messed up, whereas dragging other points did mess up their rhombus.

The girls now had a dilemma, since shapes that were not constructed were drawn. This shape could be messed up and so was not constructed. Yet, they felt that there was some sense of construction involved.

C: Hey! [laugh] I don't get that.
M: That was a kite, look that goes along there… I don't get how it can be constructed and drawn?
I: Ahh! Just say what you mean there?
M: Well, those 2 points change…{yeah} and they're constructed… And the other corner…{yeah} those 2 just...don't work [embarrassed laugh]… They like aren't constructed…they can mess it up…
C: Yeah, they can mess it up
M: It's not a messed up kite, but it is a messed up rhombus. {Yeah}
C: No it's not, because it's a square that's changed, which is…
M: I don't think it's a rhombus any more.
I: You don't think it's a rhombus anymore? {No} So what do you think it is?
M: A kite, or a quadrilateral [quadrilateral said together].

Michelle at least had recognized that the notion of messing up needed to be related to the figure supposedly constructed (line 17). They could regard the figure as the construction of a kite but not of a rhombus. This understanding was far from stable.

I: So can you mess the kite up?
C: That doesn't look like a kite any more. But if you look like that…
M: That one looks longer than that one. But sometimes it looks like a kite, and sometimes it doesn't.

The girls were still struggling with the idea that the kite sometimes looked like a rhombus. A resolution to this paradox would have been to incorporate rhombuses into their definition of kites i.e. to see rhombuses as special types of kites. It seemed though that the partitional type of definition was closer to the prototypical visual-based component of the figural concept. The pedagogical challenge here was clearly to find a means of helping the girls to focus more on defining the figure in terms of its properties. The dragging movement was raising the conflict but the resolution was not apparent to them. Their confusion was intensified when moments later yet another shape (the deltoid, or "pointy shape" as Michelle calls it in line 28) appeared:

I: Can you get it when it doesn't look like a kite?
C: Yeah, that one.
I: What would you call that?
M: Um, um… A pointy shape.

DISCUSSION

The episode illustrates the problem to which Herschkowitz (1989), Fischbein (1993) and Tall and Vinner (1981) have all alluded. Michelle and Christine clearly are constrained by referring rather infrequently to the properties of the figure. This tendency to refer to prototypes was evident despite some attention on the IWB in the teaching programme to the way in which quadrilaterals can be rotated. Their prototypes are useful resources for simple manipulations of orientation but do not support hierarchical inclusive definitions. In the teaching programme, squares were seen quite explicitly as special types of rhombi but these experiences, it seems, were insufficient to encourage Michelle and Christine to refer consistently to conceptual criteria.

It is reasonable to suppose that the figural component might have been reinforced by the visual affordance of the IWB. It is clear that task design and teacher-focus are crucial if the kinaesthetic affordance (of both the DGS and the IWB) is to be harnessed in support of the conceptual aspect of the figural concept. We do not believe that the kinaesthetic affordance was properly exploited in the first iteration of this study. We now ask ourselves whether it is reasonable to expect such children to understand the definition of a kite when they do not appreciate the utility (Ainley & Pratt, 2002) of a definition. We therefore are attracted to the approach advocated by De Villiers (1998) in which we put children in the position of being definers themselves.

We want to place emphasis on the dilemma presented by different Cabri constructions of the same (apparently) mathematical object through a task such as the following:

> Here are three figures constructed in different ways in Cabri. Which provides the best definition of a kite?

One construction might be based on Michelle and Christine's approach, which is restricted in so far as the figure disappears when the lowest point (as depicted in Figure 1) is dragged inside the circle. Another construction (as used in our interviews) might allow deltoids as well as kites. A third could be constructed in such a way that the figure disappears just as the kite is about to transform into a deltoid. All three constructions should have the property that in certain configurations the figure looks like a rhombus and in others like a square. The issue of whether a partitioned definition is better than a hierarchical definition would be raised. The same issue could be raised using a similar task based on rectangles, squares and oblongs[2].

The task would be introduced on the IWB, followed by small group work on PCs. The group work would enable children to act as definers drawing personal conclusions. The IWB would act as a public forum for opinions to be shared, conflicts raised and hopefully awareness that different definitions are possible and that utilities are attached to each possible definition.

In summary, two conjectures have emerged from this initial study and they will be tested during Spring, 2003:

(i) The visual and kinaesthetic affordances of the IWB are insufficient to encourage the fusion of conceptual and visual aspects of children's figural concepts when these

[2] In primary schools in England and Wales, "oblongs" are often referred to as shapes whose length is greater than their width i.e. rectangles which are not squares.

affordances are embodied in tasks that simply focus on the visual transformation of geometric figures, and

(ii) The kinaesthetic affordances of the IWB need to be embodied in tasks based on the utilities of contrasting definitions that draw attention to the conceptual aspect.

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DEVELOPMENT OF PERSONAL CONSTRUCTS ABOUT MATHEMATICAL TASKS - A QUALITATIVE STUDY USING REPERTORY GRID METHODOLOGY

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We can fill the gap between theory (of ambitious teacher education) and practice (in mathematics classrooms) more easily, when we take into account that teacher education is not only explicit acquisition of knowledge and methods but also development of individual personal constructs of mathematics and mathematics education. But how can we get knowledge about the development of individual implicit theories of teacher students? In this article, we propose a method to explore teacher students' individual conceptions about learning and teaching mathematics. It is based on Kelly's personal construct theory and uses repertory grid methodology for data collection as well as formal concept analysis for data analysis. The method was applied in a qualitative evaluation study of a pre-service teacher course in Darmstadt.

Various research studies have unanimously shown that classroom activities of teachers are heavily influenced by implicit subjective theories (operationalized as cognitions, beliefs, attitudes, or personal constructs, respectively). These implicit theories relate to all aspects of mathematics classrooms, especially mathematics, methods, teaching, learning, the teachers' role, and the pupils (see Thompson 1992, or Törner 1997 for overviews). The fundamental insight in the importance of implicit theories should therefore have consequences in teacher education. Sophisticated teacher training conceptions should not only focus on the explicit transmission of information, conceptions and methods for learning and teaching mathematics but also on the development of the implicit subjective theory of each teacher student. It was in 1983 already that Alan Bishop emphasized that the main aim of teacher education should be the "broadening and development of teacher's personal constructs." (Bishop 1983, p. 21). In the course of his argumentation, he refers to the theory of personal constructs developed by the clinical psychologist Kelly (1955). This approach was theoretically and methodologically deepened by McQualter who analyzed the process of "becoming a mathematics teacher" with personal construct theory (1986).

In this article, we follow the idea that it is a main task for teacher education to develop teachers' personal constructs. As a consequence, we claim for including this aspect into the evaluation of teacher training modules. Therefore, qualitative evaluation studies should also focus on the individual development of personal constructs.

INVESTIGATING PERSONAL CONSTRUCTS

The constructivist key message of Kelly's personal construct theory is that the world is 'perceived' by a person in terms of whatever 'meaning' that person applies to it (Kelly 1955, Fransella/Bannister 1977). The basis on which persons develop their personality, attitudes, and concepts and perceive reality are their systems of personal constructs.

> It is important to note that constructs are not only names, or concepts, or attitudes, or opinions. Constructs have a function for the individual. They serve as tools to replicate events in our imagination, and to make up our view of the world by continuous confirmation or disconfirmation, thus ‘to construe reality’ (the *construction* corollary). Constructs are organized in systems, often hierarchical in structure, there are superordinate constructs, core constructs, peripheral constructs, according to their importance to the individual’s life (the *organisation* corollary). We have different construct systems for different areas and realms which may even be partially incompatible or at least contradictory when involved at the same time (the *fragmentation* corollary). Constructs in principle can be changed through experience (the *experience* corollary). […] Constructs are significant characteristics of the individual (the *individuality* corollary), i. e. ‘personal’. (Scheer 1996).

We agree with Bishop (1982) and McQualter (1986) that mathematics teachers’ perceiving and acting is highly determined by their personal constructs about teaching and learning. Hence, a qualitative evaluation of teacher training modules should also focus on these implicit elements. But how can they be investigated?

In the methodology of qualitative social research, a large variety of methods have been developed to identify implicit theories and belief systems. The test procedures and methods for knowledge elicitation mainly differ in their degree of standardization. One extreme is the completely standardized questionnaire offering multiple choice answers only. The advantage of this kind of data collection is that it can be compared directly and is analysed qualitatively as well as quantitatively. On the other hand, the missing possibility for test persons to express their thoughts in their own language produces reductive results which sometimes cannot adequately explore the probands’ implicit theories. The other extreme is the free interview without any structured guidelines. This kind of knowledge elicitation is not reductive but the results are not easily comparable and the processes of interpretative analysis are too sophisticated for evaluation of teacher education.

Kelly himself developed a methodology for exploring systems of personal constructs by so-called repertory grids (1955). A repertory grid guides a form of highly structured interview, formalizing the interactions of interviewer and interviewee and putting into relations personal constructs and given objects of discourse (more details in the next section).

In a procedure described below, the person first defines the area that the test is to be applied to (“elements”), then he develops the items (“constructs”), then the test person completes the grid that is made up by the two dimensions, elements and constructs. Repertory grids try to combine the advantages of both extremes: having a structured way of data collection in order to simplify the analysis afterwards while no language in which the test persons are supposed to express their implicit theories and personal constructs is imposed.

The method of repertory grids has been applied increasingly in clinical psychology and psychodiagnostics concerned with self conceptions and social relations (cf. Fransella/Bannister 1977). In mathematics and science education research, there is an increasing number of studies using repertory grids, for example to explore teachers’ beliefs about educational principles and aims, teachers’ views of mathematics (Williams/ Pack 1997) and pupils’ beliefs on being good or poor in mathematics (Hoskonen 1999;

more studies are cited in his paper). The closest to ours is McQualter (1986) who analysed the *development* of mathematics teacher's role conceptions within a teacher education program. Non of the existing studies used our approach of starting with the mathematical tasks as an important tool to design learning environments, and all of them used different methods for data analysis.

DESIGN OF THE STUDY

We activated repertory grid methodology to evaluate the course "diagnostics of learning efficiency" for prospective mathematics teachers held by Regina Bruder in Summer 2002 in the department of mathematics education at Darmstadt University of Technology. The course focused on the design and the assessment of mathematical tasks for multi-faceted diagnostics of learning effects (for all details concerning the study see Bruder/Lengnink/Prediger 2003). We started out our investigation with the hypothesis that the course would not only increase the teacher students' explicit knowledge about mathematical tasks and their didactical functions but that it would also change and develop their implicit personal constructs about mathematical tasks. In order to evaluate this assumption, we have interviewed 16 students by means of repertory grids in the beginning and at the end of the course.

In a *first step* of these repertory grid sessions, the probands were supposed to become familiar with a set of mathematical tasks which we proposed as the elements of our study. All of these tasks are belonging to the algebraic treatment of linear equations but vary in nature and content. In a *second step*, the interviewer proposed three pairs of tasks. Then the interviewees were asked to find at least one pair of attributes which separates the tasks.

	comprehensive task	computational task	find a problem for a solution	find a solution for a problem	closed	open	formal/algorithmic task	"heuristic" task	comprehend a solution	find a solution	metacognitive task	isolated task	geometrical understanding	reality-orientation
T1		X			X		X					X		
T2	X		X		X									
T3	X					X		X						
T4					X				X					
T5	X		X		X			X						
T6		X		X	X					X				
T7						X					X			X

Fig. 1: First Repertory Grid of Test Person 1

In this step, personal constructs are specified which are relevant for the implicit theory of the probands. In a *third step* we organized the tasks and the attributes in a two-dimensional grid, the so-called repertory grid (see Fig. 1 for an example). The students were asked to add other attributes which they considered to be important for talking about mathematical tasks. Then they were supposed to fill the grid with crosses between all tasks and attributes which they considered to be related in their view.

The main idea of this approach is that the probands are free to talk about the mathematical tasks in their categories and their language. By doing so, we can investigate how teacher students think and speak about a main instrument of mathematics classrooms without a filter of prefixed possibilities of answers, i.e. we want to identify their personal constructs without imposing given constructs.

Notwithstanding this openness, the process of the construct elicitation is strongly structured: sequence of steps, the focus on separating attributes resulting from the comparison of two given tasks, the restriction to the relation “element has the attribute”, the demand to appraise all tasks with respect to their attributes. These restrictions serve as structural aids to make personal constructs explicit which the test persons are usually not conscious of.

DATA ANALYSIS

For analysing such repertory grids as ours, different methods of data analysis can be used. Often, complexity is reduced by using factor or cluster analysis. We decided to take a more cautious instrument of data analysis which has been used for repertory grids in psychoanalysis by Spangenberg/Wolff (1988) for the first time. It allows to visualize the structures of small grids in a line diagram without loss of information. Hence, the data can be represented without artefacts being produced by the analysis itself which is very important for psychodiagnostic contexts like ours.

We can read off the repertory grid of Fig. 1 from the line diagram in Fig. 2 in the following way: every task is assigned to a circle, and the related attributes can be found by following ascending lines to the attributes. For example, task 5 has the attributes “comprehensive task”, “closed”, “find a problem for a solution” and “heuristic task”. Dually, an attribute is assigned to all tasks which can be reached by descending lines.

The line diagram does not only represent the grid but it also makes its logical structure explicit. We can find logical dependencies between attributes: e.g. “find a problem for a given solution” implies “computational task” and “closed” since these attributes can be reached from “find a problem for a solution” via ascending lines. Even incompatibility become visible: In the context of the given grid, there is no task which is considered to be a “computational task” and to have “reality-orientation” at the same time.

The line diagram represents the landscape of the implicit theory of the test person and allows its systematic investigation. On this basis, the test person’s understanding of constructs like “open” and “closed” can be explored and clarified.

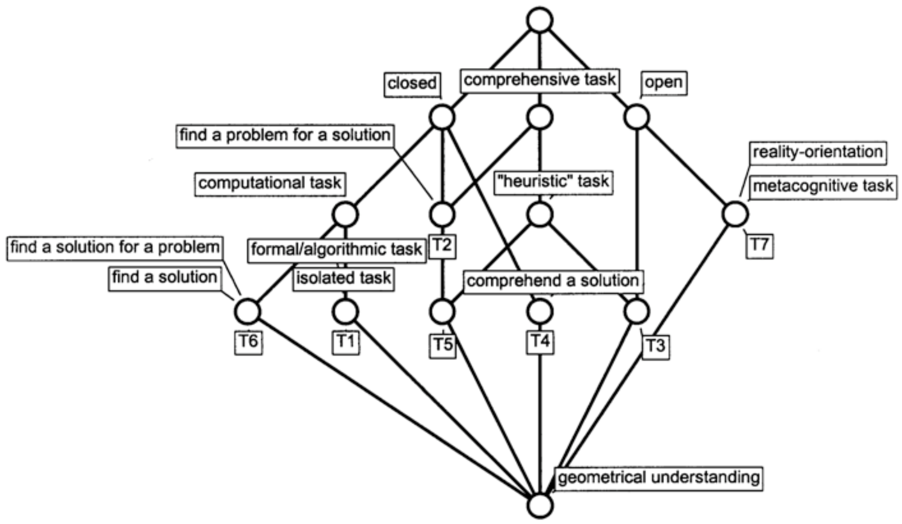

Fig. 2: Line Diagram for the first Repertory Grid of Test Person 1

RESEARCH QUESTIONS AND SELECTED RESULTS

In the evaluative study of our course, we were interested in comparisons between individuals' grids and in the development of their implicit theories during the course. Therefore, our analysis was guided by the following research questions:

- What attributes are taken by a student, how do they differ in the second interview and how do they differ from other students? Can we identify a development towards the didactical terminology offered in the course? How is it integrated into the individual system of personal constructs?
- On what categories do the different persons focus?
- How does the focus change from the first to the second investigation?
- Which degree of generality and which degree of differentiation do the given attributes have and how does it change?
- Can we specify typical patterns of development for the identified systems of personal constructs?

Due to restricted space, we can only give some hints on selected results concerning the last research question, namely the patterns of development. Our study has not only shown that the students strongly differ in their respective focus which is expressed through differing linguistic levels and categories about the tasks. Students also vary in their ways how they integrate the learned vocabulary and theories about mathematical tasks into their individual system of personal constructs.

One typical pattern of development can be seen by comparing the systems of personal constructs of Test Person 1 (see Fig. 2 and 3). His first diagram shows a clear focus on attributes concerning the categories "task-format" and "requested learning-activities". It is significant that this remains constant in the second evaluation, although the vocabulary is professionalized and the didactical terms have been integrated in the students' subjective theory.

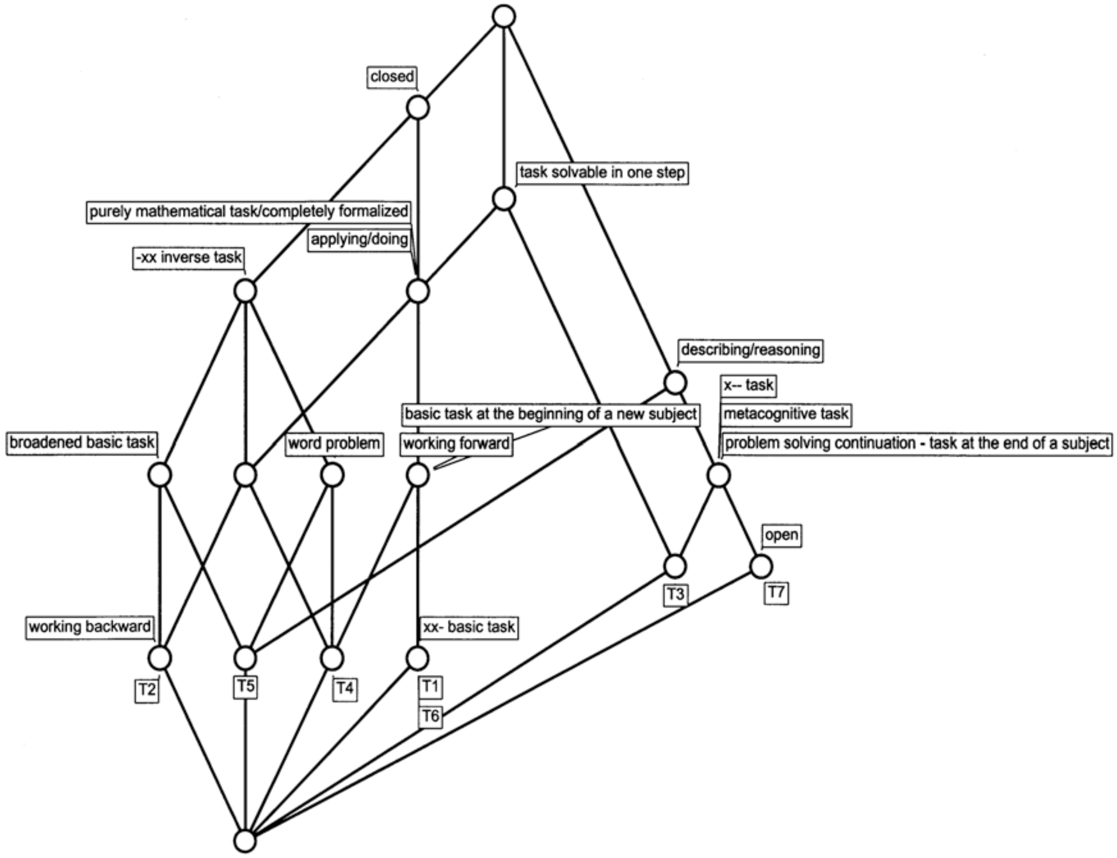

Fig 3: Line Diagram for the second Repertory Grid of Test Person 1

For illustration, consider the attributes “find a solution for a problem”, “find a problem for a solution” and “comprehend a solution” of the first diagram. In the second investigation those attributes have been replaced by a complete characterization of task-formats: In a triple consisting of (precondition, way of solution, solution) for every task it is indicated which part is given and which is searched for. This language was introduced in the course and has been integrated in this personal construct, probably because of its utility to describe the former intuitive distinction of tasks with respect to their format.

In contrast to Test Person 1 there is another type of student, who constructs an almost completely new theory about tasks during the course. As an example see the development of repertory-grid line diagrams constructed by Test Person 2 (Fig. 4 and 5). This person has dropped his former analysis of the tasks and adopted the new language used in the course. The attributes chosen for tasks in the second evaluation are certainly more general than the ones of the first evaluation. The diagram is smaller and the structure is simpler.

But even though the diagram of the second system of personal constructs is less complex than the one of the first evaluation, a learning effect can be demonstrated by the analysis. In the second evaluation, the test person focuses on three interesting categories of attributes for tasks, namely “the intention of tasks”, “the task-format” and “the requested learning activities”. Discussion of these diagrams could help this student to analyze his own learning level and to show the necessity of having a more differentiated language.

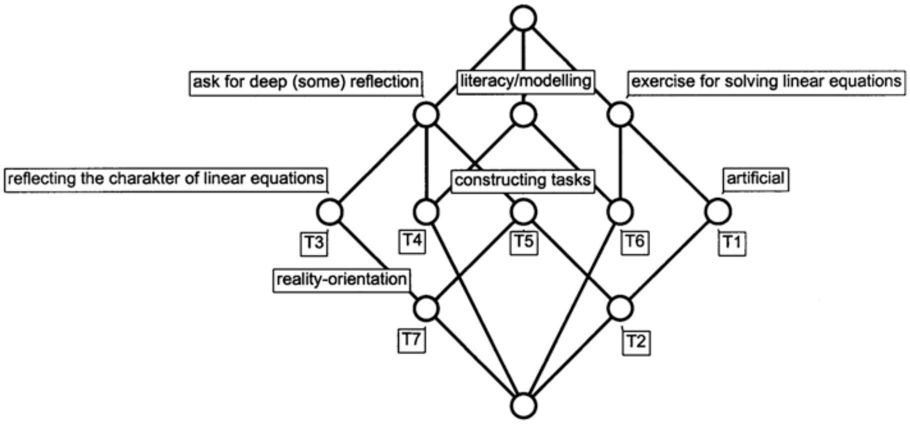

Fig 4: Line Diagram for the first Repertory Grid of Test Person 2

Besides these two types of learners, there is at least one other interesting type of student which enriches his language by some professional concepts, whereas the basic language remains almost constant. Although this categorization into three different patterns of development is idealized and some students may belong to more than one of the described types, it helped us to understand the different learning effects in didactical courses.

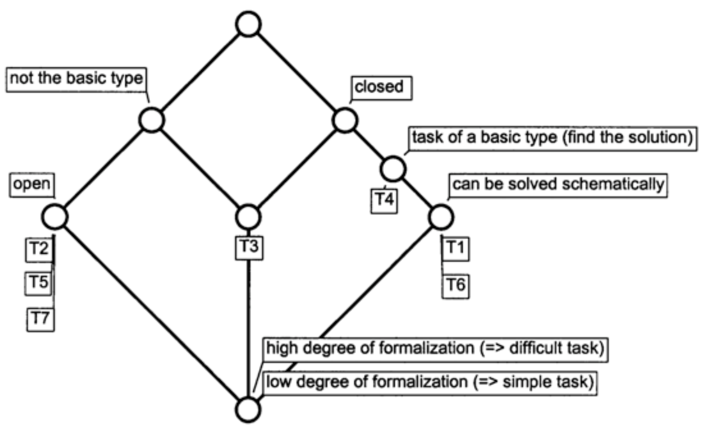

Fig 5: Line Diagram for the second Repertory Grid of Test Person 2

CONCLUSION

Repertory grid methodology, combined with formal concept analysis, has proved to be a promising tool to investigate the development of personal constructs of the teacher students. The line diagrams help to understand how teacher students think and talk about mathematical tasks in their own language. Whereas tests like classical evaluation tools usually assess the increase of explicit knowledge within the course, the comparison of repertory grids has given us the opportunity to explore the patterns of individual development on the implicit level. The gap between the official learning content and the individual way to integrate it into the system of personal constructs gives us a clue about reasons for the gap between theory (of ambitious teacher education) and the prospective practice in mathematics classrooms: Although the teacher students have learned their lessons of explicit knowledge quite well, they have only fragmentarily integrated it into their individual thinking. This is why we should pay more attention to the evolution of individuals' implicit theories.

For this aim, we have proposed a methodology which proved to be useful, but not without difficulties. According to our experience, the most critical step in this method is the specification of mathematical tasks which serve as elements for the grids. These choices heavily influence the resulting landscapes of constructs.

In our future research, the method will be complemented by direct discussion about the constructed repertory grids. This will clarify open questions of interpretation and, at the same time, it offers an opportunity for consulting the teacher students about their individual thinking and their perspectives of developing their personal constructs.

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LESSON STUDY CHARACTERIZED AS A MULTI-TIERED TEACHING EXPERIMENT[1]

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Japanese lesson study, in various adapted forms, is becoming increasingly significant in professional development of mathematics teachers in the USA. Our goal in the research reported in this paper was to investigate, in a three-tiered teaching experiment, the cycles of learning of two researchers, six teachers, and the students in three grade 4 classes and three grade 5 classes, in a six-month long lesson study project in the first half of 2002. The learning processes evolved in each of the three tiers (students, teachers, and researchers) over the course of three iterations of a lesson on measurement taught respectively in grades 4 and 5 by the three grade 4 and three grade 5 teachers. This paper documents some of these shifts in learning in each of the three tiers and assesses this form of lesson study for professional development, through the eyes of the teachers.

IMPROVING THE TEACHING OF MATHEMATICS THROUGH LESSON STUDY

Although it is recognized that adaptations are necessary in using traditional Japanese lesson study in a different country where the culture and values may not be congruent with those of Japanese society (Stigler & Hiebert, 1999), the assumption is sometimes made that this form of professional development will be universally beneficial. In our research we set out to investigate in a systematic way the processes that take place over the course of several iterations of the same lesson, with planning and debriefing sessions preceding and following each iteration, for the purpose of assessing what is learned by teachers in this form of professional development. As researchers, our own learning was a central element in the study. And for both teachers and researchers, the learning of the grade 4 and grade 5 students in the study was the reason for the project in the first place. Thus a multi-tiered teaching experiment (Lesh & Kelly, 2000), which takes account of the learning of students, teachers, and researchers, was an appropriate choice of methodology, as will be elaborated in the following sections.

Conceptual framework.

The conceptual framework of the research is drawn from theoretical and empirical fields (Brown & Dowling, 1998). In the theoretical domain, our literature base includes the books by Stigler and Hiebert (1999) and Liping Ma (1999), both of which were supplied to and studied by the team of teachers prior to the commencement of the lesson study. The research was also informed by the growing lesson study literature in the USA (Fernandez et al., 2001; Lewis, 2000; Murata & Takahashi, 2002). The conceptual framework embraced "six principles for gradual, measurable improvement through lesson study" (Stigler & Heibert, 1999):

1. Expect improvement to be continual, gradual, and incremental.

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2. Maintain a constant focus on student learning goals.
3. Focus on teaching, not teachers.
4. Make improvements in context.
5. Make improvements the work of teachers.
6. Build a system that can learn from its own experience (pp. 132-136).

Some of these principles will be re-visited in discussing the results of the study.

In the empirical field, a three-tiered teaching experiment (fig. 1) was structured as follows (adapted from Lesh & Kelly, 2000, p. 211).

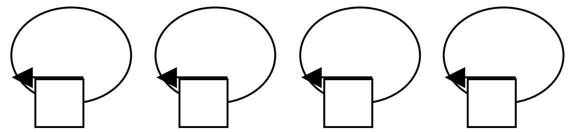
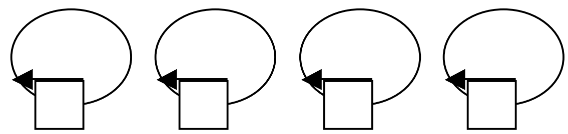
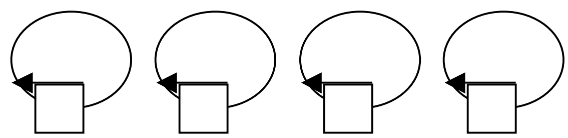
TIER 1: THE RESEARCHER LEVEL Researchers are learning progressively about the processes involved in lesson study, and the processes of teacher development and student learning.	 Several iterations of a lesson **Researcher-level teaching experiment**
TIER 2: THE TEACHER LEVEL Teachers are learning progressively through lesson study about the processes involved in the students' learning, and how they might facilitate this learning.	 Several iterations of a lesson **Teacher-level teaching experiment**
TIER 3: THE STUDENT LEVEL Students are learning progressively through working individually and in groups, on a rich mathematical task.	 One lesson iterated with different students. **Student-level teaching experiment**
DEVELOPMENT IS CYCLIC, AND THE FOCUS IS ON PROCESSES INVOLVED AT EACH LEVEL	

Figure 1: Lesson study as a three-tiered teaching experiment.

A three-tiered teaching experiment was a natural choice for the methodology of the research, because this form of research design involves teams of teachers and researchers working together, investigating a research question in the natural setting of the classroom, developing learning trajectories for the students with regard to the research question, meeting and reflecting on the outcomes of the experiment, and then repeating the whole process several times. The cyclic nature of the iterations of teaching a lesson on measurement, resonates also with the developmental research process described by Gravemeijer (1994), and the research is also consonant with the teacher development experiment of Simon (2000). Because space restrictions prohibit the reporting of details of learning in all three tiers, the specific research questions addressed in this paper are as follows.

1. What learning was reported by the teachers through participation in the lesson study?
2. Did the teachers judge this experience to have contributed to their professional development, and if so, in what ways?

Thus we are concentrating on the second tier in this paper. Because all three tiers are intimately connected, we shall also report on some of the learning that took place in the third tier, that of the students, and this whole paper details learning at tier 1 because it is the researchers who are reporting. But the focus is on tier 2, the learning of the teachers. These six teachers (three of grade 4 and three of grade 5) were chosen because they were known to one of the researchers, and because of their interest in participating in the lesson study.

Criteria of quality were addressed in the research by two forms of triangulation, namely that of multiple observers, and that of multiple data sources. The whole team of six teachers and two researchers reflected on the experiences of each iteration, and of participating in the lesson study. The results of the research are the negotiated interpretation by the whole team, of the data collected. There were six data sources, namely, notes taken during nine planning and debriefing meetings, audio recordings of some of these meetings, transcriptions of video recordings of lessons, lesson study grids drawn up by the grade 4 and grade 5 teachers in two teams, artifacts of students' work in lessons, and finally, field notes of presentations in which all but one of the team of six teachers and two researchers participated, at two conferences, one local and one regional.

The empirical setting (Brown & Dowling, 1998) and the choice of a lesson topic and problem, are elaborated in the next section.

Three iterations of a lesson on measurement.

Steps in the Japanese lesson study process are as follows (Stigler and Hiebert, 1999).

1. Defining the problem.
2. Planning the lesson.
3. Teaching the lesson (cycle 1).
4. Assessing the lesson and reflecting on its effect.
5. Revising the lesson.
6. Teaching the revised lesson (cycle 2).
7. Assessing and reflecting again.
8. Sharing the results (pp. 112-115).

While the team recognized that it might be necessary to adapt the process to US culture, these eight steps were all part of the study. The first seven steps were followed by a third cycle of revising, teaching and reflecting on the lesson, which was taught once by each of the six teachers (see Table 1). Presentations at a local conference on August 15, 2002, and at a regional conference on October 18, 2002, completed the eighth step.

Our program began with preparatory studies of two books (Ma, 1999; Stigler & Hiebert, 1999). After gaining familiarity with the lesson study approach, the group began, in February, meeting on a regular basis at the university for the purpose of selecting a topic and task and planning to teach a lesson by mid-April. In all, there were nine planning and debriefing meetings in addition to the three iterations of teaching the lesson.

Grade Level	First Iteration	Second Iteration	Third Iteration
Fourth Grade	April 18, 2002	May 8, 2002	June 12, 2002
Fifth Grade	April 17, 2002	May 8, 2002	June 12, 2002

Table 1: Dates of the teaching of the measurement lesson

At the meeting on March 6, all the participants had brought suggestions for possible mathematical content areas and specific topics suitable for a lesson that could be taught both in grade 4 and grade 5. Content areas that were suggested included geometry (perimeter, area, volume), fractions (equivalent fraction, multiplication of fractions), probability, and measurement of length. The researchers recognized that ownership by the teachers was important, and the group finally agreed on the following measurement task, set in the context of an imaginary radio competition, which seemed open and rich enough to facilitate students' learning of measurement in grades 4 and 5. The competitive nature of this formulation of the task may already be a departure from a Japanese cultural value of cooperation.

> "Walking in Sunshine" (Play a portion of this song, and fade out).
>
> Hey! Hey! Hey! You can be walking in sunshine when you enter out contest, "Steppin' to Cash Contest." 4th and 5th grade students, how would you like to walk into 500 dollars? You heard me right! 500 dollars! Well, you can if you enter the "Steppin' to Cash Contest." Here's all you have to do! Figure out how many footsteps it will take to walk from Normal to Peoria. If your estimate matches ours or is the closest, you could win 500 dollars! So come on, step up to win!
>
> "Walking in Sunshine" (and fade out).

The task lent itself to exploring issues of mathematical models for real situations, and allowed us to teach measurement within the context of problem solving.

Planning, preparing, and predicting.

By April 10, the planning had progressed to the following basic structure for the first iteration of the lesson, to be taught in grade 4 by Kelly and in grade 5 by Barry[2].

1. Announcement of "Radio Station Contest" by the teacher.
2. Whole-class discussion of ideas by students and teacher.
3. Work by individual students, each writing on a big yellow sheet of paper, deciding whether and how they wanted to take actual physical steps, to mark these out on the paper, to represent their thinking concerning the problem.
4. Work in groups of four students, again representing on a big white sheet of paper the results of the sharing of ideas and group activities.
5. Whole-class presentations and discussion of the results of small-group work.

Materials such as yardsticks and calculators would be made available. Students also had access to the information that there are 5,280 feet in a mile, and the distance from Normal to Peoria was taken to be 42 miles (or re-negotiated to be 40 miles in Barry's class).

A large part of the team preparation had involved negotiation of meanings of elements of the problem itself. What is a step? Is it different from a pace? How is it measured? It was

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[2] All names of teachers are pseudonyms.

foreseen that students might ask some of the questions they in fact did ask: Does it matter who does the walking? Does it matter if the walker is happy or sad? Does one have to walk in a straight line, and does this make a difference? Because the aspect of predicting student responses was known to be important in Japanese lesson study, the team worked out two grids, one for each of grades 4 and 5, consisting of four columns with the following headings: learning activity; expected student reaction; guidance/advice (to be provided by the teacher); and finally the actual reaction of the students (to be filled in after the teaching of the lesson). As an example, a small part of a grid for grade 5 is presented in table 2. A final grid was completed by the grade 5 teachers as a group after the third iteration, that is, after all three grade 5 teachers had taught the lesson, as a summary of "what happened" in all three iterations, taken in order. Thus the final grid shows (indirectly) the changes that took place between iterations as a result of reflections and debriefing by the whole group. Because of space restrictions, only two sections – introduction and small-group work - of Barry's lesson (that is, the first iteration) are presented in table 2. (Stages omitted are the "yellow sheet" work of individual students prior to the session in small groups, and the whole-class presentations and questions that followed the "white-sheet" work in groups.)

Issues that arose in the teachers' reflections were the role of questioning, the structuring influence of the tools that are provided (including the calculator and the yardstick), students learning through their mistakes, "allowing students to struggle with a process, rather than a focus on one correct answer or desired destination" (Barry, August 15). Some of these issues are discussed in the next section.

CONCLUSIONS FOR PROFESSIONAL DEVELOPMENT

The teachers made numerous reflective comments about the value of meeting with other teachers for the purpose of promoting student's knowledge and problem solving abilities. All six agreed that they had never had another educator in their classroom to offer them constructive ideas about helping children understand and reason through mathematics (comments from a meeting on June 12, 2002). Barry elaborated on this (August 15), "You had other colleagues there in the room with you. Usually that means they are there to watch me, and critique. But now, these others were watching what I was watching." All six teachers were encouraged that they could study the ways their students were learning within the immediate situations of their classrooms. This point illustrates Stigler and Hiebert's (1999) fourth principle for lesson study, "Make improvements in context."

In a related observation, the teachers shifted the way they participated in classroom observation. Beginning with the first round of the lesson in mid-April, the teachers who

Learning Activities	Expected Student R	**Guidance/Advice**	Actual Reaction
Introduction Pose question to entire class: “How many steps is it from Normal to Peoria?” Have class clarify things they need to know in order to solve problem What do you know? What do you need to know in order to solve this problem?	Wonder if it’s a real contest How many steps are in a mile? How many miles is it to Peoria? How big is a step?	Write question on the board. Write “40 miles” on the board We’re going to assume that’s it	*Barry’s Class:* How many steps are in a mile? What is the exact number of miles to Peoria?
Group work Students take yellow paper to the pre-assigned groups and are asked to develop a strategy to solve the problem Instructed to ask questions, share information… Materials: large pieces of white paper and a yardstick At some point during the process, teacher may want to reconvene the class to share questions that are being asked (not strategies)	Ask whose steps to measure What is a step? Some students will measure feet, rather than steps Expect students to watch other groups Actually take steps and begin to measure Some computation Begin talking about an “average” step	Redirect the original question Would _________ make a difference? What do you think a step is and why7? Show me how you’re going to walk to Peoria We want to see a visual representation, or a drawing Is that what your picture represents? Watch for inconsistencies in what they’re physically doing and how they’re representing it	How many yards are in a mile? I have a math book. What do we need to look up? How many feet in a mile? We’re going to estimate the steps in a mile Groups began taking steps Students performing calculations on their yellow papers Students counting steps on yardstick Resources being used or requested: (textbook, rulers, calculators, floor tiles) Began drawing on big sheets of paper Discussed measuring toe to toe, heel to heel and toe to heel Students decide to find an average step for their group Students jumped to simple calculations in an effort to solve quickly “It’d be easier to walk to Peoria than to go through this.”

Table 2: Part of Barry’s grade 5 lesson study grid.

were not presenting were observing students by moving rapidly from group to group, much as they might have done were they responsible for the lesson themselves. In contrast, one of the researchers focused her observations on only one of the groups of students in the classroom throughout the entire lesson. The teachers noticed this contrast in approaches, and we discussed the different purposes for each at the meeting on May 1. As a group, they decided to use this "teaching experiment" style for questioning and following one or two students closely through the duration of a lesson. After meeting on May 8 (after the second iteration), they commented on how much more they could learn about the lesson as a dynamic process by watching a specific group progress through the entire process, and chose to use the technique again on June 12 (third iteration). Barry reflected as follows,

> Instead of me figuring out all the students, we each watched a 'pocket' of that class. Here is what I saw, this is what the other teachers helped me do: I shifted from accomplishing a particular goal. I moved instead to look at what the kids are thinking and how I could help them grow. The different environment [of the lesson study approach] shifted my focus.

He attributed the growing ability to see what children need to grow in their mathematics to this particular research environment. This kind of observation can form a critical part of teachers' classroom practice, supporting and extending an "informal assessment" component of their pedagogy.

One barrier the team had to overcome was the difficulty of changing from a typical emphasis on classroom routines, and on the sequencing of student exercises into the substantive issues for lesson study. We came very slowly to this latter emphasis. It took a long time and much effort to ask new questions: how do children think about a mathematical idea, how does that idea fit in the curriculum, and what kind of strategies do children use, or need to use to investigate that mathematical idea? The six teachers in our group were initially focused on crafting a lesson together. But our group progressed quite slowly into the substantive work of anticipating students' reasoning and strategies related to the mathematical concepts. Resonating with Stigler and Hiebert's second and third principles, "Maintain a constant focus on student learning goals," and "Focus on teaching, not teachers," lesson study only succeeds where teachers genuinely shift to assessment of the students' thinking within a classroom where a lesson is being taught without so much attention to the words and actions of the teacher. In a collaborative teaching experiment such as this, the lesson comes to be seen as belonging to the entire group, not to any one individual teacher: critique is then not of an individual, but an attempt to improve the lesson that then belongs to all.

FINAL WORD: WHERE ARE WE GOING?

Barry voiced it well (notes, August 15 presentation):

> A practical area that arose was that of how this process and these changes in lesson preparation and presentation impact classroom management, especially in the areas of timing and assessment. As teachers we want to work towards a point where we are less focused on "neatly wrapping up the lesson" in the allotted time, and more focused on the process and what the students are learning through that process of mistakes, conversation, questioning, self-evaluation, etc. We also want to work towards a point where we can find ways of assessing this process and find ways of making that assessment work within the boundaries and confines of our current evaluation system.

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CALCULATORS, GRAPHS, GESTURES AND THE PRODUCTION OF MEANING

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In this paper we report an analysis of a teaching sequence in which Grade 11 students were asked to produce some graphs corresponding to the relationship between time and distance of a cylinder moving up and down an inclined plane. The students were also asked to carry out the experience using a TI 83+ graphic calculator equipped with a sensor, and to discuss and explain the differences between their own graphs and the ones obtained with the calculator. We analyze the students' processes of meaning production in terms of the way diverse semiotic resources such as gestures, graphs, words and artifacts become interwoven during the mathematical activity. Our findings suggest that a complex relationship between gestures and words allow the students to make sense of the time-space graphic expressions.

INTRODUCTION AND THEORETICAL FRAMEWORK

In an artifact-mediated classroom activity, Grade 11 students were asked to investigate and graphically express the relationship between time and distance of a cylinder moving up and down an inclined plane. Strictly speaking, the temporal-spatial relationship of the cylinder's motion cannot be seen, even if an experiment is materially carried out. Indeed, in such a case, crude perception merely allows one to see the cylinder going up and down the inclined plane. Although motions of this kind were only systematically mathematized in the early 17th century, and since then there may be a certain intuition that "traveled distance" and "consumed time" –to use Galileo's words– bear a certain mathematical relationship, the graphic expression of such a relationship (which was not conceived until many years after Galileo's work) is certainly much less intuitive. In fact, the temporal-spatial mathematical relationship of a body's motion is an abstract, conceptual and cultural entity. To render this relationship apparent in the classroom requires a fine understanding of space, time and movement. In particular, the graphical account of motion may require students having recourse to diverse semiotic resources, such as gestures, words, drawings, coordinate systems, artifacts, etc.

Recent research has shown the cognitive import of gestures, words, and artifacts in the production of graphical as well as algebraic symbolic expressions (Arzarello and Robutti 2001, Roth and Lawless 2002, Robutti and Ferrara 2002, Radford 2002, 2003). The reported research, as well as other research carried out in other scientific fields like linguistics and psychology, indicates that, in the students' talking and gesturing activity, words and gestures play a substantial role, even if the specific role of words and gestures may vary according to the adopted theoretical perspective. For instance, in the early 1980s Kendon contended that gestures express underlying cognitive representations as words supposedly do (Kendon 1981, p. 38). Following this line of thought, McNeill suggested that gestures and speech share the same psychogenetic source (McNeill 1985, see also Crowder 1996). In a more social, interpersonal perspective, gestures and words can be seen as semiotic means that students use to objectify knowledge (i.e. to make things and relations apparent in their universe of discourse). It is within the latter perspective that the analysis of the students' activity will be conducted in this paper. Considering gestures as (a loose type of) signs, our intention is to investigate how gestures, words, and artifactual actions are mobilized by the students in order to objectify and endow with meaning the emerging mathematical content (i.e. the referent) of the sign-graph expressing the conceptual mathematical spatial-temporal relationship of the cylinder motion. More specifically, our goal is to investigate what we want to call "semiotic nodes", that is, pieces of the students' semiotic activity where action, gesture, and word work together to achieve knowledge objectification.

METHODOLOGY

Data Collection: The data presented here come from an ongoing longitudinal classroom-based research program where teaching sequences are elaborated with the teachers. As the research program unfolds, theory, data, and hypothesis are cyclically generated. Usually, in these sequences the students work together in small groups of 3; then the teacher conducts a general discussion allowing the students to expose, confront and discuss their different solutions. In addition to collecting written material, tests and activity sheets, we have three video-cameras filming three groups of students. Subsequently, transcriptions of the video-tapes are produced. These transcriptions allow us to identify salient short passages that are then analyzed in terms of interaction and students' use of semiotic resources. In this paper, we will focus on one of the small groups.

The Teaching Sequence: A two-day mathematical activity based on a hands-on investigation of motion along an inclined plane included different tasks and questions. The instructional design rested on the premise that the mathematical investigation of spatial and temporal relationships in motion problems supposes the cognitive capability of conceptualizing motion from different mathematical reference systems. Bearing this in mind, we will discuss only 3 questions here.

In Question 1, the teacher propelled a cylinder (called cylinder A) upwards, from the bottom of the inclined plane. The students saw the cylinder go up and come down. The students could repeat the experiment as many times as they wished. The teacher then provided the students with an activity sheet and asked them to produce a graph (called graph A) representing the relationship between time and space of cylinder A's motion. The students were given no information concerning the initial point (or point zero) from where the distance should be (qualitatively) measured. We expected the students to locate the point zero on the bottom part of the plane, that is, the point where the cylinder was put in motion (a point that coincides with the body's position). The teacher also asked the students to sketch a second graph (Graph B) in the same coordinate system where graph A was drawn for a hypothetical cylinder (cylinder B) put in motion on an identical inclined plane one second after cylinder A started moving (in other words, cylinder A starts at $t_s = 0$ and cylinder B starts at $t_s = 1$). (Cylinder B's motion was hence a "thought experiment").

In Question 2, the students were asked to perform two experiments (motion starting at t=0 and motion starting at t=1 sec) using a TI 83+ calculator and the Calculator Based Ranger (CBR –the motion detector). In the calculator-based experiments, they were instructed to place the CBR at the top of the inclined plane. The students, who had previous basic experience with the graph calculator and motion sensor detector, had to compare their graphs A and B to the ones they obtained with the calculator.

Finally, in Question 3, the students were asked to study the graph shown in the right corner of Figure 1. The graph was accompanied by the following instruction: "A group of students drew the following curve to represent the relationship between time and space when a cylinder is propelled upwards on an inclined plane. This group placed the distance origin around the center of the inclined plane. Is this curve correct? Explain in detail your answer."

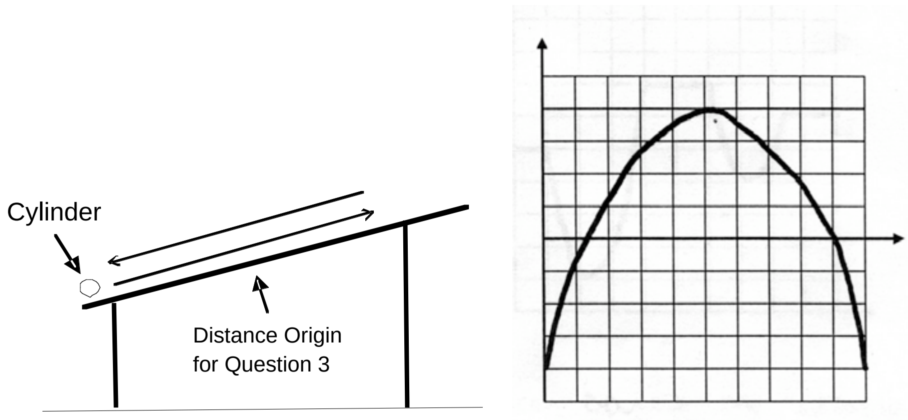

Figure 1. Left: Inclined Plane or Table showing distance origin for Question 3. Right: Accompanying Graph for Question 3.

RESULTS AND DISCUSSION

Question 1: As expected, the students produced graphs starting at distance D=0 (see Figure 2). Key words with which the students gave meaning to the graphs here were the "initial point" (which was equated to point zero of traveled space), "going up", "maximum point", "going back".

> Judith: The initial point is at zero and goes up to the maximum point, then, then …
>
> Vanessa (interrupting): [it] continues to fall to point zero
>
> Judith: (adding) while time runs out.

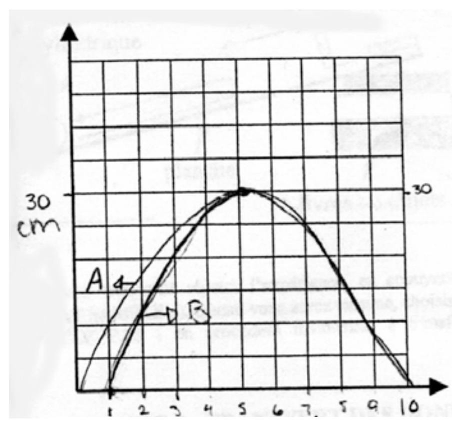

Figure 2. Graphs A and B

Question 2: As mentioned previously, in this question the students were asked to put the CBR at the top of the inclined plane. Let us focus here on the discussion concerning the students' comparison between their delayed motion graph (Graph B in Fig. 2) and the calculator's graph (called Graph C and shown in Figure 3).

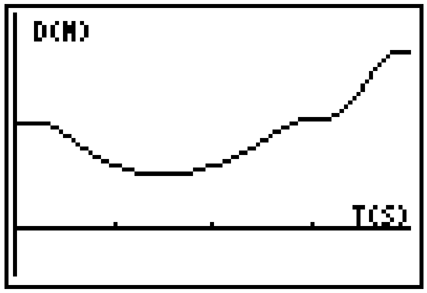

Figure 3. Calculator's Graph C

The students noted several differences, among them the following: (1) Graph C was not perfectly curved in the part after its minimum value, (2) contrary to Graph B, in Graph C the value of the variable D (distance) in the ending points is not the same (i.e.$D_f > D_o$), and (3) Graph B starts at D = 0 and its shape is different from Graph C.

Difference (1): This difference was explained by a slight turn of the cylinder when it was rolling upwards on the inclined plane.

Difference (2): This difference was more difficult to understand. After discussing different ideas:

> Judith: … (looking at the inclined plane) This thing there [the cylinder], does it go further? (the other two girls turn to see the inclined plane which was behind the students' desks) … like this … (she makes a gesture with her right arm; the gesture starts with her arm extended in front of her body and moves back, miming the cylinder motion in its coming back down trajectory) does it measure the …? Oh!
>
> Vanessa: What?
>
> Judith: You started on the table [i.e. the table that served as the inclined plane for the experiment], right? (Vanessa□Yes) And when it was rolling it fell off the table (with a similar gesture her arm is bent again and goes beyond her desk, as the falling cylinder did during the final part of its motion when it fell off the inclined plane and was caught by the student)… I don't know…
>
> Vanessa: It has nothing to do with that.
>
> Judith: It does have something to do with that […] That's the curve, right? Here (she points to the horizontal segment of the left part of Graph C on the calculator screen) suppose this is when you started on the table and when you finished (she points now to the horizontal segment of the right part of Graph C), you've finished further, that's further. […] Let's say that your distance here would be 30, and 45, that's the error! […] Now why it started there (initial point of Graph C) … I don't have any clue…

In Lines 1 and 3 Judith makes an "iconic gesture", that is, a gesture that bears a resemblance with its referent. The iconic sign-gesture enacts the falling trajectory of the cylinder. It allows Judith to call her group mates' attention to a specific part of the phenomenon. The iconic gesture affords a segmentation of the phenomenon and operates a choice of what has to be taken into account. Thus, the iconic gesture does not stress speed, time, accurate distance and other elements. What it stresses is the fact that the cylinder went off the table. However, the students mobilized more semiotic resources than gestures. There is, in fact, a coordination of gesture, gaze, and words. Along with gestures, Judith uses locative words and time-related expressions to achieve a coordination of time, space, and movement. This is an example of semiotic node (see Figure 4).

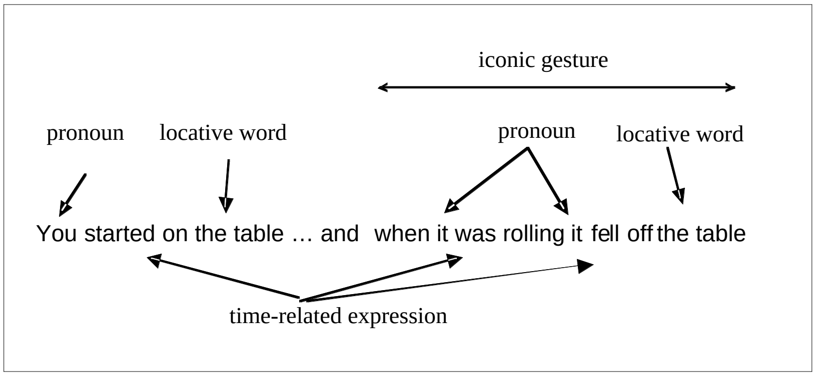

Figure 4 . Example of a "semiotic node" where word and gesture achieve coordination of time, space, and movement.

In Line 5, Judith has recourse to an “indexical gesture”: pointing with her finger, she indicates two parts of the calculator graph on the screen. In this case, numbers (30 cm and 45 cm) come to play the role of the iconic gesture that has previously shown the cylinder falling off the table. The first number represents the students’ estimated distance from the cylinder’s maximum point to the bottom of the table. However, the cylinder never went 15 cm off the table (i.e. 45-30), for it was caught in the air as it fell off. Numbers are not accurate, and the students do not worry –accuracy is not at stake.

Difference (3): As the previous excerpt intimates, Judith was able to provide an interpretation for Difference (2), i.e. why Graph C starts and ends at different values of the variable D. Nevertheless, the students’ understanding of the relationship between time and distance was still vague. The reference point for the distance remained ambiguous. What the students understood was that the cylinder traveled more distance (absolute distance) in its falling back trajectory than its moving up one. The students kept discussing without success why Graph C does not start at D=0. When the teacher came to see their work, he did not provide an answer. His presence, however, catalyzed the students’ ideas, which at the end he reformulated using a metaphor –the “eyes metaphor”:

> Carla: (talking to the teacher and pointing to the initial point of Graph C) We don’t understand why it didn’t start at zero […]
> Vanessa: It’s because it started the other way around, right? […]
> Teacher: Ah! I don’t understand […] we have always rolled the cylinder from the bottom to the top… (he makes a gesture as if he is rolling up the cylinder)
> Carla: (talking at the same time as the teacher) Is it because you’re further from the thing [i.e. the CBR]?
> Judith: (Understanding Carla while the teacher is still talking to Vanessa) That’s true … Ah! Yeah! I get it! It is like we watched the cylinder leave and arrive like this (she puts her hands on the bottom of the desk) when it was at the bottom of the table … but now (she makes a complex gesture: with her left hand placed far from her she signifies the position of the CBR

and with her right arm extended and then bending it she mimes the movement of the cylinder coming back to the bottom of the table) … it's the thing [i.e. the CBR] that is at the top!

Vanessa: (Understanding the other girls) Ah! Well we weren't looking from the point of view of the thing, it's because of that! O.K.

Teacher: O.K. Well there, the point of view … your eyes (he points to his eyes) … it's the CBR. For one of the graphs [Graph B] your eyes were at the bottom [of the inclined plane] (he puts his right hand in front of him and close to his body to signify closeness) and for one of the graphs [Graph C] your eyes were on … (he makes a gesture putting his hand in front of him and far from his body to signify the top of the inclined plane) […]

Judith: (Understanding) O.K. It's the same thing as that but from a different point of view.

We consider the gesture-word systems of Line 5 and Line 7 as two supplementary examples of semiotic nodes. In each case, indeed, a new kind of awareness is made apparent. In Line 5, the semiotic node serves to make sense of the fact that D_f is greater than D_o in Graph C. Epistemologically speaking, this semiotic node has a sense-making constructive dimension. In Line 7, the semiotic node brings to a higher degree of awareness the importance of the position of the spatial origin. It provides the students with a way to better interpret graph motions and to understand what has experimentally happened. Let us now turn to Question 3.

Question 3:

The students remarked that, in the graph, some values of the distance axis "D" are negative. They argued that negative distances are impossible.

Judith: No because your distance can't become negative […] It moves away from you or it comes close to you but (inaudible).
Carla: Well on our graph it does both.
Judith: It is because it doesn't go beyond the point? (the word «point» is accompanied by a gesture of both hands indicating an imaginary point in front of the body) Let's say that this is zero, zero is here (she turns her body to the right and places her right hand at the bottom of the right part of her desk to indicate the zero point; there is a coordination of the gesture and the deictic word "here". She is imagining a distance axis having an origin at the bottom of her desk, where her hand is) and it doesn't go negative because it doesn't go beyond (she moves her left hand from the top to the bottom of her desk and her left hand goes beyond her right hand that is still signifying the origin. She is implying that, in this reference system, points to the right of this zero point –i.e. points falling beyond the desk– are negative).
Vanessa: I don't know if it is because of that, but what you say makes sense.
Carla: (Carla is not convinced. She interprets the bottom edge of Judith's desk as the horizontal axis of time in their graphs. She says:) Yeah, but those are the seconds (after a relatively long pause of approximately 2.5 seconds she waves her hand and draws in the air a concave graphic similar to Graph C while saying:) On the graph it goes like this … that's the seconds (she gestures a horizontal line) … it goes up and comes back down (she makes again a concave graphic similar to Graph C), the distance ... (she makes a vague gesture in the air that tries to locate a position for the distance; she falls silent for a relatively long pause of approximately 3 seconds while she and the other girls think)
Judith: Like your distance starts at zero (zero is again emphasized using a gesture that indicates a point on the desk close to her. Of course, this assertion is true if the position of the CBR coincides with the body's position). […]

Carla: Like the first [Graph A, i.e. the case where the distance is measured from the bottom of the inclined plane]… the closer it goes to the CBR it will be negative because we started here (indexical gesture pointing to the actual bottom of the inclined plane) … [this] started at the top (the word 'top' refers now to the initial point of the concave graph C that she reproduces here with a right hand gesture), the lower we were on the x axis … or whatever … the lower we were, the closer we were to the CBR. If we had gone beyond 0.5 m (i.e. the approximate maximum distance that the cylinder could travel from its maximal position on the inclined plane to the bottom of the table) it would be negative.

Judith: Exactly, it doesn't go beyond the point.

This excerpt stresses the students' difficulties in conceptualizing the difference between the spatial origin of the cylinder motion and the mathematical spatial origin. While the first one was perceptually seen, the second, in contrast, requires the students taking into account a theoretical perspective. As Line 1 makes plain, body position provides a powerful perspective ("it moves away from you or it comes close to you"). But this perspective has to be shifted in order to make apparent (or objectify) the phenomenon from other perspectives. Despite the success of the "eyes metaphor" in the previous question, the students could not elaborate a conceptual idea for the point zero distance. In Line 3, Judith mentions the word "point" and accompanies it with an indexical gesture. The concrete point on the bottom of the desk becomes the origin. "Zero is here", she says, and keeps her right hand there. Her left arm (initially extended) starts traveling –like the cylinder– from a far position towards the bottom of the desk. And while she is saying that "it doesn't go negative because it doesn't go beyond" her left hand does go beyond the supposed point zero. Here the complex system of iconic and indexical gestures contradicts what is uttered. In a sense, Judith is providing us with the enactment of a gestural-and-word-proof by contradiction. And Vanessa finds it meaningful (Line 4). In Line 5, Carla, talking to herself as much as to the other girls, makes an iconic gesture. This time the content of the iconic gesture is not the motion of the traveling cylinder but the calculator-produced graph. Carla's iconic gesture hence has a different referent from Judith's in Line 3. However, in referring to Graph C, the CBR (i.e. the distance origin) should be located at the top of the inclined plane. In the following line (Line 6), Judith says that the distance starts at the bottom. We see then the students talking about two different origins. The misunderstanding is not clarified. On the contrary, in Line 9, Carla refers to Graph A, switching thereby the origin albeit seemingly without being aware of it. This confusion allows her to interpret Judith's argument and, in the end, consensus is wrongly reached.

Line 3 exhibits another example of semiotic node. Line 5 does not. In the latter, the gesture-word system has a heuristic role but it does not produce any novelty in terms of knowledge objectification or meaning production.

CONCLUDING REMARKS

The analysis of the students' semiotic activity carried out in this paper sheds further light on the students' conceptual strategies in understanding motion problems. In our analysis, we paid particular attention to the word and gesture system. Our theoretical construct of semiotic node allowed us to locate specific points in the students' semiotic activity where gestures and words achieve a coordination of time, space, and movement leading to the social objectification of abstract mathematical spatial-temporal relationships. The fact that the detected semiotic nodes were strongly oriented to the objectification of the mathematical space origin and the actual motion of the cylinder may explain, to some extent, the students' failure in securing a good mathematical understanding of the problem at hand. Indeed, in these semiotic nodes, time was rarely mathematized. In the students' discussions, time appeared mostly as marking the starting and ending points of the cylinder motion or else it was considered in a very rough qualitative way (as in Line 3 of the students' dialogue related to Question 1 or as in the first example of semiotic node; see Fig. 4). It is true that, in Figure 2, the beginning of Graph B correctly shows the characteristic type of delayed motions, but Graph B ends at the same time as the non-delayed motion Graph A! It may be true, as Koyré (1973) remarks, that it is more difficult to think in terms of time than in terms of space. A suggestion for teaching would be to encourage students to pay due attention to the time variable and to incorporate it in a more sustained way in the analysis of spatial-time relationships.

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THE EFFECTS OF NUMERICAL AND FIGURAL CUES ON THE INDUCTION PROCESSES OF PRESERVICE ELEMENTARY TEACHERS

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In this report, we address the following questions: What aspects of information do preservice elementary teachers rely on when performing inductive reasoning? What contexts enable them to perceive the inherent invariant relationships from a finite sample and, thus, formulate viable generalizations? To what extent are they able to justify inductive results noninductively? Our responses are based mostly on inductors' ability to perceive similarity between compared entities that they compute over numerical and figural cues. We tested a model of similarity and examined predictions of the model in the case of two problem tasks. In the model, we assume that both numerical and figural cues contribute to similarity leading to induction. An analysis of the induction processes of the 42 participants suggests that even if relationships among numerical values have had greater contribution to similarity than did figural ones, those who induced figurally acquired a better understanding of the generalizations they constructed.

PURPOSE

The question concerning how concepts are formed is central in almost all work in cognition and learning. Since inductive reasoning plays a significant role in the study of patterns, including object categorization and classification, there has been, and still is, a need to systematically explore processes that enable successful induction. Inductive reasoning, or generalizing knowledge from a finite sample of particular instances, is a common activity in school mathematics. All students need to acquire proficiency in performing inductive reasoning because its predictive function helps reduce the amount of time and energy that would be needed if all cases were to be investigated one at a time. Further, induction promotes generalization and abstraction, two key processes that are necessary and highly valued in mathematical discourse (Schoenfeld & Arcavi, 1988; Skemp, 1971).

In this research report, we deal with the following issues: What aspects of information do preservice elementary teachers rely on when performing inductive reasoning? What contexts enable them to perceive the inherent invariant relationships from a variety of particular instances and, thus, formulate (viable) symbol-based generalizations? Further, to what extent are they able to justify inductive results noninductively, that is, by other means of explanation that have not been drawn by force of surface appearances or by mere arbitrary speculation? Our responses are based mostly on inductors' ability to perceive similarity between compared entities that they compute over numerical and figural (e.g. by way of illustrations) cues. For example, when prospective teachers look for a pattern in order to describe in symbols the number of regions formed by connecting points on a circle, some might compare from the numbers or values they generate from specific cases rather than establish them geometrically through relationships that could be drawn from the figures.

In this work, we developed and tested a model of similarity and examined predictions of the model in the case of problem tasks in which inductive relations were, in the final stage, expressed algebraically. In the model, we assume that both numerical and figural cues contribute to similarity leading to induction.

CONCEPTUAL FRAMEWORK: SIMILARITY AND INDUCTION

Recent psychological research has focused on understanding mechanisms that underlie children's use of similarity in induction processes because they relate to tasks that involve, at the very least, classification and categorization of objects, the formulation of common properties, and the establishment of word meanings (Davidson & Gelman, 1990). The underlying principle behind similarity is that it is not static: it changes based on the context in which it is used (e.g., the type of task influences the judgment made about similarity (Hahn & Ramscar, 2001)). Furthermore, children and adults' ability to engage in similarity continues to evolve as a consequence of intellectual maturation, personal experiences, and "increases" in "domain knowledge" (Gentner & Rattermann, 1991, p. 226).

While much work has focused on children and adults' symbolic abilities to use induction in natural cases (such as breathing practices of different kinds of fish), little attention has been paid on how they would perform induction within the context of mathematical tasks which are governed mostly by symbol systems that are artifactual in nature (i.e., human-made with specific rules of engagement). The conceptual framework that we have developed in this study is used primarily to ascertain the extent to which insights on similarity and inductive reasoning in natural kind categories also apply to mathematical situations that involve induction.

Gelman (1988) shares Simon's (1981) claim that "natural kinds are susceptible to scientific study, whereas artifacts have traditionally not been. … (I)t is possible to study artifacts scientifically …. However, the scientific study of artifacts differs fundamentally from the scientific study of natural kinds" (p. 69). Gelman highlights the significance of "properties that emerge from the interaction between an object and its environment" (ibid) which may affect an individual's ability to induce. Studies have yet to determine if "expertise" in the use of artifact obeys the same structure as inducing from natural kinds. Gelman astutely points out that while natural kinds are "constrained by their genetic or molecular structure" (p. 70), artifacts are characterized by "functions" that change or could be modified. What this means in the context of this study deals with the possibility that inducing constructs in mathematics (treated as artifacts) may not be the same as inducing constructs in natural settings. Attempts to induce attributes in mathematics may likely be constrained by domain-specific factors such as artifactual relationships borne of the technical language and symbols that enable the construction and existence of the attributes which may not be perceived by an individual to be of the natural kind. That is, attributes that are not found in nature and, hence, possess a structure that is strange to common sense and intuition and may require more elaboration (Tall, 1986) and reflective abstraction (in Piaget's sense).

Gentner and Rattermann (1991) distinguish between relational similarity (analogy) and object-based similarity (mere appearances) resulting from studies which claim that individuals perform similarity based on either the attributes observed in objects or the relational attributes that structure the objects. In one of Gentner's (1988) works, he demonstrates the "relational shift" phenomenon that takes place among children: young children tend to perform similarity on objects while older children and adults tend to perform similarity on relations. There is as well a developmental view which makes a distinction between lower- and higher-order "relationship commonalities" (Gentner & Rattermann, 1991, p. 228), whereby the acquisition of formal operations (in Piaget's sense) marks the shift from the lower to the higher order (Inhelder & Piaget, 1964).

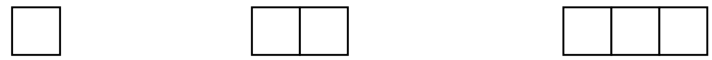
Markman (1989) and Gelman (1988) both advance the notion that "homogeneity" is a property that can very well explain (successful) categorization of objects. Homogeneity pertains to traits shared by category members (such as the striped character of all zebras or family resemblances from within classes of objects). If categories are hierarchically organized based on complexity and specificity, children are found to be capable of inducing "basic-level" homogenous categories first (Rosch, Mervis, Gray, Johnson, & Boyes-Braem, 1976). Then they learn superordinate categories, albeit with relative difficulty (for e.g., "chair" is basic while "furniture" is superordinate (Markman & Callanan, 1984); see also Skemp, 1971, pp. 19-34). Basic level objects in a category are "overdetermined" in the sense that they share common features or common parts or common functions (Markman, 1989). Superordinate objects in a category possess few common properties and "are more inclusive, with greater perceptual dissimilarity" among objects (Markman, 1989, p. 73), making them difficult for younger children to discover. Comparing the manner in which younger and older children categorize, Horton and Markman (1980) point out that facility and efficiency in language use are likely to affect the way basic and superordinate categories are acquired. Further, Callanan (1985) claims that parents tend to use basic level terms with young children, while Gelman (1988) finds that adults who induce depend on factors other than homogeneity. This idea of homogeneity is significant in the construction of mathematical tasks that involve induction because it provides a structure for classifying tasks based on the level of homogeneity, including the kinds of tasks that are explored by students in actual instruction.

METHOD (PARTICIPANTS, DESIGN, AND PROCEDURE)

Participants in this study included 42 undergraduates (34 women, 8 men) who took the test for extra credit. They were enrolled in an introductory course for elementary mathematics teachers in a public university in northern California. Their ages ranged from 19 to 55, with a mean age of 23.42. Racial profile is as follows: 15 Caucasian Americans, 4 African Americans, 11 Asians and Asian Americans, and 12 Hispanic Americans.

Four induction tasks were prepared with each task consisting of three figures accompanied by three numerical values. Each triad was constructed so that the second and third figures and numerical values were related to the first and second figures and numerical values, respectively. Each task required all participants to either draw or compute values for two additional cases before they were asked to obtain a generalization of the task. Due to limitations in time and space, we discuss results obtained from the two tasks given in Table 1 below.

1. Consider the problem below.



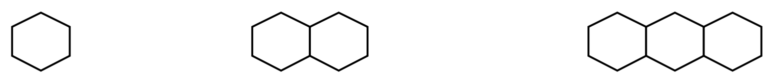
Number of squares	1	2	3
Number of matchsticks	4	7	10

How many matchsticks are needed to form 4 squares?

How many matchsticks are needed to form 5 squares?

How many matchsticks are needed to form n squares?

2. In the figures below, one hexagon takes 6 toothpicks to build, two hexagons take 11 toothpicks to build, and 3 hexagons take 16 toothpicks to build.



Number of hexagons	1	2	3
Number of toothpicks	6	11	16

How many toothpicks are needed to form 4 hexagons?

How many toothpicks are needed to form 5 hexagons?

How many toothpicks are needed to form n hexagons?

Table 1: Two Induction Tasks

Each participant was tested individually by the presenting author. Each interview lasted between 25 and 45 minutes and was audiotaped. Each participant was asked to read the problem and to simultaneously think aloud and write down what they were thinking.

RESULTS AND DISCUSSION

The primary task that this study was aiming to accomplish is to articulate structures of prospective teachers' induction processes in the case of mathematical problems whereby the inductive results are all expressed in variable form. In particular, when aspects of information have been provided, we wanted to find out if they induced from the figures (i.e., figural similarity) or from the values (i.e., numerical similarity). In the case of successful inductors, we were interested in determining the contexts that enabled them to see through features that remain unchanged and how they employed induction to capture the invariance. These contexts, we contend, would play a significant role in the manner in which generalizations are justified inductively and noninductively as well. Tables 2 and 3 present the data according to success level and the type of similarity employed. At the outset, the data confirmed our claim that the participants induced numerically rather than figurally, with a mean of 64.5% (counting correct and incorrect responses). Both quantitative and qualitative data (i.e., interview transcripts) also confirmed an assumption we made in our model of similarity, that is, prospective teachers performed similarity from among the already known and computed numerical values and paid little attention to the figures in which the numbers were actually derived.

	Figural	**Numerical**
Correct Response		
3n + 1	5	5
4 + (n – 1)3	1	
Partially Correct Response		
4 + 3n	4	1
n (or x) + 3	5	16
Incorrect Response		
4n		1
n + (n – 1) = #n – 1		1
n x 4 - 1	1	
n + 3 = 16		1
Unable to Generalize		1

Table 2 Summary of Responses from Problem 1 (n = 42)

	Figural	**Numerical**
Correct Response		
5n + 1	4	6
6n – n + 1		1
6 + (n – 1)5	1	
Partially Correct Response		
6 + 5n	4	1
n (or x) + 5	5	17
Incorrect Response		
n + (n – 1) = #n – 1		1
26 + n = 31		1
Unable to Generalize		1

Table 3 Summary of Responses from Problem 2 (n = 42)

ANALYSIS OF RESPONSES

Prospective teachers who gave correct and partially correct responses (mean of 56%) to the two induction tasks saw that numerical values provided a better clue to similarity than the figures that accompanied the numbers. This result is not surprising considering the fact that they accommodated new knowledge in accordance with their prior experiences which oftentimes asked them to obtain formulas from sequences of numbers using algebraic methods such as finite differences. In obtaining a variable expression for each of the two problems, the most frequent response that used numerical similarity was, however, partially correct. On average, 42% of the teachers responded by saying that the similarity relation among the numbers in Problems 1 and 2 could be expressed by the formula “n + 3” and “n + 5,” respectively. When prompted to justify the viability of the expressions, 69% of them pointed to common differences which they obtained from the sequences without linking the numerical differences to the figural differences between any consecutive pair of figures. Further, due to lack of notational fluency, the variable “n” in “n + 3” and “n + 5” was defined to mean “the number of sticks before it.” What they meant, of course, were the following two recursive relations: “$a_n = a_{n-1} + 3$” and “$a_n = a_{n-1} + 5$.”

Another partially correct response using numerical similarity involves the expressions “4 + 3n” and “6 + 5n.” In the case of “4 + 3n,” because the participants established numerical similarity from among the sequence of dependent values (4, 7, 10, 13, 16) without having considered how those values were related to the squares being formed, the variable “n” has been assumed to take on values beginning with 1. A similar reasoning holds in the case of the expression “6 + 5n.”

Among the correct responses that used numerical similarity (with a mean of 15%), all the participants used “guess and check” to generate the expressions “3n +1,” “5n + 1,” and “6n – n + 1.” In the case of “3n + 1,” for instance, some teachers constructed a two-column table showing number of squares in the first column and number of sticks in the second column. They then obtained the common difference between two successive values in the second column, wrote down “3n” and observed that each term was “always 1 more than 3 times n.” The same process was made in the case of “5n + 1.” A second numerical similarity strategy for obtaining “3n + 1” involves the painstaking process of trial and error. One of the participants, Jose, started out with the expression “4n - 1” and computed the value for n = 1. Because the value obtained was 3, he then tried “4n – n” and evaluated this expression for n = 1. Seeing that he needed 1 more to obtain the first term, 4, he added 1 to “4n – n.” Once again he evaluated “4n – n + 1” and saw that it worked for n = 2 and 3. He then simplified the expression to “3n + 1” and checked to see if the numbers 4, 10, 13, and 16 did obey the rule. Jose employed the same method in figuring out the expression “6n – n + 1” for the second problem. Another participant, Raina, used a different numerical similarity method in obtaining the expressions “3n + 1” and “5n + 1.” First, she used the given table of numbers and column by column computed the difference between the independent and dependent values in the same column. Because a pattern emerged from the differences which she computed for the first three cases, she then extended the table by simply following the pattern. This enabled her to determine the number of matchsticks for the next two cases. However, in establishing the formula, she simply relied on the first difference, 3, and guessed that by adding 1 (pointing to the first term in the first row of values) to 3n, she would be able to generate all the dependent terms of the sequence. Prospective teachers who induced using numerical similarity methods such as common differences, guess and check, and trial and error were unable to justify how their formulas were related to the problems they were solving.

In the case of prospective teachers whose induction processes involved figural similarity, we observed that the formulas they generated reflected the manner in which they interpreted the figures drawn, including the ones they were asked to construct. The generalized formula was an indication of the process of construction which for them remained uniform and invariant throughout. In thinking about the expression “3n + 1,” Shelly, for instance, started out by computing the common difference, 3, and explained that 3 was the number that determined the “difference between one figure to the next” since forming a new square meant adding 3 new sticks:

> You’re trying to make ahm a full square with four matchsticks and if you already have one side, then you would be adding 3 more on to it depending on the number of squares that you wanna make ‘coz that’s how many you’re gonna put, that’s how many 3s you’re gonna add on.

Prospective teachers who induced by figural similarity clearly understood how symbols played out and what they meant in explicitly expressing generalized relationships. In the transcript below, Chuck explained how he established the expression “4 + (n – 1)3” for the first problem.

> How many matchsticks are needed to form four squares? So ahm I’m looking for a pattern. For every square you add 3 more. So let’s see. So that would be four plus 3 for 2 squares. Plus three more would be for 3 squares. So it’s ten matchsticks. So you have 4. So there would be 13. So 13 plus 3 more is 16. … So for three squares, it would just be two 3s. So there’d be two 3s, three 3s is for four squares, and four 3s for five squares. For n squares, it would just be ahm n minus 1 3s.

GENERAL DISCUSSION

In this study, prospective teachers analyzed induction tasks that contained both figural and numerical cues. Relying mostly on their prior mathematical experiences, their induction processes suggest a preference for numerical similarity strategies. It seems that, on average, six out of ten participants saw invariant attributes in numbers rather than in figures. An analysis of their written responses also indicates that it did not matter for inductors who employed numerical similarity to obtain a large number of cases because they would usually make an attempt to generalize only from the known cases. Although the evidence is still weak, we observed that inductors who employed numerical similarity seemed less capable in justifying their results noninductively, while those who employed figural similarity provided sufficient noninductive justifications due, in part, to the manner in which they connected the symbols and variables they used to the patterns that were generated from the figures. Further, it seems that those inductors who used numerical similarity employed processes and established results that contained fallacies and contradictions. Raina’s induction process, for instance, only made sense within the context of the two problems that she actually solved and not for other problems. Overall, inductors who employed figural similarity were more relation-oriented, while those who employed numerical similarity were more object-oriented since the generalizations they developed were justified solely in terms of how well they fit the information already known and available to them.

The question concerning contexts that enable prospective teachers to determine invariant characteristics from a finite sample of particulars is addressed by considering (1) the level of homogeneity of the numerical values or figures, (2) the nature of the property of invariance being induced, (3) the typicality of the induction tasks being performed, and (4) especially in the case of prospective teachers, the kind of mathematical knowledge that they bring with them to induction. Our data shows how important it is for prospective teachers to have a proper understanding of symbols and variables because they affect the manner in which invariant features are expressed. The two problems presented in this report could be classified as comprising basic-level objects, that is, the figures and numbers were homogenous in that they shared many common perceptual attributes which, thus, encouraged similarity leading to induction. Our data suggest that the prospective teachers found it easier to perform similarity and induction from basic-level homogenous objects than superordinate ones. They experienced tremendous difficulty in determining invariant characteristics of figures and/or numbers when they became too diverse from each other and when the induction tasks became rather untypical.

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FIVE KEY CONSIDERATIONS FOR NETWORKING IN A HANDHELD-BASED MATHEMATICS CLASSROOM

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Handheld devices, most familiar to educators today in the form of graphing calculators, are rapidly improving their interface, computational, and communication capabilities. Communication capabilities allow participants to rapidly share mathematical objects among their handhelds, potentially contributing to improved classroom discourse. We have had the opportunity to explore the pedagogical uses of these new capabilities by extending our SimCalc technologies and curriculum with two significantly different forms of networked handheld computers. The contrast helps us to understand several pedagogically relevant distinctions among types of electronic communication. In this report, we describe list five key networking considerations and illustrate them with three classroom activities that have proven productive.

INTRODUCTION

Graphing calculators have become deeply integrated in the mathematics curriculum, supporting reform objectives, and allowing the NCTM to state that "technology is essential," without economically limiting reform to those schools that can afford computers (National Council of Teachers of Mathematics, 2000). Graphing calculators have succeeded by democratizing computation, but importantly, they also democratize access to powerful mathematical representations (Doerr & Zangor, 2000). The technology is continuing to evolve rapidly, featuring better displays, faster computation, and, importantly, new wireless communication options.

Mathematical discourse and communication is an important theme in research on mathematics learning (Cobb, Yackel, & McClain, 2002), and we expect that these new wireless, communication options for handheld devices will support pedagogy that engages students in classroom discourse more deeply. Indeed, a growing community of researchers (Stroup et al., 2002) has written about the potential of new classroom networks to improve classroom learning, and an PME-NA discussion group has been formed on this topic. The discussion group met in October 2002 and identified wide ranging uses and possible benefits of the technology (see Davis, 2002; Kaput, 2002; Owens, Demana, & Abrahamson, 2002; Roschelle & Pea, 2002; Wilensky & Stroup, 2000). The discussion at the 2002 conference drew a large audience, and raised as its most significant issue the question of understanding succinctly what functionalities new classroom networks offer, and how those functionalities support pedagogy.

In this report, we seek to respond by describing five key classroom networking considerations. To serve the interests of the PME audience, we seek to focus only on those aspects of networking that are most pedagogically relevant. To judge pedagogical relevance, we have drawn upon our joint research with SimCalc software and curriculum in networked classrooms with two remarkably different technical configurations. Moreover, throughout our effort, we have also kept abreast of related projects (e.g., in the PME-NA Discussion Group).

Our account here is descriptive, seeking to summarize networking considerations that have become increasingly prominent in our research as we have observed classrooms over two years of experimentation, data collection, analysis, and research discussion. After briefly orienting readers to the SimCalc Project, we describe three pedagogical activities that have been repeatedly successful in our classroom field sites. We use these to draw out five key design considerations that relate communication infrastructure and pedagogy. Reports in preparation and existing publications describe more extensively the design tensions (Tatar, Roschelle, & Vahey, submitted), curriculum research (Hegedus & Kaput, 2001; Kaput, 2002), and classroom outcomes in our classroom experiments.

THE SIMCALC PROJECT

For the last decade, the SimCalc Project has focused on increasing students' ability to learn the mathematics of change. Through iterative design experiments, we have developed an approach to the concepts of rate and accumulation that builds upon piecewise functions, expressed in position and velocity graphs that students can directly manipulate, and motion simulations that result from the graphs (Roschelle, Kaput, & Stroup, 2000). Our curricula and software have been successful used with students in a wide variety of middle school, high school, and university settings.

In our recent work (supported by National Science Foundation grant #0087771), the SimCalc Project has focused on handheld devices in order to provide access to our software in more classrooms. TI-83+ graphing calculators have been a primary target, because of their availability in American high school mathematics classes. In addition, we have explored color Palm OS devices, to gain a better sense of what future handhelds (with better displays and stylus-based interaction) might offer. As we will discuss below, the TI and Palm products offer very different styles of networking. Thus we have two teams using roughly the same representations and curriculum with different devices and networks. The contrast between our devices and settings makes the relevant pedagogical distinctions among networking capabilities more evident.

ILLUSTRATIVE ACTIVITY STRUCTURES

Major classroom benefits of networked handhelds are mediated by the forms of activity in which they are used. Hence, we illustrate how the considerations we have discussed above play out in classrooms by discussing three of our most successful activity types.

The Exciting Sack Race

In the "exciting sack race", students create both a position graph for a character who is racing alongside a given second character and a narrative story for the race. The story and graph are supposed to be as exciting as possible. Typically students make races in which their character is ahead, then falls behind (perhaps even going backwards or stopping), eventually catching up and ending in a tie.

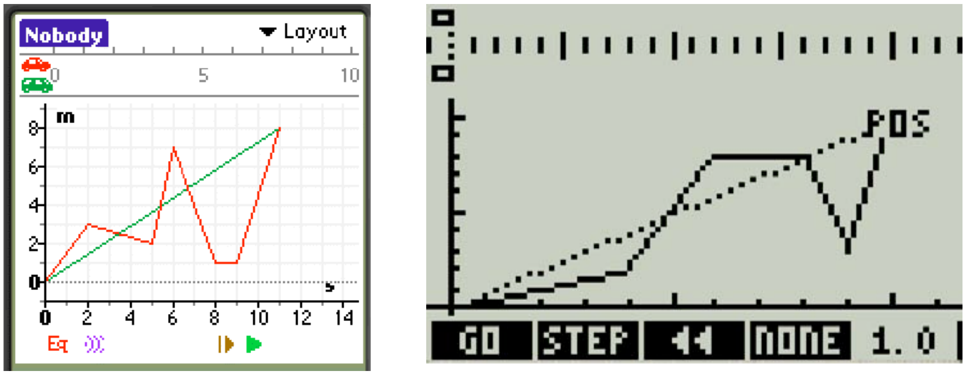

Figure 1: Exciting Sack Race on Palm (left) and TI (right).

The exciting sack race works well on both technologies (Figure 1). The communicative infrastructure is primarily used to distribute an initial setup and then collect each student's race. Collection on the TI is accomplished by "grabbing" a single student's work for display on the teacher's public display. In contrast, "collection" in the Palm classroom is a matter of asking a student to walk to the single classroom projector and place their Palm under it. It is important that students can keep their story private until they are ready to share it, although the eventual sharing is not anonymous. In some variants, students exchange stories (via paper) and then create a new graph that fits the story. They can then compare different graphs that fit the same story.

We have observed that this activity engages students in exploring slope-as-rate (e.g. how to make a position graph in which a character "catches up", "stops", or "goes backward"). With the right teacher provocation, it can also lead to improvement of mathematical description of the features of graphs (e.g. "negative slope" not "bends down"). Every time we have used this activity structure, it has proved exciting and engaging for the students. In fact, in each of the four Palm classrooms, at least one student has literally jumped up and down and called others to come over and look when he or she discovered that negative slope meant that the simulation moved backwards.

Match My Graph

In Match My Graph, one student ("checker") makes a hidden mathematical function describing a motion (e.g. a linear function). A second student ("guesser") contributes a function to the first student, constituting their guess. The checker then provides a verbal clue to the guesser, explaining how the two functions are different (e.g. "your slope is too steep"). The guesser than makes a revised guess and resubmits. The students iterate until the functions are the same. Then they exchange roles. Match My Graph may be played in many variations: different function or graph types may be used; the guesser and checker may see different representations; the checker may only see the guesser's motion, without a corresponding graph.

We have done this activity with both Palm and TI technology. On the Palm, student-student beaming is used to contribute a guess, which is collected for comparison as an overlay on the same Cartesian plot. The teacher may distribute sample language for clue via a big piece of paper posted at the front of the class, and clues are communicated socially. On the TI, cables are used to contribute guesses to the linked machine.

A key functionality of the technology in both cases is to keep some information (i.e., the hidden function) private, while facilitating comparison within a single graph. The technology's support of multiple representations is also key: in some Match My Graph challenges, the guesser submits their function using a position graph but the checker compares the function using its corresponding derivative on a velocity graph. This forces students to use mathematical language based on how velocity and position functions relate rather than superficial features of graphs. One striking difference is in the differences in the communications infrastructure supported by the different technologies. Because a physical cable links the calculators, the activity must occur in dyads, and the groups are not easily malleable. In contrast, the beaming communication of the Palm handhelds allows any number of guessers, in a variety of configurations.

In our analysis of Match My Graph classroom activities, we have seen students who are highly motivated to solve the puzzle grapple with language, and sometimes become frustrated at the ambiguity of everyday expressions when attempting to describe a mathematical situation. Through the introduction of a set of mathematically defined "clues" students begin to see the relevance of mathematically precise language. Although we do not expect students to adopt mathematically correct language immediately, they do begin to appreciate more precise language, and begin to adopt mathematical language in the creation of a bridging language (Herbel-Eisenmann, 2002).

Aggregation of Parametrically Varying Functions

In this class of activity, the teacher asks every student to construct a mathematical object with one or more common properties (e.g., make a line that intersects the point 6,3). Further, each student may be given a unique parameter (e.g. one student is assigned a y-intercept of 1, another is assigned 2, etc.). The students build their functions, and then contribute them to a central location. The contributions are aggregated on one coordinate system (Kaput & Hegedus, 2002).

We have only tried Parametric Variation as a whole class activity using the TI technology, as it is best supported a network that reaches the whole classroom. (We also plan to try it in March 2003 on Palm technology, with four students submitting functions to a fifth Palm which will show all four contributed functions.) A powerful way to integrate social and mathematical structures (c.f. Stroup et al., 2002) is to break the class into numbered groups and have each student count off within their group, yielding 2 numbers that can be used as unique "personal" parameters for each student.

In classroom experiences, we have noted many virtues of the parametric variation activity. It engages all students, as each is responsible for making a contribution. The teacher can easily see if some students are not participating, creating an environment of accountability and high attention. Further, the aggregated result quickly reveals if any student missed the mark; with most functions passing through a single point, any functions that do not satisfy the requirement are visually obvious. But most importantly, the aggregate becomes an important mathematical object in its own right, and a source of classroom discussions. Students can be asked to describe how their value of the slope co-varied with their y-intercept (e.g. increasing y-intercepts required decreasing slopes) and to reason about which lines had positive vs. negative slope. As an advanced exercise, they can be asked to write a single parametric equation that models all the lines.

COMMUNICATION AND PEDAGOGY

We now consider at a somewhat more abstract level how the communication infrastructure used in these classroom activities relates to pedagogy.

Consideration 1: Network Extent and Generality

Networks are usually used to enable students to connect to *resources outside the school*. In contrast, the activities we have focused on *within-class* networks that exchange mathematical objects among students and the teacher. The extent and generality of the network infrastructure brought into a classroom has fairly dramatic pedagogical consequences because it intimately interacts with learning and teaching.

For example, the Internet is currently the most common computer networking communication platform. The Internet is a general purpose form of networking: it supports interactions from any device to any other device worldwide, and is agnostic with regards to the content and types of exchanges. The news media has amply documented the potential dangers of bringing the full generality of the Internet into the classroom: students can be distracted by email and instant messaging; they can view undesirable content; they can use the technology to cheat (Pownell & Bailey, 2001).

The within-classroom networks we have explored exchange messages only within a classroom; these networks are not general-purpose and do not support email, instant messaging or web browsing. Classroom communication centers on the exchange of mathematical functions and their manipulation. In our scenarios, most mathematical content discussed in the classroom is created within the classroom, and shared among participants in ways strongly tied to pedagogical purposes; we have observed little disruptive use of the network.

Consideration 2: Network Topology

A major design distinction among within-classroom networks is in the messaging topology supported. TI Navigator[1] primarily supports a *hub and spoke* topology, where all student messages travel ONLY to and from the teacher hub. Indeed, communication flows in TI Navigator are primarily teacher-initiated (the teacher "grabs" student constructions or "broadcasts" a starting point to all students), which encourages network communication to follow a conventional call-and-response cycle. The Palm product[2] supports a neighbor-to-neighbor topology. Using beaming, students can communicate only with their spatial neighbors at a time of their choosing. Consequently, the TI product naturally supports teacher-led synchronized full class interactions, along the lines of a conventional call and response cycle. The Palm topology better supports dyadic and small group interactions, enabling different groups to proceed at their own rates. However, a teacher using Palms can distribute something to the whole class by purchasing special broadcasting hardware and software, or via a cascade, in which she seeds a few students, who pass it on to a few more students.

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[1] See http://education.ti.com/us/product/tech/navigator/features/features.html for product information.

[2] See http://www.palm.com/education/ for education-related product information.

We have noted that a benefit of both of these topologies is that it is easy for the student to decide *where* to send a message. A Palm user points the device at the recipient. A TI user always sends only to the teacher or to the partner connected with a physical cable, so no selection is needed. A topology that matches the structure of typical classroom exchanges is thus a powerful simplification.

Consideration 3: Anonymity and Group Display

The above topologies differ in their implications for how and when information travels from private to public. Neighbor-to-neighbor beaming more naturally leads to an identification of an idea with a person. Hub-and-spoke designs allow the student's contribution to the whole to remain anonymous when desired. In situations where students feel vulnerable, anonymity can be crucial in helping students risk answers without being singled-out (Davis, 2002; Owens et al., 2002). However, when discourse is to be mediated by the students themselves, detailed knowledge of the position held by the other may be required and beneficial.

A related pedagogical issue is the use of a public display; usually a LCD projector driven by a computer that is connected to the classroom network. A public display can be used for teacher demonstrations, to feature or compare students' work, to show aggregated work (as described above), and for displaying the results of instant polls and quizzes. Participatory simulations (Wilensky & Stroup, 2000) and formative assessments (Owens et al., 2002) particularly rely on a public display which becomes the center of attention; the main purpose of the handhelds is to provide input to the public display. On the other hand, many successful peer and small group activities do not require a central classroom display, which is worth noting since LCD projectors are costly and not generally available in classrooms.

Anonymity and the group display, under control of the teacher, interact in subtle but important ways. In actuality, all students may log in and the teacher may be able to determine exactly who contributed what. Nonetheless, it can be beneficial to avoid labeling individual students' contribution in the public display so that students focus on the mathematics rather than who produced it. In other cases, personal identification with a publicly displayed object can generate strikingly intense student attention and make classroom discussions very fruitful.

Consideration 4: Types of Network Functions

We have found it useful in our classroom experiments to think about the network as accomplishing pedagogically useful transformations of the data available throughout the classroom, instead of merely sending or receiving this or that. The network operations we have found pedagogically useful include:

- *Distribute*: Sending the same starting document to every student (TI, Palm).
- *Differentiate*: Sending different parameter settings to each student, in a systematic pattern (TI).
- *Contribute*: Transmitting a mathematical function or data point constructed by a student to a peer or the teacher (TI, Palm).
- *Collect*: Forming a group of related but distinct functions or data constructed by multiple students; often viewed as side by side contrasts (TI, Palm).

- *Aggregate*: Combining related functions or data into a single overall construction, often then displayed publicly, with or without anonymity (TI).

Two additional network operations would be useful, but are hard to implement on the technology available to us:

- *Look:* It would be useful to a teacher to capture a view from a student's screen without disrupting the student, for example while walking around the classroom
- *Exchange:* Swapping information so as to continue to the next step of a symmetrical process, for example grading each other's work.

Consideration 5: Features of Representational Integration

Finally, it is crucial to the pedagogical uses we have found for classroom networks that communication and representational functions are tightly integrated. Generally speaking, students contribute mathematical functions or points via the network, and these become visible within graphs or other representations. We have found it pedagogically useful to manipulate how transmitted functions become visible in a receiver's display. For example, a function, f, constructed as a position graph on one handheld may show up as f' in a velocity graph on another, requiring students to think about their relationship. Further, it is sometimes crucial to hide transmitted data from the student, so a "secret" can be passed from machine to machine, and only revealed when guessed by another student. Finally, some activities group students by the kind of contribution they are asked to make; it is then useful to layer contributions by the identity of the group.

CONCLUSION

Classroom networking is still at an early stage of exploration, so it is not yet possible to say what the most important uses of classroom networks in a mathematics classroom will be or what the relationship between special and general purpose classroom configurations will be. Further, available networks differ markedly in their capabilities, supporting widely different participation structures. It will take the efforts of many researchers over an extended time to tease out the best practices and uses of these new capabilities, and how varied goals and purposes can be integrated for practical use in the classroom. The five key considerations in classroom communication infrastructure have had high pedagogical relevance in our studies. By attending to these considerations, researchers may more clearly explore the emerging design space for improving classroom discourse.

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AN INTERPRETING GAME IN A THIRD GRADE CLASSROOM

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The paper analyzes the dialogical interaction between a group of third graders and their teacher in one classroom episode. The semiotic and cognitive activity of the dialogical interaction is considered to be an "interpreting game". The episode exemplifies how in dialogical interaction the back-and-forth loops of interpretation, intention, and linguistic expression contributed to the emergence of meanings carry out by the equal sign. Dialogical interaction is considered to be an interpreting games in the sense of being a playful semiotic and cognitive activity between teacher and students constituted by "interpreting cycles" as their most elementary units of analysis. The interpreting game analyzed here is constituted by four interpreting cycles.

INTRODUCTION

The purpose of the paper is to analyze the dialogical interaction between teacher and students as the students interpret a number sentence in which a number is missing. This dialogical interaction is analyzed here as an "interpreting game". It is labeled a "game" because of its elements of playfulness, creativity, surprise, and unpredictability. The dialogical interaction is considered to be an "interpreting game" because the interpretation of the participants stimulates and promotes the dialogue. In interpreting games, the intentions of the participants (to either communicate, guide, explain, justify, or convince, etc.) are anchored in former interpretations. The interpretations, intentions, and linguistic expressions of the third classroom participants mingled with one another in an interrelation characterized by mutual influence and transformation from which the construction and refinements of meanings of the equal sign emerged.

THEORETICAL RATIONALE

Like any other discourse, discourse in mathematics classrooms is essentially a semiotic activity that shapes and is shaped by the cognitive activity of the participants— the teacher and the students. According to Wood, "Communicating in or about the field of mathematics involves taking part in mathematical discourse, whether by reading, writing, listening or speaking. Discourse is a broader concept than language because it also involves all the activities and practices that are used to make meaning in a particular profession" (2000, p. 1).

Discourse in mathematics classrooms inherits, in a certain way, those characteristics of written discourse in textbooks. This discourse is first and foremost constituted by two symbolic systems that are mainly conveyed through verbal and/or written means: natural language and mathematical notation. Most of the time, students interpret mathematical meanings from written mathematical records. Initially, mathematical notations appear to the students only as physical marks or as empty things on a writing surface. However, through processes of interpretation, those written marks cease to be perceived as empty things when they begin to acquire limited meanings. They slowly attain the status of symbols with more generalized meanings and uses. Those emergent symbols carry only those mathematical meanings abstracted by the students from different meaning-giving contexts in which the marks are used. Unfortunately, complete meanings cannot be transmitted directly from the teacher to the student because, as Hersh (1979) points out, physical marks do not embody completely all the mathematical meanings they are intended to convey. Therefore, in the classroom, the construction of mathematical meanings from written notation is an interpreting process. Students engage in this process with the guidance of the teacher who presents appropriate meaning-giving contexts to illustrate mathematical meanings denoted by the written marks.

The interpretation of mathematical notation is a personal process. Bauersfeld (1995) points to the fact that students in the classroom are usually left alone in their own constructive acts of interpreting, reflecting, integrating, and understanding and some students are not able to go beyond the physical characteristics of mathematical notation.

The semiotic and cognitive experience of teacher and students and their classroom discourse is carried out by their interpretation of all manner of signs—social, cultural, linguistic, and mathematical in nature. Or as van Oers (2001) puts Bakhtin's ideas, "people's utterances in a communication process are not only regulated by the processes that occur in direct interaction, but also by the historically developed style of communicating in that particular community of practice" (p. 68). Mathematical symbols are special kind of signs that in addition to their physical shapes, they have names and variations of meanings according to the context in which they are used. Adding to this complexity is the variety of descriptions in natural language of the mathematical meanings that symbols are intended to convey (Sáenz-Ludlow, 2002). These peculiarities of mathematics discourse make the interpretation and abstraction of mathematical meanings a sophisticated semiotic, cognitive, and social activity.

Interpreting games are in essence sign activity, namely the sign activity of the initiator (teacher/student) and the interpreter (student/teacher). Interpretation, intention, and expression are the basic elements of interpreting games and therefore the elements of "interpreting cycles", which are considered to be the most elementary units of interpreting games. Let us consider a cycle in which the teacher is the first acting participant. What the teacher intends to convey to the student is intimately related to the teacher's interpretation of a mathematical concept. In turn, the student's interpretation, of the deliberate action of the teacher, generates an intended response (to communicate, explain, justify, convince, etc.) on the part of the student(s). The cycle continues when the teacher interprets the intended response of the student(s) and generates another intended response that is again interpreted by the student(s) who in turn generates another intended response. And so on, the cycle continues until some kind of interim closure is arrived at. Interpreting cycles remain open to possible refinements of meaning. In each interpreting cycle, every intention is anchored in some prior interpretation within a particular mathematical context. That is, in interpreting games a sequence of interpreting cycles lead to an eventual convergence of the meanings; meanings that are co-constructed by the students and the teacher (Sáenz-Ludlow, 2002). Through interpreting games, the written marks of mathematical notation undergo a sequence of interpretations from which mathematical meanings emerge and become more structured and decontextualized. As Peirce (1931-1948) puts it, "every symbol is a living thing in every strict sense…the body of the symbol changes slowly, but its meaning inevitably grows, incorporates new elements and throws off old ones” (2.222).

METHODOLOGY

A group of third graders (6 girls and 8 boys) from an at-risk school participated in a yearlong teaching experiment. The teaching-experiment methodology predominantly focuses on students’ conceptual constructions over long periods of time. The teacher concentrates on the students’ mathematical constructions as indicated by what they say and do as they engage in mathematical activity (Cobb and Steffe, 1983; Steffe, 1983; Steffe and Thompson, 2000). On the one hand, the interaction between teacher and students dialectically determine and orient students’ mathematical thinking. On the other hand, the teacher should constantly make hypotheses of students’ cognitive paths, and also interpret and assess students’ mathematical actions (Steffe and Thompson, 2000). The teacher’s process of making hypotheses, interpreting, and assessing students’ mathematical actions provide ample opportunities to vary questions and arithmetic tasks according to the students’ ways of conceptualizing and interacting to foster and sustain the evolution of their understanding. The teaching-experiment in which the third grade participated focused on students’ sign-interpreting, sign-making, and sign-using processes as they re-conceptualized number, place-value, and operations with numbers in collaboration with the teacher and the other students.

To analyze the evolving classroom arithmetic activity, the changes in students' cognition, and their sign-making, sign-using, and sign-interpreting processes, the lessons were videotaped daily and field notes were kept; also, the task pages and students' scrap papers were collected and chronologically filed.

Instructional tasks were specially generated for the teaching experiment and they underwent changes according to the cognitive and arithmetic needs of the students. New arithmetic tasks were generated as a result of students’ interactions in teaching episodes. In general, the generation of instructional tasks and the research activity co-evolved in a synergistic manner.

ANALYSIS

The task analyzed in this paper was posed to the students when the teacher experiment was in its fifth month. At this time, students had achieved a certain degree of flexibility to add numbers mentally.

The teacher wrote the question on the board, gave the students time to think, and then started a whole class discussion that is analyzed here as an interpreting game. Teacher and students engaged in an interpreting game when they interpreted each other intended verbal expressions. In this game, the teacher pursued students’ interpretations and their lines of reasoning to help them construct a new meaning for the equal sign emerging from their own cognitive activity. In this process, one of the students started to have an insight into the commutative property of addition.

In the following dialogue T stands for teacher and the other abbreviations for the names of the students. Such abbreviations are italicized in the body of the paper.

Which number will make the number sentence true? 246 + 14 = _____ + 246

1 T: Da, please read the question on the board.
2 Da: Which number will make the number sentence true?
3 T: All right Da. Now read the number sentence for me.
4 Da: Two-hundred forty-six plus fourteen equals…
5 T: …something…
6 Da: plus two-hundred forty-six.
7 T: Kr, what does that equal sign mean?
8 Kr: Equals…it equals something?
9 T: Sh, what does "equals something" mean?
10 Sh: It's…it's when you add something. The equal sign is there so you can put the answer by the equal sign.
11 T: So, are you telling me that on the other side of the equal sign you have to have the answer?
12 Sh: Well yeah, because the equal sign is like when you add something up and the equal is there so you can put the answer down.
13 T: Okay. Does anyone else have an explanation?
14 Ka: The equal sign is the sum. It's like if you add two-hundred forty-six plus fourteen the sum is two-hundred sixty.
15 T: Mmm hmm…So, is that what the equal symbol means here?

Interpreting Cycle #1. The teacher interprets, through the interactions with the students, how they interpret the equal sign in the given equality. Lines 4-5-6 indicate the teacher's interpretation of Da's difficulty in reading an empty space in the context of arithmetic. Instead, lines 7-14 show how the teacher guides the dialogue using the interpretation of each student to pose questions. The teacher's intentional questioning leads her to understand that the students are far away from interpreting that the equal symbol in the given equality stands for the fact that the order of the addends is not significant in the result of the addition. The teacher closes the cycle (line 15) by reassuring herself that the students interpret the equal sign as a command to find the answer and the blank space as the place to "put the answer down".

Two questions come to mind. How will the teacher contribute to the evolution of the students' interpretations? Will the students progressively influence each other and finally arrive to a consensus on the meaning of the equal sign in this context? The following cycle in the interpreting game indicates how the teacher skillfully interpreted the students' interpretations and intentionally used them to pose questions and to involve other students in the discussion.

16 T: Da, you want to say something, what is it?
17 Da: Umm, I think that the equal sign is asking you something like what is six plus six.
18 T: What if I say six plus six equals six plus six? Is this a true sentence?
19 Sh: No.
20 T: So, six plus six does not equal six plus six!
21 Da: Actually it does. It's kind of the same.
22 T: Kind of the same!

23 Ka: It does. I can prove it because that's how much it equals up to. Six plus six equals twelve and you could say that six plus six equals six plus six because they both equal the same amount.

24 T: Teacher writes on the board 6 + 6 = 6 + 6.

25 Ka: And that equals the same thing.

26 Sh: I disagree.

27 T: Tell me why Sh.

28 Sh: Because equal doesn't mean you put six plus six again. You're supposed to add the numbers up and put the answer down. That's what equal means.

29 T: Okay, so are you saying that equals means you have to have an answer on the other side?
So, six plus six does not equal six plus six.

30 Sh: Yeah.

31 Ka: Yes, it does because both sides equal the same amount.

32 T: Can I write 6 + 6 = 6 + 6?

33 Ka: Yes.

34 Mi: Yes

35 Sh: No. Six plus six equals twelve.

36 T: Sh, six plus six is twelve (covering the left side of the equality). What is this six plus six (covering the right side of the equality)?

37 Sh: Twelve.

38 T: (Teacher writes on the board) 6 + 6 = 6 + 6 12 12
Are you telling me that twelve does not equal twelve?

39 Sh: Yes … no. I don't get it. That's equal…But how do you do the six plus six? Six plus six equals six plus six? You can't do that cause six plus six equals twelve. If you write six plus six you're just repeating over six plus six again.

Interpreting Cycle #2. The teacher modifies the example introduced by Da and puts it into the context of the initial question (line 18). Sh's negative answer and Da's answer in uncertain terms is argued by Ka with her own numerical "proof" (lines 23, 25, 31). Ka is able to see the validity of the equality 6+6 = 6+6 keeping in mind the result of the addition and without the need to see the result written down. Regardless of Ka's numerical argument (line 23), Sh continues struggling with her own dilemma (line 28). For Sh repeating the numbers is not the same as performing the addition and writing down the answer. The teacher interprets Sh's dilemma and uses the argument given by Ka to try to convince Sh of the truth of the equality 6+6 =6+6 (line 32). Once the teacher interpreted the cognitive need of Sh, she involves her as a participant in the argument that could have been presented by the teacher in a unilateral manner (lines 35-38). As a result Sh comes to doubt her own interpretations although her own dilemma is not resolved yet (line 39).

One question comes to mind. Will Ka or another student come up with a reasoning that will help Sh and other students to determine the missing number and the truth of the number sentence? The following cycle indicates that Ka is able to use an argument by contradiction to try to convince her classmates that the number on the blank space should be 14.

40 T: What is the number that will make true the equality 246+14 = ____+ 246?

41 Ka: (Ka goes to the board) You could put the answer right here (Ka writes 260 on

the blank space of the original equality) 246 + 14 = 260 + 246. Now, it would not be the same on both sides of the equal symbol because two-hundred sixty plus two-hundred forty-six is not the same as two-hundred forty-six plus fourteen. But if instead of 260 you write 14 then that would be the same thing.

42 T: So, is two-hundred forty-six plus fourteen equals two-hundred sixty plus two-hundred forty-six a true number sentence?

43 Ss: No. That's not a true statement.

44 T: Well, how can we make this (the equality written by Ka on the board) a true statement?

(T erases the 260 that Ka wrote on the blank space)

45 Sh: By putting two-hundred forty-six again or fourteen either one.

46 T: Why?

47 Sh: Six plus six equals six plus six. Two-hundred forty-six plus fourteen equals two-hundred forty-six plus two-hundred forty-six.

48 Ka: But if you put 246 in the blank space, then 246 + 14 = 246 + 246. If you put these two together (she refers to the numbers on the right side of the equality) then it's going to be four-hundred ninety-two.

49 Sh: You don't add them!

50 T: Yes you do; it says plus. The left side is two-hundred sixty; we know that. Ka says that the right side is four-hundred ninety-two. Is this a true number sentence?
246 + 14 = 246 + 246
492

51 Ke: Can I show you something?

52 T: Uhh huh.

53 Ke: (Ke goes to the board and erases the 246 in the blank space) All you're doing is to equal up to two-hundred sixty to make this one (Ke is referring to the ight side of the equation) equal up to two-hundred sixty. Then, all you're doing is just putting 14 backwards (Ke writes 14 in the blank space) 246 + 14 = 14 + 246

54 T: So, now you're trying to tell us that two-hundred forty-six plus fourteen is the same as fourteen plus two-hundred forty-six?

55 Ke: Yes.

Interpreting Cycle #3. The teacher, aware of the students' interpretations, intentionally turns the attention of the class to the initial question. Ka comes up with a numerical argument by contradiction (line 41). This argument was a tremendous insight considering that children are never presented with this type of logical argument. We also have to consider the progress in Ka's thinking as we compare this intervention with her initial interpretation (line 14). In line 42, the teacher again uses Ka's new argument to pose questions to the students in order to sustain the dialogue. Through questioning, the teacher comes to realize that Sh has over-generalized the number sentence 6+6 = 6+6 (lines 45 and 47). However, it is Ka that deals with Sh's over-generalization. Ka, with her natural ability to tinker with arguments by contradiction, takes on Sh's interpretation of repeating one of the numbers and recreates again her argument by contradiction (line 48). This time she is a bit more explicit. In line 50, the teacher uses Ka's argument with the intention to make the contradiction even more explicit (line 50). Sh made no immediate intervention. In line 53, Ke, a student who has no intervened in the discussion up to now, comes with the idea of using 14 in the blank space to "equal up" the result of the addition of the left side of the equality; this, for him, is the same as writing the same addends on the right side of the equality "backwards". Up to this moment, Ke is the only student that comes to see simultaneously the change in order of the addends, the actual addition, and the quantitative balance between the two sides of the equality. In line 54, the teacher closes the cycle by verbally summarizing Ke's argument.

In this cycle we not only see the progressive interpretations of Ka to counter Sh's cognitive dilemma but also the interpretation of Ke who had remained quiet during the discussion up to this moment. It is also apparent that the sophistication of Ka's logical argument by contradiction and Ke's comprehensive interpretation of the equal sign would have not been possible without the interpretations (valid or invalid) of the other students. In other words, progress was the product of dialogical interaction.

Two questions still linger in our minds. Will Sh modify her interpretation of the equal sign as a command to perform an operation to represent the preservation of a quantitative balance while changing the order of the addends? The other question is whether or not there were other students actively making their own interpretations although they have not intervened in the dialogue up to this point. The answers to these questions are in the positive as indicated in the following dialogue.

56 T: (The teacher sees Me raising her hand) Let's see what Me has to say.

57 Me: I think this. Two-hundred forty-six plus fourteen equals fourteen plus two-hundred forty-six. So, I say the same as Ke. It is fourteen.

58 T: Why do you think it is fourteen? You are the third person who says that. Three people said fourteen and two people said two-hundred sixty.

59 Me: Well other people think that it's two-hundred sixty. Umm…I don't mean to disagree but I disagree.

60 T: Why? Why do you disagree?

61 Me: Well, what I think you guys are thinking is that when you guys put these two together

(Me is referring to the numbers on the left side of the equal symbol) it's two-hundred sixty; so you guys think you put two-hundred sixty right here (referring to the blank space on the right side of the equal symbol) and then two-hundred sixty plus two-hundred forty-six will be two-hundred sixty. That's what I think some of you guys are thinking. But I think that fourteen should be in the blank space.

62 Sh: May I say something?

63 T: Huh uhh.

64 Sh: (Sh goes to the board) It's like this. Two-hundred forty-six plus fourteen is two-hundred sixty. If we put two-hundred sixty right here (Sh is referring to the blank space) then we have to plus two-hundred-sixty and two-hundred forty-six and that would be five-hundred six. Like this

$$\begin{array}{c} 246 + 14 = 260 + 246 \\ 260 \quad 506 \end{array}$$

65 T: So do you think this is a true statement? Will you put two-hundred sixty in the blank space?

66 Sh: I don't agree with that. It's kind of like (Sh erases 260 and replaces it with 14). It's kind of like the equal sign is down here and you put it right here. It's kind of of like you're just separating this

$$\begin{array}{c} 246 + 14 = 14 + 246 = 260 \\ 260 \qquad 260 \end{array}$$

67 T: All right. She has an interesting idea because she wants to see what she considers to be the answer.

68 Sh: Yeah, I like to see the answer.

Interpreting Cycle #4. One could have assumed that the interpreting game had ended in line 55 given that the answer to the question posed by the teacher was attained. However, the teacher continued another cycle because some other students were still willing to participate. Me not only agrees with Ke's conclusion but she also makes her own interpretation of the interpretations of other students (lines 57 and 61). Sh also wanted to participate again and what a rewarding surprise it was for the teacher. Sh modified her initial interpretation and now she starts assuming that the number in the blank space is 260 and recreates Ka's argument by contradiction. Finally, Sh concludes that the number in the blank space should be 14 (lines 64 and 66). Sh even goes a step further and creates a chain of equalities. Such a chain indicates that Sh has come around to solve her cognitive dilemma. Sh adds the numbers to be consistent with her initial interpretation of the equal sign as a command to "find and write the answer down" and she also uses the equal sign to symbolize a quantitative balance.

It is important to note here that a first analysis of this teaching episode made the research team aware of the need to generate sequences of instructional tasks to consolidate students' extended meaning of the equal sign to symbolize quantitative balance and the commutative properties of addition and multiplication as special cases. Space restrictions do not permit the analysis of such sequences and they will be analyzed elsewhere.

CONCLUSION

The analysis indicates that these students' initial interpretation of the equal sign was formed through their initial use of it in the performance of simple arithmetical operations. The students' struggles and successes in finding the number that would make the number sentence true indicate that the initial meaning of the sign as a command to perform an operation needs to take on a meaning of quantitative balance while still keeping implicit the former meaning.

The extension in meaning of the equal sign was mediated by the dialogical interaction between teacher and students and it necessitated the explicit communication of students' interpretations and the teacher's interpretation of the students' interpretations to allow the teacher to guide the dialogical interaction in accordance to the needs of the students.

The analysis of the dialogical interaction in this episode as an interpreting game constituted by interpreting cycles in which interpretations, intentions, and linguistic expressions intermingle and influence one another allows us to see how dialogue plays an important role as a mediational means in the dynamic transformation of the written mark "=" into a mathematical symbol in the mind of the students. In this episode only one student indicated that he has started to decontextualize the equal sign to interpret the commutative property of addition that was implicitly expressed in the number sentence.

References [A reference list will be provided by the presenter at the session. It will also be available from "Saenz-ludlow, Adalira" <sae@email.uncc.edu>]

THINKING IN IMAGES AND ITS ROLE IN LEARNING MATHEMATICS

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In this theoretical essay four stages of functioning of thinking in images are distinguished and thoroughly described: 1) creation of a primary image (on the basis of some visual material), 2) creation of a secondary image based on memory, 3) operating with images, 4) creative formation of new images. The role of thinking in images at the solving of geometric problems is revealed and illustrated by examples.

In pedagogical psychology usually three types of thinking of the child, which "sequentially substitute each other, being indicators of growth development" (Yakimanskaya, 1980) are distinguished.

In the early stages of its development the thinking is inseparable from practical operations – the visual-active thinking develops. It means, that at the presence of a problem the child has to immediately investigate real objects. Later such mental operations as the analysis of conditions, statement of the purpose, evaluation of the correspondence of results of transformations to the purpose of study etc. start to be formed. Thus, the basic feature of visual-active thinking is that a real situation serves as the object of direct transformations. This form of thinking is the first step and at the same time the basis for the development of other forms of mental activity. In the process of the development the child encounters with more and more complicated problems. There is a necessity to plan the operations, predicting their results beforehand. For this purpose the child needs the skill "to act mentally". However, the development of such skill is very complicated process. Before the thinking of the child will be able "to come off" the concrete reality, it should pass through the phase of visual-pictorial thinking.

The development of this form of mental activity is very important for the further shaping of thinking of the child. As O.K.Tihomirov (1969) marked, "...The visual-pictorial thinking plays the important role in the formation of child's understanding of processes of the change and development of things and phenomena".

In the opinion of A.V.Zaporozhets (2000), "... The mind of the person in whom in a childhood the visual perception of the reality and the visual-pictorial thinking have not been generated properly, can receive afterwards one-sided development and excessively abstract, separated from the concrete reality, character".

With time the child realizes the presence of interior, hidden connections between various phenomena, and on the basis of visual-pictorial thinking conceptual thinking arises. At this stage the whole system of mental operations is actively formed. The child's mind distinguishes in concepts individual and general attributes. As a result the thinking gets the inductive and deductive character. Furthermore, the reasoning in terms "as if", becomes possible, i.e. the probabilistic-hypothetical judgements can be used. The generated conceptual thinking assumes the possibility of arbitrary self-regulation of the person's intellectual activity.

The traditional selection of three stages of development of thinking has resulted, in the opinion of I.S.Yakimanskaya (1980), in the underestimation of the independent role of thinking in images in the intellectual development of pupils. In particular, "... It was not taken into account, that the thinking in images develops itself, that it is the equivalent form of intellectual activity, has rather complicated forms of display and various functions".

By many researchers the process of development of thinking is understood not as a sequential change of the forms of mental activity listed above, but rather as the gradual growth of complexity of mechanisms of processing of information. S.L.Rubinshtein (1958) wrote: "...Genetically earlier forms of visual thinking are not replaced but be transformed into to the superior forms of visual thinking". O.K.Tihomirov (1969) also argued that "... These three forms of thinking coexist and function at the solving of various problems by the adult". In the opinion of V.V.Mader (1994), "... The thinking can not be pointless and only abstract — it needs to be supported by concrete images".

Thus, one should not conclude that in the process of the development of logic (conceptual) thinking visual-active and visual-pictorial forms of thinking become rudimentary fragments in the general structure of person's thinking. The efficiency of processes of thinking in many respects depends on the level of development of both logic and visual components, and also on the degree of their integration.

The important research problem arises — to determine a role and place of pupils' thinking in images in the geometry learning in secondary school. For this purpose one has to analyze features of thinking in images, namely stages of its functioning and appropriate for these stages mental operations.

STAGES OF THINKING IN IMAGES.

In the opinion of I.S.Yakimanskaya (1980), "... The thinking in images should be considered as a complex process of transformation of sensual information". Perceiving some real object, the person distinguishes details and attributes in it, and the certain emotional attitude to this object arises. As a result the mental representation of its image is formed: a mental "picture", reflecting the object's most essential features (the term "picture" is rather conditional here, as the images may be not only visual, but also acoustical, motive, emotional etc. and frequently they may combine in themselves some of these qualities. Further this image can vary, being generalized, and finally it will reflect features not only of a concrete object, but also of a whole class of phenomena. Consider in more details, how the process of creation of images and thinking on their basis (i.e. the process of functioning of thinking in images) takes place.

The systematization of research in the area allows to distinguish four stages of functioning of thinking in images: 1) creation of a primary image (on the basis of some visual material), 2) creation of a secondary image based on memory, 3) operating with images, 4) creative formation of new images. Consider each of these stages in more detail.

The creation of a primary image at the level of sensual perception is not simply "mental photographing of a real object and "absolute" reflection of its properties. In an image those attributes of object are fixed which the perceiving subject considers (consciously or unconsciously) as the most important. As a result any image reflects in itself properties and attributes of objects selectively, depending on conditions of a task and personal preferences, on beliefs of a person. And this set of the chosen attributes and properties may freely vary during the further analysis. Therefore, the image becomes a dynamical and multidimensional reflection of a real situation.

In the learning of mathematics (especially geometry) this stage of functioning of thinking in images appears to be very important. Many typical errors of the pupils are born just at the stage of creation of primary images, appropriate to investigated geometric concepts

At the following stage of thinking in images the creation of a secondary image happens. As a rule, it is based on memory (at the absence of a real object of perception or in conditions of the conscious refusal of its use as a visual support). Thus the secondary image will be more "general" than primary one. Really, some properties of an object reflected in a primary image "are lost", and only the general, essential attributes are reproduced. As a result the secondary image reflects attributes of a whole class of objects, that is, essentially, comes nearer to concept. For example, in the opinion of N.S.Podhodova (1997), "...The concept "grows" from the "preconcept which, in turn, is based on images".

This stage is also very important for the process of learning mathematics, in a particular, geometry. However it is much more difficult to connect this stage with concrete activities of the pupils. What the pupil should do? How can the pupils' activity be controlled? How the teacher can rule the process of creation of a generalized image? — All these are open methodological problems.

At the third stage — operating with images — the active transformation of the images, created or reproduced in memory, happens. The direction and the modes of these transformations are determined by a problem situation (requirements of a task) and personal beliefs of the perceiving subject. Nevertheless, all transformations of an image save its basic attributes ("freedom of operation" is limited, for example, by conditions of a task). It means that essentially one cannot say at this stage about creation of new images — the images are the same; only combinations of their components may vary.

All the teaching of geometry at the secondary school takes place in conditions of operating with images. If two previous stages had not received due attention, there may be serious difficulties in the management of the process of learning at this stage.

At the stage of the creation of new images mental operations sometimes become the main purpose. The images will be transformed under the influence of some associations, analogies etc. As a result the new images possessing frequently completely unexpected qualities are born. In this case it is possible to speak about creative imagination. In the opinion of A.I.Gibsh (1995), just "... Possession of spatial representations and the presence of spatial imagination... is one of the basic criteria of educatedness in the field of mathematics".

It is clear, that this stage of functioning of thinking in images is the most significant in the learning of geometry at school. However, one should not think that such concepts as spatial representations and spatial imagination are connected only with this stage. All four listed above stages are necessary steps on the path of their formation, and each stage is connected with certain skills, methods and exercises.

OPERATIONS OF THINKING IN IMAGES

One can distinguish mental operation specific for each stage of thinking in images. At the stage of creation of primary images basic mental operation is the analysis (structuring) of visually perceived object, including: distinction of its separate elements, correlation of the elements with the background etc. Note that L.S.Vygotsky (1934) considered the skill to arbitrarily choose a figure and hum noise as the basic indicator of sensibility and arbitrariness of attention: "If I am able to see a thing only in a way dictated by its structure, my attention is extremely involuntarily. If I am able to see a thing so that I can make any element of this thing a center or a figure and make all the rest the background, my attention becomes extremely arbitrary".

At the stage of the creation of a secondary image the basic operation is the generalization. On the first sight there can be an impression, that an image, in the contrast to a concept, is a reflection of an individual object, i.e. that the possibilities of an image from the point of view of generalization are minimal. However, actually advanced "thinking in images allows to reach rather high levels of generalizations of phenomena, but in the specific pictorial form" (Yakimanskaya, 1980). Really, only a primary image is a reflection of an individual real object, on the basis of perceptions of which it is created. And even to this initial stage the term "a reflection" can be applied only conditionally. As already it was noticed above, the primary image created on the level of sensual perception reflects properties of an object very selectively, conceptually: usually only the most essential properties are distinguished. In the opinion S.L.Rubinshtein (1934), the image created with the help of memory, in the absence of a visual basis, "is released from "the attachment to an individual object and can be the generalized image of the whole class or category of similar subjects ".

The third stage — operating with images — is analysed in the psychological literature basically from the point of view of transformation of visual (spatial) images.

In the work of I.S.Yakimanskaya (1980) on this occasion the following is said: "Mental operations of thinking in images correspond to basic geometric transformations". In the opinion of I.A.Kaplunovich (1996), "the structure of spatial thinking... is determined by the operations corresponding to... appropriate basic mathematical transformations... Namely: reflection of two- or three-dimensional Euclidean space into itself and affine transformations of the graphs of functions". In particular, I.A.Kaplunovich (1996) selects the following "... Basic operations fulfilled in imagination with images of spatial figures: "Parallel translation, rotation, central symmetry, axial symmetry, symmetry with respect to a plane, homothety, parallel projection, orthogonal projection, graphic interpretation of operation of addition of functions, graphic interpretation of operation of multiplication of a function to a number, compression and dilation of the graphs of functions".

Except for separate operations at this stage of functioning of thinking in images I.S.Yakimanskaya (1980) revealed three types of operating with images (in order of increasing of their complexity): “transformations resulting in the change of a spatial position of an image (I-st type); transformations changing the structure of an image (_-nd type); long and repeated performance of transformations of first two types (_-d type)”. Thus, operations listed above form a multilevel system.

Probably, the description of a system of mental operations, “homomorphic to the basic mathematical structures” really represents an effective model of thinking in images (this approach is correlated with the well-known conception of J.Piaget, according to which the structure of mental operations performed by a person corresponds to basic mathematical structures). However, explanation of one abstraction (psychological) through another (mathematical) not always can clarify the essence of mental processes.

At the stage of creative formation of new images the active transformation of initial data happens. It means that all operations of the previous stage, though in specific conditions, are used. Firstly, basically operating on the II-nd or III-d type occurs, and secondly, the direction of operating is set not by any exterior requirements, but by analogies and associations, arising on the basis of available images, and as a result new images emerge.

THE ROLE OF THINKING IN IMAGES AT THE SOLVING OF GEOMETRIC PROBLEMS

The thinking in images of the pupils plays the important role in learning mathematics in the secondary school, in particular, in learning geometry. In the opinion of A.Ya.Tsukar’ (1998), without thinking in images “... The successful study of a geo-metric material is impossible... because there the skill to read images of figures, to mentally imagine required objects, to keep in sight several objects simultaneously and to operate with them is continually required...”.

Further we will analyze displays of features of thinking in images during the solving of geometric problems.

There is a lot of works in which in some form the process of the solving of problems including geometric ones is described.

Since the long time V.A.Gusev and his disciples were engaged in researching so-called skills of the solving of geometric problems and paths of the realization of these skills in practice.

Concerning a role of thinking in images in a realization of the generated skills of solving of geometric problems, its special role is exhibited at the realization of skills in the distinction of figures appropriate for the given element of a problem. More precisely, these are the following skills:

- Construction of a drawing appropriate to the text of a problem:

- Distinction of figures appropriate for the given element of a problem, i.e. those figures available on a drawing, which will participate in the solution of a problem (the additional constructions are also possible).

At the solving of problems the pupils should mentally imagine a spatial figure described in the condition of a problem. In the book of G.Polya (1965) in this occasion the following is said: “If we deal with a geometric problem, we should consider some geometric figure. This figure we can either present in our imagination or to represent on the paper as a drawing. In some cases it may appear to be better to imagine a figure, but not to draw it”.

Consider the situations when it appears to be useful “to imagine a figure” before the direct performance of the drawing. In each of these situations the condition of a geometric problem suggests the existence of several essentially different cases of the disposition of figures, each of which requires the separate approach to a solution.

Problem 1. On the straight line a points *A, B* and *C* so that $m(\overline{AB})$= 5cm and $m(\overline{BC})$= 7cm mare marked. Find length of the segment *AC*.

The disposition of points *A, B* and *C* on the straight line is not indicated in the condition, and it must be imagined.

Two alternatives are possible: the point *B* lays between points *A* and *C* (fig. 1a) or the point *A* lays between points *B* and *C* (fig. 1b).

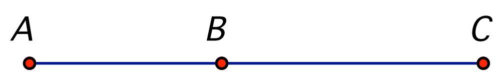

Figure 1a.

Figure 1b

It means that the given problem has two different solutions: in the first case $m(\overline{AC})$= 12 cm, and in the second case $m(\overline{AC})$ = 2 cm.

Problem 2. Two angles have a common side and their degree measures are 65˚and 35˚, respectively. What angle their incoincident sides can form?

In this problem the process of “the imagination of the disposition of angles” is broader, as is connected with the possibilities of disposition of angles not only on a plane but also in space. Considering this problem in a similar way as previous one, we see that there are two possibilities of the disposition of angles: the side of the smaller one is located 1) inside the greater angle (fig. 2a) or 2) outside it (fig. 2b). Accordingly, the angle between the incoincident sides will 30o or 100o.

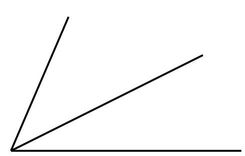

Figure 2a.

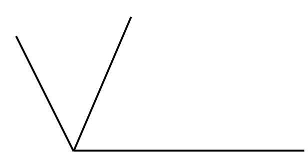

Figure 2b.

However, in the condition it is not indicated that all three rays (sides of the given angles) lay in one plane (fig. 2b). It means that the problem has infinitely many solutions, more precisely, the required angle can have any magnitude from 30˚ up to 100˚.

Problem 3. Bisectors of angles A and D of the parallelogram ABCD divide its side __ into segments of lengths 5cm, 2cm and 5cm, respectively. Find the perimeter of a parallelogram.

In this problem the initial figure is the given parallelogram, and the solution depends on its form and size. There are two possibilities of the disposition of bisectors of angles at the base: these bisectors may be intersected inside the parallelogram and outside it (fig. 3 a, b).

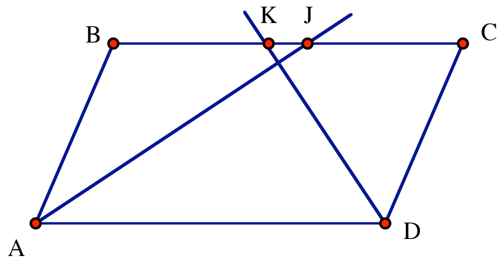

Figure 3a.

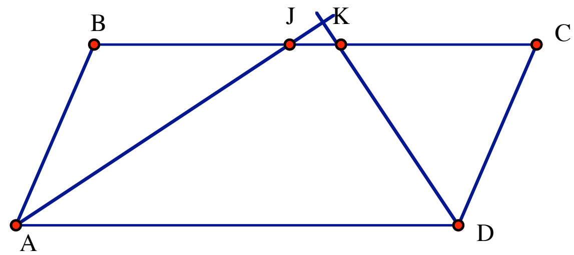

Figure 3b.

The idea of the solution of this problem is the same for both cases (it is based on the fact that triangles ABJ and DCK are isosceles). However, the answers will be different: the perimeter is either 34cm or 38cm. Pupils frequently restrict themselves to considering only one situation fixed by a drawing that they have constructed. As a result the solution appears to be incomplete. That is why the appeal "to imagine a figure before the performance of the drawing" is useful, and the methodology of solving geometric problems should separately elaborate this question.

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PRIMARY TEACHERS' CONCEPTIONS ABOUT THE CONCEPT OF VOLUME: THE CASE OF VOLUME–MEASURABLE OBJECTS

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In this paper part of the results obtained by a research project called "Primary teachers' thinking about the concept of volume and its teaching", performed from 1997 to 2001, are reported. This paper focuses in one of the two main objectives of the aforementioned research: To describe the mental object volume of the participant teachers. The experience took place in a five-session workshop directed to in–service teachers and mathematics advisors of the Public Educational Bureau (Secretaría de Educación Pública). The sessions were audio and video taped and a set of questionnaires was posed in the first session. The transcriptions of these recordings, as well as the answers obtained by the questionnaires, were analyzed in order to attain the planned objectives.

The literature related to teachers' thinking and mathematics education is extensive since the 1990's decade. Researchers that also have under their charge the education of future teachers have developed a great part of these studies. Between the results obtained some indicate that teachers' conceptions, as well as their experiences when they were school–aged children, influence their professional practice (see Civil, 1996; Thompson, 1992).

In accordance to the aforementioned results, the importance of studying teacher's conceptions and beliefs about the different elements in the schools' curriculum is manifested. If some of their conceptions are wrong or incomplete, they will teach them to their pupils in the same way. This paper is focused mainly on teachers' conceptions about the concept of volume, and the results presented are some of those obtained by the general research.

In the mathematics' teaching research literature volume is a scarcely studied concept; it appears more frequently in sciences' teaching literature. Although, some researches centered in teachers' conceptions about the concept of volume exist (see Enochs & Gabel, 1984), anyway most of these studies focus their attention in children .

The first results related to the learning processes of the mathematical concept of volume are, probably, Piaget's quantity and matter's conservation studies (Piaget, Inhelder & Szeminska, 1970). Some others of further researchers followed them. In any case, as Vergnaud (1983) observes, the volume's conservation studies do not accomplish the volume subject matter. He, in collaboration with other researchers developed a study where the central point is volume's arithmetical processes. From other point of view, Janvier (1997) among others, obtained interesting outcomes about the teaching of volume.

Due to the problematic involved in the research that origins this paper, a particular mention is deserved for Potari & Spiolotopoulou's (1996) study in which attention is focused on the influence that certain objects' characteristics –both geometrical and physical– have on children's conceptions about the concept of volume. In preliminary studies and in the theoretical frameworks' construction, the influence that the meaning deserved by people to the term volume had on their personal domain set for the function volume (the set of things they consider as volume–measurable objects) was perceived.

In this paper, geometrical aspects are considered to be those elements, processes and procedures that take into account the shape of objects, for example their concavity or convexity. Numerical or arithmetical aspects are those centered in measurements, size and numerical operations, while physical aspects are considered to be those related to matter's condition such as solidity or density, for example.

Authors like Rouche (1992) had paid attention to volume–measurable objects from a didactical point of view. Nevertheless, we consider that, because of the consequences it may have on the teaching and learning processes of the concept of volume, one of the main contributions of the general research "Primary teachers' thinking about the mathematical concept of volume and its teaching", from which this paper is a fragment, is the call on attention about people's volume conceptions influence on their perception of objects to be volume–measured.

According to Freudenthal (1983), concept's construction should not be the main teaching purpose, even when they are the "backbone of our cognitive structures" (op. cit., p. x). He affirms that children learn what things are, for example a chair or a table, without teaching them the concept of chair or the concept of table.

To understand the difference Freudenthal established (1983) between the expressions concept and mental object we may consider that a mathematical concept is used in a variety of contexts. For instance, the concept of volume may be comprehended as the amount of space claimed by a solid, or as a free space inside a closed surface, or as the amount of material that fills a recipient or like a number of cubic units, or as the space displaced by a solid when placed into a liquid and in some other ways. The usage's rules of volume in all these cases are different. The totality of these usages, in all contexts, is what may correspond to the concept of volume, or using Puig's (1994) terms: "the semantic field of volume" or "the encyclopedia knowledge of volume".

> Now, the subject reading a text or interpreting a message does not operates in the set of the encyclopedia– that is to say, the totality of usage produced in a culture or episteme– but in a personal semantic field, which has been elaborated producing senses –senses that become meanings if the interpretation is adequate– in situations or contexts that demand new uses [of the mathematical concept of interest] (Puig, 1994, p. 9).

The scrutiny of the literature related to teachers' conceptions and beliefs showed that it is not easy to obtain agreements respect to the use of the terms conceptions and beliefs (see Thompson, 1992; Törner, 2000 for example). Recently, Pehkonen & Furinghetti (2001) pointed out that a great amount of the encountered discussion in the reviewed papers would be avoided if in mathematics a distinction were made between concepts and conceptions.

> The word 'concept' (sometimes replaced by '*notion*') will be mentioned whenever a mathematical idea is concerned in its 'official' form, as a theoretical construct within the 'formal universe of ideal knowledge'. [Whereas] the whole cluster of internal representations and associations evoked by the concept –the counterpart of the concept in

> the internal, subjective 'universe of human knowing'– will be referred to as 'conception' (Sfard, 1991 in Pehkonen & Furinghetti, 2001, p. 649).

As it can be seen, Freudenthal's characterization of the expressions *concept* and *mental object* coincide with Sfard's given meanings of *concept* and *conception* respectively. In this manner, one of the main objectives of the general research can be posed in these terms: to characterize *volume mental objects* of the observed teachers. In this paper a part of the results related to this objective are presented.

THEORETICAL AND METHODOLOGICAL FRAMEWORK

The theoretical framework was constructed using the guidelines of the Local Theoretical Frameworks Theory of Filloy (1999). This theory, is described as a "theoretical and methodological framework for the experimental observation in mathematics education" (Filloy, 1999, p. 1). In it "the object of study is focused by means of four related components: (1) teaching models, (2) cognitive processes models, (3) formal competence models and (4) communication models" (Filloy, 1999, p. 4). For the research project herein reported the theoretical framework has been partially exposed in Sáiz & Figueras (2000); its detailed description appears in Sáiz (2002). In the next paragraphs a general description of each of the components of the local theoretical framework is presented.

The formal competence models component puts together the elements that allow to describe and decode what is observed, in the case of the research herein reported, this component contains all the knowledge that an ideal user should have in order to understand and decode the messages' exchange that takes place when volume related tasks are performed. It is worth saying that this component does not correspond to the concept in the sense formerly explained, but that it conforms the observer's mental object, which is, somehow, a mental object between the concept and the teachers' personal semantic fields, for whom a characterization is pretended.

The construction of this component followed two axes. The first one consisted in a documents scrutiny in mathematics' history books in order to identify the historical process of development of the mathematical knowledge related to the concept of volume. The second one took as its initial point the phenomenological analysis applied to the area and volume concepts by Freudenthal (1983) albeit, not every aspect of his analysis was taken into account. As he states, the approaches to the mental objects area and volume constitute a large variety and the analysis required to decide if they form a consistent whole or not, and if all of them lead to the same result "*require efforts that surpass anything that can be asked or realised, say, at the highest secondary grades...* " (Ibid. p. 381). Nevertheless, the analysis has been extended to other considerations such as the relation of volume with other magnitudes like capacity and weight; the role volume plays in physics science and the geometrical and physical characteristics of bodies when they are considered as volume–measurable objects. This component was a theoretical and methodological framework not only for the observation's organization but for the data analysis as well.

For the construction of the teaching models component, a scrutiny and a comparative analysis of volume teaching models used in Mexico in a period of one hundred years (1898–1997) was applied; in this process the national educational program for the primary school, in use since 1993, was considered as the “pattern” to compare with the different volume teaching models. The formal competence models component and some specific guidelines for the teaching of volume sustained by experts of different countries were useful too.

Some empirical experiences with in service teachers in proficiency programs revealed evidence, which showed that –frequently– teachers have difficulties in solving volume problems and that they possess misconceptions related to the concept of volume. The former facts, between others, explain the need of the cognitive processes models component, performed with results derived from research linked to children’s cognitive difficulties related to the concept of volume (see Piaget, 1960; Vergnaud, 1983).

DATA ACHIEVEMENT

The communication models component is related to the data’s attainment processes. For the general research project, the challenge was to provoke a discourse that reveal information about what was being investigated, in the particular case of what is herein reported: the teachers’ mental object volume.

It was essential to design a communication process that was not a knowledge test; there were evidences that this kind of examinations make teachers feel nervous and assaulted. It was necessary to obtain a free and spontaneous talk. Hence, the application of some carefully designed questionnaires and the design of activities and problems for a workshop related to volume and its teaching, directed to in–service primary teachers, was the way that allowed the obtaining of appropriate data (see Sáiz, 2000).

Twenty-two primary teachers participated in the experience, some of them were in service teachers and others were assessors in “teachers’ centers” (locations where primary and secondary level teachers, from 1st to 9th level, may go to find books, didactic materials and assistance). All the sessions were audio and videotaped. The transcriptions of the recordings and the set of answered questionnaires contained the whole information for the analysis.

ANALYSIS METHODOLOGY

The data obtained by means of the communication processes are the teachers’ ideas expressions in the appearance of words, phrases, sentences, paragraphs and other fragments of their discourse. This discourse is expressed by means of a mathematical system of signs (Filloy, 1999) and, in its usage, signification and sense production processes are manifest. The analysis must recover these meanings in order to get elements allowing the teachers’ mental–object volume description.

Briefly, it can be said that the initial analysis categories are naturally inferred from the local theoretical framework’s components. In a first moment data was classified in three large categories: cognitive processes, teaching models and formal competence; in a later analysis, data was sub classified in different sub categories, for example, qualitative aspects and quantitative aspects. The different versions of analysis categories were modified by the scrutiny and analysis accompanying the reiterated lectures of the data. These processes show the great amount of data obtained and the difficulties to work on it;

in the last stage of the analysis a software tool was used, and a definitive analysis categories' taxonomy was obtained.

The results of the analysis are organized in different ways; teachers' mental object volume may be described by exhibiting different classes of results related to: 1) the set of objects considered as volume–measurable by the participant teachers; 2) the procedures used by them for measuring, comparing and obtaining volume; and 3) a set of meanings and results obtained by the measurement and calculus processes performed by the teachers On the next paragraph, examples of the aforementioned classes of results are presented.

DISCUSSION AND RESULTS

1. Volume–measurable objects are those for which three lengths can be obtained.
Maybe teachers are thinking in the measuring of large, wide and height. With these measurements, they say, that volume may be calculated. (In this and the next presented results, an example of the evidence that sustains each conclusion follows; the first number represents the row in the document where the evidence appears; letters M and V, followed by a number, are codes for teacher identification).

(Answers to the question: Can you obtain the volume of an auditorium?)

64 M2: [Yes, because] we can see three dimensions.

65 M4: An auditorium. Yes. It is measurable.

117 V5: An auditorium. Yes, generally they have three linear magnitudes and with them you can obtain it.

2. Bodies perceived by teachers as surfaces are not considered volume–measurable objects.
This result is related to the former one. Since teachers perceive thin objects such as paper sheets and handkerchiefs, as surfaces, they think that there's no width to measure; there is a missing data in order to obtain volume by multiplying three numbers.

(Answers to: Do you think that you can obtain volume of the following things?)

69 M4: A handkerchief. No. It has a minimal third dimension.
97 M9: A sheet of paper. No. It is an area measure.
120 M14: A handkerchief. No. It is planar.
144 V10: A sheet of paper. No. It has area and perimeter; it's a planar figure.

3. Teachers do not consider some daily things as volume–measurable objects, due to their irregular shape.
It seems that, from the participant teachers perspective, objects that are not identifiable with scholar geometric solids (cube, prism, pyramid) cannot be volume–measurable objects. Maybe the absence of a specific formula, or the lack of imagination to apply break-make transformation (Freudenthal, 1983), or few experience in using measuring tools conduct to the belief mentioned in result number 3.

(Answers to: Do you think that you can obtain volume of the following things?)
115 V5: A chair. No, because of its shape, it is not just one shape it has several ones.
119 V5: A female screw: No, because its irregular shape and tiny dimensions.
125 V5: A spinning top: No, it's difficult because of its irregular shape.

4. In non-scholar situations teachers manifest the usage of different meanings related to the word *volume*; the dominant one is that of volume as a number.

In the research herein reported some different meanings related to the word *volume* were established, some of them arose during the construction of the theoretical framework. For example, volume as 'internal volume' or the number of cubic units that constitute a body, was taken from Piaget *et al* (1970); volume as 'capacity' came into view from capacity's meaning as the volume of matter that fills a container; volume as 'enclosed volume' or the free space enclosed in a closed surface emerged from the analysis of the data; teachers mentioned and distinguished it from the 'internal volume' aforementioned. Another meaning is 'volume as a number', the one that was expected to emerge more frequently; although statistics performed by the software did not point out this meaning as the more frequently found in the data, anyway, it appears in an implicit manner in many situations. In fact, the previous cited results (1 to 3) show an implicit 'number' meaning associated to the term *volume*; that's why teachers cannot see some objects as volume-measurable, because their shape, or size, gives place to an "incomplete" numerical information, and so volume (a number) cannot be obtained.

In one of the workshop's sessions, a list of objects was presented to the teachers; just two of the elements in the list were selected by all of them as volume–measurable objects: an orange, associated to a sphere, and a tank related to a cylinder shape. Even those who select thin or tiny objects as volume–measurable, when asked about the process they will use to obtain such a measurement answered that they will measure three lengths and then multiply them. For instance consider the paragraph that contains the responses of three teachers to the questions: "Can you obtain the volume of...?" and "How will you do it?"

190 V5: ...a sheet of paper? Yes, obtaining the area and multiplying by the sheet of paper's width.

192 V9: ...a pond? Yes it has a geometrical body shape, by formula.

193 V10: ...an auditorium? Yes, by multiplying length, width and height.

194 V10: ... a pond? Yes, by multiplying length, width and height.

5. Some teachers have constituted a mental object associated with the bodies' characteristic of 'having three dimensions'.

In teacher's answers to the question: "What means volume for you?" a definition – included in textbooks of the decade of the sixties and former ones –apparently emerges: "Volume is everything that can be measured in three dimensions: large, width and height" (Sánchez, 1960, p. 26).

(Answers to: What is volume?)

13 M4: ...it's the third dimension.

16 M11: ...it's the bodies' characteristic of having three dimensions.

At the present time, the importance of giving sense to concepts such as length, area and volume and not to center measure teaching in formulas is emphasized in national and official programs and textbooks used in Mexico. Anyway, teachers' conceptions and the influence of former teaching models may continue to obstruct these tendencies. Although teachers know, repeat and, even, agree with the new educational discourse, their conceptions and beliefs betray them. Sometimes teachers have difficulties solving volume related problems and misconceptions linked to this concept. That's why teacher's education and proficiency programs must consider measuring contents in order to improve these subjects teaching, particularly in the case of volume.

It is worth mentioning that the acquired results are connected with the case of the observed teachers, and albeit they are announced as general results it is not our pretension to consider them representative from a statistical point of view. We think that, even though applying questionnaires and to analyze the performance of teachers does not constitute determinant evidence to know how volume is being taught in classrooms, the results here exposed, as well as those of the general research project, may be considered as a first approach to what happens in school in the case of the volume concept.

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ASSESSMENT PRACTICES IN SCHOOL MATHEMATICS: ACTING AND DEBATING

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Assessment practices influence and are influenced by a number of educational as well as social factors. The present study looks at these practices of five primary teachers displayed in the mathematics classroom and in a relevant pedagogical discourse developed in the context of an interview. The results show that, on the whole, their practices were rather conventional and did not differ much between the two contexts, demonstrating weak frames of practice and interpretation.

INTRODUCTION: SOME THEORETICAL ISSUES

The assessment of the educational process is one of the most significant functions of an educational system. Its significance in the educational process lies in the classroom itself and the school environment in general, as well as in the educational policy, which characterizes and is being characterized by the political-ideological framework in which it is exercised. In the school context, assessment is used for the shaping of an effective learning environment, but it also determines the degree and the form of support pupils should be given on an individual as well as collective basis. At the same time, the results of educational assessment constitute strong evidence for the evaluation of educational policies. The formal procedures and practices of assessment are used in classifying children, while the results of assessment constitute the basis upon which decisions about their future are taken. Thus, along with the educational aims it serves, assessment, it could be argued, performs a number of political and social functions in contemporary societies.

The above points highlight the complexity of the assessment process as well as its significant contribution to the formation of the identity of the educational system, the teacher and the student. More importantly though, they show the necessity to consider assessment practices in the school as well as in the wider social environment where they are exercised, the role of the latter being catalytic in the formation of these practices. In this perspective, it is deemed important to study systematically the procedures of assessment a teacher adopts in the classroom. The focus is on the teacher, since he/she is the most powerful, undisputed participant in the educational process and therefore, the main carrier and formal 'factor' in these procedures (Filler & Pollard, 2000).

Assessment and teaching practices are closely related to the teacher's identity. This is constructed from his/her inner, personal views and ideas as shaped by external cultural influences and the expectations of specific groups operating in the wider society (e.g. parents, educational officials etc). This "internal-external dialectics" (Jenkins, 1996) determines the practices of teaching and assessing a teacher adopts in the classroom. However, these practices are also shaped to a great degree by the conceptions teachers form concerning their pupils and the school environment. Consequently, the features that characterize teachers' practices as well as their understanding of their students should be examined in conjuction with the differentiated answers provided by the latter in a given school environment (Filler & Pollard, 2000).

This differentiation in the students' answers can be interpreted in the light of an epistemology beyond the traditional, according to which there is not necessarily a relation between a student's 'text' and the meanings the teacher, as a reader of the text, constructs. On the contrary, the messages formed depend on the features that the reader distinguishes in the text. These features vary according to the pedagogical discourse within which the text is interpreted, and the positionings adopted by the teacher-reader within this pedagogical discourse along with his/her previous experience. Kress (1989) argues that the text itself constitutes an "ideal reader", presenting a reading position from which there appear to be no problems and is "natural". However, readers do not necessarily adopt the "ideal" position. Consequently, there is no guarantee that the interpretations teachers as assessors offer are exactly the same as those of their pupils (Morgan, 2002). In other words, the teacher proceeds to assessments subjectively. The differentiation in assessment results seems to be related to the degree of subjectivity in judgements, since teachers do not judge as a whole in the same subjective way. Moreover, they do not have the same subjective judgement in all cases and in all subjects in the school curriculum. The degree of subjectivity differs not only between teachers but also between subjects. As a result, assessment seems at times to be an informal, spontaneous function and at others a formal, clearly defined operation, which has probably been taught or pointed out. In studies concerning teachers' assessment practices, in both formal and informal settings, emphasis is put on the fact that the features teachers consider as contributing to the validity of their assessments can only be discovered through examining the practices of teaching and assessing their students (Morgan, 1996).

Assessment in mathematics is often seen to be equivalent to an evaluation of the level of understanding achieved by the pupils. Morgan (2000) advocates that this approach to assessment in mathematics rests on two hypotheses: a) pupils have characteristics such as skills, abilities and knowledge that can be identified and measured and b) the principal role of assessment is to reveal and measure these characteristics. Both these hypotheses are based on the belief that, theoretically speaking, there is a fundamental "truth" in mathematics to be discovered and measured; however, this emphasis on the measurement of children's achievements is very restrictive, as it does not allow for the complexity and entirety of the assessment process to be appreciated. That is, it does not allow for the pupils' work to be understood in relation to the power structures developed in the classroom, the school and the wider society. Hence, in looking at assessment in the everyday classroom, the social nature of mathematical behavior, theories of pedagogical discourse and communication, as well as a sociological analysis of the role of education, mathematics and assessment all need to be taken into account. This highlights the importance of carefully examining the practices of assessment utilized by the teacher and identifying his/her informal assessment behavior, which constitutes a significant element of the entire assessing function.

THE STUDY

The study presented here constitutes part of a larger project which aims at studying teaching and assessment practices in school mathematics as well as the way in which these practices are related to teachers' conceptions of mathematics, its learning and teaching. In this paper, the focus is narrowed down to an examination of the assessment practices employed by teachers within the mathematics classroom and the pedagogical discourse developed in interpreting these practices. In particular, an attempt is made to address the following research questions: a)"What are the main features characterizing assessment practices both in the classroom and in teachers' pedagogical discourse in school mathematics?" and b) "How do these features differ in the two contexts?".

The sample consisted of five primary teachers (four males and one female), all of whom had 15 to 23 years in service. Two of the subjects were teaching senior classes (5th, 6th grade), two intermediate level classes (3rd, 4th grade) and one a junior class (1st grade). All were graduates of Pedagogical Academies (two years courses) and had attended a number of in-service training programs, while some had participated in a number of innovative projects in education.

Two research tools were used for the collection of the data: observation of mathematics lessons and interviews with each teacher. Specifically, five lessons were observed and videotaped, one for each of the five teachers. Teachers were told to act "as normal" and try to follow their "usual" approach. The interviews were semi-structured and thematically divided into three parts. The first concerned the teacher's educational background (undergraduate, postgraduate studies, in-service training), his/her professional and more general scientific profile (participation in conferences, research projects, etc) and their teaching experience. The second part of the interview concerned issues related to the nature of mathematical knowledge, the teaching and learning of mathematics (11 questions), while the third part concentrated on aspects of assessment in school mathematics (11 questions). Each interview lasted for an hour and 45 minutes, and took place outside the school environment. Both the observations and interviews were carried out in the spring term. The videotaped lessons and tape-recorded interviews were transcribed; the transcripts constituted the research data.

DATA ANALYSIS AND DISCUSSION

Teachers' assessment practices were studied on the basis of their actions while teaching mathematics, whereas the pedagogical discourse they develop around this issue was examined via their answers to the questions of the interview. In addition, the subjects' views and conceptions concerning mathematics, its teaching and learning as expressed in the interviews or as they emerged in the classroom, were also used to "fill in the gaps" and provide more flesh to the data.

In order to analyze the assessment practices employed in the classroom and displayed in the interviews of the five teachers, Tsatsaroni et al's (1997) suggested categorizations were adopted; namely: (a) *emancipatory-implicit criteria (E)*: assessment, if not rejected as a reproductive mechanism, operates in a combined way, for both the educational process and the class itself. It contributes to the acquisition of a critical attitude towards knowledge, (b) *informal-implicit criteria (I)*: assessment focuses on the learning process and procedures rather than on the individual child, and (c) *formal-distinct criteria (F)*: pupils' assessment takes place individually and is based on the use of an "objective" scale. It is characterized by distinct criteria and focuses on performance.

In the following, some distinct features of the assessment practices revealed by each of the five teachers in the sample, first in their teaching and then in their interviews, are briefly presented.

a. Assessment practices: teaching sessions

Orpheas: Assessment takes both an oral and written form. It allows for a revision of what the students have been taught. The students are asked to solve exercises similar to those done in class (*formal-distinct criteria*).

Notis: Assessment is carried out through solving the exercises in the textbook as well as other exercises provided by a supplementary book. The teacher checks the answers to the solved exercises and signs the pupils' notebooks. The latter do the exercises individually and address the teacher only when they have finished solving them (*formal-distinct criteria*).

Aris: The teacher gives his students handouts with exercises. Each student finds the answer on his own and then goes up to the teacher to announce it in a secretive way (*formal-distinct criteria*).

Sophocles: The level of the students' understanding appears to be closely related to the result (*formal-distinct criteria*).

b. Assessment practices: interviews

Orpheas: "The student's assessment in a written form is standard: either he has done the operations or solved the problem correctly, …or he has made a mistake in solving it, in which case the teacher sees that the student hasn't understood it…. They know most of the criteria. Based on what I ask, the students know that they will be assessed in the very same things…. Assessment in written form is more important especially in mathematics. While being assessed in writing, the student is focused on the question; he/she will think hard enough to answer and what he/she will answer, stays" (*formal-distinct criteria*).

Christine: "When I have to hand in grades formally, then I am obliged to have a standard line of reference according to which I will grade the students...what they can achieve as a whole….Then I seek another way of assessment" *(formal-distinct criteria)*…. Each time a teacher should assess the students individually, according to what each of them can do…. I believe the students observe the teacher's behavior and sooner or later they figure out what the teacher wants" (*informal-implicit criteria*).

Sophocles: "I think that assessment in written form is more effective…. One of the things I take into account when assessing is how the student has performed on his/her tests …. Every two weeks the students take a test I prepare to see how they are doing … I return the tests with a grade on" (*formal-distinct criteria*)…. Oral assessment takes place daily, in order for the teacher to have a complete picture for his students; oral assessment complements the written form of assessment….I believe that as the teacher gets to know his/her students, in the same way they get to know their teacher, what points he/she insists on, what his/her demands are, what he/she likes to listen to" (*informal-implicit criteria*).

In order to increase their validity, the data were analyzed separately by the two researchers. A detailed discussion concerning the meaning of the different categories took place beforehand. Based on the results of this "double" data analysis, which hardly differed, table 1 was constructed.

Table 1: Teachers' classification according to assessment practices

Source of data	**ASSESSMENT PRACTICES**		
	E	**I**	**F**
Teaching sessions			
		A, S, C	A, S, O, N,
Interviews			
		A, S, C, N+	A, S, O+, C

Note: 1. A = Aris, S= Sophocles, C=Christine, N = Notis, O = Orpheas

2. "+": The evidence for the specific classification is clear/ Absence of "+": The evidence is satisfactory.

As table 1 shows and confirmed by the interviews, the teachers present a quite distinct and stereotypical picture regarding mathematics and its teaching. With slight variations, they appear to see mathematics as a static body of knowledge, which the pupil is asked to 'conquer'. Their teaching actions suggest that they believe the teacher to be in possession of a specific number of clearly defined criteria, known to students, which are used to assess whether they have acquired the "right" mathematical knowledge. It is worth noting that two of the teachers (Aris and Sophocles) employed assessment practices both of an "informal" and "formal" type. This could be attributed either to their uncertainty and/ or possible confusion with respect to more contemporary practices of assessment, or to the fact that these teachers had come to realize how limited the formal means of student assessment are, and they were consciously trying to broaden their scope.

On the whole, the way the teachers appeared in the interviews did not vary significantly from the way they appeared in the teaching sessions. However, there are a few "shiftings" worth mentioning. In particular, two of the teachers (Christine and Orpheas) seemed to interpret assessment practices in a slightly more 'conservative' way compared to what they actually did in the classroom. An explanation of this could lie in the particular characteristics of these two teachers. Orpheas is the oldest of the teachers involved in the study, very confident, with some in-service training and a strong interest in local authority politics. Christine, on the other hand, is in her late thirties, with a fair amount of hours of in-service training, very enthusiastic and eager to learn, but with little time to spare, as she is very devoted to her family. They are both very articulate and have strong, on the whole traditional views about mathematics and its teaching and learning. Thus, it could be argued, the limited training opportunities of these two teachers in conjunction with their strong personalities did not allow them to essentially 'update their pedagogical discourse'. However, Christine's great interest in the educational process motivated her to explore alternative assessment practices in the classroom; this is less the case with Orpheas, who is not as flexible and seems to have lost interest in classroom matters. On the contrary, one teacher's pedagogical discourse (Notis) indicated a more relaxed 'reading' of the assessment practices compared to those he exercised in the classroom. It could be that in the interviews, the teachers, removed away from the reality of the classroom, might express more "open-minded" views and be more receptive to alternative practices compared to the ones they use in reality.

The preceded analysis offered an overview of the assessment practices employed or debated about by the teachers in this study. However, it does not allow for an appreciation of the particular features of these practices. To this purpose, and in order to increase the validity of the interpretations provided above, the data for each of the teachers involved in the study were analyzed at a more detailed level, thus providing a kind of profile for each of them. In the following, the profiles of two representatives of the sample are presented in short.

Aris

Teaching: For Aris, the assessment criteria focus on the individual performance of each student through frequent oral questions, mainly requiring recollection from memory. He asks his students to give "*mathematically logical answers*", "*to work as if they were mathematicians*", but does not explain to them what a mathematically logical answer means to him. Consequently, the students are not given the opportunity to familiarize themselves with the rules that would allow them to work and behave in ways the teacher expects them to. The assessment criteria he uses are often so indistinct that can only be seen through careful examination of his practices. Assessment takes place both orally with questions and in writing with activities.

Interview: Aris believes that he states directly to his students principles and assessment criteria, and he puts greater emphasis on oral assessment. He argues that there are times when the student's ability to use language appropriately interferes in the assessment of the mathematical knowledge the student has acquired. Written assessment is considered formal but he regards the oral form as more effective. According to Aris, the level of understanding can be detected in the students' answers. The announcement of the assessment results in class is done verbally and concerns mainly each student's performance. Aris states that he avoids announcing indirectly the assessment results to his students but does not deny the fact that this may happen at times. In his assessment practice outside the classroom, he makes comparisons between students, while he takes into account other elements such as the effort the student makes, his participation in the lesson etc. Finally, he believes that mathematical knowledge cannot be measured.

Orpheas

Teaching: For Orpheas, assessment is carried out both orally and in writing. Written assessment includes exercises similar to those found in the textbooks. This confirms his view – articulated in many ways in his interview - that revision plays a decisive role in the successful acquisition of mathematical knowledge.

Interview: Orpheas believes that assessment measures the students' ability to handle and apply the things they have been taught. He considers formative assessment to be more important for students. He uses oral assessment during the lesson, asking questions to his students. He depends on the students' answers to assess how much they have understood. The students' behavior and attitudes influence Orpheas to a great extent when he starts to get to know his students, but this impact becomes less as he gets to know them better. He considers written assessment more important and appropriate, especially for mathematics. He insists on the importance of the written assessment because he believes that the student has time to think in order to answer, and his answer is considered evidence, while in oral assessment, the student has to think very fast. Summarizing, Orpheas shapes the content of the mathematical knowledge his students are asked to acquire and he, himself, states to what degree they have understood. Consequently, he is in a position to point out which of his students' work is mathematically correct. Finally, he believes that assessment can measure the mathematical knowledge that has been grasped by the pupils.

CONCLUDING REMARKS

The analysis of the data showed that the assessment practices employed by the teachers of the sample in the classroom did not vary significantly from teacher to teacher, and reflected rather conventional views: mathematical knowledge is difficult; students are expected to either simply reproduce it or to acquire it, and whatever the case, the results can be measured. To assess the level of mathematics understanding achieved by the pupils, the teachers frequently but informally used oral questioning. Written assessment was less popular but considered more formal in character. In all cases, tasks requiring mere recollection from memory or straightforward application of mathematical knowledge dominated. The assessment criteria were never clearly articulated to the students and the results of the assessment were poorly communicated to individual pupils. As a result, students were uncertain as to the value of these results, and tended to either eventually ignore them or to interpret them as a way of comparing themselves to their fellow students. Furthermore, there were teachers whose assessment practices were not always consistent to one another, indicating confusion and / or uncertainty with respect to their stand to this.

This picture did not change very much in the interviews, thereby underlining its stability and power. The teachers' pedagogical discourse concerning assessment practices was more or less compatible with the actual practices they exhibited in their teaching sessions. The explanation of this situation should be sought in the past (and current) pre-service and in-service training systems, especially with respect to mathematics, as well as in the wider educational and social contexts. These systems appear to be failing in helping teachers to develop a clear and deep understanding of theoretical and empirical issues concerning the subject matter and its teaching and learning. As a result, teachers become insecure in handling and articulating a well-structured pedagogical discourse that would assist them to activate alternative frameworks of interpretation for the related educational phenomena, and develop alternative practices in the classroom. This is predominately the case with assessment practices, as they have immediate and often overwhelming effects on the pupils' lives and future careers. The strict, centralized, often oppressive and impoverished educational and social frameworks within which today's schools operate, stabilize and even worsen this reality. It is clear that more research on the ways in which teachers' knowledge and conceptions of mathematics, its teaching and learning are formed and influence the development of flexible and effective practices in the classroom is needed.

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INFLUENTIAL ASPECTS OF DYNAMIC GEOMETRY ACTIVITIES IN THE CONSTRUCTION OF PROOFS

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We present results from a study which investigated the influence of dynamic geometry-based activities in the development of proving skills (in geometry) in high-school students (15-16 years of age). After a 12-week course on Cabri and the writing of conjectures and proofs, students were asked to write and prove conjectures based on their observations in six Cabri-based activities. We analyzed the written data in the light of Balacheff's work (1987, 1999). Although progress is shown on the level of the development of knowledge, in that the objects and their relationships become more meaningful, there are still many difficulties in decontextualizing the activities and in the development of a functional language, necessary for the passage from pragmatic to intellectual tools.

INTRODUCTION AND THEORETICAL FRAMEWORK

For several years we have been investigating the role that new technologies play in the stages of the development of a proof in mathematics education. The use of technological tools bring the possibility for different types of conceptualizations of mathematical objects which may help or hinder the processes involved in the development of proofs. Most of the research regarding the use and influence of technological tools in proving has been done with regards to dynamic geometry environments (Hoyles & Jones, 1998; Balacheff, 1999; de Villiers, 1998, 2002; Jones, 2000), and in this paper we will present data from a study involving dynamic geometry-based activities for the construction of geometrical proofs; however, it is worth noting that it not all the research is exclusive to geometry or to that tool (e.g. Sacristán & Sánchez, 2002).

Some researchers have called for using proofs to create a meaningful experience; that is, as a means to help students understand why results are true (Hanna,1998), by providing an opportunity for exploration, discovery, conjecturing, refuting, reformulating, explaining, etc. which technological tools such as dynamic geometry environments can provide (De Villiers, 2002). Computer tools can be used to gain conviction through visualization or empirical verification, but as De Villiers (2002) points out, proofs have multiple functions that go beyond mere verification and that can also be developed in computer environments: such as *explanation* (providing insight into why it is true), *discovery* (the discovery or invention of new results), *communication* (the negotiation of meaning), *intellectual challenge* (the self-realization/fulfillment derived from constructing a proof), *systematization* (the organization of various results into a deductive system of axioms, concepts and theorems).

Balacheff (1987, 1999), in particular, has introduced certain ideas that we consider important for our analysis. First of all, he distinguishes between the terms *explanation,* and two terms which are generally both translated as *proof* into English: the French terms *preuve* (proof) and *démonstration* (mathematical proof) to . For him, an explanation is a discourse that aims to clarify the true makeup of a proposition or result. *Preuve* (proof) is an explanation that can be accepted by the community and which leads to the existence of a common system of validation. Finally the *démonstration,* or mathematical proof, is an accepted proof by the community of mathematicians (Balacheff, 1987). Balacheff has also proposed a distinction between *pragmatic proofs* and *intellectual (*or *conceptual) proofs*; emphasizing the role of language in the passage from the former to the latter. Pragmatic proofs are those based on effective action carried out on the representations of mathematical objects. They lead to practical knowledge that the subject can use to establish the validity of a proposition. Intellectual proofs demand that such knowledge is reflected upon, and their production necessarily requires the use of language that expresses (detached from the actions) the objects, their properties and their relationships. In other words, pragmatic proofs are based on action, while the use of a functional language (which includes a specific vocabulary and symbolism) and a "mental experience" (where actions are interiorized) characterize the transition to the intellectual ones. The transition from pragmatic proofs to intellectual proofs culminating in mathematical proof, involves three components: the *knowledge* or *levels-of-action component* (the nature of knowledge: knowledge in terms of practices — "*savoir-faire";* knowledge as object; and theoretical knowledge); the *language* or *formulation component* (ostentation, familiar language, functional language, formal language); and the *validation component* (the types of rationale underlying the produced proofs: from pragmatic, to intellectual, to mathematical proofs).

The development of a functional language involves processes of "detemporalization" and "decontextualization". That is, it should be a language for talking about mathematical objects and for communicating ideas related to them, independently of the situation, school context, or of the persons with whom the communication takes place (e.g. the teacher).

In the next section, we present results from a teaching experiment using Cabri-Géomètre, that is part of our research in investigating the role that dynamic geometry environments can have in the passage from pragmatic to intellectual proofs in geometry.

A TEACHING EXPERIMENT USING CABRI-GÉOMETRE

The aim of the research presented here, was to investigate alternative types of activities, in this case using dynamic geometry software, for improving proving skills in geometry in high-school students. (It is worth noting, however, that in two other parts of our research we have also investigated the same activities with teachers, and a group of students different than the one reported here). To select our subjects, we applied a diagnostic questionnaire, based on the questionnaire devised by Healy & Hoyles (1998), to a group of 40 Mexican high-school students of ages 15-16 years, in order to evaluate their understandings of aspects linked with mathematical proofs. We marked the correct answers in order to get a general evaluation of each student according to the number of right answers. The 8 top students from the results (which in this case coincided with the students that their teacher considered the best in mathematics) then participated in an experimental 35-hour course consisting of 3 hour-sessions once a week. These students, as in general students of this educational level in Mexico, have studied some themes of Euclidean geometry, but not a full course.

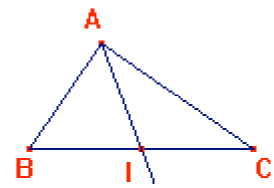
The aims of the course were to instruct students in three aspects: (i) the use of the Cabri-Géomètre software (knowledge of commands, construction of objects, etc.); (ii) a review of basic geometrical knowledge (simple propositions on parallel lines, angles and triangles); and particularly (iii) the writing of *conjectures* (with an emphasis of having students recognize that a conjecture is a proposition formed by two parts: an antecedent and a consequent) aiming to have students learn to express explanations, verifications and, if possible, proofs of geometrical theorems and properties. We are aware of concerns that explorations in dynamic geometry environments, where conjectures are confirmed by testing the figures through dragging may reduce the perceived need in students for deductive proof (Chazan, 1993; Hanna, 1998; Hoyles & Jones, 1998), and that is one of several reasons why we placed emphasis on the third aspect of the instruction process.

The didactical design of the course consisted in activities similar to those reported in this paper: the teacher explained how to construct objects in Cabri, let the students explore them freely, then there was a joint discussion on the writing and proving of conjectures derived from the activity. At the end of each session, students were given an activity to carry out on their own. During the last session, students were asked to work on six activities, and for each one write and prove a conjecture.

Each of the activities (all related to medians and areas of triangles), was meant to lead into a specific proposition. These propositions constitute a system in the sense that they can be deduced from previous propositions. The background needed to construct the proofs was simply the formula for the area of a triangle (base x height /2) and propositions constructed or derived from the system of six activities. In this paper we will only talk of the first two activities as they are representative of all the others.

Activity 1

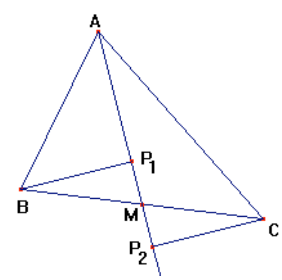
• Draw a triangle, label the vertexes with A, B, C. • Mark the midpoint of segment BC; call it I. • Define the triangles ∆AIB and ∆ACI. • Get the areas of the triangles ∆AIB and ∆ACI. • Grab any vertex A, B, or C and drag it to new position. Observe what occurs with the areas. • Write a *conjecture* about your observations and write the corresponding proof.



Activity 1 was designed to lead to the construction of a proposition related to the theorem: *If in the triangle ∆ABC, AM is a median, where M is the midpoint of BC, then Area (∆AMB)=Area (∆ACM).*

Activity 2

• Draw a triangle ∆ABC (label the vertexes). • Mark the midpoint of segment BC (call it M). • Draw the ray AM • Draw the perpendicular line to AM that passes through the vertex B • Define the point of intersection between AM and the above perpendicular; call it P_1. • Draw the perpendicular line to AM that passes through the vertex C • Define the point of intersection between AM and the above perpendicular; call it P_2. • Hide the perpendicular lines BP_1 and CP_2 • Draw the segments BP_1 y CP_2 • Get the length of the segments BP_1 y CP_2 • Drag the vertexes A, B, or C and observe what happens with the lengths of the above segments • Write and prove the *conjecture* about your observations.



Activity 2 was designed to lead to the construction of a proposition related to the theorem: *If in a triangle ∆ABC, AM is a median, then the distance from B to AM is the same as the distance from C to AM.*

SAMPLE DATA AND INTERPRETATION OF SOME RESULTS

We used the written propositions of the students as a window to get insights into their abilities to understand the fundamental ideas involved in the proof, as well as evaluating their skills to formulate and write conjectures and proofs. From a detailed analysis of the propositions written by the students we deduce that, on the one hand, through the exploration activities in the Cabri environment, students do develop a certain "phenomenological" knowledge of the elements and notions involved; this shows some progress on the level of the knowledge component. On the other hand, the written data also shows the difficulties that students have in expressing the propositions; this suggests little progress on the level of the language component. Most of the "proofs" that students gave were really *explanations*, since few of them could be classified on the level of intellectual proofs; all the others would be classified as pragmatic proofs.

The first example is from a student called Alexandra:

> **Alexandra's Proposition - Activity 1**: In every triangle, when drawing the [median[1]] the height and the bases will measure the same and therefore will have the same area. Since we found the midpoint for BC, the two bases will measure the same and because the height is perpendicular to the base, both triangles will have the same height.

Anyone unfamiliar with the activity would be unable to understand it. There is an absence of a reference to the two triangles that get formed when sketching the median of a triangle; this makes the statements "the height and the bases will measure the same and therefore will have the same area" very confusing. She also uses the symbols (she uses the reference BC for the base of the triangle) referred to in the activity, but doesn't properly introduce them in her proposition. There are similar problems in her proposition for the second activity:

> **Alexandra's Proposition - Activity 2**: In a triangle where we draw the [median[2]], the perpendicular segments to it will measure the same if they are the height of the triangles that are being formed. In the triangle ABM the segment BP_1 is its height because it is perpendicular to the base in the same way for the triangle AMC.

This second proposition is even more confusing than the first: it mentions the segments perpendicular to the median, but it doesn't specify that they each pass through one of the two remaining vertexes. Nevertheless, in spite of the problems with the written proposition, it is worth noting that Alexandra was one of only two students who was able to link the result of the first activity with this second one: in her "proof" she seems to be using the congruence of the area of the two triangles that are formed with the median. In a previous study to the one reported here, none of the students linked the two activities in this way. Independently of that, it is clear from her written work that she has a good understanding of the content of the proposition she makes, and although her statement is

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[1] The quote has been translated from the original Spanish. In Spanish, Alexandra used the term *mediatriz* which is the perpendicular bisector, but she was referring to the median because that is what she constructed in the activity. Nevertheless, her mistake shows a confusion (very common due to the similarity) between the terms *mediana* and *mediatriz* (perpendicular bisector).

[2] As in the previous activity, Alexandra used the term *mediatriz* to refer to the median.

faulty, she does present the correct elements for the proof. Like Alexandra, most students were able to identify the elements and relationships needed for the proof, but they were unable to properly articulate them in their written propositions; nevertheless this is important in making the activities meaningful (progress on the level of the knowledge component).

In Alexandra's case there is also an attempt to use symbolic references and to generalize her propositions ("in every triangle"). However, although attempts to generalize were found in most students, the use of symbolic notations wasn't as common or was misused (see further below), and students very often relied on familiar language.

In fact, students in general didn't pay much attention to be rigorous in the writing of the conjectures they observed. They centered more on the explanation and "proving" than on clarifying the propositions they discovered through the Cabri-based activities. This is in spite of the fact that during the instruction phase the emphasis was put on the writing of the conjectures. In the written work from Activity 1: only three of the eight students made any attempts of writing down the proposition, and only one of those three cases can be considered almost correct (there was no reference to the original triangle). All the other students wrote directly an explanation or a proof. In the second activity the situation is even worse: only one student (Alexandra) tried to give a complete proposition with the problems already discussed; three other students only gave the conclusion without stating the conditions and hypothesis of the proposition; all of the other students directly wrote an explanation or proof. It is also worth noting that most of the students (see sample data) write both their proposition and their proof in a single paragraph, again indicating that their focus was more on explaining than on writing a formal proof.

> **Zaira's Proposition - Activity 2**: The length of P_1 is equal to P_2, because both are perpendiculars to AM with respect to a point, but M is the midpoint of the triangle and because of that it is the same measure that goes from M to B than the one from M to C.

Before the first coma, she tries to state the "result" (that the segments are equal), followed by a failed attempt to prove it. We suggest two reasons why students avoid writing their conjectures:

- The difficulties in describing and articulating the geometric elements that come into play
- The school setting: i.e. the habit of having the activity given by the teacher and knowing that "he/she knows what is being talked about." It is therefore enough to just give some indication that the activity was carried out and understood.

An aspect which points to the difficulty in the appropriation of a functional language is the fact that Zaira uses the symbols P_1 y P_2 in reference to the segments BP_1 and CP_2 (Barbara does the same); this suggests that Zaira sees the symbols only as labels that can simplify the writing ignoring how they can refer to the structure of an object and its properties. (In this sense, another student used MP not in reference to a segment MP in the figure, but simply as an abbreviation for the term midpoint).

However, it is important to mention that there may also be a cultural dimension to the difficulties that students have in the writing and formulating of the propositions. General writing difficulties have been found almost everywhere in the world, and we shouldn't be surprised that even more difficulties arise in the writing of conjectures and proofs. It is evident from the written work of the students as described above, that these students are not used to expressing mathematical ideas (whether formally or informally) despite the emphasis given to that in the teaching experiment. But it is important to note that in the Mexican Mathematics School Curriculum there are no explicit objectives to teach proof. Furthermore, although we have yet to substantiate the following statement, we have observed a general lack of encouragement in schools for students to express mathematical ideas with rigor and formality: the emphasis tends to be simply on producing the correct solution. Thus the above deficiencies are even less surprising, and cultural aspects may enhance this problem.

Problems of decontextualization

In the theoretical framework we mentioned the importance of decontextualization for the development of a functional language. We found problems of decontextualization in students in two ways: first, as mentioned above, in that they assume that the reader of their propositions (the teacher) knows what they are talking about. There is another level of “situatedness” which is more specific to the dynamic geometry environment. Take, for example, Barbara’s proposition:

> **Barbara’s Proposition - Activity 1**: The triangle has the same base and the same height and because it is divided in two by the median, thus when I move one of the points of the triangle they always have the same measure for their area.

It is worth noting that as the problems become more complex, references to actions carried out with the software (e.g. “if I drag…”, “when I move…”) become more and more common. This could be interpreted that as the problem is more complex, the more difficulties students have in the decontextualization. We dare say that these “situated” propositions by the students are akin to what Noss & Hoyles (1996) have called “situated abstractions” where the generalizations that occur are still in terms of the actions and the situation in which they arise. In terms of Balacheff’s theories, we would say that they are still on the level of pragmatic proofs.

The construction of meaning through the DGE activities

Nevertheless, we find that the use of the dynamic geometry software, helped students make progress on the level of the knowledge component. Students seem to grasp the meaning of some of the theorems thanks to the phenomenological approach that the use of Cabri makes possible. There is a fundamental difference in the construction of the geometrical figure between doing it with paper-and-pencil and doing it in a dynamic geometry environment: whereas in the first one it is the construction of a particular case, in the latter one it is actually the construction of a “general case”. The construction in Cabri also requires the definition of structural elements (e.g. the midpoint is intrinsically defined as the midpoint); failure to do so will result in the figure “falling apart” through dragging. In the first activity, for example, by dragging and experimenting they can observe on the screen that the areas of the two triangles that get formed when dividing a triangle with a median, are always equal: this gives students a clear reference of what the proposition is about. When they are asked for a proof, they look for an explanation and they are generally able to observe the necessary elements that are needed for the proof, suggesting the fundamental idea of the proof.

In terms of the theoretical framework and Balacheff's ideas, we can say that the Cabri-based activities seem to favor the knowledge component, while the development on the level of language and formulation remains a difficult area. The benefit of the software seems to be in that the construction activities produce appropriate meanings for the objects and their relationships, but progress on the level of the validation component depends on the other two components: both the knowledge and the language ones.

The written work that we have analyzed in particular in this paper, reveal the ways in which some of the contextual elements involved in the implementation of the tasks are difficult to abstract in order to produce an approximation to, not even a written proof, but also to a modest written mathematical explanation of the geometrical propositions. The students are still far from being able to produce written texts explaining the results observed in an activity but that are independent of the situation of the activity.

The role of language is crucial in the intellectual work involved in geometry and our research points to its complex nature. Geometrical activities involve two types of treatments of the geometrical objects: the visual treatment and the discursive treatment. The software favours the visual treatment, since it helps avoid the discursive treatment and use of symbolic language; this in turn, by permitting an immediate access to "concrete forms of abstract entities" (Laborde & Laborde, 1995, p. 243), helps develop a phenomenological knowledge of the mathematical objects in a much faster way that in paper-and-pencil activities.

But, on the other hand, the theoretical knowledge of geometry, that which supports a mathematical proof, is developed in the transition between pragmatic and intellectual proofs, and demands the development of a functional language. It seems that the software-based explorations, without complementary activities specific to the development of language, do not provide many opportunities for the development of first a descriptive language and then a functional one.

We consider this one of the limitations of dynamic geometry environments where objects are controlled through the mouse, as opposed to other computer tools, such as Logo, where the objects are controlled through the symbolic notation of the programming code.

Finally, although a profound discussion of this issue is beyond the possibilities of this paper, it is important to mention a further problem that we have identified in the use of Cabri. We find that an important activity in the interpretation of a figure in order to discover a conjecture and deduce its proof, is that which consists in the distinction and regrouping of elements or subfigures of a given geometrical figure; what Duval (1995) calls *reconfiguration*. Because of the sequential way in which figures are constructed in Cabri, the process of reconfiguration is not facilitated, as we have observed in all our research subjects. We believe this may be a reason why most students did not link the propositions derived from one activity with the others, as Alexandra did (see above).

Despite the limitating influences of the use of Cabri in the development of functional language and in facilitating reconfiguration, we believe that the advantages in creating meaningful knowledge provide an important foundation for the passage and understanding of mathematical proofs and deductive reasoning. And further research may help develop complementary activities to assist students overcome those limitations.

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STUDENTS' USE OF TECHNOLOGY IN MATHEMATICAL PROBLEM SOLVING: TRANSFORMING TECHNOLOGICAL ARTIFACTS INTO MATHEMATICAL TOOLS

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Can textbook exercises be transformed into problem solving activities that encourage students to develop mathematical thinking? This study documents what high school students showed when they were explicitly asked to use technological tools to examine and solve a routine problem from different angles or perspectives. In this process, students dealt with several representations that were important to analyze and quantify concepts of variation or change attached to the problem, in terms of models of solutions.

When does a technological artifact become a mathematical problem-solving tool for students? What process of appropriation do students go through in order to use such tool in mathematical practice? What type of mathematical resources and strategies do students need in order to transform the use technological artifacts into mathematical tools? What process do students take in order to meaningfully employ those tools in problem solving activities? What types of mathematical representations are enhanced through the use of technological tools? These are important questions that we use as a guide or reference to evaluate the potential of technological tools in students learning of mathematics. In particular, we are interested in documenting what high school students exhibit when are asked to systematically use dynamic software, excel, and symbolic calculators in problem solving activities. In this context, we recognize that there are multiple ways in which technology can be employed by students and we are interested in characterizing those ways in which technology becomes a powerful tool for students to identify and explore conjectures, to quantify and analyze (graphically or numerically) particular phenomena, and to identify patterns or relationships through the analysis of distinct representations. In this study a routine problem that often appears in first calculus course is used to identify and analyze different types of models that students construct to solve the problem. In this process, the use of technology becomes a powerful tool for students to represent and examine relationships through the use of resources and concepts that appear traditionally in domains such as algebra, geometry, functions, and trigonometry.

CONCEPTUAL FRAMEWORK

There are different learning trajectories for students to take in order to achieve mathematical competence; however, a common ingredient is a need to develop a clear disposition toward the study of the discipline. Such a disposition includes a way of thinking in which students value: (a) the importance of searching for relationships among different elements or components of the tasks in study (expressed via mathematical resources), (b) the need to use diverse representations to examine patterns and conjectures, and (c) the importance of providing and communicating different arguments (Santos, 1998). Thus, it becomes important to encourage students to think of the discipline in terms of dilemmas or challenges to be met and resolved. This means that they need to conceptualize their learning experiences in terms of activities that involve posing questions, identifying and exploring relationships, and providing and supporting their answers or solutions (NCTM, 2000). It is necessary to value the students' participation and persuade them about the power of reflecting on what they do, in mathematical terms, during their interaction with tasks or mathematical content. "To be able to guide students' inquiry toward the learning of the mathematical content in the syllabus, teachers must first convince students that inquiry is a legitimate, safe, and productive way to learn in school" (Lampert, 1995, p. 215). Here, students' view of mathematics involves accepting that it is more than a fixed and static body of knowledge; it includes that they need to conceptualize the study of mathematics as an activity in which they participate actively in order to identify, explore, and communicate ideas attached to mathematical situations.

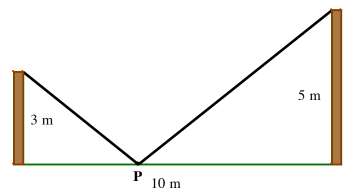
> …Students themselves become reflective about the activities they engage in while learning or solving problems. They develop relationships that may give meaning to a new idea, and they critically examine their existing knowledge by looking for new and more productive relationships. They come to view learning as problem solving in which the goal is to extend their knowledge (Carpenter & Lehrer, 1999, p. 23).

Students need to use different representational media to express their ways of thinking while dealing with tasks or problems. The students' constructions of powerful representational systems play an important role in developing distinct artifacts to understand and explain complex systems (Lesh, in press). The use of different tools offers students the possibility of examining situations from perspectives that involve the use of various concepts and resources. As a consequence, each representation might become a platform to identify and discuss mathematical qualities attached to the process of solution. Thus, during this process, a table might shed light on trends displayed by discrete data while an algebraic approach focuses on continuous behavior and general tendency (infinity). Geometric and dynamic approaches to the problem might provide a means for students to visualize and examine relationships that are part of the depth structure of the task. Specially, dynamic constructions help students focus their attention on common properties that appear while moving elements within the same configuration or representation. Lesh (in press) argues that "useful ways of thinking usually need to develop iterative and recursively, with input from people representing multiple perspectives". In this context, solving the task goes beyond reporting a particular solution, it is a process of constructing, investigating, representing, applying, interpreting, and evaluating several ways to solve the problem (Schoenfeld, 1998).

GENERAL PROCEDURES AND THE INITIAL PROBLEM

Twenty-four students (grade 12) worked on a series of tasks that included textbook problems (in addition to assignments of the course). Students met twice a week during 2.5 hours. In this report we document features of mathematical thinking that students exhibited while interacting with one problem. In general, each student had access to a computer (excel and dynamic software) and a calculator. In each session, students had opportunity to work individually, in small groups, and as a part of the whole group discussions. Some students had some experience in using the tools and often those students taught other students during and out of the regular sessions. The example used to illustrate the students' ways of thinking (models) involves a routine problem that regularly appears in calculus textbooks. Thus, models that students exhibit during their interaction with this task illustrate mathematical processes and content that appear when students use distinct representations to explore mathematical qualities attached to various methods of solutions. Ideas from arithmetic, algebra, geometry, and calculus emerge naturally as a means to analyze relationships that appear in each student's approach. It is important to mention that there is no intention to provide a detailed analysis of students' work, rather each students' approaches to the task is shown to highlights the type of representation used to solve the problem. In particular, attention is paid to the variety of ideas and strategies that emerged when students are encouraged to use different technological tools to represent and approach even routine exercises. Thus, the initial nature of the task can be transformed into sequences of students' mathematical explorations.

The Initial Problem[1]. The distance between two telephone poles is of 10 m as shown in the figure. The length of each pole is 3 and 5 meters respectively. To support the poles, a cable from the top of each pole will be tied to a point on the ground between the two poles. Where should that point be located in order to use the minimum length of cable?



Two poles and a cable

Students worked on this problem first individually and later as a part of a group of three students. When some small groups presented their approaches to the entire class, it was common that new ways to solve the task emerged from the class discussion and students had opportunity to rewrite their initial approaches. At the end of each session, the teacher directed the class to discuss advantages and limitations of what students had presented. So, in general, students became aware not only of the power of their own methods but also of the strengths of other students' approaches.

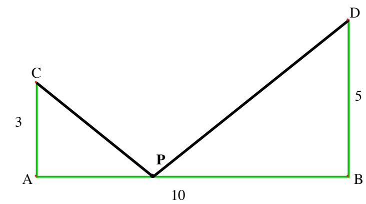
Students' construction of models to solve the problem

The term model is used to characterize ways in which students identify and employ ideas, concepts, representations, operations, and relationships to solve problems. So the construction of models is a process that involves constant exchange and refinements of students' ideas.

Students' initial interaction with the problem focused on identifying key ideas to detect particular relationships. Thus, understanding the task involved the introduction of a representation and notation that led students to discuss a set of questions.

[1] The general statement of this problem is known as Heron's Problem. It is often stated as "given two points on the same side of a line, find a point on that line such that the sum of its distances to the given points is minimal".

(i) An important mathematical idea embedded in this task is to recognize that the length of the cable varies when P is moved along the segment between the two poles. Here, it is also important to quantify that change. Students, in general, introduced particular notation that helped them identify key elements of the task.



Representation and a notation

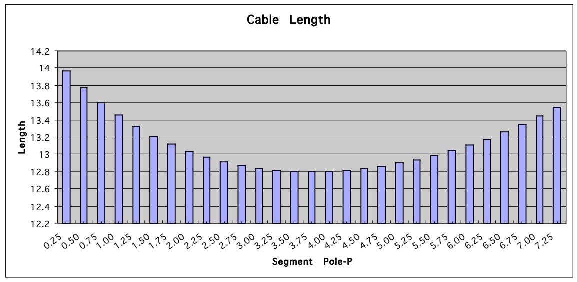
(ii) How can we know that the length of the cable change when point P is moved along the line between the two poles? How can we determine the distance between one pole and point P? What data do we have? Can we use the Pythagorean's Theorem? These were initial questions that helped students identify relevant information and explore relationships between the length of the cable and location of point P.

I. A Discrete Model. Some students focused on calculating particular cases that emerge when point P is moved along segment AB. Although they initially divided the segment of length 10 into two arbitrary segments, later they organized the lengths systematically into a table arrangement. AP represents the distance from the length of the shorter pole to point P; PB the length between point P and the other pole. P1 and P2 represent the lengths of the poles, D1 and D2 the corresponding hypotenuses and D1+D2 the length of the cable. A table that includes a refined partition of segment AB and the hypotenuses of the two triangles are shown next.

AP	PB	P1	P2	D1	D2	D1+D2
1	9	3	5	3.16227766	10.2956301	13.4579078
1.25	8.75	3	5	3.25	10.0778222	13.3278222
1.5	8.5	3	5	3.35410197	9.86154146	13.2156434
1.75	8.25	3	5	3.473111	9.64689069	13.1200017
2	8	3	5	3.60555128	9.43398113	13.0395324
2.25	7.75	3	5	3.75	9.22293337	12.9729334
2.5	7.5	3	5	3.90512484	9.01387819	12.919003
2.75	7.25	3	5	4.06970515	8.80695748	12.8766626
3	7	3	5	4.24264069	8.60232527	12.844966
3.25	6.75	3	5	4.4229515	8.40014881	12.8231003
3.5	6.5	3	5	4.60977223	8.20060973	12.810382
3.75	**6.25**	**3**	**5**	**4.80234318**	**8.0039053**	**12.8062485**
4	6	3	5	5	7.81024968	12.8102497
4.25	5.75	3	5	5.20216301	7.61987533	12.8220383

Students observed that the length of the cable decrease up to some value and then increases again. Here, they identified that when the point on is 3.75 from the point A, then the cable gets the minimum length.

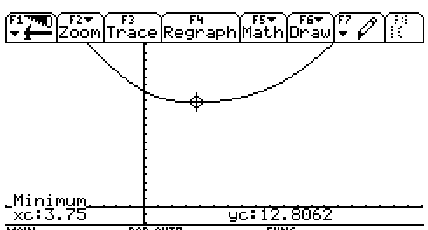
A Visual Representation. After students generated a table, they presented the corresponding graph. Here, they observed that when the point gets closer to one of the poles the length of the cable gets larger. Indeed, they observed that when the point was 3.75 the length of the cable reaches the minimum value. Here, students asked: When or under which conditions does the midpoint of the segment determine the minimum length of the cable? Students reported that when the lengths of the poles were the same then the midpoint of AB was the point at which the cable reaches its minimum length. Otherwise, the point will be located on the segment and close to the point with shorter length.



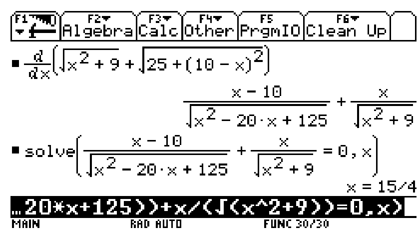
VISUAL APPROACH

The key ingredients of this model include the idea of analyzing particular cases, the process of refining the segment partition, the use of the Pythagorean theorem, the use of the tool (excel) and the visual representation of the data.

II. A Symbolic Model and the Use of a Calculator. A small group of students expressed the hypotenuse in each right triangle as $\sqrt{x^2+9}$ and $\sqrt{25+(10 - x)^2}$ respectively and the length of the cable as the sum of these two expressions. That is, $l(x) = \sqrt{x^2+9} + \sqrt{25+(10 - x)^2}$. How can we find the minimum value of $l(x)$ in this expression? Can we graph this function? With the help of a calculator, some students graphed the function $l(x)$ and identified the value in which the minimum value is reached. Other students, who decided to follow algebraic procedures to find the minimum value of $l(x)$, realized that they could contrast their results with those obtained through the use of the symbolic calculator. So the calculator functioned as a monitor of the students' work.



Graphic representation

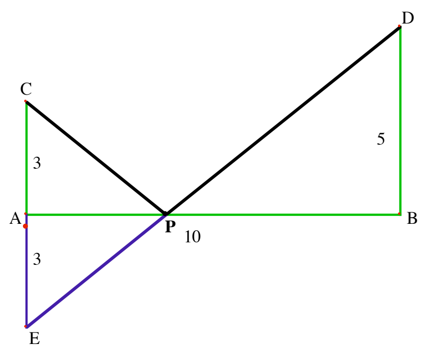


Symbolic approach

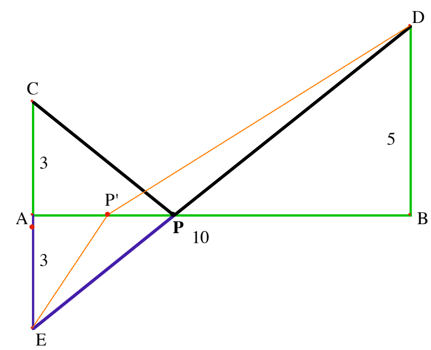
In this model, students focused on representing the function that related the position of the point to the length of the cable. Here, the fundamental components of this model include the use of particular notation, algebraic procedures, and establishing connections between the graph representation and the problem. The calculator had a dual purpose: to graph the function and identify the solution and to verify algebraic operations (derivative, roots of equation).

Iii.a geometric model. Another method suggested by the instructor was to examine the case in which one pole was reflected on its vertical line with respect to segment that joins the two poles (figure below). They observed that angles APC and APE are congruent.

Students recognized that in this case the segment ed that intersects AB at P is the minimum length of the cable. They argued that any other point different p' will generate a triangle EP'D in which the sum of EP' and P'D will be always longer than ED (figure below). The argument was based on using the triangle inequality. That is, they showed that in ΔEP'D, EP' P'D ED.



Reflection method



Justification of the solution

To find the distance AP, Some students recognized that triangles APE and BPD were similar. Therefore, the corresponding sides held proportionality, that is, $\frac{x}{3} = \frac{10 - x}{5}$ which led to $x = \frac{15}{4}$.

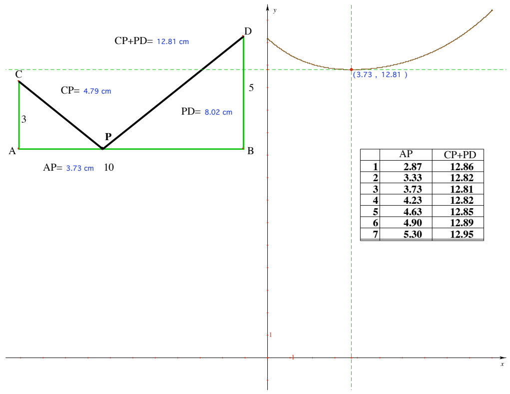
Slope Approach. Some students also realized that the minimum distance of the cable is obtained when the slopes of the two lines CP and PD are the same but with opposite sign. That is, when the angles APC and BPD are congruent. Here, they introduced a coordinate system with A as its origin point. Thus, they calculated the slopes of the line that passes by (0, 3) and (x, 0) and the line that passes by (x, 0) and point (10, 5).

$m_1 = \frac{-3}{x}$ and $m_2 = \frac{5}{10 - x}$; to hold the condition students observed that:

$$\frac{-3}{x} = -\left(\frac{5}{10 - x}\right) \Rightarrow -30 + 3x = -5x \Rightarrow x = \frac{15}{4}$$

The components attached to this model involve the use of properties of triangles (congruence and similarity) to identify the solution. Supporting the solution was also a key part of this model. In addition, the use of a coordinate system played an important role to introduce basic ideas of analytical geometry.

IV. A Dynamic Model. Yet another approach that students showed while dealing with this problem was to represent the problem through the use of dynamic software. This software allowed students to determine graphically the relationship between the distance AP and the length of the cable (CP + PD). Here, by moving point P along segment AB a graph of the behavior of the length is generated. It is also important to mention that a table including some values of distance CP and the corresponding value of CP+PD can also be obtained. In this case what students reported was an approximation of the minimum length of the cable, that is, 12.81. In addition, students could drag basic parameters and generalize their results (for example, varying distance between poles or poles lengths).



	AP	CP+PD
1	2.87	12.86
2	3.33	12.82
3	3.73	12.81
4	4.23	12.82
5	4.63	12.85
6	4.90	12.89
7	5.30	12.95

A dynamic representation

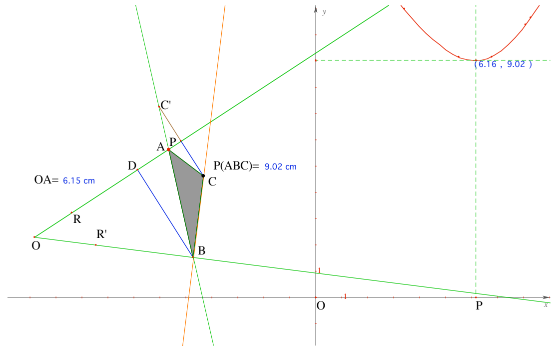
A key component of this model was to represent dynamically the relationship between the point of the segment and the length of the cable. The use of a coordinate system to show the graphic representation of that relationship was also an important ingredient. In addition, students in this model could explore easily other cases in which they change parts of the initial representation (lengths of poles and segment).

Students had the opportunity to discuss advantages and disadvantages attached to each model. In particular, they noticed that geometric and dynamic models did not involve algebraic procedures to find the solution. To close the session, the teacher posed the following related problem:

> Let C be a given point in the interior of a given angle. Find points A and B on the sides of the angle such that the perimeter of the triangle ABC is a minimum.

Students' first approach was to represent the problem with the use of dynamic software. In using the software it was also important to introduce a particular notation (figure below). Let OR and OR' rays with a common point O and C a point on the interior of angle ROR'. Thus, their first goal was to find the minimum length from point C to any of the angle side. They drew segment CB from point C and perpendicular to ray OR'. Now, from C they drew a perpendicular line to OR and from B a perpendicular segment BD perpendicular to OR. They recognized that segment DQ and segments QC and DB represented an *analogous case* to the original problem. That is, the objective was to identify a point on DQ such that BA + AC was minimum. Using that the line joining BC' (CQ = QC') intersects DB at P. and this point determines the minimum distance. Therefore, the triangle with the minimum perimeter is triangle CPB.

An extension: An angle, an interior point and a triangle with minimum perimeter, and graph showing an approximate solution using dynamic software.



REMARKS

When students openly search for various approaches while working on mathematical tasks it is common to identify different types of representations that help them examine and use different problem solving resources and strategies. Some tasks or problems that often appear in regular textbooks can be taken as platform to engage students in mathematical practice. In particular, the use of technology became a powerful tool to explore properties and relationships that did not appear in paper and pencil approaches. It was evident, that students' ideas about solving routine problems get enhanced when explicitly they search for various ways to represent and solve the tasks. That its, routine problems are seen as a means to encourage students to extend and reflect on their mathematical thinking. Thus, teachers might use initially some of their textbook problems as a way to engage students in the search of powerful representations to elicit and refine their previous mathematical ideas. These ideas eventually are transformed in models that are useful to solve problems.

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ALGEBRA IN ELEMENTARY SCHOOL[1]

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Increasing numbers of mathematics educators, policy makers, and researchers believe that algebra should become part of the elementary education curriculum. Such endorsements require careful research. This paper presents the general results of a longitudinal classroom investigation of children's thinking and representations over two and a half years, as they participate in Early Algebra activities. Results show that 3rd and 4th grade students are capable of learning and understanding elementary algebraic ideas and representations as an integral part of the early mathematics curriculum.

BACKGROUND AND EARLY RESEARCH QUESTIONS

Early research about algebraic reasoning highlighted shortcomings such as students' (1) limited interpretations of the equals-sign (Booth, 1984, 1988; Kieran, 1981, 1985; Vergnaud, 1985); (2) misconceptions about the meaning of letters standing for variables (Kieran, 1985; Kuchemann, 1981; Vergnaud, 1985); (3) refusal to accept an expression such as "3a +7" as an answer to a problem (Sfard & Linchevski, 1994); and (4) difficulty in solving equations with variables on both sides of the equals sign (Filloy & Rojano, 1989; Herscovics & Linchevski, 1994). Many researchers originally attributed such findings to developmental constraints and the inherent abstractness of algebra.

Over time, however, data from innovative classroom activities began to support a new view. Davis (1985, 1989) gave convincing examples of how algebra could be introduced in 5th grade mathematics classrooms. Successful work in the former Soviet Union (Davydov, 1991, Bodanskii, 1991), with even younger children, came to the attention of researchers. Mathematics educators began to find common ground between arithmetic and algebra (Bass, 1998; Carpenter & Franke, 2001; Carpenter & Levi, 2000; Carraher, Schliemann, & Brizuela, 2000, 2001, 2003; Davis, 1985, 1989; Kaput & Blanton, 2001; Schifter, 1999; Schoenfeld, 1995; Schwartz, 1995). Approaching the introduction of algebra in the early grades from various, occasionally overlapping, perspectives (generalizing arithmetic, moving from particular to generalized numbers, focusing on mathematical structures common to sets of algorithms, introducing variables and co-variation in word problems, focusing on the concept of *function*, to tie together isolated mathematics topics, etc.), they identified previously overlooked opportunities to explore the algebraic character of early mathematics. These recent studies suggest that shortcomings of instruction may have had a decisive role in the gloomy results from early

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studies of algebraic reasoning among adolescents (Booth, 1988; Schliemann & Carraher, 2002).

In this paper we describe the general results of a longitudinal study we developed in 2nd to 4th grade classrooms. We will argue that, given the proper conditions and activities, elementary school children can reason algebraically and meaningfully use the representational tools of algebra. We hope that our empirical research on Early Algebra will help make advances regarding issues that most researchers in mathematics education recognize as important, such as development versus learning, the role of contexts, and the role of representational systems.

OUR APPROACH

Our approach to the introduction of algebraic concepts and notations in elementary school was guided by the following ideas about learning:

> (a) Cognitive deficits and cognitive difficulties with algebra may result from the limitations of the mathematics curriculum elementary school children have access to; (b) Mathematical understanding is an individual construction that is transformed and expanded through social interaction, experience in multiple meaningful contexts, and access to mathematical symbolic systems and tools; and (c) Children need to be socialized into the symbolic systems but also need to make them their own. To this goal, students benefit from opportunities to begin with their own intuitive representations and gradually adopt conventional representations as tools for representing and for understanding mathematical relations.

Furthermore, it rests on the following ideas about mathematics:

A. Concerning arithmetic, algebra, and their interrelations

> (a) Opportunities to explore the algebraic character of elementary mathematics are present throughout existing curricula, though rarely seized upon; (b) Algebra is both a notational system and a field of mathematics devoted to the study of mathematical structures; (c) Arithmetic is a part of algebra, namely, the part that deals with number systems, the number line, numerical functions, and so on; (d) generalizing lies at the heart of algebraic reasoning; and (e) Arithmetical operations can be viewed as functions.

B. Concerning symbolic representation

> (a) Mathematical concepts (algebra included) are closely identified with four *key symbolic systems* (natural language, number, geometry, and algebraic-symbolic notation); each of these has important roles to play already in early mathematics education; (b) Each of these systems has its own expressive rules and internal logic; and (c) A central problem in mathematics consists in moving back and forth between diverse representations, often across these key symbolic systems.

Our approach focuses on algebra as a generalized arithmetic of numbers and quantities. This highlights the shift from thinking about relations among particular numbers and measures toward thinking about relations among sets of numbers and measures, from computing numerical answers to describing and representing relations among variables. This requires engaging students in specially designed activities, so that they can begin to note, articulate, and represent the general patterns they see among variables.

In our longitudinal study, we worked with 70 students in four classrooms (three mainstream and one bilingual education classroom), as they learned about algebraic relations and notation, from grade 2 to 4. Students were from a multiethnic community (75% Latino) in Greater Boston. From the beginning of their 2nd semester in 2nd grade to the end of their 4th grade, we implemented and analyzed weekly activities in their classrooms. Each semester, students participated in 6 to 8 activities, each activity lasting for about 90 minutes. The activities related to addition, subtraction, multiplication, division, fractions, ratio, proportion, and negative numbers. The project documented, in the classroom and in interviews, how the students worked with variables, functions, positive and negative numbers, algebraic notation, function tables, graphs, and equations.

In our classroom work we generally began not with a polished mathematical product, such as a number sentence or a graph, but rather with an open-ended problem or a statement such as "Maria has twice as much money as Fred." After holding an initial discussion about the situation, we ask students to express their ideas in writing. We discuss their representations and introduce a conventional representation, such as two parallel number lines drawn on the floor (on which Maria and Fred's amounts correspond to positions); the new convention is often foreshadowed in students' own drawings. We may ask them to show how the representations (their own and the conventional one) must be updated to account for new information, such as amounts changing while some properties remain invariant (the relation "twice as much" continues to hold). In the specific case of a Cartesian coordinate graph and of the problem involving Fred and Maria, we would then rotate one of the number lines by 90 degrees and ask the students to plot themselves as points at the intersections of ordered pairs for Maria and Fred's amounts. A "Human Graph" is thus built and becomes the object of discussions. We explore the relations between the mathematical representation and the problem-situation in terms of correspondences between the graph and the underlying situation. What do you notice about the way in which students/points are patterned? Why are they falling on a straight line? (When each point is expressed through coordinate notation) what do you notice about the relation between the two numbers? If I choose a position on Fred's line/axis, can you predict where the Fred-Maria point will land? If Fred has a really large amount of money can you say anything about where the Fred-Maria point will be located? (See Schliemann & Carraher, 2002; Schliemann, Goodrow, & Lara-Roth, 2001a; and our website--earlyalgebra.terc.edu--for detailed information and partial results of our Early Algebra activities).

RESULTS

Partial results of our longitudinal study have been reported in detail at PME meetings and published elsewhere. Here is a general description of our analysis of classroom activities:

- Young students (9-10 years of age) can learn to think of arithmetical operations as functions rather than merely as computations on particular numbers (Carraher, Schliemann, & Brizuela, 2003; Schliemann & Carraher, 2002; Schliemann, Carraher, & Brizuela, 2003). They can also work with mapping notation, such as $n \rightarrow 2n - 1$, and realize that the algebraic expression constitutes a rule according to which one set of input numbers maps onto another, output set (Carraher, Schliemann, & Brizuela, 2003).
- For young learners, the number line can be a meaningful tool for representing numbers and operations and to solve problems (Carraher, Brizuela, & Earnest, 2001; Carraher, Schliemann, & Brizuela, 2001; Peled & Carraher, 2004 in preparation).
- Even young students can easily learn to accept the idea of negative numbers. However, operations involving negative numbers pose special challenges when dealing with word

problems, number lines, graphs, and other contexts (Peled & Carraher, 2004 in preparation).

- The idea of "difference" (corresponding initially to the expression, $|a-b|$, and later to (a-b) is important for appreciating the algebraic character of additive structures; yet it takes on subtle differences in meaning across the contexts of number lines, measurement, subtraction, tables, graphs and vector diagrams (Carraher, Brizuela, & Earnest, 2001).
- Certain large-scale activities where children enact mathematical objects and relations, can play an important role, helping to introduce new algebraic concepts. Such activities provide meaning for children to interpret mathematical notation and solve algebraic problems (Schliemann & Carraher, 2002; Schliemann, Goodrow, & Lara-Roth, 2001a).
- Children often complete function tables by treating each column as a number sequence task to be solved in a downward fashion regardless of the values contained in the other column. This led us to seek and successfully implement alternative ways to pose problems so as to highlight the functional relations across columns (Schliemann, Goodrow, & Lara-Roth, 2001b).
- There is an inherent ambiguity in how letters represent quantities in algebraic expressions. Students must recognize that letters may refer to particular values or instances, but also to sets of possible values or variables. This shift in meaning has generally not been addressed in curricula. We found that, given appropriate activities, 3rd graders can grasp the meaning of variables, as opposed to instantiated values (Carraher & Schliemann, 2002; Schliemann, Carraher, & Brizuela, 2002).
- Graphs of linear functions are within reach of 3rd grade students, contributing to their initial understandings of function and of multiplicative structure concepts (Schliemann & Carraher, 2002; Schliemann, Goodrow, & Lara-Roth, 2001a).
- After participating in our activities, 4th grade students were able to solve algebraic problems using multiple representation systems such as tables, graphs, and written equations with variables on both sides of the equality (Brizuela, 2002; Brizuela & Schliemann, 2003).
- The symbolic systems used in algebra are an inherent and important part of learning algebra; children need both access to these systems as well as opportunities to constantly represent algebraic concepts in multiple ways, both conventionally and idiosyncratically.

At the end of 4^{th} grade, three to four weeks after our last class, the students were individually interviewed and asked to solve problems. A control group (CG) of 26 children, at the end of their 5^{th} grade, who had been taught by the same teachers in the school, were also interviewed and compared to our experimental group (EG). Here are some preliminary results:

<u>Question 1:</u> Is 6 + 9 = 7 + 8 True or False?

Here we explored students' understanding of the equivalence in an equation, treating both sides symmetrically, instead of holding an "input – output" conception and thinking that they should do an operation on the left side and get a result on the right.

We found that 85% of the 4th grade students (EG) responded correctly that the equation was true while only 65% of the fifth graders (CG) did so.

<u>Question 2:</u> Below [A drawing of two boxes and another box plus 9 candies was shown under the question.] there are boxes of candy [each containing an unspecified amount]

and loose candies. Each box has the same number of candies. Which would you rather have? Two boxes of candy? Or one box of candy and 9 loose candies? Why?

Here we explored whether the students could relate and operate on an unknown quantity. Could they generalize and integrate their part-whole schema with a comparison schema in a situation with two piles, each composed of different parts and involving unknown amounts? Could they handle situations in which there was no "one answer", but rather a set of solutions, or a variable that depends on the changing value of an amount in the situation (the independent variable)?

We found that a larger proportion of children in the EG (44%) could handle situations with unknown quantities, answering that their choice would depend on the number of candies inside each box, while 31% of the children in the CG did so.

Question 3a: The children were asked to complete a function table representing the following problem: Mary has three times as much money as John. Column 1 was labeled 'John' and column 2 was labeled 'Mary'. After completing the table, we continued: If we don't know how much money John has, we can say that he has N dollars. If John has N dollars, what could you write to say how much money Mary would have?

How would the students fill up the function table? Would they continue focusing on isolated columns as was the case in 2nd grade? Or would they focus on the functional relationship between the two variables?

We found that 65% of the EG filled the table working with functional relations and 50% of the CG did so. To represent that Mary had three times as much money as John, 70% of the EG accepted to represent John's amount of money as N and Mary's amount as John's amount times three. In the control group, only 29% of the children did so.

Question 3b: Which of the graphs [three linear function graphs were shown as possible answers] shows that Mary has three times as much as John? How do you know you chose the correct graph?

Would the children be able to identify a specific linear function relationship in a graph. And if so, how would they justify their choice? Would they focus on isolated points or would they identify the general properties of the function depicted in the graph?

Here, 78% of the EG chose the correct line while only 46% of the CG did so. Note that the CG, had also received instruction on drawing graphs by their regular teachers. Of the children choosing the correct line, 39% in the EG provided general justifications that took into account any possible pair of numbers (e.g., "Because when you times John's money by 3 it tells Mary's number of money" Or "Because if I would times 3 all the bottom numbers it would be on that line."). In the CG 25% adopted this general approach.

Question 4: In the last part of the interview, children were asked to represent in writing and to solve the following problem: "Harold has some money. Sally has four times as much money as Harold. Harold earns $18.00 more dollars. Now he has the same amount as Sally. Can you figure out how much money Harold has altogether? What about Sally?" Each step in the problem was presented gradually.

In this problem, we wanted to see if children could accept to work with an unknown amount, how they would represent the unknown amount, and whether they would use equations and the syntax of algebra to find a solution to the problem.

Of the 63 EG children who were interviewed, 56% represented Harold's initial amount as N, X, or H and 49% represented Sally's amount as Nx4. For Harold's amount after earning 18 more dollars, 35% of the children wrote N + 18. 17% of the children wrote the full equation N + 18 = N x 4 and 27% of the children correctly solved the problem. However, only 6% (four children) systematically used the algebra method to simplify the equation. Two children, when prompted, correctly explained the algebra method. Apparently, as the children worked in their written representations, they easily inferred that Harold's starting amount was 6, without the need to use the algebra method. As Albert stated, "I thought about six because it just popped in my head." In the CG, 23% of the children solved the problem but no one wrote an equation or used algebra methods.

DISCUSSION

During the last few years, we have made certain strides forward in expanding students' mathematical reasoning and in helping them develop and use algebra notations and tools to solve problems. However, we did not explore the limits of children's capabilities regarding algebra. As we focused on discrete quantities, linear functions, and graph spaces in Quadrant I, we may have underestimated children's potential to learn algebra.

The issue of sustainability of learning is also still open. In our research we worked with the students in their classrooms, but met with them for only six to eight times per school term. We believe, and this is what we want to test in our next study, that much more can be achieved if children participate in early algebra activities on a daily basis, as part of their regular curriculum.

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GOAL SKETCHES IN FRACTION LEARNING

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To examine how conceptual knowledge about fraction magnitudes changes as students' learning progresses, 5th and 7th-grade students were asked to solve fraction magnitude problems that entailed finding a fraction between two given fractions and then to evaluate solutions for similar problems that were modeled for them. When the given fractions share a common denominator or numerator, a simple strategy is to keep the common value and choose an intermediate value for the other component. 5th graders used this strategy on both common-numerator and common-denominator problems, and judged it "very smart" when it was modeled. 7th graders typically converted common-numerator fractions to a common denominator and often judged the strategy of picking an intermediate denominator “not smart."

INTRODUCTION

While many U.S. students, and often even their teachers (c.f., Ma, 1999), think of mathematics learning as primarily a matter of learning computational procedures, educational psychologists and mathematics educators agree that mathematics learning needs to be conceptually grounded (Hiebert, Carpenter, Fennema, Fuson, Wearne, Murray, Olivier, & Human, 1997; Kilpatrick, Swafford, & Findell, 2001). Indeed, contemporary research in cognitive science indicates that the growth of mathematical understanding and the growth of computational skill are mutually facilitative (Rittle-Johnson, Siegler, & Alibali, 2001; Kilpatrick et al., 2001). In particular, conceptual knowledge appears to play a fundamental role in the development of problem-solving strategies (Siegler, 1996; Siegler & Crowley, 1994). For example, there is evidence that students can make sound judgments about the merits of a new strategy even before they themselves have begun to use it. When Siegler and Crowley asked students to rate a variety of strategies that were modeled for them, the students rated as "very smart" strategies that were more advanced than the ones they themselves used. Siegler (1996) postulated that knowledge structures called goal sketches, “specif[y] the hierarchy of objectives that a satisfactory strategy in the domain must meet" and thus provide a conceptual basis for evaluating possible strategies and for generating new ones (p. 194).

On this account, students’ judgments about alternative problem-solving strategies are a potentially rich source of information about their conceptual knowledge of the problem domain for which the strategies are proposed. The research to be reported here applied this reasoning to the examination of students’ developing knowledge about fraction magnitudes. This domain was selected for study because there is abundant evidence that students have difficulty understanding the magnitude relations between fractions with different denominators (e.g., Peck, Jencks, & Chatterley, 1980). Thus, the objective of the present research was to examine student’s goal sketches about fraction magnitudes, particularly in relation to fractions with different denominators, and to examine how goal sketches change with age and years of schooling.

To separate conceptual knowledge from learned computational rules, problems on which students were not likely to have been directly instructed were used. On each problem, two fractions were presented and the task asked of the child was to generate a third fraction that was in between the two in its value. After completing all the problems, the student was asked to make judgments about different solution methods modeled by the experimenter on a similar problem set.

We focus here specifically on problems in which one of the two components of the fractions—numerator or denominator—was the same across the two fractions presented (e.g. 3/13 & 4/13, a common-denominator problem, or 1/4 vs. 1/5. a common-numerator problem). These are of particular interest because it is not necessary to do precise calculations in order to identify with certainty a fraction

that is of intermediate magnitude. A straightforward solution strategy is to preserve the common numerator or denominator while generating an intermediate value for the component that differs across the two given fractions (e.g. 2/9, for a fraction between 1/9 & 4/9). This type of strategy (which shall henceforth be termed the "intermediate-component strategy") is interesting, particularly in relation to common-numerator problems, because it uses an understanding of the problem constraints to avoid the need for the relatively complex computational procedure of finding a common denominator for the fractions.

METHOD

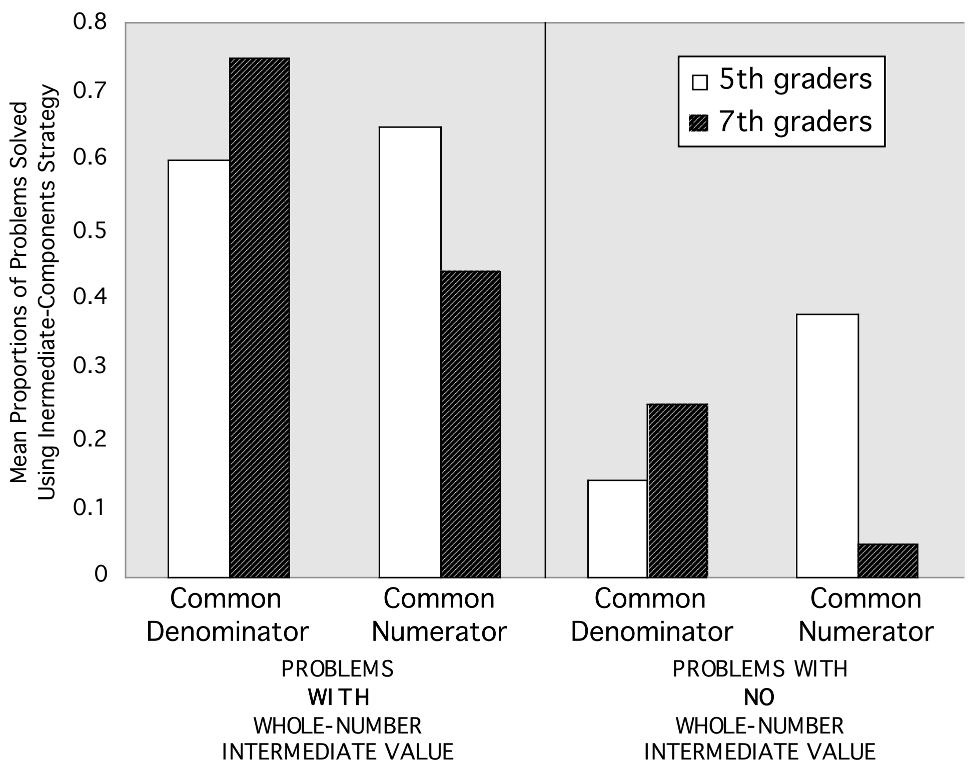
Twenty-four 5th-grade children (11 female, 13 male) and 10 7th-grade children (7 female, 3 male) have participated in the study to date (the samples will be equal in size when data collection is completed). A female experimenter tested students individually in a single session lasting about 20 minutes. First, students received 12 problems on which they were asked to find a fraction whose value was in between the two fractions they were given. Four of these were common-denominator problems and four were common-numerator problems; the rest involved pairs of fractions that differed in both their numerators and their denominators. Within the common-denominator and common-numerator problems, half the problems involved numerators or denominators (whichever differed) that differed by 3 (e.g., 2/9 vs. 5/9), while the other problems involved numerators or denominators that differed by only 1 (e.g., 1/4 vs. 1/5). This distinction is significant for the intermediate-component strategy because, in the latter case, there is no whole-number intermediate value. It is therefore necessary, if an intermediate-component strategy is to be used, either to use a fractional value for numerator or denominator or to convert the fractions to equivalents that contain larger (and more widely-spaced) numbers, e.g., by multiplying numerator and denominator by 2.

In the second part of the experiment, the experimenter modeled solutions for 10 problems: three common-denominator problems, three common-numerator problems, and four problems for which the given fractions differed in both their numerators and their denominators. On the common-denominator and common-numerator problems, three strategies were modeled by the experimenter: (1) an intermediate-component strategy resulting in a whole-number value (when the component that was not in common differed by more than one; e.g., e.g. 2/7 was found for a fraction between 2/6 vs. 2/9), (2) an intermediate-component strategy resulting in a fractional value (for problems on which the component that was not in common differed by only one; e.g. 1 over 4-1/2 was found for a fraction between 1/4 vs. 1/5), and, (3) an intermediate-component strategy involving conversion to equivalent fractions (e.g. 2/9 was found for a fraction between 1/4 vs. 1/5). Each of these strategies was modeled once on a common-denominator problem and once on a common-numerator problem. After each problem the child was asked to judge whether or not the experimenter's answer was correct and also to rate it as "very smart," "kind of smart," or "not smart."

RESULTS & DISCUSSION

Figure 1 summarizes students' use of the intermediate-component strategy in their own problem solving. There was a sharp drop in use of the intermediate-component strategy between the problems for which a whole-number intermediate value was available (plotted in the left panel of the figure) and those for which only a fractional intermediate value could be generated without converting to equivalent fractions (plotted in the right panel). Clearly, the availability of a whole-number intermediate value made it much more likely that students would adopt the intermediate-component strategy. Additionally, usage of the intermediate-component strategy changed with grade level, as can be seen

Figure 1

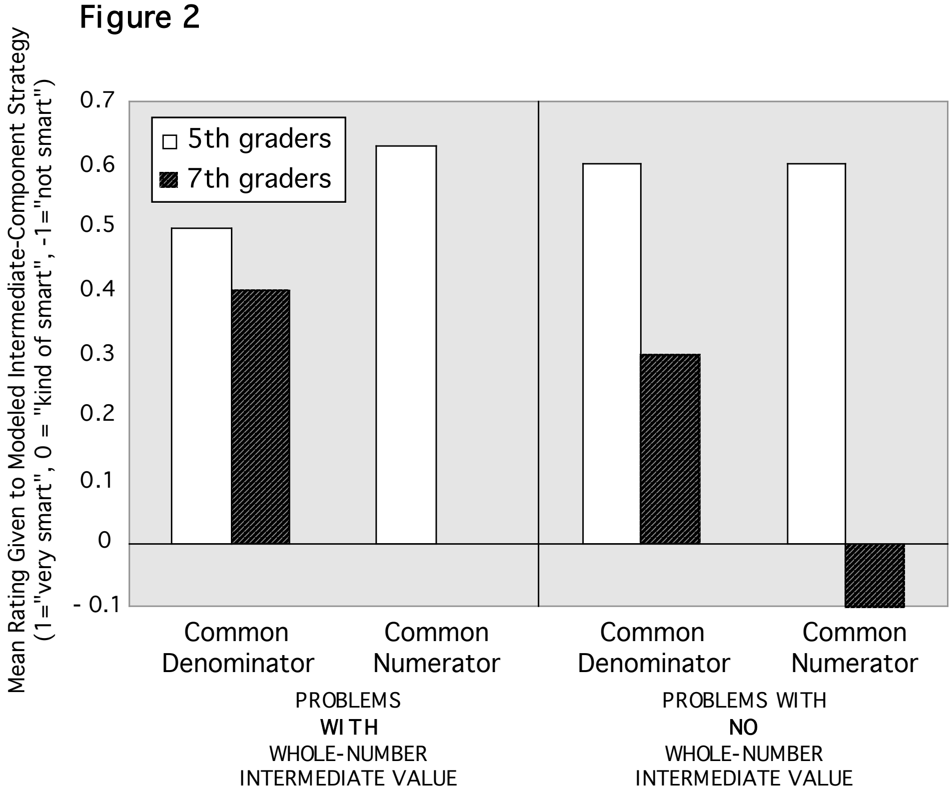


by comparing the white versus shaded bars within each panel. The 7th graders used the intermediate-component strategy more often than the 5th graders did on common-denominator problems, but less often than the 5th graders on common-numerator problems. Instead, they solved 45% of the common-numerator problems on which there was a whole-number intermediate value between the two denominators, and 55% of those on which there was not, by converting the fractions to common denominators. In contrast, none of the 5th graders even once attempted to convert different-denominator fractions to a common denominator in order to solve the problems. Thus, the 7th graders appear to have extended the school-taught procedure of finding a common denominator for different-denominator fractions to the problems presented here—displacing a simpler solution strategy that was widely used by 5th graders.

Figure 2 summarizes the judgments students made about the intermediate-component strategy when it was modeled by the experimenter. 5th graders gave much more positive ratings overall than 7th graders did—in part because the younger students were reluctant to judge anything the experimenter did as “not smart”. The pattern of ratings within each grade level, however, illuminates more telling differences between the groups.

Among the 5th graders, the pattern of judgments diverges from the pattern of strategy use in that applications of the strategy that result in fractional numerators or denominators are judged no less smart than ones that result in more conventional whole-number values. This is the only aspect of the data in which we see the kind of divergence between

Figure 2



students’ own strategy use and their judgments about modeled strategies that provided evidence for goal sketches in research in other mathematical domains. Although the 5th graders tend not to use the intermediate-component strategy when it results in a fractional numerator or denominator, they acknowledge it to be a fairly smart way to solve the problems when the experimenter models it.

Among the 7th graders, the pattern of judgments closely resembles the pattern of strategy use observed in the first part of the study. Thus, just as the 7th graders were more likely to use an intermediate-component strategy on common-denominator problems than on common-numerator ones, they also deemed it “smarter” when the experimenter applied it to common-denominator problems than when she applied it to common-numerator problems. Likewise, they deemed it “smarter” when it yielded a whole-number value for the intermediate numerator or denominator than when it yielded a fractional value for that component. The close correspondence between the strategies 7th graders use in their own problem-solving and their judgments about strategies modeled by the experimenter suggests that there is no longer a gap between their goal sketches and their actual problem-solving, at least with respect to the strategies studied here.

Insofar as goal sketches guide strategy development, then, it appears that acquisition of the computational algorithm of finding a common denominator has derailed rather than stimulated the process of strategy development. In learning the procedure of converting fractions to common denominators, students apparently came to believe that that is the only correct way to work with different-denominator fractions. That 7th graders were able to extend the algorithm of converting to a common denominator to new problems can be seen as positive in that it indicates the generalizeability of their learning. However, the fact that this strategy for solving the present problems displaced simpler but equally effective alternative strategies, so that they were not even judged positively when modeled for the students, underscores concerns about the dominance of procedural over conceptual aspects of student learning. In focusing on common denominators, they failed to recognize the possibility of drawing conclusions about magnitude relations among different-denominator fractions—surely an important element of understanding fraction magnitudes.

Thus, while procedural learning has been found to be facilitative of conceptual understanding in previous research (e.g., Rittle-Johnson, et al., 2001), the present findings underscore concerns that it can also have an adverse effect. Clearly, a fundamental problem for educators and psychologists alike is to clarify the conditions under which its impact is positive versus negative. To clarify this issue, research on students’ goals sketches should be combined with detailed examination of the instruction those students are receiving.

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. THE IMPACT OF PREPARING FOR THE TEST ON CLASSROOM PRACTICE

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For the past three years, we have been investigating the implications of testing on students in elementary mathematics classrooms throughout the state of NJ. Our research, and the work of others, indicates that the impact of testing depends on many factors, not the least of which involves how teachers prepare their students for the test. In this particular report we focus on the ways in which teachers engage in what is commonly referred to as "teaching to the test". Our results indicate that this can manifest itself in many ways ranging from drill on test taking mechanics to the incorporation of more complex problem activities into everyday instruction. We explore how each plays out in the context of classroom instruction, and the implications this may have for students.

BACKGROUND AND THEORETICAL FRAMEWORK

This paper[1] builds upon previous publications in which we discussed the broader implications that testing has had on teaching practices at the fourth grade level throughout the state of NJ[2] (Schorr & Firestone, 2001; Schorr, Bulgar, Monfils & Firestone, 2002; Monfils et al 2002; Firestone, et. al. 2002; Schorr et al, in press)[3]. In our previous papers we summarized our findings with respect to the overall changes that teachers were making in their regular classroom practice in response to the test. Briefly stated, we noted that teachers are now increasing their use of hands-on manipulatives, small group instruction and real life problem activities. However, we also reported that these increases have not been accompanied by some of the deeper changes advocated in state and national standards in mathematics (NCTM, 2000; NJ Mathematics Curriculum Frameworks, 1996) including closer attention to children's mathematical thinking, increased classroom discourse, and an approach to mathematics instruction that is more focused on learning with understanding. We have also reported that many of the teachers believe that they have either introduced or increased their coverage of topics that had been addressed with low historical frequency in the 4th grade math curriculum (such as, probability, statistics and data analysis).

This particular paper explores a related, but different set of issues, namely the specific techniques that teachers use to prepare their students to do well on the state test, and how

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[1] The work on this paper was supported by two grants from the National Science Foundation. The opinions presented here are those of the authors and are not necessarily shared by NSF, Rutgers University or Rider University.

[2] Our previous presentations and publications (PME 2002; PME NA 2002 and 2001) present our findings with respect to the strategies that teachers have incorporated into their classroom instruction as a result of the test. This paper addresses an issue not heretofore addressed: the issue of "teaching to the test" and how that manifests itself in the context of classroom instruction.

[3] We are grateful to William Firestone and Lora Monfils who did much of the data collection and analysis for this research.

this manifests itself in classrooms. The test that we examine is the NJ Elementary School Performance Assessment (ESPA), introduced in 1999. This test was designed to be consistent with state and national standards, and measure student achievement in several content domains including mathematics. The test combines multiple choice and constructed response items, but does not involve hands-on performance assessments. Further, this test was implemented with relatively low stakes attached to it. More specifically, at the student level, test scores may be used in-house for identifying students in need of instructional support, but are not used for grade retention or promotion. Although test scores may be used in state monitoring of districts having special needs and receiving supplemental state funding, the more general effect is local competition among schools and districts motivated by the annual release of school scores in local newspapers and state websites. The teachers we interviewed, generally speaking, also reported that they perceive the test to be of low stakes. In fact, despite reservations about the test's length and the time taken away from instruction, teachers generally view the test as a positive force for improving instruction. Because the stakes are relatively low, the NJ context provides a useful study of what teachers will do in terms of "teaching to the test" even when they do not have a strong accountability system motivating them to increase their students' performance on the test.

Before continuing, it is useful to provide a review of the arguments surrounding the impact of testing on classroom instruction, especially as it relates to test preparation to provide a context for considering our analysis. To begin, many supporters of standardized testing see testing as one way of using the authority of the state to ensure that all students are exposed to the same instructional standards. They maintain that such tests can prompt teachers to revise their instructional approaches to conform to the expectations set forth by the test (and standards that they are based upon), and that sort of teaching to the test can be good if the test is well designed. (O'Day & Smith, 1993). There are others, however, that believe that extensive testing will encourage measurement of less relevant skills, and reinforce more traditional approaches to teaching (McNeil, 2000). Many contend that tests can and do influence the actual content taught in classrooms. For example, untested content either falls out of the curriculum or gets put off until the end of the year and after the test (Corbett & Wilson, 1991; McNeil, 2000). The types of items that are placed on the test are also claimed to influence the types of problems teachers use in class. The argument is that by including items that require students to solve more complex problems, teachers will be more likely to provide students with the opportunity to do the same in class. Indeed, one reason given for the great interest in various forms of performance assessment and portfolios in the 1990s was the hope that tasks requiring students to show their work and explain their answers would promote inquiry-oriented instructional approaches (Resnick & Resnick, 1992; Rothman, 1995). Many tests do indeed combine both multiple-choice formats with other formats intended to measure higher order thinking and problem solving abilities. However even when tests employ formats where students construct responses, researchers note that some of the same types of instructional reactions that are typical of the more traditional formats have been found to occur (Smith, 1996; Stecher & Barron, 1999). Regardless of the format, some researchers report that the evidence that testing promotes instructional change remains unconvincing or inconclusive at best (Newmann, Bryk, & Nagaoka, 2001; Smith, 1996).

It is therefore important to study and document the implications of testing on actual classroom practice, both in contexts where the stakes are high and low. NJ, as mentioned previously, provides a useful case study of a situation in which the stakes are relatively low.

This paper will report on some issues associated with teaching practices related to preparation for the test. The research question we address here is the following: What is the relationship between actual instruction and test preparation? In brief, we will report that an increased focus on preparation for the test occurs widely and is multidimensional ranging from embedded approaches that emphasize the use of test-like activities and items to drill on low-level mechanics.

METHODS

Data Collection and Sample

The qualitative aspect of our study involved collecting data through direct observation and interviews for a total of 78 teachers. The teachers came from districts that were representative of the state in terms of geography and demography. There were 63 teachers observed in 2000, (58 of those were observed twice, the others only once). There were 27 teachers observed in 2001, (26 of the 27 were observed twice, the other only once). Of the total of 78 teachers, 12 teachers were observed in both years.

After each observation, teachers were interviewed about such issues as the content of their lessons, the nature and goals of the instruction and the effect of the ESPA upon various aspects of their classroom practice. A researcher, who kept a timed running record of all activity that took place in the classroom, particularly noting the activity of the teacher and the students, collected the qualitative data. The field notes included problems and activities, materials used, the questions and responses of students or teachers, the physical layout of the rooms, the overall sense of classroom community, and any other elements of the class that could be observed. All observations were coded by at least two independent raters (for more information on the coding system and results, see Firestone, Schorr and Monfils, in press)

INTERVIEWS

After each observation, teachers were interviewed and asked several open-ended questions about their instructional practice. These questions were related to the lesson itself, asking about the objective of the activity, how it fits into the math curriculum and the instructional methods that were used. Teachers were asked reflective questions such as how successful they felt the lesson was and what they would change when doing this lesson in the future. They were also asked about their own professional development and about how the 4^{th} grade test has impacted their practice in terms of curriculum and specific test preparation.

Interview data were transcribed and entered into a qualitative data analysis software package. Interviews were sorted by question. Responses were analyzed in clusters, as there was considerable overlap in responses given to individual questions. Within each cluster, responses to specific questions on test preparation practices were reviewed and coded according to emergent themes. Responses were counted within each code.

RESULTS AND DISCUSSION

Our results indicate that the teachers we observed and interviewed are indeed making changes in their instructional practices in order to help their students do better on the test. Throughout this section we will describe how this manifests itself in classrooms.

The changes that teachers were making in response to this low stakes test were in most instances, embedded into their everyday classroom practice, or more focused on specific procedures and practices in the periods just prior to the actual test[4]. Some of the more common practices that occurred in the weeks just prior to the test included drill on "test-besting" skills (ways to learn what types of questions are routinely on the test, and how the test is organized and scored in order to increase the likelihood of answering a question correctly) and the use of test-like items either for practice sessions throughout the year or as part of classroom activity during the month before the test. For example, in the year 2000 post-observation interviews, most teachers (37 out 58) noted that they used either practice books or sample test problems they received from the State or downloaded from the Internet. Twenty specifically stated that they use commercially available ESPA prep books. The amount of time spent using such workbooks ranged from strictly the month before the exam to once a week starting in September (the beginning of the school year). Another 17 said they use their own sample problems or sample problems or tests they received from the state or downloaded from the state's website for the same purpose.

Most of the teachers involved in our study noted that they like to use the sample questions or tests so that their students can become familiar with the format and style of exam when it is administered (in late spring). Some teachers use ESPA-like questions as a "problem of the day" throughout the year, while others only use them closer to the actual time of the exam as a means by which their students can practice a "timed" test. In our 2001 interviews, when we asked specifically about test prep in the few weeks before ESPA, an overwhelming majority (19 of the 27 teachers) responded said that they used commercially prepared materials or sample problems released by the state, another 12 stated that they taught test-taking skills. Only 10 (of the total of 27) of those questioned stated that they reviewed the curriculum as a way to prepare their students for the test.

To better understand our findings we looked to the interview data to see what teachers thought about the effects of ESPA on their actual teaching as it relates to preparing students for the test. Many teachers described a more embedded approach to test preparation that included the use of more open-ended types of questions throughout the year. For example, one teacher stated the following.

> I like the way the questions are challenging and make them think. I think it certainly has affected the way that I've taught, I teach, and that I'm very, I'm always looking for opportunities to have an open-ended question somewhere and that's good.

The following quote from one teacher is typical of what teachers across the state told us.

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[4] See also Monfils, et al, 2002

> The benefits of the test are that I think it is making district sit up and take notice that you know what, we can't teach the way we used to teach because the kids are not getting the skills they need for real life. …the main thrust of the test is to change the way teachers are teaching in the classroom to match the skills that are needed for the students of today. We can't teach knowledge. We have to teach skills for getting knowledge. That was what the test was supposed to drive was this kind of teaching in the classroom. …

Most teachers made reference to some form of an embedded approach to preparing their students for ESPA, using phrases like "teaching life skills", "teaching that reflects the standards", "critical thinking", "yearlong preparation", and stating that by covering the curriculum and corresponding standards, they were indeed preparing their students for ESPA.

> My classroom is set up to work with ESPA, not separate that I'm gonna stop and train for. My writing folders, right from the beginning my math portfolios--everything is open-ended questions, rubrics are used from the beginning, language is used. It's just part of our way of life in here.

Many of the teachers who claim to use a more embedded approach to test preparation still use practice items from commercial materials, especially the open-ended items. They state that they do this to familiarize their students with the format of ESPA and/or as an end of unit review. One teacher described how she used practice problems from the "Coach" book for each content domain or cluster that appeared on the test (content domains or clusters are organized around such topics as patterns, operations on numbers, etc.) after she felt that her students had learned most of the skills as part of their regular instruction from the math text.

> So what I do is I take that review part at the end of each cluster, and when I feel like the class has, you know, that we've done many of the things in that cluster, then I'll pull out that review and so they'll get used to some of the types of questions that are on the ESPA. And the biggest ones that I do are those open-ended questions…

The open-ended items were mentioned by almost all of our teachers as providing opportunities for worthwhile instruction in terms of getting their students to think more about mathematics, and explain their answers orally or in writing. These teachers tended to talk more about teaching their students about process rather than particular content just for the test. One teacher stated the following.

> I don't believe in teaching to the test. But the kinds of skills -- if I can incorporate something in a lesson that they need on a test, to me that's a life skill; it's not a test skill

Lessons typical of this embedded approach to test preparation were observed in a number of classrooms. In one classroom, observed just a few weeks *after* ESPA, students worked in groups on an ESPA-like problem (used as a problem-of-the-day). The teacher said that students had been taught to use rubrics to score their own work, and it was clear throughout the lesson that the students were aware of the rubric criteria for a high score. The teacher also stated that the practice of using an ESPA problem of the day was something that she did throughout the year, and chose to continue even after the test. In fact, she even encouraged students, as a homework assignment, to create their own problem of the day.

Other teachers emphasized a more direct approach to test preparation. For instance, we observed one teacher who used practice items from a test preparation book as a review to bring closure to a math lesson on standard and metric units of measurement. In this lesson the teacher demonstrated conversion between scales of measurement with water and containers and asked the students discrete, single response questions. There was little room for student inquiry, and students were not given the opportunity to explore by working with the materials themselves. In the last 5-10 minutes of class, the teacher used multiple-choice practice items from the test prep book he had distributed earlier to ask for responses to narrow questions about the units of measurement taught in the day's lesson. In another observation, the teacher used test preparation booklets but did not discuss the answers or attempt to have all students understand the concepts behind the questions.

Teachers in districts that experienced a drop in scores often referred to a more decontextualized test preparation, one that involved use of commercial materials in the period preceding ESPA. There was a great emphasis on the use of questions which they believed to be similar in nature to test questions, and the direct drilling of students on test-besting techniques. In one district a third grade teacher said:

> We got our results back from last year. Apparently, the district wasn't very happy with them…. So, ah, because of that, they ordered us books. So I guess we're going to be asked to teach ESPA stuff from an ESPA book.

The overarching belief in such situations was that the use of commercially prepared books could help to increase scores on the test. One teacher bemoaned the fact that when the scores came back (they were expected in January) the teachers would undoubtedly be forced to begin the test preparation activities. She said that the scores would come "Just in time to start test-besting again.". Another teacher who used test like items to get the students ready for the test openly stated that she did not like the term "teaching to the test", rather she said "and it's not that you're teaching to the test, but you're prepping for the test".

CONCLUSIONS

The question that we have tried to address is: "What is the relationship between actual instruction and preparation for the test? Our findings suggest that teachers are indeed making choices about curriculum and practice, which are motivated solely by the test. Almost all teachers used some form of test preparation in their instruction. In some cases, this means "teaching to the test" by drilling students on test mechanics and/or simplistic notions about how to get higher scores by out-guessing the test makers. In other cases it means incorporating some embedded strategies into everyday instruction. This includes, for example, using more open-ended problem activities that resemble the test questions throughout the year, or in the period just prior to the test.

In sum, our findings reveal that even though the stakes are relatively low, the test is indeed causing teachers to consider ways to get their students to increase their scores—many of which, we do not believe contribute to learning with understanding, especially in cases where the emphasis is on "out smarting" the test makers and scorers. On a brighter side, we have noted that some teachers have incorporated more open-ended activities into their regular classroom instruction as a way to familiarize students with the test-like questions. These attempts did not always result in increased opportunities for learning either. However they may hold greater promise as we consider ways to capitalize on teachers' good intentions given the presence of such testing.

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AN ANALYSIS OF MENTAL SPACE CONSTRUCTION IN TEACHING LINEAR EQUATION WORD PROBLEMS

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This study is to explore the potential of Fauconnier's mental space theory in understanding the teaching and learning processes of mathematics. Notions from cognitive linguistics have been introduced to research in mathematics education recently, but there have been little studies about what potential the framework of mental space has in understanding mathematics teaching and learning. This paper will present an analysis of classroom data based on that framework. The data was collected at lessons of linear equation word problems of a Japanese seventh grade classroom. It is argued that the mental space theory will provide a useful framework for understanding the process of mathematics instruction, and that the idea of mental space blending could be a powerful tool for analyzing mathematical problems.

Introduction

The framework of cognitive linguistics has been introduced in recent research in mathematics education. Especially, the notions of metaphor and metonymy have been pointed out as essential in teaching and learning mathematics by several researchers (e.g., English, 1997; Presmeg, 1992). Also, "Mathematical Idea Analysis" in the embodied cognition perspective (Lakoff & Núñez, 2000; Núñez, 2000) has showed the power of metaphor in understanding and creating mathematical ideas.

This paper is to explore the potential of Fauconnier's mental space theory (Fauconnier, 1994, 1997; Fauconnier & Turner, 2002) in understanding the teaching and learning processes of mathematics. The mental space theory has provided an interface for understanding the connections between meanings and language expressions, without falling into formalism like previous linguistic theories. It provides a framework for understanding meaning construction *process* in the cognitive linguistics, as well as theorizing the notions of metaphor and metonymy as essential tools of meaning construction. This paper will attempt to show some of its potential for mathematics education by using classroom data.

THEORETICAL FRAMEWORK

Human experiences are always mediated by conceptual structures, and human beings construct realities through conceptual structures, and give meanings to the experiences. Thus, conceptual structures play fundamental roles in human cognition. These are called cognitive models (Lakoff, 1987). The basic components of cognitive models are categories and schemas. Categories are mental constructs resulting from human's practice of grouping experiences. Schemas are mental constructs resulting from experienced patterns of activities, events or phenomenon. Cognitive models are constructed through various projections from these categories and schemas.

To understand the *process* of meaning construction, it is necessary to have tools to represent how these cognitive models are employed in meaning construction. Fauconnier's theory of mental spaces and mappings (Fauconnier, 1997) provides such tools. A mental space is a field of thinking, which is either temporally constructed (e. g., "today's weather"), or relatively stable. Meaning construction is explained as making connections ("mapping") between mental spaces, and producing a network of mental spaces. There are three kinds of mapping (for details, see Fauconnier, 1997):

(1) Schema mapping: This to directly map a schema, frame, or model to a mental space.
(2) Projection mapping (analogy or metaphor): This is to project a structure of a mental space to another mental space.
(3) Pragmatic function mapping (metonymy): This is to connect two spaces based on a pragmatic function.

There is another important operation in mental space construction, "conceptual blending." It is a mental operation to connect more than one mental space and produce an integrating mental space. It is a fundamental and powerful tool for human beings to develop imaginative and creative conceptualization. Fauconnier and Turner (2002) illustrate it by using a solution of the following "Buddhist monk problem":

> A Buddhist Monk begins at dawn one day walking up a mountain, reaches the top at sunset, meditates at the top for several days until one dawn when he begins to walk back to the foot of the mountain, which he reaches at sunset. Make no assumptions about his starting or stopping or about his pace during the trips. Riddle: Is there a place on the path that the monk occupies at the same hour of the day on the two separate journeys? (p. 39)

One way to solve this problem is to imagine the monk walking down from the top at the dawn of the same day when another monk (a double role) walks up the mountain: the monk must meet himself somewhere on the path.

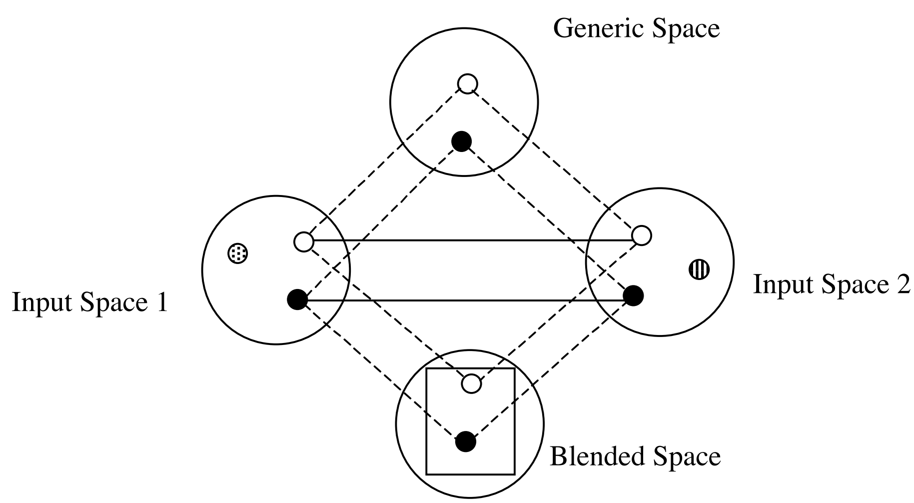

Figure 1. Blended Space (from Fauconnier & Turner, 2002, p. 46).

Fauconnier and Turner (2002) analyzed this solution by making a network of four mental spaces (Figure 1):

> Input Space 1: The journey to the top of the mountain. Input Space 2: The journey back to the foot of the mountain. Generic Space: Journey at the mountain.
> Blended Space: A situation where the monk begins walking down from the top at the dawn of the same day when another monk (a double role) walks up the mountain.

Input Spaces are the sources to be connected and integrated. They have some elements corresponding to each other (e.g., moving individual, mountain, motion): the correspondence forms a "cross-space mapping." Generic Space is a space that has a structure common among Input Spaces. Blended Space ("Blend") is a space onto which the structures of Input Spaces are partially projected. It is not just a mixture of projected structures. The most important feature in the Blend is that a new structure emerges that is not in Input Spaces. In the Buddhist monk problem emerged is that two monks are moving toward opposite directions.

DATA COLLECTION

The data collection was conducted in mathematics lessons of a teacher in a public junior high school in Japan. The teacher had a master's degree in mathematics education and was very interested in improving his teaching. His seventh grade mathematics class was the target of data collection. The class had 19 students. Almost all the mathematics lessons had been observed and audio-and-video recorded from November 2001 to February 2002. Interviews were also conducted with the teacher and students. The lessons contained a unit of linear equations and a unit of proportional, and inversely proportional functions. Those units were chosen because they contained important ideas of mathematics, equation and function, and mathematically rich activities. The present paper uses data from only the lessons about word problems in the linear equation unit.

ANALYSIS

In Japan, the use of letter in mathematical expressions is introduced early at the seventh grade. The linear equation is taught after that. At the beginning of the lessons of linear equation word problems, the teacher prepared large worksheets, called "thinking sheets." Each sheet contained one word problem. Students were asked to write their solutions, their thinking processes, and the solutions discussed in the class on their sheets.

Single-schema-mapping problems

The first word problem the teacher used in the worksheet was as follows:

> "We went shopping carrying one thousand yen. We bought six pencils and one 480-yen pencil case, and 280 yen remained. How much was the price of one pencil?" (Nov. 21, 2001)

This problem could be solved by mapping the following a "shopping payment" model on to the mental space of the problem (Figure 2):

> The amount of money you possessed before shopping subtracting the amount of money you spent results in the amount of money that remained after shopping.

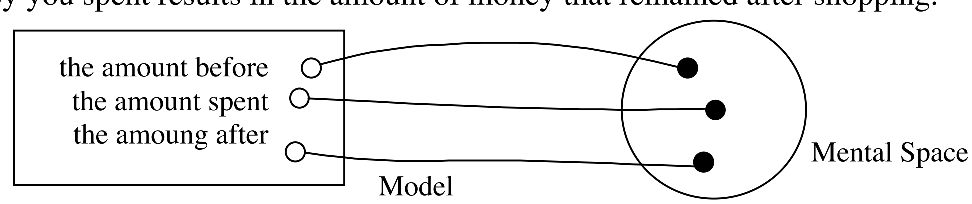

Figure 2. Single-schema mapping.

There are several models equivalent to this, all of which are structured by the part-whole schema. By projecting one of these to the mental space of the problem, we can give an overall structure to the latter. Since the price of one pencil is unknown, we may assign a letter x to it. By using a shopping payment model, an equation "$1000 - (6x + 480) = 280$" is obtained.

At the beginning of the word problem lessons, each problem was solved by mapping single models like the above. Another model appeared was a "total payment" model: The total of the number of each product one bought multiplying its unit price results in the total payment.

Blending problems

After working on several single-schema-mapping problems, the teacher delivered a new worksheet containing a "surplus-shortage problem": "There are children. If we distribute three oranges to each child, then eight oranges remain. If we distribute four oranges to each child, then we are five oranges short. Find the number of children."

The students worked on it individually. They were then allowed to consult with other students. After that, the teacher asked a student Kawata [pseudonym] to write his solution on the chalkboard and explain it before the class. When he finished the explanation, the teacher went to the board and made a detailed explanation. The teacher did not treat this problem as single-schema-mapping problem.

Before discussing the teacher's explanation, let us see how a process of solving this word problem could be explained using the framework of blending. First, the four mental spaces are constructed:

> Input Space 1: Giving three oranges to each child, leaving eight oranges.
> Input Space 2: Giving four oranges to each child, with five oranges short.
> Generic Space: Distributing each person an equal amount of things
> Blended Space: A situation where oranges are given to children, allowing of two ways of distribution: (1) Giving three oranges to each child, leaving eight oranges, (2) Giving four oranges to each child, with five oranges short.

In this linear equation word problem, the blended space is a space that satisfies the both scenarios projected from the two Input Spaces. An emergent structure in the blended space is that there is only one solution, whereas in either Input Space, solutions are indefinite. In the Buddhist Monk problem mentioned above, it is inferred from common sense (but, mathematically, from the continuity of the path) that two persons walking the same path from the opposite ends meet at one point on the path. In linear equation word problems, it is a mathematical principle that there is only one solution that satisfies two (non-equivalent) linear conditions simultaneously (cf. Fauconnier & Turner, 2002, p. 53).

To make an equation, at least the following processes would be necessary:

(1) To identify two quantities common in both Input Spaces: the number of children, and oranges.
(2) To assign letter x to either of them, say the number of children.
(3) To express the other quantity (the number of oranges) by using the x. Going back to the two Input Spaces, two different expressions are obtained.
(4) To connect the above two expressions because they both express the same quantity.

Thus, an equation $3x +8 = 4x -5$ is obtained. (If we assign x to the number of oranges, then another equation $(x - 8)/3 = (x + 5)/4$ is obtained.) The idea of using common quantities in Input Spaces is essential in this equation making. These common quantities are each considered identical in the cross-space mapping.

Let us return to the teacher's explanation. His discussion of this word problem (Dec. 4, 2001) fits the above analysis very closely. He drew diagrams during explanation. While talking about the condition "If we distribute three oranges to each child, then eight oranges remain," he drew the upper half of Figure 3. Saying "On the other hand," he began to talk about the second condition "If we distribute four oranges to each child, then we are five oranges short," while drawing the lower half of Figure 3.

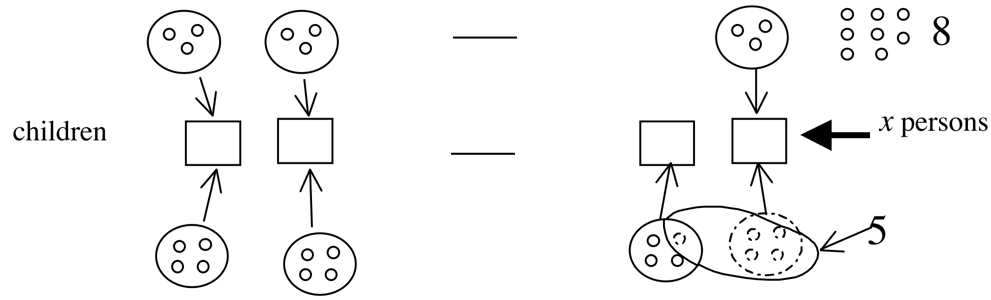

Figure 3. The teacher's diagram.

From his drawing, it is clear that the teacher was introducing the two Input Spaces during his explanation. Also, his contrasting phrase "On the other hand" indicates that he considered both conditions as sharing the "distribution schema."

His attempt to integrate two Input Spaces is apparent in Figure 3. His diagram clearly indicates that the number of children is common in both spaces. After assigning letter x to it, he pointed out that using the letter x, the number of oranges could be expressed. He returned to the Input Spaces, and discussed how to express the number of oranges in each space. He then wrote expressions in words on the chalkboard (Figure 4).

the number of oranges
- 3 x the number of children + the amount of surplus
  8
- 4 x the number of children - the amount of shortage
  5

Figure 4. The teacher's writing on the chalkboard.

He pointed out that the number of oranges did not change in both conditions, and connected the two expressions by an equal sign. Thus, he obtained a linear equation.

The teacher's explanation thus fits the mental space blending very well. What about students? As mentioned, before the teacher explained a solution of the above problem, a student Kawata had explained his solution in front of the class. Immediately after Kawata described how he obtained the expressions "$3x + 8$" and "$4x - 5$," the teacher stopped him and asked questions:

T (the teacher): Wait, wait. What does $3x + 8$ represent in the end?
Kawata: [Showing no hesitation]The number of oranges.
T: Ah-huh. What about $4x - 5$?

Kawata: [Showing no hesitation]The number of oranges.

T: The number of oranges, okay. So, what're equal, what're equal?

Kawata: Huh? The number of oranges.

Kawata was clearly aware that each expression represented the same quantity, the number of oranges. Though he did not draw a diagram like the teacher, his explanation was consistent with the teacher's.

Some of the other students failed to use the common quantities in the Input Spaces in making equations, however. They relied on a pragmatic function mapping between mental spaces of mathematical operations and ordinary phrases. It connected division, addition, and subtraction to "distribute," "remain," and "short," respectively. Using this mapping, they replaced phrases in the problem with mathematical operations, and obtained two (wrong) expressions, $x \div 3 + 8$ and $x \div 4 - 5$, and connected them by an equal sign to make an equation. This is the well-known "direct translation strategy" (Chaiklin, 1989).

The above connections were motivated from learning of arithmetic word problems. Students seem to have developed close associations between phrases in a word problem and mathematical operations since elementary schools. Mathematical symbols functions as shorthand of phrases. The resulted expressions did not represent quantities, but sequences of events. In single-schema-mapping problems, the schemas used consisted of sequences of events. The teacher's initial use of these problems might have encouraged students' use of direct translation strategy.

The students were assigned a variant of "surplus-shortage problems" later in the class: "We are going to distribute cookies to children. If we give 3 cookies to each child, 12 cookies remain. If we give 4 cookies to each child, 3 cookies remain. Find how many cookies there are." (Dec. 13, 2001). Also, I asked several students to solve similar "surplus-shortage problems" in interview sessions. The above two types of solving processes appeared again among the students.

In the third worksheet, the teacher discussed another well-known type of problems "compound motion" problems:

> A boy left home to a railroad station which was located 2*km* from home. 12 minutes later, his elder brother left home after the younger brother by bicycle. The younger brother was walking 70 *m* per minute, the elder brother pedaled 280 *m* per minute. How long did it take for the elder brother to catch up with the younger brother? (Dec. 5, 2002)

In the discussion of this problem, the teacher's explanation again followed the blending. He advised the students to "think *two boys separately*," drawing a diagram, constructing two separate mental spaces for each boy. Pointing out that the distances two boys advanced were the same when the elder brother caught up with the younger brother, the teacher blended the two spaces and made an equation.

DISCUSSION

Meaning construction is the heart of understanding of mathematics. This study tried to show how useful the framework of mental space theory would be in understanding the meaning construction process in mathematics classrooms. The paper analyzed actual classroom data of teaching linear equation word problems.

It is not possible to assess the full potential of cognitive linguistic analysis in just one small study on linear equation word problem instruction. The analysis presented in this paper was to provide one demonstration of the use of framework and tools of the mental space theory in understanding mathematics teaching and learning. Lakoff and Núñez (2000)'s Mathematical Idea Analysis focused on mathematical concepts. The current paper showed that the cognitive linguistics would provide useful tools to analyze word problems in mathematics, too, though some of the analysis would be familiar to researchers. Especially, the idea of blending mental spaces seems to be very powerful in understanding word problems. The teacher's explanations of several typical word problems were found to be very consistent with the blending process of mental spaces. His ways of explanations are not exceptional. Actually, many of explanations of linear equation word problems in published textbooks fit well with the explanation by mental space blending except single-schema-mapping problems. The mental space theory could be used for analyzing not only "correct" mathematical thinking, but also students' "wrong" thinking. The well-known strategy of solving word problems, "direct translation," could be considered one instance of the use of a pragmatic function mapping.

Comprehension of algebra word problems has often been approached by decomposing problem texts into atomistic formal propositions (e. g., Nathan, Kintsch, & Young, 1992). However, such approach seems to be a long way from explaining meaning construction processes in the classroom because its decomposition is often artificial and loses the sense of reality of classroom discourse. Analysis by the mental space framework starts from the situation where one constructed for understanding, thinking, or explanation: What mental spaces are introduced? How are they introduced? What connections are made between them? and so on. Therefore, the mental space approach would be more compatible with the situated cognition.

There are many other theoretical issues the paper did not address, however. For example, how mental space construction and formal mathematical expressions were related to each other in general. More detailed analysis and theoretical elaboration are necessary to put the theory into practice.

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LANGUAGE USE IN A MULTILINGUAL MATHEMATICS CLASSROOM IN SOUTH AFRICA: A DIFFERENT PERSPECTIVE

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This paper presents an argument that language-use in multilingual mathematics classrooms in South Africa is as much a function of politics as it is of cognition and communication. It draws from a wider study focusing on language practices in intermediate multilingual mathematics classrooms in South Africa. In the study the notion of cultural models (Gee, 1999) is used as an analytic tool to describe and explain the language practices in a multilingual Grade 4 mathematics classroom where learners learn in English, a language that is not their main language. The main argument of the paper is that in a context like South Africa, where mathematics and English have symbolic power, and where procedural discourse dominates over conceptual discourse in mathematics teaching and learning, a practice is forged wherein it is difficult to move mathematics beyond procedural discourse.

INTRODUCTION

This paper explores the complex relationship between language and the teaching and learning of mathematics in multilingual classrooms in South Africa. Learning mathematics has elements that are similar to learning a language since mathematics, with its conceptual and abstracted form, has a specific register (Pimm, 1987, 1991). Mathematics, however, is not a language like French or Xhosa, therefore communicating mathematically requires the use of an ordinary language, the language in which mathematics is taught and learned. A majority of learners in multilingual mathematics classrooms in South Africa learn in a second language. In these classrooms the language of learning and teaching (LoLT) is English, one of the eleven official languages in South Africa. How is mathematics learning enabled or constrained in these multilingual classrooms? What kinds of mathematics discourses are dominant and why? Embedded in these questions are pedagogical issues about language and learning, and political questions about language and mathematics and about language-in-education policy (LiEP).

In this paper I draw on a wider study to explore the above questions. I begin with a brief description of the current language-in-education policy (LiEP) and the school mathematics curriculum context in South Africa. Through this description I highlight the dominance of English as a LoLT and the emphasis on mathematical communication in the school mathematics curriculum. I then point to research done in relation to language and communication in bi/multilingual classrooms. This discussion will highlight the significance of language as power in mathematics education settings, and thus the need for research into the relationship between language and the teaching and learning of mathematics in South African classrooms to consider the political aspects of language. These discussions provide a theoretical context for what follows: a description and analysis of a research project focusing on language practices in intermediate multilingual mathematics classrooms. From these empirical and theoretical discussions I present the main argument of the paper that in a context like South Africa, where mathematics and English both have symbolic power, and where procedural discourse dominates over conceptual discourse in school mathematics assessment, a practice is forged wherein it is difficult to move mathematics beyond procedural discourse.

THE CURRENT LANGUAGE-IN-EDUCATION POLICY (LIEP) AND THE SCHOOL MATHEMATICS CURRICULUM CONTEXT OF SOUTH AFRICA

The current language in education policy recognises eleven official languages. Previously I have argued that while this policy is intended to address the overvaluing of English and the undervaluing of African languages, in practice English still dominates (Setati & Adler, 2001; Setati, Adler, Reed and Bapoo, 2002). Although it is the main language of a minority, English is both the language of power and the language of educational and socio-economic advancement, that is, it is a dominant symbolic resource in the linguistic market (Bourdieu, 1991) in South Africa. The linguistic market is embodied by and enacted in the many key situations (e.g. educational settings, job situations) in which symbolic resources, like certain types of linguistic skills, are demanded of social actors if they want to gain access to valuable social, educational and eventually material resources (Bourdieu, 1991). In this paper I consider what this dominance of English mean for communicating mathematically in multilingual classrooms where learners learn in English, a language that is not their main language?

According to the South African school mathematics curriculum, learning to communicate mathematically is central to what it means to learn mathematics (DoE, 1996, 1997). Learners are expected to participate in a variety of mathematical talk and written practices, such as explaining solution processes, describing conjectures, proving conclusions, and presenting arguments. The official description of the mathematics learning area emphasises the role that language plays in the expression, development and contestation of mathematics.

This focus on the communication of mathematics raises questions about the language used for communication and how mathematics teachers find a balance between initiating learners into ways of communicating mathematics and making language choices in their multilingual classrooms.

TEACHING AND LEARNING MATHEMATICS IN BI/MULTI-LINGUAL CLASSROOMS

The complex relationship between bilingualism and mathematics learning has long been recognised. I will not rehearse the discussions here as they have been described in detail elsewhere. Dawe, 1983; Zepp, 1989; Clarkson, 1991; Stephens, Waywood, Clarke & Izard, 1993; Setati, 1998; Adler 2000 and Setati & Adler, 2001 have all argued that bi/multilingualism per se does not impede mathematics learning.

Most research on the teaching and learning of mathematics in bi/multilingual classrooms has presented the learners' main languages as resources for learning mathematics (e.g. Addendorff, 1993; Adler, 1996, 1998, 2001; Arthur, 1994; Khisty, 1995; Merritt, *et al*. 1992; Moschkovich, 1996, 1999, 2002; Setati, 1996, 1998; Setati and Adler, 2000; Ncedo, Peires & Morar, 2002). These studies have argued for the use of the learners' main languages in teaching and learning mathematics, as a support needed while learners continue to develop proficiency in the language in which they learn mathematics at the same time as learning mathematics. All of these studies have been framed by a conception of mediated learning, where language is seen as a tool for thinking and communicating (Mercer, 1995).

Language, however, is much more than a tool for communication and thinking; it is always political (Hartshone, 1987; Reagan & Ntshoe 1992; Mda, 1997; Friedman, 1997; Heugh, 1997; Granville; Janks; Mphahlele; Reed; Watson; Joseph and Ramani, 1998; Gee, 1999). It is one way in which one can define one's adherence to group values. Decisions about which language to use, how, and for what, are not only pedagogic but also political. This political role of language is not dealt with in the literature on bi/multilingualism and the teaching and learning of mathematics.

In the study reported in this paper the work of Gee (1999) was central in exploring and explaining the language practices of teachers in multilingual mathematics classrooms not only from the pedagogic and cognitive point of view but also the political. His work was particularly relevant because he sees language as always political. He argues that when people speak or write they create a political perspective; they use language to project themselves as certain kinds of people engaged in certain kinds of activity. The teachers' decisions about which language to use, how and when do not only reflect curriculum and pedagogic decisions, but also the political context of their practice together with the identities and activities they are enacting.

In the study described in this paper the notion of cultural models (Gee, 1999) was used as an analytic tool to explore and explain the language practices of teachers in multilingual mathematics classrooms. Gee uses this notion of cultural models in socio-linguistics as one of the tools of discourse analysis. He describes cultural models as our 'first thoughts' or taken-for-granted assumptions about what is 'typical' or 'normal' (1999: 60). They do not reside in people's heads, but they are embedded in words, in people's practices and in the culture in which they live. They are learned from and shared with other humans through the media, written materials and through interaction with others in society.

THE STUDY

The study was qualitative and initially involved six intermediate phase mathematics teachers. The findings presented in this paper are from an analysis of one teacher's data. Her name is Kuki[1]. She is multilingual and shared a main language (Setswana) with her Grade 4 class in which she was observed. Data was collected over two years and it included: teacher interviews, lesson observations, learner interviews, a focus group interview and a reflective group conversation with teachers. The classroom observation data presented in this paper are drawn from an analysis of Kuki's lesson 5[2]. To enable a rigorous and focused analysis the transcript for Kuki's lesson 5 was divided into 9

[1] Kuki is her real name and it is used at her request. For a detailed discussion on methodological issues that emerged in the study see Setati (2000). At the time of the study Kuki had a Senior Primary Teachers' Diploma (SPTD) and a B.A degree. She had been teaching for 10 years.

[2] Lesson 5 is selected for focus in this paper because the richness of Kuki's language practices and mathematical communication were best illustrated in her teaching during this lesson.

stanzas[3]. The following questions were asked for each stanza to guide the analysis of the cultural models that source Kuki's language practices: What cultural models are relevant? How consistent are the relevant cultural models? Are there competing or conflicting cultural models at play in Kuki's language practices during teaching? What could have given rise to Kuki's cultural models? To guide my exploration of Kuki's cultural models, and to ensure a focus on language practices, I paid specific attention to the language(s) and mathematics discourse used in each stanza.

THE FINDINGS

The table below gives a summary of the discourses and language(s) used, together with the cultural models that were active in each of the stanzas in Kuki's lesson 5. The table is followed by a discussion, with empirical evidence on how Kuki used the mathematical discourses, the LoLT and the learners' main language in her lesson 5.

Table 1: Discourses, Languages and Cultural Models in Kuki's Teaching

	Maths discourses		**Non-maths discourses**		**Languages used**		**Cultural Models Dominant model: ENGLISH IS INTERNATIONAL**								
Stanza	Procedural	Conceptual	Regulatory	Contextual	English	Setswana	English is the language of authority	Learners' main language is the language of solidarity	Learning mathematics is about communication	English is the LoLT	School is about English	Learners' main language is the language of conceptual discourse	English is the language of procedural discourse	English is the language of mathematics	Procedural discourse is the discourse of assessment
1			√		√	√	√	√							
2	√				√		√		√				√		
3			√			√		√							
4				√		√				√	√				
5				√		√				√	√				
6	√				√				√				√	√	
7		√				√						√			
8		√				√						√			
9	√		√		√	√		√					√	√	√

Procedural discourse refers to discourses that focus on the procedural steps to be taken to solve the problem and conceptual discourse refers to discussions in which the reasons for

[3] Stanzas are 'clumps' of tone units that deal with a unitary topic or perspective, and which appear (from various linguistic details) to have been planned together (Gee, 1999: 89).

calculating in particular ways and using particular procedures to solve a mathematical problem also become explicit topics of conversations (Cobb in Sfard *et al.*, 1998: 46). These two discourses are both crucial in mathematics learning and develop different kinds of mathematical knowledge. Thus fluency in mathematical discourse requires ability to engage in both procedural and conceptual discourses. Regulatory and contextual discourses are non-mathematical. Regulatory discourse refers to discussions that focus on regulating the learners' behaviour. Contextual discourse focuses on the context of the task.

As summarised in Table 1, Setswana was used in seven out of nine stanzas in Kuki's lesson. However, it was used largely for the non-mathematical discourses (regulatory and contextual). Mathematically, English was dominant. English was used in four stanzas and three of those were in procedural discourse. The power of mathematics in Kuki's class was thus through procedural discourse and in English. Below is an example of how procedural discourse typically occurred in Kuki's class. This example is appropriate because it shows how Kuki not only used procedural discourse in her class but how she also encouraged her learners to use it.

<u>Stanza 2</u>

(The teacher, Kuki chooses Mpho to do the solutions of the following problems on the board: 113 X *22 and 141* X *22 with the first group.)*

12 Mpho: two times three?
13 Group: six
14 Mpho: two times one?
15 Group: two
16 Mpho: two times one?
17 Group: two
18 Mpho: two times three?
19 Group: six
20 Mpho: two times one?

TH	H	T	U
	1	1	3
	□	2	2
	2	2	6
+ 2	2	6	
2	4	8	6

The interaction in the above stanza is in English, abbreviated and procedural. It is dominated by the kind of talk that Arthur termed 'final draft' (1994). This occurred in all the stanzas in which English was used for mathematical discourses (Stanzas 2, 6 and 9) suggesting that the fact that the interaction was in English contributed to the form of the mathematical discourse. The mathematical discourse that took place in stanza 2 is categorised as procedural discourse, because the learners' discussion focussed on a particular procedure and not on why that particular procedure was used and why it worked. Mpho decided on the procedure and then asked the learners questions that would give answers to calculations like 'two times three'. She did not ask them which procedural process to follow. She assumed that all the learners in the group knew and understood the procedure that was to be followed, and also that there was only one procedure to be followed. This is evident in the stanza because the group members did not ask Mpho to justify the procedure she was using, and there was also no discussion of other procedures that could be used to solve the same problem. Mpho transformed the task from one in which the learners were supposed to decide on the procedure for calculating *113* x *22* and *141* x *22,* to one in which they completed particular steps of a given procedure. So the task as done on the board by Mpho involved simple multiplication, and addition of single-digit numbers. For example see utterances 12, 14, 16. Procedural discourse was dominant also in Kuki's assessment of her learners. This was evident in the learners' books and a test she gave during lesson 5. In the extract below, from the reflective interview, Kuki explained why she encouraged procedural discourse.

> Researcher: … in your lessons … you call learners to the board, to work out problems and you emphasise that they should "talk to the class".
>
> Kuki: … in a nutshell I am trying to encourage them to communicate.
>
> Researcher: Do you encourage them to communicate in a particular way?
>
> Kuki: … kids do imitate. I think we have seen that, even at your home, if you can do something or the way you talk, if you have got a daughter they will imitate you. So I believe that kids like imitating, so maybe they are trying to imitate their teacher (Reflective interview, 1999).

Kuki's purpose was to encourage communication in her class. The above extract suggests that Kuki was not necessarily concerned with the nature of the discourse in her class as long as the learners were communicating. This view resonates with the cultural model that emerged from her interactions during the focus group interview and the pre-observation interview: *learning mathematics is about communication*. This cultural model emphasises the fact that learning is about communication and children need to talk in order to learn. While this way of talking gives the learners an opportunity to communicate, it does not teach them how to communicate mathematically. This is a weakness, particularly in a mathematics class where learners have to be initiated into the mathematics discourse in English, a language that is not their main language.

As Table 1 shows, conceptual discourse was used mainly in stanzas 7 and 8. In both stanzas the dominant language was Setswana. Below is an extract from stanza 7 which shows how Kuki typically used questioning to engage her learners in conceptual discourse in Setswana.

> Teacher: Hundred and fourty four. Mara jaanong go tlile jang gore re tshwanetse gore re di tymse ka gonne nna nka nne ka nagana gore mare why re sa re twelve plus twelve? [But now, how did you know that we are supposed to multiply, why are we not saying 12 plus 12?]

Here she was expecting the learners to explain how they knew that they had to multiply. The problem that Kuki was working on with the learners stated that, *'In the SPCA are 12 cages; in each cage are 12 dogs. How many dogs are there altogether?'* The words 'how many' and 'altogether' in the above problem suggest multiplication or repeated addition and this is what Kuki wanted the learners to highlight. The learners gave two responses. The first was that "Because re batla di answer tsa rona di be right. *[Because we want our answers to be correct]*". The second was based on the diagram that Kuki had drawn on the board to represent the context of the problem. In her explanation, which was given in English, the learner counted the dogs in each of the cages drawn. In my view, both these responses are procedural, they do not explain why multiplication was the appropriate operation to use. Kuki's use of Setswana for conceptual questions and revoicing of the learner's responses in conceptual discourse in Setswana emphasised the role of the learners' main language (Setswana) as the language of conceptual discourse; the language in which explanations and justifications are asked for and are provided.

The distribution of discourses, languages and cultural models across the stanzas mirrors the conflicting cultural models and identities that emerged in Kuki's interviews and teaching in lesson 5. Throughout the lesson analysed, Kuki switched from one language to another. Switches in discourses (mathematical and non-mathematical), cultural models and identities accompanied her language switches. As discussed earlier, both procedural and conceptual discourses are crucial in acquiring fluency in the mathematical discourse. Thus the engaging learners in conceptual discourse is important. In Kuki's case, however, conceptual discourse was not seen as valued mathematical knowledge. It was only spoken and not assessed. While assessment was not the focus of this study, it is important to note here that assessment communicates to the learners what is valuable mathematical knowledge. The absence of questions demanding fluency in conceptual discourse in the class test thus suggests their unimportance. While not deliberate, by presenting procedural discourse as valuable mathematical knowledge Kuki also gave English a higher status than Setswana because procedural discourse in her class was in English. Thus emphasising the cultural models *English is the language of procedural discourse* and thus *English is the language of mathematics*.

IN CONCLUSION

This paper has described a study in which the notion of cultural models was used as a mechanism for describing and explaining language practices in a Grade 4 multilingual mathematics class. The study has shown that in a context like South Africa, where mathematics and English both have symbolic power, and where procedural discourse dominates over conceptual discourse in school mathematics teaching and learning, a practice is forged wherein it is difficult to move mathematics beyond procedural discourse.

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MIDDLE SCHOOL STUDENTS' THINKING ABOUT VARIABILITY IN REPEATED TRIALS: A CROSS -TASK COMPARISON

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This paper summarizes the thinking of 84 middle school mathematics students' about variability in three stochastics tasks that involve repeated trial. Differences in students' acknowledgement of variability were found, depending on whether the task was from a sampling environment, or a probability environment. Students' tended to neglect variability in the probability environment. We conjecture that the way that probability is normally introduced to students is part of the cause of this phenomenon.

INTRODUCTION

Prior to several years ago there had not been much previous research focused on students understanding of variation. The concept of variability was proclaimed to be a missed opportunity in research on students' understanding of data and chance (Shaughnessy, 1997). Much of the previous work on students' understanding of data and chance has concentrated on means (e.g. Mokros & Russell, 1995) or intuitions on probabilities of outcomes and comparisons of relative likelihoods of outcomes (Fischbein & Schnarch, 1997; Konold et al, 1993). Questions about variability tend to involve possibilities for repeated outcomes from sampling, or data from repeated trials of a probability experiment, or shapes of distributions of outcomes. In this paper we report on middle school students acknowledgement of variation across three 'repeated trials' tasks.

RECENT RESEARCH

In the past four years some initial research into students' thinking about variability has begun. Shaughnessy, Watson, Moritz, & Reading (1999) found a variety of types of student thinking about variability in a repeated sampling environment. When presented with a known mixture of colored objects (say 50% red, 50% other colors), most of a sample of over 700 middle and secondary mathematics students from three countries acknowledged variability in the numbers of reds that will be obtained when repeated samples were pulled from the mixture. However, students differed in the way they presented variability in their predictions, and in their reasons for their predictions. When six samples of size ten (with replacement and mixing in between each sample) are drawn from a 50% red mixture, some students predicted a 'reasonable' spread around the expected value of 5 reds in 10 (e.g., 4,7,5,8,6,5—"because they will be around 5, but not exactly"), while others predicted 'high' (6,8,7,6,9,10—"because there are more reds") or 'wide' (4, 0, 10, 2, 9, 3—"because anything can happen"). These researchers also found that upper secondary students who had studied probability had a greater tendency to disregard variation in such predictions on sampling (5,5,5,5,5,5—"because 5 is the most likely outcome each time) than middle school or lower secondary students. Similar results have been reported in an analysis of interviews on sampling situations obtained from students aged 9 to 18 by Reading and Shaughnessy (2000). Recently, Watson has reported results of younger students thinking about variation (Watson, 2002). Watson and her colleagues have also worked on developing a scheme for describing and measuring levels of students' understanding of variability (Watson, 2000; Torok & Watson, 2000, Watson et al, to appear).

THIS RESEARCH

Each of the three tasks reported in this research involves predicting the results of repeated trials: predicting outcomes of repeated samples from a mixture; predicting the distribution of outcomes for repeated rolls of a die; and predicting the results of repeated samples of spinner trials. The questions we are interested in investigating include: 1) What differences, if any, occur in the way students predict results from repeated trials across the three tasks? 2) What reasons do students give for their predictions of outcomes on repeated-trials tasks? 3) How do their reasons differ across task environments?

PROCEDURES.

In the Fall of 2002 survey data was gathered to investigate students' acknowledgment, description of, and reasoning about variability. Tasks involving variability in three environments— sampling, probability, and data sets— were administered to over 300 students in ten classrooms from six schools, two middle schools and four secondary schools. Five of the six schools were located in a large metropolitan area of the United States (2 urban and 3 suburban schools) with the sixth school from a rural location. Each of the six schools involved in this research project has one research project class, in which we are gathering survey, individual interview, and whole class video data. Four of the schools have contributed an additional comparison class, in which only the survey data is being gathered. In this paper we will focus on some initial survey results of the middle school students' thinking about variability in the outcomes from repeated-trials tasks in the sampling and the probability environments. This research is part of a multi-year research project[1] to investigate the development of secondary and middle school students' conceptions of variability.

Eighty-four middle school students in three classes (2 Grade 6 suburban, 1 Grade 7 urban) were administered a written survey investigating their thinking about variability on tasks involving the sampling, probability, and data set environments. The three repeated trials tasks of interest for this paper are given below. We will refer to them respectively as "T1. The Sampling Task", "T2. The Dice task", and "T3. The Spinner task" for the purposes of discussion. In each task, there were several questions that preceded the ones given below to help launch the environments with the students (e.g. "How many reds would you expect to get in one sample of 10? Would it be the same everytime? What would surprise you? What is the chance the spinner lands on the shaded area on one spin? Does 1 or 6 have a better chance of being rolled, or are they the same? Why?)

T1. The Sampling Task

Suppose you have a container with 100 candies in it. 60 are red, and 40 are yellow. The candies are all mixed up in the container. You pull out a handful of 10 candies and count the number of reds.

Suppose six of your classmates did this experiment, each of them pulling out 10 candies. (After each pull, the candies are put back and remixed).

a) What do you think is likely to occur for the numbers of red candies that each classmate would pull out? (Write the numbers of reds in the spaces).

______ ______

______ ______

______ ______

b) Why do you think this?

T2. The Dice Task. Consider rolling a normal six-sided die.

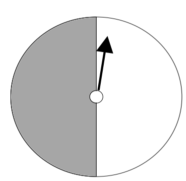
Imagine you threw a die 60 times. Fill in the table below to show how many times each number might come up. Why do you think this?

Number on Dice	How many times it might come up
1	
2	
3	
4	
5	
6	
TOTAL	60

T3. THE SPINNER TASK A CLASS USED THE SPINNER BELOW.

Suppose that you were to do 6 sets of 50 spins. Write a list that would describe what might happen for the number of spins out of 50 the spinner would land on the shaded part in each of the 6 sets of 50 spins.

______, ______, ______, ______, ______, ______



Results and Discussion. Each student was assigned a code indicating whether they acknowledged variation for the outcomes on each task, and how they acknowledged it. Responses were coded R (Reasonable), H (High), W (Wide) L (Low) or with the numeral 6 (for T1), 10 (for T2), or 25 (for T3) if students wrote all 6's or all 10's or all 25's on a list. This type of coding is similar to ones used by previous researchers on repeated trials tasks. The number of students in each class who responded with strings of identical results, such as 6,6,6,6,6,6 for the numbers of reds in the six pulls of the Sampling Task, or 10,10,10,10,10,10 for the number of each of the outcomes from sixty rolls in the Dice Task is recorded in Table 1. For example, the entry 4 – 21 – 5 in the Grade 7 column indicates that there were 4 students who responded 6,6,6,6,6,6; 21students who responded 10,10,10,10,10,10; and 5 students who responded 25, 25, 25, 25, 25, 25 to the tasks T1, T2, and T3 respectively in that class. Also recorded in Table 1. are the numbers of students who predicted a "reasonable" spread in the outcomes for at least one of three repeated trials tasks. In the Sampling Task, outcomes with a range of ≤ 7 for the numbers of reds were considered "reasonable", while responses like 1, 7, 4, 10, 9, 0 were coded as "Wide". If all six outcomes on the list were numbers ≥ 6, the response was coded "High". If all six outcomes were numbers ≤ 6 the response was coded "Low." Similar decisions were made for the other two tasks. For example, a response list with 5 ≤ "numbers"≤ 15 for the frequencies of the die outcomes was considered a "reasonable" spread, as was a response list with a range from 15 to 35 "shaded landings" for the six sets of 50 trials of the Spinner Task.

Classes→	G7 N=29	G6 N=25	G6 N=30	Totals N=84
Tasks →	T1- T2 -T3	T1- T2 -T3	T1- T2 -T3	T1- T2 -T3
6 – 10 – 25 "No Variation" Totals[1]	4 – 21– 5	3 – 13 – 4	0 – 12 – 4	7 – 46 – 13
R – R – R "Reasonable Variation" Totals[2]	19 - 7 - 13	16 - 8 - 14	24 -10 -14	59 - 25 - 41

Table 1. Frequencies of "No Variation" and "Reasonable Variation" responses for each task in each class

1. 6-10-25 indicates the number of students who responded 6,6,6,6,6,6, or 10,10,10,10,10,10, or 25,25,25,25,25,25 respectively for the results of six trials on that task in that class.
2. R-R-R indicates the number of students who responded with a 'reasonable' spread respectively for the results of six trials on that task in that class.

Table 1. indicates that there was a very strong tendency for these students *not* to acknowledge variation when predicting the frequency distribution of outcomes for the dice problem. More than half the students predicted 10,10,10,10,10,10 for the frequencies of the six outcomes for 60 rolls of the die. On the other hand, most of the students *did* predict lists of outcomes for the repeated trials of the Sampling Task and the Spinner Task that had some sort of spread in the repeated outcomes (91% for the Sampling Task and 85% for the spinner task). Over 70% of the students predicted "Reasonable" spreads for the repeated outcomes on the Sampling Task, while only 48% predicted "Reasonable" spreads for the Spinner Task, and only 30% for the Dice Task. These results are quite

consistent across all three classes, and both grade levels. These middle school students clearly felt that the results of the Dice Task should behave quite differently than the results of the Sampling or Spinner Tasks. A comparison of the students' reasons for their decisions on the Sampling and Dice Tasks may help us to understand the differences in their thinking about the two tasks.

Student A: (On T1) "5,6,5,4,6,7…I'd expect 6, a lil more and a lil less." (On T2) 10,10,…,10…This is reasonable since each number has 10 chances."

Student B: (On T1) "6,7,8,5,9,4…Because there are more red". (On T2) "10, 10, …, 10…They all have an equal chance of winning."

Student C: (On T1) "6,5,4,3,6,5…You're not always going to get 6." (On T2) "10, 10, …10, They all have an equal chance of rolling."

Student D: (On T1) "3,4,5,6,7,8…There are more reds." (On T2) "10, 10, …,10…Each number has a one out of six chance."

Student E: (On T1) "6,10,0,5,8,9....Students can pull any number." (On T2) " 10, 10,…,10…Each number has an equal chance."

Student F: (On T1) "6,7,5,6,6,5…Most of the candies are red." (On T2) "10, 10, …,10…There is only one of each number so each number has the same chance."

In their responses to the Dice Task, the majority of the students were focusing only on the theoretical probability of a single outcome for one roll of the die, 1/6 for any number, whereas they were much more likely to consider a range of possible outcomes in either the Sampling Task or the Spinner Task. Previous research has indicated that the teaching of theoretical probability for single outcome events might interfere with students' attention to variability in the results of repeated trials (Shaughnessy et al, 1999). It is likely that these students have had experiences with calculating theoretical probabilities for the outcomes of rolling one or two dice in the past. They know they should expect a probability of 1/6 for any number on one toss. On the other hand, their responses on the survey also indicate that they know that the chance that the spinner lands on the shaded part on one spin is 1/2. Knowing the probability for the spinner does not cause them to predict 25 shaded spins out of 50 every time anywhere near as often as they predict 10 for each number on 60 die tosses. To these students, the die is "supposed" to come out fair. What else could one possibly mean by the word "fair die?"

Furthermore, these students were not consistent across the three task environments on their predictions for the variability in outcomes of repeated trials. Of the 84 students in the study, only 14 of them (16%) predicted reasonable (R) lists of outcomes for *all* three tasks, and only 5 of the students indicated no variation at all on *all* three of their predicted lists (predicting all 6's, all 10's *and* all 25's respectively on T1, T2, and T3). Students' responses across the three tasks were all over the place, with the most frequent coding triad being R – 10 – R for 16, about 19%, of the individual students. These 16 students expected a "reasonable" spread of results around 6 for the Sampling Task, and around 25 for the Spinner Task, but doggedly held that all 6 numbers on the die would occur 10 times.

Based on the results of this study, we conjecture that students are likely to predict constant results for repeated trials in a familiar probability situation like the Dice Task, and to neglect the issue of variability in the frequencies of individual outcomes. Although some students did acknowledge variability on the Dice Task (Student G: “12, 11, 10, 12, 9, 9…These numbers are all *around* 10”), these students were in the minority. We believe that part of the reason for such student reasoning may be the way that probability is taught in our schools. All too often we rush our students to calculating the probability of individual events or probabilities of particular outcomes, without consideration for the variation in results that can occur in actual repeated trials. We rarely give our students opportunities to develop their intuition for a likely “range of outcomes” in repeated trials situations, especially when there is a convenient probability model, like the uniform distribution for the Dice Task, to tap. We conjecture that if we want students to attend to variability *across* a variety of environments, we will need to raise *explicit* attention to variability in those environments in our work with students. This is particularly true of probability tasks, like the Dice Task. Rather than ask, “What is the probability of getting a 6” we might better ask “If we rolled the die 25 times, how many sixes do you think you would get? Now, suppose four students each rolled the die 25 times? What would the list of the numbers of sixes each of them obtained look like?” It is not just the exact probability of an outcome that is important in data and chance, but perhaps even more so, how that outcome is situated within the distribution of outcomes for an experiment, and what the “likely range” of outcomes for the experiment will be.

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STUDENTS' UNDERSTANDING OF Z_N

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In this study, we explore six students' conceptions of $\mathbf{Z}_n$ *in an effort to understand students' conceptions of quotient groups in general. We discovered that there were three different ways our students thought about* $\mathbf{Z}_n$*, namely as infinite sets, element-set combinations, and representative elements. We explore how* $\mathbf{Z}_n$ *might be conceived of in terms of these three cognitively different interpretations, as well as our students' difficulties in working with* $\mathbf{Z}_n$ *using each of these interpretations. We conclude that* $\mathbf{Z}_n$ *is not a trivial example of a quotient group, and provide recommendations for teaching* $\mathbf{Z}_n$ *and other quotient groups.*

INTRODUCTION

The first course in abstract algebra is considered by mathematics students and teachers alike to be a troublesome and often disappointing experience. There may be many reasons why students struggle in abstract algebra; the course requires skills in proof, conceptual understanding of abstract structures, facility with complex notation, and mathematical breadth sufficient to understand examples developed using functions, matrices, complex numbers, and permutations. The course usually moves quickly and presupposes a good deal of mathematical maturity. Because of the difficulty of the course, students often leave with negative feelings toward proof, abstraction and formal mathematics

In our own experience with teaching abstract algebra, we have found that student attrition rises steeply when students encounter the concept of quotient or factor group. The relatively little research that has been done on students' understanding of abstract algebra seems to confirm our experience, but does not help us understand why students struggle with quotient groups. Dubinsky, Dautermann, Leron and Zazkis (1994) suggest that students' struggles are due to the complexity of the quotient group concept, building as it does on notions of subgroup, normality, and group operations. However, they provide little insight into exactly how students are thinking of these underlying constructs, and how their understanding contributes to or undermines their attempts to make sense of quotient groups. Asiala, Dubinsky, Mathews, Morics and Oktaç (1997) observe that students who were successful in working with quotient groups often computed coset products using the representative method (i.e., writing the coset aH as a). However, they note that students' use of this type of notation gave little information about how the students were thinking about cosets and quotient groups.

To better understand why students struggle with quotient groups, we decided to investigate students' conceptions of $\mathbf{Z}_n$, the cyclic subgroups of order n. We chose this collection of quotient groups because they are typically used in abstract algebra texts as straightforward, unproblematic first examples of quotient groups, in hopes that their mathematical familiarity will help students understand the more abstract case (cf Herstein, 1999). We explore in this paper why $\mathbf{Z}_n$ as a prototype of a quotient group is nevertheless difficult to understand, and what this might imply for understanding quotient groups in general. In particular, we address the following research questions:

1. What are students' conceptions of $\mathbf{Z}_n$ as a quotient group?
2. What difficulties do students have in understanding $\mathbf{Z}_n$ as a quotient group?

METHODOLOGY

The data we report here were collected as part of a larger study designed to investigate students' conceptions of quotient groups. The study consisted of two parts: the compilation of case studies (Stake, 1998) of six undergraduate students enrolled in an undergraduate introductory course in abstract algebra taught by Williams; and an analysis using grounded theory (Strauss & Corbin, 1998) of the cases and other class data to identify conceptions, themes and problems common across students in the class. The six students who participated in the case studies were recruited based on their willingness to participate and selected so that there were an equal number of men and women. Siebert conducted six 45-minute, semi-structured interviews with each student during the semester. Tasks in these interviews focused on students' understanding of groups and quotient groups, including $\mathbf{Z}_\mathbf{n}$ for different values of n. These interviews were videotaped, and careful fieldnotes were created for each interview. Additional data were collected from the class as a whole, including videotapes of class instruction, detailed fieldnotes of class, and copies of the written work of all students in the class.

Our attempts to understand students' conceptions of $\mathbf{Z}_n$ represent a dialectic between a "top down" and "bottom up" approach. Our study was "top down" in the sense that we used our own conceptions of what is important to know about quotient groups to select tasks that would reveal whether or not students possessed the understanding we valued. At the same time, however, we were sensitive to the way students made sense of and approached tasks. In this way, our study was also "bottom up," in that students' solutions often caused us to rethink what ideas and images were important to understanding quotient groups. Thus, students solutions not only informed us about how they thought about $\mathbf{Z}_n$, but also led us to change our own understanding of $\mathbf{Z}_n$. Our newfound understanding often led to the development of new tasks leading to new data, and further revisions of our model of understanding $\mathbf{Z}_n$. We refer to this cyclic process as *grounded content analysis* (Lobato, personal communication, 1999), and recognize that content analysis cannot be conducted in the absence of the students to whom we wish to teach the content.

The second part of our study—the analysis using grounded theory of the cases and classroom data for common themes, conceptions, and problems—began during data collection. Important themes and problems were identified from class fieldnotes and from the journal entries Siebert wrote after each student interview. Subsequent interview questions pursued these themes and problems. Once all of the data were collected, we compiled descriptions of each target student's understanding of $\mathbf{Z}_n$ and quotient groups in general. We also identified and transcribed relevant segments from class instruction on $\mathbf{Z}_n$, and reviewed all the students' homework and test responses related to $\mathbf{Z}_n$. What emerged from these comparisons of correct and incorrect solutions was a framework of concepts that were foundational for our students to understand $\mathbf{Z}_n$, as well as insights into the difficulties students had with these concepts. Due to the space constraints of this paper, we only present our findings concerning three interpretations of $\mathbf{Z}_n$ and students' difficulties in perceiving $\mathbf{Z}_n$ in these three different ways.

THREE INTERPRETATIONS OF $\mathbf{Z}_n$

Based upon our analysis of students' thinking, we propose that there are three different ways that students might reason about $\mathbf{Z}_n$: as infinite sets, as element-set combinations, and as representative elements. We illustrate these three interpretations in Table 1 using $\mathbf{Z}_4$ as an example. Note that each of these three conceptions involves a unique representation of cosets and coset operations. However, the differences between these three interpretations of $\mathbf{Z}_n$ is more than mere notational variation. Each of these three conceptions involves an algebraic group composed of cognitively and experientially different, albeit mathematically equivalent, objects. In other words, while one may assert that there is no real mathematical difference between the structures of the three algebraic groups, we suggest that there is a vast difference in how students think about and operate on the objects that comprise each group. We briefly explain each one of these interpretations below.

Table 1: Three Interpretations of Z_n illustrated with Z_4.

	Infinite Sets	Element-Set Combinations	Representative Elements
Elements	{…, -8, -4, 0, 4, 8,…}	$0 + 4\mathbf{Z}$	0
	{…, -7, -3, 1, 5, 9,…}	$1 + 4\mathbf{Z}$	1
	{…, -6, -2, 2, 6, 10,…}	$2 + 4\mathbf{Z}$	2
	{…, -5, -1, 3, 7,11, …}	$3 + 4\mathbf{Z}$	3
Operation	Add all elements of one set to all elements of another	$(a + 4\mathbf{Z}) + (b + 4\mathbf{Z})$ $= (a + b) + 4\mathbf{Z}$	$a + b = (a + b)\text{mod}4$

Infinite Sets

Under this interpretation, the group $\mathbf{Z}_n$ is comprised of a collection of infinite sets. The group operation is set addition, defined to be the collection of all possible sums created by adding single elements from one set to single elements from the other. To form the sum, students need to add several single elements from one set to several single elements from the other set until they see a pattern and are able to write the infinite set that contains all possible sums.

Element-Set Combinations

In this second interpretation, group objects are comprised of an element and a set. The element serves as an operator on the set, because it shifts all of the elements of the set along the number line the same number of spaces as its value. For example, $1 + 4\mathbf{Z}$ is all the multiples of 4 shifted right one position. The group operation for these objects is grounded in the infinite set interpretation of cosets. To add shifted sets, we use the same type of addition as for infinite sets. However, this is equivalent to shifting the original set by the sum of the two element operators. For example, $(1 + 4\mathbf{Z}) + (2 + 4\mathbf{Z})$ is the same as

$(1 + 2) + 4\mathbf{Z}$, because when we add any multiple of 4 shifted 1 to the right to any multiple of 4 shifted 2 to the right, we get a multiple of 4 shifted 3 to the right.

Representative Elements

In this third interpretation, group objects are single elements. The group operation consists of adding elements modulo n, also referred to as clock arithmetic. In this interpretation, the set aspect of the group elements is hidden by the operation. Thus, students are able to do calculations without ever invoking the image of a set.

Relative Strengths of the Three Interpretations

Each interpretation provides our students with unique conceptual insights into $\mathbf{Z}_n$. The infinite sets interpretation brings to the foreground the set nature of cosets. Students cannot avoid thinking about the group operation in $\mathbf{Z}_n$ as operating on an object comprised of a collection of elements. Furthermore, the infinite set interpretation is needed to provide a satisfactory conceptual justification for why the group operation for the element-set interpretation is defined the way it is, and why it is not defined as $(a + n\mathbf{Z}) + (b + n\mathbf{Z}) = (a + b) + 2n\mathbf{Z}$. On the other hand, the representative interpretation of $\mathbf{Z}_n$ brings to the foreground the element operators on the original subgroup, and drastically reduce the cognitive load required for computation. Finally, the element-set interpretation balances both a set interpretation and an element operation interpretation. As such, it plays a critical role in proofs and conceptualizing the structure of $\mathbf{Z}_n$.

STUDENTS' UNDERSTANDING OF THE THREE INTERPRETATIONS OF $\mathbf{Z}_n$

Although students varied in their ability to work successfully with the three different interpretations of $\mathbf{Z}_n$, we were nonetheless able to identify common trends across the six target students in our study. We present these trends below for each of the three interpretations of $\mathbf{Z}_n$.

$\mathbf{Z}_n$ as a Group of Infinite Sets

We found that all six students were able to write examples of $\mathbf{Z}_n$ as a collection of infinite sets and perform the group operation on those sets. However, students universally had a different understanding of the group operation than we had intended. To add infinite sets, the students took one element from one set, added it to one element in the other set, and then identified the infinite set that contained the sum of the two elements as the sum of the two infinite sets. While this method of operating always leads to correct results, it is different from thinking about operating on infinite sets as whole objects. This led to problems when students were asked to explain why they could add whole sets by just adding two elements. For example, David tried to justify this method for adding two cosets in $\mathbf{Z}_{10}$ by listing the two sets and adding elements that were vertically lined up, as shown in Figure 1. David was unable able to recover once he recognized that his result was actually an infinite set from $\mathbf{Z}_{20}$, not $\mathbf{Z}_{10}$. In fact, we found that students' most common methods for trying to add two infinite sets were either vertical addition of elements, as David did, or set union, despite having seen the instructor demonstrate the correct method for adding infinite sets in class. In the interviews, only two students were able to produce the valid group operation for adding whole infinite sets, and then only after a great deal of experimentation.

$$
\begin{array}{r}
0+10\mathbf{Z} = \left\{\ldots, -20, -10, 0, 10, 20, \ldots\right\} \\
6+10\mathbf{Z} = \underline{\left\{\ldots, -14, -4, 6, 16, 26, \ldots\right\}} \\
\left\{\cdots, -34, -14, 6, 26, 46, \ldots\right\} = 6+20\mathbf{Z}
\end{array}
$$

Figure 1: David's incorrect addition of infinite sets.

$\mathbf{Z}_n$ as a Group of Element-Set Combinations

We found that all six of our students could write examples of $\mathbf{Z}_n$ in the form of element-set combinations and perform the group operation correctly. Furthermore, students were generally more successful in justifying the group operation for element-set combinations than they were for infinite sets. Four of the six students noted that subgroups of $\mathbf{Z}$ are normal, and thus by a theorem they had investigated in class, the coset operation was well-defined. However, despite these successes, there were occasional lapses in students' attention to and understanding of the set part of the element-set combination notation. These lapses showed up in students' notational mistakes. For example, when given the problem of determining what $4\mathbf{Z}+6\mathbf{Z}$ produced, several students interpreted $4\mathbf{Z}$ and $6\mathbf{Z}$ as the cosets $4+\mathbf{Z}$ and $6+\mathbf{Z}$, and then added the element parts of these cosets to get $10+\mathbf{Z}$, which they wrote as $10\mathbf{Z}$. Occasionally students inserted elements into the set part of the coset notation, as Mandy did when she wrote cosets in $\mathbf{Z}_{12}/(4_{12})$ in the form of $n+4_{12}$ instead of $n+(4_{12})$. These mistakes suggest that students were often thinking of cosets in terms of elements and not as sets.

$\mathbf{Z}_n$ as a Group of Representative Elements

In general, our students were most successful with the representative interpretation of $\mathbf{Z}_n$. Students were adept at working with the clock arithmetic group operation for representative elements. We looked for evidence of misconceptions concerning students' understanding of this interpretation of $\mathbf{Z}_n$ and were unable to find any. Students were able to correctly solve all problems involving the representative element interpretation of $\mathbf{Z}_n$. However, as demonstrated above, their success with this interpretation cannot be interpreted as an indication of solid understanding of $\mathbf{Z}_n$. In particular, by operating with representative elements, students are not required to address the complexity of thinking of $\mathbf{Z}_n$ as a collection of sets.

Students' Flexibility with the Three Interpretations

For the purposes of proof and conceptual understanding, the element-set interpretation is likely to be the most powerful. However, it derives its power from coordinating the infinite set interpretation with the representative element as an operator. In other words, unless students are able to maintain meaning for both the element and the set aspects of the element-set objects, they become little more than formal symbols. We found that in many instances, the set component of the element-set objects became merely a notational baggage with little meaning, as demonstrated above with students' syntactical errors in writing cosets in element-set notation.

We hypothesize that students often lacked meaning for the set component of the element-set interpretation because they did not understand how to operate on infinite sets. In other words, while students knew the definition of the operation on element-set objects and generally recognized that the element-set operation was well-defined, this did not help them understand why $(a + n\mathbf{Z}) + (b + n\mathbf{Z}) \neq (a + b) + 2n\mathbf{Z}$. Naturally, an instructor might provide mathematical arguments justifying the element-set interpretation by appealing to group closure or the operation being well-defined, but these are not cognitively satisfying. Students' experience with adding algebraic expressions suggests that they should add like terms, so that $(a + n\mathbf{Z}) + (b + n\mathbf{Z})$ should yield $(a + b) + 2n\mathbf{Z}$. When students discover that this is not how element-set objects are added, then it is difficult for them to maintain meaning for the set component, because it does not receive the same status in computations as does the element. In other words, the element-set objects become essentially representative elements with the set appended at the end. Thus, the operation for element-sets does not contain any more explanatory power than the operation for representative elements.

CONCLUSIONS AND RECOMMENDATIONS

The cyclic groups $\mathbf{Z}_n$ are in many ways the prototypical examples of quotient groups. It is often assumed that because they are formed from a familiar set by a simple relationship, they are easily accessible to students. However, our analysis of students' thinking about $\mathbf{Z}_n$ suggests that $\mathbf{Z}_n$ is a cognitively complex algebraic structure that involves three different cognitive interpretations. Because of this complexity, abstract algebra instructors cannot assume that their students will naturally and easily grasp the complexities of $\mathbf{Z}_n$ as an example of a quotient group.

We found that students were able to do computational problems within $\mathbf{Z}_n$ without difficulty by thinking of it as a collection of representative elements under clock arithmetic. Indeed, our students tended to reduce the objects in $\mathbf{Z}_n$ into single elements whenever possible. While it is often useful to think about group objects as single elements when working with quotient groups, it is also important to be able to flexibly return to thinking of the group objects as sets when necessary. To move beyond computational facility, our students needed to think of $\mathbf{Z}_n$ in terms of element-set combinations. Such an understanding is built upon an understanding of elements of $\mathbf{Z}_n$ as infinite sets – specifically, as sets created by "shifting" the subgroup n**Z** by adding an integer to each element. Thus, full understanding of $\mathbf{Z}_n$ as element-set combinations must take into account both the infinite set n**Z** *and* the element that shifts it. Given this understanding, the definition of addition in $\mathbf{Z}_n$ becomes natural, and students' ability to flexibly deal with problems and proofs is greatly enhanced. Our data suggest that a full understanding of $\mathbf{Z}_n$ as element-set combinations that allowed for such flexibility was not common among our students.

Recommendations for Teaching $\mathbf{Z}_n$

Students will need substantial help in understanding $\mathbf{Z}_n$. In particular, instructors will need to help their students understand the three cognitively different interpretations of $\mathbf{Z}_n$, because the ability to work with $\mathbf{Z}_n$ as a quotient group often requires students to work flexibility within and between these three different interpretations of $\mathbf{Z}_n$. Furthermore, our research suggests that the most difficult part of understanding and coordinating the three interpretations of $\mathbf{Z}_n$ is helping students to focus on and understand the key role that sets play in the construction of $\mathbf{Z}_n$. Our students tended to lack this flexibility in moving from thinking of group objects as elements to thinking of them as sets again. We suggest that students need specific experiences in working with $\mathbf{Z}_n$ as a collection of infinite sets and exploring all the different possible group operations to identify an operation that yields group structure. Students also need opportunities to move back and forth from element-set combination and representative element interpretations to the infinite set interpretation.

Recommendations for Teaching Other Quotient Groups

Upon reflection, we feel that understanding a quotient group in terms of sets, element-sets, and representative elements would be helpful when studying any quotient group, not just $\mathbf{Z}_n$. Each of these interpretations is not only applicable to other quotient groups, but also emphasizes different aspects of a quotient group, and thus enhances understanding. As with $\mathbf{Z}_n$, the set interpretation of the quotient group brings to the foreground the set nature of cosets, and also motivates and justifies the operation on element-set objects. In contrast, the representative element interpretation reduces the cognitive complexity of the group structure by associating them with their element operators, perhaps allowing students to more easily perceive and recognize emergent properties of the resultant quotient group. Finally, an understanding of the element-set interpretation supports and in turn is supported by an understanding of the other two interpretations. The element-set interpretation is particularly crucial because it is the representation that coordinates both the set and element operator aspects of cosets. For these reasons, we recommend that instructors of abstract algebra consider addressing these three interpretations of quotient groups for any of the quotient groups their students study.

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TEACHERS' MATHEMATICS: CURIOUS OBLIGATIONS[1]

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W e explore the nature and consequences of teachers' problem solving through an example of a teacher's mathematical problem solving as it was occasioned by a student's mathematics. This illustration demonstrates the value of an interpretive framework that points to the mathematics of the classroom as a collective. Arising from this exploration is our core assertion, that the mathematics teacher has an obligation to be curious about mathematics. Our research has significant implications for teacher education as it points to the significance of the teacher's mathematics within the classroom collective and the possibilities for the growth of the teachers' understandings within that collective.

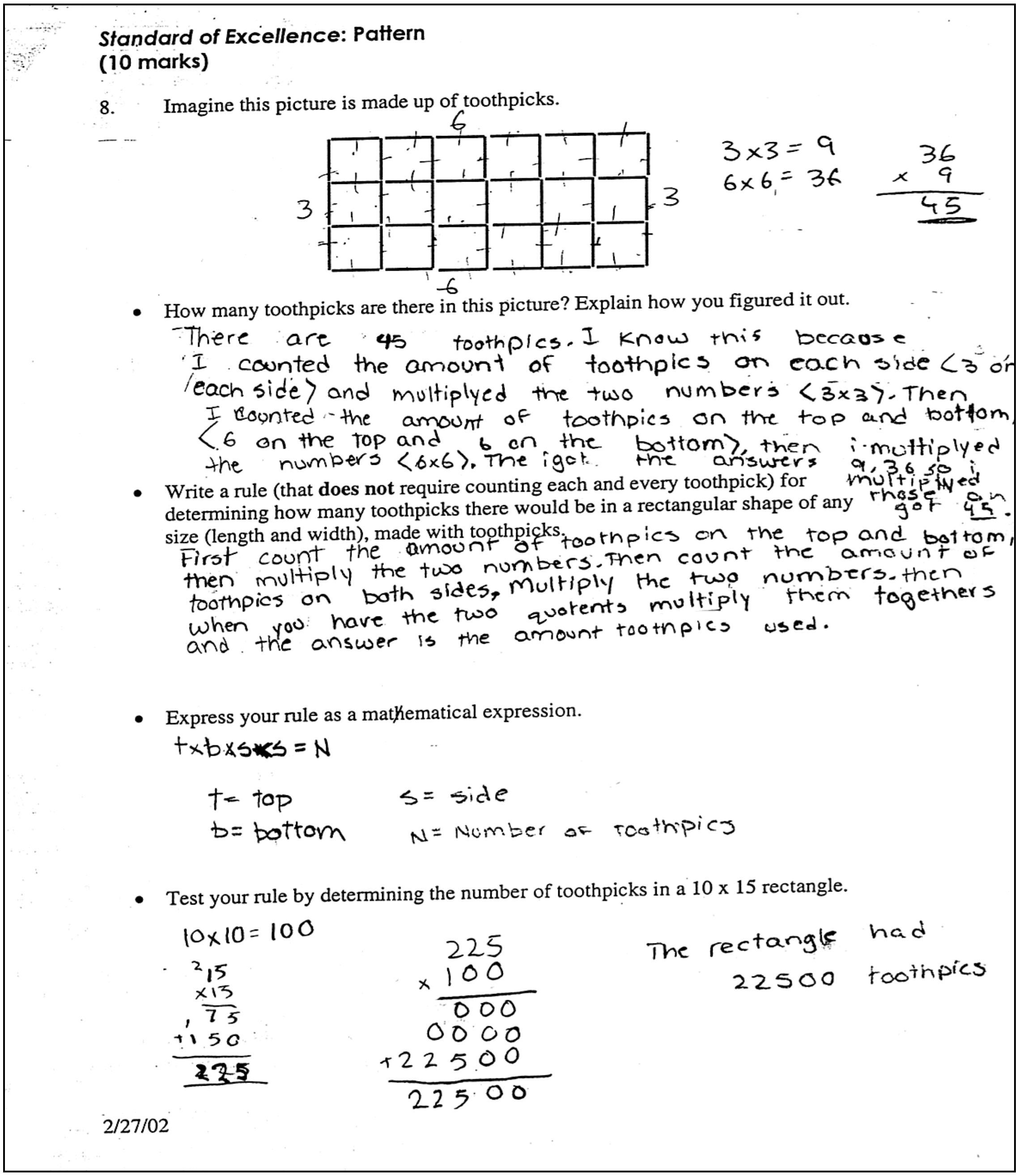
THE WAY TEACHERS NEED TO BE WITH MATHEMATICS

Every now and then, the mathematics teacher is compelled to engage in mathematical problem solving. We use one such event to frame this paper. In a recent classroom-based study, a test question, a student's response to that question, and the teacher's response to the student's response prompted us to rethink some of our own assumptions about the way teachers need to be with mathematics.

We borrow this idea from Ball and Bass (2002) who argue that the teacher's knowledge of mathematics is of a different sort than that of the research mathematician. Briefly, the research mathematician's work might be characterized in terms of the formulation of increasingly powerful generalizations. One might say, such efforts are oriented toward the compression of ideas. By contrast, the grade school teacher's responsibilities are more about decompressing mathematical ideas and student responses to those ideas. The teacher, that is, must have well-honed abilities to "pull apart", "unpack", and otherwise interpret the mathematizations that she encounters.

The focus of this writing is another aspect of the way a teacher needs to be with mathematics. We argue the teacher also has an obligation to be curious about the subject matter that she teaches—which, although clearly an issue in psychology, is not a topic that is often addressed an the contemporary psychology of mathematics education literature. The idea does find support in related literatures, however. For example, the importance of curiosity in research mathematics is a common theme (e.g., Burton, 1999a). As well, as Damasio (1994) develops from a neurological and cognitive scientific perspective, curiosity is among the many emotions that are necessary to the development of the rational faculties.

As a route into our discussion, we invite you to review, first, a seventh grade student's response to an exam question that was posed after a unit on patterns and relations (Figure 1), and then the teacher's log entry in response to Arlene's answer.



Standard of Excellence: **Pattern**
(10 marks)

8. Imagine this picture is made up of toothpicks.

6

3 3

6

3x3 = 9
6x6 = 36

36
x 9
45

- How many toothpicks are there in this picture? Explain how you figured it out.

There are 45 toothpics. I know this because I counted the amount of toothpics on each side (3 on each side) and multiplyed the two numbers (3x3). Then I counted the amount of toothpics on the top and bottom, (6 on the top and 6 on the bottom), then i multiplyed the numbers (6x6). The i got the answers 9, 36 so i multiplied these an got 45.

- Write a rule (that **does not** require counting each and every toothpick) for determining how many toothpicks there would be in a rectangular shape of any size (length and width), made with toothpicks.

First count the amount of toothpics on the top and bottom, then multiply the two numbers. Then count the amount of toothpics on both sides, Multiply the two numbers. then when you have the two quotents multiply them togethers and the answer is the amount toothpics used.

- Express your rule as a mathematical expression.

t×b×s×s = N

t = top s = side
b = bottom N = Number of toothpics

- Test your rule by determining the number of toothpicks in a 10 x 15 rectangle.

10x10 = 100

15
x15
75
150
225

225
x 100
000
0000
+22500
22500

The rectangle had 22500 toothpics

2/27/02

Figure 1: Arlene's Response

THE TEACHER'S JOURNAL ENTRY

In marking this exam paper, I scanned the student's response and found myself looking at a "correct number" with an equation for finding that number that I did not recognize. It was neither like my solution nor any of the student solutions that I had already marked. My immediate response to Arlene's work was to wonder, "Is her computation correct?" Within her solution there are some errors; however, it is the case that $6^2 + 3^2 = 45$, the correct number of matchsticks for the case given.

I turned to her written work and could not find anything in it that helped me understand why summing the squares of the length and the width would produce the correct result. I speculated that Arlene's generalization certainly would not work for the 10 x 15 case and went about testing it only to find out that it does lead to a correct answer. (Her work was not of much help to me given the mistakes and apparently incorrect reasoning.) Finding that the 10 x 15 case could also be calculated by adding the squares of the dimensions was highly bothersome for me since I had carefully (or thought I had carefully) selected the "test" case. I was deliberate to select a second example that was unrelated to the first case—unrelated in the sense that I worried about it being a multiple of the first case for example, thus making the second one a case of the first.

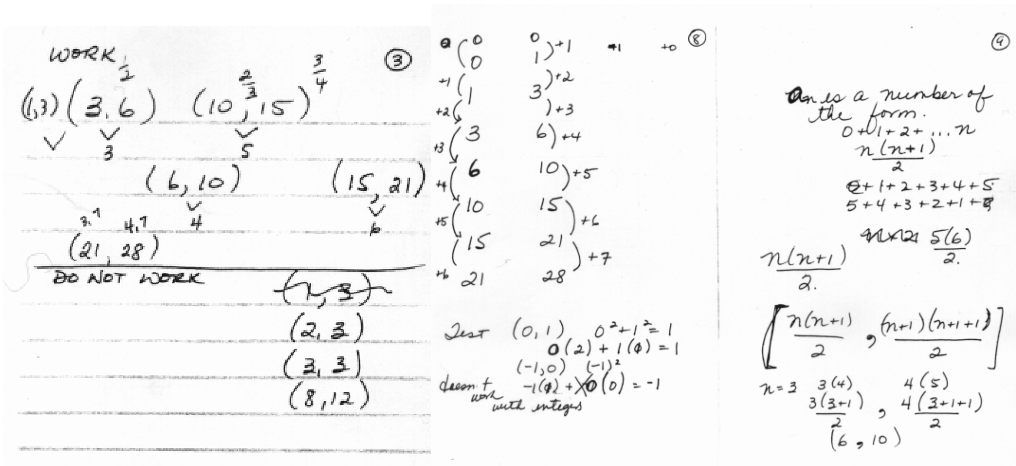


Figure 2: Extracts from the teacher's specializing and generalizing

Seeing two cases that both could be solved with a model that I could not yet "see" related to the problem I realized that in order for Arlene's method to work that it must be equal to the expression I wrote to model the problem. I wondered, "Is $l^2 + w^2 = l(w+1) + w(l+1)$ in general?" With only two lines of algebraic manipulations it was clear this is not the case. Hence, it could only be true under special circumstances and clearly a 3 x 6 rectangle and a 10 x 15 were two of those special cases.

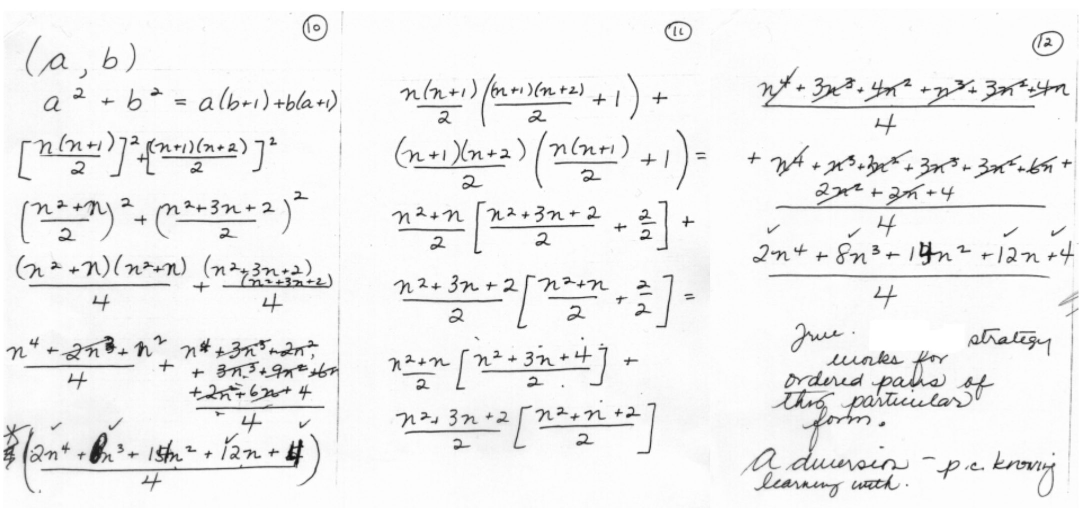


Figure 3: The teacher's proof

Muddling my way through this new problem and trying to make sense of the situation I was now in I specialized and put the dimensions back into the two equations and saw where they

> came together but it did not help much. So, I tried yet another case. This time I tried a 2 x 3 rectangle. I now had a case that did not work with Arlene's model. Now my curiosity was fully engaged. I needed to know: "Under what conditions does $l^2 + w^2 = l(w+1) + w(l+1)$?"
>
> After 4 pages of work in which I moved from specializing to generalizing and back again I learned that $l^2 + w^2 = l(w+1) + w(l+1)$ when l and w are two numbers, n and n+1, such that n is the sum of the first x whole numbers and n+1 is the next such number. When shown the problem, a colleague noticed these were triangular numbers; only then did I recall that fact.

As I researcher, I reflect back on my response to Arlene's mathematics and wonder, How was I with the mathematics?

A TEACHER'S OBLIGATIONS

One of the major themes to emerge in the mathematics education literature over the past few decades is the matter of teacher attendance to students' efforts to represent their emergent understandings. The topic of teachers attending to learners has been particularly prominent within the constructivist literature, as radical and social constructivists alike have highlighted the issue. See, for example, Pirie and Kieren's (1994) model for observing the growth of mathematical understanding and the Cognitively Guided Instruction Group's (Franke & Kazemi, 2001) strategy of engaging teachers in careful observation of students' mathematical activity.

More recently, Ball and Bass (2002) have tied this issue of teacher attendance to the issue of teacher knowledge of mathematics. This research focuses mainly on the sorts of conceptual competencies and interpretive abilities that are necessary for one individual to make sense of another's understandings. In the interpretations that follow, we concern ourselves more with the classroom collective than with individual understandings. Our intention is to examine the contribution to the collective of teachers' responses to student work. (Note that, as demonstrated in the example presented, we are not limiting the discussion to teachers' responses to students themselves. Our more general concern is the matter of their responses to student work, whether or not the substance of those responses are represented to students.) As elaborated below, we view the individual student as just one of a number of nested complex learning systems, and we wonder if the individual learner is the learning system that should be the primary focus of the teachers' attentions. In effect, we are trying to ask, if we change our assumptions about the individual as the locus of mathematics learning in school classrooms—and instead focus on the classroom collective as the learning system of interest—then what becomes the obligations of mathematics teachers?

In the case presented, the teacher's mathematizations were neither brought back to the student nor represented to other members of the class. At first pass, then, it might seem odd that we have framed our core assertion—that mathematics teachers have a sort of obligation to be curious about the subject matter—with this particular narrative. After all, the most common rationale for the teacher to engage in mathematical inquiry is so that she or he can better model for students what it means to engage with mathematical problems and processes. For us, the rationale of modeling is hinged to some deeply engrained but problematic assumptions. In particular, the notion that teaching is a modeling activity seems to be rooted in the assumption of radical separations among persons in the classroom. The teacher models, the learner mimics, but their respective actions are seen to be separable and to spring from different histories, interests, and so on.

We favor a different interpretation of human interaction, one that posits more profound intertwinings of identities and intentions. Framed by ecological theories (e.g., Bateson, 1979) and complexity science (e.g., Johnson, 2001), we argue that the reason for the teacher to engage in mathematical inquiry—or, more specifically, the reason the teacher is obligated to be curious about mathematics—has to do with her or his role in the emergence of a mathematical community, not with modeling. In terms of classrooms, whereas the imperative for the teacher to model mathematical engagement appears to be rooted in the assumption that the classroom is a collection of learners, we believe that the teacher needs to be mathematically curious because he or she is a part of the collective learning system of the classroom. This shift in phrasing is more than a rhetorical gesture. Academically speaking, it corresponds to a recent elaboration of established interests in the psychology of learners toward an interest in the psychology of social systems. (See, e.g., Burton, 1999b.) This conception of a classroom, in terms of a collective character rather than a collection of individual characters, is consistent with insights from the emergent field of complexity science.

Elsewhere we have discussed some of the common ground and some of the divergences of complexity science (and related discourses, such as enactivism) and many of the theoretical perspectives that currently figure prominently among mathematics education researchers, including radical constructivism and social constructionism (see Davis & Simmt, in press; Gordon Calvert, 2001; Towers & Davis, 2002). As such, we do not address that topic here. Instead we use complexity as a window into the role of teachers' mathematical curiosities in the project of school mathematics.

"SITUATED IN" VERSUS "PART OF"

In the main, when matters of individual learning and collective groupings are both addressed, the discussions tend to be framed in terms of what it means for learning agent(s) to be situated in particular social context(s). The locus of learning, that is, is generally assumed to be the solitary human who is cast as a sort of fundamental particle of cognition. Complexity science challenges the deeply engrained cultural assumptions that underpin this habit of interpretation. For the complexivist, learning is a broader notion, coterminous with the idea of evolutionary adaptation. Whenever a coherent system undergoes transformations in a manner that enable it to maintain its coherence within its dynamic circumstances, in complexity terms, it can be said to have learned. Events of learning thus include such diverse phenomena as the formation of the European Union, adjustments in stock market values, the rise of life on the earth, and the emergence of consciousness in a species. In terms of the project of modern schooling, some relevant learning systems include societies, mathematics (understood in terms of the intertwined activities of a mathematical culture), schools, and classrooms—in addition to individuals. Culturally speaking, this recent assertion of complexity science might be interpreted as a remembering of an ancient intuition. There has long been a tendency to discuss and describe each organization's level in this range of phenomena in terms of bodies (e.g., a body of knowledge, a student body) that grow and adapt.

This shift in frame is what prompts us to speak differently about individuals and classrooms—not as agents in situations, but as coherent forms that are parts of coherent forms (that are parts of coherent forms, and so on). The classroom collective unfolds from and is enfolded in learners and teachers. This frame undercuts many of the binary oppositions that are so often used to characterize learners and schooling—most obviously, perhaps, the common contrast of teacher-centered and student-centered instruction. There are no centers to complex systems.

There is a problem with this manner of characterization, especially when applied to something as deliberate as mathematics teaching. In our experience, it is rare and unusual that a classroom collective emerges around the generation of mathematical knowledge. Rather, the collective project (and when humans gather together, there is always some manner of collective project, even if is mutual destruction) seems most often to be organized around matters of social positioning. For us, the interesting question—and the common feature of our varied research efforts over the last decade—has to do with the emergence of collective possibility around the mathematics itself. With regard to the topic at hand, it is here that the issue of a teacher's personal engagement with mathematics takes on a particular relevance. The example of Arlene and the teacher provides a good example of what we mean by the expression, "a classroom collective emerges around the generation of mathematical knowledge". As we examine the 11 pages of the teacher's mathematizing, we encounter several mathematical claims that are new to us. To mention one, in terms of a theorem, for every pair of consecutive triangular numbers, a and b—but only for pairs of consecutive triangular numbers—the following is true: $a^2 + b^2 = 2ab + a + b$.

Whose insight is this? Following the conventions of contemporary research culture, and assuming it hasn't already been published, it clearly belongs to the teacher. However, an attention to the events that surround the emergence of the insight reveals that, in fact, the idea arose in the cogitations in a few overlapping communities, which themselves are hooked into still broader communities. For instance three key events that contributed to the emergence of the idea, and without which the insight might never have arisen, are Arlene's erroneous response, the teacher's accidental selection of two pairs of numbers that satisfy the theorem, and a colleague's casual mention that the numbers generated by the teacher are triangular.

An error by a student, a coincidental choice by a teacher, and a comment from a colleague. Such elements are not the typical fare of mathematical progress. Or are they? A close examination of the strew of interests represented in current mathematics research suggests that the there is something troublesome about the classic definition of 'progress' as a linear movement toward a perceptible goal. Progress seems to be neither linear nor directional, but more about the pursuit of interests that unfold as individuals and collectives negotiate their ways through constantly shifting interpretive backgrounds. We might further highlight the recursively elaborative nature of the event to emphasize the nonlinear, nondirectional nature of mathematical insight. Culminating in this paper, this writing might be described as the interpretation of a group of mathematics education researchers to an interpretation of a teacher-researcher to the interpretation of a colleague to the interpretation of the teacher-researcher to the interpretation of a student's interpretation of a test question. These qualities of complex intertwinings, recursive elaborations, and unforeseeable ends are what prompt us to argue that the teacher must be mathematically curious. This curiosity cannot be framed in terms of causal influence in one's teaching. It is more a matter of necessary contribution. To underscore this point, we would hazard the claim that, in our combined recollected experiences as researchers, teachers, and learners, every "teachable moment" that we've encountered has been dependent on (but, of course, not determined by) the teacher's expressed curiosities.

Teachable moments, we believe, are moments of complex emergence—that is, moments in which diverse agents cohere into collectives with shared purposes and insights. And although we argue that teacher curiosity is a critical element, we would not want to diminish the significance of student's individual interpretations, their private interactions, the texts and other artifacts that are made available, and so on. However, we do feel that teacher curiosity stands out as a critical element in the mathematics classroom. As the above example demonstrates, it compels teacher attendance to student articulations, it opens up close-ended questions, and (as described elsewhere, see Davis & Simmt, in press) it can trigger similar contributions from learners. Indeed, we would go so far as to suggest that one of the biggest problems facing contemporary school mathematics is that, generally speaking, mathematics teachers are not curious about the subject matter.

TEACHER EDUCATION

Many of our preservice teacher education students arrive to our classes with a genuine curiosity about the subject matter, particularly at the secondary level. At the other extreme, many arrive with a fear of the subject matter, especially at the primary levels. A few arrive neither curious nor fearful, having opted into the route of a mathematics teacher because the subject matter seems so easy to teach. It is, after all (and in their opinions), unambiguous. Over the years, it has been curious to observe that, contrary to prominently expressed opinions in the mathematics education literature (see, e.g., Ernest, 1991), it seems the aspects that most frame these diverse students' engagements with the subject matter in our classes are not their beliefs about the nature of mathematics, but the extent to which they can become curious about the subject matter. Platonist and social constructionist alike can be profoundly engaged or frustratingly detached.

Our experience has also shown that curiosity is not an innate proclivity, but can be learned, to some extent at least. Shifts in attitudes can be occasioned as prospective teachers take part in mathematical activities—and, in particular, in those sorts of activities that require them to be participants in a learning collective. (Such activities are not difficult to design. For instance, it can be done by extending almost any mathematical activity with the simple task of formulating a new question to pose to—and hopefully stump—classmates.) A key seems to be that such activities are neither teacher-centered nor learner-centered, but mathematics-oriented. The agents are brought together by common activity with shared purposes—which, psychologically- and sociologically-speaking, is a critical element in transforming a collection of me's into a collective of us (see Johnson, 2001). We thus support the current and widespread practice of structuring courses for preservice mathematics teachers around mathematical activities, and would advocate for an elaborated practice of framing those activities in terms of collective or joint inquiries. This suggestion stems from our belief that, like mathematical knowledge, curiosity is a collective phenomenon, even when it is expressed in private pursuits.

To return to the example that we used to frame this paper, we cannot say how the teacher's response to the student's response affected the course of activity in the classroom. But it is precisely the fact that we cannot specify the consequences that prompts us to argue that curiosity around the subject matter is an obligation, not an option for the teacher. Complexity does not tell us how a teacher's attitudes and activities contribute to the collective, only that they do.

Note

1. This paper is based on data collected in a year-long teaching experiment (funded by the Social Sciences and Humanities Research Council grant 410-2000-0500) in which Simmt taught a grade 7 mathematics class. That teaching experiment was part of a collaborative research project in which Towers, Gordon and Simmt are exploring the implications of high activity and interaction rich mathematics classes.

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LOGICO-MATHEMATICAL ACTIVITY VERSUS EMPIRICAL ACTIVITY: EXAMINING A PEDAGOGICAL DISTINCTION

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I present a theoretical distinction that may prove useful in conceptualizing mathematics teacher education (and graduate education) and research on mathematics teacher education. Further, the distinction can contribute to developing frameworks on the design of mathematics curricula. The distinction between empirical activity and logico-mathematical activity focuses on the nature of a mathematical concept and how that concept develops, key issues in the quest to teach mathematics for understanding.

A primary goal of the mathematics education reform in North America during the last 15 years has been to promote students' learning of mathematics with understanding. This goal is in response to a widespread perception that too many mathematics students learn mathematics as a collection of disconnected and meaningless (to the learner) facts and procedures. This reform effort has been fueled by and has continued to require re-conceptualization of the nature of mathematics, what it means to do mathematics in school, how mathematical concepts are learned, and how mathematical concepts can be taught. In this article, I explicate a pedagogical distinction that could prove useful in conceptualizing the design of mathematics lessons and the education of mathematics educators. The theoretical distinction presented is grounded in a Piagetian empirical framework. Examples of data and author-generated lessons provide the basis for examining this distinction.

Over the last 6 years, my colleagues and I have been engaged in a research project, the Mathematics Teacher Development (MTD) Project[i]. The purpose of the project has been to understand the mathematical and pedagogical development of K-6 teachers (inservice and preservice) as they participated in a comprehensive reform-oriented teacher education program. This research has resulted in a set of distinctions about the pedagogical thinking that underlies the practice of teachers participating in the reform (cf., (Heinz, 2000; Simon, 2000; Tzur, Simon, Heinz, & Kinzel, 2001). In this article, I explore another distinction, deriving in part from the MTD research, that involves conceptualization of the nature of mathematical concepts, what it means to do mathematics in school and how mathematical concepts are learned.

One characteristic of classrooms and curricula guided by participation in the reform is an emphasis on students' active involvement in the development of new (to them) mathematical ideas. Different modes of active involvement have often been articulated (e.g., problem solving, looking for patterns, representing, explaining, justifying, finding counter examples). In the two lessons that follow, the first from MTD data and the second from one of the recent NSF-supported curricula, a similar lesson structure is used that makes use of pattern recognition. After describing these lessons, I will make a case for what I consider to be problematic aspects of the pedagogical conceptions underlying these lessons. I will then exemplify and briefly describe a contrasting framework for conceptualizing mathematics concept development and lesson design.

IVY'S LESSON ON AREA OF TRIANGLES

The MTD data that I describe in this section were included in a detailed analysis of Ivy's practice (Heinz, 2000). That analysis focused on the underlying structure of Ivy's practice. Subsequent observations, including situations that were not part of the MTD project, have led to a re-examination of these data and articulation of a new distinction.

Ivy, a sixth grade teacher (students age 11 years), was in her sixth year of teaching when she designed and taught this lesson on the area of triangles.

Ivy wanted her students to

> find the formula . . . I really believe that they forget what we just tell them and that they will remember what they figured out. And if they don't remember it, they can figure it out again and maybe faster the next time.
> . . . I want them to understand it.

Mathematical relationships that Ivy was aware of were the basis for her lesson design.

> We are building off those right triangle ideas because that is where the formula builds from, which is actually from rectangles. So I am trying to take them from rectangles to right triangles to non-right triangles to see how it is all related to the rectangle itself.

Following is an outline of Ivy's lesson:

1. Ivy led a review of how to find the area of a rectangle on a geoboard.
2. Students worked in small groups to find the area of a 2x3 right triangle.
3. The whole class discussed their strategies and results for step #2.
4. Students worked in small groups to find the areas of all of the right triangles they could make on their geoboards and recorded the measures of the base, height, and area for each triangle.
5. Students shared their data from step #4 with the whole class while Ivy recorded the information in a 3-column table
6. Students examined the table to come up with a formula.

Ivy's instructions for step #6 were:

> Look at how these numbers are in this chart with our areas . . . and see if you can figure out a pattern that you can use every time using the numbers [measures of base and height] to come up with the area. . . . There is something that you can do to these [measures of] the bases and the heights to get the area.

A PUBLISHED LESSON ON EQUIVALENT FRACTIONS

In the United States, mathematics educators often consider the state of the art in reform-based mathematics education curricula to be represented by recent National Science Foundation supported curricula.[ii] It is my experience that the lessons within each curriculum, although generally superior to those found in preexisting curricula, are uneven in quality. One explanation for this phenomenon might be the multiple authors involved in writing each of the curricula. However, I would argue that a more important reason is the lack of or inadequacy of explicit frameworks for guiding lesson design. This latter point suggests work to be done in mathematics education. The pedagogical distinction that I explicate in the next section may prove useful in curricular design efforts.

I include the first 7 steps of the "At a Glance" (*Math trailblazers: A mathematical journey using science and language arts (K-5)*, 1999) that summarizes the lesson on equivalent fractions.

1. Ask students to use their fraction chart from Lesson 3 to find all of the fractions that are equivalent to 1/2. List these on the board or overhead.
2. Ask students to compare the numerators and the denominators of the equivalent fractions in order to look for patterns.
3. Ask students to suggest other fractions that are equivalent to 1/2.
4. Write number sentences on the board or overhead showing the equivalencies.
5. Students look for patterns in the number sentences.
6. Students use the patterns (multiplying or dividing the numerator and the denominator by the same number) to find fractions equivalent to 3/4, 1/3, and 2/5.
7. Students use the patterns to complete number sentences involving equivalent fractions.

ANALYSIS OF THE TWO LESSONS AND DEVELOPMENT OF DISTINCTIONS

The two lessons, just described, have similar goals and structure. The goals involve the generation of a computational strategy (generalization) or formula with "understanding."[iii] The structure involves generating a set of examples, finding the numerical pattern (relationship) among the parts of the examples, and establishing that pattern as a generalization for computing the missing number in further examples.

Lessons of this type, if criticized, are generally criticized on the basis of issues of justification. That is, although examining a set of examples to find a pattern is appropriate for generating a conjecture, it does not constitute mathematical proof that the relationships involved are true for all cases of the type being considered. There remains a need for deductive justification. This is an important mathematical issue, but not the one that I focus on here.

Let us consider what students might learn from these lessons. In Ivy's lesson, students are likely to learn that there is a fixed relationship among the base, height, and area of a triangle and that it can be represented as A=bh/2. Similarly, in the lesson on equivalent fractions, students might learn that there is a numerical relationship among equivalent fractions. To produce an equivalent fraction, one can multiply the numerator and denominator by the same number (not zero and not necessarily an integer). Is this what we mean by "understanding" in mathematics? I argue that it is not.

Understanding is a broad term, and a single definition is unlikely to capture all significant meanings (cf., Piaget, 2001; Sierpinska, 1994; Simon, 2002). However, for the purpose of analysis and contrast with the lessons described above, I offer the following characterization of understanding. *Mathematical understanding is a learned anticipation of the logical necessity of a particular pattern or relationship(s)*[iv].

In the lesson on area of triangles, the reader might see how the work with the geoboard could result for *some* students in an understanding of the logical necessity of the relationship among the base, height, and area of a triangle. However, the derivation of the formula from the numerical pattern structures the lesson towards learning *that* the formula is appropriate as opposed to *why*. Likewise, the lesson on equivalent fractions does not foster an anticipation of the logical necessity of the patterns found. In the next section, I will discuss how a lesson that begins as Ivy's lesson did could be designed to foster anticipation of the logical necessity.

PROMOTING ANTICIPATION OF LOGICAL NECESSITY

Contrasting Lessons

In this section, I present lessons on the same two topics. These lessons are meant to provide a useful juxtaposition, allowing examination of the underlying pedagogical constructs and how these constructs are related to the development of mathematical understanding (as I defined it above). The lessons that follow are not necessarily appropriate for any particular group of students.

<u>Area of a triangle.</u> For brevity and because it is sufficient for my purpose, I will describe a lesson that develops only a generalization for the area of a right triangle. The lesson begins in a similar way and is based on the assumption that students understand the relationship between the area of a rectangle and the measures of its sides.

1. Students are asked to find the area of particular right triangles using geoboards and to justify their approach. (It is anticipated that students add rubber bands to make the right triangle into a rectangle.)
2. Students are given a ruler, asked to find the area of right triangles that have been drawn on plain paper, and asked to justify their approach. The drawings involve right triangles whose legs are not parallel to the sides of the paper. This is meant to preempt overgeneralization that could result from work with the geoboard. (It is anticipated that students will draw two sides to complete a rectangle. Some students may have already abstracted the relationship from step 1.)
3. Students are asked to anticipate (without drawing) what they would do with a triangle of side measures 3. 4, and 5 units to find the area and what the area would be? Likewise for a triangle of side measures 5. 12, and 13 units. (It is anticipated that students will think about drawing a rectangle and then consider how the side measures of the triangle would give them information about the size of the rectangle).
4. Students are asked to write a generalization for how to calculate the area of a right triangle given the measures of the sides.

<u>Equivalent fractions.</u>

Again for brevity and because it is sufficient for my purpose, I will describe a lesson that develops only a part of the concept involved. This lesson promotes a generalization for making equivalent fractions when the new fraction is expressed in terms of smaller fractional parts (e.g. making 1/2 into 4/8) and for which the new numerator is the unknown. Students are assumed to have an understanding of whole number multiplication and division and knowledge of multiplication/division number facts through 10x10. Further, they are assumed to have a basic understanding of fractions, including representation using area diagrams and the meaning of the numerator and denominator.

1. Students are asked to draw a rectangle with 1/2 shaded. They then are instructed to draw lines on the figure so that the figure is divided into sixths and to determine 1/2 =?/6.
2. Students are asked to draw a rectangle with 2/3 shaded. They then are instructed to draw lines on the figure so that the figure is divided into twelfths and to determine 2/3 =?/12.
3. Students are asked to draw diagrams to determine the following:
    a. 3/4=?/8
    b. 4/5=?/15
    c. 1/4=?/20
4. Drawing diagrams to solve equivalent fractions problems is not much fun when the numbers get large. For the following do not draw a diagram. Rather think about what would happen at each step if you were to draw a diagram. Use that thinking to answer the following:
    a. 2/9=?/90
    b. 7/9=?/72
5. Use a calculator to calculate the following. Write down each step that you do and the result you get. Justify each step in terms of how it is related to cutting up a rectangle.
    a. 16/49=?/147
    b. 13/36=?/324
6. Write a calculator protocol for calculating a problem of the form a/b=?/c.

UNDERLYING PEDAGOGICAL PRINCIPLES

In Ivy's lesson and the published lesson, the students engage in an *empirical* process. Students are involved in collecting a set of results and identifying a pattern in those results. The process does not require any insight into why that pattern is produced (the logical necessity). What is it about the latter set of lessons that has the potential to foster understanding as the anticipation of logical necessity?

Let us look more closely at the lesson on equivalent fractions. Students begin by using an activity sequence that they already have available (further subdividing a rectangle) in service of a goal that they have established[v] (to determine the numerator of the equivalent fraction). The students determine how many subdivisions must be made in each of the original fractional parts to change their initial diagram to one that will portray the equivalent fraction (e.g., to convert thirds into twelfths, each fractional part must be subdivided into 4 parts). The student then is able to examine the diagram to determine the number of subdivisions in the shaded region (e.g., total = 8), the new numerator. If it does not happen spontaneously, Problem 4 is designed to focus the students on the relationship between the subdivisions of all the original parts and the resulting subdivisions of the shaded region.

The implied claim that students can pay attention to (perceive) this relationship is worth examining. Piaget's(1977) central construct of *assimilation* maintains that a learner can only attend to that for which s/he already has the assimilatory schemes to structure the experience. In the example of using the drawing to solve 2/3=?/12, the student intentionally subdivides each of the thirds (including those that are shaded) into 4 parts. It is therefore well within her/his capacities to come to anticipate that, as a result of subdividing, there are 4 times as many small parts as there were larger parts in the shaded region.

To describe this process more generally, the students' activity (subdividing by a particular number) produces particular effects (an augmentation of the shaded parts by a factor of that number). Through repeated use of the activity to accomplish a goal, the students are able to pay attention to the effects of their activity and eventually to see a pattern in the relationship of the activity and its effects (reflective abstraction). This mechanism for explaining the learning of mathematical concepts is developed in greater detail in Simon, Tzur et al. (2000; 1999).

Note, that this mechanism is further applied in the design of Problem 5. Here the task is designed to encourage abstraction based on the activity of determining the factor relating the original denominator and the new denominator.

I leave it to the reader to go through a similar analysis of the lesson on the area of a right triangle.

POTENTIAL SIGNIFICANCE OF THIS PEDAGOGICAL DISTINCTION

I have used examples from data to articulate a pedagogical distinction between lessons that engage students in *empirical activity* and lessons that involve *logico-mathematical* activity. I use these terms[vivii] because, although the distinction is not equivalent to Piaget's (2001)) distinction between empirical and reflective abstraction, the distinction can be thought of as analogous to it. I emphasize that the distinction is not simply one of the need for deductive justification. Rather this is a distinction that is fundamental to what is meant by a mathematical concept and the process by which concepts in mathematics are learned. The distinction highlights the difference between a mathematical generalization (e.g., theorem) and a mathematical concept. The former can be arrived at and proved without development of an anticipation of its logical necessity. A concept involves understanding and thus anticipation of logical necessity. The distinction between these two types of activity is potentially useful in conceptualizing the design of effective mathematics lessons and the education of mathematics educators (teachers, researchers, curriculum developers). I expand on each of these points.

In recent years, mathematics students have benefited from curricular efforts based on mathematics education research conducted in the last 30 years. As I mentioned above, although overall curricula are improving, there is still considerable unevenness between and within curricula. The distinction offered in this article is intended to contribute to the development of useful frameworks for guiding lesson design. It also provides a lens for viewing existing curricula. One established curricular effort that consistently builds on students' activity in a logico-mathematical process is the Dutch Realistic Mathematics Education (Gravemeijer, 1994). Using the distinction I have presented, their notion of *model of* becoming a *model for* can be understood as a technology for representing students' activity as a basis for students' reflection on the relationship between their activity and its effects.

Data from our research (e.g., Ivy's lesson) and analysis of recent curricula (e.g., lesson on equivalent fractions) suggest that some educators who intend to teach mathematics for understanding are generating lessons that engage students in only empirical activity. This distinction is one through which teacher educators and graduate educators can look at the prospective and practicing educators with whom they work. A useful (and ambitious) goal for the education of mathematics educators would be to promote their understanding of mathematics conceptual learning as built on a logico-mathematical process.

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[ii] There are approximately a half dozen such curricula at each of the elementary, middle school, and high school levels.

[iii] "Understanding" is in quotes because of the lack of shared meaning for the term. Discussion of this point is up coming.

[iv] Although this articulation of "understanding" is my own, it is consistent with the ideas of others, most notably Piaget (2001).

[v] Although the teacher poses the problem, each student's activity is based on the goal that s/he sets. It is anticipated that the students' goals will be compatible with the intention of the teacher.

[vi]

[vii] I request that readers who find this choice of terminology to be problematic and/or who have ideas for other terminology, to communicate with me. If you find the terminology to be appropriate, I would be interested in knowing that as well.

THE PROVISION OF ACCURATE IMAGES WITH DYNAMIC GEOMETRY

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Results of a study involving pre-constructed, web-based, dynamic geometry sketches in activities at the secondary school level revealed that the provision of accurate images is an issue. Many students do not automatically understand that an onscreen image is accurate. Others inadvertently create a special case by dragging, then generalise from this static but unsuitable model. The research suggests that, for students to reap the benefits of working with an accurate image, they need to set aside, during the exploration phase, the traditional attitude of suspicion towards diagrams, and to recognize that dynamic geometry diagrams offer valuable and accurate visual evidence.

INTRODUCTION

The study that informs this paper was undertaken to evaluate the benefits and limitations of the use of pre-constructed, web-based, dynamic geometry sketches in activities related to proof at the secondary school level. From the analysis of the data two main themes were identified: 1) the relationship between the activities and the development of geometric thinking skills, and 2) the relationship between the design of the materials and the exploration process. Underlying these was an important sub-theme--how students responded to a visually accurate image.

Many researchers have recommended an increased emphasis on the use of visual reasoning in mathematics (cf., Duval, 1998; Goldenberg et al, 1998; Presmeg, 1999; Dreyfus, 1991). With respect to computer images, Sutherland and Balacheff (1999) reported that visual images displayed on computer screens allow students to gain access to mathematical knowledge by “rendering more visible the nature of the objects with which a student is engaging” (p. 2), and in a 1986 study of visualisation in high school students, Presmeg found that dynamic imagery--although used by only a few "visualisers"-- was effective in helping students generalise. The implication is that visually accurate images are beneficial in helping students understand geometric ideas.

Visual geometric images whether printed or onscreen, fall into three categories: 1) special case, 2) general case, and 3) inaccurate. One might argue that the last category is unnecessary, however, an examination of most secondary texts will reveal many diagrams that include measurements but are not drawn to scale. In keeping with tradition, teachers warn their students that the diagrams are not necessarily accurate and that they are to focus only on making logical deductions. The results of this study suggest that the ‘diagram bias’ thus created acts as a roadblock when students examine a pre-constructed, and ‘accurate’ (i.e., to within a measurable error), dynamic sketch.

Pre-constructed sketches

Pre-constructed sketches created with *Cabri Géomètre* (Baulac, Bellemain, and Laborde, 1992), or *The Geometer's Sketchpad* (Jackiw, 1991) as well as pre-constructed, web-based sketches created with *JavaSketchpad* (Jackiw, 1998), or Cinderella (1999, Richter-Gebert and Kortenkamp) can be used as an alternative to having students construct their own dynamic diagrams. In all pre-constructed dynamic sketches, points can be dragged; pre-set relationships, such as measurements and ratios, update as a consequence of dragging; and action buttons to hide or show details, to move and to animate objects can be included. Web-based sketches created with *JavaSketchpad* do not permit the user to construct or delete objects; however, those designed with Cinderella can provide options for constructing a limited number of objects. Angles and lengths in these sketches are represented accurately to within a small error, unlike those in textbook diagrams.

Interpretation and theoretical framework

Whether a pre-constructed sketch is web-based or not, it presents a geometric situation to the student in visual format. Since the creator of an image knows details that are hidden from an ordinary viewer, interpreting a pre-constructed sketch is similar to interpreting a picture that someone else has drawn. Measurements or measurement tools are provided, but students must apply their own organisation to the information, and draw conclusions about how items are connected—a difficult task because mathematical pictures and diagrams contain a great deal of information represented in a concise but "nonsequential" format (Goldenberg, Cuoco and Mark, 1998).

Dynamic sketches include several options for motion including animation capabilities and the dragging provision, which allows the student to explore an object in motion, at a controlled speed. In 1998, Arzarello, Micheletti, Olivero, Robutti, Paola, and Gallino classified modalities of dragging as: “Dragging test”, “wandering dragging”, and “lieu muet” (dummy locus). They found that students who produced good conjectures made use of "lieu muet" dragging, a purposeful mode which "can be seen as a wandering dragging which has found its path" (p. 37).

Extensive studies of Cabri have shown that a geometry problem cannot be solved simply by perceiving the onscreen images, even if these are animated. The student must bring some explicit mathematical knowledge to the process (p.32). That is, an intuition about a generalization involves more than observed evidence (Fischbein, 1987). This study, on the use of pre-constructed, dynamic geometry sketches, found that the basic task of perceiving detail, which involves noticing lengths, angles, measurements, labels, markings, then noticing the change in these as a consequence of dragging, is difficult for many students. I contend that this is in part due to traditional but limiting attitudes towards diagrams that students bring to the dynamic environment.

DESCRIPTION OF THE STUDY

The research used a case study approach and multiple sources of information -- observation field notes, videotape, audiotape, a student questionnaire, and interviews with teachers. Collected data was transcribed, then analysed by coding, developing categories, describing relationships, and applying simple statistical tests where appropriate.

The transcripts were coded in several ways to allow analysis of student actions and thinking, and to link these to particular labsheet questions or sketch features. During this process, students’ uses of and responses to the visual images were examined (see Sinclair, 2001 for further detail).

Three mathematics classes from two different secondary schools participated in this study. The 69 students were enrolled in the Ontario grade twelve advanced mathematics program (replaced in 2002), which covered topics in algebra, geometry, analytic geometry and trigonometry (Curriculum guideline, 1985). The study focused on congruence and parallelism, the first section in the geometry unit. Although the students had done introductory work on deductive geometry related to congruence and parallelism in grade 10 and on similarity in grade 11, none had worked with dynamic geometry software.

Three 75 minutes sessions or four 45 minutes sessions were held with each class. During this time, students worked in pairs on four tasks. An additional task was done as a whole class activity. In each class, several pairs were studied in more depth by audiotaping or videotaping their activities.

JavaSketchpad, was used to prepare four web-based, dynamic geometry sketches for student pairs to explore during the sessions, two extra sketches for those who finished early, and one sketch for a group discussion. The labsheet that accompanied each sketch provided directions for opening and manipulating the sketch, a statement of the problem, and questions related to the task.

Problems chosen as the basis for the web-based sketches were similar in difficulty to those in the student text, *Mathematics: Principles and Process, Book 2* (Ebos, Tuck, and Schofield, 1986) and related to triangles and quadrilaterals.

Each of the sketches supported the possibility of arriving at a solution from a transformation perspective as well as from a straightforward application of congruency theorems. The intention was to allow students to use symmetry considerations, a) to visually confirm or negate conjectures, and b) to develop a new perspective on geometric relationships.

In the pre-study interview the three study teachers identified difficulties that their students experience in the geometry strand. For example, teachers mentioned that students constructing congruency proofs frequently select sides or angles that do not correspond to one another, or, in fact, do not even belong to the subject triangles. They noted that this problem usually occurs when figures overlap or are presented in rotated, reflected, or translated form. These student difficulties reveal an inability to “see” each overlapping figure separately or to mentally transform a figure to a new orientation to compare it with another. To address these difficulties, sketches included action buttons or provisions to highlight particular figures, to toggle details on and off, and to rotate or reflect shapes so that they could be superimposed, or viewed from the same orientation.

Overview of a session task

A very brief overview of one task is included here to help the reader understand the context for the discussion.

Day 2, task 1.

This task gave students the opportunity to apply properties of parallel lines and to investigate a problem using a rotation.

The triangles to be proven congruent were coloured to attract student attention. When a vertex of quadrilateral ABCD was dragged, AD and BC appeared to remain equal and parallel, as did AB and DC. When the "Show Given Information" button was used, students could deduce that ABCD was indeed a parallelogram since opposite sides were marked equal and measurements were given.

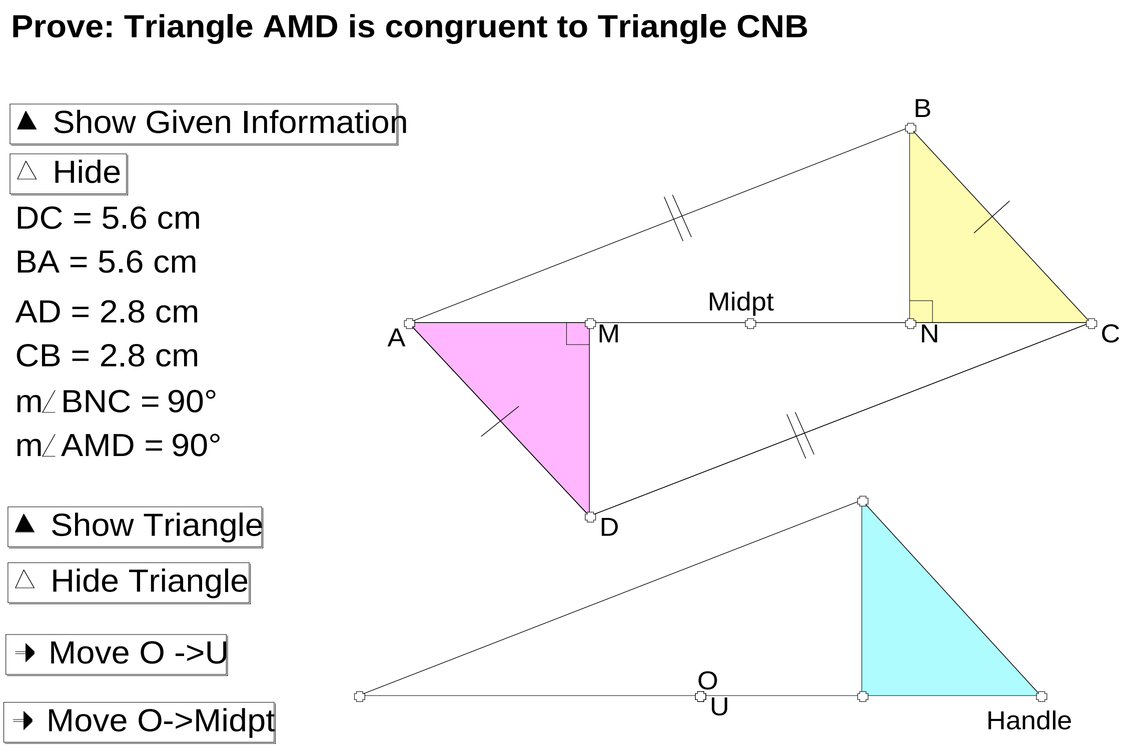

Figure 1: Day 2, task 1—After selecting: "Show Given Information," "Show Triangle" and "Move O→U."

It was expected that students would use ASA (angle, side, angle congruency theorem) to prove that △AMD and △BNC were congruent: AD = BC (given), ∠DAM = ∠BCN (parallel line law), and ∠MDA = ∠NBC; however, students could also investigate the relationship between the two by superimposing an additional given triangle over △ABC and then rotating it to fit over △CDA. This movable triangle was a tool for testing whether △AMD and △BNC were congruent; however, it could also be used to demonstrate the fact that congruent triangles have congruent altitudes (i.e., △ABC and △ADC are congruent, which implies that AM must equal BN). Questions on the labsheet such as: “What do you notice about the new triangle?” and “How can the information provided by these images be used to explain why DM = BN?” were aimed at helping students notice and address the information provided in the sketch.

DISCUSSION

The particular responses (or non-responses) to the provision of an accurate visual image were related to three broad categories: noticing details, use of dragging, and entertaining alternative methods.

Noticing details

The following (unconnected) comments show that students noticed details in the sketches. (Note: all students are identified by pseudonyms.)

Doug: Angle BED--hey!…Angle BED is 72.455.

Katy: Um, uh the angle shadings They're the same angles. Yeah, I would say that. The angle shadings mean that they're congruent angles. So, congruent sides and congruent angles.

Dave: They match. It matched it with the other one. It shows us that they're congruent.

Bea: Oh, so the yellow and the purple

Familiar markings and colour drew attention--students noticed items that were coloured or marked and sometimes missed those that weren't. Colour was also used as a simple and effective means of referencing objects in discussion as shown by Bea’s comment. On the other hand, despite Doug's comment, the transcripts show that measurements were often ignored.

It is not clear whether some students did not notice the measurements (in *JavaSketchpad* lengths and angles are in a list and not attached to the object), or whether they were so attuned to the “rules” of deductive geometry that they did not expect to use measurement data. The ability to display an accurate image is commonly assumed to be a benefit of dynamic geometry software--it seems reasonable to conclude that the task of noticing and interpreting relationships between objects is easier if figures are drawn to scale. However, the study results showed that many students either do not realize or ignore the fact that the onscreen image is accurate.

The tendency of study students to gloss over measurements is in stark contrast to their awareness of colour and markings. It is of concern because the ability to explore how a figure has changed requires focused attention to details that update under the operation of dragging.

Use of dragging

Although initially intrigued by the ability to drag points, study students usually stopped dragging after a short time and concentrated on interpreting a static figure. This led to mixed results.

Some students treated the onscreen image as if it were a pencil sketch--as if the diagram represented objects and their relationships, but was not drawn to scale. For example, two above average students made the following (unconnected) comments:

Barb: Maybe cause it's slanted you can't tell it's a square.

Sue: If this is equilateral these sides would have to be equal. [In this instance, the triangle was clearly not equilateral].

I hypothesised that such responses might stem from prior use of textbook diagrams. Geometry teachers frequently warn their students not to make conclusions based on the appearance of diagrams that are not necessarily accurate. These students used their knowledge of deductive theorems to correctly solve the problems, but gained nothing from being able to use an accurate model.

In the following example, abandoning dragging led to an erroneous conclusion. Doug and Sal were looking at a triangle in Day 1, task 2 (not shown). Angle BEA may have been very close to a right angle on their diagram, but if they had dragged the sketch they would have seen it change.

Doug: .. BEA--angle E is 90 degrees..

Sal: There's no thing [referring to the symbol for a 90 degree angle]

Doug: Well, you can't see it.

Sal: That's right

Doug: Well, I'm thinking this is an[sic]--cause it looks like it, right?

In this case, Doug and Sal, two average students, actually did treat the sketch as accurate! Along with some other students they persisted in drawing conclusions based on what the angles or sides looked like. Certainly many informal conjectures suggest themselves to mathematicians because they “look like” they are true. Some withstand further investigation; others prove false. The students’ problem was not in basing a guess on the visual evidence before them but in failing to realise that they were observing a specific case.

Entertaining alternate methods

Some student responses draw attention to the fact that we often focus on methods that are more suited to symbolic rather than visual analysis. This tendency forces the student to turn away from the image medium to compose a proof. For example, in Day2, task 1 (see Fig 1), the sketch allowed students to use rotation to explain why two segments were equal, instead of deducing the result via a triangle congruency proof. The question was: "How can the information provided by these images be used to explain why DM equals BN?" Students intuitively understood that when the triangle was rotated, DM would fall on BN.

Clara: Because the triangle fits--the triangle fits both.

Despite this comment Clara did not follow up with a step-by-step analysis. She felt that the result needed no further explanation, but she was unable to compose a written transformation-based proof. Instead she used a traditional congruency proof.

If Clara wanted to justify her conclusion without abandoning the image that made the congruency clear, what language would she use? I do not think most teachers are able to step back from their deductive geometry experiences to provide a simple, clear proof that uses transformation concepts and takes advantage of the visual reality offered by an onscreen image.

Examples such as this also highlight students' unfamiliarity with describing visual information in precise terms. We cannot intuit if we do not perceive and I contend that students are not taught to perceive visual details—they are taught to select information from diagrams—even if this information is false to the eye.

CONCLUSION

In their 1996 summary analysis of research on computer-based learning environments in mathematics, Balacheff and Kaput note that one of the ways in which the computer makes its primary impact is by "changing the relationships between learners and the subject matter and between learners and teachers—by introducing a new partner" (p. 495). The resulting environment is didactically complex. In addition to the usual teacher-student interactions, there are interrelationships among the student, the computer and the task (Sutherland & Balacheff, 1999). The nature of this student-computer interchange is shaped by the unique characteristics of the software and its objects. This paper has briefly presented how the accuracy of pre-constructed, dynamic geometry sketches affects and is affected by the experience of the secondary school geometry student.

The study students who treated dynamic sketches like textbook models missed visual evidence that might have provided support for a deeper understanding of geometric relationships. Having always been provided with diagrams that carefully displayed a general case, some students did not recognize the pitfalls of creating and analyzing a special case. And when students were able to bring only traditional proof techniques to bear on dynamic problems they failed to experience the true power of dynamic software.

As educators we need to be aware of the biases that our students bring to their work. In the case of accurate and interactive images we need to explicitly focus students' attention on the differences between textbook diagrams and dynamic geometry sketches, and help them find ways to mine the benefits of visual reality.

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AESTHETIC VALUES IN MATHEMATICS: A VALUE-ORIENTED EPISTEMOLOGY

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Much has been said on matter of aesthetics and mathematics—perhaps most commonly articulated in relation to properties such as 'beauty' and 'elegance,' which are used to distinguish good from not-so-good mathematical products. Despite its importance in the work of mathematicians, it has been argued that aesthetics cannot be incorporated into school mathematics given students' difficulties with basic problem-solving skills (Dreyfus and Eisenberg, 1986; 1996). In contrast, this paper argues that it is both possible and desirable to incorporate aesthetic concerns into the mathematical activities of students. The argument is based on a re-articulation of both the nature and purposes of the aesthetic in school mathematics that extends beyond the objective, product-oriented interpretations more commonly discussed.

The evaluative role of the aesthetic pervades the world of the professional mathematician where terms such as 'elegant' are regularly used to judge 'good' theorems, solutions and proofs. Some researchers, such as Dreyfus and Eisenberg (1986), propose that students should also develop aesthetic mathematical competencies, in addition to their cognitive ones, in order to appreciate the 'elegance' of certain solutions and proofs. Their proposal follows from the tenet that "one of the major goals of mathematics teaching is to lead students to appreciate the powers and beauty of mathematical thought" (p. 2).

In order to determine the feasibility and sensibility of such a goal, Dreyfus and Eisenberg designed a study in which they investigated whether college students were able to appreciate elegant solutions. They find that the students were unable to find elegant solutions to these problems. Furthermore, the students did not find the solutions which were later presented to them—and more elegant, according to "experts"—any more attractive than the ones they had come up with on their own. Dreyfus and Eisenberg conclude that their students were incapable of aesthetic appreciation. Might the researchers be wrongly equating the lack of agreement between students' and mathematicians' judgements with students' lack of aesthetic sense?

The issue is partly an epistemological one. When mathematicians evaluate entities such as proofs and solutions, they do so for two reasons: one, to establish personal value; and two, to establish collective value. As Alibert and Thomas (1991) note, these two purposes are often absent in school mathematics, where "the subject matter is presented as finished theory, where all is calm… and certain" (p. 215). Students approach mathematics as something to be accepted and learned while mathematicians approach it as something to be evaluated and negotiated. Alibert and Thomas are particularly concerned with the problems this epistemological disconnect produces when it comes to proofs. While mathematicians use proofs to convince (both themselves and others), students see proofs as difficult, formal, and sometimes arbitrary things. Ironically, Alibert and Thomas believe that the perceived need to preserve the precision and the 'beauty' of mathematics—by emphasising the rigour of formal proofs in the classroom—may compromise students' concern for meaning and value as well as their appreciation for the functional role of proof. In other words, why would students develop preferences for one proof over another if proofs only have already-established truth values? That would be an epistemological category mistake.

In their study, Dreyfus and Eisenberg suggest the possibility that proofs can have an aesthetic value, but they take an objectivist view of aesthetics—that a certain solution is elegant in and of itself, independently of human perceivers—and, in considering students' evaluations, look for aesthetic preferences that match those of professional mathematicians. Since they do not find these, they conclude that students do not show aesthetic appreciation. Yet perhaps these students are showing and developing quite different aesthetic preferences, which suit their own current goals and needs.

For example, Brown (1973) describes what might be called a "naturalistic" conception of beauty manifest in the work of his graduate students. He recounts showing them Gauss' apocryphal encounter with the famous arithmetic series: $1+ 2 + 3 + \ldots + 99 + 100$. He asks his students to investigate variations of the general scheme (that the sum of the first n numbers is $(n +1) * n /2$). They come up with many geometric and algebraic approaches, each equivalent but expressed in various ways. Brown asks them to discuss their approaches in terms of aesthetic appeal. Surprisingly, many of his students prefer the rather messy, difficult-to-remember formulations to Gauss' neat and simple one. Brown conjectures that the messy formulations do a good job of encapsulating the students' personal history with the problem as well as its genealogy, and that the students want to remember the struggle more than the neat end product. This is in clear opposition to the way that mathematicians like to present their results: they are almost always devoid of any of the guesses, supporting sketches, and history of the solving process. Brown's observation highlights how the contrasting goals, partly culturally imposed, of the mathematician and the student lead to the use of different aesthetic criteria.

The question that must be addressed is whether the goal of nurturing aesthetic preferences is to align them with those of professional mathematicians. In contrast with Dreyfus and Eisenberg, who want to initiate students into an established system of mathematical aesthetics, I propose that educators nurture students' development of aesthetic preferences according to the animating purposes of aesthetic evaluation in school mathematics. The starting point is not to train students to adopt aesthetic judgements that are in agreement with "experts" but, rather, to provide students with opportunities in which they want to—and can—engage in personal and social negotiation of the worth of a particular idea. A student's aesthetic capacity is not equivalent to her ability to identify formal qualities such as economy, cleverness, brevity, simplicity, structure, clarity or surprise in mathematical products. Rather, her aesthetic capacity is her ability to combine information and imagination when making purposeful decisions regarding meaning and pleasure—this is a use of the term 'aesthetic' drawn from interpretations such as Dewey's (1934).

Thus far I have focused on solutions and proofs—the 'ready made mathematics'—as objects of aesthetic evaluation. However, as Le Lionnais (1948/1986) points out, mathematicians will also judge the aesthetic value of many other mathematical entities including definitions, diagrams, theories, methods and algorithms—entities that students encounter far more frequently than finished solutions and proofs. But as with proofs, students are presented with definitions, methods, and algorithms, as if, echoing Alibert and Thomas, they were finished—all being calm and certain. Students accept the division algorithm and the definition of a quadrilateral without being invited to consider questions such as: Is it good? Is there a better one? Does it have value?

In the following example, I invite a group of four middle school students to consider these questions about a method for constructing a square. Their responses suggest that students do indeed, with little guidance, show a strongly developed aesthetic capacity when considering the value of mathematical entities.

AN ILLUSTRATIVE SKETCH FROM RESEARCH

The following sketch is taken from the observation of four grade eight students in a western American city who were working independently on a geometry course using *The Geometer's Sketchpad*. It is from one site of an ongoing research programme being carried out to study the aesthetic dimension of student mathematical activity. This is the third class of the semester and the students are attempting to re-construct an iterated image involving many squares (for more details, see Sinclair, 2002).

The four students Aleah, Becca, Sara and Zhavain are attempting to construct their first square with Sketchpad. *Constructing* a square in Sketchpad is not a trivial matter; one must first know what defines a square, and then know how to use the appropriate tools. Most students start by using the segment tool to draw four equal sides (the salient property of the square for them) and then attempt, when the time comes, to put the segments at right angles (the more tacit property). I let the students draw squares using only the segment tool, and then show them how to use the circle tool to construct equal segments. Since they have already learned to construct perpendicular and parallel lines, they are then able to construct their squares. Except Aleah. She is stuck on her horizontal segment, insisting on "turning it" up to a vertical position—not wanting perhaps to bother with circles and perpendicular lines. I show her how to turn her segment using the rotate command. Once she has completed her square, she proudly shows the technique to Sara.

I ask Sara which technique she prefers: the rotation method or her "compass and straightedge" one? Sara thinks the rotation method much easier and quicker to perform (given the grammar of Sketchpad's tools at least, where rotation is a one-step action). But, she describes the compass and straightedge method as "more perfect and more mathematical." I ask her what she means by "perfect," and she tells me that the compass and straightedge gives a better construction because she knows that the "points are at the right place." Sara manages to convince Becca and Zhavain of her opinion, but not Aleah.

I then show the students how to create a custom tool that will allow subsequent squares to be created effortlessly. Sketchpad's custom tools are accompanied by scripts, which provide a symbolic representation of the steps involved in the construction associated with the tool. Aleah thinks that the brevity of her script—"look, only six steps!" (compared with Sara's ten)—will help convince her classmates of the rotation method's superiority, but alas, they still prefer their "perfect" method.

This short episode shows a few different aesthetic preferences emerging from the students' negotiation. Sara seems to have a 'classical' orientation, preferring the Euclidean approach to constructing a square. But perhaps she had been enculturated into believing that things which are more technical, more complicated, are in turn more mathematical. In fact, I was initially surprised at Sara's answer, convinced that she would prefer Aleah's method. But Sara also seems convinced that the compass and straightedge construction is somehow more precise. This may be due to the sense of determinacy that points of intersection provide; after all, she constructs each vertex by finding the point where a circle and a perpendicular line intersect.

The rotation method does not have the same sense of determinacy though, of course, it also provides a precise location for each vertex. Since Aleah never actually followed Sara's method, she may not have experienced that sense of determinacy that comes from finding a point of intersection.

Aleah's penchant for the rotation method has several sources. First, the rotation method is hers; she is the one who discovered it. Second, the rotation method grew out of her unique way of seeing a square. Even when I showed the students how to use the circle as a compass, Aleah had a specific idea about how the square should evolve that did not involve circles and perpendicular lines. She knew she wanted to rotate; all I did was to show her that Sketchpad could help her accomplish her goal. So not only was Aleah's method her own, but it answered the particular question *she* had about making a square. In contrast, since the other students hadn't seen the square as Aleah did—as rotated segments—, Aleah's method answered a question they did not ask.

Lastly, Aleah seems to adopt a familiar mathematical aesthetic for simplicity and economy; she uses both as criteria for evaluating the two solutions. Her rotation method is simpler because it does not require using the circle as a compass tool, and it essentially repeats the same step over and over, instead of requiring several constructions that are not transparently related to the square (what does a circle have to do with a square?). Her solution has more economy because it literally takes fewer steps, and saves her from having to hide extra geometric objects (in Sara's method, the circles and line segments used to determine the vertices need to be hidden and replaced with segments).

Possibly, Aleah prefers her method because it is hers and merely appeals to the criteria of simplicity and economy as less subjective-looking reasons. Professional mathematicians might be accused of doing the same thing—ultimately preferring their own discoveries and solutions. In fact Wells (1986) raises this issue, suggesting that they might indeed be "aesthetically biased, as many artists seem to be, towards their own fields and their own works" (p. 39). As with Aleah, professional mathematicians might also invoke aesthetic criteria such as simplicity and economy when trying to convince colleagues of the significance of their work.

The first and second sources of Aleah's preference for the rotation method match Brown's findings that students prefer their own solutions. However, where Brown emphasises the students' solution process, and the attachment they feel to their own solution paths, I believe Aleah's preference is not so much about the process as it is about the relationship between the problem and the solution. Aleah's method *fit* her problem, which happened to be slightly different than her classmates' problem. Here the aesthetic plays a slightly more experiential role since the aesthetic 'sense of fit' straddles the process of inquiry instead of operating only at the final evaluative phase.

CONTEXTS OF NEGOTIATION

The four students do not come to a consensus, however, each student develops a sense of the value of their different square-constructing methods, thus establishing a personal connection with some of the mathematical ideas they will continue using. This episode shows quite clearly that middle students can draw on some of the aesthetic criteria used by professional mathematicians. More provocatively, it suggests that students, if allowed to, naturally behave in ways that are just as aesthetic as they are cognitive or affective—a suggestion consistent with the theories of a wide range of scholars (see Dissanayake, 1992; Dewey, 1934; Johnson, 1993). The accessibility of students' aesthetic sense may seem surprising given Dreyfus and Eisenberg's research. However, the apparently conflicting conclusions reveal different assumptions and goals for aesthetic appreciation.

In Dreyfus and Eisenberg's study, students who had already found their own solutions for a problem were presented with an outside, "expert" one; this set up a very different context of negotiation than the one in the episode related above. The "more elegant" solutions were essentially presented as the *right* solutions, eliciting responses from the students such as "Oh, that's how you do it." The students even became defensive "My way works too." Furthermore it is not clear that the students really understood the "aesthetically superior" solutions; in fact, Dreyfus and Eisenberg report that the students wanted to "pick up the pencil and start working—without first reflecting upon different solutions paths" (p. 7). This slight condemnation reflects Dreyfus and Eisenberg's belief that aesthetic judgements can be made based on objectively accessible features that determine the aesthetic merit of a solution, which would be agreed upon by the "experts."

However, Wells (1986) has shown that the "experts" do not in fact always agree on their aesthetic judgements, and that furthermore, mathematicians take into account their personal experiences with solutions when making aesthetic judgements—sometimes having to 'live through' the solution or proof again. By wanting to pick up their pencils, Dreyfus and Eisenberg's students were showing that they needed to better familiarise themselves with the different solution path before being able to compare it with their own: they could not make spontaneous value judgements.

Instead of a right/wrong context of negotiation, the students in the episode above were invited into a value-oriented context of negotiation. Through the process of negotiation, they were given the opportunity to familiarise themselves with each other's methods, instead of having to make immediate judgements. I have claimed that these students appealed to aesthetic criteria that are similar to ones used by mathematicians but, for two reasons, I have not focused on whether "experts" would have agreed with any of the students' preferences. Firstly, I question whether there is in fact an agreed-upon "expert" opinion and whether there is an aesthetic metric—some hierarchical combination of aesthetic criteria—that could produce an "expert" opinion. Secondly, students usually work with mathematical ideas that are so familiar and evident to mathematicians that they fail to elicit aesthetic responses for them. But even if a professional mathematician is not surprised by or drawn to the 'magical' properties of the number 9, that does not mean the idea is unworthy of aesthetic consideration. Surely, educators cannot hope to help students appreciate the 'elegance' of the Pythagorean proof of the infinitude of primes without first helping them make value judgements on their own mathematics.

I have argued that a primary goal of inviting aesthetic evaluation into the classroom is to encourage students to develop a value-oriented sense of mathematics. In addition to presenting students with a more genuine image of mathematics as professional mathematicians practice it, a value-oriented sense of mathematics can help engage students at a more personal, humanistic level, thus making their experiences in the classroom more memorable and meaningful. After all, as Johnson (1993) writes, the aesthetic provides the very "means by which we are able to have coherent experience that we can make some sense of" (p. 208). A value-oriented sense of mathematics should also provoke meta-cognitive activity since aesthetic evaluation draws on reflections of one's feelings and beliefs about mathematical ideas.

Wells (1986) offers yet another reason for inviting aesthetic evaluation into the mathematics classroom; he points out that teachers might have much to gain in probing students' aesthetic judgements by helping them adapt classroom teaching towards their students' perceptions. For instance, based on Aleah's perception of squares, a teacher might invite Aleah to construct other shapes using transformational geometry. In probing students' aesthetic judgements, teachers might also gain insight into students' ways of thinking and feeling, which can help them adapt the conditions of classroom learning.

SUMMARY

It is tempting, for one who takes pleasure in and values the 'beauty' of certain mathematical entities, to view aesthetic appreciation as a goal in and of itself. Taken into an educational context, the temptation can turn into a conviction that students should be able to take pleasure in and value the 'beauty' and of school mathematics. By showing that students often fail to perceive supposed beauty, Dreyfus and Eisenberg's study suggests that students may not be able to engage in aesthetic evaluation for the same purposes that motivate many professional mathematicians. In contrast, the example above shows that students can and do behave aesthetically in the mathematics classroom, but that their aesthetic behaviours have very functional, yet pedagogically desirable, purposes: establishing personal and social value.

The challenge for educators will be to find ways of productively evoking and nurturing the aesthetic capabilities of their students. But since aesthetic considerations are appropriate in a much wider range of mathematical situations than previously thought, the opportunities will extend beyond problem solving activities. The challenge for researchers will be to determine how students' aesthetic capabilities can also contribute to the *process* of mathematical thinking, particularly in comparison to the way in which aesthetic capabilities contribute to the process of mathematical inquiry for professional mathematicians (Papert, 1978; Poincaré, 1908/1956).

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FROM COGNITIVE SCIENCE TO SCHOOL PRACTICE: BUILDING THE BRIDGE[1]

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The paper is focused on recent researches in neuroscience and developmental psychology regarding mathematical abilities of infants. A model that tries to explain these findings is developed. The model underlies the mental operations that could be systematically trained to generate efficient school learning. The model is built from a cognitive constructivist perspective on learning, is transferable into teaching-learning materials and its development has direct implications in structuring the curriculum and reorienting teaching practice.

HOW MUCH IS THE NEWBORN BRAIN SHAPED FOR FURTHER DEVELOPMENT?

In the early childhood, through the comprehensive learning of maternal language, the child is crossing an extremely dense period of mental accumulation. Synthetically described, this is a recursive categorical learning; the child acquires category knowledge through a few-steps-inductive process and he/she is able to retrieve prototypes or simply exemplars on a similarity-based analysis. For example, the young child is able to identify a table even this table is made from wood or metal, rectangular or oval, covered with food or not, present in a 3D space or just painted on a piece of paper; he/she internalized a category the table belongs to. This process of learning, which commonly has a single mentor – the family environment, and a single “mental medium” – the maternal language, is time dependent and self-regulating. From this view, perception is equivalent to categorizing, with its double process: identifying the class a specific object belongs to, and recognizing that object as a prototype of that class. Category perception is a complex mental process, which supposes recognizing and classifying and it is learned, even in early childhood, through numerous crossings in between various levels of concreteness and hierarchies that implies abstracting.

The last three decades research in cognitive science, neuroscience, and developmental psychology has been significantly concentrated on understanding children’s capabilities to learn in the first year of life. In many researches, the perception of grouping and separating similar objects has been interpreted in relation with the number sense. For example, Wynn (1990, 1992) showed that 5-months infants seam to be able to compare the cardinals of two sets of objects up to three and to react when the result of putting together or taking away is not the right one. Infants showed longer looking at arrays presenting the wrong number of objects, even when the shapes, colors, and spatial location of the objects in the displays were new (Simon, Hespos and Rochat, 1995). Other experiments with older infants using different response systems (manual search/locomor choice) (Feigenson, Carey and Hauser, in press) led to the same

[1] I wish to thank Fulbright Commission for the opportunity of developing this research program.

conclusion. In addition, infants successfully discriminated 16 from 32 dots or 8 versus 16, but failed to discriminate 8 versus 12, or 16 versus 24 dots. These results have been confirmed using various sensory modality (visual or auditory) or format (spatial versus temporal) (Xu and Spelke, 2000). It seams that the sense of approximating natural numbers found in adults appears to be present and functional in 5 to 10-months-old infants. To measure perception, experimenters used the Weber's law: as numerosity increases, the variance in subjects' representations of numerosity increases proportionately, and therefore discriminability between distinct numerosities depends on their difference ratio. Representations of large approximate numerosities show a variation of ratio in Weber law between 1.5 and 2 for infants (Spelke, 2002) and 1.15 for adults (van Oeffelen and Vos, 1982).

A PROMISING HYPOTHESIS FROM THE EDUCATIONAL PERSPECTIVE

How could we explain these findings? Although the theories differ in important ways, they share en emphasis on considering that infants possess amazing competencies that biologically predispose them to learn. As far as we are navigating among hypothesizes, we should focus on the less restrictive ones and the most productive ones from the perspective of mathematics learning. The claim I argue for is that the mind is generically endowed with some categories of basic mental operations. The extensiveness, the deepness, the ramifications of these operations are subjects to further development through education, but the roots of these operations are inherited and they permit to infants to contact the real world and to think about it. I will call these roots *inner operations*. In the early childhood, the inner operations permit to build classes of objects on a similarity base, and to develop the extensions of these classes to more abstract mental constructions.

The categories to which the inner operations belong to will be nominated in operational terms in order to underlie their dynamics. These categories are: *Associating, Relating, Algebraic operations, Logical operations, Topological operations, Iterating, Generating*. The names used to identify them are just labels – they do not express by themselves the complexity of each operational category. A synthetic description of these basic mental operations, based on clustering, is presented below.

The operational category generically called ***Associating*** includes operations described as connecting two entities based on a one-to-one correspondence. The capacity of building correspondences one-to-one evolves from its primitive form of *matching* objects one-to-one, to associating through isomorphisms various representations (see Singer, 2001, for a detailed presentation). As an inner operation, associating permits to infants to discriminate between one and two objects. The infants automatically match one or two objects "one-to-one" and are surprised when "the partner" is missing, but they could only exceptionally do this for three or more objects, due to limited carrying capacity of memory at this age. The replication of some experiments confirms this. For example, utilizing the looking time technique, Uller and Leslie (2000) showed that 12-month-olds understand what "exactly two" means, but they have difficulties to pass over 3 when adding objects. Uller (2002) concluded that this capacity of differentiating up to four objects (sometimes called subitizing) should be considered primitive and foundational, perhaps at the core of cognition.

The operational category generically called ***Relating*** contains operations described as connecting an entity to one or more others, based on a relationship. As an inner operation, *relating* permits to infants to compare one specific object to others around them in order to assess their similarities and differences. Within this inner operation, the child realizes that there is a difference between the mother's face and the father's face; there is a difference between one object and two similar objects. From another perspective, while associating emphasizes symmetry, relating could emphasize asymmetry.

The first operational category described above (*Associating*, which could be modeled by bijective functions) is included in the second one (*Relating*, which could be modeled by mathematical relations), but there is no meaningful to study this inclusion; this is why the two categories are separately listed. While associating suppose *matching* (bilateral connection) as a representational task, relating suppose *mapping* (network connection).

The category of ***Algebraic operations*** contains operations dealing with quantities that are combined in a specific well-defined way and the result of this combination is analyzed from a quantitative perspective. In school, one studies the binary operations (defined on a Cartesian product with two factors), such as addition and multiplication. As inner operations of the algebraic category, *Pre-arithmetical operations* refer mainly to a list of general operations that, quantitatively expressed, lead to addition, subtraction and so on, such as: grouping, taking away, magnifying, reducing, adding, combining, etc. These permit to infant to distinguish between the situation when an object is added onto a scene and the one in which an object is taking away. The inner operations of this category assure also a sense of increasing and decreasing quantities.

While the first two operational categories (associating and relating) act by building relationships between two or more entities without modifying their nature, the algebraic operations suppose an intervention to obtain from the given sets of entities another one, whose cardinal (measure) is to be determined.

The category of ***Logical operations*** is completing the category of algebraic operations by giving a formal meaning to it. In the young child mind, rudimentary elements of logic are present as inner operations; they consist in relating two facts through conjunction or disjunction and perceiving the result of the two as a third fact. For example, when mother *and* father are coming, the child perceives that they are coming *together*, comparing to the situation when mother and father appears separately in space/time and the child is expected to see mother *or* father. Moreover, very early in life the child is able to react to the "*don't*"-s. Another logical operation refers to inferring in the format of "If p, then q". This type of reasoning appears in the early years mostly in simple causal inferences, associated to conditioned reflexes.

The category of ***Logical operations*** is extending the current mathematical meaning of logical reasoning to the general capacity to formulate logical inferences for different types of reasoning. For example, different patterns are involved in deductive reasoning and non-deductive reasoning. In addition, different patterns are involved in different types of *deductive reasoning*, which could be: *conditional, consecutive, causal, modal, normative, procedural.* Different patterns are also involved in the following types of non-deductive reasoning: inductive reasoning (in which, based on examination of a number of cases from a class of objects, a conclusion is formulating about the entire class) and analogical reasoning (in which, a conclusion about an individual case is formulated as a consequence of observing one or more individual cases.). The daily reasoning and argumentation actually mix many of these types, but when analyzing the mechanisms that underlie understanding, it is important to describe them as precisely as we could, in order to develop appropriate training.

The category of ***Topological operations*** has a pervasive presence on development in the first years of life. The topological operations permit to identify boundaries, to relate them with discrete components, to globally perceive objects, to pass the frontier between discrete and continuous, to have an intuition of infinity. They account for the convergence of thinking and for global perception. The term *topology* usually refers to a domain of mathematics that started from geometry and has as central idea that continuous geometric phenomena can be understood by the use of discrete invariants. One of the strengths of algebraic topology has always been its wide degree of applicability to other domains. Nowadays that includes physics, differential geometry, algebraic geometry, and number theory. In the context here, I use an extension of the mathematical term. In a wider sense, in this category are included mental operations dealing with the idea of infinity (convergence to infinite). The topological operations permit to associate to a thing those properties that are in a geometrical sense the most permanent - the ones that will survive distortion and stretching. From a topological perspective, the geometrical distances and angles are not relevant, the geometrical shape neither, what it is important is conserving the continuity. The primitive topological property of mind permits to infants to discriminate numerosities when they are significantly different and permit later to make numerical approximations with different orders of magnitude and various degrees of precision. They also conduct to globally perceive continuous surfaces. This type of perception was identified in some researches, while checking the subitizing paradigm. For example, Clearfield and Mix (2001) reported that, when systematically manipulated contour length or aria, 6-to 8-month-old infants dishabituated to a change in either contour length or aria, but not to a change in number. They concluded that infants might actually be using continuous quantity rather than number to discriminate between displays.

While algebraic and logical inner operations are dealing with finite and discrete quantities, topological operations are addressing to infinite and continuous properties. The nature and mechanisms of the inner operations in the topological and algebraic categories are completely different; and from this perspective, the results found by Daehene, Spelke and others during the last decade are completely predictable and they constitute verification for this theory.

Iterating is described as perceiving regularities in variability; this operation allows initially for imitation, but further for identifying and developing patterns. In recent psychological research, mimicking (Fisher and Bidell, 1998) is seen as an ability to overcome the skill level by manifesting a behavior analogous to the next level; even it is less consistent comparing to the advanced level, this ability stimulates progress in learning at the early ages. In this way, imitation is seen as a primitive form of *perceiving and developing patterns*. Iterating as an inner operation accounts for the recursive property of mind.

Generating could be described as an operation-category whose elements create new entities, previously unknown, starting, eventually, from entities already known. A special element in this category is the unconditioned generating: *Grasping*. To be more specific, *Grasping* could be defined as perceiving an entity or its essence instantaneously, without proceeding discursively in space or time (i.e. by passing from one bit of information to another). As a basic mental operation, *Grasping* plays also an important role in the emergence of spurts and other types of discontinuities recorded in the experiments done in micro-development (Fisher and Bidell, 1998). *Generating* category assures a kind of readiness to start. This operational category help the mind to build the leap to learning, it creates *the innate dimension of the motivation to learn.*

Therefore, on short, the operational-categories identified as foundational for learning are: *Associating, Relating, Algebraic operations, Logical operations, Topological operations, Iterating, Generating*. To have a shorter reference in the following, I will use the initials and I will call this list the ARALTIG model. Apparently, most of the operations on the list look to be mathematically defined. This is true only in the measure in which mathematics conceptually offers powerful tools to accurately define and build categories.

In the newborn mind the ARALTIG operations are less differentiated. Through training, they reproduce and ramificate while internalizing and self-structuring new knowledge. They are incorporated in and act for each part of this knowledge. The specific operations of a class share, as representatives, the property that defines the class. That means they are functioning on the same mental scheme, no matter the level of abstraction involved by the entities the operations apply to. This invariance creates conditions to foster patterns. The ARALTIG operations naturally combined in various ways predispose the human brain to learn language and to develop through language. These categories of operations are necessary and sufficient to nest and sustain the brain-mind development. They are the mold that permits and assures architecturing the development; the other operations the mind is processing result from the ARALTIG model through relating the basic operations in various ways.

STRUCTURING PREDISPOSITIONS – A WAY TO OPTIMIZE LEARNING

The born mind starts building category representations from the beginning, and for doing this, it is endowed with operations (or properties) that are self-developing in interaction with the environment. The ARALTIG model is built in terms of operations and not in terms of properties for three reasons: the elements of these categories ***act*** as operations, ***could be trained*** as operations and ***constitute the basis*** for complexes of operations. The strength of this model consists in its connectivity to an effective and creative learning. If until now I tried to build arguments, I will try next to explain the consequences. I will use, as before, conclusions of various experiments, then I will try to schematize some principles for an operations-centered training, principles which have been applied for building dynamic structures of thinking (Singer, 1996; 2001).

First of all, a short insight into a question: why arithmetic learning is so difficult? Though usually children begin verbal counting in their second or third year of life, they arrive to understand the meaning of the counting routine late, after two or more years. The experiments done by Wynn (1990, 1992) provide evidence that children understanding of counting develops in four steps. In the first stage, the child is able to understand “one” as referring to an object, but he/she is not able to associate another numeral with a number of objects. He/she can only dissociate one and more. At this moment, one, two, three,… are just words-label, not words-meaning. After 9 months of counting experience, Wynn's children entered the second stage of comprehension, in which they were able to correctly identify "two" with various arrays of two objects and to make comparison between one, two and more. After three further months of training, on average, children showed they learned the meaning of "three". Finally, in the fourth stage, they proved comprehension for all the ten numbers in the counting routine. In this experiment, children needed about 1 to 1 and 1/2 year of training/practice to associate the counting routine with its meaning. Is this a long term? Considering the complexity of the task, it is not.

In what consists the complexity of this task? The children have to internalize and automatize a double invariance: invariance of the class the particularly object belongs to (the specific physical elements of the set of which the cardinal is determined) and the invariance of the class of classes the object belongs to (conservation over nature and order of objects and successive inclusions). The counting routine adds a complex association to this: associating to the ordered sets their cardinals, and a recursion (transforming cardinals into ordinals). To be more explicit, a suggestive non-standard representation for this 4-levels of complexity (double invariance, association and recursion) is given in Fig. 1. When "2" appears, it is restructuring the relationship between "1" and its set; the same, when "3" appears, it is restructuring the relationship between "1" and "2" and their set; the process is going on recursively, for 4, 5 and so on.

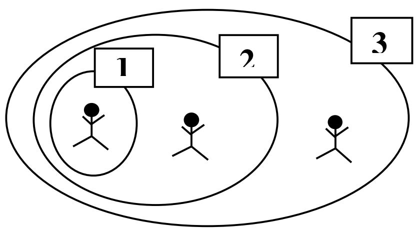

FIGURE. 1: RESTRUCTURING

In terms of computational representations in the brain, as far as the cardinal is increasing, an already created connection has to be successively interrupted (disconnected) and replaced by another, different connection. The recursion of this process: is essential, is difficult to be learned because the process continuously restructures connections, and constitute the fundamental difference from the nature of the subitizing process. At the same time, this seams to be possible due to the human mind innate ability of iterating and identifying patterns.

This short insight gives just a snapshot about the nature of difficulties to face when describing the process of learning. The ARALTIG model has the advantage of showing the few meaningful directions on which the training could be focused with the purpose to foster the natural predispositions and to avoid low significant redundant information (the so-called noise in complex systems).

TRAINING FLEXIBILITY – A WAY TO OPTIMIZE UNDERSTANDING

I will repeat a truism: a consistent training should be based on a strong understanding of the initial conditions. In this context, the initial conditions consist in *mental operations* applied on *information* that have different *degrees of complexity* and involve various *levels of abstraction*. In describing the information complexity, I used three independent vectorial dimensions: *the nature*, *the structure*, and *the procedure*. Concerning the *nature* of information, the criterion was the *level of generalization*: information could be organized on a scale from general (entities, laws, theories, etc.) to particular (examples, case-studies, etc.). The *structure* of information is described by the *connections among concepts*, which organize information on different stages from unstructured (non-systemic – poor connections, no hierarchies) to structured (systemic – strong connections and hierarchies). Concerning the *procedure*, the criterion was the *dynamics of connections* and from this perspective, information could be organized from reproductive to creative procedural actions.

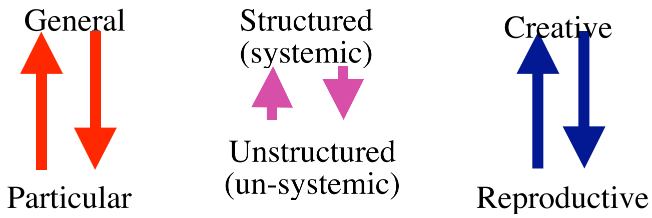

Figure 2: Structuring the information degrees of complexity

Differentiating these dimensions is significant for organizing training. Thus, to make abstractions incorporable in mental structures, a lot of passages from particular to general and vice versa need to be practiced. It is also important to pass through as many organizational stages as possible, as well as to face both algorithmic and creative tasks. The schema in Fig. 2 stress that, as far as we identified significant stages on each of the three scales, the training needs to focus on the transfer in-between various stages, more than to focus on each one separately. This is removing the didactical approach from a "horizontal" way of perceiving teaching: islands of information to be transmitted, to a permanent process of "vertically" restructuring students' knowledge by incorporating the new elements into a dynamic structure. Particularly, this implies systematically practicing the basic operations by creating patterns of variability. This kind of training supposes identifying and developing optimal individual pathways in a multidimensional network. These pathways could be incorporated in a learning technology of the future, aiming at creating fluent thinkers. Here, technology is used in a very broad sense, as strategic mean to achieve a purpose; the didactical technology I am referring to includes printed materials for teachers and students and eventually software, but its most pervasive component is didactical intelligence.

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CONNECTING THEORY AND REFLECTIVE PRACTICE THROUGH THE USE OF PERSONAL THEORIES

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This paper outlines a way of connecting theory and reflective practice in mathematics teacher education. The construct of personal theories is put forward as an innovative pedagogical tool for connecting theory to practice through reflective writing. The cognitive process of 'noticing and naming' emerged as way of theorising practice and helped to track a pathway of professional and personal growth. Personal theories highlight how a narrative and reflective orientation to teacher education can enhance the process of learning to teach mathematics by acknowledging the affective factors involved in developing an autonomous mathematical identity. It is argued that connecting theory and reflective practice can be particularly useful in developing a reflective disposition and should begin with eliciting and analysing personal beliefs and theories.

DEVELOPING A THEORETICAL PERSPECTIVE

The term 'reflective practice' is a common term within the discourse of teacher education, but it is rarely illustrated in practical terms. Recently, research on learning to teach has begun to focus on the need to provide opportunities to make explicit, and build on, prospective teachers' existing knowledge, beliefs and attitudes about teaching and learning because "…what has to be learned is intimately connected to what is already known" (Beattie, 2000, p. 19). In mathematics teacher education, it is acknowledged that changing the beliefs and practices of prospective teachers so they reflect a more constructivist approach to teaching and learning is problematic (Grouws & Schultz, 1996; Mewborn, 1999). Rather than focusing on *changing* beliefs and practices, Korthagen & Kessels (1999) describe a more innovative approach to pedagogy for teacher development where the prospective teacher and their personal theories become a *starting point* for teaching programs. They argue for an emphasis on the development of reflective practice, inquiry-oriented activities, and interaction amongst learners. Support can then come from teacher educators who could provide 'fruitful' and meaningful theories (public theories) that relate to student teachers' practical problems and personal theories. Korthagen & Kessels (1999) describe this as a 'realistic approach' that has strong parallels with theories of mathematics learning and teaching.

The use of a theory/reflective practice interactive model of teacher education can provide opportunities for collecting, examining and beginning to codify the 'wisdom of practice' (Shulman, 1987) that emerges from prospective teachers' personal theories about their lived experiences. As Shulman (1987) reminded us "the neophyte's struggle becomes the scholar's window" (p. 103), therefore, exploring prospective teachers' reflective narrative texts related to the process of becoming a teacher of mathematics could help bridge the well-documented 'gap' between theory and practice in mathematics teacher education courses and shed further light on the influence of personal beliefs about mathematics in relation to the type of pedagogical practices that are adopted to teach mathematics (Korthagen & Kessels, 1999; Mewborn, 1999).

For this study, reflective narrative texts refer to written texts elicited from prospective teachers for the explicit purpose of thinking about beliefs and actions with a view to critical reflection and improvement (Hatton & Smith, 1995, Sullivan & Mousley, 1998). In this study, the texts can be thought of as a pedagogical tool that took the form of personal theories (Bullough & Gitlin, 1995) about teaching, learning and assessing students in mathematics. The writing of personal theories provided opportunities for theorising and sharing experiences and beliefs through verbalising, clarifying and recording reflective thoughts related to theory and practice.

Personal and public theories

An important aspect of the theory/reflective practice connection is that the word 'theory' refers to both *personal and public theories*. For some, this notion will require a paradigm shift in how theory is perceived. *Personal theories* are grounded in a person's 'appreciation system' that is based in beliefs and consists of a repertoire of values, knowledge, theories and practices (Schon, 1983). Eliciting *personal theories* becomes particularly important for the professional growth of prospective teachers of mathematics because 'the level of consciousness about beliefs influences their disposition to realize change" (Cooney & Shealy, 1997, p. 91). On the other hand, *public theories* represent all other forms of traditional research that are represented in textbooks, journals and other related literature, and also through the theories chosen for study by a teacher educator. Both personal and public theories can be thought of as the 'living, intertwining tendrils of knowledge, which grow from and feed into practice' (Griffiths & Tann, 1992, p. 71).

The notion of *personal and public theories* was influenced by Mason's (1998) description of 'inner research' and 'outer research'. To talk about inner research is to talk about awareness of, and being explicitly articulate about personal and practical sensitivities, experiences and ideas (personal theories) that may affect mathematical attitudes and identity. These sensitivities require *introspection*, a metacognitive process of observing oneself from the inside (Mason, 1998). The need to maintain a personal awareness and sensitivity to the teacher within can be thought of as maintaining a reflective disposition. The writing of personal theories represents a constructive way of talking to ourselves that helps develop the authority and identity to teach through personally conceptualising mathematics teaching (Cooney & Shealy, 1997; Sullivan & Mousley, 1998). Concomitantly, *outer research* (public theories) represents a more public type of theory that attempts to explain rather than to experience. This form of research requires a distancing from practice where there is an attempt to objectify activities. Perhaps most important is the fact that 'outer and inner research do not displace or replace each other, but rather are complementary' (Mason, 1998, p. 375). This complimentarity between inner or *personal theories* and outer or *public theories* is needed to generate and support authentic and legitimate reflective texts.

Reflective practice

The term reflective practice describes the nexus between *reflection* and *practice*. If the term *practice* encompasses both the practice of teaching and the practice of learning, then practical experience becomes a site for learning. However, this will only occur if the learner possesses a disposition to be reflective. Perkins, Jay & Tishman's (1993) 'dispositional theory of thinking' helps to frame the need for a reflective disposition that has three qualities. A reflective practitioner would need to have: the *inclination,* or felt tendency to be reflective about their practice; *sensitivity* or awareness of a personal stance, and to occasions or opportunities when reflection is warranted; and the *ability* or know how to follow through with reflection in order to develop future practice (Perkins, et al., 1993). Reflective practice (of both teaching and learning) could then lead to the reframing of personal theories that assimilate public theories and possibilities for future action. The underlying premise of reflective practice is that any reflection requires thought which leads to action that is dependent on the result of the thinking that occurred. Mewborn (1999) suggested that action (practice) and reflection can be seen as a 'bridge across the chasm between educational theory and practice' (p.317). Mewborn (1999) also highlighted the importance of both individual (introspective) reflection and shared reflection so that prompting and probing can foster the reflective process.

The synergy between personal and public theories and reflective practice is now explored in more detail to illustrate how a reflective disposition can be developed to cognitively enhance the process of learning to teach mathematics. Following Perkins, et al. (1993) reflective practice does not just require the ability, or skill building to be reflective. We also need to consider the more affective qualities of sensitivity and inclination within the teacher if reflective ability is to be enacted. Acknowledging the importance of sensitivity and inclination in the development of reflective practice can help to 'close the gap between ability and actual behavior' (Perkins, et al., 1993 p. 12). It is also possible to imagine that the development of a reflective disposition could raise awareness of the importance and place of theories in actual practice. In mathematics teaching in particular, a reflective disposition can assist prospective and practicing teachers to maintain a heightened and critical awareness of their pedagogical practices and an open mind towards more authentic and constructive approaches to teaching, learning and assessing.

CONTEXTUAL SETTING AND INQUIRY METHOD

This study took place in a rural university in New South Wales, Australia. Although fifty-five participants engaged in writing reflective narrative texts during a year-long subject (Assessment and Diagnosis in Mathematics) in their final year of an undergraduate education degree, this paper reports on a case study (Stake, 2000) conducted with one participant. Prior to the present subject, the participants had engaged in three mathematics subjects that emphasized and modeled constructivist principles of teaching, learning and assessing mathematics in the first three years of their course.

In particular, an 'instrumental case study' was chosen as an illustrative technique. Stake (2000) suggested that this type of case study is chosen to advance the understanding of an external interest, in this case to provide insights into the nature and role of personal theories in the theory/reflective practice cyclic approach to mathematics teacher education. Consequently, Elizabeth's story is not necessarily representative of all the participants in the study, but is told to illustrate how theory and reflective practice can coexist in a mathematics teacher education program. The use of a narrative approach (Connelly & Clandinin, 2000) to tell Elizabeth's story is consistent with the philosophy behind the use of reflective narrative texts such as *personal theories* to enhance the development of a reflective disposition. The use of narratives in mathematics teacher education allows for the extensive use of participant voice to tell stories of personal and professional growth that can heighten our awareness of the dispositions that shape our practice (Cooney & Shealy, 1997; Mewborn, 1999; Sullivan & Mousley, 1998).

ELIZABETH'S STORY: A NARRATIVE ANALYSIS

Elizabeth is a self confessed 'struggler' when it comes to learning. But what I sense in her descriptions of her learning experiences is the struggle to make sense of things and find herself as a learner in order to find herself as a teacher. Elizabeth grew up with a supportive family and enjoyed happy childhood experiences at elementary school. Her high school experiences were where she 'lost the plot' and 'got in with the wrong crowd because I was a follower instead of a leader'. Elizabeth sought a 'second chance at learning' when she enrolled in a college to do her final year of school again. She attributed her success there to a renewed desire to learn, being surrounded by adults and realising, finally, that she was the only one who could make her learning successful.

Gaining entry into university fulfilled a life-long goal for Elizabeth. Becoming a teacher was something that she had always thought about, her learning journey had turned itself around and she was ready to move forward. Two years at university saw her achieve mixed success and she felt she needed to take a break from learning to decide what she really wanted. After six months, she was 'back on track' and knew for certain that she wanted to become a teacher. Academically, Elizabeth found a new desire to learn. Her grades improved considerably and she had a renewed sense of self as she embarked on her final two years of teacher preparation.

Initial personal theories

Prospective teachers were asked to document their personal theories about teaching, learning and assessing mathematics at the beginning of the year and again at the end of the year. Elizabeth's initial personal theories reflected a 'socially-critical orientation that needed to give students a voice in their learning'. An example of the theory/reflective practice interface occurred as I read Elizabeth's reference to student voice and recommended a journal article for her to read that related to listening to children's voices in the classroom. This recommendation emerged from the process of reflective writing and created a meaningful and purposeful connection between Elizabeth's personal or inner theory and a more public or outer theory.

In mathematics, Elizabeth always needed to know 'why $2 + 2 = 4$' but was never told. She attributed her lack of success in school mathematics to being 'treated like idiots' and never being told 'what we were learning or even why it was important that we learnt it'. This perceived lack of communication became an underlying theme of all Elizabeth's reflective texts during the year. She felt 'cheated by her teachers' in the past because she often 'took time to catch on to a concept' and saw herself as a 'feeler rather than a thinker'. Experiences from Elizabeth's past clearly affected her beliefs and attitudes towards teaching mathematics in the future. Not surprisingly, her initial personal theories reflected that students should learn through a 'three-way process of modelled, guided and independent learning'. She believed that 'students and teachers learn together and in the same way'. That is, when they are: 'challenged and motivated, able to build on knowledge skills and values; able to make connections between knowledge and experiences; able to see a purpose in what they are doing; and acting, reflecting and interacting with each other in a supportive environment'. Elizabeth acknowledged in her initial personal theory that her 'learning and values are changing continuously, even after two weeks back at Uni'. It seems clear that Elizabeth wanted learning to be different to what she had experienced as a student at school, and communication with students was obviously going to play an important role.

I have drawn on the work of Belenky, Clinchy, Goldberger & Tarule (1986) to describe the personal and professional growth evident in the conceptualising of personal theories. Very briefly, Belenky et al. (1986) used the metaphor of voice to describe five different perspectives in which to view knowing: *silence* describes a voiceless stance that relies totally on external authorities as a way of knowing; *received knowing* refers to the perspective that knowledge can be received or even reproduced from an all knowing external authority, but individuals are not capable of creating their own knowledge; *subjective knowing* describes a shift in the authority of knowing to view it as personal and private; *procedural knowing* refers to a more reasoned reflection related to an investment in learning and applying objective ways of knowing; and *constructed knowing* that views all knowledge as contextual and created by using both objective and subjective strategies for knowing (p. 15).

Like many of her peers, Elizabeth's initial theories reflected a received way of knowing theory. Essentially, her theories reflected a reproduction of other people's theories to help her frame her own theories. Risk-taking and creativity were not evident. Her reference to 'modelled, guided and independent' learning reflected a more public theory that had been shared during her course. Similarly, Elizabeth's description of how students and teachers learn reflected the flavour of a number of learning theories that had been shared and critiqued during her study. While this is not necessarily a negative aspect of theory development, following Belenky et al., I would argue that a more constructed way of knowing would have included more creative and contextualised descriptions of theory that were supported by personal and meaningful practices that had been developed as a result of multiple experiences. Elizabeth's received way of framing her personal theories appeared to reflect the nature of her past experiences learning mathematics. For Elizabeth, knowledge was given by an external authority (teacher) and she was expected to reproduce it with little or no interaction or shared negotiation. While Elizabeth's personal theories appeared to describe a more dialogic approach to teaching mathematics, she provided no elaboration of how this could be achieved in her initial theories.

Final personal theories

The personal theories written by Elizabeth at the end of the year-long subject took on a remarkably different form to her first attempt. She presented her theories as a 'learning journey I have undertaken over the year through personal experiences, lectures, case stories, theories and reflection'. Elizabeth's personal theories took the form of a learning portfolio that made many connections to cartoons, quotes and pictures to 'contribute to what I believe'. It could be said that Elizabeth's final personal theories reflected a more constructed way of knowing because she had used a blend of other people's theories to construct her own personally meaningful theories. Two examples of her personal theory development will help to illustrate a constructed way of knowing that blends personal (subjective) and public (objective) theories together and explicates the process of *noticing and naming* that appeared to contribute to meaningful ways of conceptualising teaching.

The process of ‘noticing and naming’

One of Elizabeth’s personally created theories was named ‘Talking to know’ and is described in the following entry in her learning journal:

> It’s fine to read about it and see it but doing it and feeling it is how I learn. No that’s not completely true I think I learn more from talking about it. My name for this is ‘talking to know’…in the end I work it out from talking and making sense. It’s strange, it’s not listening to someone, although sometimes they trigger my thinking off, it’s more about me talking, making sense. I’m still discovering how I learn but I am enjoying my journey. This is what I mean by ‘talking to know’. I am talking to you and trying to sort out stuff in my head. This subject is like the missing piece of the puzzle. I am making connections. However, you’re very clever Tracey. I am going to call you Queen Epiphany from now on because you are allowing us to open our minds and create something that is ours. Understanding. I’ve never felt more at ease with my thinking. I wonder if everyone else is feeling the benefits of this? I am making this point to you Tracey because when you have a conversation with me or anyone I guess and they’re trying to make sense of something and then they do and it clicks for them then you are having what I call a teacher moment. (Personal theory entry October, 2001)

The conversational quality of this entry epitomises a significant aspect of prospective teachers’ writing that I have identified as *noticing and naming* and was adapted from Mason’s (1998, p. 366) term ‘noticing and marking’. Elizabeth’s reflective writing provided an opportunity to heighten an awareness of her beliefs and *notice* what was important to her. *Naming* the learning process ‘talking to know’ allowed me to listen into and then join her conversation through my written feedback that was potentially meaningful, and part of a shared process of ‘coming to know’. The process of *noticing and naming* emerged from one of our classroom conversations about listening to your inner voice and to other people’s conversations (public theories) and *notice* what resonates with you as a learner and prospective teacher, then *name* that resonating moment so it becomes a part of your personal theories for the future. I would argue that the *noticing and naming* process is an authentic learning experience that leads to teachers authoring their own learning and empowers them to be theory generators as well as theory users. Moreover, I believe the cognitive process of *noticing and naming* enhances the development of a reflective disposition in prospective teachers and helps to build a more positive and autonomous mathematical identity.

The second theory that Elizabeth noticed and named was the ‘It’s like theory’. The following entry explains the development of this theory:

> The ‘it’s like’ theory is about making links. It’s about learning and connecting with other schema in my head. I then thought about when a child did this in my classroom and they used an allegory to make their point. That was a real teacher moment! …they are a great way for both adults and students to achieve “it’s like”…but how do we as teachers get the students to think this way or get to this point? Well, maybe the teacher can ask the children if anyone has had a similar experience? I know that I have done this possibly daily when teaching, but I never really made the connection before of why this is one way that a child can learn. Well, I’m not that stupid, sub-consciously I would have been aware but I guess saying it makes my connection more powerful. (Personal theory entry, November, 2001).

Elizabeth’s description of the ‘it’s like’ theory illustrates how reflective writing can both capture and theorise a personal learning experience such as the use of analogies or allegories and connect it to a pedagogical practice for the teaching of mathematics.

During a presentation to her peers, Elizabeth explained her 'it's like' theory in practice. She used an anecdote from her internship to describe how a young boy in her class made a connection between writing in their mathematics learning journal and the posters they had created as a class. The young boy said that 'the journal is like the posters that tell how we think about something in mathematics. You want to know how we think about maths'. In this anecdote, Elizabeth's implementation of an alternate assessment strategy (journals) was a practical example of her personal theories that had been developed as a result of exposure to more public theories that were shared during the year-long subject she had participated in.

There are noticeably common elements between Elizabeth's initial personal theories and her final theories. For example, the 'it's like theory' suggests the use of analogies and allegories that might provide an opportunity for 'modeled, guided and independent learning' referred to in Elizabeth's initial theory, but the *noticing and naming* of the 'it's like' theory provided a personal commitment and elaboration of the strategies behind the theory. This process of elaboration is an important aspect that would therefore suggest a more constructed way of knowing that highlights the nexus between personal and public theories that were the result of the writing and sharing of personal theories.

IMPLICATIONS FOR TEACHER EDUCATION

In a sense, this search for finding words, speaking for oneself and feeling heard by others are all part of a 'discourse of becoming' (Britzman, 1991), a search for voice that enhances agency and advocates that teacher educators adopt an approach of *listening more and telling less*. A reflective orientation to teacher education can be achieved through the use of related reflective narrative stories such as those illustrated in Elizabeth's case story. The writing of personal theories provided an opportunity for framing and reframing beliefs about the teaching and learning of mathematics by stimulating teacher reflection (Sullivan & Mousley, 1998). The articulation of personal theories appeared to enhance the development of a reflective disposition that was supported by the key process of *noticing and naming*, which provided opportunities for teacher educators to *listen more* to personal theories rather than simply selecting and telling predetermined public theories. In addition, the process of *noticing and naming* appeared to signal a more *constructed knowing* (Belenky et al., 1986) that led to visions of practice that were more personally meaningful and authentically developed.

Such a narrative approach has been relatively unexplored in mathematics teacher education. The explicit use of narrative stories in the form of personal theories can provide an innovative pedagogical tool for empowering teachers to explore and direct their own learning and make meaningful connections to more public theories. Personal theories have the potential to contribute to the growing body of knowledge in mathematics teacher education that sheds light on the process of becoming a mathematics teacher and how existing beliefs and experiences can affect and shape who we are as teachers. Moreover, *noticing and naming* our own personal theories that are connected to more public theories means that theory need not be dispensed in a language separated from the teacher's reality, which can then open the door wider for prospective teachers to become theory generators as well as theory users. As Elizabeth often said 'you have to love it to use it in mathematics'. When theories are personally generated (through the process of noticing and naming), they are more likely to be 'loved' and used.

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TWO MEANINGS OF THE 'EQUAL' SIGN AND SENSES OF COMPARISON AND SUBSTITUTION METHODS

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In this paper we analyze the meanings of the 'equal' sign as generated by sense production of the methods of substitution and comparison for solving problems and systems of equations of two unknowns. These methods are usually introduced through an extension process of the syntax and meanings recently learned by students in order to solve problems using linear equations with one unknown. Through this process some users were able to confer sense to the methods and thus generate the new meanings required.

In Filloy (1991) we introduced the notions of ***meaning*** and ***sense*** for analyzing the learning processes and the creation of rules which allow to coordinate the actions performed for solving one-unknown problems through "concrete models" (see Filloy and Rojano, 2001; and Filloy, Rojano and Solares, 2002). In this paper we use these notions for studying the transition from one-unknown representation and manipulation to the representation and manipulation of one unknown given in terms of another unknown—In fact, this transition corresponds to a ***didactic cut*** (see Filloy and Rojano, 1989; Solares, 2002). The new representation of the unknown is used in the ***comparison and substitution methods*** in such a way which allows the **reduction** of a two-unknown problem to a one-unknown problem and making possible to apply the previously learned syntax in order to solve one-unknown linear equations. In the particular case of the system $(S_1):\begin{cases} y = 12 - x \\ 5x - 6 = y \end{cases}$ the performance of the ***comparison method*** entails the equalization of two operation chains for one of the unknowns and for the data which allow to calculate the other unknown's value. That is, two ways for calculating the value of one of the unknowns are equalized. And in the case of the system $(S_2):\begin{cases} x + y = 12 \\ 5x - 6 = y \end{cases}$ to perform the ***substitution*** entails replacing the '***y***' in the second equation in the first and through this operation chain find the '***x***' value. Thus, a chain of operations is substituted into another.

We will analyze the meanings of the 'equal' sign generated by children between 13 and 14 years old when using the comparison and substitution methods in two-unknown equations' solving process. For the students interviewed the ***sense*** of this methods is given by the linking of all actions performed. At the beginning of the learning process these action chains are not yet provided of sense. The increasing syntactical complexity of the relations between the data and the unknowns, the changes in the data's or solutions' numerical domains, for example, obstruct both the use of the methods and the spontaneous solution strategies. At that stage, readings from more concrete strata of the new Mathematical Sign System do not allow to identify the changes in the problematic situation as members of the same kind of problems. Only when the sense conferred through the sequence of mathematical texts in the Teaching Model is acquired these strata will be identified as members of the same kind of problems –susceptible of being solved through the same process, or chain of actions. That is the moment when the new notions –such as the new notion of equality– will become stable (see Matz, 1980; Kieran, 1981; Kieran and Sfard, 1999; Drouhard, 1992).

Here, it is useful to see once again the way in which Mt. (Filloy and Rojano, 1989, pp. 21-22), one of the subjects interviewed, generated the meaning assigned to the 'equal' sign when learning the syntax for solving one-unknown linear equations: **IMt26. 10x – 18 = 4x + 6...**

> **Mt:**…if I obtain the value of ***x*** and I perform that operation (points towards "***10x – 18***") I obtain one result. That result has to be equal to this (points towards "***4x + 6***")

THEORETICAL AND METHODOLOGICAL FRAMEWORK

For the experimental design a **Local Theoretical Model** (Filloy, 1990) was built up in order to explain, upon the semiotic notion of **Mathematical Sign Systems** (MSS) the empirical observations obtained through videotaped clinical interviews.

From the theoretical perspective of the **Local Theoretical Models** each specific object of study is analyzed through four interrelated components: (1) the **Formal Competence Model**; (2) the **Teaching Model**; and (3) the **Cognitive Process** and (4) **Communication Models**. Below, the specific characteristics of these components in our study will be described.

FORMAL COMPETENCE MODEL

In order to construct **the Formal Model** component, we used the syntax model for simple algebraic expressions and equations developed by Kirshner (1987) and completed by Drohuard (1992). Besides, we incorporated the semantic elements proposed by Drohuard (1992) in order to study the meanings of the algebraic writing. These studies on algebraic syntax and semantics render important results for teaching –such as Drouhard's definition of *automathe formel* for defining subjects who center their attention on the rules that have to be applied (*sens*) and not in the results obtained (*dénotation*)–, these studies do not incorporate to the analysis the spontaneous usage that learners give to already learned elements of the algebraic language in order to solve new problems.

The **Formal Competence Model** that we designed allows us to study the syntactical complexity of the algebraic substitution and comparison methods used for solving equation systems. Substitution method results more complex.

TEACHING MODEL

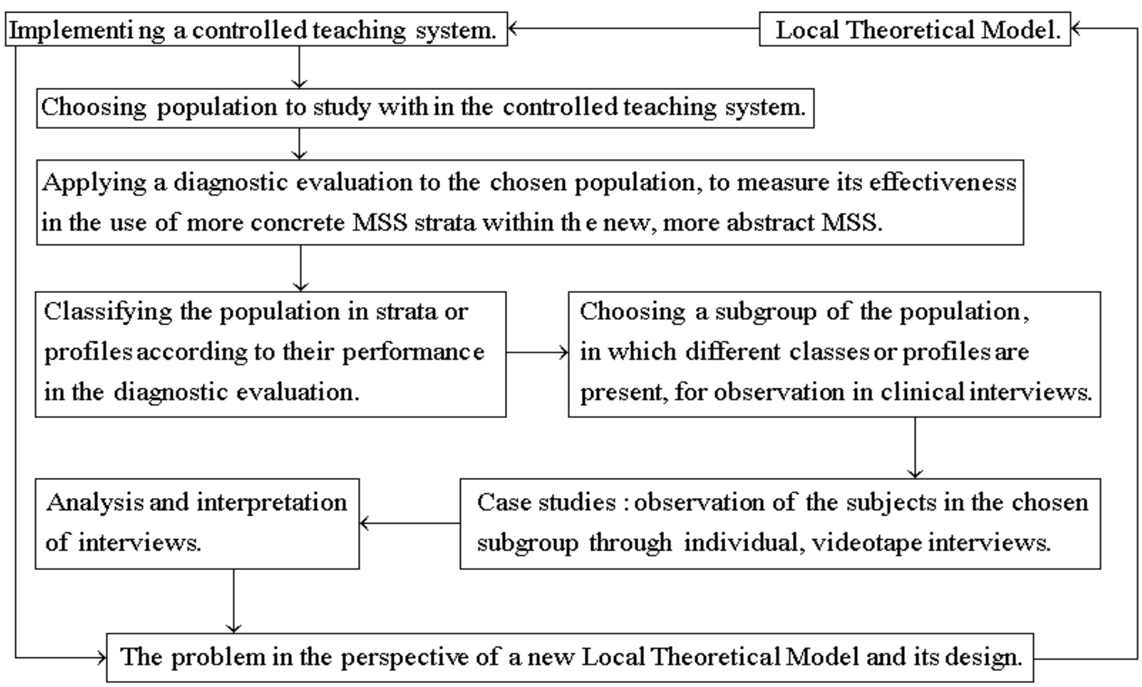
Upon the base of the analysis performed at the formal level we adopted the following **didactical route** for introducing these methods –coming from the previously acquired competencies for solving one-unknown linear equations: (1) reduction of the two-unknown and two-equation system to a one-unknown equation through the application of comparison or substitution; (2) solution of the one-unknown equation applying the previously learned syntax; (3) substitution of the numerical value found in one of the two equations; and (4) solution of the equation through the application of the previously learned syntax.

COGNITIVE PROCESSES MODEL

For the definition of the **Cognitive Processes Model** we used the cognitive tendencies list provided by Filloy (1991). In the current study the following items are particularly relevant (we follow the original enumeration given in Filloy's list, 1991): (2) conferring intermediate senses; (3) returning to more concrete situations upon the occurrence of an analysis situation; (5) focusing on readings made in language strata that will not allow to solve the problem situation; (8) the presence of inhibitory mechanisms; (9) the presence of obstructions arising from the influence of semantics on syntax and vice versa; (11) the need to confer senses to the networks of ever more abstract actions until they become operations.

For the present study we worked with 12 students from the "Centro Escolar Hermanos Revueltas", in Mexico City. Video-taped clinical interviews were carried out with children who already had been instructed in pre-algebra, and who already had been introduced to topics of elementary algebra through the solution of one-unknown linear equations and of word problems associated to these equations. However, they had not yet been introduced to the systematical use of open algebraic expressions nor of linear equation systems.

The following scheme describes the development process of our research:



INTERVIEW ITEMS

The items list is divided in two sections: word problems and syntactic tasks. The items list proposed to each particular individual may change depending upon the cognitive tendencies found during the interview.

Nine word problems were proposed. The followings are some examples:

> The sum of two numbers equals 90. If one of these numbers is added to 16 and then multiplied by 5 the result is 360. Which are these two numbers?

The difference between two numbers is 27. We know that seven times the minor number plus 30 is 12 times the minor number. Which are these two numbers?

A rectangle's perimeter is five times its width. Its length is 12 meters. ¿What is the width?

The syntax questions were:

S.1	$x + 2 = 4$	**S.6**	$14 + x = 37$	**S.11**	$3x + 4 = 22$	**S.16**	$y - 6 = 3x + 20$
	$x + y = 8$		$4 - y = 28$		$4x + 2y = 34$		$5y - 4x = 64$
S.2	$x + y = 10$	**S.7**	$45 - x = 17$	**S.12**	$3 \square (8 + x) = 6$	**S.17**	$3x + 8y = 84$
	$x - y = 4$		$x + y = 41$		$2x + y = 23$		$8x + 3y = 59$
S.3	$x + y = 9$	**S.8**	$x + y = 60$	**S.13**	$2 \square (3 - x) = 6$	**S.18**	$4x - 3 = y$
	$2x + 3y = 23$		$3x = 171$		$4x + 3y = 12$		$6x = y - 7$
S.4	$x + y = 12$	**S.9**	$2 \square (x + 6) = 84$	**S.14**	$4 \square (3 - x) = 4$	**S.19**	$3x - 2 = y$
	$5x - 6 = y$		$x + y = 104$		$x + y = 13$		$5x = y + 8$
S.5	$x + 33 = 48$	**S.10**	$4 \square (x - 8) = 72$	**S.15**	$x - y = 1$		
	$x + y = 73$		$x + y = 17$		$x + y = 5$		

THE EMPIRICAL STUDY: OBSERVATIONS

Because of the space constrains, in this paper we describe two high level performance cases –Mn and Mt– and one average level –L. Through these cases we describe different ways of assigning meaning to new algebraic objects: those obtained through sense production with the comparison and substitution methods.

1. **The "trail and error" strategy**: It appears as a spontaneous solving strategy in all the cases analyzed, and it is linked to the spontaneous readings performed by the learners when they are faced for the first time to two-unknown equation systems. The following is the "strategy" used by one of the cases (L), which illustrates cognitive tendencies 2 and 3:

<table>
<tr><td colspan="2">SL.3. x + y = 9 2x + 3y = 23</td></tr>
<tr><td>L writes: 0 + 9
1 + 8
2 + 7
3 + 6
4 + 5
5 + 4 then

Crosses the last line and then points out line by line, starting at the first one. She stops at "4 + 5" and writes:
4 5
8
15
23

L: The numbers are 4 and 5.
Interviewer: 4 and 5?
L: Yes. First thing I did was to obtain the possible additions which gave 9 as a result and then, by "trail and error": 2 multiplied by 1 is 2 (points out "1 + 8") for checking if they lead to the correct amount.</td><td>Observations: L interprets these two equations as "linked", that is, as equations in which "x" and "y" have the same value in both equations. L writes the different ways for obtaining 9 through the addition of two positive integers and then performs the operations upon the unknowns indicated in the second equation until she finds those values for the unknowns, with which she obtains 23.</td></tr>
</table>

As we will see below these spontaneous readings and strategies can obstruct the learning of new general solution methods such as comparison and substitution (Cognitive tendencies: 5, 8, 9).

2. **Assigning Sense to Comparison and Substitution and the Meaning of the 'Equal' Sign. Difficulties.** There exist two obstacles which obstruct the application of the comparison method: (A) the reading of objects (unknowns and data) and of operations within the context of positive integers, and (B) a lack of the knowledge required to establish the new equality: the algebraic equivalence. (Tendencies 2, 5, 8, 9).

SMt.17. ***4x □3 = y 6x = y □7***	
Mt wants to obtain the value of '***y***'. So, she transforms the proposed system into: $$4x - 3 = y$$ $$6x + 7 = y$$ but instead of performing the equalization of the two expressions, she looks for the solution through the "trail and error" method using only the positive integers. Mt: In here (***4x - 3 = y***) says that four times '***x***' minus three equals '***y***'. And here (***6x + 7 = y***) says that six '***x***'… plus seven!, equals '***y***'. This (***6x + 7 = y***) has to be bigger than this (***4x - 3 = y***).	*Observations: Mt is able to solve one-unknown equations, regardless of numerical domains of the operated numbers nor of the solutions or the complexity of the equations' algebraic structure.* *Besides, she uses comparison in the case of equation systems derived from verbal problems in which both equations have the same unknown solved and the solutions are positive integers.*

The difficulties related to sense assigning in algebraic substitution are related to the readings of the unknown representation in terms of another unknown. Such readings are performed from different abstraction levels and are accompanied by the inhibition of the use of algebraic substitution. Besides, as expected from the analysis performed with the **Formal Competence Model**, we found that the meaning assigned to the 'equal' sign when used for the equalization of two operation chains which allow the calculation of the same value, is different from the meaning assigned to this sign when algebraic substitution is applied.

3. **Different Abstraction Levels.** The **Communication Model** allows to establish the difference between the readings by the interviewer and the pupil. When a competent user applies the algebraic substitution method he or she uses the equivalence between two expressions knowing the sense of the method, knowing that this method leads him or her to find an unique value for '***x***' and '***y***', if there is any. For example –in the system (S2) that was presented at the introduction, a competent user considers as equivalent the expressions "***4x – 3***" and "***y***", for him or her; these are representations —or names— of the same object: the unknown ***y***. But, when questioned, the learner's spontaneous readings focus on the operations chain performed for calculating the value of the unknown '***y***'. This type of difference in the meanings assigned to algebraic expressions is present in the whole algebra teaching process, in which the teacher is a competent user and the pupils are learners, generating difficulties such as the one described in here. (Tendencies 2 and 5).

4. **The Criss-Cross Method**. Mn invents a solution method which allows him to solve –in combination with the "trail and error" method– all the equation systems proposed. His solution method: ***the criss-cross***, entails the sum of the left member of one of the equations to the right member of the other; the simplification through term elimination and, when possible, and solves the simplified equation for one of the unknowns. This method keeps the equivalence between members as well as the value of the unknowns. Mn is able to perform additions between entire algebraic expressions. The following is an example of how Mn solves an equation system through the combination of *the criss-cross method* and his "trail and error" strategy:

SMn.7. 3x + 8y = 84 8x + 3y = 59
Mn: …There is "***8x + 3y + 84 = 3x + 8y + 59***", so I am mixing them both. If I add this (points towards the left member of the second equation) to this (points towards the right member of the first equation) it is going to be the same as if I added this (the left member of the first equation) to this (the right member of the second equation) because they are both equivalents. If I eliminate from here (***8x + 3y + 84 = 3x + 8y + 59***) "***3x***"… Well, and I also eliminate "***3y***" I get, by the way, that "***5x + 84 = 5y + 59***"… I get "***5x + 25 = 5y***" from which I deduce…Then, if I divide everything by ***5*** I obtain that "***y = x + 5***". From which I deduce that x is minor than y by ***5***… now I have to see, which can the equivalence be? Which is the value of ***x*** and which the value of ***y***? For example, if I say that ***x*** is –just for giving a value– ***4***… No, it can't be ***4*** because suddenly I don't… well if the value of x is four then the value of y is obviously ***9*** and then I get (in ***3x + 8y = 84***) that ***12*** (***3*4***) plus ***9*8*** it is going to be ***72***, ***12 + 72 = 84***, which is satisfactory. Now I will verify with the other (***8x + 3y = 59***), that says ***8x***, that is, ***32*** plus ***27*** (***3*9***) equals ***59***. I check and I think that they are both correct, so I deduce that ***x = 4*** and ***y = 9***.

Mn does not have problems with the numeric domains of the equations and systems that he has to solve. His elimination and simplification strategies are closely linked to the meaning of the 'equal' sign established in a equation. His "trail and error" strategy it is based upon his mental calculation abilities; on his mechanisms for anticipating the numeric values to be obtained; and on his coordination of the actions performed (cognitive tendency 11). In an extensive article we will analyze the potential of ***the criss-cross method*** and the possibility of using it in the regular school teaching processes.

5. **Inhibition of Substitution Usage.** The interview design is directed towards the students' usage of the substitution method, as it can be seen by the last items of the interview in which one of the unknowns is solved for them. However, even Mn –who has enough syntactical competence as to generate a new equalization method: *the criss-cross*– is far from using the substitution method. (Cognitive tendencies 5 and 8).

NEW PERSPECTIVES

This last observation would seem to avail teachers' belief that substitution method's syntax –more difficult to tackle than the comparison method's syntax– is the cause of the *inhibition* presented at observation 5. However, using an ad-hoc formal model for describing each of these methods' syntax, it can be seen that the dialectic between syntax and semantics is the main obstacle in the event of errors when following a rule for which it is necessary to use one or more rules previously and competently used. This is the field in which our present investigation is focusing.

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USING AN EMPOWERMENT PROFESSIONAL DEVELPOMENT MODEL TO SUPPORT BEGINNING PRIMARY MATHEMATICS TEACHERS

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This is a case study report from a larger study that focused on how an empowerment professional development model influenced the mathematics pedagogical practices and beliefs of Australian primary school teachers during their first year of teaching. The research used an interpretive approach for analysis of data from interviews, observations, reflective journals and group meetings. Initially, the challenges of classroom realities led the beginning teacher of this case study to discard the 'theoretical', 'modern', learner-focused ideas for mathematics teaching she had learned about in pre-service teacher education. Through the support of the empowerment professional development model she began to develop ways to navigate between theory and the realities of classroom practice.

PURPOSE OF THE STUDY

The directions that mathematics education has been endeavouring to follow in the last decade have been influenced by many factors. Curriculum guidelines developed to articulate and support new directions present mathematics teaching and learning as engagement in inquiring, meaning-oriented, challenging, purposeful and relevant activities (e.g. Australian Education Council, 1994; National Council of Teachers of Mathematics, 2000). This perspective is a departure from 'traditional' mathematics teaching in which teachers tell students how to think about concepts and how to perform particular procedures, with students then expected to practise skills and reproduce ideas. The two perspectives are based on different assumptions and beliefs about the nature of mathematics and effective mathematics teaching and learning. Hence, the reform efforts associated with current curriculum documents set ambitious goals for schools, teachers and students. They challenge 'traditional' views about mathematics and the appropriateness of related 'traditional' teaching and learning practices.

The new directions promoted for mathematics education therefore require changes in beliefs, values and practices on the part of teachers and students. These changes are not likely to happen without the support and guidance of appropriate professional development (Borko & Putnam, 1998). It was this concern that motivated the study upon which this report is based. Thus, within the context of mathematics education reform, the overall study (Sparrow, 2000) aimed to examine:

- the potential impact of an empowerment model of professional development, later named the 'fellow worker' model.

The research is of significance in reporting on beginning primary mathematics teachers' dilemmas and challenges, the related choices they make, how the foci or nature of their decisions change as they develop professionally, and how a 'fellow worker' can play a valuable role in this development and professional empowerment (also see Sparrow & Frid, 2002). In addition, the research highlights how beginning teachers struggle to make links between the theories of mathematics education espoused in their pre-service education program and the realities of teaching practice as encountered in their classrooms.

THEORETICAL FRAMEWORK

The nature of teacher professional development

Teacher professional development, as conceived of in this study, aligns with constructivist philosophy in that learning to teach is viewed as a process of personal growth that is influenced by personal beliefs, commitments, and ways of operating in and interpreting one's world (von Glasersfeld, 1995). Means by which this growth can be facilitated are not fully understood and articulated within the educational literature, although various models for professional development have been designed and implemented. These models can be loosely grouped into three categories according to how the notion of 'change' or 'growth' is incorporated into the model (Sparrow, 2000). The three groups –Transmission, Partnership, and Empowerment – although inter-related, vary in the degree to which an external expert acts as a director or decision maker. Since the intent of this study was to support teachers in ongoing and long-lasting professional development, design of the professional development model needed to be consistent with what the literature reported as 'effective' principles and activities for changing beliefs, knowledge, and teaching practices. Thus, design of the professional development model drew predominantly on the ideas and mechanisms of the Empowerment group.

Principles for an empowerment model of professional development

Key within an empowerment model of professional development is that teachers' professional learning needs should be driven by their concerns, interests, and the realities of their daily classroom and school experiences. They should have ownership of their professional development so that 'coming to know' as a professional is based on their own reasoning processes, and so that their own ideas and voices can be effectively integrated with those of others (Cooney, 1996). The ideas and related practices may come from many sources, however, what is key is that the teachers themselves mediate the ideas, construct meanings from them, and act according to their own values and decisions (Richardson, 1994). Teachers are then empowered to develop a professional identity, and be analytical and attentive to context. Further, they are empowered to recognise themselves as their own change agents.

The 'fellow worker' professional development model of this study was designed to support these ideas of empowerment. More specifically, it was designed to use reflection and experimentation to actively involve teachers and provide for personal choices in their actions (Clarke & Peter, 1993). The researcher became the 'fellow worker' – a more experienced colleague who could support the beginning teachers in the process of reflection, consideration of options related to teaching practices, and small-scale experimentation with options. This role meant the fellow worker was a hybrid of a critical friend, colleague, listener, mentor, resource person, and supportive, interested person. His role was also as a catalyst by which the beginning teachers could reason about and learn from their own teaching experiences. In this context it was recognised that, as Easen (1985) noted:

> You cannot change other people, nor can they change you; people can only change themselves. The best anyone can do is to provide a structure, which helps others to change, if that is what they want to do. (p. 71)

Consequently, the fellow worker had to be more concerned with choice than with change, and be wary of reinforcing his self-image of mathematics teaching irrespective of the needs and wishes of the beginning teachers. Key features of the structure that framed the fellow worker professional development model are outlined in Figure 1 along with some of the literature sources from which these principles were developed.

- The fellow worker will be an experienced teacher.
- Reflection on practice and beliefs about mathematics and mathematics teaching and learning will be emphasised.
- The beginning teacher will decide which issue will be the focus of attention, and what will happen in the classroom or elsewhere.
- Options for action will be decided between the fellow worker and beginning teacher, with experimentation in the classroom encouraged to provide data for reflection.
- Meetings with other beginning teachers will be established.

(Clarke, 1994; Clarke & Peter, 1993; Eason, 1985; Feiman-Nemser, 1992; Knowles, 1984; Prawat, 1991; Pultorak, 1996) (Adapted from Sparrow, 2000, p. 72)

Figure 1. Key features of the fellow worker professional development model

METHOD

Since its aim was to understand the nature of a professional development learning process, this study was designed as a naturalistic, interpretive inquiry. Hence, qualitative methods, with their capacity to emphasise contexts, meanings and individuals' interpretations, were adopted (Marshall & Rossman, 1995). More specifically, the research involved case studies of four beginning primary teachers with the researcher (author 1) in the role of the fellow worker of the professional development model.

The four beginning teachers were volunteers for the research and they all had recently completed a 1-year Graduate Diploma in Education in Western Australia after prior completion of a 3- or 4-year Bachelor's degree. Data were collected from interviews, teacher and researcher journals, group meetings, and classroom observations. Interviews occurred at least twice in each of the four 10-week school terms, while group meetings took place at the end of Terms 1, 2, and 3. Interview transcripts were the initial data analysed, with the other data sources used to substantiate and expand themes identified in the interview data. Hence, data analysis proceeded inductively, with NUD*IST as a data handling tool (Qualitative Solutions & Research, 1997). Initial nodes for use in NUD*IST were selected from factors identified from the literature as relevant influences upon beliefs and pedagogy.

Tiffany, the beginning teacher of this case study report, was 23 years of age when she began teaching in a combined grades 2 and 3 class of 34 students. The school, located in a country town about two hours drive from a major city, had an enrolment of approximately 180 students in kindergarten to grade 7. The student population included children from farming as well light industrial and professional families, and also included many Aboriginal children.

FINDINGS AND DISCUSSION

The following discussion focuses on the case of Tiffany as an exemplar of how use of the 'fellow worker' empowerment professional development model was a process that impacted upon teachers' practices and beliefs. Detailed discussion of the one case only provides a complete, coherent, and authentic account of one teacher's experiences without fragmenting the findings amongst more than one teacher. In this way, a picture of Tiffany's professional development pathway and its context are presented, with related commentary on how key components of the professional development model – reflection and experimentation – appeared to be influential in this pathway.

Initial reflections and related beliefs and teaching practices

Early in the year, Tiffany's reflections displayed conflicts between the ideas she had from her teacher education program and her underlying beliefs about children and mathematics learning. In particular, she knew about using concrete materials and activity-based teaching strategies for mathematics, and had tried to use them, but their effectiveness in practice was not evident. Instead, she had behaviour problems with the children that led her to adopt "the easier option of worksheet maths":

> … back to the first lesson I did, I think I used blocks and things like that, and that got pretty out of hand. That put me off for a while, so I went back to things that were easier to control.
>
> … For this reason I choose maths experiences that involve the children sitting and working on paper. To tell you the truth, it is just plain easier to teach maths in this class in this way.

What Tiffany witnessed in practice was that when you do "like we were taught, to try to make it interesting and use concrete things … [then] they don't seem to be learning". Her interpretations of her children's responses to her mathematics teaching were that students did not enjoy or learn from other than the "sit down and … just do the straight kind of work". This caused her confusion because it conflicted with the theories and teaching strategies she had learned about in her pre-service program:

> I've had a sit down where we've had lists of sums [written computations]. … They just loved doing it, and they'll do pages and pages of that. … And it was amazing, they just seemed to like it, and they've actually told me that's what they prefer to do. … It was weird because we are sort of taught, you know at college, that you do all these open-ended things. But it doesn't really work with my kids.

Hence, Tiffany found herself in a dilemma. She expressed a desire to use more hands-on, investigative, or game-based activities to "get a balance", but she was not convinced that this would support mathematics learning:

> Like it's good to do. I've tried a few number recognition games and stuff, but really I don't see any concrete evidence that they've learnt anything from it.

At this time, early in the year, it was not clear what Tiffany's beliefs were about what constitutes mathematics learning, particularly because she could not clearly articulate her ideas while she was experiencing much confusion. Her beliefs appeared to be related to notions of children sitting quietly, staying on task, and accurately completing worksheets and written calculations. She did not appear to have the mathematics knowledge or pedagogical knowledge to see what learning might be embedded in other types of learning activities. This, along with what appeared as a lack of pedagogical experience for managing children's behaviour when implementing alternative activities, led Tiffany to initially adopt a 'traditional', rote-learning, teacher-centred approach for her mathematics teaching practices. She appeared to believed that her children needed to learn by sitting down, being told what to do, and then remembering and practising procedures.

Ongoing reflections and experimentation

The role of the fellow worker became vital as Tiffany continued reflecting on her classroom experiences and what they might mean in relation to her teaching practices, children's learning, what she had learned at university, and what she wanted to try in her teaching. He was able to support her to explore some of her ideas and to then see related successes. Through explicitly asking Tiffany to reflect upon positive things that happened in her mathematics teaching, no matter how small, he was able to assist her to examine what in fact children were showing about their mathematics thinking and learning. At the same time, Tiffany was supported in taking small risks to try new things or re-try things that had not worked earlier.

Examples of how the fellow worker supported Tiffany to experiment with new teaching practices arose in relation to the use of concrete materials, calculators, games-based activities, open-ended tasks, and alternative assessment (i.e. techniques other than paper and pencil tests). The use of concrete materials will be briefly discussed here as an exemplar of how Tiffany was empowered via experimentation and reflection to develop insights into children's learning, as well as the skills and confidence to implement new teaching strategies.

Tiffany repeatedly expressed a view that mathematics learning should involve concrete materials and related activity-based tasks, although it was not clear if this commitment came from personal beliefs or from beliefs that the theory she had been taught at university must contain some truth. The fellow worker prompted her to talk about her experiences with concrete materials, encouraging her to reflect upon what had or had not happened and why this might be:

> Well if you want to try a bit more, let's try to pick the eyes out of it if you like, and see ways in which it might go forward, but little ways. (Fellow Worker)

Subsequently, once options had been examined with the fellow worker, Tiffany experimented, with varying degrees of success, with using concrete materials:

> I actually intend to do it again with some different material. I think they can do it. I mean, I said last time they couldn't do it, but sometimes they can.
>
> Because I thought, well I shouldn't give up on it. … It's a matter of changing the way you do it and adapting to what they can handle, and then I can sort of do it.
>
> They seem to have gotten on to it quite well [using concrete materials for fractions]. Like I thought, first, they're writing fractions and they're not going to have a clue. But the way we did it, I did the concrete things with the pattern blocks … a lot of them seemed to have understood the concept of a fraction.

Without the support of the fellow worker, one might speculate that Tiffany would have continued to see the use of concrete materials as too difficult to pursue, abandoning her efforts as she had done at the beginning of the year. Instead, the reflection and experimentation process, supported by the fellow worker, appeared to be a catalyst to both initiate and facilitate change:

> [The journal] … it's actually quite good because it makes me think about it as I'm writing. As I was writing one, I thought, well then you could have done something different that you wouldn't have had that problem with.
>
> I think you get stuck in a mindset. Like once you get into the practical, teaching that way [traditional method], it's hard to get out of it. You know, like you are just thinking all the one way. But if you start thinking, reflecting, then you can in fact probably think of a million things to do.

Tiffany's words in these interview excerpts display her increasing awareness and appreciation of the value of reflection. She simultaneously showed development of her capacities for both breadth and depth in reflection, demonstrating an increased capacity to analyse her own pedagogical practices and the nature of children's mathematics learning and related learning needs:

> Sitting them down by themselves, not working in groups, basically they all did the same thing. Which wasn't really catering for their different abilities. But they all did the same thing, and the ones who could do it probably finished early, and I didn't really cater for extending them.
>
> They just need more realistic practice. I don't know exactly what I mean by that. I guess not just a subject on a page is what I was meaning. They need experience at counting real things, in the room and outside. And like, with those games, that was sort of a real thing, and they need more practice at that sort of stuff … to apply the principles from there to the real world.

As the year progressed, Tiffany started to develop options and solutions herself, relying less on the fellow worker to offer possibilities. At all times, even initially when the options were largely generated by the fellow worker, possible solutions were considered in the context of Tiffany's own classroom experiences, teaching capacities and beliefs. That is, both reflection and experimentation were grounded in the specifics and realities of practice rather than the generalities of theory. In this way Tiffany was able to develop personal meanings for and ways of operating with the theories of mathematics teaching and learning that she had learned about at university.

Changes in practices and beliefs

While Tiffany was supported by the fellow worker in navigating between theory and practice, she shifted in her teaching practices as well as her beliefs about teaching and learning. Specifically, her practices developed to include more use of concrete materials, games-based activities, open-ended tasks, calculators, and observation and portfolio assessment. When non-traditional teaching strategies were implemented, though sometimes with only a limited degree of success, Tiffany began to trust the worth of alternative methods that she had previously viewed as the theory from university that "doesn't really work with my kids". Thus, positive experiences in experimenting with putting theory into practice, in the specific context of her own classroom, began to affect her beliefs about mathematics teaching and learning. She explicitly noted late in the year that she now recognised she had been struggling with a belief that mathematics teaching is "telling":

> You know what I think it is? I think it was because I had in my mind what maths should be. I'm standing there and they are all sitting there. … so when they are actually learning by themselves, I think that's just a time filler. It's not a maths lesson. But really it is. But I had the view that I should be up there teaching all the time.

At the end of the year Tiffany was thinking much more of the mathematics teacher as a coach, rather than a director with a pre-determined script. She now believed that children needed guidance to obtain their best performance, and she still appeared to believe there was a performance to be refined, and thus, a correct way to do mathematics. However, she had begun to recognize value in allowing learners to make some use of their own thinking:

> Well I think I should be a coach, sort of like, because they need the guidance, and then they need, they can't totally be left alone to find things out. They need some sort of encouragement. … I don't think you should be telling them all of the time. Get them on the field and then they do their own thing.

CONCLUSIONS AND IMPLICATIONS

Tiffany's journal notes as well as comments in meetings with the fellow worker indicated she found herself in an ongoing struggle with what she felt should be done in mathematics, what she tried to do, and what the children preferred or were in fact able to do successfully. Her pre-service teacher education program had presented a picture of how one might teach, but early attempts to implement this style did not work. However, she was able to make changes, from a traditionally-oriented teacher to one who was experimenting with a variety of teaching and assessment styles. Whether this development can be attributed solely to the empowerment professional development model is debatable. However, there is evidence to suggest that is was influential in that with the assistance of the fellow worker instances of 'good' practice were supported and further developed.

These findings suggest three main things concerning mathematics teacher education and related research:

- More mentoring or other forms of support are needed for beginning teachers as they deal with the many challenges they face in their classroom management, curriculum planning, teaching and assessment practices, and professional beliefs and identities.
- The impact of pre-service teacher education programs needs to be examined, particularly with regard to how to bridge the gaps between what is promoted in mathematics education studies and what are the realities of schools and classrooms.
- In pre-service education, teachers need to further develop their skills and confidence as reflective practitioners who can examine the inherent complexities and conflicts of teaching alongside a wide array of potential resolutions and how they might be put into practice.

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GETTING AT THE MATHEMATICS: SARA'S JOURNAL

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In this paper, we discuss issues in planning and conducting research into mathematics learning. We emphasize two central themes: (a) the learners' mathematics (especially the issues and ideas, in given problem situations, that learners choose to think about and to present, and (b) the kinds of knowledge that learners may be building (including their ideas about what mathematics is, and how people do, learn, use, communicate and understand it). While the first theme is at least partly mathematical, the second interweaves cognition and epistemology. Anchored in student data from an extended classroom teaching experiment in the mathematics of change, we focus on key choices needed to build narratives to help researchers capture, in detail, how students understand their own engagement with their mathematics.

According to Brousseau (1997), for example, learning takes place (or perhaps fails to take place) in the course of student-teacher interactions. These are seen as governed by (frequently tacit) understandings (the *didactic contract*) about what is to be learned, how learning can be demonstrated, and about the respective responsibilities of learner and teacher. To understand an act of learning, in this view, one needs to understand not only what was said and done, but also the nature of the underlying contract. And to change the way learning takes place—which is critical to implement needed reforms—the contract often must be renegotiated. Such negotiations, as we know, can hinge on central questions about mathematics: how we do it, how we learn it, what it is about, how we talk, display and write about it, and what constitutes proficiency. In this paper we will emphasize the *mathematics*, most especially to suggest possible ways, as researchers, to think about and work with students' presentations of their mathematics. We will focus here on one student's journal, and on how to build what we will call *composite narratives*, based on that journal and a careful choice of additional supporting data.

The present paper introduces several key ideas that will guide a group of papers now in progress, each one devoted to the work and thinking of one of six undergraduates (all majors in the arts) over three months, who participated in an extended classroom teaching experiment on representing and reasoning about change (Designing Calculus, BYU, 2002), led by the first author.[1]

Sara is a senior, concentrating in ballet. She began by working on a task designed earlier by diSessa and his coworkers, called the *desert motion* (diSessa & Sherin, 2000; Sherin, 2000). In Figure 1 we reproduce top third of first page of her journal, which begins with her own statement of the problem.

[1] In addition to both authors, the research team includes Marin Turley, Tiffini Christensen and Joe Curtis.

Math
9/10

#1 - Desert Motion

A woman was driving on a long road in a desert. She sees a cactus with a water faucet poking out. [When she realizes this] she slams on the breaks, backs up, stops at the cactus. Then she jumps out of the car & gets a glass of H^2O, jumps back into the car & take off down the same road (same way).

- Analize this problem -

Figure 1. Sara's Journal: first page, top, September 10, 2002.

Sara's journal offers vivid presentations of her thinking. Later, she and two other students, by inventing a series of graphic, physical and kinesthetic representations, would discover ways to make connections between distance, speed and time, so that speed as rate of change, and distance as area under a graph of speed, became transparent to them. Figure 2, the rest of Sara's initial page, describes how she *began* to think about the crux issue of the desert motion problem: *how to represent the motion of the car*.

At first I was concerned with solving this problem - I never really understand problems like (~~pace~~ rate x time = distance). It was hard for me to think of the problem in terms of mathmatical concepts. But when I thought about it in terms of movement, and the sequence of events - movement & change, it became easier. We started by drawing pictures.

Figure 2. Sara's journal: first thoughts, September 10, 2002.

To provide a starting point for analysis—and to illustrate the way we work with student presentations—we will trace how each such presentation (here a sequence of pictures, Figure 3) often serves two fairly distinct purposes (Dörfler, 2000): to present part of a learner's thinking to *herself*, and to provide a tool for communication to *others*. The journal entry shown in Figure 3 began as a collaboration, on the classroom whiteboard, with another student (Krista), as they modified initial sketches to be able to encode more information. This encoding was unpacked in class, triggered by other students' questions, before Sara drew her final version in her journal, based in part upon these interactions.

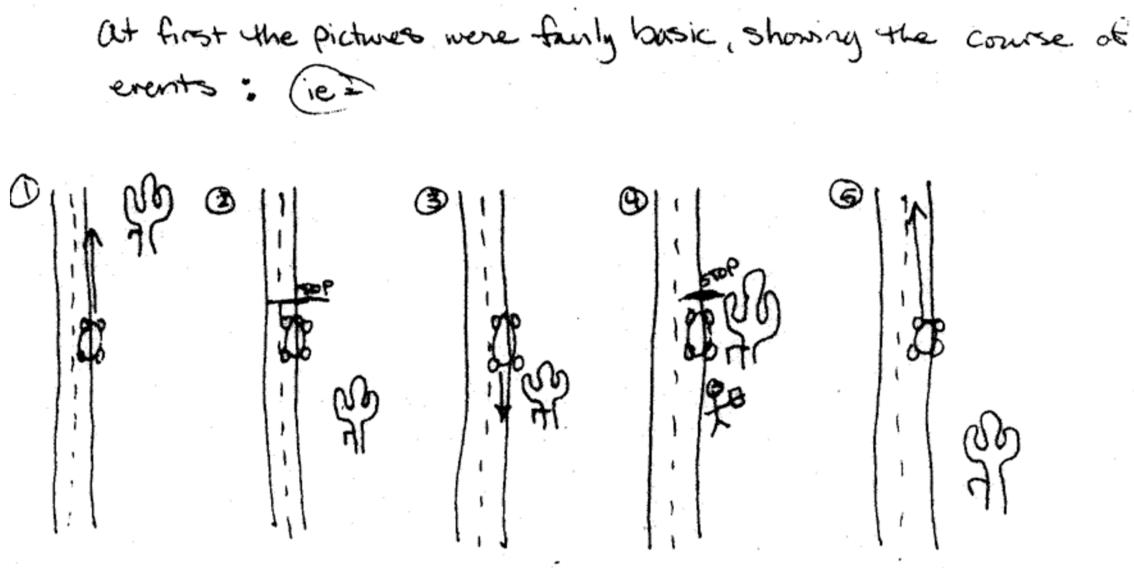

Figure 3. Sara's journal: first pictures, September 10, 2002.

Our theoretical perspective begins with mathematical activities (such as the desert motion) that are rich enough to elicit the construction of mathematically important processes and understandings. Here, as Sherin and diSessa found (diSessa & Sherin, 2000; Sherin, 2000) with sixth-grade students, the desert motion task quickly led to cycles of construction, critique, and reconstruction, leading to increasingly detailed representations of the car's motion. As side effects, representations were proposed, modified, discarded and reshaped, culminating in the collective reinvention of what we would call the standard graphs, using (a) slopes to present rates of change and (b) areas defined by graphs of rates to present total changes. Following Dörfler (2000), we see such activities as *prototypes* (more precisely, *mathematical* prototypes) for the construction of particular mathematical meanings, in this instance local rates of change and global changes.

This emphasis on prototypes has implications for our research methodology. Because we focus on the building and the use of prototypes in the development of mathematical understanding, we need to follow student presentations over time, and trace the variety of ways in which the students work with them, both personally and in collective discourse. It is therefore natural to build narratives, in which students' thinking can be traced through close analysis of presentations where they demonstrate the reasons for their actions.

To build such narratives, we first locate critical events (Maher & Martino, 1996), then trace the student presentations made in them to build a timeline that will anchor a *composite narrative* (Speiser, Walter & Maher, 2003). In Sara's case, the timeline centers on selected entries from her journal. This initial timeline is next supplemented with selected videotaped segments of activities to which the journal (perhaps implicitly) refers, to produce a *basic narrative*. This basic narrative is annotated, somewhat in the style of open coding (Strauss, 1987), to identify emerging themes in student work and discourse, and to initiate more theoretical descriptions.

Based on Sara's journal, the desert motion and two later tasks provided prototypes, not just for mathematical constructions, but also for *new ways of thinking*. In November, for example, two months into the teaching experiment, she wrote:

> —How mathematics has changed my way of thinking…[2] In dance especially I feel like I think about movement in a more clear, analytical way. I think that when I am given a phrase of movement that travels across / through the space to music, knowing that RxT = D really makes a difference in the way I approach the movement. I know that if I need to travel a longer distance in the same amount of time, the Rate of Speed of my movement needs to proportionally increase. This has been a real revelation for me!

We shall call an activity an *epistemological prototype* when it anchors a *new way of thinking* (as arguments about block towers were for children in Maher's Kenilworth study, serving first as prototypes for particular proof strategies, but then as prototypes for the *idea* of proof).

As researchers, bearing in mind the different purposes that prototypes can serve, we approach the *explicit* mathematics that Sara discusses in her journal with a view toward building up descriptions of her learning process, based primarily on how she wrote about it. For Sara, the formula connecting distance, speed and time, referred to as problematic in her first journal entry (Figure 2) evolved into a major unifying theme throughout the course. Hence our descriptions of her mathematics would need to build from what she writes and does to understand *this* formula.

We return now to the larger research context. Sara's narrative is one of six. As our developing analysis has shown (to be explained in later papers), each student's narrative reflects its own unique trajectory, in which personal reflections interweave with group collaboration in different but closely related ways. As in (Speiser & Walter, 1997, 2000; Speiser, Walter & Maher, 2003), we could trace the classroom action as a single conversation, a collage of many voices leading to collective opportunities for understanding. But to gain more detail about the mathematics being built, and most especially how learners think *about* it, we will need to work in detail with the variety of students' personal reflections, thinking of journals and other narrative material as inscriptions which can serve the dual functions (a) of clarifying issues for oneself and (b) making such reflection vivid for an analytic reader. In this way prototypes arrive not just with features but as parts of learners' lives and histories.

In Sara's case, detailed analysis (to be reported later) of the composite narrative based on the journal entry quoted above, including its supporting video, indicates that graphs emerged for her as tools for anchoring new ways of thinking about *her own motion*, with the desert motion task as prototype. Thus the desert motion task (as carried through by our six learners) has functioned both as a mathematical and an epistemological prototype, in significantly different ways for different students. To the extent that it may also anchor an emerging understanding of the learning process, we might call it a *pedagogical*

[2] Her ellipsis dots.

prototype. Our understanding of our students' mathematics, in particular, emerges from the students' presentations (such as Sara's journal and the classroom discourse that surrounds it in our narrative). Through these, we can learn about the mathematics that the students worked with, and about the kinds of knowledge that they built.

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EMERGENCE OF MATHEMATICAL KNOWLEDGE STRUCTURES. INTROSPECTION[1]

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In the study, a part of longitudinal research focused on the emergence of mathematical knowledge structures in a learner's mind is presented. It concentrates on the analysis of introspective data gained from the author's study of a non-standard arithmetic structure in terms of the model of abstraction in context (Hershkowitz, Schwarz and Dreyfus). Some episodes are described by factual and interpretative accounts. It is shown that the above model can be applied to introspective data and used for their interpretation. The parallel between the model and Duffin and Simpson's idea of understanding is discussed.

This study is a part of longitudinal research aimed at describing the emergence of mathematical knowledge structures in a learner's mind. It is a very broad field of research and we concentrate on the construction of a new structure as an analogy to an existing structure. The process of constructing an internal mathematical structure is a mental activity, i.e. it is not directly observable. The methodology we used consists of think-aloud interviews with university students and the author's introspection. Introspection has been chosen because we believe that by studying ourselves from the inside we can infer about mental processes of other people, we "develop sensitivity" (Mason, 1998). "By introspection we mean constantly seeking to discern our individual perceptions of experiences, both past and present, and our reactions to them" (Duffin & Simpson, 2000a). Some of the research results have been reported elsewhere (Stehlikova & Jirotkova, 2002; Stehlikova, 2002). Here we will concentrate on the introspective part.

THEORETICAL FRAMEWORK

One of the central aspects of learning mathematics are the processes of the emergence of mathematical knowledge structures. Several theories are available which have a common goal: "They aim to provide a means for the description of processes during which new mathematical knowledge structures emerge" (Dreyfus & Gray, 2002). For our analysis, we have chosen the model of abstraction in context.

Hershkowitz, Schwarz and Dreyfus (Dreyfus, Hershkowitz & Schwarz, 2001a; Hershkowitz, Schwarz & Dreyfus, 2001; Schwarz, Hershkowitz & Dreyfus, 2002) have recently proposed a model of dynamically nested epistemic actions for processes of abstraction in context which has since been elaborated (e.g. Dreyfus, Hershkowitz & Schwarz, 2001b; Tabach, Hershkowitz & Schwarz, 2001; Tabach & Hershkowitz, 2002; Tsamir & Dreyfus, 2002). The proposers of the theory characterise abstraction as "an activity of vertically reorganising previously constructed mathematics into a new mathematical structure". By reorganising into a new structure, they mean the establishment of mathematical connections (making a new hypothesis, inventing or reinventing a mathematical generalisation, a proof, or a new strategy for solving a

[1] The paper was supported by grant GACR 406/02/0829.

problem). On the other hand, neither learning to mechanically perform a mathematical algorithm nor rote learning qualify as abstractions.

The authors of the theory also claim that abstraction strongly depends on context, on the history of the learner and on artefacts available to them and in this sense structure is internal, "personalised" (Schwarz, Hershkowitz & Dreyfus, 2002). Thus hereinafter by a structure, we will mean a mental image which a person holds in his/her mind about a mathematical structure.

The authors of the theory call mathematical methods, strategies, concepts, etc. structures. We would have preferred to reserve the word 'structure' for more complex knowledge and simply call what is being built 'mathematical knowledge'. Similarly, the term abstraction has a more specific meaning in mathematics for us, thus instead of 'processes of abstraction', the term 'processes of construction of knowledge' seems to us to be more appropriate.

The genesis of abstraction is seen as consisting of three stages (Hershkowitz, Schwarz & Dreyfus, 2001):

1. A need for a new structure,
2. The constructing of a new abstract entity in which recognizing and building-with already existing structures are nested dialectically, and
3. The Consolidation of the abstract entity facilitating one's recognizing it with increased ease and building-with it in further activities.

Three epistemic actions which are constituent of abstraction are (Schwarz, Herhskowitz & Dreyfus, 2002):

> *Constructing* is the central action of abstraction. It consists of assembling knowledge artefacts to produce a new knowledge structure to which the participants become acquainted. *Recognizing* a familiar mathematical structure occurs when a student realizes that the structure is inherent in a given mathematical situation. *Building-With* consists of combining existing artefacts in order to satisfy a goal such as solving a problem or justifying a statement.

STUDY

The tool of our investigation of an internal mathematical structure is an arithmetic structure $A_2 = (\underline{A}_2, \oplus, \odot)$ which we call *restricted arithmetic* (hereinafter RA) [2] where $\underline{A}_2 = \{1,2,3,...,99\}$ is the set of z-numbers. The gate to RA is the mapping $r: \underline{N} \to \underline{N}$, $n \mapsto n - 99 \cdot [n/99]$, which we call *reducing mapping*; here $[y]$ is the integer part of $y \in \underline{R}$. Reduction can also be introduced as an instruction illustrated by several concrete examples[3]: *Perform a 'double-digit sum' operation on a natural number until you get a one or two digit number. A double-digit sum operation is similar to a digit sum operation but instead of adding digits, we add two digits at a time*. For example, $r(682) = 82 + 6 = 88$, $r(7\,945) = r(45 + 79) = r(124) = 24 + 1 = 25$.

[2] It was elaborated by Milan Hejny.

[3] The isolated models of reduction, e.g. the numerical examples, are introduced to the student at the same time as its verbalised universal model (for the theory of isolated and universal model see Hejny, in press).

The reducing mapping r is used to introduce binary operations of z-addition $\oplus$ and z-multiplication in $\underline{A}_2$ as follows: $x, y \in \underline{A}_2,\ x \oplus y = r(x + y)$ and $x \odot y = r(x \cdot y)$. For instance, $72 \oplus 95 = 68$, $72 \odot 95 = r(6\,840) = 9$.

Note: This context has been chosen as a tool of our research and not a different part of mathematics because it presents a fresh part of mathematics, not elaborated elsewhere, it is suitable with respect to the author's mathematical knowledge and abilities and the analogy with ordinary arithmetic allows her to pose questions and develop solving methods herself.

The only **participant** of this part of our research is the author, a 30-year-old researcher. The **data** available for analysis consists of the author's detailed notes of her solutions to problems. The notes are dated and besides the solutions themselves also contain her comments on them as they occurred to her at the time of writing. In addition, the author used, if possible, different pens at different times for writing her notes. When the problems were considered to be solved, a descriptive table was made which consisted of: the task, its solution, its interpretation by the author. The table together with all the notes was subject to analysis.

The **problems** solved by the author which we will describe here consist of the study of squares, of general powers in RA and of looking for multiplicative groups in RA. The study spanned four months and used some results which the author found out previously. The author's investigations will be divided into several episodes which will be described by factual accounts (written in first person singular in italics) which are shorter versions of our comments in the descriptive table and interpretative accounts. The interpretation will be mainly done in terms of abstraction in context.

FACTUAL AND INTERPRETATIVE ACCOUNTS OF INVESTIGATIONS

Study of Squares

x^2	x
1	1, 10, 98, 89
4	2, 20, 79, 97
9	3, 30, 36, 63, 69, 96
16	4, 59, 95, 40

The study of squares was motivated by my effort to get an insight into the solutions of quadratic equations for which I needed to know which z-numbers were squares. Using the notion of squares from ordinary arithmetic and the fact that additive inverses have the same square, I made a list of squares (a part of it is in fig. 1). I noticed anomalies and regularities (e.g. $45^2 = 45$, $55^2 = 55$, $22^2 = 88$ and $88^2 = 22$; $(ab)^2 = (ba)^2$, $a^2 = a0^2$, where a, b are non-zero digits) and gradually 'chains' and 'cycles' emerged. For instance, $4^2 = 16$, $16^2 = 58$, $58^2 = 97$, $97^2 = 4$, etc. The list was not illustrative enough and I tried to represent it in a graphical form. I decided to use arrows for the relationship 'being a square of' and redrew the list several times. Finally, I got a visual diagram consisting of nine clusters (one of them is in fig. 2). I had a strong sense of satisfaction because the diagram was pleasing to the eye and it consisted of clusters of certain shapes and I felt confident that I would be able to get new information from it.

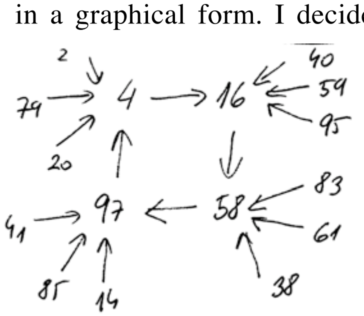

Figure 2.

Next, I felt prepared to prove some of the found regularities. While doing so, my attention was attracted by the fact that the numbers in the diagram made various sets. It seemed an obvious choice to check them for group

> properties with respect to z-multiplication. I checked the following sets: numbers from the inner 'rectangle' (e.g. numbers 4, 16, 97, 58 from fig. 2); all numbers from the cluster; numbers which have the same square.
>
> While checking the last sets, I became interested in the questions how the numbers with the same square are connected. It was very easy for NN numbers[4]. To solve the problems for zero-divisors was more difficult but I managed to find a rule which worked.

What has been constructed here, is the structure (in the author's head) of squares, their mutual relationship and of regularities (we will call it S1) which has quite a simple visual representation (its part is in fig. 2). The need for the new structure was given by our study of quadratic equations. Using the notion of squares from ordinary arithmetic and the fact that additive inverses have the same square (recognising previously constructed knowledge and building-with it something new), a list of squares was constructed. By studying the list and noticing anomalies and regularities, 'chains' and 'cycles' were constructed (recognising & building-with the knowledge of the relationship 'being square of'). By further recognising and building with the relationship 'being square', with observed regularities and with the idea of using nodes and arrows for the visual representation, the visual diagram was constructed in a rather raw form and by several redrawings, the diagram consisting of nine clusters was constructed and S1 was consolidated[5]. We think that the structure S1 was also consolidated when the author proved regularities because she had to reflect on the properties of squares. The suggestive clusters of the diagram led naturally to distinguishing some subsets and investigating them for group properties (recognising & building with sets of numbers which seemed to be mutually connected and with the knowledge of group structure).

The last part of the study was motivated by the author's natural curiosity leading her to the question if there was a simple rule connecting all the numbers in a cluster. It contributes to the understanding of the structure S1 and enriches it. This raises the question of what the construction is. Shall we say that a structure has been constructed if after some time we will find out that we have missed some of its important characteristics? Or shall we say that the structure is being constructed until all the characteristics have been found? It would then require an external authority which would judge that the learner has discovered everything about the structure and it has thus been constructed. As we deal with introspection, it seems natural to speak about construction when the solver feels that he/she constructed something which he/she did not know previously and that he/she understands it in terms of the definition of understanding given by Duffin and Simpson[6]: "When I understand, I feel comfortable ; I feel confident; I feel able to forget the detail, confident that I can reconstruct it whenever I need it; I feel that the thing belongs to me; I can explain it to others" (Duffin & Simpson, 2000b). The

[4] By NN numbers we will mean non-zero *z*-numbers which are not zero divisors.

[5] An indication that it was consolidated is that when the author had to draw the diagram again after a long time, she was able to reconstruct it without having to go through the same process again.

[6] As we only deal here with introspection, we will not consider the outer demonstrations of understanding the authors provide.

process of construction is thus an iterative process and it belongs to a person, is 'personalised'.

Study of Sequences

k	*a*
30	2, 50, 68, 83, 5, 20...
15	4, 25, 16, 31...
10	8, 62, 17, 35,...

> The discovery of some groups in the study of squares motivated me to go back to my study of sequences {a^k, a $\underline{A}_2$, k N}. I already knew that structures ({a^k, a□ NN, k N},) are groups. However, I was not satisfied with this result and I wanted to explore the structures more and to see what the situation with zero divisors was. I made a table for NN (its part is in fig. 3): k□s the number for which a^k = 1 (I knew from Euler's theorem that k□must be a divisor of 60). It was not difficult to fill in the table because I used the knowledge that (a) inverse numbers generate the same set (e.g. 2 and 50) and (b) if I know that e.g. 2^{30}=1, then 4^{15}=1 and similarly 8^{10}=1. A similar table was made for zero divisors where also the length of the pre-period (if any) was given. I investigated sets from the table for group properties and discovered some groups. I made an important 'discovery' of a different multiplicative neutral element other than 1 – number 45.

What is being constructed (or rather what has started to be constructed) here is <u>the structure of multiplicative subgroups</u> (we will call it S2). The need for it stemmed from our feeling that there had to be an inner organisation of subgroups which had been so far found in a rather haphazard way. Within it another construction was made – the construction of <u>the structure of sequences</u> (we will call it S3). The structure S3 was constructed via recognising and building with some knowledge taken from ordinary arithmetic and adapted for RA (see (a) and (b) above).

As for the neutral element, it was not a discovery as such. During her university studies, the author met many examples of different neutral elements other than those found in ordinary arithmetic. However, the time showed that this knowledge was not immediately available when needed (as the knowledge of e.g. group properties was). Some previous knowledge gained at university was consolidated through a rediscovery in a different context.

Study of Third Powers

> I wanted to find more subgroups and realised that if I studied third powers similarly to squares, I might find some new ones. I made a list and a diagram of third powers. I was surprised to see that unlike squares, third powers 'repeated' ($x^3 = (x + 33)^3 = (x + 66)^3$), and that they occurred 'in threes' (8, 9, 10; 17, 18, 19, etc.). From the visual diagram I was able to discover a set {1, 10, 89, 98} (NN for which $a = a^3$) which together with z-multiplication formed group. By further investigating sets of numbers with the same third power, two more groups were found of order 3.

The process of the construction of S2 continued, new groups were identified. The structure of squares was further consolidated as it was used as a basis for the construction of the structure of third powers

Further Study towards the Construction of S2

The lack of space does not allow for the detailed description of the rest of the process. Let us only say that other subsets of *z*-numbers were investigated for group properties and subgroups were organised in a table according to their order.

> I felt the need to summarise all the found subgroups to see if the list was complete. I noticed the relationship between the order of subgroup and the order of group and I remembered Lagrange's theorem. I listed all possible orders of subgroups and filled in a table. Two types of subgroups were missing – of order 12 and 20. I went over all my notes trying to find a clue even though I did not know what I am really looking for. Suddenly I saw that the set $\{4^k$, k N) equals the set of all squares which are NN. It occurred to me that something similar could be true for third powers. When I checked the set of all third powers which were NN, I got a multiplicative group of order 20. Then I investigated the fourth powers but it did not bring anything new. (I started to use Maple at this stage.) When I investigated the fifth powers, I discovered the last subgroup – of order 12. I really felt a sense of accomplishment.

The solver felt that the construction of S2 had finished (even though later, she went on with the study and developed S2 further). This time the author built with the theory (Lagrange's theorem) and with all the knowledge she had constructed so far in her investigations. An important part of the whole process was the chance recognition of the equality of the two sets. From then on, she could build with the knowledge that if we investigate the third, fourth, fifth, etc. powers, other subgroups might be found. Lagrange's Theorem was re-constructed similarly to the neutral element above.

The structure of subgroups was later consolidated when the author had to present it to others and describe it verbally. Moreover, this consolidation also went on when the author carried out the presented analysis of her own work! She had to reflect on S2 even more deeply than when investigating it earlier.

DISCUSSIONS AND CONCLUSIONS

Visual representations: The structures S1, S2 and to a certain extent S3 too are specific in that there is a visual representation available for them (the visual scheme and tables) and thus it is possible to analyse them more easily than structures with no visual representation.

Introspection: We accept that introspection as a research method is rather controversial. In agreement with Duffin and Simpson (2000a), we take into account that introspection should be complemented by other techniques in order for us to get more creditable results, and thus we made an attempt to accompany it by co-spection (Duffin and Simpson characterise it as "sharing of our own personal reactions to experience"). In our case, it is the sharing between the author and a colleague of hers to whom she presented her account of her work and who tried to find flaws and inconsistencies in it.

One of the dangers when using introspection is that the researcher may reinterpret (albeit unconsciously) his/her former reasoning in the light of what he/she knows at the time of analysis. On the other hand, when analysing other people's solutions, we naturally use our own experiences and interpret them accordingly. We believe, that no two researchers will analyse a solution in the same way. Thus, the findings from the introspection will be complemented by results from interviews with university students.

Abstraction in context: The model of abstraction in context was used for different kinds of data than previously. As far as we know, the model has been used for (a) an interview with a single student, (b) an interview with a pair of students, (c) a series of interviews with a single student, (d) a series of interviews with a pair of students. Here we applied the model to introspective data. Moreover, we showed that the consolidation of new knowledge can also be done when one is analysing one's own work, not merely reflecting on it. An illustration was given as to the consolidation of the knowledge gained some time ago. The question was raised if the consolidation can come about when proving.

The model of abstraction proposed by the theory of abstraction in context seems to be able to account for the part of data of our research on structuring mathematical knowledge presented above. It remains to be seen how the theory of abstraction in context can be used for other data from our research and for results which have already been found in terms of the grounded theory approach, procept theory or the theory of isolated and universal models. Besides, our study brought to light some problems with terminology which we had when using abstraction in context.

Understanding: When analysing the data, we could see a parallel between the processes of abstraction and the processes of building understanding. If we have constructed something, we understand it, understanding is in the very heart of constructing. Duffin and Simpson (2000b) define understanding as "an ongoing process of the development of connections (building), a state of the available connections at a given time (having), and the act of using the connections in response to a problem (enacting)". They characterise the act of knowledge construction as follows: "Indeed, in solving a substantial problem, a learner may use some recalled facts, enact some of their understanding, get stuck, find, and resolve conflicts by building new connections, enact the understanding inherent in those new connections, bring in more recalled ideas, and so on." This characterisation resembles the characterisation of the knowledge construction given by the model of abstraction in context. The connection between these two theories will be further studied.

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GRADE-RELATED TRENDS IN THE PREVALENCE AND PERSISTENCE OF DECIMAL MISCONCEPTIONS

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Over a period of about 3 years, 3204 students in Grades 4 to 10 completed 9862 tests to identify and track their interpretation of decimal notation. Analysis of the longitudinal data demonstrates that different misconceptions persist among students to different degrees and in different patterns across the grades. Estimating the prevalence of misconceptions is complex due to the nature of longitudinal data. Best estimates are provided of grade prevalences and the proportions of students affected during schooling.

Many students have difficulty understanding decimal notation. The reasons for this lie both in the nature of the mathematical and psychological aspects of the task and in the teaching they receive. Understanding decimal notation is a complex challenge, which requires the co-ordination of many ideas and draws on previous learning and fundamental metaphors of number and direction both to advantage and disadvantage (Stacey, Helme & Steinle, 2001). As a consequence, there are a wide variety of erroneous ways in which students interpret decimal numbers, often referred to as decimal *misconceptions*. This paper reports results from a study that has both cross-sectional and longitudinal components, so that the prevalence of different misconceptions can be determined and the paths that students take between these misconceptions can be traced over some years. This quantitative work is set in a context where we have also examined more closely particular misconceptions and have provided some explanations in terms of how the students may be thinking about decimal notation, drawing on interview and written data from school students and teacher education students (Steinle & Stacey, 2001; Stacey, Helme & Steinle, 2001) and examined the effectiveness of targeted teaching, although these aspects are not discussed in this paper.

This paper reports two aspects of the longitudinal data. Firstly, we report the prevalence of certain groups of misconceptions by grade level and show that there are interesting variations amongst the patterns in how these misconceptions persist amongst students of different ages. We also discuss how an estimate of the lifetime prevalence of these misconceptions might be obtained from the longitudinal data. In previous papers, we have made preliminary reports on the prevalence of expertise and misconceptions, as well as the paths that students commonly follow on the way to expertise (Steinle & Stacey, 1998, Stacey & Steinle, 1999a and 1999b). The present paper extends these analyses by using a larger dataset collected over a longer timeframe and by using more sophisticated analyses. We intend that the results will be useful to researchers working on students' understanding of decimals and to those interested more generally in the aetiology of students' ideas. We also intend that the discussion of the technical difficulties of analysis of longitudinal data will be of broad interest.

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DATA COLLECTION

Sample and Procedures

The Decimal Comparison Test (described below) is used to classify students' thinking about decimal notation. The test, which takes less than 10 minutes, was administered by volunteer class teachers to intact classes of students in 13 volunteer schools in 6 suburban regions of a large Australian city. Researchers analysed the tests, allocating a code that indicates the students' thinking, and returned all results to the classroom teacher with explanations. By including both secondary schools and their "feeder" primary schools, not only could students be tracked from grade to grade within a school, but also across the primary-secondary transition (Grade 6 to Grade 7). The results of individual students were traced throughout the study and the longitudinal results were analysed with the computer program "STATA".

In total, there were 3204 students in Grades 4 to 10, who completed 9862 tests between 1995 and 1999. Schools were requested to test students at six monthly intervals but not all teachers volunteered and others delayed testing for various reasons. These procedures have important effects on the data which need to be managed: there are a large number of students who are tested only once (or whose later tests could not be confidently identified), many students have broken sequences of tests due to their own absence from school on the day of the testing or their class not having been tested in a given semester. Overall, however, a very rich data set was collected: the maximum number of tests completed by a student was 7 (49 students), the average number of tests per student was 3.1, and the average time between tests was 8 months, offering an unprecedented opportunity to see the development of students' ideas on one topic over time. The regions represented all socio-economic groups, but the sample is only representative and voluntary, not randomly selected.

Decimal Misconceptions and their Diagnosis

It is well established that the task of choosing the larger of a pair of decimal numbers (or ordering a larger set) is very useful for revealing how a student interprets decimal notation. Some students consistently choose the longer of the pair (e.g. they will say 4.63 is a larger number than 4.8) and others will choose the shorter (e.g. they will say 5.62 is a larger number than 5.736). We label these behaviours as Longer-is-larger (L) and Shorter-is-larger (S). There are many different patterns of thinking which lead to these behaviours, which have been explored by Fuglestad (1998), Resnick et al (1989), Sackur-Grisvard and Leonard (1985), Stacey and Steinle (1998), Swan (1983) and others. Consequently, although L and S are sometimes referred to in the literature as misconceptions, they are actually both behaviours arising from clusters of misconceptions, which space precludes us from discussing here.

The behaviours L and S were identified on the basis of the pattern of responses to the Decimal Comparison Test, which has been based on similar tests reported in the literature. This test consists of one sheet of paper with 30 pairs of decimal numbers (referred to as 30 items) with the instruction: *For each pair of decimal numbers, circle the larger*. The thinking pattern of the student is diagnosed by a detailed analysis of the pattern of responses to the 30 items, and the test is allocated one of 11 "fine codes". One of these (A1) represents expertise on this task with the student being correct on all item types, nine represent misconceptions which are reasonably well understood and one is for test responses with unclassified patterns. The patterns of thinking corresponding to these "fine codes" are grouped according to whether they exhibit L or S or other behaviour as explained below. Because of space limitations, the analysis presented in this paper is at the level of the L and S groups of misconceptions (the "coarse codes"), not the finer classification.

Table 1 provides the ten core items from the Decimal Comparison Test that serve to identify the L and S students. For convenience, the larger decimal is always given first in Table 1, but not on the test. The code A (Apparent-expert) indicates that a student has answered correctly on these ten core items, and the code U (Unclassified) is used on papers that do not fit the A, L or S response patterns. To allow for "careless errors", the A, L or S code is allocated even if there is one deviation from the expected pattern. So, for example, a student with 0 or 1 correct on Type 1 and 4 or 5 correct on Type 2 would be classified as L. The ten core items have been carefully chosen as "normal" items to avoid the complications, many of which arise from visible and invisible zeros (Steinle and Stacey, 2001), which distinguish between the fine classifications. Note again that A, L, S and U are behaviours, rather than particular ways of thinking. A student could be classified A, for example, by being truly an expert or by simply selecting the number with the larger digit in the tenths column as the larger, in which case they may not be able to choose the larger from 7.942 and 7.94.

Behaviours	Type 1 items					Type 2 items				
	4.8 4.63	0.5 0.36	0.8 0.75	0.37 0.216	3.92 3.4813	5.736 5.62	0.75 0.5	0.426 0.3	2.8325 2.516	7.942 7.63
Longer-is-larger (L)	X	X	X	X	X	√	√	√	√	√
Shorter-is-larger (S)	√	√	√	√	√	X	X	X	X	X
Apparent-expert (A)	√	√	√	√	√	√	√	√	√	√
Unclassified (U)	none of the above									

Table 1: Expected responses by students with particular behaviours

Figures 1 and 2 illustrate how students' responses fit the A, L, S groups. Figure 1 shows the 36 possible scores on Type 1 and Types 2 items if students answered randomly with probability being correct on any item of 0.5. The majority of students would, if guessing, have scores of 2 or 3 on both Types 1 and 2. (Note that the (5,5) corner corresponding to A is hidden in this graph, because the expected numbers are very small.) A choice other than one half for the probability of being correct or unequal probabilities for Type 1 and Type 2 will shift the "mountain" of probabilities from the centre, but not change the overall single peaked mountain shape. The contrasting data in Figure 2 comes from the 3531 tests completed in 1997 by students from Grades 5 to 10. There are three clear peaks. A considerable number of students answer all 10 items correctly hence the column at the back corner (5,5) in Figure 2 is considerably taller than all other columns (and has been truncated to 800 from 1664). The very low columns in the centre of the graph demonstrate that students, on the whole, are not choosing randomly on this test. Secondly, the very tall columns at the side corners, corresponding to L and S behaviours, confirm the validity of the L and S constructs. In fact, if only the 1867 papers with one or more errors in these 10 items are considered, then 59% are coded as L or S. Hence, Figure 2 shows that the sets of Type 1 and Type 2 items are internally homogeneous but different to each other and hence that the classifications to L, S and A are indeed meaningful.

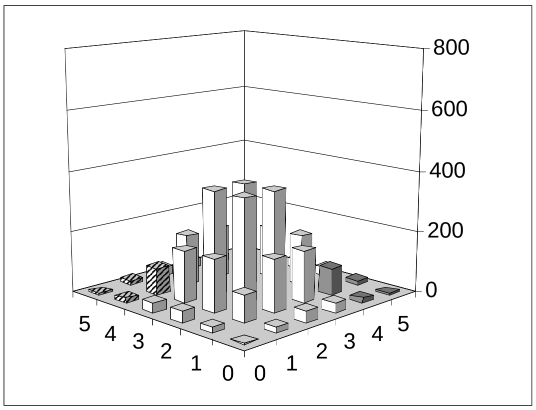

Figure 1: Distribution of A, L, S, U assuming answers selected at random.

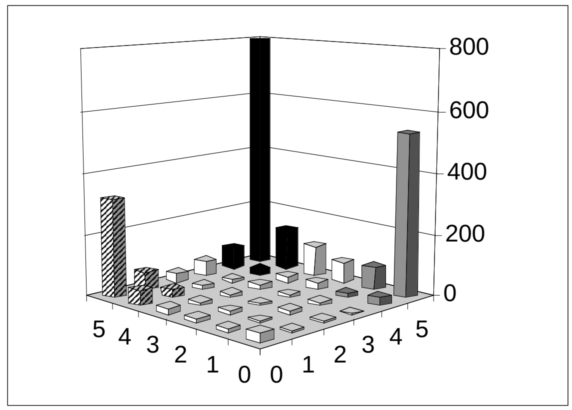

Figure 2. Actual distribution based on data from 1997 (Truncated at (5,5) corner).

(Key: A black, L grey, S striped, U unshaded)

PREVALENCE ACROSS THE GRADES

An important use of the longitudinal data is to report on how many students are likely to be affected by each of the decimal misconceptions. This simple notion requires careful thought. One useful measure is the percentage of students who demonstrate the behaviour at any given grade and another is the percentage of students who are likely to demonstrate the behaviour at some time in their schooling. In defining useful measures, we draw on epidemiology (Hope et al, 1998, p 791) where the *point prevalence* of a disease is defined as the number of cases at a point in time divided by the population at risk and the *period prevalence* is the number of cases at any time during the study period, divided by the population at risk. This concept moves closer to the notion of lifetime risk for a disease.

The measure which corresponds to point prevalence in our context is “grade prevalence”, which is intended to indicate the percentage of students in a given grade who are likely to exhibit the behaviour. The naïve measure for this would be the percentage of tests from a given grade receiving a certain code. However, we have used only the 3204 first tests that students have done, so that individuals are not included twice (especially in the data from one grade) and to avoid the repeat-test effect. Data shows that students who undertake the test repeatedly improve more than others, most probably because teachers who take the effort to test their students are more likely to give decimal notation an emphasis in their teaching. Additionally, these students probably have above average attendance at school. The grade prevalence results are affected by the constitution of our sample, as well as the strictness of the rules for classifying behaviour (as mentioned above, we permit one error per type, for example). The data set is of sufficient size that over 250 first tests have been available at each grade.

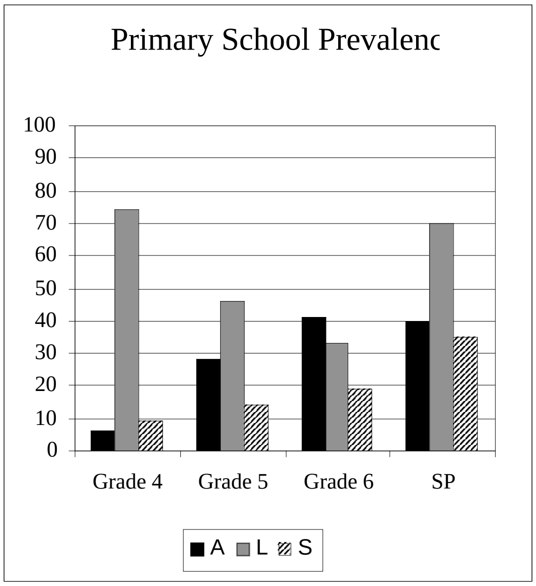

Figure 3. Primary school grade and schooling prevalences for A, L and S.

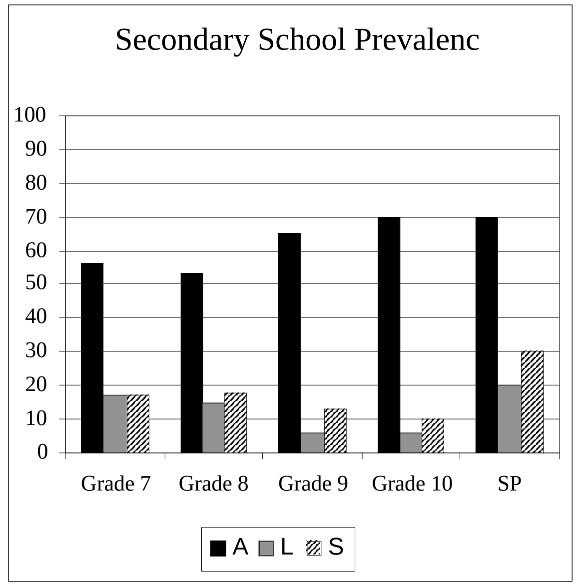

Figure 4. Secondary school grade and schooling prevalences for A, L and S.

The grade prevalences for Grades 4 to 10 are shown in Figures 3 and 4 (along with the schooling prevalence (SP), which is discussed below). The grade prevalence of L drops steadily (often nearly halving across successive grades), indicating that L is principally a misconception of younger students. In contrast, the grade prevalence of S stays reasonably constant, with 10 to 20% of students in the middle grade levels exhibiting this behaviour. In the next section, we will show that there are also very different patterns of students' movements into and out of these misconception groups. The percentage of students coded as A increases, as would be expected. However, it falls significantly short of 100%, rising only to 70% at Grade 10, indicating that problems with understanding decimals continue beyond the compulsory years of schooling for a significant group of students. Data not shown in Figure 3 shows that at all grade levels, about 12% of the tests coded A are not A1. This indicates that about 12% of students who can deal with the Type 1 and 2 items, cannot deal with those with zeros after the point, the same digits in the tenths column or other complicating features. They may have no real understanding of decimal notation, despite being coded as A.

PERSISTENCE OF MISCONCEPTIONS ACROSS THE GRADES

The calculation of grade prevalence above uses only cross-sectional data. In this section, we use the longitudinal data to describe how persistent each misconception code is. Figure 5 shows the probabilities of a student retesting in the same code on the next test that they complete. This data is based on an average of 332 students per data point with a maximum of 1202 students (Grade 7 A to A) and only two data points based on less than 100 students (Grades 9/10 L and S).

Figure 5 shows that students testing as A have a high probability (around 90%) of retesting as A on the next test that they do, regardless of their initial grade level. Although not shown, the retesting probabilities for A1 are similar. The probabilities for retesting as an L reflect the decreasing numbers of students in this code, and the fact that the L code is populated mainly by young students (Figures 3 & 4). The data for S demonstrates that students about Grade 8 (based on 195 students) are especially likely to be retained in this group (for explanation see Stacey, Helme, & Steinle, 2001). The different trends at Grades 9 and 10 may be due to the smaller sample or it may reflect the fact that students at this level who are still in the L or S codes have special learning difficulties.

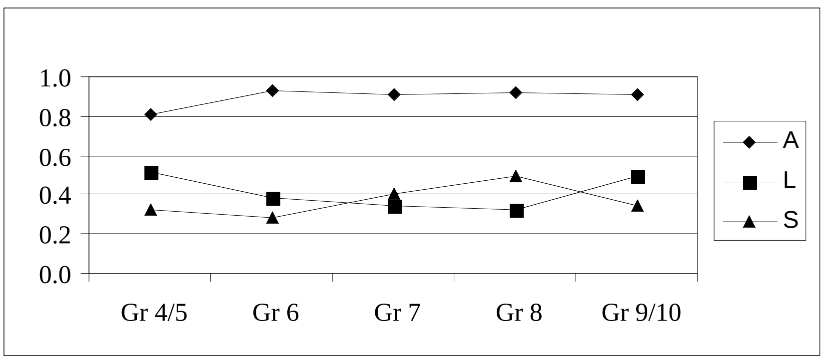

Figure 5. Probability of retesting in the same code, by grade of initial test.

Whereas the behaviours of groups L and S seem similar in Figure 5, Table 2 reveals the differences. The probabilities in the second row reflect the data in Figure 5, combined across grade levels. However, the first row gives the probability that a student testing in a given code had tested in that code on their immediately prior test. (Note that the data sets are necessarily not quite the same: row 1 is based on second to final tests, row 2 is based on first to penultimate tests etc). Whereas only one quarter of the L students came from another code on the preceding test, on average, half of the S students were previously in another code (mainly L or U). To oversimplify, many students begin in the L group at Grade 4/5 and then they move out at some stage, whereas code S recruits across the middle years of schooling, with students moving in and out with reasonable probability. This data is affected by the length of the study, which may have concluded before the student returned to the code and hence it provides a lower limit on the real data. This phenomenon is central to discussion of the schooling prevalence in the next section.

	A	A1	L	S	U

Probability of an immediately previous test of same code	0.77	0.74	0.73	0.48	0.71
Probability of an immediately following test of same code	0.91	0.89	0.44	0.38	0.71

Table 2. Probabilities of a student having previous and following tests in same code

SCHOOLING PREVALENCE

In addition to the grade prevalences, it is also of interest to know how many students are affected by the misconception during their school lives (or at least between Grades 4 and 10). This quantity, which we call *schooling prevalence*, is analogous to the lifetime risk for a disease. A first proposal is to calculate the percentage of the 3204 students who were allocated a given code on any test that they completed during the course of the study. This would be reasonable if all 3204 students had been completely tracked from Grades 4 to 10. However, it is inappropriate for our sample, because students enter and leave the study at all stages from Grade 4 to Grade 10. Students who were in Grade 8 when they entered the study, for example, may never test as L even though they were L beforehand. Calculating the schooling prevalence therefore requires care. A first decision is that the data better supports calculation of primary schooling prevalence and secondary schooling prevalence separately. The 3 years of the longitudinal data nearly matches the complete timespan of interest (Grades 4 – 6 and Grades 7 – 10). Happily, this information is useful, matching the separate professional concerns of primary and secondary teachers.

Table 3 shows the schooling prevalences based on the sample of primary (secondary) students who have completed at least 4 tests in the primary (secondary) school. The calculated prevalence is the number of students who have at least one test in the code, as a percentage of the restricted sample. No student is in both samples. Table 3 data for both L and S are compatible with the grade prevalences (Figures 3 and 4) and the persistence patterns. Comparison with the grade prevalences in Figures 3 and 4 shows that the prevalence for A is very much higher. This is because of the repeated test effect discussed above. Given this and the pattern of the persistence data (Figure 5), our best estimate for the schooling prevalences of A (and A1) is obtained from the maximum of the grade prevalences, say 40% for primary and 70% for secondary. Round numbers are used to emphasize that these are estimates. These best estimates have been graphed as SP in Figures 3 and 4.

	A	A1	L	S	U
Calculated primary schooling prevalence (n=333) *(Best estimate bracketed)*	68% *(40%)*	60% *(30%)*	71% *(70%)*	35% *(35%)*	44%
Calculated secondary schooling prevalence (n=763) *(Best estimate bracketed)*	89% *(70%)*	83% *(60%)*	20% *(20%)*	27% *(30%)*	29%

Table 3: Schooling prevalence calculated from restricted sample (students with 4+ tests in primary (secondary) school) and overall best estimates (*brackets*).

CONCLUSION

This paper has reported best estimates for the percentages of students in each of the Grades 4 to 10 likely to exhibit certain types of behaviour (L, S and A) related to misconceptions about decimal numeration. The different behaviours exhibit different prevalence, persistence and patterns across grades. Using the longitudinal data and knowledge of the prevalence and persistence, estimates have been made of the percentage of students who will exhibit the given behaviours at some time during their primary schooling (effectively after Grade 4 for this topic) and secondary schooling. These results are summarized in Figures 3 and 4. The results show the importance of addressing this decimal numeration explicitly. More generally, the paper has demonstrated some of the challenges and potential of dealing with longitudinal data and has provided a case study of the development of students' ideas on a topic across the grade levels.

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BEING EXPLICIT ABOUT ASPECTS OF MATHEMATICS PEDAGOGY

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It is conventional wisdom that contextualising mathematics tasks can make them more meaningful for students, and that open-ended questions create opportunities for student engagement. Yet concerns are emerging that strategies such as these may exacerbate the disadvantage of some. We report data from a project that seeks to address such concerns by encouraging teachers to be explicit about aspects of their pedagogy. When teachers were explicit about aspects of the pedagogy, the students responded in the direction intended.

We are reporting data from a project entitled Overcoming Structural Barriers to Mathematics Learning1. The project is motivated by concerns that reforms in the teaching of mathematics have failed to address the obvious disadvantage of some groups of students. In particular we draw on two commonly employed teaching strategies—the use of contexts, and the use of open-ended questions. We sought to identify and describe approaches to overcoming factors that may inhibit successful implementation of these strategies in classrooms.

The term *contexts* refers to real or imaginary settings for mathematical problems that illustrate the way the mathematics is used. Meyer, Dekker and Querelle (2001) discussed the use of contexts in mathematics curriculum, drawing on examples from five recent curriculum documents developed in the United States, all incorporating "pervasive use of context" (p. 522). They suggested that contexts can be used to motivate, to illustrate potential applications, as a source of opportunities for mathematical reasoning and thinking, and to anchor student understanding.

Open-ended questions and approaches are also increasingly popular. Open-ended tasks can engage students in productive exploration (Christiansen & Walther, 1986), enhance motivation through increasing the students' sense of control (Middleton, 1995), encourage pupils to investigate, make decisions, generalise, seek patterns and connections, communicate, discuss, and identify alternatives (Sullivan, 1999) and contribute to teachers' appreciation of mathematical and social learning of students (Stephens & Sullivan, 1997).

Concerns about contexts and open-ended approaches

Notwithstanding this support, more recently there have been some concerns about the effect of context based and open-ended strategies in mathematics teaching and assessment. In a comprehensive review of the national testing system in the United Kingdom, Cooper and Dunne (1999) found that contextualising mathematics tasks created particular difficulties for working-class students, so much so that they performed

[1] This project is funded by the Australian Research Council, and the Victorian Department of Education and Training, but the views expressed in this paper are those of the authors.

significantly poorer than their middle-class peers on these tasks whereas performance on decontextualised tasks was equivalent. Similarly, Zevenbergen and Lerman (2001) argued that some students are better able to decode mathematics tasks expressed as contextualised problems and that the capacity to answer questions appropriately corresponds somewhat with student social background. Lubienski (2000), monitoring the implementation of a curriculum and materials based on open-ended contextualised problems, found that target students who preferred the contextualised trial materials were all from high socio-economic backgrounds (SES), whereas the majority of target pupils who preferred typical non-contextualised mathematics were low SES. Many of the low SES students claimed to be worse off as a result of the style of the materials, and none found the contextualised materials easier. Lubienski stressed that some of the low SES students experiencing difficulty with the trial materials were high achievers. These studies pose new considerations in understanding how some practices in school mathematics may create unintended barriers to success for some students.

It has been suggested that the different responses of pupils is a product of the school curriculum. For example, Anyon (1981) found that what she called working-class schools tended to emphasise rote learning strategies while the schools with professional class pupils spent more time building up knowledge relationally. It has also been argued that the receptiveness of pupils to mathematics is social-class based. Cooper (1986), for example, implemented a program that aimed to improve the teaching of mathematics in a school serving a lower SES community and found that the pupils' view of what the school intended to do adversely affected the way in which they learnt. Mellin-Olsen (1981) argued that both the educational and social context influence the learning goals and strategies adopted by the pupils.

Lerman (1998) proposed that differences between the classroom expectations and the students' aspirations may exacerbate disadvantage and that classroom discourses both "distribute powerlessness and powerfulness" (p. 76). Likewise, Zevenbergen (1998) argued that pupils from backgrounds where there are discontinuities between linguistic registers and societal aspirations of home and school have difficulty interpreting some aspects of classroom processes, and that teachers should make socio-cultural norms of pedagogy explicit to students. This can mean that for children whose cultural norms are similar to those embedded within mainstream pedagogical practice, the mathematics is more accessible than for students whose culture does not fit the dominant classroom routines. Delpit (1988) proposed that schools should seek to teach their usually *implicit* values, and that to pretend that schooling (and society) is democratic actually denies groups outside the mainstream access to the opportunities that schooling is intended to provide.

In other words, there are factors inherent in the culture of schooling that may constrain the potential to engage some students in learning—one aspect of what Bernstein (1996) termed invisible pedagogy.

Seeking a solution

We are proposing that teachers who adopt strategies that seek to be inclusive of all students need to be *explicit* about factors that may inhibit participation. One framework that may be useful for guiding the development of such approaches was proposed by Dweck (1999), who argued that people's views of intelligence, whilst stable without intervention, are malleable to suggestion. Dweck suggested that pupils who have a fixed view of intelligence predominantly seek affirmation from external sources, including from the teacher. Their positive self image relies upon success on tasks. When experiencing difficulties, such pupils lose confidence in themselves, tend to denigrate their own intelligence, develop negative approaches, have lower persistence, and exhibit plunging expectations and deteriorating performance. There are other students, according to Dweck, whose view of intelligence is not so fixed, and who are more mastery oriented. The latter pupils do not view failure as an indictment on themselves, but as a necessary part of learning, and when experiencing challenge they are likely to increase their focus on task goals and utilise a range of strategies. Being within a socio-economic or cultural subgroup within a school is thought to compound the effect of these different perspectives on intelligence. Dweck argued that teachers can influence the way that students are likely to respond by teaching specific behaviours and attitudes such as decoding tasks, perseverance, seeing difficulties as opportunities, and learning from mistakes. Importantly, they can explain the purpose of particular aspects of pedagogy such as grouping, use of contexts, the type of solutions sought, the modes of communicating responses, and aspects of responses that will be valued.

It is this theme of explicitness that underpins our project. Rather than accepting concerns expressed about contextualised or open-ended strategies and thus suggesting that teachers should avoid these, we believe that teachers can take action to make *explicit* various aspects of hidden pedagogy that we term *implicit*. Our aim is to create environments that assist all students to participate fully in learning mathematics.

OVERCOMING BARRIERS TO MATHEMATICS LEARNING

One of the goals of our project, called *Overcoming Barriers to Mathematics Learning*, is to design, implement and describe teaching approaches that incorporate attention to explicit pedagogies.

The process used to identify and describe aspects of implicit pedagogy was described in Sullivan, Zevenbergen and Mousley (2002). An outcome was the production of a manual that lists a range of particular strategies that teachers could use to make implicit pedagogies more explicit (Sullivan, Mousley, & Zevenbergen, 2002) and so address aspects of possible disadvantage of particular groups.

Essentially the project uses multiple case studies to examine the classroom implementation of explicit mathematics pedagogical strategies. The data are based on a model, developed from Clark and Peterson (1986), that has teacher beliefs and understandings interacting with opportunities and constraints, each influencing (and influenced by) the teacher's intention and actions.

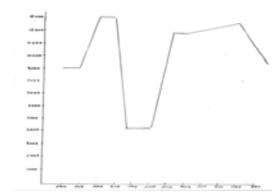
The teachers in our project had three days of professional development where we outlined the project and our expectations of their participation, presented the research-based manual of advice about explicit pedagogies, and considered a range of key elements of contextualised open-ended teaching approaches for primary mathematics classrooms. The teachers from each of the schools involved also carried out initial planning for teaching their first unit of work at this stage.

Data were collected by a questionnaire on aspects of their beliefs and understandings, and through interviews on what they saw as some opportunities and constraints. A planning survey was used to gain insights into their intentions for teaching a unit of mathematics. Structured observations of teaching were undertaken then descriptions of the teachers' actions and classroom interactions were compiled.

The data presented below are from just three of the lesson observations. A variety of naturalistic and structured lesson observation schedules had been trialled, and the instrument selected was adapted from that used by Clarke et al (2002). This consisted of (a) a form for the collection of overview comments; (b) a naturalistic report using two columns, one for a record of what happened and one for the observers' impressions; and (c) a framework to structure observer's post-lesson reporting of their immediate impressions. These notes from the lesson observers were used as the source of the data presented below. There were two observers for these three lessons.

ONE LESSON THREE TIMES

To present the sense of the extensive and rich data and some issues associated with the implementation of the explicit pedagogies, the following are selected excerpts from the observations of one of the three lessons. The teachers of Grade 6[2] in this school worked as a team, planning together and essentially seeking to teach the same lessons in their own classrooms. The focus here is on the extent to which they chose to be explicit about some of the aspects of pedagogy outlined in the manual.



The lesson was described in the plan as follows:

> Present children with a line graph. Tell students this is a graph of the number of worms in the worm farm. Write a story about it.

The task material consisted of a graph as shown[3]. The axes were labelled with months, and numbers of worms, in thousands. The text on the page read: "Look at the line graph. The graph represents the population of worms in our worm farm at particular times throughout the year. Write a story about it. (Use up all the space below the graph.)" Pre and post lessons were implemented that supported this lesson. This ensured that students were familiar with the content and processes needed in this lesson. In each of the three teachers' versions of this one lesson there were three phases: a review of the previous lesson, the main story-writing activity and reporting back, then a follow up task.

The first teaching of the lesson was by Shelley[4], who appeared to emphasise the mathematical aspects of the lesson, but who made only limited effort to make explicit aspects of the pedagogy. For example, the observation notes recorded that, during the introduction, in which Shelley reviewed the strip and pi graphs, she:

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[2] 7th year of schooling, with children aged about 12.

[3] The gradient of some parts of the graph is not mathematically ideal, but this did not affect the quality of the experience.

[4] All names in the paper are pseudonyms.

… was most explicitly probing their thinking. "Can you see the different sections?" Could you tell me how many pieces of passionfruit?" etc. There was even significant extension by asking what else could be shown.

Shelley posed the task for the lesson using an A3 page on which the line graph and text as shown above were presented. She explained that the graph showed the amount of food in the worm farm over time. The lesson observation recorded that:

> She used an open ended prompt of "If there was not writing what might the graph be?"

While this was a useful mathematical extension, not only focusing attention on the key features of the graph but also alerting the students to the possibility of transfer of the information to other contexts, it is a relatively abstract way to pose the task.

Shelley was explicit about the intention for them to create their own story, as was recorded:

> … attention to the fact there is no right or wrong answer, but they are just writing their story.

The students then were set to work:

> They chose their own partner, collected sheets from the teacher and went to the tables. There was some initial puzzling over the task. Interestingly, even though they were paired up for this aspect, there was quite little talking initially. This did build up. … Clearly they are required to take responsibility for their own management and behaviour.

Even though instructed to work with a partner, there was no explanation of why they should do this, or in what ways the partner was intended to assist. It was also noted that

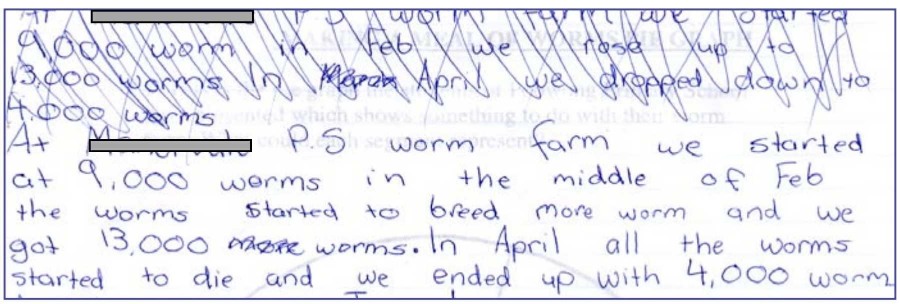

Figure 1: A shift from a descriptive to a more creative outcome.

the mathematical aspects of the task were made clear, but not the practical or even the creative. As was noted by one observer:

> On looking at the work, they seemed to have done mathematically correct descriptions of the graph, but they were not particularly creative in their interpretations. The issue of their initial lack of creativity was interesting. Perhaps they have too little scope in their knowledge of worm farms to be creative. Alternatively perhaps they did not link the creative interpretation with maths.

One observer casually turned to a pair of students nearby. They had started giving a mundane decontextualised numerical description of the graph. The observer asked them whether there might be reasons for the graph changes. The students adapted their work as shown in Figure 1. In other words, it took little prompting for them to respond in a more creative way. This perhaps highlights the need for making this expectation explicit in the first place.

Nearing the end of the lesson, Shelley asked each pair to work with another pair to share their stories, again without any explanation of why they should do this, or what they should look for. She then led a final review that focused on the explanations. As was noted:

> Teacher made insightful comments drawing their attention to important mathematical elements.

In the evaluative notes, the two observers agreed that this teacher was explicit about the openness of the task and that the students could choose their own style of answer, but was not explicit about the purpose of the task, any linking to the curriculum, the sequence of activities, the process for undertaking the activity, the reasons for grouping, or any mathematical terminology. These were all elements that had been part of the professional development undertaken by the teachers involved in the project. There was no obvious acknowledgement of differences between students' capabilities or backgrounds, although none appeared to be necessary.

The data gathered on the second teacher, Greta, indicated that she was explicit about more aspects of the pedagogy, although she had a less mathematical approach to the teaching, resulting paradoxically in more creative mathematical responses from the students. One difference from Shelley's lesson was that a similar line graph to the above was drawn on the board, but with no labelling. Her students were asked to suggest what the graph might represent, and suggested hospital charts, bank account balances, etc. The observers later agreed that the absence of the labels provided a better prompt for the students than asking them to imagine the labels not there in that creativity, use of graphical clues, and aspects such as scale were appreciated explicitly. In giving directions for the story writing, Greta was more explicit about the opportunity for creativity, the connections to the given context and the need for realistic contextualised explanations. Somewhat in contrast, she was more directive about how they should work, and actively monitored their products, but did not make any statements about the purpose of having the children work with partners. In the review stage of the lesson, Greta was explicit about a number of aspects of the pedagogy, but was not explicit about the mathematics and did not pursue any mathematical opportunities, and did not evaluate the children's responses. The observers both collected evidence to show that aspects of pedagogy about which Greta was explicit had a positive effect on corresponding aspects of the responses of the students.

Like Greta, the third teacher, Simone, was explicit about aspects of the open-ended approach and about the types of responses being sought. Simone had repeatedly stressed the potential for creative and realistic dimensions of their stories. In commenting on the review, an observer noted that:

> They came back together. Again (Simone) emphasised the possibility of multiple correct answers. Even though it is a line graph there is much we can find out.

Throughout the lesson, but especially in the review stage, she emphasised the mathematics and its connection to the context. For example, as one student read a story Simone traced the line graph on the board with her finger. As a result, the children's attention was focussed on specific mathematical ideas and language. As with both of the other teachers, the majority of the students seemed to react to the explicit directions that were given by responding in the way the teacher intended.

DISCUSSION AND CONCLUSION

We are reporting here on ways that three teachers used an open-ended question set in a practical context. These three teachers planned together an interesting lesson directly connected to a class project, that of managing a worm farm. The open-ended nature of the set task allowed potential for use of appropriate mathematics and student creativity. The lessons were structured to ensure students' involvement, and it seemed clear enough to the students what they were expected to do. Clearly it was possible for the teachers to be explicit about a range of aspects of the pedagogy, and this explicitness contributed to the students' experience. For example, all three teachers were explicit about the possibility of multiple responses, Greta and Simone emphasised the potential for the students to be creative, and these same two teachers incorporated the context of the worm farm explicitly. Shelley and Simone emphasised the mathematical dimension of the task both in posing the task and in the review of student products. The students generally responded to this explicitness in both their written work and reporting back. It seemed to be effective in shaping the nature of their learning. In other words, it seemed that there was a direct link between the intentions that the teacher communicated, the ways the students responded, and the understandings that they developed about the nature of the task.

There were other aspects of the pedagogy that were not made explicit by any of the three teachers. These included:

> (1) the purpose of working with a partner; (2) the expected nature of communication with the partner; (3) the purpose of working on an open-ended question; (3) the criteria for evaluating student responses; (4) the rationale for using a context, and that context in particular; (5) the aspects of mathematics it was intended the students would learn; (6) expected strategies for organising and pacing of the work; and (7) expectations regarding use of specific mathematical terminology.

All of these aspects of pedagogy remained implicit, and students were expected to do their own interpreting. There was no evidence that students worked any of these out for themselves or that they saw the need to do so. While the children would be seen as working in a self-directed way in all three classrooms, they merely followed the teachers' expectations, picking up any subtle hints offered.

In summary, the findings of the project to date include aspects of implicit pedagogy that may be more apparent to some students than to others, and we now have a better understanding of both explicit and subtle ways that teachers' expectations are conveyed in mathematics classrooms. We believe that we have learned more about how to create environments that will assist students to participate more fully in the learning of mathematics. In the lessons observed, including the three reported above, making aspects of the pedagogy more explicit resulted in productive responses by most of the children. Interestingly though, even where teachers were explicit about aspects such as creative use of the mathematical information and the use of the given context, some students did not heed that advice. It seems that being explicit about particular requirements does generally produce positive outcomes, but it is not a guarantee.

Further, in most lessons observed the students worked on tasks fairly independently of teachers, and the teachers gave little progressive feedback or further assistance to groups of students from minority groups. Other than a few specific reactions to anticipated discipline problems, there was no differentiation between students that was evident to the observers. This was disconcerting because one of the assumptions in our project—and a focus of the professional development that it included—was that pedagogical strategies commonly associated the use of contextualised open-ended tasks are likely be clearer for some students than others. This implies that some particular actions are needed to address the needs of specific students. The teachers neither planned nor taught in a way that indicated their acceptance of this contention. This aspect of our project requires further examination.

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SELF-EFFICACY IN MATHEMATICS AND STUDENTS' USE OF SELF-REGULATED LEARNING STRATEGIES DURING ASSESSMENT EVENTS

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High stakes summative assessment has been a significant element of educational policy in England and Wales for more than a decade. However, research evidence indicates that formative assessment is more likely to raise student achievement. Although some students are able to use summative assessment formatively, devising their own self-regulated learning strategies, which will support them in lifelong learning, many students fail to do so and for them an emphasis on summative assessment often leads to a reduction in self-efficacy and motivation. This paper reports on an investigation into students' beliefs about themselves as learners of mathematics and the strategies they use before and after assessment. Although most students believe in the value of revision, they often fail to employ effective revision strategies and many fail to use assessment results formatively.

FORMATIVE OR SUMMATIVE ASSESSMENT?

Educational policy intended to raise student achievement in England and Wales, has included high stakes summative assessment as a significant aspect. Over the last decade, the use of summative assessment over and above that required by the National Curriculum has grown rapidly (Harlen & Deakin Crick, 2002) and target setting for both teachers and pupils is often based on such data. However, there is evidence to suggest that it is the process of analysing one's own strengths and weaknesses before and after assessment, rather than the act of target setting, which leads to improved learning (Tanner and Jones, 2000). Proponents of regular, summative assessments claim that they motivate students to learn. However, this is not true for all students. In fact, a recent systematic review of the impact of summative assessment suggested that target setting based on summative assessment often results in students (and teachers) emphasising extrinsic motivation at the expense of intrinsic, and a focus on test performance rather than understanding. This often leads to shallow rather than deep learning, and damaged self-esteem for failing students, thereby resulting in a reduction in self-efficacy and effort for a significant proportion of students (Harlen & Deakin Crick, 2002).

On the other hand, a review of research by Black and Wiliam (1998) demonstrated very clear evidence that formative assessment, which is designed to be integral to the learning and teaching process, can lead to significantly improved performance. Moreover, our aims for education and lifelong learning must demand more than simply improving student performance in standard tests. Our aim should be to develop autonomous, self-regulating learners who are inclined to choose to learn and improve their knowledge and skills throughout their lives. Formative assessment, and particularly self-assessment, clearly has a part to play here.

The terms formative and summative are not used consistently in the literature (Black and Wiliam, 1998). The terms are often used to describe types of assessment with some writers using the term formative to describe classroom assessment and summative to describe externally imposed assessment or examination. However, we use the terms here to describe the *functions* rather than the *types* of assessment (Brookhart, 2001) and recognise that any assessment event, either classroom based or externally imposed, could be used to satisfy formative and/or summative ends. We define an "assessment event" to include: the preparation for the assessment by both the teacher and the student, the feedback from the assessment offered by the teacher, and the impact of the assessment on the subsequent learning behaviours of the student (cf Brookhart, 2001). The ultimate end user of assessment information should be the student and to be considered formative, assessment should be used to improve student performance (Black and Wiliam, 1998).

The literature often refers to the need for clear, shared, learning goals so that students are able to compare actual with desired performance and then try to close the gap. Formative assessment is thus intended to provide feedback information about performance to assist the student in identifying the gap and may suggest strategies to facilitate development (Gipps, 1994; Black and Wiliam, 1998). Unfortunately, many students fail to understand how to interpret or respond to the feedback which teachers offer (Sadler, 1998); indeed, when feedback emphasises marks or grades, formative comments are often ignored (Butler, 1987). However, there is some evidence to suggest that successful students are able to use even summative assessment information in a formative manner, formulating their own goals and monitoring their development towards them (Brookhart, 2001).

Developing self-regulated learning

Much of the literature about formative assessment, naturally emphasises the role of the teacher because teachers are responsible for planning and administering assessment, but in fact, the demand for improved performance should move the focus to the behaviours of the learner. Ideally students should share the teacher's learning goals and plan to close any gaps, monitor their own progress, and evaluate the success of their learning. The ultimate goal of feedback should be to teach students to regulate their own learning (Sadler, 1998; Gipps, 1994; Black and Wiliam, 1998). Successful students have usually learned to do this and are often able to use intermittent assessment events which have been designed to provide summative information for their own formative purposes. They review work and engage in self-assessment as a regular ongoing process and use feedback information in both summative and formative ways simultaneously (Brookhart, 2001). On the other hand, weaker pupils often believe that the purpose of such assessments is to make them work harder rather than differently. In fact when such students repeatedly gain low marks, it may generate "a shared belief between them and their teacher that they are just not clever enough". In such circumstances, the self-regulation which occurs may be to "retire hurt" to protect their self-esteem (Black, 1998: 43-44). Low self-efficacy is often an unintended outcome of summative assessment.

Students' self-efficacy for mathematics may be defined as their judgements about their potential to learn the subject successfully. Students with higher levels of self-efficacy set higher goals, apply more effort, persist longer in the face of difficulty and are more likely to use self-regulated learning strategies (Bandura, 1977; Wolters & Rosenthal, 2000).

Research also highlights the importance of metacognition if students are to regulate their own learning effectively. Metacognition includes three components: a) the awareness that individuals have of their own knowledge, their strengths and weaknesses; b) their beliefs about themselves as learners and the nature of mathematics; and c) their ability to regulate their own actions in the application of that knowledge (Flavell, 1976; Tanner and Jones, 1994; 1995; 1999; 2000).

The development of autonomous, self-regulated, lifelong learners of mathematics depends on the interaction of three linked psychological domains of functioning: the affective, the cognitive and the conative (Bandura, 1977). The affective domain includes students' beliefs about themselves and their capacity to learn mathematics; their self-esteem and their perceived status as learners; their beliefs about the nature of mathematical understanding; and their potential to succeed in the subject.

The cognitive domain includes students' awareness of their own mathematical knowledge: their strengths and weaknesses; their abstraction and reification of processes; and their development of links between aspects of the subject (Tanner & Jones, 2000).

The conative domain links the affective and cognitive domains to pro-active (as opposed to re-active or habitual) behaviour. It includes students' dispositions to strive to learn and the strategies that they employ in support of their learning. It includes their inclination to plan, monitor and evaluate their work and their inclination to mindfulness and reflection. In particular it includes the strategies, which they are inclined to use when reviewing or revising their work (Snow, 1996).

Using summative assessment events to improve learning

The political pressures on schools to use intermittent summative assessments of students' learning are unlikely to reduce in the near future. As we indicated above, several negative impacts on learning may result from on over-emphasis on summative assessment. However, some students have learned to overcome these negative aspects and are able to use intermittent summative assessment to support their own self-regulated learning. Research suggests that the following conditions are necessary for this to occur.

Firstly, the students' self-efficacy must be high. They must believe that their mathematical ability is not fixed. They must attribute their success or failure in mathematics to controllable factors such as effort or revision, rather than uncontrollable factors such as bad luck or lack of ability (Bandura, 1977; Black, 1998; Wolters and Rosenthal, 2000).

Second, they must have metacognitive knowledge of their own mathematical abilities. For example: they must be aware of their strengths and weaknesses, how their knowledge compares with the potential demands of the assessment, what they understand and do not understand, and the errors they are most likely to make (Tanner & Jones, 2000).

Third, they must be aware of, and be inclined to use effective strategies for reviewing and revising their work and analysing their successes and failures. For example, they need to have strategies for planning and engaging in revision, identifying key features of their work and anticipating potential difficulties in questions in advance of assessment. After assessment they require strategies for analysing and evaluating their performance. And finally, they must be inclined to implement these strategies in the belief that their performance will improve (Gipps, 1994; Tanner & Jones, 1994; Brookhart, 2001).

METHODOLOGY

The research reported here was funded by the General Teaching Council for Wales. The aim of the research was to investigate year nine (aged 13-14) students' self-efficacy in mathematics and their use of self-regulated learning strategies during assessment events.

A questionnaire was designed based on the three domains of functioning outlined above to ensure face and construct validity. The aim was to allow the collection of a large data set with minimal disruption to normal teaching or potential for researcher bias or interference. The questionnaire was trialled on a one-to-one basis with a small number of similar students prior to administration to all the students in the study. The final instrument consisted of 47 statements, to be answered using a five-point Likert-type scale, running from “strongly disagree” to “strongly agree”. The questionnaire was divided into three sections: self-efficacy, metacognition and learning strategies.

The self-efficacy scale was divided into two sub-sections. The first included nine statements and addressed students' beliefs about themselves as learners of mathematics. In particular it focused on their beliefs about the fixed or changeable nature of mathematical ability. Typical statements included:

> Some people just can't do maths
>
> Working hard leads to success in maths
>
> You can't change your maths ability

The second self-efficacy sub-section included 12 statements and addressed students' attributions of success and failure in mathematics examinations. Typical statements were:

> If I do well in a maths exam it's because:
>
> I was lucky with the questions
>
> I worked hard

A parallel set of questions was phrased negatively and included statements such as:

> If I do badly in a maths exam it's because
>
> The questions were too hard
>
> I have no natural ability in maths

The next section of the instrument focussed on students' metacognitive knowledge. There were ten statements such as:

> I know which parts of maths I don't understand
>
> I know in advance if I am going to get a question right
>
> I know which mistakes I am most likely to make in maths

The final section included 16 statements and identified strategies which students might use for learning mathematics. Students were asked how often they used each strategy and the five-point scale offered the options: “Never”, “Almost never”, Sometimes”, “Almost always”, “Always”. It included statements such as:

> If I make a mistake I try to find out where I went wrong
>
> I always revise for maths tests
>
> Before a maths exam:
>
> > I find a quiet place to revise
> >
> > I read through my maths book

I try to predict what the questions will ask

When I get my exam paper back:

I work out where I went wrong

I only look at my mark

I make sure that I understand my mistakes

The sample consisted of two year nine classes in each of six comprehensive schools. The schools were necessarily an opportunity sample of those teachers and schools willing to participate in the research, but comprised a reasonably representative stratified sample of comprehensive schools in South Wales, including examples of affluent suburban, inner city, rural, and industrial locations. The 303 students involved spanned the ability range.

THE RESULTS

Students' responses to the statements were coded from1 to 5 for analysis with 5 representing a strongly positive attitude or an effective strategy which was always used. The total scale proved to be reliable (Chronbach's alpha = 0.89). The sub-scales also had adequate reliability (self-efficacy, 0.76; metacognition, 0.53; strategies, 0.89).

On average students were positive in their beliefs about themselves and their potential to learn mathematics (self-efficacy), their metacognitive knowledge and their use of self-regulated learning strategies (see Table 1). However, the higher standard deviation in the strategies sub-section reveals a greater variability in their use of learning strategies.

	N	Mean	Standard deviation
Self-efficacy	303	3.59	0.41
Metacognition	302	3.40	0.41
Strategies	299	3.45	0.70

Table 1: Means and standard deviations of the sub-scales

The vast majority of students thought it was worthwhile to try hard in mathematics (93%) and thought it was worthwhile to revise for examinations (90%). Most disagreed that only intelligent people could do mathematics (84%) and that mathematics ability was unchangeable (75%). However, although 50% of the sample disagreed with "Some people just can't do maths" worryingly, a hard core of 28% agreed with this statement. Similarly, 21% agreed with "You can either do maths or you can't"

Success in mathematics examinations was generally attributed to hard work (84%) or doing lots of revision (71%). However, other attributions of success included uncontrollable factors such as "luck with the questions" (61%), "easy questions" (46%) or "a good memory" (46%). Failure in examinations was attributed to not doing enough revision (74%) or not doing enough work (56%). However, a hard core blamed their failure on uncontrollable factors such as poor memory (24%), being unlucky with the questions (20%) or having no natural ability in mathematics (14%).

The self-efficacy results indicate that a significant minority of students have beliefs about themselves as learners of mathematics which would impact negatively on their future performance. However, most students have positive beliefs about their potential to succeed in the subject if they work hard and revise. However, the results of the next two sub-scales indicate that many students do not know how to apply their efforts efficiently and unfortunately only 51% of students claim to know a variety of ways to revise.

Students' claims for metacognitive knowledge were generally positive, with 86% claiming to understand some parts of mathematics completely and 75% claiming to know which parts of mathematics they don't understand. Yet only 57% claim to know which mistakes they are most likely to make. Similarly, 47% report that they often get a question wrong but don't understand why. Only 40% have the confidence to claim that they know when they have got a question right and just 23% claim to know in advance that they are going to get a question right. This suggests that their metacognitive knowledge is not as good as their initial claims might suggest and accounts for the relatively low reliability of this sub-scale compared with the others.

The strategy sub-scale reveals that, in line with their beliefs about the value of hard work and revision, most students claim to revise before mathematics tests (66%) and try to find out where they went wrong if they make a mistake (59%). However, most of the learning strategies they employ are of limited value. Only 25% make a note of the main points learned in every lesson and just 23% "like finding bits of maths which link together".

The efficiency of their revision strategies is dubious. For example, the most popular strategy (73%) is to "read through my maths book" which is unlikely to be of great value for learning. 71% claim to try to work out what they don't know, but given the evidence in the metacognitive sub-section, the extent to which this is successful must be doubted. Only 53% control their environment by finding a quiet place to revise.

More effective revision strategies were used by only a minority of students: making revision notes (44%) doing lots of questions (41%) writing some questions to test myself (39%) highlighting the most important parts of my work (34%) or trying to predict what the questions will ask (20%). All these strategies require some degree of metacognitive knowledge and it is perhaps not surprising that they were used more infrequently.

After assessment had taken place most students claimed to make sure that they understood their mistakes (69%); worked out how to do better next time (65%); and worked out where they were going wrong (61%). However, given that 55% claimed to only look at their mark when getting their examination paper back, the validity of their claims must be dubious.

DISCUSSION OF RESULTS

Although most students have beliefs about themselves as learners of mathematics, which would support future self-regulated lifelong learning, a substantial minority holds beliefs, which are likely to have a negative impact on future learning. Students who attribute their success or failure in mathematics to uncontrollable factors are unlikely to apply effective learning strategies. However, even though the majority of students express positive beliefs and attitudes towards revision, the results of the metacognitive and strategic scales suggest that they are unlikely to apply their efforts to best effect.

The failure of most students to claim appropriate metacognitive knowledge is of concern, as it suggests that their revision is likely to be inefficient. A very substantial minority of students lacks the detailed knowledge of their strengths and weaknesses, which would be necessary for them to regulate their learning effectively. Furthermore, only half the students have knowledge of a range of effective revision strategies to apply.

The majority of students appear to believe in the value of hard work and revision, but either do not know, or are not inclined to apply, effective revision strategies. The most commonly used strategies are passive or reactive in character rather than pro-active. Reading through their books does not require active processing and is unlikely to impact significantly on their learning. Similarly, although most students realise that they should learn from their assessments, the majority "only look at their mark" (cf: Butler, 1987). In such circumstances their desire to "work out how to do better next time" is likely to be restricted to unfocussed targets like "try harder" or "be more careful".

Clearly self-efficacy, metacognition and the use of self-regulated learning strategies are closely associated. There was a strong correlation (0.51, $p<0.01$) between students' self-efficacy scores and their "Strategies" scores. There was a moderate correlation (0.42, $p<0.01$) between students' self-efficacy scores and their claims to metacognitive knowledge. Similarly, a moderate correlation (0.41, $p<0.01$) was found between "Strategies" and "Metacognition". Causality is probably complex, but students who believe that their mathematical ability is not fixed and that their performance is due to controllable factors are more likely to employ learning strategies than those who attribute success or failure to luck or lack of ability. The more effective learning strategies such as "working out what I don't know" and "highlighting the important parts of my work" contribute to the development of metacognitive knowledge. Students with good metacognitive knowledge are likely to revise more effectively. Successful application of learning strategies is likely to encourage the development of self-efficacy. A virtuous circle may become self-perpetuating.

Unfortunately, for many students a vicious circle may develop, with lack of self-efficacy leading to a failure to apply effective learning strategies, or develop metacognitive knowledge. Repeated failure in assessments may then reinforce "a shared belief between them and their teacher that they are just not clever enough" (Black, 1998). Breaking into this vicious circle is difficult, but we would suggest that attempts be made to teach all students the self-regulated learning strategies, which are currently only known by the successful minority of mindful and metacognitively skilled students.

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THE ASSESSMENT OF MATHEMATICAL LOGIC: ABSTRACT PATTERNS AND FAMILIAR CONTEXTS

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Undergraduate students' written exams were analyzed from a freshman-level mathematics course that emphasized, among other topics, the study of mathematical logic. Findings indicate that on questions related to the negation of a conditional sentence, students performed much better when given natural-language contexts than they did on questions presented in abstract form. However, students showed improvement over time only on the abstract questions. These results raise doubts about the validity and efficacy of using natural-language examples to promote and test the understanding of symbolic logic.

INTRODUCTION

This report describes significant findings about learning and teaching logic. The data are from a larger study of the nature of skills and understandings developed in an undergraduate course entitled "The Language of Mathematics." Among other mathematical topics, the course emphasized logical connectives, truth tables, and the abstract logical forms of mathematical and non-mathematical sentences. Findings indicate that, when quiz and test questions attempted to measure the students' understanding of logic, the context of the questions was very significant. A substantial difference in students' responses was identified between problems presented in natural-language contexts and parallel problems in abstract mathematical contexts. These results raise doubts about the validity and efficacy of using natural-language examples to promote and test the understanding of symbolic logic.

Many texts use natural-language examples from familiar contexts to illustrate results from symbolic logic. The form "H ⇒ C" (Hypothesis implies Conclusion) can be discussed by considering the example, "If you liked the book, you will like the movie," and asking what can be deduced from "You did not like the book." (Bennet and Briggs, 2002). However, when examples and questions are phrased in natural language, students' prior knowledge may interfere with, and perhaps even eliminate, the intended instructional focus on logical form (Gregg, 1997). It has long been recognized that humans have an understanding of conditionals in a social context that may not be closely related to their understanding of conditionals in a more-abstract context such as mathematics (Devlin, 2000).

Judging by the overwhelming dominance of natural-language examples of logic in "liberal arts" mathematics texts, one might suppose it has been demonstrated that analysis of natural-language examples illuminates the logical forms themselves and is an effective way to teach symbolic logic. In spite of the fact that logic technically addresses the abstract forms of sentences and not their meanings, our findings suggest that responses to natural-language examples are strongly confounded by the students' prior knowledge of the context and that these answers may provide minimal information about the students' grasp of the abstract logical forms supposedly being considered.

The findings of other research studies of students' use of deductive reasoning also indicate that certain characteristics of the tasks, other than their logical structure, influence the students' responses. Gregg (1997) describes a high school classroom exchange in which students appeared to assess the truth of a sequence of statements dealing with a "real life" situation on the basis of personal opinion and experience. Kücherman & Hoyles (2002) found that many 13 year-old students preferred to reason through particular examples rather than on the basis of logical structure when given tasks based on familiar number theory relationships. The presentation of information as a sequenced set of statements may also affect responses. Ferrari & Giraudi (2001), working with 10^{th} graders, and Barkai et al, (2002), in a study of elementary school teachers, documented that the subjects tended to regard each of the statements in the set as an isolated topic rather than as being logically related to each other.

The study reported here adds to this body of knowledge by investigating how students responded to logic statements presented in different symbol/language systems, and with varying degrees of contextual familiarity. In addition, the study examines the often unexpected nature of information obtained from classroom assessment.

THE STUDY

The sub-set of findings that are of interest here were taken from the students' work during the second half of the semester. The emphasis during this time was on the study of logic for mathematics, including logical connectives and equivalents, and ways of expressing mathematical generalizations and existence statements. Examples of logical statements and expressions were presented in pure logical form, algebraic language, natural language, and through truth tables and Venn diagrams.

The "Language of Mathematics course was offered at the freshman level to 35 students. Six were elementary education majors pursuing a mathematics option. Twelve, who were enrolled in health-related fields, were fulfilling a prerequisite for an introductory-level statistics course. The remaining sixteen students were taking the course to complete a minimum math core-curriculum requirement for graduation.

The data for the larger research study consisted of the students' written responses to all the quizzes, the four chapter tests, and the final exam. A small subset of the data that was found to be particularly striking is analyzed here. It consists of four test questions that pertain to the negation of conditional sentences. These problems each present a sentence with the form "H $\rightarrow$ C" and request its negation.

The four questions are presented below. The date on which each test was administered and the instructional focus prior to that exam are noted. Examples of common student responses, their underlying logical form, and the percentage of students giving that type of answer are also provided.

A) Exam 3, April 5 (after truth-table logic) *Simplify: not* [(*not A*) $\rightarrow$ *B*]

Correct answer: (not A) and (not B). [*H and not C*] 11.4%

Typical incorrect answers:

A and (not B) [*not H and not C*], A and B [*not H and C*]. 22.9%

A or not B [*not H or not C*], A or B [*not H or C*], not A or B [*H or C*]. 14.3%

A $\Rightarrow$ not B [*not H* $\Rightarrow$ not C], A $\Rightarrow$ B [*not H* $\Rightarrow$ *C*]. 48.6%

B) Exam 3, April 5: *How can this law be broken? "Don't drive if you're intoxicated."*

Correct answers: **71.4%**

You're intoxicated and you drive. **[*H and not C*]** **34.3%**

Driving while/when intoxicated. **[*not C and H*]** **31.4%**

Driving intoxicated. [*not C and H*] 5.7%

Typical incorrect answers:

You cannot drive and be intoxicated. [*not (not C and H)*] 2.9%

Drive if you're intoxicated. [*H* $\Rightarrow$ *not C*] 17.1%

Don't drive if you're not intoxicated. [*not H* $\Rightarrow$ *C*] 5.7%

Not H would be drive [*defined negation of C*] 2.9%

C) Exam 4, April 26 (after studying quantifiers)

Negate and put in positive form: "All who wander are lost."

(Students knew that "positive form" requested that the "not" be eliminated or, at least, moved inside the statement as far as conveniently possible.)

Correct answers: [[$\exists$*x st. H and not C*] 31.3%

There exists a wanderer who is not lost. 28.1%

There is one who wanders and knows where he is going. 3.1%

Less than fully correct answers: 28.1%

Not all who wander are lost. [*not (H* $\Rightarrow$ *C): not simplified as requested.*] **21.8%**

You wander and are not lost. [*H and not C*: *no quantifier given.*] 3.1%

There exists one that is not lost. [$\exists$*x such that x* $\Rightarrow$ *not C*] 3.1%

Incorrect answers: 4 0.6%

There exist people who are lost but do not wander. [$\exists$*x s.t. C and not H*] 3.1%

All who wander are not lost. [*H* $\Rightarrow$ *not C*] 28.1%

Not all who wander are not lost. [*not (H* $\Rightarrow$ *not C)*] 6.2%

Not all who are lost wander. [*not (C* $\Rightarrow$ *H)*] 3.1%

D) Final Exam, May 9: *Here's a sentence: If x* $\in$ *S, then f (x) = 0. What is its negation?*

Correct answer: $\exists$x such that x $\in$ S and f(x) $\neq$ 0. [$\exists$*x st. H and not C*] 5.9%

Less than fully correct answer: x $\in$ S and f(x) $\neq$ 0. [*H and not C*] 29.4%

Incorrect answers:

If x $\in$ S, then f(x) $\neq$ 0. [*H* $\Rightarrow$ *not C*] 38.2%

If x $\notin$ S, then f(x) $\neq$ 0. [*not H* $\Rightarrow$ *not C*] 14.7%

If x $\notin$ S, then f(x) = 0. [*not H* $\Rightarrow$ *C*] 2.9%

Other answers 8.8%

ANALYSIS

For each question, each student's response was examined to determine its underlying logical form. This sorting revealed six general Answer Categories.

- (6) mathematically correct,
- (5) "close" response having only a minor error,
- (4) incorrect response that included the connective "and,"
- (3) incorrect response that included the connective "or,"
- (2) "if, … then" statement,
- (1) incorrect response that did not use a logical connective.

The numerical designations for the six categories reflect a subjective measure of the mathematical "correctness" of a student's response. The "close" category is only relevant for the last two problems (*C* and *D*) because these questions are mathematical generalizations, and their negations must be stated as existence statements. For these questions, student responses were categorized as "close" if they were in the form "H and not C" but were not written as existence statements. A response for question *C* was also considered "close" if it was written in the form "not (H □ C)." That is, the negation operation was stated but not carried out. Table 1 shows the percentage of responses classified by Answer Category for each question.

Answer Category □	6	5	4	3	2	1
Abstract: A	11	0	23	14	49	3
Natural Language: *B*	71	0	3	3	23	0
Natural Language: *C*	31	28	3	0	38	0
Abstract: *D*	6	29	0	0	56	9

Table 1. Percentage distribution of Answer Categories for each question

FINDINGS

Two research questions were addressed. 1) Does the distribution of student responses vary with the form and content of the question? 2) How does the distribution of responses vary over time? In other words, did some types of questions present more difficulty to students, and was there improvement in their measured knowledge as the course progressed? In order to address these questions, the distributions of the different types of student responses were compared across problems, and differences in the form and content of the four problems were characterized.

Distribution of Student Responses

At the level of the entire class of students, a comparison of the distribution of responses shown in Table 1 reveals a "quick" answer to the two research questions. More students did well on the two natural-language questions than on the other two problems. For the natural-language problems *B* and *C*, over half of the students wrote correct or nearly correct answers (71% and 59% respectively). In contrast, for the more abstract questions *A* and *D*, only 11% and 35% wrote correct or nearly correct answers.

Overall, the improvement in the percentage of correct and "close" (good) responses over time was a function of the type of question being asked. Responses to the two natural-language questions fell from 71% (*B*) to 59% (*C*) between Exam 3 and Exam 4. On the other hand, the percentage of good answers to the two abstract questions improved from 11% (*A*) on Exam 3 to 35% (*D*) on the Final Exam. The fact that the students improved on the abstract questions but not on the natural-language questions suggests that their responses to the natural-language questions continued to be based throughout the course on contextual sources of information. The natural language contexts appeared to overwhelm any gains that the students made in their study of logic.

Characterization of Problem Form and Content

A comparison of the four questions examined five aspects of problem form and content; 1) the symbol system used to express the logical statement, 2) the context of the statement, 3) the level of abstraction of the objects symbolized, 4) whether the underlying logical form was implicit or explicitly given in the statement, and 5) the complexity of the problem related to a straightforward "translation" of the logical form into that particular context. These aspects are summarized in Table 2. The total percentage for the correct and close answers is shown for each question.

Formal, Abstract Symbol Systems	Natural Language, Experience-Based
A: *Simplify: not* [*(not A)* □ *B*] (11%) Symbol system: Mathematical logic Context: Logical objects Level of abstraction: High – Objects are abstracted as letters, operations as symbols Form: Explicitly stated Complexity: Moderate – the hypothesis is given as "not A"	*B*: *How can this law be broken?* (71%) *"Don't drink if you're intoxicated."* Symbol system: Natural language Context: Rules of driving Level of abstract: Low – experiential world Form: Somewhat implicit – "If" present in statement *Complexity: Low /context - rules are familiar phrases. Moderate / form – hypothesis, conclusion in reverse order*
D: *Here's a sentence: If x* □ *S, then f (x) = 0. What is its negation?* (35%) Symbol System: Algebra Context: Sets and functions Level of abstraction: Semi-abstract - but algebraic objects are generalizations Form: Explicitly given by the structure "if…then" - implicit generalization Complexity: Moderate –objects in the hypothesis and conclusion are abstract generalizations	*C*: *Negate and put in positive form:* *"All who wander are lost."* (59%) Symbol system: Natural language Context: Adage-like statement Level of abstract: Low – experiential world Form: Implicit logical form, generalization explicitly expressed by "all" Complexity: Moderate – negation difficult to express in fluent English

Table 2. Form and content analysis

It can be seen that, as a class, the students tended to do less well on the two questions that were expressed in a formal, abstract symbol system, even though in these questions, the logical form under examination was explicitly available to the students. It is interesting to note that the most difficult question (*A*) and the easiest one (*B*) were sequential problems on the same exam. These two questions epitomize the key factors that appeared to affect problem accessibility – abstract logical form and contextual familiarity.

Negation

The use of abstract patterns or familiar context also factors into the mechanics of generating an answer to a given question. The ability to produce the correct response consists of two parts – knowing how to represent the negation of either the given hypothesis or conclusion, and also knowing the correct logical form to use with this negation. An examination of the students' responses shows that the symbol system in which the conditional statement was expressed influenced the students' abilities to successfully create an appropriate negation.

Negating the component expressions in the conditional statement appeared easier for the students to express in the abstract symbol systems than in the natural-language statements. These differences may be the result of the lack of ambiguity inherent in the abstract systems, as compared to the variety of English phrases that can be used to express a single thought.

In the two abstract symbol systems (logical form and algebra), students negated objects by simply placing "not" in front of a letter or by drawing a slanted line through the symbols "□" and "=". In contrast, negating several of the natural-language phrases appeared to present problems for some students. In question *B* it was much easier to talk about "driving intoxicated" than it was in *C* to describe wandering while "not lost." In *B*, only one of the students (2.9%) stated the negation but did not carry it out, writing a response in the form "not (not C and H)". In contrast, ten of the students (31.1%) wrote similar forms for question *C*; "not (H □ C)," "not (H □ not C)," and "not (C □ H)". None of the students gave such responses for the two abstract questions.

CONCLUSIONS

There is not enough evidence in the data analyzed for this particular study to draw firm conclusions. Our findings do support Devlin's (p. 118, 2000) claim that "people reason much better about familiar, everyday objects and circumstances than they do about abstract objects in unfamiliar settings, even if the logical structure of the task is the same." In addition, the potential impact of our findings on the way that mathematical logic is taught are sufficiently significant that further study is called for. Our results suggest the following interpretations.

One way to make sense of the variations in the students' responses is to consider the kind of knowledge that the students might have been applying as they answered each question. From this perspective, it can be conjectured that, for many students, the responses to the abstract questions *A* and *D* were based on their ability to memorize or learn a particular abstract pattern of logical symbols, while the answers to the natural-language questions *B* and *C* were created from the students' knowledge of experiential relationships. The wide variation in the answers to question *A*, which was expressed in pure logical form, suggests that many students had not yet mastered the appropriate pattern for this form. Yet, these same students were able, on the very next test question (*B*), to produce a logically correct statement through their knowledge of the rules of driving. We suggest that these differences in responses are linked to the way that the conditional statement is expressed either by abstract patterns or within a familiar context.

It appears that the study of the forms of mathematical logic during the second half of the course had an impact on the students' ability to correctly respond to abstractly stated conditional statements, but not to such statements when they were written in a natural-language context. As was noted, aspects of the natural language may have had more impact on the students' inability to produce a correct response than did any lack of understanding of logical form.

At issue, is whether and to what degree the four problems can be considered to represent examples of mathematical logic. Does a knowledge of the forms of mathematical logic reside only within symbolic systems, or can its applications be found in everyday experiences expressed in natural language? The percentage distribution of Answer Categories across the four problems and the form and content analysis of each question suggest instructional and assessment challenges. It appears that different kinds of knowledge are being tested across the set of problems and that this knowledge is related to the nature of the symbol system or context in which each question is posed.

Gregg (1997) wrestled with this question in his analysis of a classroom episode from a tenth grade geometry class. Even when informed that the statement, *If a* student *likes geometry, then he or she will pass,* was assumed to be true, the students assessed the truth of the resulting contrapositive, *If a student doesn't pass, then he or she does not like geometry,* on the basis of personal opinion rather than on logical equivalence. To the students, the statement was meaningful on what Gregg termed a "semantic level." Gregg (p. 547) concluded that the use of "real-life" examples may "actually contribute to difficulties in the traditional approach to teaching conditionals and the notion of logical equivalence."

In this report, the detailed examination of the students' responses, in conjunction with the form and content analysis of the four statements, highlights the complex task of developing appropriate concept images for topics of mathematical logic. There is more to this task than simply mastering a set of truth-table definitions or sequences of logical patterns. Logical form needs to be encountered in many contexts, with explicit links made between the form and the, sometimes overriding, focus of a given context. In addition, care must be taken in the kinds of inferences that are drawn from particular types of assessment questions. It is important for teachers to be aware of the alternative sources of knowledge that students may bring to bear when confronted with particular types of examples.

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THE ROLE OF REPRESENTATION IN TEACHER UNDERSTANDING OF FUNCTION

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It is becoming widely recognised that teachers' content knowledge has an important influence on their pedagogical content knowledge, and hence on the learning of students. In secondary schools function is one of the fundamental concepts of mathematics. This paper considers the understanding of function exhibited by a group of teacher trainees in response to various representational presentations. The results show that there is a wide range of differing perspectives on what constitutes a function, and that these perspectives are often representation dependant, with a strong emphasis on graphs.

INTRODUCTION

The idea that a teacher's content knowledge base will influence the quality of the understanding that students develop in an area of mathematics has received support from research findings (Ball & McDiarmid, 1990) and curriculum reform documents (National Council of Teachers of Mathematics, 2000). This not particularly surprising since one might expect both lesson goals and structures to be contingent on teacher understanding of the subject matter. After introducing the concept of pedagogical content knowledge for these types of activities by teachers, Shulman (1986) and, later Leinhardt (1989), went on to link the development of mathematics teachers' content and pedagogical content knowledge, suggesting the existence of links between content knowledge and explanations and representations generated during teaching. Confirmation of this was found in a study of the effects of content knowledge in algebra by Menzel and Clarke (1999, p. 371), which noted that teachers with a weak content knowledge "lacked the detailed knowledge needed to both identify specific student difficulties and construct situations that might assist students to overcome their difficulties." The extent of the difference which teachers can make is shown in the large-scale study of Sullivan and McDonough (2002). Their conclusion about the influence of teachers on learning was that "The data presented here suggested that the differences [in improvement of student learning] between the most effective and least effective teachers are substantial." (p. 255).

Function is a fundamental concept of school mathematics and hence, a teacher's content knowledge of function is likely to be crucial to providing a positive learning environment for much of secondary school mathematics. Even's (1998) research with college students emphasised the importance of representations in understanding of function, finding that students had difficulties in flexibly linking different representations and finding a link with pointwise and global approaches to function problems. Indeed in a research project where the relationship of an experienced teacher's conceptions of function to his practice was examined, Lloyd and Wilson (1998) found that he valued explorations of multiple representations of problem situations and that these offered students increased opportunities to understand. Furthermore, his content knowledge structures did influence his teaching, so that the "teacher's comprehensive and well-organised conceptions contribute to instruction characterised by emphases on conceptual connections, powerful representations, and meaningful discussion." (p. 270).

The contrast between the simplicity of the function concept and the complexity of its manifestations, and the concept images it may evoke (Vinner, 1983) has been well described by Akkoc and Tall (2002). They found that some students were unable to see and apply the fundamental (simple) definition of function, instead relying on almost arbitrary aspects of examples they focussed on. Comparing the function concept maps of eight professors having PhDs in mathematics with those of twenty-eight university mathematics students, Williams (1998) found that the student maps portrayed an emphasis on minor detail, such as the variable used, algorithms, and the idea that functions are equations. In contrast she found that "none of the experts demonstrated the students' propensity to think of a function as an equation. Instead, they defined it as a correspondence, a mapping, a pairing, or a rule." (*ibid,* p. 420). Chinnappan and Thomas (2001) found that their trainee teachers had a strong tendency to think of functions graphically and procedurally, separating algebra from functions (which they saw as graphical) in their thinking, and displaying gaps in their knowledge of function.

Hence this research sought to understand further prospective teachers' thinking about functions and its relationship to function representations and the formal concept.

METHOD

This research comprised a case study analysis using a group of thirty-four pre-service secondary mathematics teacher trainees at The University of Auckland. The teacher training at this institution is a graduate programme and so all had a degree with a substantial component of mathematics and had done some teaching. Each of the teachers was given a questionnaire comprising 13 questions. In each case they were presented with either an algebraic, graphical, ordered pair, or tabular representation and were asked to say whether or not it could be seen as a way of representing a function, giving a reason for their answer (See Figure 1 for a summary of the representations). Some 'grey' areas and other key properties of functions were deliberately targeted in the questions. Hence the choice of representations presented to the teachers included the issues of values where the function is not defined (e.g., Q1), the lack of an explicit statement on domain (e.g., Q6, 8), absence of information on whether y or z is a function (e.g., Q4, 8), and acceptance of a function of two variables (Q9), etc.

Another specific purpose behind the choice of these representations, apart from a consideration of the role of the representation itself, was to examine the level of the perceived need for an equation or formula with two explicit variables in order to have a function. During informal discussions with teachers, many were not happy to describe the algebraic form x^2 as a function, stating that it had to be written as $y=x^2$ before it could be a function. However, it emerged that they were comfortable with seeing forms such as e^x, $\sin x$ and $\cos x$, etc as functions. The idea of requiring two variables appears to arise from the requirement in the formal definition of function of specifying the domain and co-domain before the relation or 'rule' defining the function is presented. The motivation in the research was to try and understand and document aspects of the subjects' concept image of function, and how it is influencing their mathematical thinking, and hence their teaching.

1. $f(x) = \dfrac{e^{-x}\ln(1+x)}{\sin 2x}$

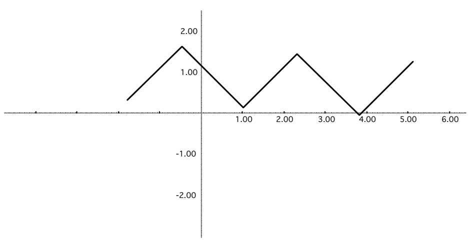
2. $(x, 2x)$, where x is a real number

3. $\begin{pmatrix} p & 0 \\ 0 & 0.5 \end{pmatrix}$, $p \in Z$

4. $\dfrac{d^2y}{dx^2}$

6. {(1,3), (2,5), (3, 4)}

5.



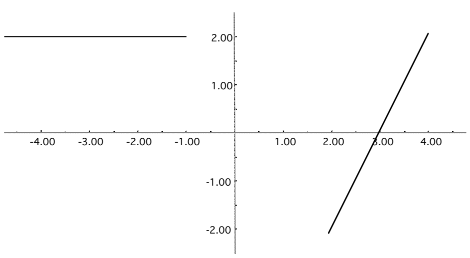
12.

x	y
1	9
2	16
3	25
4	36
5	47

7. $g(y) = \begin{cases} 1 & y \text{ rational} \\ y & y \text{ irrational} \end{cases}$

8. $\lim\limits_{h \to 0} \dfrac{f(z+h) - f(z)}{h}$

10.



9. $(a, b) \to a + b$

11. $\displaystyle\int_1^x \frac{1}{t}\,dt \quad x \in R$

13. $\{(x, y) \mid x \in \mathrm{N}, y = 1\}$

Figure 1. The representational formats used in the questionnaire.

RESULTS

As may be seen from Table 1 there was no question on which the teachers were unanimous about whether the given format represented a function or not. In every case except question 3 (which was split 41% Yes, 47% No) there was a majority considering that it was a function, although in many questions there was a significant minority, between 23.5% and 29%, disagreeing. If the teachers' decisions were based solely on the application of a basic function definition then one would expect a more tightly positioned distribution, with values something like those in Q5, Q10, and Q12.

However, on closer inspection it became clear that as well as the lack of unanimity, there were also some interesting reasons given for the answers, as noted in the responses below. As expected, based on the example x^2 mentioned above, some of the replies indicated that the representation could not be a function unless there was an equation, or

Table 1: *Number of Teachers Describing Each Representation as a Function*

Question Number	Response to Whether a Function or Not (N=34)		
	Yes	No	No Answer
1	23	9	2
2	26	8	0
3	14	16	4
4	21	12	1
5	30	4	0
6	24	10	0
7	21	9	4
8	21	13	0
9	16	11	7
10	29	3	2
11	21	8	5
12	30	4	0
13	19	10	5

two variables explicitly given. For example, responding to Qs 2 and 6, which had ordered pairs, T26 (teacher 26) wrote "no variable" as the reason for it not being a function. However, for Q13 which also has ordered pairs but in a set format, he was happy to accept it as a function "mapping y=1 for all $x \in N$."

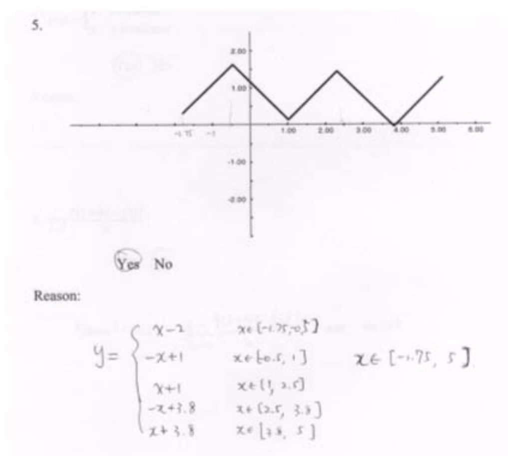
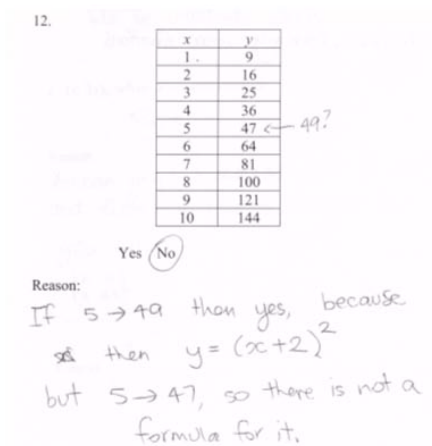


Figure 2. Need for formulas to define a function.

Similarly, T8 wrote that Q2 was not a function, because it "only contains x values", yet was happy to describe Q11 as a function, since "each x value has a corresponding y", even though there was no y given in the representation. For Q7 T1, explaining why it is not a function wrote "It is not a formula or relation of 2 or more variables but it gives only one variable and the value of the variable is determined by a number and itself." Thus she did not see the notation $g(y)$ as sufficient for specifying the range, but wanted another explicit variable in the equation. T3 took the idea of wanting an equation for the function to extremes in Q5 providing a formula for each of 5 sections of the split domain function (see Figure 2). In Q13, where two variables were given but in the context of an ordered pair representation, T7 said that it was not a function because "It is merely a set of 2 values, and they do not affect the value of others." However, for Q6 she said that it was a function because in the given domain "Each x has a corresponding y value." This may indicate that it was not the ordered pair representation that was the problem for her in Q13 but the format or the context of the notation.

The tabular representation in Q12 proved to provide a context in which a number of the teachers were looking for a formula that would fit the table. However, it had been arranged that the value of y at x=5 was 47 not 49 in order to ascertain how they would cope with this if they wanted a formula. While some teachers, such as T1, T3, and T26 either 'corrected' it to 49, assuming it was an error, or missed or ignored it, so that it fitted the formula y=(x + 2)2 that they had modelled to the data, others such as T28 said it was not a function because "What's the eqn [equation] used to get the value of y?" T19 also thought that it was not a function unless the 47 was made into 49 so that a "formula for it" could be modelled (see her reasoning in Figure 3). Hence the table clearly evoked the concept of a formula or equation for function in a number of the teachers, supporting the observation of Williams (1998).


12.

x	y
1	9
2	16
3	25
4	36
5	47 ← 49?
6	64
7	81
8	100
9	121
10	144

Yes No

Reason:

If 5 → 49 then yes, because

then $y = (x+2)^2$

but 5 → 47, so there is not a

formula for it.

Figure 3. Algebraic formula/equation needed for a tabular representation of function.

In contrast, eight of the teachers used an argument for Q12 based on some form of reasoning to do with mapping of variables, more akin to the experts in Williams' (1998) study, to establish it as a function. These were generally more successful in arguing for a function, making comments such as: "Every element *x* in the domain is linked to only one element in the range." (T6); "each *x* has a corresponding *y*" (T8); "Each *x* value has only one *y* value." (T14); "each *x* value has only 1 corresponding *y* value." (T20); "One *x* only one *y*." (T24); "There is a relation that maps each value of *x* to *y*" (T30); and "Each *y* is different. So for each *x* there is at most one *y* i.e. it's a function." (T16). While these are not all complete arguments (e.g., saying both that every *x* has a value and each *x* has only one *y* value), or have extraneous detail (e.g., 'each *y* is different') they *are* using aspects of an informal definition. However, they did not always correctly apply this line of reasoning based on the fundamental definition of function, with more emphasis being placed on the mapping nature than on the actual formal definition. For example, T9 wrote

"For every $f(x)$ there is an x value", which makes it an onto relation, but not necessarily a function.

Not surprisingly the table representation in this question was linked to a graph by several teachers. T2 and T29 stated that it was a function because it was a "list of points in an x,y plane" and "Because this table of values can be plotted on a set of axes". They seem to have constructed the idea that any planar graph drawn from a table represents a function. The graphical link also evoked the erroneous consideration that the function must be continuous, with T23 saying that it was a function because "there exists such function such that connects all of these points" and he drew a continuous graph. This is no doubt linked to the idea of getting a formula or equation to model the data, since in their experience most of the graphs of algebraic functions (often polynomials) would have been continuous. Nine of the teachers gave no reason for their answer on Q12.

The concept that we have a function if a graph can be drawn arose in other questions too. T8, for example, wrote that Q4 was a function because "you can draw a graph" and gave similar reasons for Q10 and Q12 (where the graphs had been drawn). It may be that a lack of experience of graphs of non-functions has caused them to forget that not all 2-dimensional graphs represent functions.

Some students were very strongly constrained to a graphical perspective on function, reducing most examples, where they could to a graph. T33, for example, attempted to use the 'vertical line test' wherever she could, since this was clearly the dominant idea of function for her. So for Q2, Q5, Q9, Q10, and Q13 she employed this method, stating in Q2, "increasing straight line cuts vertical line test." In Q13 (see Figure 4) she managed to draw the graph with discrete points (a 'point graph' as she calls it) and see that any vertical line would cut it at most once. However, there were some problems applying this test, and she struggled to use it for Q3, stating "I'm not sure what this vector looks like on 2 dimensional plane □ therefore not a function."

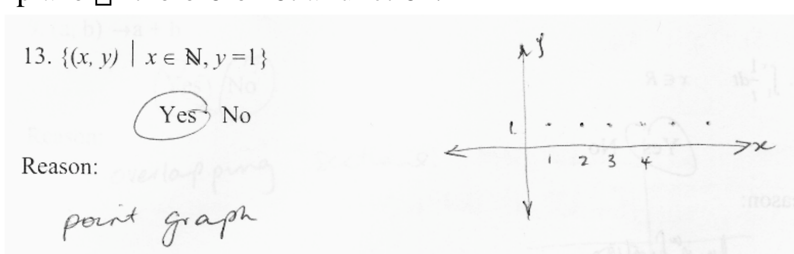


Figure 4. Relating an ordered pair and a graphical representation.

A reliance on graphs was also seen in the answers of T20, who drew graphs for Q2 and Q6 (see Figure 5) but wrestled with questions such as Q3 (not answered, and a '?' placed by it), Q7 and Q9 (both not answered) where the graphs proved too difficult. It is interesting that in Q2 he linked the ordered pairs given (in terms of x) to a graph and then to the equation $y=2x$ in order to make a decision. However, in Q6 he linked the ordered pairs (without the x present) to a graph and then used reasoning based on a correspondence between variable values. In each case he accepted it as a function. Clearly the portion of the function concept image evoked varies subtly depending on slight variations in the representational content.

T20 was among 4 of the teachers (T1, T16, T20 and T22) who thought that the graph in Q10 was not a function since it was discontinuous.

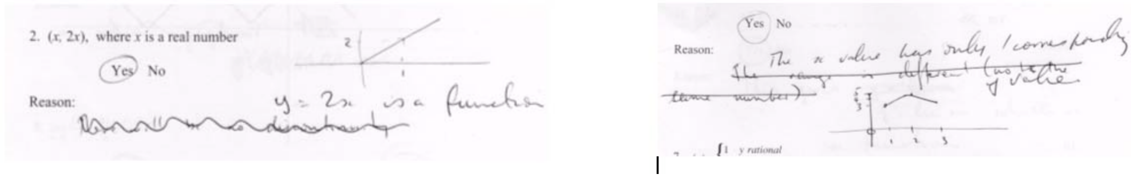


Figure 5. Use of graphs leading to differing reasoning to decide on function status.

No teacher mentioned the fact that the function in Q5 was not differentiable. This may be because they thought it irrelevant or that they did not consider it at all.

There was some evidence of a procedural approach to the learning of function among these teachers. Attempting to learn in an instrumental manner can easily lead to errors. In one case T29 had tried to learn the vertical line test for a graphical representation of a function, but had mis-remembered it as a horizontal line test. As Figure 6 shows she used it in Q10 to say that it would not represent a function "if a horizontal line crosses both lines", and so was not a function.

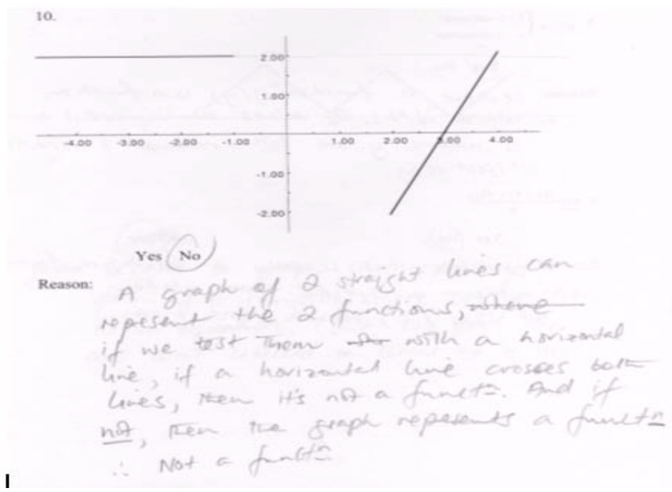


Figure 6. Instrumental learning of a rule goes wrong.

CONCLUSION

There is no doubt that for some of the teachers the questionnaire highlighted the shaky nature of their hold on an understanding of function, including some principal elements of the function concept missing from their concept images, and this probably led to the lack of unanimity. Supporting this, when the teachers were invited to make a general comment at the end of the questionnaire, T9 wrote 'I suppose I realised how unsure I am about what makes a function a function." and T29 added "Thanks for reminding me what a function is. But I still couldn't remember what it is exactly."

The most success in terms of giving supportable reasons for whether the representations were of functions came from those who, like the experts in Williams' (1998) study based their thinking on relationships or mappings between values of variables, even though they did not directly refer to a formal definition of function. There is also some evidence here that rather than seeing function as a concept that crosses representational boundaries some of the trainee teachers are engaging in shifting their concept image focus depending on the representation they are engaged with, as reported by Lauten, Graham and Ferrini-

Mundy (1994). Throughout the questions the graphical perspective had a strong dominance for a number of the teachers. While visual imagery is often a useful asset for assisting mathematics learning, Aspinall, Shaw and Presmeg (1997) have described how an uncontrolled use of it can have negative implications for application of functions in calculus, and Chinnappan and Thomas (2001) present examples of this too. It would seem that the data presented here support the view that for many teachers the graphical representation of function is becoming dominant to such an extent that it could hinder a growth in inter-representational understanding. Certainly the teachers, and hence their students, would benefit from development of stronger inter-representational thinking about function.

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EFFECTIVE TEACHING WITH VIRTUAL MATERIALS: YEARS SIX AND SEVEN CASE STUDIES

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This paper describes an approach to mathematics teaching and learning using teacher-constructed "virtual" materials. In this approach, virtual copies of real materials are manipulated with mouse movements to replicate the traditional physical manipulation of real materials. The paper reports on three case studies where the approach was used to teach different mathematics topics (e.g., equivalent fractions and decimal fraction scales) to a Year 6-7 and a Year 7 class. It describes and analyses the virtual materials in relation to how they relate to prior teaching with physical and pictorial materials. It speculates on the strengths and weaknesses of the approach, and the effect differences between physical and mouse manipulations have on learning.

Over the last 10 years, educational authorities have invested billions of dollars placing computers in classrooms with the expectation that integration of technology and instruction would enhance learning outcomes by changing the nature of teaching and learning in terms of content, delivery, teacher-student interactions, and the roles of both teachers and students. However, many teachers who have not grown up with computer technology have developed high levels of stress (technophobia) when faced with a teaching future that is inexorably leading to the integration of learning technologies (e.g., Morton, 1996). These teachers have difficulty accessing mathematics software that suits their students' needs (Becker, 1994). However, of more concern is the prevalent belief that mathematics cannot be taught effectively with computers (Sarama, Clements & Jacobs-Henry, 1998; Norton, 1999).

To help teachers overcome their technophobia and beliefs about the efficacy of computers in mathematics instruction, the authors have established a Virtual Mathematics Program in which they work collaboratively with teachers in their classrooms helping them construct and implement their own mathematics activities using virtual copies of concrete materials. This paper explores three examples of this use of virtual materials.

Virtual materials and their role in teaching. As argued in Baturo & Cooper (2001), most activity with real or concrete materials in mathematics involves *sliding, joining, separating, grouping, ungrouping, partitioning, turning* and *flipping* actions. All of these actions are available on computer through mouse movements and images of the materials ("virtual materials") using the commonly available generic "office" software (e.g., *MicroSoft* Office, *ClarisWorks*). As described in Baturo and Cooper (2002), virtual activities reflect a variety of options. They can be simple "click and drag, copy and paste" activities (e.g., representing numbers with MAB) through to ones that have capacities for actions and representations not easily available with concrete materials (e.g., modifying the polygon shapes). In particular, shapes can be enlarged by specific amounts, or turned by specific degrees. Some virtual activities have mouse actions that closely imitate the physical actions with real materials (e.g., sorting shapes, turning the hands of a clock). Others have mouse actions that are very different and nowhere near as richly kinaesthetic as the physical actions (e.g., flipping a shape).

From a teaching perspective, virtual materials activities can be "debugged, reconstructed, transformed, separated and combined together" (Healey & Hoyles, 1999, p. 59) and saved for later reuse by the same or other students. As well, virtual activities enable students' manipulations to be saved and stored for later assessment, providing teachers with unique knowledge of all students' proficiency with all components of the manipulations. The strength of a teaching approach that builds virtual materials into its repertoire of activities is that it is multi-representational (providing visuals, language & symbols) and dynamic (showing transformations and changes as well as relations). In this way, as Healy and Hoyles stated, virtual materials use the visual, symbolic and operational power of the technological media and provide another pedagogical and didactical tool for the media. Initial findings from the trials in the Virtual Mathematics Program (Baturo & Cooper, 2001) are indicating that, for technophobic teachers, virtual materials provided a bridge from the acquisition of computer skills to the implementation of classroom activities; the teachers found virtual activities easy to develop, did not require specialist software, and promoted positive learning outcomes. It seems as though, because virtual activities have comforting similarities to concrete activities, teachers are more able to recognise opportunities for translating their traditional teaching activities to computer activities. Furthermore, in every class trialed thus far, the teachers have been impressed by their students' excitement, prolonged engagement, and natural collaboration that have been provoked by the virtual activities.

Teaching of mathematics. Mathematics consists of things, relations between things, and transformations of things (Scandura, 1971). Within this paradigm of mathematics, importance lies in the relations and transformations not in the things; yet, within our research experience (Baturo & Cooper, 2001; 2002), teachers tend to focus primarily on the "things" and neglect or downplay transformations that often give rise to patterns and therefore relationships. Mathematics learning is about the refinement, abstraction, and integration of concepts and processes, and mathematics teaching is about facilitating this process of refinement, abstraction, and integration. Current pedagogical beliefs emphasise that the abstraction of concepts and processes is best served by a combination of work with appropriate manipulatives and reflection with peers and teacher (English & Halford, 1995).

According to Halford (1993), understanding mathematics involves representing one mathematical structure by another and determining what is preserved and what is lost between the structures. Students translate *external representations* (concrete, virtual, pictorial, diagrammatic, written symbols, spoken words) to *internal representations* (e.g., mental models/perceptions of the external representations). Similarly, they store the kinaesthetic actions (physical and mouse movements) undertaken to represent the relations and transformations in memory. Thus, the use of external representations (real and virtual) should provide a mental image to scaffold the concomitant concept development and abstract symbolism. The current mathematics syllabus and curriculum documents for Queensland schools draw heavily on Payne and Rathmell's (1975) model of concept development that has three main components (representations, language, symbols) and six concomitant interactions, all of which they claim are essential for full concept construction. Whilst all three components can be thought of as different representations of the concept, it is useful to separate them when planning teaching and learning activities. Figure 1 adapts the model to include a continuum of materials with respect to degree of abstraction.

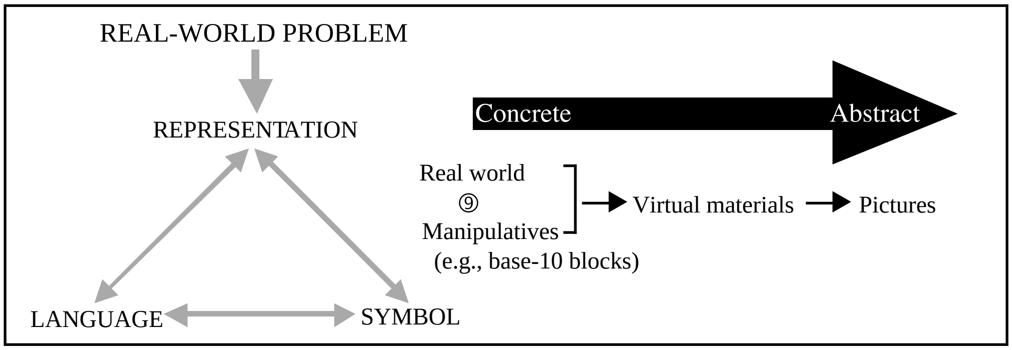

Figure 1. Adaptation of Payne and Rathmell's (1975) model components and interactions required for concept construction (Baturo & Cooper 2002)

Noss, Healy and Hoyles (1997) contended that, although students can mentally replicate (in their schemas) the relations and transformations represented by concrete material and abstract this mental replication to symbols and mental models, *there is a gap between action and expression that is difficult to bridge*. Baturo and Cooper (2002) argued that virtual materials are more abstract than concrete materials but less abstract than pictorial representations and therefore are able to help *bridge the gap from concrete to pictorial representations and, then, to abstraction*. Concrete materials are less abstract than virtual materials because they are multisensory (i.e., they can be seen, smelt, moved, picked up, touched, weighed) whilst real/concrete materials are essentially bisensory (seen and moved). They argued that, although the multisensory nature of concrete materials may develop more detailed memory structures (schema), for example, a tactile memory, the more abstract bisensory virtual materials develop deeper mathematics understandings.

CLASSROOM STUDY

The virtual material activities reported in this study were trialed with a Year 6-7 and a Year 7 class in a state primary school in a regional city in Queensland. The school's students were predominantly from low socio-economic backgrounds. The school's performance in mathematics was low even when compared to schools of similar background.

The trial of the virtual materials was part of a collaborative action research project (Kemmis & McTaggart, 1988) in which University mathematics-education lecturers and teachers worked together to improve the teachers mathematics-teaching practice in terms of enhancing students learning outcomes. The collaboration involved the development of units of mathematics instruction that represented exemplary practice. Therefore, the units began with students exploring physical materials to develop the required concepts, processes and principles and then moved on to virtual materials.

The Years 6-7 and 7 classrooms contained nine computers linked to the Internet (three in the Year 6-7 classroom and six in the Year 7 classroom). The teachers of the Years 6-7 and 7 classes, while not technophobic, had limited knowledge in how to use computers to teach mathematics. They, therefore, made little use of the computers for mathematics teaching and, in fact, for any teaching. As the teachers admitted later, the computers in their classroom were used for less than one hour a day.

The two teachers were experienced with good skills in teaching mathematics in a traditional way. They had excellent behaviour management and relationships in the classroom appeared to be based on mutual respect. The Year 6-7 teacher relied more heavily on textbook pages for instruction than the Year 7 teacher. The Year 7 classroom was more used to class discussions than the Year 6-7. For the action research collaboration, he two teachers chose to work as a pair, undertaking the same instructional units, planning together and sharing the development of new materials.

INSTRUCTIONAL UNITS

There were three instructional units in which virtual materials were used.

Unit 1: Equivalent fractions. The two teachers developed a unit on fractions as a prerequisite to a major unit on percent. Part of this fraction unit was the introduction of equivalent fractions. This was done through manipulating real world materials (e.g., chocolate bars, cakes), physical materials (e.g., paper folding showing one-half is the same as two-quarters), pictures (e.g., fraction mats composed of length representatives of halves, thirds, quarters, fifths, and so on aligned so that equivalence can be easily seen vertically) and patterning materials (e.g., fraction sticks – Popsicle sticks with numbers placed regularly as in Figure 2). The virtual materials used were a copy of the fraction sticks. Equivalence was first introduced as a capacity (i.e., it is possible for one-third to be equal to two-sixths) and then as a pattern (i.e., six-ninths is equivalent to two-thirds because both the numerator and denominator are three times larger than for two-thirds).

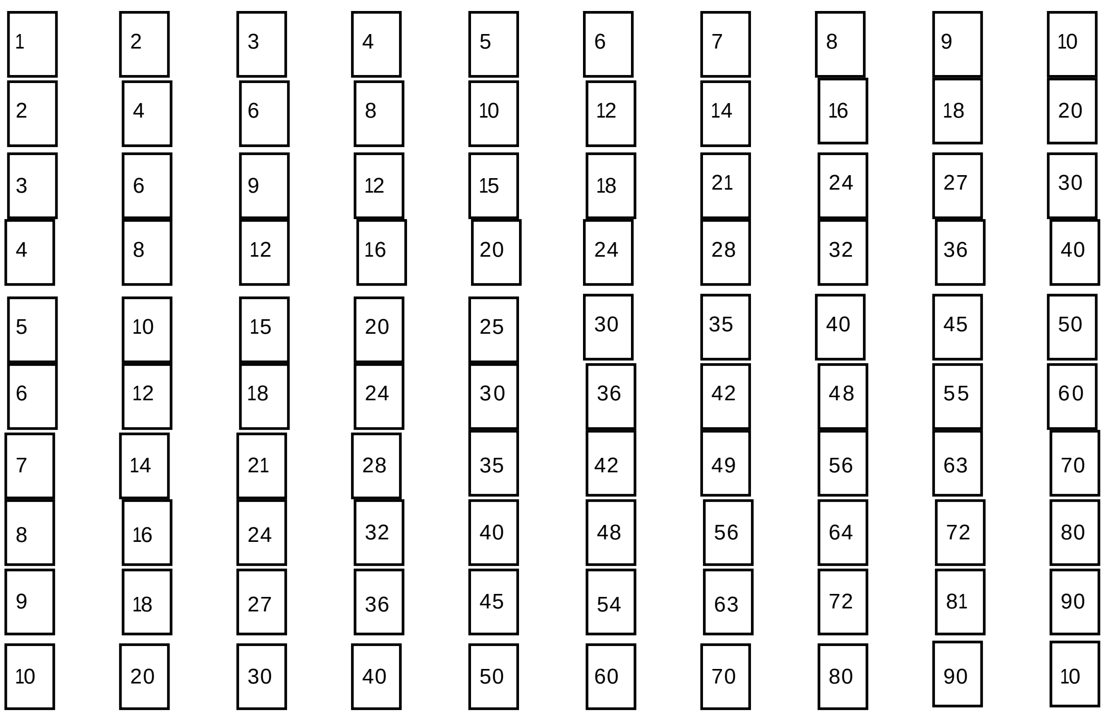

Figure 2. Fraction sticks

The fraction sticks provide the final step in the learning process for equivalent fractions. A representation for, say, two-fifths consists of the 2 stick (naming the stick by its left-most number) placed above the 5 stick. All the fractions equivalent to two-fifths are then displayed for equivalence-relation pattern to be identified, and for the fraction to be compared with, added to or subtracted from another fraction of unlike denominator (as in Figure 3).

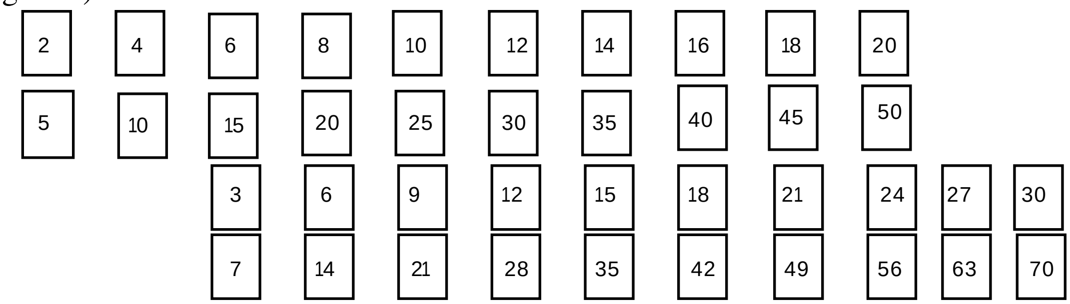

Figure 3. Comparing two-fifths and three-sevenths with fraction sticks

Unit 2: Scales. Both Years 6-7 and 7 students were performing poorly on items that required the reading of scales, especially if there were decimals to be inferred. A unit of instruction was developed that involved the students constructing and interpreting scales. Instruction moved from *complete whole-number scales* with all numbers shown to *incomplete whole-number scales* with partitions not numbered and gradations every fifth or second unit to *incomplete decimal scales*. At the end, virtual materials were used in which positions on incomplete decimal scales could be interpreted and constructed. These attempted to emulate real world instances of scales (e.g., measuring cylinders, odometers, syringes).

Unit 3: Slides, flips and turns. Both Years 6-7 and 7 students had experienced restricted space activities. They had not yet been taught any transformational geometry concepts and processes. A unit of instruction to introduce translations (called *slides* in the Queensland mathematics syllabus), reflections (*flips*) and rotations (*turns*) and to study their properties was developed using tracing paper and Miras as the physical material. This was followed by a series of virtual activities that used mouse movements and the Draw toolbar to practice flips, slides and turns. The virtual activities also used the flips, slides and turns to develop art designs.

RESULTS

The action-research collaboration between lecturer and teacher was based on Lesh and Kelley's (2000) multi-tiered approach where the teacher acts as the major researcher with respect to students' outcomes. The following results with respect to the three units are, therefore, based on accounts given by the teachers on students' responses to the virtual materials. The teachers reached their positions on the virtual materials by observing the students on the computers, talking to the students about their perceptions of the materials after computer use and detecting changes in understanding from tests, homework and textbook activities. The students attempted the virtual materials through a rotation system that enabled students to use computers in turn. They worked on the computers in pairs.

Teaching Unit 1 (Equivalent fractions). Both teachers reported that the virtual use of fraction sticks had been unsuccessful. Their students had found the materials difficult to use and had quickly lost interest in using the computers. Many students said that the activities were "boring". However, both teachers reported that the use of the physical fraction sticks had been very successful and were well liked by the students. They had

therefore focused on the physical materials and not continued with the virtual activities.
A closer analysis of the virtual fraction sticks supports this finding. The virtual sticks and the physical sticks are nearly identical and the mouse movement to pick up and move a stick is similar to the physical movement. However, the virtual movements are more complex. Moving sticks with fingers is a simple task. However, since the sticks are simply lines and numbers, selection of a virtual stick involves quite delicate and precise mouse movements, as the arrow must be placed directly on a line. It is also easy to lose attachment to a virtual stick and once mixed, it is hard to select one of the virtual sticks. The sticks were also unattractive. Taken all together, it is no wonder that students preferred the easy manipulation of physical sticks than the frustratingly delicate manipulation of virtual sticks, particularly when the virtual sticks gave no learning or engagement advantages.

Teaching Unit 2 (Decimal scales). Both teachers were delighted with the virtual scales and reported that the students had both enjoyed the activities and learnt from using them. Unlike the teaching of equivalent fractions, where there were many physical materials available, the teaching of scale had been a slow concise development of knowledge across more and more complex number lines. Students read the values off the number lines or marked the values onto the number lines.
Thus, the move to computers was much more of an attraction in the scale lessons than it was in the fraction lessons. As well, the virtual scales had authenticity (they were moving drawings of real measuring instruments) and movement (e.g., the syringe reading was changed by using the mouse to move the plunger, while the odometer reading was changed by using the mouse to rotate a pointer). They were an extension of what was being done in class not a copy and they gave an *engagement* advantage.

Teaching Unit 3 (Slides, flips and turns). Again both teachers were delighted with this virtual activity and reported that the students both enjoyed the activities and learnt from using them. This is interesting because, with flips and turns, virtual materials move into an area where the mouse movement are very different to physical movements. A tile may be turned with a rotation of a hand or flipped with a movement of the hand upside down, but a virtual tile requires an icon on the Draw toolbar to be activated and a round handle to be dragged in a circle for a turn and the Draw menu to be activated and either “flip vertical” or “flip horizontal” to be selected for a flip. In particular, flips are very different; they do not require a movement but rather a selection from a menu and they can only be flipped in two directions. Any other direction for a virtual flip requires turning as well as flipping.

However, the virtual materials are very different to the physical materials for sliding, flipping and turning. Using tracing paper to trace, move and retrace is detailed work and is often inexact, as hand movements cannot replicate the perfection of abstract mathematical movements. It also lacks colour and is slow, making designs difficult. On the other hand, virtual slides, flips and turns, although different, are exact, can involve colourful materials, are quick and easily lead to attractive and complex designs. Furthermore, in virtual materials, the sliding, flipping and turning actions have to remain separate and be done in a measured way; hands often slide, flip and turn physical materials at the same time thus confusing the actions. Thus virtual slides, flips and turns are not only extensions of physical slides, flips and turns, they provide greater exactitude, allow for detailed experience of each action in isolation and enable easy preparation of attractive complex designs; they have both *learning and engagement* advantages.

DISCUSSION AND CONCLUSIONS

As the tool of this generation's time, computers should be utilised whenever and wherever possible in the educational arena (Baturo & Cooper, 2002). However, like physical materials, the case studies in this paper have shown that virtual materials are not effective simply by their presence. They have to add something in terms of engagement or learning to the instruction, and this from the students' perspective. Virtual materials, along with physical materials, have to be understood in terms of what the students perceive from the images and what they do with them (Baturo & Cooper, 2002).

Student manipulation of virtual materials is a very different use of computers in mathematics education than that commonly seen in Queensland schools. It has all the virtues of digital material; programs can be saved for later reuse by the same or other students and there is the opportunity for all students' manipulations to be saved and stored for later assessment (Baturo & Cooper, 2002). It has the strength of being multi-representational (providing visuals, language & symbols) and dynamic (showing transformations and changes as well as relations). In this way, it uses the visual, symbolic and operational power of the technological media and provides another pedagogical and didactical tool for the media (Healy & Hoyles, 1999). Its abstract nature gives it capacities for actions, activities and representations not easily available with physical materials; for example, shapes can be enlarged by specific amounts, or turned by specific degrees (Baturo & Cooper, 2002). It is less time consuming in terms of preparation, particularly with respect to space activities, requiring only one template that can be downloaded for individual student's use. It is a colourful, vibrant and dynamic way to teach. It extends and amplifies existing technologies, modifying, reshaping, and blending the ways in which humankind works mathematically and enhances the teaching and learning value of physical materials, particularly when integrated with manipulation of physical materials (Kaput & Rochelle, 1997). On top of this, the closeness of virtual to physical materials makes it more comforting for mathematics teachers (and more familiar to students) (Baturo & Cooper, 2001).

However, as we have seen with the fraction sticks, closeness with the physical materials with which it is integrated may be a factor in ineffectiveness as well as effectiveness. In these cases, some added ability in terms of authenticity, speed or attractiveness may be needed to make the use of virtual materials worthwhile. Of course, difference is inherent between virtual and physical materials in terms of the form of manipulation –mouse and hand. This difference can be positive, as Unit 3 shows; sliding, flipping and turning with virtual materials requires each of the three actions to be separately carried out and enables the consequences of each action to be differentiated from the others.

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OBSTACLES FOR MENTAL REPRESENTATIONS OF REAL NUMBERS: OBSERVATIONS FROM A CASE STUDY

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To probe beyond declarative knowledge about real numbers in secondary school, the authors interviewed students in Grades 9, 10 and 12. The main question seems to be whether the length of a decimal expansion is "indefinite" or "infinite". It blurs the mental representation of rational numbers as well.

INTRODUCTION

A central goal of mathematics instruction in secondary school is a conceptual understanding of numbers, not only as an important ingredient of the basic numeracy, but also as a foundation of continuous mathematics (e.g., the calculus) - and even as a major step in the history of ideas. This goal is by no means trivial, since the concept of real numbers intertwines numerical, arithmetical, algebraic, and topological strands.

The motivation of this study were confused and contradictory statements about irrational numbers made time and again by university students who had successfully taken at least one analysis course, and were competently studying fairly advanced mathematics. We wanted to find the source of this confusion.

Of course there had been previous studies of the subject, notably the recent ones by Fischbein et al. [FJC 95] and Peled and Hershkovitz [PH 99]. The former found that the irrational muddle was caused neither by the inability to conceive of incommensurable lengths nor by the notion that the rationals took up every bit of space on the number line. The second one, surveying pre-service teachers, found that many had trouble placing certain items like 0.33333... and $\sqrt{5}$ on the number line - although $\sqrt{5}$ occurred as the diagonal of a 2×1 rectangle in a geometry problem they solved - but that their *declarative* knowledge of irrationals was quite satisfactory.

Since quantitative investigations are largely limited to this declarative aspect, we looked for a qualitative approach, and decided to use the form of videotaped interviews (subsequently transcribed) with small groups of students, which would prompt the participants to discussions among themselves and allow the interviewer to follow up with questions formulated on the spot. The conversation in the interviews was guided by the queries and results mentioned in [FJC 95] and [PH 99].

With this aim in mind, we collected evidence on four levels: Grades 9, 10, and 12, as well as prospective teachers in their fourth year of university. The results so obtained point to consistent difficulties, which are still demonstrable in the prospective teachers' understanding: the insufficient internalization of the notion of irrational number (and thus, *the problems with real numbers* mentioned throughout the literature) *are already visible in the inconsistent mental representation of rational numbers*. It is the weak conceptual tie-in between a number like 22/7 and its theoretically equivalent decimal counterpart 3,142857… According to the students on all our levels, the latter representation is flawed by a connotation of inaccuracy. Considering such " infinite" decimal expansions as legitimate and complete mental objects lies outside the naive range of acceptance: to regard such serpents of digits as "rational" goes against the grain of the students.

This sobering realization recalls the distinction found in [Kl 28], where Felix Klein differentiates between *approximation mathematic*s and *precision mathematics*[1]. The numerical evaluation (!) of 22/7 as 3,142857, plus perhaps another few hundred places which interest nobody, is a natural problem of approximation mathematics; but the idealizing step of accepting this as the representation of an infinite, periodic expansion implies a major shift of paradigm toward precision mathematics, a shift which apparently Plato and Aristotle already argued about. It seems inevitable to us, that Klein's statement ([Kl 32], *ibidem*) "that the concept of irrational number belongs certainly only to precision mathematics", must be extended to include the mental objects known as infinite periodic decimal expansions.

As far as the role and treatment of rational numbers in school mathematics is concerned, our observations clearly entail and support certain consequences, which are discussed in the literature again and again (cf. Stowasser [St 79], Groff [Gr 94]) but go beyond the scope of this paper.

HISTORY

Klein's differentiation between *approximation mathematics* (AM for short) and *precision mathematics* (PM for short) appears to us to be the key to understanding the process of concept formation and the nature of concept images (for the terminology, cf. [TV 81]). To avoid misunderstandings, we hasten to add that there is nothing imprecise about AM: it means "approximate but refinable to any desired degree". On the other hand, PM means "totally precise", i.e., zero-tolerance for error.

The opposition of those two styles goes back at least 25 centuries, to about the time when the torch of scientific innovation passed from Babylon to Greece. Toeplitz ([To 63], Ch.I, §4) says that Plato had a strong preference for PM, while his student Aristotle favored AM - another good reason for these acronyms. However, it would be misleading to confuse this distinction with that of Applied versus Pure Mathematics. Courant and Robbins ([CR 41], Ch. I, § 6) point out that PM has brought with it a "tremendously simplified description of physical phenomena", and Klein calls it an "indispensable support" for the development of AM itself. One page later, he finds nevertheless that school is not the place to deal with it, since it "would hardly be adapted either to the interest or the power of comprehension of most of the pupils."

The difference between the two styles can be most clearly explained by an example: the result of dividing 144 by 233 is, in decimal notation, 0.61802575... Being the quotient of integers, it must of course be periodic, but its period happens to be of length 232 - too

[1] He later devoted a whole book to it: Elementarmathematik vom höheren Standpunkt aus. Band III: Approximations- und Präzisionsmathematik. Berlin: Springer.

long for practical purposes. AM will stand ready to work out some more places: 0.618025751073..., even more; but PM will insist on treating it as 144/233, not losing a crumb. To PM the Golden Mean is 1/2 of $\sqrt{5}-1$, while AM is content with 0.61803398875..., ready to work out some more places when required. To call one of them "rational" and the other one not, would not even occur to AM. Even PM does not *depend* on that distinction - it only makes it possible.

PM seems to have been a Greek invention, while the Babylonians always worked in "AM-mode". The glories of Greek mathematics are known well enough not to need any further praise, but it is interesting to see ([To 63], p.17) how its greatest master, Archimedes, struggled to get π stuck between the rationals 223/71 and 22/7, while Ptolemy, still working in Babylonian mode 4 centuries later, produced an entire table of sines - all of which (except one) are irrational.

Of course, the Babylonians worked in base 60 (as we still do with minutes and seconds) and that tradition was amazingly durable - even in cultures whose spoken numbers were thoroughly decimal. As late as 1250, Fibonacci displays the solution of a certain cubic equation as 1 plus 22 minutes, 7 seconds, 42 thirds, 33 fourths, 4 fifths, and 40 sixths [To 63, p. 15], which differs from the today's answer by only 0.00000000003 in decimal notation.

This method was translated into the decimal system in 1585 by the Flemish engineer and mathematician Simon Stevin. In the preface to his slender booklet *De Thiende* (The Tenth), he says that he did not invent but only found it, and urges all people having to measure and calculate, be they astronomers or merchants, to use it. He makes no reference to the Babylonians, but the point of this whole excursion is that, with Stevin's modification, their system might be heading for a revival by the present "calculator generation" - to whom the distinction between rational and irrational is an anachronism, and only finite decimal numbers are really "real".

RESEARCH QUESTION AND METHODOLOGICAL CONSIDERATIONS

The over-all research question, then, was: *what is the mental image of rational and irrational numbers in the present generation of students?*

Our study was carried out in the early autumn of 2002, in Northwestern Germany. The first participants were two prospective teachers (Stan and Sue) in their 7th semester at the university, whose ideas and conceptions will play a role in our conclusions, and about a dozen students of Grade 9, who by their vivacious reactions mainly helped articulate our procedure. Our principal sources were three groups of senior secondary students: 4 females (pseudonyms begin with F), later 4 males from Grade 10 (pseudonyms with G), and finally 4 males from an enriched mathematics course in Grade 12 (pseudonyms with M). The "field work" was conceived as a series of open interviews, which were videotaped and are now available in transcribed form. The conversations varied in length between 40 and 70 minutes. Their general themes were known to the students, but there was no sign of any systematic preparation.

As a warm-up, they were asked to make crosses wherever appropriate in the empty fields of an 8×4 table, which was basically copied from [FJC 95]: the columns were labeled by the properties “number, rational, irrational, real” and the rows by the seven symbols in [FJC 95], to which we had added ÷-5 as an eighth. Our intentional questions - as against those, which came up spontaneously - were as follows.

1. When did you encounter these notions, and how did they strike you?
2. Can you give some examples?
3. Are these properties inherent, or are they aspects of the representation?
4. What is the relative abundance of rationals versus irrationals?
5. How can you recognize a periodic decimal?

Most of the time, however, was spent on the unexpected question of whether fractions, by being represented decimally, could give rise to irrational numbers. It is important to note that the interviewer did not somehow insinuate this. The reader can ascertain this at www.pims.math.ca/~hoek/interviews/ where all transcriptions will be posted. The consistency of the high school students' responses might also raise the suspicion that they all got the wrong message at school. In fact, they were from two different schools, whose common curriculum introduces irrationals in Grade 9, with a textbook of exemplary clarity. We know their teachers as very competent, both mathematically and pedagogically.

OBSERVATIONS, RESULTS AND INTERPRETATION

Because of the volume of our material, we must reduce our exposition to very few aspects of our exploration. In particular, we leave aside the students’ ways of wrestling with Question (4) - which led to reasonable conclusions more often than not. The furthest off the mark was Stan, who did *not* wrestle with it, but simply voiced his opinion that irrationals were “stop-gaps” and relatively rare. Nor shall we have enough space to discuss how the “finiteness” of the string of digits for a number is seen to depend on the base (e.g., 2, 10, 60) of the place value system used - cf. Question (3).

On the epistemology of understanding the concept of number.

To begin with, the observer is surprised that the conceptual equivalence of common fractions and periodic decimal expansions is by no means evident for these students. For one thing, their primary contact with numbers is through calculator displays. For another, methods for moving between periodic decimal and fractional forms (possibly calculator-assisted) are no longer widely known: since the New Math, they are mentioned only in passing. This experiential deficit has the immediate epistemological consequence that fractional and decimal representations are not on the same ontological level.

Even for Stan (4th year university), 1/3 is more accurate than 0.33333… (“*For me, 1/3 is more precise. When I compute with 1/3, I think back to the world of the Greeks, and I calculate more accurately*”). His classmate Sue tries to repair the perceived imperfection of the latter by writing it as $\sum_{i=1}^{\infty} 3/10^i$, in other words, by *completing* the trans ition to infinity.

The four young men of Grade 10, after asserting that irrationals are “*infinite behind the decimal point”* and although coaxed to recognize the equally infinite 0.33333… as rational, flatly declare that 1/3 is irrational, too, and stand by that opinion. The young women in the same grade are less certain and less unanimous. When asked, whether periodic decimals could be irrational, Fanny says: "*yes, I'd say that*", while Flora says about 0.99999…: "*we have learned that it is also 1*", thus avoiding a direct assertion. In the end, three of the four women vote that 1/3 is irrational, while Flora remains doubtful. In both of these Grade 10 groups, the lead-in question about irrational numbers had produced the same reaction "*not imaginable*" (Fanny, Gilbert). More specifically, they have "*many*" (Frieda), even "*infinitely many*" (Guido), places after the period, and cannot be "*determined exactly*" (Fiona) because they are "*apparently unending*" (Flora). But this feature of irrational numbers taints periodic decimals as well.

In the Grade 12 group, similar ideas come to the surface: *"… an irrational number is … not determined, goes on and on, … is not anchored to a certain point. The point can be encircled, infinitely close, but cannot be grasped."* (Marco). "*Put simply: an irrational number has infinitely many places after the period. I think that some fractions, under certain circumstances, can also be irrational numbers* … (Moshe). It is striking that Moshe brings up the word "fraction" which had not been mentioned previously. Manfred summarizes part of the ensuing discussion as follows: "*in the case of a fraction, you still have a computation to do, a fraction is not yet a finished number. When you have a decimal number with 0. and a great many digits behind it, you know that you have got a number, no further computation is necessary.*" It is clear from the rest of the conversation that his "many" means "*finitely many*". Michael agrees that 2/7 is "*only the symbol*" for a number.

Let us emphasize that these students have no problem in interpreting a (finite) decimal fraction and in locating it on the number line, at least in principle. They argue (in their own words) that such a number contains an explicit and understandable locating algorithm: after finitely many steps, you arrive at the correct point on the number line. This perspective shows that they look at these entities as generalized natural numbers. In their view, there is no essential difference between integers, decimal fractions, or even common fractions and roots (!), as long as the latter two are regarded as "placeholders" for numbers yet to be computed. It is amazing to see how closely this understanding correlates with Stevin's notion of number from more than four centuries ago. [Ge 90]

To us, the conclusion seems inescapable that these high school students stand firmly on the ground of what Klein calls approximation mathematics. Precision mathematics is far from their minds: 2/7 is not seen as a number, but is relegated to the world of symbols. The two university students do know the difference between rationals and irrationals, but this knowledge is still more declarative than fully realized.

Verbal description and conceptual content

It is well known that everyday language impinges on mathematical notions even if these have been carefully defined (cf. continuity of a function). This phenomenon appears to affect the notion of "rational" number more strongly than we had expected. However, our students are in good company: according to Klein, astronomers dealing with planetary orbits would consider 2/7 as rational but 2021/7053 as irrational ([Kl 32], p. 36).

In our Grade 12 group, the literal meaning of the word "rational" formed an additional obstacle to understanding what is meant. Early on Marco declares: "*Ratio is that which is given by the structure of thought. Ratio has to do with reason. And for me, a reasonable* *number is one that can be nailed down, that can be defined clearly*." The conversation turns quite philosophical. Much later, Moshe says: "*... Generally speaking, I think that a number is rational, when I can represent it in its entirety, i.e., when I can work with it. This I can do only with numbers that can be converted into natural ones. They need not be natural or whole; they can also be decimal numbers, which have an end. That is, when I have a number with an end, I can work with it, then it is rationally understandable …*"

These linguistically induced conceptions make it difficult to accept infinite (periodic) decimal fractions as rational numbers. Official teachings notwithstanding, prompted only by their daily experience with calculators, these students do, in fact, exactly what Simon Stevin had recommended.

Affective Components

Just like Stan (cf. the quote in 4.1 above), Fiona finds 1/3 more dependable than 0.333…: *"I find that 1/3 is more reliable, because you can depend on it. You know in the end, what the result will be. With 0.333…, that is 3-period, I always feel: can I really depend on that …?"* Fanny finds 1/3 *"friendlier"* than its decimal expansion. Marco expresses his incredulity that anybody could really imagine an infinite expression by saying: "*If anyone can imagine* that*, I'd be really impressed."* More importantly, all the students interviewed have a positive, open attitude toward mathematics, and do some honest, serious thinking. Nobody tries to fake it. At the end of the interview, the M-group even agrees it had been "fun".

Criteria of rationality

To explore our "intended" question (5), we handed out a sheet with 5 decimal fractions between 0 and 1, each with 288 places printed out, and asked which ones were rational. Initially, we were working under the assumption that only short periods would be recognized as rational, and therefore tried to test the students' understanding by confronting them with long ones. Even when dealing with the first two, which had moderate periods, it dawned on every group of students, that their rationality was *not* decidable: a viscious teacher could always derail it at the 300^{th} place. In their Platonic existence as fractions or algebraic numbers, the third one had a period of 256, the fifth one of 294, and the fourth one had none. Instead of clarifying the difference between rational and irrational numbers in decimal form, these observations only deepened the students' distrust of infinite (!) decimal representations.

The serpent of nines

The question about the relation between 0.9999… and 1 comes up again and again, as mentioned by many commentators. Tall[2] says that "the primitive brain notices *movement*", and Zeno of Elea might have thought similarly. In our survey, the keyword "asymptote" is brought into the open by Marco: "*… I come infinitely close to it as to an asymptote, but I can never say:* that *is the number*." Since asymptotes play a considerable role in analytic geometry, these Grade 12 students would have been taught that the function $f(x)=1-10^{-x}$ never quite reaches the constant $g(x)=1$. However, this very function produces 0.9, 0.99, 0.999, and so on, for x=1, 2, 3, etc. This remarkable *inconsistency* of

[2] www.warwick.ac.uk/staff/David.Tall/themes/limits-infinity.html

language and imagery (!), between functions, which never quite make it and numbers which are eternally there, erects an additional obstacle.

CONCLUSIONS AND OUTLOOK

We must admit that our expectations were modest when we began to probe the understanding of irrational numbers on these various grade levels. After all, we were aware of Felix Klein's opinion (quoted in Section 2) about the "pupils" appetite for irrationals, and had diverse other reasons (in part also quoted above) to be skeptical.[3]

Nevertheless, we found our observations surprising. We had believed that an adequate treatment of the real numbers was made difficult mainly by the massive influx of mysterious irrationals into the orderly system of good, clean rational numbers. This was clearly not the view of the students we questioned: their horizon was that of Klein's AM, and their working environment was that of Stevin's decimal fractions. Thus, what we had believed to be a *conceptual problem* concerning irrational numbers turned out to be a *notational* one which covered must of the rationals as well.

It is well known that mathematically equivalent statements are not always didactically equivalent. The same is true for representations, for instance common fractions versus decimal ones. For some one equipped with a calculator, the *workability* of the decimals further tips the balance in their favor (remember Moshe's "… *when I can work with it*.") In some sense AM is the mode of action, PM that of contemplation. Conceptual problems are abundant in the latter, while the former is more easily affected by problems of notation and language. In it, the fact that infinite decimal expansions cannot be written down produces a major cognitive obstacle, turned into mockery by the common meaning of the word "rational" – as shown by the students' affective utterances.

In retrospect, our surprise has given way to the realization that a *shift in the mental image* of "number" was to be expected, when the actual contact with numbers had shifted from relatively sparse markings on paper or blackboard to very explicit displays on calculators and computers.

Sparse as it was, the older mode of communication probably left more room for the imagination (of the lucky few) to delve into the immaterial realm of PM, while the modern explicitness holds it back in work-a-day world of AM - which might be just as well, according to Klein.

With hindsight, it all makes sense - but much more research would be required to corroborate our conclusions on a large scale. If this should happen, it would clearly imply that curricula stay with Stevin and in AM as long as possible. This does not mean avoiding all references to common fractions (especially those with finite decimal

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[3] Toeplitz hides his dim view in the folds of the following convoluted comment : *If a high school graduate were asked, what exactly was a number according to the mathematics he had been taught, he would no doubt be able to consent to the suggestion that he say, what he had so far understood to be a number was an infinite decimal fraction*. Unfortunately, this sentence was not included in the English translation [To 63] of the German edition of 1949, where it appears near the top of page 15.

expansions, or at least small denominators) or shirking all irrationals. But, as Stowasser observes [St 79]: "*The measurement of continuous magnitudes provides no sensible motivation for the calculus of fractions*." So, we might make it at least as far as first Year College without it.

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TEACHER AND STUDENTS' JOINT PRODUCTION OF A REVERSIBLE FRACTION CONCEPTION

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Within an elaborated social-constructivist perspective, I conducted a teaching experiment with two fourth graders to study how a teacher and students can jointly produce the content-specific reversible fraction conception. Ongoing and retrospective analysis of the data revealed the non-trivial process by which students can abstract multiplicative reasoning about fractions without using the activity of splitting. The study contributes to the psychology of mathematics education by articulating a culminating advance in a developmental sequence of iteration-based fraction conceptions and the teacher's role in fostering such an advance in students.

In this study I addressed the problem of how teachers and students jointly produce mathematical conceptions (Bauersfeld, 1988). In particular, I generated a grounded, content-specific explanation of students' learning of a *reversible fraction conception* as a transformation in their iteration-based fraction conceptions and of the teacher's role in this process. Content-specific explanations are critical for utilizing general models of growth in mathematical understandings while teaching particular ideas to particular students (Tzur, 2002). Generating content-specific explanations is particularly important in the domain of fractions, because, as Davis, Hunting, and Pearn (1993) asserted, "The teaching and learning of fractions is not only very hard, it is, in the broader scheme of things, a dismal failure" (p. 63).

CONCEPTUAL FRAMEWORK

The study drew on a social-constructivist approach (Cobb & Bauersfeld, 1995), which coordinates social and psychological perspectives for research. The explanation of concept formation that I used is an elaboration of the psychological perspective and of the possible role for teaching. In this section I briefly present key aspects of the general and the content-specific constructs that, combined, provided a framework for the study's design and data analysis (see Simon et al., 1999; Tzur & Simon, 1999).

General Constructs. In this framework, a mathematical conception is considered a dynamic, mental relationship between an activity and its effects ('conception' and 'activity-effect relationship' are used interchangeably). That is, activity generates and is a constituent of a conception—it is not just a catalyst to the process or a way to motivate learners. Abstracting a new conception is made possible by the mechanism of reflection on activity-effect relationship, which is an elaboration of Piaget's notion of reflective abstraction (note that 'reflection' does not imply learner's awareness). Two qualitatively distinct stages in abstracting a new conception—participatory and anticipatory—are identified on the basis of the nature of the learner's anticipation. At the participatory (first) stage, the learner anticipates effects of an activity and utilizes them in problem situations. However, the learner cannot independently call upon the activity (hence the activity-effect relationship); he or she must somehow be cued for which activity to use (e.g., chance, engaging social interaction). At the anticipatory (second) stage of a new conception, the learner can independently call upon and utilize that conception proper to the problem situation. The content of the anticipated relationship is the same in both stages; they only differ with respect to the availability of the relationship in a given situation. Teaching is viewed as a cycle of 3 principal activities: inferring learners' current conceptions, hypothesizing a learning trajectory (Simon, 1995) from current to intended conceptions, and selecting/using tasks (problem situations) that learners can assimilate and solve as a means to form the intended conceptions. Teacher tasks are geared toward: (a) engaging learners in setting goals and initiating activities toward those goals and (b) orienting learners' noticing of effects of activities and their reflection on the designated activity-effect relationship.

Content-Specific Constructs. Researchers identified two fundamental activities that constitute fractional conceptions, splitting (Confrey, 1994) and iteration (Steffe, 2002). Splitting refers to recursively acting on the results of previous activities (e.g., part of a part of a part, etc.) and was considered to generate multiplicative structures, such as a geometric sequence, better than the iteration of units (e.g., repeated addition). Yet, Kieren, Mason and Pirie (1992) noted that rational numbers are both additive and multiplicative quantities, and splitting activities such as paper folding are insufficient for generating the additive meaning. For example, one cannot expect splitting activities alone to generate the critical understanding of non-unit fractions such as 6/11 (let alone the 'improper' 13/11) as the effects of iterating 6 (or 13) abstract units of size 1/11 each (Behr et al., 1992). Tzur's (1999; 2000) identification of the partitive and iterative conceptions indicated how the activity of iteration, which is available via whole number conceptions, allows students to abstract unit fractions (e.g., 1/11) and non-unit fractions (e.g., 6/11, 13/11). Moreover, he showed that iteration could generate fractions as a multiplicative relation: the child conceives of, say, 13/11 as a unit that is 13 times as much as 1/11, where 1/11 is itself not just part-of-whole but any unit which fits-in-a-given-whole exactly 11 times. The study reported here was set to address the content-specific problem of how students transform (reverse) those iteration-based conceptions of non-unit fractions so they can decompose such fractions. In the analysis section I will show that such a critical conceptual advance—abstraction of a reversible operation in Piaget's sense—is not as trivial as it might appear.

METHODOLOGY

The study was part of a larger constructivist teaching experiment in which, together with 5 researcher-teachers, I taught fractions to two fourth graders, Linda and Jordan. During that year, we conducted 29 videotaped teaching episodes once or twice a week, about 30 minutes each, and collaborated in the ongoing analysis/planning sessions between every two consecutive episodes. Retrospective analysis of students' conceptions and the teacher's role in fostering them consisted of rigorous, line-by-line interpretations of participants' language and actions: asking questions about the data, making grounded hypotheses about possible explanations for these questions, and systematically searching for confirming/disconfirming evidence.

To support activities needed for students' learning of fractions the study utilized a computer microworld called Sticks, which was developed along with and informed by the teaching experiment. In Sticks (see Figure 1), using Draw, one could produce on the computer screen a linear figure (a 'stick') of various lengths and replicate any such stick as many times as desired using Copy. One could mark a stick vertically using Marks and erase or move those marks, cut any stick at a desired point using Cut, and join one stick to any other stick on the screen using Join. One could also partition any stick into a desired number of equal pieces (2-99) using Parts, break a marked or partitioned stick into pieces using Break, and iterate a stick (plain or marked) using Repeat—a coordination of Copy and Join. One could pull pieces of a marked or partitioned stick using Pull-Parts and measure any stick by using Measure. One could also fill a stick or any part of it with 10 different colors using Fill, use Label to attach a fractional symbol to any stick, and cover or uncover sticks or parts of them.

ANALYSIS

In this section I explain how Jordan, Linda, and I (the researcher-teacher) jointly produced the reversible fraction conception. I focus on how their work oriented my teaching, how my teaching occasioned their abstraction of the reversible conception, and how their advances occasioned my articulation of the abstraction process.

Prior to our work on the reversible fraction conception, Linda and Jordan had established two major iteration-based conceptions—partitive and then iterative—as reorganization in their whole number conceptions (Tzur, 1999; 2000). Using these conceptions, they were able to generate and operate on symbolized unit and non-unit fractions (i.e., anticipatory stage). For example, in a playful context where we considered ourselves workers at a pizza stand, they could independently and mindfully explain the solution to problems such as, "How much of a pizza would you sell if four customers bought 3/10 of a pizza each?" Therefore, I decided to present a task that required decomposition of a non-unit fraction in the pizza-stand context.

Initial Assessment of Reversibility. I began episode 26 (May 3) by drawing an unmarked stick, told the children that it represented 5/8 of a pizza but the original pizza was lost, and asked if they could rebuild the original pizza. Immediately, both Jordan and Linda activated Parts to solve the problem by partitioning the unmarked 5/8 into 8 rather than 5 parts. This was a clear indication of calling upon their prior, anticipatory iterative conception, so I decided not to pursue the problem further and moved to working on tasks of selling and buying pieces of a 10-slice (10/10) pizza.

Accidentally, but fortunately for the goal of the study, Jordan ruined and erased the original 10-slice pizza. My on the spot reaction was to utilize this as an opportunity to further examine the children's thinking about rebuilding the pizza from parts of that pizza (1/10, 2/10, 9/10) that were still available on the screen. I asked a question of the type "Think and tell me what would you do" in order to both enable each child time to think of a solution and examine their anticipation of activity-effect relationship. After a few seconds Linda said that she would copy and repeat the 2/10 five times and Jordan said he would copy and join the 9/10 and 1/10. Then, each child implemented his or her

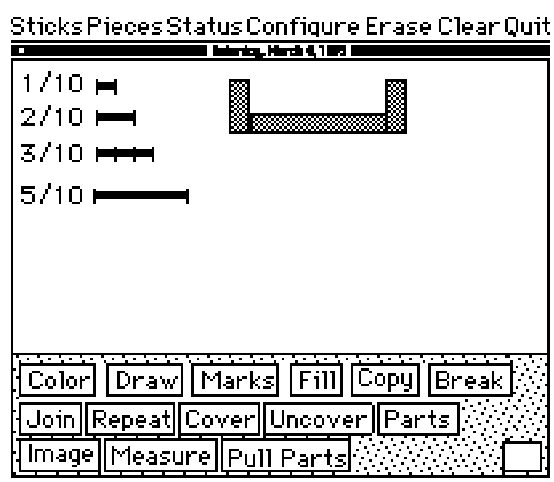

Figure 1. The initial screen of Episode 27

plan.

Each child's response indicated that they anticipated the effects of both activities: iterating 2/10 five times or adding 9/10+1/10 must reproduce a whole like the original. While reflecting on their anticipations and my task I realized that they have just "taught" me how they might decompose a non-unit fraction (i.e., iterating non-unit fractions). My critical role, then, was to devise a task that would transform their independent use of that anticipatory conception into the reverse activity. Eventually, I designed a task for them to set the same goal (rebuild the whole) in the same context (pizza-stand). I chose units (1/10, 2/10, 5/10) that would lead them to bring forth iteration and also a marked unit (3/10) that would require further activity of partitioning (see Figure 1). The hypothesis was that even with the 3/10 being marked, the children would begin with 5/10 (doubling), then use the 1/10 (iterating a unit fraction), then 2/10 (anticipated, non-problematic iteration of a non-unit fraction), and finally use the 3/10. Using the 3/10 was likely to bring forth its decomposition, at least once, which I would then use to orient their reflection on and foster further utilization of such activity.

Reversing an Available Anticipatory Conception. I posed the task two days later, in episode 27 (May 5). Their goal was to rebuild the pizza so it would fit in the 'oven,' using one of the pieces at a time. I asked them to think and tell what they planned to do before actually doing it. As hypothesized, Jordan said he would use the 5/10 twice; Linda said she would use the 1/10 ten times. Both children seemed confident that their solutions must work. Then I asked which of the other two pieces they could use next. Linda immediately and excitedly said she would use the 2/10, then implemented her anticipated action while saying: "Cop-I … (she copies the 2/10 and repeats it 4 more times while counting the non-unit fractions) 2, 3, 4, 5." Linda's work indicated that she anticipated the effects of iterating the 2/10 five times by using multiplication in conjunction with her iterative fraction conception. To convince me that Linda solved the problem properly Jordan said, "two plus, umm, 2-times-5 is 10," and Linda said that she thought of the same reason. The groundwork was laid for the designated transformation.

Excerpt 1 (Episode 27, May 5)

Res.: That's great. You see, what I like very much is that you have different ways of doing things. N-O-W (intonation emphasizes "here's the real challenge"), we have only one piece left.

Linda: (Echoing Res.) One – Piece - Left. (Then, enthusiastically) I know how.

Jordan: (Immediately following Linda) I know how. (Excited, to Linda) It's my turn. (Utilizes the microworld efficiently in a seemingly well-planned solution. Using Copy he makes 4 replicates of the 3/10, breaks the last one into three pieces of 1/10 each, joins the three pieces of 3/10 first, then also the 1/10, and trashes the remaining two pieces of 1/10.)

Res.: (To Jordan) Very nice. (To Linda) Do you have another way to make the original pizza?

Linda: (Excited) YES. (Using Copy she replicates the 3/10 once, then using Repeat she iterates it three more times while counting in "threes") 3, 6, 9, 12. (She searches for Cut in order to decompose the 12/10 into 10/10 and 2/10 without breaking the entire 12/10. Because Cut is not available, she breaks the 12/10 into 1/10 parts, then joins only 10 pieces into a 10/10 whole.

Res. and Jordan: (Watch Linda's work silently)

Linda: (Trashes her 10/10-stick, copies the 3/10 again, repeats it only 3 times into 9/10, clicks on the trash to bring out the 1/10 she trashed earlier, and joins that last piece to the 9/10.)

The children's work supported our hypothesis: the previous iteration of a unit fraction (1/10) and of non-unit fractions (5/10, 2/10) engaged them in utilizing their iterative fraction conception and thus oriented their reflection on the activity (iterating 1/10) that generated the non-unit fractions. In this sense, the social interaction prompted them for the possibility to decompose the non-unit fraction by reversing that activity. Yet, the decomposition was made only at the end of the children's activity sequences. To foster the transformation further, I continued to challenge them.

Excerpt 2 (Episode 27, May 5)

Res.: Can you solve it in a different way?

Jordan and Linda: (Each takes turns to utilize a different sequence of 'buttons' in the microworld, always decomposing the 3/10 last.)

Res.: Okay. You always started, when you take the 3/10, by making more [of those] pieces. Can you think of making it smaller instead of bigger? Start by making it smaller, then make the 10?

Jordan: (Holds the mouse, then asks) What do you mean, like ...

Linda: Copy!

Jordan: (Copies the 3/10 and says) Like this?

Res.: (To Jordan) Now you have the 3/10, right?

Jordan: (Nodding yes) Mm-hmm ...

Res.: Can you think of a way that, using it, making it smaller?

Linda: (Excited) I do, I do.

Jordan: (Hesitantly) Smaller?

Res.: Smaller than the 3/10 and then reproduce the 10/10.

Jordan: (Pulls the first 1/10 out of the 3/10 and trashes the 3/10 and repeats the 1/10 ten times.)

Linda: That was my idea.

One might say that my researcher's cue ("make it smaller") funneled the children to the intended action. However, Jordan's difficulty to interpret the cue indicates that this was not trivial. Rather, I suggest the following inference on the basis of both children's work. My cue oriented their reflection on the sequence of mental actions that they used in their previous solutions, when anticipating reproduction of a 10/10 whole with decomposition of the 3/10. Through reprocessing the decomposition of 3/10 last they noticed two effects anew: this decomposition created a desired unit fraction (1/10) and, being last in the activity sequence, the decomposition could match the goal ("make smaller first") set by my cue. In this sense, Excerpts 1 and 2 demonstrate the joint production of the initial transition to a reversible conception. Via reflection on their actions in the previous episode, I contributed a task and follow-up prompts that (a) occasioned their utilization of available conceptions and (b) oriented their reflection onto the designated (reversed) activity-effect relationship.

To foster further reflection on that relationship I posed the 5/8 task again. I cleared the screen, made a new 'oven' and an unmarked 5/8 of a pizza, and told them: "My dear workers! You lost your pizza, which comes in 8 slices. But you still have one piece left that is 5/8 of the original pizza." Linda immediately and eagerly said that she knew how to do this even before hearing a question. Jordan, who always grabbed the mouse and began working if he knew the solution, hesitated and asked several times about the task. For him, even understanding the task was not a trivial repetition of the previous one. He did not yet construct even the provisional, activity-dependent anticipation of a new participatory conception, in which the compositions of a non-unit fraction by means of iteration is to be reversed. As Excerpt 3 indicates, Linda also needed to initiate and reflect on some actions before she could clearly abstract the new relationship.

Excerpt 3 (Episode 27, May 5)

Linda: (Activates Parts, dials down from 8 and says to herself) Why not ... eight ... and Parts ... oh, five ... (Mistakenly, she dialed down to 3 so she says) OOPS. (She dials back to **8** and almost clicks on the 5/8-stick, but then changes her mind, says to herself) No (and dials to 5 while saying to herself) I want 5. (She mistakenly turns Parts off so she turns it on, purposefully dials to 5, partitions the 5/8-stick into 5 parts, pulls 3 parts from the partitioned stick while counting them) One, two, three. (She joins the 5/8 with the 3/8 while counting the pieces with the mouse and says) Two, three, four, five; Six, seven, eight.

Res.: Now isn't that nice?!

Linda: (Puts the 8/8-pizza into the "oven" and humming to herself, proudly) Doo-doo-dee-doo. (Releases the mouse as if saying "I'm done and I know that it is the right solution.")

Linda's "OOPS" experience is a critical point in her abstraction of the participatory reversible conception. Her initial excitement about the new task was rooted in an anticipation of the need to partition the unmarked non-unit fraction but not yet a clear anticipation of the number of parts needed. I suggest that by reflecting on her nearly executed activity of partitioning the 5/8 into 8 parts and on her mental run of this action to its effect, she noticed that such effect would not match her goal. This reflection led Linda to change her plan and to the desired transformation. She related, in reverse, the activity of partitioning the non-unit fraction with the number of iterations (5) used to produce it from an imagined unit fraction (1/8), as indicated by her purposeful actions and private speech when Parts was accidentally turned off.

Before the end of episode 27, after observing Linda's solution, we only had enough time for Jordan to reproduce another original pizza by partitioning the unmarked 5/8-stick first and then doubling it and joining 3 more pieces (3/8). However, it was unclear to what extent he abstracted that rudimentary, provisional anticipation himself. To test this, a week later (episode 28, May 12), I took advantage of Linda's late arrival and posed the following task: presented with an 'oven' and an unmarked stick labeled 7/10, I asked him if he could reproduce the whole pizza from that piece. Again, for a few minutes, Jordan struggled to make sense of the task itself. Then, he took action.

Excerpt 4 (Episode 28, May 12)

Jordan: (Copies the 7/10-stick once, activates Parts, dials to **10**, and actually partitions the 7/10. The researcher indicates nothing, but Jordan himself seems at unease. He thinks for 10 seconds then says "OOPS" and trashes that stick. He makes one more copy of the 7/10-stick and stops to think. After 20 seconds of silence, he says to himself "Oh," stops for about 2 more seconds, activates Parts, dials immediately to 7, and partitions the unmarked 7/10. At this point he seems to work purposefully and smiles, while pulling 3 more parts from the 7/10 and joining them with the 7/10. Finally, he drags the 10/10 stick into the 'oven' as if to check that it's right, then drags it out of the oven and leaves the mouse to indicate "I'm done.")

Jordan's recurring difficulties to make sense of the task and his first actual partition of the 7/10 into 10 parts highlight that following Linda's solution in the previous week was not sufficient for his abstraction of the new conception. His partitioning of 7/10 into 10 parts is not surprising. It is an example of a predictable 'interference,' implied by the participatory/anticipatory distinction. If a conception has only been established at the participatory stage, the learner cannot independently call upon it so he or she calls upon conceptions established previously at or beyond the anticipatory stage.

Most importantly, Jordan's work provides a significant 'window' to the invisible functioning of reflection on activity-effect relationship. Jordan's uneasiness indicated that his actions created effects he did not anticipate. I suggest that then, during the first 10-second silence, he reflected on mental runs of his action for plausibly producing the desired effect (whole). This reflection led to his realization, indicated by his "Oops" utterance: partition into 10 parts is inappropriate. This realization is compatible with Linda's, when she almost partitioned the 5/8 into 8 parts. Underlying both realizations is noticing of their coordinated actions in the anticipatory production of a non-unit fraction: Partitioning the whole and then iterating the unit fraction. This last inference is supported by the 20-second period of silence: Jordan independently anticipated the need to partition the 7/10, but he still had to figure out the number of parts. The long pause and the resulting confident manner in which he used Parts to select 7 indicated that he did not merely use trial-and-error. I infer that to figure out how many parts he further reflected on and differentiated the process he **had available** (producing 7/10) into separate actions. This reflection led to **his** realization of the need to reverse the sequence: begin by partitioning the 7/10 into the 7 parts of 1/10 each that were previously used (last in the sequence) to compose the 7/10. In turn, Jordan's difficulty and elaborated process brought forth my reflection on and distinction of the non-trivial abstraction of the content-specific, reversible fraction conception. The following episode (29) was last in that year and I devoted most of it to converse with Linda and Jordan about their overall experience during the year. Thus, I can only claim that both had abstracted the reversible conception at least at the participatory stage.

DISCUSSION

This study addressed the problem of how a teacher and students jointly produce the iteration-based reversible fraction conception. Analyzing the process of abstracting such a conception and the teacher's role (including useful tasks) contribute to the ongoing effort to identify content-specific understandings and the sequence in which they might develop. A transformation of the partitive and iterative conceptions (Tzur, 1999; 2000), the reversible fraction conception culminates the abstraction of multiplicative reasoning with fractions without/before using the activity of splitting. This helps to further explain two critical issues in the psychology of mathematics education: (a) why is learning of fractions such an obstacle for many students (Behr et al., 1992; Davydov & Tsvetkovich, 1991; Kieren, 1988; Streefland, 1991) and (b) how might teachers support meaningful learning in this domain.

The study also highlights an important aspect of the teaching-learning process. Small group and whole class discussions, alongside teacher tasks and prompts, greatly contribute to learners' abstraction, via orienting learners to their available conceptions—goals they can set, activities they can call upon, effects they can notice, relationships (anticipations) they can form. However, as the example of Jordan's learning with/from Linda demonstrated, these four basics of conceptual progress and the mechanism of reflection that ties them together reside within the learner. Thus, to understand a particular idea in mathematics each learner must abstract the constitutive activity-effect relationship for himself or herself.

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ON THE SEARCH FOR GENDER-RELATED DIFFERENCES IN DUTCH PRIMARY MATHEMATICS CLASSROOMS

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This paper reports on a study carried out in the Netherlands which is aimed at finding an explanation for the curious evidence that the reformed approach to mathematics education appears to have a less positive influence on girls' results. Over and over it turns out that girls' achievement scores in primary mathematics are lower than those of boys. The research started with 5000 schools and ended up with 4 schools. The paper focuses on the last part of this zooming-in research in which the classroom observations took place. By means of these observations a number of classroom characteristics, which make the teaching less optimized for girls and may cause their mathematics scores to be lower, have been traced.

INTRODUCTION

Since 1987, the National Institute of Educational Measurement (Cito) investigates the mathematics achievements of Dutch primary school students on a national scale. This takes place with a five year interval. As well as making visible the developments in the yield of education, these so-called PPON studies, have served to emphasize gender differences in mathematics scores. It turns out that boys are systematically outperforming girls. See the PPON results reported by Wijnstra (1988), Bokhove et al. (1996) and Jansen et al. (1999). It is unclear how long these differences have existed. Research (Wijnstra, 1982) had already indicated in the early 1980s that girls were behind in mathematics in primary schools in the Netherlands. Internationally the situation is different. Although the findings of international research are not always unequivocal, on the whole the situation is that the further along one moves in education, the more the boys outperform the girls. In general, no differences are found in primary education, and if they are, they are usually to the advantage of girls (e.g., Hyde et al., 1990; Leder, 1992; Geary, 1994). In addition the latest development is that in several countries girls are starting to do better in secondary education as well (e.g., Shaw, 2002). However, this is not the case in the Netherlands. The atypicality of our results was proven yet again in TIMSS (Mullis et al., 1997). For example, significant gender differences in grade 4 were only found in the Netherlands, Japan and Korea.

Girls' lagging result lead to the intriguing question of whether the Dutch approach to mathematics education, called "Realistic Mathematics Education" (RME), is equally suited for both girls and boys.

The foundations for the RME approach were laid by Freudenthal and his colleagues in the early 1970s. A brief overview of the philosophy and principles of RME can be found in Van den Heuvel-Panhuizen (2001). The significance of RME lies in its focus on mathematics that is worthwhile to learn and makes sense to the students. RME tries to achieve these goals by making mathematics experientially real for the students and having them actively involved in the learning process. In short, the RME approach means that mathematics education starts with rich context problems that can be solved in different ways. By means of interactive classroom discussions and the use of models, the inital context-connected strategies gradually evolve to more general, formal solutions reflecting a higher level of understanding.

THE MOOJ STUDY

Although there had been indications for a long time that girls did less well in mathematics in primary school than boys, it was not until the mid 1990s that research was done into these gender differences. In 1995, the Freudenthal Institute of Utrecht University and the Center for Study of Education and Instruction of the State University of Leiden received a grant from the Dutch Ministry of Education to start this research. It resulted in a collaborative project in which Cito was involved as well. The study was called the MOOJ study, and lasted until 1999. For a detailed report, see Van den Heuvel-Panhuizen & Vermeer (1999).

The purpose of the MOOJ study was twofold: (1) getting an overview of the size and nature of gender differences in mathematics achievements in primary school; (2) finding mechanisms in mathematics classrooms that contribute to these differences. The way the MOOJ study was set up was rather different from the customary design in educational research. The study consisted of three phases, with Stage I starting with around 5000 schools, while Stage II zoomed in on 14 schools and Stage III zoomed in even further on only 4 schools. The focus in the present research paper is on this final part of the study. Before going into this in more detail, first a short summary of the first two parts.

Stage I was meant mainly to map gender differentiation and to identify schools where the differences were higher or lower. To accomplish this, the mathematics scores in the Cito End of Primary School Test in 1993, 1994 and 1995 were analysed for gender differences. In these years, this test was administered in about 70% of the sixth-grade classes (the students are 12 years old at that point) , which led to a data base of mathematics scores from approximately 100,000 students. The results that were gained from the analysis of these scores were presented at PME 21; see Van den Heuvel-Panhuizen, 1997). In total, gender differences at three analysis levels were found. The first finding was that in each of the three years the avarage total scores of boys were about 6% points higher than the avarage total scores of the girls. This difference is about a quarter of the standard deviation. The second finding was that the test items showed remarkable gender-specific characteristics. Particular problems (called “boys problems”) were always done better by the boys and some other problems were done relatively well by the girls (called “girls problems”). The third finding was that in half of the schools the boys outperformed the girls (these schools were called “boys schools” or “B-schools”). In the other schools the average score of the girls was equal to that of the boys or higher (these schools were called “girls schools” or”G-schools”).

In Stage II, the study zoomed in to 7 B-schools and 7 G-schools to collect additional information about teachers and students. One of the main findings of this part of the study was that there were strategy differences between boys and girls. (For a more detailed report, see Van den Heuvel-Panhuizen & Vermeer, 1999).

MOOJ STUDY _ STAGE III

Stage III was the most crucial part of the MOOJ study. In this part, observations took place in 4 sixth-grade classes to see if any specific patterns exist that may explain the gender differences in mathematics achievement. The observations took place in April 1997 in two B-schools and two G-schools. These schools were selected from the Stage I and II data. They were "extreme" schools, which means that in the B-schools the differences were most to the boys' advantage and in the G-schools the differences were to the girls' advantage or were smallest. The rationale for selecting extreme schools is that in these schools the chance of observing gender differences and related factors is largest. Such a selection is in agreement with the idea of purposeful sampling that is characteristic for the "Grounded Theory" approach as formulated by Glaser and Strauss (1967) and others.

To answer the question of why the girls get the same score as boys in particular schools and why they do not in other schools four lessons were observed in each of the four grade 6 classes. The teachers were free to choose a particular content for these lessons (with the exception of the fourth lesson) and were not informed about the gender-specific aim of the observations. In the request to the teachers to be allowed to make the observations in their classrooms, they were told that the researchers wanted to gain knowledge about classroom practice. The observations had a double-focused set up. During each lesson observations were made from two different perspectives. The Leiden team had a general didactical perspective and collected data about general aspects of verbal communication within the classroom. The Utrecht team observed from a domain-specific didactical perspective and focused mainly on the characteristics of the learning situations that occurred within the lessons. After both sets of observations were finished, the findings of the two teams have been linked to each other. For analysis of the data, use has been made of the "Constant Comparative Method" (Glaser & Strauss, 1967; Strauss & Corbin, 1990) which implies repeatedly moving back and forth between the data that were found in the different classes and by the different observers, the observers thoughts about the data, and the conjectures made about it, finally resulting in some conclusions with which the whole team could agree.

The domain-specific didactical perspective

The observations of the Utrecht team were focused on mathematical content and the teaching methods in the lessons. The observations were carried out by four experienced mathematics educators and researchers Adri Treffers, Leen Streefland, Koeno Gravemeijer and the author of this paper who also prepared the observation format.

The theoretical base that was taken as the starting point for the development of the observation and analysis points consisted of:

- *Didactical characteristics of RME*; such as offering learning opportunities by (a) paying attention to different strategies and their relation, (b) developing number benchmarks, (c) developing knowledge about daily-life measures, (d) developing estimation strategies.
- *Gender-specific interaction characteristics* (Jungwirth, 1991, 1996) that are related to (I) determining competence such as (a) undoing completeness vs. teacher's echo, (b) authority insistence vs. argumentative insistence, (c) emerging failure vs. concealing of failure; or that are related to (II) learning opportunities such as (a) blocking task constitution vs. tasks constitution, (b) blocking outside reference vs. demonstrating everyday knowledge.

- *Class climate characteristics* (e.g. Cobb, Wood & Yackel, 1991) related to (a) learning (what are the social norms regarding responsibility and autonomy of students?); (b) subject matter (what are the socio-math norms regarding e.g. different strategies, estimations, real world references); and (c) evaluation (what are the social norms and the socio-math norms regarding who is determining what is correct/incorrect?).

The procedure for the observations was as follows. The four observers each attended one lesson of each of the four teachers. During the lessons they made notes about significant episodes based on the above list of characteristics that they were free to apply based on their own insights. After observing the lesson they made a report consisting of three parts: (1) a description of the lesson, (2) theoretical memos about what they saw in the lesson, and (3) a prediction of the classroom type (B-classroom or G-classroom) including the arguments this was based on.

With the exception of the author who was the project leader of the MOOJ study and had to make the arrangments and schedule for the observations, the observers were not aware about the classroom type.

After all observations had taken place, the observers studied each other's reports, and after a short intermezzo to let everything sink in, a meeting was held during which the observers reacted to their respective findings, and where finally conclusions, with which all four observers concurred, were formulated.

The general didactical perspective

For the observaties by the Leiden team use has been made of the FROG tool (Dolle-Willemsen, 1997). This is a computer-based observation tool for classroom interaction that covers several categories of classroom activities. The most important categories are: explaining or demonstrating by teacher, asking questions by teacher, pausing, giving turns to boy/girl, answering questions by boy/girl, taking initiative by boy/girl, reacting by teacher. The computer screen shows the categories and the observer has to score the category each time the category changes. The lessons were all scored by one observer who had a large experience with this tool, and who also was not aware of whether the lesson was taking place in a G-classroom or a B-classroom. For the analysis of the scores, both the frequency and the amount of time spent on specific categories was taken into account.

RESULTS CLASSROOM OBSERVATIONS

Because it is impossible to dicuss all results in this paper, they will only be summarized here.

Results from the Utrecht observations

One of the points to come out of the Utrecht observations was that regarding *the class climate characteristics*, the following points were seen as being positive for girls' learning achievements: security, mutual respect and an ordered atmosphere with clear social rules. The classrooms that fulfilled these characteristics were nearly always characterized as G-classrooms.

As can be seen in Table 1, the predictions of the *classroom type* almost completely agree with the actual nature of the classroom. In total ten out of twelve possible predictions

	Predictions of the classroom type and the observers' arguments

	Classroom #2 (G)	Classroom #7 (G)	Classroom #13 (B)	Classroom #6 (B)
Observer U1	--	**G-Classroom** –less uncertainty –no challenge –no reason to be afraid of mistakes –focused on routines	**probably B-Classroom** –insufficiently safe atmosphere –strong focus on smartest solution –ego-focused –more participation boys	**probably G-Classroom** –focus on procedures –focus on CITO-like abilities –focus on achievement (= disadv. girls) –ego-focused (= disadv. girls)
Observer U2	**G-Classroom** – knowing strategies in advance –large degree of clarity in organisation –large degree of clarity in assignments –honest teacher's report that strategies were emphasized because of the research	**G-Classroom** –good atmosphere –good organization –structured way of working –boys made the mistakes	**B-Classroom** –instruction is rule-directed, but the speed is high –the lack of sufficient learning opportunities –problems with concentration (mixed-grade class) –textbook series with few possibilities to apply routines –didactical deficits of the teacher –more questions from boys	**B-Classroom** –high speed of the instruction –verbal character of explanations
Observer U3	**clear G-Classroom** –quiet atmosphere –structured teaching –teacher shows respect –didactics aimed at rules overall impression: affective + cognitive +/–	**clear G-Classroom** –good atmosphere, paternal teacher shows respect –didactics aimed at rules overall impression: affective + cognitive +/–	**B-Classroom** –unordered, chaotic–rather impersonal approach – very little mutual respect overall impression: affective – cognitive +/–	**B-Classroom** –aimed at indidual work; not enough collective moments –not enough learning support –students addressed too much on their individual qualities –shortcomings in didactics overal impression: affective +/+,– cognitive –
Observer U4	**G-Classroom** –secure and orderly classroom atmosphere –a lot of social room –instrumental explanation –no clear appeal to individual input (=indirect adv. girls) –leaving learning opportunities unused (=disadv. girls) –little initiative from girls (= disadv. girls)	**G-Classroom** –safe social atmosphere; teacher respects the students –didactically safe atmosphere; structured didactics and fixed approach –little emphasis on initiative, reflection, critical attitude (=indirect adv. girls)	**B-Classroom** –complex organisation; order problems and inefficiency in explanations–limited didactic quality –influence of realistic method (certain sums, models and approaches) that boys may be gaining more from	**B-Classroom** –not a really secure atmosphere –instrumental expalanation, but failing didactics (for example in estimation) –tasks are ambivalent –not always understanding children and not really looking into what they have found

Table 1: Summary of the observers' predictions of the classroom type (G or B) and their arguments

were correct (twelve predictions could be made, the four made by Observer U4 do not count, since this observer was aware of the type of classroom), Although this high correlation needs to be relativized because of the low number of degrees of freedom, especially the agreement in argumentation gives good indications for answering the research question.Based on the observations from the Utrecht team, few to no conclusions could be drawn regarding the *interaction characteristics*. This was caused in the first place by Jungwirth's interaction characteristics only being recognized in a very limited way during the lessons. Furthermore there was no gender specific pattern. See, for example, the following two observation, both from classroom #7, a G-classroom, and both relating to a boy.

Undoing completeness:	[Classroom #7; Lesson 1; U4] Harry gives a good answer to an exercise straightaway. The teacher did not expect this and starts to explain to the whole class how to get to this answer. (By explaining so elaborately how Harry got this answer, the suggestion is made that Harry himself cannot do this.)
Teacher's echo:	[Classroom #7; Lesson 1; U4] The teacher corrects Fred's answer and adds that Fred made a mistake. (Because the teacher says that Fred only made a slip of the tongue, he in facts indicates that Fred is competent.)

Despite the fact that Jungwirth's gender-specific characteristics were only found to a certain degree, it became clear that they can expose interesting mechanisms which can have an unmistakable influence on mathematics achievements, for example, by having an effect on the arising or not arising of learning opportunities (e.g. blocking task constitution vs task constitution) and by influencing how both students themselves and others regard their competence (e.g. undoing completeness vs teacher's echo).

When taking *didactical characteristics* as the point of view, the shortfalls in the implementation of realistic didactics stood out especially (a more detailed look at this, based on classroom vignettes, will be taken in the presentation at the conference). Classrooms where didactics fell short according to the observers were often classified as B-classrooms. Classrooms with a lot of structure and instrumental explanations were often classified as G-classrooms.

In short, the observations of the Utrecht team contain indications that a socially and cognitively secure atmosphere worked immediately to the advantage of the girls. An additional indirect advantage for the girls was that in the G-classrooms own cognitive input (such as knowledge of measures) and social input (for instance, taking the initiative) were not expected.. This was called an indirect advantage because it meant that boys could distinguish themselves less.. In the B-classrooms on the other hand, the lack of a clearly secure atmosphere worked directly to the disadvantage of the girls. Indirect disadvantages for the girls here were didactic shortcomings and insufficient learning opportunities (for example teaching wrong strategies, dismissing correct solutions, not teaching how to estimate, not building knowledge of measurements). More and more the perception arose that a bad implementation of RME is more disadvantageous for girls than for boys.

Results from the Leiden observations

It became apparent from the analysis of the Leiden team observation data that more thinking questions were asked in the two G-classrooms and that there were more breaks to think than in the B-classrooms. Especially classroom #7 devoted a relatively large amount of time to thinking breaks. This fits the pattern of cognitive support that was recognized, on the basis of the Utrecht observations, as an important characteristic of the observed lessons in this classroom. Furthermore, a more detailed analysis of the frequency of the categories showed that on the whole the students were taking a more active role in the learning process in the two G-classrooms. This point also arises from the observation that generally speaking more questions were asked in the G-classrooms.

DISCUSSION

The context of the research question at the basis of the MOOJ study is very complex and giving *the* final answer is difficult. The research has not only resulted in many new research questions, but also in a number of very penetrating points for discussion; especially regarding the role of RME. The RME approach to learning may be better teaching than the traditional education that existed in the Netherlands twenty-five years ago, but measured by mathematics achievements it is apparently not the best way to teach girls. As is suggested by the MOOJ observations, this might be caused by the fact that the RME approach is hard to find in classroom practice. It turns out that this especially hits girls. Badly implemented RME means that students have to rely, in a way, on their own abilities, which has as a result that boys, given their 'natural' abilities which better fit RME, do better in this situation than girls. Maybe girls, more so than boys, are more explicitly depending on education. Applying smart strategies, acquiring estimation strategies, developing knowledge of measurements etc, must not be pursued only as goals, but schools should offer sufficient learning opportunities and enable active participation by students.

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REMEDYING SECONDARY SCHOOL STUDENTS' ILLUSION OF LINEARITY: A TEACHING EXPERIMENT

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Previous research has shown that many secondary school students improperly apply linear models when solving non-linear problems involving lengths, area and volume of similar plane figures and solids. This phenomenon is called the "illusion of linearity". This paper presents a teaching experiment in which we developed and tested a learning environment to help students overcome the illusion of linearity. The experiment was successful in improving students' performance on non-linear problems, but this improvement was disappointingly small. Moreover, there was a decline in students' performance on linear problems. So, the experiment was not successful in developing in students a profound conceptual understanding of proportional and non-proportional relations, which includes the disposition to distinguish between situations that can and cannot be modelled linearly.

THEORETICAL AND EMPIRICAL BACKGROUND

According to Freudenthal (1983, p. 267), "linearity is such a suggestive property of relations that one readily yields to the seduction to deal with each numerical relation as though it were linear." This phenomenon is often referred to as the "illusion of linearity". It has been exemplarily reported in students of different ages and in different domains of mathematics and science education, such as elementary arithmetic, algebra and physics, and has been systematically studied in geometry (for an overview, see De Bock, 2002) and recently also in probability (Van Dooren, De Bock, Depaepe, Janssens, & Verschaffel, 2002).

In the domain of geometry, a systematic line of research by means of paper-and-pencil tests has shown a very strong tendency among secondary school students aged 12-16 to overgeneralise the linear (or proportional[i]) model to problems about the relationship between lengths and areas/volumes of similarly enlarged or reduced geometrical shapes (see, e.g., De Bock, 2002; De Bock, Verschaffel, & Janssens, 1998). These studies have shown that even with considerable support, such as self-made and ready-made drawings or metacognitive hints, only very few students made the shift from incorrect proportional to correct non-proportional reasoning. In a recent study with individual interviews (De Bock, Van Dooren, Janssens, & Verschaffel, 2002), information was obtained on the problem solving processes and explanatory factors underlying this tendency to produce linear answers. First of all, the results showed that the majority of the students use the linear model in a spontaneous, almost intuitive way, while some students really are convinced that linear functions are applicable "everywhere". Second, students have particular shortcomings in their geometrical knowledge (e.g., the belief that irregular figures have no area, or that a similarly enlarged figure is not necessarily enlarged to the same extent in two dimensions), prohibiting them from discovering the correct solutions for some non-linear problems. Third, many students have inadequate habits, beliefs and attitudes towards solving problems in (school) mathematics, leading to stereotyped and superficial mathematical modelling.

The next stage of the research program – which will be the focus of the current paper –involves the design, implementation and evaluation of a learning environment aimed at overcoming the illusion of linearity, more specifically in the context of the enlargement/reduction of plane figures and solids, and the effect on lengths, (surface) areas and volumes. We aim at developing in students a deep conceptual understanding of proportional and non-proportional relations and situations, the adequate geometrical knowledge base to solve this type of problems, and a more mindful approach towards mathematical modelling.

ORGANIZATION OF THE TEACHING EXPERIMENT

A series of 10 one-hour experimental lessons – including all teacher and learner materials□ was developed for use with 13-14-year old students. With respect to the purposes of the lesson series, the results and the conclusions of earlier studies (e.g., De Bock, 2002; De Bock et al., 1998, 2002) discussed above were taken into account. The development of the learning environment was moreover strongly inspired on the principles of realistic mathematics education (see, e.g., Gravemeijer, 1994). First, the lessons were interspersed with various realistic problem situations aimed at challenging students' mathematical (mis)conceptions, beliefs and habits that lead to stereotyped and superficial modelling. Second, the problem situations and tasks allowed rediscovering of the required mathematical notions, building on students' own productions and informal knowledge. Third, the learning environment relied on a combination of instructional techniques that have proven to be successful in enhancing students' deep understanding and higher-order thinking skills (e.g., articulation and reflection) (see also Collins, Brown, & Newman, 1989). A fourth characteristic was that multiple representations of the learning contents (such as drawings, schemes, tables, graphs, formulas, and words) were used and their reciprocal relationships were accentuated to enhance deep-level learning (NCTM, 2000).

During the 10 lessons, the following topics were successively addressed: recognizing and constructing similar figures/objects, proportional relations and their properties, linear growth of the lengths and perimeter in similar figures, quadratic growth of the area and cubic growth of the volume. The lesson series ended with a project about the "Life and Work of the Gnomes" (Poortvliet & Huygen, 1976), in which the students were engaged in attractive, challenging and authentic problems involving the combined application of all learnt contents. Examples of learning activities and exercises from the experimental lessons are given in Figure 1.

RESEARCH METHOD

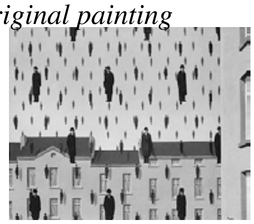
Two comparable groups of secondary school students (8th graders, aged 13-14) were involved in the study. The experimental group of 18 students followed the experimental lesson series, while the control group of 17 students followed the regular lessons (in which none of the contents under consideration was explicitly treated). All lessons in the experimental group were videotaped and all student notes were collected. Moreover, during the lesson series students' perceptions and evaluations of the lessons were registered by means of a questionnaire after the fourth and eighth lesson.

The learning gains in both groups were assessed by means of a word-problem test consisting of 2□proportional items (about the perimeter of an enlarged square or circle) and 4 non-proportional items (about the area or volume of an enlarged square/cube or irregular figure). Table 1 gives an example of a proportional and a non-proportional item. Three parallel versions of this test were constructed, which were, respectively, administered to the experimental group before the intervention (pretest), after the intervention (posttest), and three months afterwards (retention test). For practical reasons, the control group received only the pretest and the retention test.

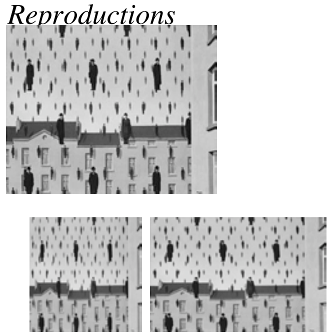
Similar figures/objects (lesson 1-2)

Which reproductions are similar to the original painting?

Original painting



Reproductions



Afterwards: examination of similarity of real objects (cans, envelopes, bottles, …)

Proportionality/Linearity (lesson 3)

A

B

	picture A	picture I
/idth of picture	… mm	… mı
iagonal of picture	… mm	… mı
eight of the cupboard	… mm	… mı

Put the data from the table in a graph and explore that graph. (e.g. "an object with a height of … mm on picture A is … mm high on picture B")

Perimeter: linear growth (lesson4)

Is a big cola bottle of 1.5 litres similar to a small cola bottle of 0.5 litres?

If you strip the labels of the bottles, the label of the small bottle is 5 cm high by 20 cm wide. The label of the big bottle is 7.3 cm high. When both bottles are similar to each other, what should be the width of the label of the big bottle?

Area: quadratic growth (lesson 5)

It's Anne's birthday and her mother is going to make pancakes, using three pans of different sizes. Anne asks her friend: "You may choose between two big pancakes (30 cm diameter), four regular ones (20 cm diameter) or six small ones (15 cm diameter)."

Her friend reasons as follows: "You better choose six small pancakes because 2 □ 30 cm = 60 cm, 4□□0 cm = 80 cm and 6 □ 15 cm = 90 cm"

What do you think about the reasoning of Anne's friend?

Volume: cubic growth (lesson 6-7)

An apple grower sells two sizes of apples: The first one has an average diameter of 6 cm and costs 10 eurocent and the second one has an average diameter of 9 cm and costs 20 eurocent.

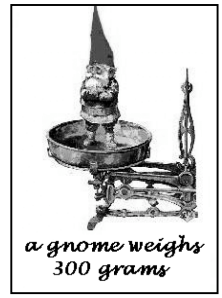
→ Compare both sizes of the apples. What is the enlargement factor (k)?

→ The apples have a similar shape. How much more weighs a big apple compared to a small apple?

→ Which apple size is the most economical to make apple sauce?

Another apple grower has other prices for the two kinds of apples: "The apples with an average diameter of 6 cm costs 1/kg, those with an average diameter of 9 cm costs 1,20/kg". Which apple is the most economical now?

Integrative project (lesson 8-9-10)



Assuming that a gnome is similar to a human being, is it possible than that a gnome with length 15 cm weighs 300 grams?

How long is the belt of a gnome?

What is the area of the sole of a gnome?

How much coffee is there in a cup for gnomes?

Figure 2. Examples of materials used in the lesson series

Table 2. Examples of word problems used in the test

Proportional item (perimeter)	*Non-proportional item (volume)*
Steve needs 10 minutes to dig a ditch around a square sandcastle with a side of 50 cm. How much time will he approximately need to dig a ditch around a square sandcastle with a side of 150 cm?	In his toy box, John has dice in several sizes. The smallest one has a side of 10 mm and weighs 800 mg. What would be the weight of the largest die (with a side of 30 mm)?

All problems were offered as open questions, and students had to write down their answer as well as their calculations. Answers were scored either as correct or as incorrect, and incorrect answers were further categorized in terms of one of the following categories, based on a scrutinized analysis of the students' solution steps: application of proportional methods to non-proportional items, application of non-proportional methods to proportional items, other errors.

RESEARCH QUESTIONS AND HYPOTHESES

The goal of this study was to test whether a learning environment with the above-mentioned characteristics could cause a substantial reduction in students' tendency to produce linear answers in situations where they are not correct. Based on our previous studies (De Bock, 2002; De Bock et al., 1998, 2002) we expected that on the pretest, experimental and control group students would generally respond correctly to the proportional items and incorrectly on the non-proportional items, because of their tendency to apply proportional strategies for these latter items too. Due to the learning experiences in the experimental lessons, we expected a significant progress in the performance of the experimental group on the posttest – more specifically on the non-proportional items – and that this progress would largely persist on the retention test. For the control group, no significant evolution from pretest to retention test was expected, because these students were not involved in any learning activities specifically addressing the linearity misconception.

RESULTS

A 2 × 2 × 3 repeated measures ANOVA was conducted with ‘group’ (experimental vs. control), ‘item’ (proportional vs. non-proportional) and ‘test’ (pretest vs. posttest vs. retention test) as independent variables and the performance on the word problems as the dependent variable. Based on our hypothesis, we expected a significant ‘item’×‘group’×‘test’ interaction effect. The results of the ANOVA confirmed this expectation, $F(1,488) = 4.80$, $p = 0.0290$. An overview of the percentages of correct answers is given in Table 2. Because of the significant three-way interaction effect, all pairwise differences in this table were statistically tested by means of post-hoc Tukey tests (correcting for multiple comparisons).

Table 3. Percentage correct answers (and standard deviations) of the experimental and control group on the proportional and non-proportional items at the three test moments

	Proportional items						**Non-proportional items**					
	Pre		**Post**		**Retent**		**Pre**		**Post**		**Retent**	
	%	SD	%	SD	%	SD	%	SD	%	SD	%	SD
Experimental	83.3	7.8	58.3	7.8	52.8	7.8	29.2	6.2	61.1	6.2	50.0	6.2
Control	85.3	8.1			73.5	8.1	13.2	6.4			16.1	6.4

As expected, on the pretest there was a significant difference between the performance on the proportional items (which were solved very well) and the non-proportional items (which were mainly solved incorrectly), both in the experimental group, $t(488) = 6.56$, $p = 0.0001$, and in the control group, $t(488) = 8.48$, $p = 0.0001$. Again, this is evidence for students’ overwhelming tendency to produce proportional answers in non-proportional situations. Moreover, at the pretest there was no significant difference between the experimental and control group, indicating that both groups were indeed comparable. Both groups performed almost the same on the proportional items, and the difference for the non-proportional items was also not significant[ii].

We will first discuss the results of the control group. Afterwards, we will examine more closely how the performances of the experimental group evolved from pretest to retention test.

With respect to the control group, we did not expect a significant evolution from pretest to retention test. In line with this expectation, we observed a very small, non-significant increase in the performance on the non-proportional items (from 13.2% to 16.1% correct answers) and a non-significant decrease in the number of correctly answered proportional items (from 85.3% to 73.5%) from pretest to retention test. A qualitative analysis of the protocols in this group showed that, first, as expected, the percentage of answers resulting from an improper application of linearity on the non-linear items on the pretest and retention test was about 80%. Second, an increase in the number of overgeneralisations of non-linear strategies to linear items could be observed (from about 11% to 18%). Probably as an effect of retesting, some students started to apply non-proportional solution methods to the proportional problems they solved correctly before. As will be explained below, this is similar to observations made in earlier studies (De Bock, 2002; De Bock et al., 1998, 2002).

In the experimental group, there was a significant improvement in the performances on the non-proportional items from pretest (29.2%) to posttest (61.1%), $t(488) = 3.09$, $p = 0.0001$, followed by a non-significant decrease in the performances from posttest to retention test (to 50.0% correct answers) This means that students in the experimental group made a significant progress in their performance on the non-proportional items, and this progress persisted over several months. However, this improvement in performance was not as high as we had expected. Contrary to the results for the non-proportional items, the score of the experimental group on the proportional items decreased from 83.3% correct answers on the pretest to 58.3% on the posttest, $t(488) = -2.62$, $p = 0.0090$, and went further down from posttest to retention test (although not significantly) to 52.8%. Apparently, in line with our earlier studies, when these students discovered that some problems can not be solved by applying proportions, they started to apply non-proportional solution schemes to proportional problems too (De Bock, 2002; De Bock et al., 1998, 2002). An additional qualitative analysis of the answers of the experimental group revealed first of all that on the pretest, about 70% of all the solutions on the non-proportional items could indeed be characterized as linear. This number of unwarranted linear answers strongly decreased in the posttest to about 18%, while in the retention test, the percentage raised again to about 30%. But students who no longer applied linear solutions to solve non-linear problems, did not always perform better than before: in the posttest and retention test they made errors in applying non-linear solutions on these non-linear problems (such as confusing area and volume, just taking the square of one of the given numbers, ...). The qualitative analysis also confirmed the overgeneralisation effect: while on the pretest only 13% of all the solutions to linear items could be characterised as an application of non-linear strategies, this number raised to 36% on the posttest and retention test.

The results of the experimental group on the posttest and retention test revealed the fragile and unsteady nature of these students' emerging non-proportional reasoning scheme. A careful analysis of the videotapes of the experimental lessons supported these conclusions. Certain fragments indicated that non-linear relations and the effect of enlargements on area and volume remained intrinsically difficult and counterintuitive for many students. For example, there were students who at the same time understood that the area of a square increases 4 times if the sides are doubled in length (since the enlargement of the area goes "in two dimensions"), while they had difficulty in understanding why this does *not* hold for the perimeter (which also increases in two "directions", according to a student).

CONCLUSIONS AND DISCUSSION

In general, the results of this study confirmed our hypothesis. Initially, both the experimental and the control group performed well on the proportional items but often failed on the non-proportional items, due to the application of linear methods. After the experimental lesson series, the experimental group applied linear solution methods less often on the non-linear items on the test. Apparently, the illusion of linearity was broken in these students. However, a considerable part of the non-linear items on the posttest and retention test were still solved erroneously, due to linear reasoning or to mistakes in the application of non-linear strategies. Moreover, at the posttest and retention test, the experimental group made more errors on the proportional problems, because they started to overgeneralise the newly learnt non-proportional strategies to the proportional problems they previously solved very well. It seems that after the lessons, the students still experienced serious difficulties in knowing which model they had to use in which situation.

Therefore, we can hardly argue that the lesson series has reached its goal. The experimental lessons were unable to develop in the students a deeper understanding of proportionality and non-proportionality, and a disposition to distinguish between situations that can and cannot be modelled proportionally. The findings indicate that the experimental students' emerging non-proportional reasoning scheme remained fragile and unsteady.

A first possible reason is that our intervention involved 13-14-year olds, i.e. students at an age where the proportional reasoning scheme was already well established and practiced – explaining why non-proportional relations were experienced as counter-intuitive by some students. It seems, therefore, important that, at the first time when students meet proportional relationships in their mathematics curriculum, they are also confronted with counterexamples (situations where linearity does not work). As a second reason, it seems that an intervention of only 10 hours – that was moreover separated (and considerably different from) the regular mathematics curriculum – was not satisfactory to change the students' habits and beliefs contributing to a superficial modelling process, while these habits and beliefs are an important facilitating factor in the occurrence of improper linear reasoning (see De Bock et al., 2002).

The results on the tests and the (ongoing) analysis of the videotapes offer many valuable indications for the development of an improved version of the current learning environment, which will be implemented and tested on a larger scale in the near future.

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[i] The terms *linear* and *proportional* are here used as synonyms, referring to relations of the form $f(x) = cx$, for which the properties $f(a + b) = f(a) + f(b)$ and $f(ka) = k\,f(a)$ hold, graphically represented by a straight line through the origin.

[iiii] Despite the non-significant outcome of the Tukey test, the difference in both groups' performances on the non-proportional items seemed to be meaningful. Therefore, an additional analysis of covariance (ANCOVA) was performed, predicting the performances on the non-proportional items on the retention test on the basis of the group (experimental/control) *correcting for the performances on these items on the pretest* (and thus cancelling out any differences between the groups at the pretest). The corrected means of both groups at the posttest (44.3% and 22.3% for the experimental and control group respectively) were still statistically different, $F(1,32) = 6.42$, $p = 0.0164$.

THE VALUE OF WENGER'S CONCEPTS OF MODES OF PARTICIPATION AND REGIMES OF ACCOUNTABILITY IN UNDERSTANDING TEACHER LEARNING

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Ongoing efforts to include social contextual dynamics in research on mathematics education require theoretical frameworks that allow researchers to zoom their focus out to include the social sphere (Lerman, 1998). This is especially important when studying teacher development, as teachers learn in a wide variety of social contexts. Here we illuminate two key concepts that are helping us better understand the affects of various communities on the development of our cooperating teachers. These concepts are Wenger's (1998) modes of participation *and* regimes of accountability. *By looking specifically for variation among different communities' regimes of accountability, and the modes of participation they engender, we are able to better understand the development (or lack thereof) that we are seeing take place in our participants.*

INTRODUCTION

In a recent special issue of *Educational Studies in Mathematics*, Kieran, Forman, and Sfard (2001), and their contributing authors, present examples of, and potential directions for, the ongoing movement toward inclusion of the social in the study of learning mathematics. This movement, which entails efforts to coalesce cognitive views of learning with social and cultural views of learning in communal practice, has led to much rich theoretical and methodological development. The editors note, however, that this social turn in education research presents a great difficulty. This difficulty stems from the complexity that considering potentially innumerable social factors brings to the task of developing coherent theoretical and methodological frameworks from which to satisfy the need for scientific trustworthiness. Lerman specifies some of those factors:

> The mathematical practices within a class or school, the way in which they are classified and framed, the state/community/school values which are represented and reproduced, and teachers' own goals and motives, form the complex background to be taken into account by the research community (Lerman, 2001, p. 101).

We believe that the study of teacher learning presents an especially perplexing case of this problem. Whereas the learning of school mathematics happens for students within an individual classroom community where these other factors can be considered as (certainly essential) "background," we are beginning to understand the great extent to which these other factors are actually part of the foreground for the teaching of mathematics. That is because mathematics teachers participate as adults in the communities formulating the factors Lerman lists, and the teaching of mathematics happens at the intersection of these communities. In our attempts to understand how, why, and what teachers learn, we are increasingly understanding these communities and the multitude of entailments they encompass as the necessary targets of, rather than as a supplementary resource to, our research. We follow others (Zeichner, 1985; Raymond, 1997; Stein, Silver, & Smith, 1998) in adjusting the zoom of our gaze (Lerman, 1998) out to include a broader range of social factors in our research on teacher development.

In this paper we use a case to illuminate two concepts from Wenger's (1998) work – modes of participation and regimes of accountability – that have been pivotal in helping us do just that. Our goal is to illustrate how these concepts have aided our understanding and to propose broader adoption of them as a means of adding to the theoretical coherence of the field of socially-oriented research on teacher development.

MODES OF PARTICIPATION & REGIMES OF ACCOUNTABILITY

Our desire to understand the affects of a range of different communities on the learning and practices of mathematics teachers has led us to operationalize, in a framework for our analytic coding, Wenger's (1998) ideas about the social nature of learning in community. We have done this with a conceptual framework we call mathematics teacher identity (Bohl & Van Zoest, 2002). With this framework we've attempted to delineate and organize the capacities that individuals develop relative to teaching mathematics in a way that makes clearer the relationships between those capacities and the social contexts within which they are learned. The framework places different forms of cognition and social capacity along a continuum from cognitive to social. On the extreme social end are what Wenger (1998) calls modes of participation. These are the ways that people act and interact relative to the endeavors of a social group.

Wenger proposes three forms of activity that comprise a community's modes of participation. The first is mutual engagement. A teacher does this directly in the community of the classroom with her students or in school department meetings with other teachers. This also happens indirectly when teachers act in a particular way as a result of identification with a specific broader community, such as a university's reform mathematics education community (see, e.g., Skott, 2002). The second mode of participation is understanding and fine tuning the jointly-held conception of what is to be done. This happens as participants negotiate their jobs individually and together with the community's end goals in mind. This is particularly relevant to current work in reform mathematics education as the ideal classroom practice, i.e., the end goal, is still relatively ill-defined and open for negotiation. The third mode of participation is developing a shared repertoire of resources, discourses, and styles for accomplishing the necessary tasks. This regards the "how" of actually implementing the types of practice that will support a community's goals.

Modes of participation are learned through engagement with others and, perhaps more importantly, are also entirely embedded within acts of engagement with others. One's competence, or lack thereof, within a community (be it a classroom, a professional organization, or a cohort group of teachers) is related to the ways that one can use knowledge gained from other participation experiences to develop a set of modes of participation that is evaluated positively within the standards of the community.

This concern with the evaluation of individuals' participation is where the concept of regimes of accountability comes in. With this idea, Wenger folds into his conception of communities issues of discursive power and the ways it impacts social activity. A community's regime of accountability is the set of discursive characteristics and administrative arrangements that serve to regulate people's activities by providing criteria and systems for their evaluation. These are both explicit and implicit. Explicit parts of a school's regime of accountability might be a set of rules regarding teachers' need to adhere to a specific regional curriculum guide to determine what content needs to be taught and when. This can come to bear on teachers through either a regime of official school sanctions related to teaching according to the guide, or through unofficial channels like variations in administrative support. An example of an implicit aspect of a regime of accountability might concern the types and quantity of homework that a community deems appropriate (e.g., as in Forman & Ansell, 2001). Such values might come to affect a teacher through either colleague's differential levels of willingness to engage in conversation about other homework options, or through differential levels of support provided the teacher by administrators when parents question new and different homework assignments.

THE CASE OF SILVIA

To illustrate the way in which we have found the ideas of modes of participation and regimes of accountability useful in the interpretation of our data, we offer a case scenario from our recent research. This case serves two purposes: it illustrates many of the issues that we have seen in our own and others' research studies, and it allows us to frame these concerns within the conceptions described above. Although the case concerns a new teacher, Silvia, many of the issues apply to experienced teachers we've worked with as well.

Silvia graduated from a pro-reform mathematics teacher preparation program and was part of a special project that involved her in professional meetings and development opportunities during the final two years of her program. She came to her methods courses believing strongly in the reform effort because she felt she had not developed much understanding of mathematics herself during her "successful" history of mathematics courses. In terms of actual teaching experience, Silvia participated in several extended periods of classroom participation prior to her internship. She also interned with a reform-supportive veteran teacher who was herself in the process of formulating what a reform-oriented classroom community would look like. Her internship specifically required that she plan lessons using a reform-oriented textbook, and participate in both team teaching and solo instruction over the course of three months. During that time, she established herself as capable and earned the praise of her mentor.

Through her reform-based university coursework, internship in a reform-supportive classroom and school, and many experiences working with reform-oriented teachers, Silvia developed a strong facility with the language and beliefs of the movement. This became apparent in her first job when she found herself serving, relatively successfully, as a spokesperson for her department's reform efforts when parents or the school board needed explanation. She remained active in the broader professional communities which she had joined before beginning teaching, thriving on the sense of mutual engagement toward the goal of improving mathematics education, and drawing much personal strength from her understanding that her efforts to teach in a reform manner were part of a broader national movement.

Within her classroom, however, Silvia had great difficulty developing a repertoire of methods that would support the discourse-oriented classroom community that she described as her goal in several interviews before and during her internship. Her students did not willingly participate in the whole class discussions and activities. When she attempted cooperative group investigations, students spent much time off task, and when they were working on mathematics they tended to be working individually and using each other only to check the correctness of their answers. There did not seem to be much student development of understanding going on in any of numerous observations.

Troublingly, even though she had a facility with the language of reform and extensive experience evaluating teaching and curricula from a reform perspective, Silvia did not appear to realize the gulf between what she had envisioned earlier and how her current classes were running. She did mention that she was disappointed that she felt she had to tell some students the answers before they would be satisfied. However, she did not perceive that authority problem as dominating her class and turning her whole-class discussions into instances of her dispensing information. She also did not appear concerned about the amount of time students were off-task, or by the fact that there was little substantial conversation going on in her student groups. Indeed she viewed herself as doing a relatively good job of implementing reform, and maintained the belief that she only needed to continue to make small improvements in order to reach her ideal.

In short, her modes of participation as a classroom teacher, specifically her understanding of the task and her repertoire of methods, were very much out of sync with her beliefs both as professed and as exhibited by her modes of participation outside the classroom. Further, her evaluations of her own teaching were much more positive than those of outside observers trained to measure reform instruction. Thus, Silvia seemed to be functioning under a different regime of accountability when in her classroom.

EXAMINING DIFFERENCES AMONG COMMUNITIES

Inconsistencies like those between Silvia's in-classroom and extra-classroom participation were disappointing. She and others had been involved in intense, early, and ongoing participation in multiple communities that supported reform, and had reflected often over a two-year period about what they were seeing in others' classrooms, and what they hoped to achieve in their own. Why was she, then, able to teach in a reform manner less than even moderately well, and only slightly better than others who had not participated in the extra professional and reflective experiences and thus were less versed in reform? Although we are still completing an analysis aimed at addressing this question, our initial findings are indicating to us how the ideas of regimes of accountability and modes of participation provide a coherent framework for considering differential community impact. Here we apply these ideas to Silvia's case as an illustration.

During her teacher preparation, Silvia had been a successful participant in many communities where she had been required to converse and reflect about the tenets of, teaching methodology and curriculum for, and justifications underlying reform teaching. These communities varied in their makeup, but most included both developing and veteran teachers (both reform supporters and reform skeptics), school administrators, and researchers. The modes of participation in these experiences where largely conversational in nature and the regimes of accountability were fairly undemanding. For the most part, successful participation required only reflective, thoughtful conversation about teaching and learning from a reform perspective, and an open mind concerning what it entails. When Silvia moved on to the extra-classroom communities at her job site—her department and school, the local parents, and the broader professional community—she found regimes of accountability requiring the same conversational modes of participation for her success. Even in situations that involved new dynamics (e.g., discussions with school board members), she was able to utilize her competence because such situations called for modes of participation similar to those she had experienced, and existed within regimes of accountability similar to those she had successfully operated within previously.

In order to understand why a relatively high level of success in these various communities did not translate into a reform classroom practice, it helps to realize that a teacher's own classroom environment is very different from these other communities in terms of what is required for success. This is because the teacher serves as the architect of that community's regime of accountability, within which students function, rather than as an equal co-participant. She acts within the modes of participation that she establishes for herself within that regime, but she is actually not held accountable to it in any real sense. The regimes of accountability within which she acts (or better, to which she is responsible) are those of her department, her school's administration, and the reform community.

Silvia came to her classroom with some experience maintaining a relatively reform-oriented classroom with the help of an overarching authority (the classroom teacher in her field experiences), and no experience either establishing such a community or maintaining one as the sole authority over a group of students. She knew how to discuss the ideal of a reform classroom, but had an insufficiently developed repertoire of resources, discourses, and styles for establishing and orchestrating a regime of accountability that would promote the types of student modes of participation she envisioned. Furthermore, there were no supports in place to assist her in that development and, in particular, no community within which to directly engage in the joint activity of improving the participants' classroom teaching.

Having established some sense for why Silvia had not developed a strong reform program in her classroom, we need to better understand why it is that she was not better able to perceive her shortcomings. We work under the assumption that individuals generally want to experience themselves as being successful, and want others to view them as successful, at the things they choose to do. It follows, then, that individuals will respond in some way to the regimes of accountability of the communities wherein they participate, as it is through those regimes that one's level of success is determined. The immediate regime of accountability to which Silvia felt most answerable was that of her school administrators. The requirements of her administrators where fairly rudimentary, including coverage of material, maintaining a controlled classroom environment, and maintaining positive relationships with her students. However, she wanted to also satisfy the regime of accountability of the greater reform community of which she self-identified as an active member. Further, she needed to accomplish this with a classroom community that she was comfortable orchestrating. In the end, she used "cooperative groups" and "investigations" where students actually worked individually sitting next to one another with the goal of getting correct answers; had students "justify their thinking" by showing all of their work; and "respected students mathematical understandings" by having excelling students show other students how to get answers. In short, she recontextualized (Bernstein, 2000; Ensor, 2001) the meanings of these reform mathematics terms so that they helped her fulfill several necessary criteria for her success in these various communities. These are: the need to maintain a quiet, controlled classroom where she could cover sufficient material as specified by the regime of accountability of her administrators; the need to be able to describe her practice using the language of reform, as supported by the regime of accountability of the broader reform community; and her need to create a classroom regime of accountability for her students that she and her students felt comfortable working within. This latter regime ended up looking very much like those of the communities with whose modes of participation she'd had extensive practice—the traditional classes of her own mathematical training.

FURTHER CONSIDERATIONS

What we are seeing is that our participants have learned much about the modes of participation that they actually practiced within in our programs, and were better able to transfer that learning to communities where the regimes of accountability paralleled those of the communities wherein they were required to put those modes of participation to use. This is similar to what Boaler (1998, 2002) has documented relative to student learning in different types of classroom communities. She points out that it is not so much the case that students in different types of classrooms (traditional versus reform) develop different *amounts* of capacity, but rather that they develop different *types* of capacity that are related to different types of applications and contexts. Although our tentative finding seems obvious to us in retrospect, we would have been blind to it had we not zoomed our gaze out to include our participants' actions as teachers outside of their classrooms as well as the characteristics of the communities in which they were participating.

Our initial results are helping us develop a sense for the ways that variations in regimes of accountability and modes of participation relate to the transferability of conceptions and action among communities. Although our understanding of the mechanisms of those relations is still tentative, it is clear that the types of community situations where our participants are showing the clearest development are those that are highly similar to the learning situations we provided. We think of this similarity in terms of levels of isomorphism between the learning and application community contexts. The more isomorphic the learning and application contexts were, the more the learning was apparent in application. This is hardly a novel idea; brought to the fore in the current discussion by Brown and his colleagues (Brown, Collins, & Duguid, 1989) two decades ago, this idea has been explored extensively relative to the learning of both mathematics and mathematics teaching (see, e.g., Newmann, Secada, & Wehlage, 1995; Bobis, 2002). The social focus provided by Wenger, though, offers a new perspective that highlights different characteristics of learning contexts that might need to be taken into account in the design of learning opportunities for teachers. As we continue our analysis we will be working to delineate the modes of participation and regimes of accountability of both teacher education and classroom contexts. Our hope is that the resulting knowledge will aid us in our work as professional developers to create communities and activities with regimes of accountability and modes of participation that will improve the likelihood of the learning that takes place becoming a resource for future classroom teaching.

IN CLOSING

Regarding the effects of communities on development, Lerman states that,

> As a person steps into a new practice, in social situations, in schooling, in the workplace, or other practices, the regulating effects of that practice begin, positioning the person in the practice. (2001, p. 98)

The social turn in research challenges us to understand those regulating effects and learn how to use them to the advantage of the development of classroom teachers. With this paper we have attempted to illustrate how our own work is moving in that direction. The main point is not so much about the very tentative conclusions we are beginning to draw from our research, but rather about the illumination of the theoretical concepts that are pointing us to them. Wenger's conceptions of modes of participation and regimes of accountability have helped train our gaze on aspects that our analysis suggests are critical when studying teacher development from a socio-cultural perspective. They provide what we believe is a lens appropriate to the task of understanding the complex situation of teaching, where contexts (i.e., communities) vary in form and objective, and overlap to create a hybrid at their intersection in classroom practice. In the introduction we commented on the call to develop a coherent theoretical framework for tying the social into studies of learning. Our hope is that these concepts will become part of that effort, and that other researchers will join us in exploring the implications of the new concerns that they bring into view.

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COMPUTERS IN THE PRIMARY CLASSROOM: BARRIERS TO EFFECTIVE USE

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The computer has been in the world's mathematics classrooms for some period of time now, but with varying degrees of implementation in the promotion of student learning. In this paper we describe longitudinal research over seven years which has investigated how primary school teachers actually make use of computers in their mathematics classroom, and what they perceive to be the major obstacles to improving or extending such use. The results show that while there are now many more computers available to primary teachers and there is some change in the software used, there are still a number of key barriers to increased use of computers as mathematical learning tools.

INTRODUCTION

Increasingly research literature over the last ten years has described the implementation of computers in school learning as rather patchy, and somewhat sparse (Ely, 1993; Askew and Wiliam, 1995; DFE, 1995). Recently Ruthven and Hennessey (2002) surveyed reports of school computer use and conclude that "Typically then, computer use remains low, and its growth slow." (p. 48). Considering possible reasons behind this situation Veen (1993) has argued that teacher factors outweigh school factors in the promotion of computer use, and Maddux (1994) agrees that computers will continue to be seen as relatively unimportant until most teachers incorporate them into their teaching. A number of reasons for the failure of many teachers to use the computer have been described (e.g., Thomas, Tyrrell & Bullock, 1996), including the upheaval resulting from the computer's presence; an unwillingness to change classroom management techniques; not wanting to lessen teacher control in the classroom; and an inability to focus on the mathematics and its implications rather than the computer.

Considering the primary school situation in particular, a survey on types of software use by Niederhauser and Stoddart (2001) found that primary teachers use both skill-focused and open-ended software. Considering reasons why teacher do not use the computer, Dunn & Ridgway (1991, 1994) found that, among student primary teachers, a lack of teacher confidence was the major reason why the computer was not employed in class, with 40% of one sample saying they were unsure or anxious about using computers. The study by Thomas (1996) agreed with this finding, citing 55% of primary teachers as giving the same reason.

This paper reports on a study which tracked computer use in the primary classroom at two discrete points seven years apart and sought to identify both patterns of usage and perceived barriers to use, with the aim of recording advancements in implementation and changes in attitudes to obstacles preventing it.

METHOD

In 1995 a questionnaire on computer use was sent out to every primary/intermediate school in New Zealand and replies were received from 480 of the 2471 schools (19.4%) in the primary/intermediate sector. Within each of these schools a number of teachers responded and so 1500 individual teacher replies were obtained. Some of the results of this survey of computer use were published by Thomas (1996). In 2002 it was decided to follow up the earlier survey in order to gain some longitudinal data on how the situation might have changed in schools over this time. While there are well over 2000 primary schools in New Zealand many of those in rural areas are very small, sometimes with only one or two teachers. These small schools, while very important in New Zealand, were not considered typical examples of the social and economic activity occurring in a larger school which would have many more teachers, a larger budget and probably more computers. Hence in the follow up study we chose to survey Auckland, a major centre with one third of the student population of the country, which is likely to be more representative of these types of schools. On this occasion 300 questionnaires were sent to the ICT or/and mathematics leader in all the primary and intermediate schools in the Auckland area, and 87 (29%) were returned. This level of response, which is acknowledged as quite good for a postal survey, should give a representative sample. Many of the questions asked the leader to fill in answers for all the computer use they were aware of in their school, while others asked for the teacher's personal professional opinion. Only a single teacher was employed because in the intervening years since 1995 it has become apparent that teaching has become an even more stressful and demanding profession in many ways, and particularly in terms of demands on time. Teachers are more reluctant than ever to spend their valuable time filling in forms or research questionnaires, and hence we targeted the person we thought would feel the greatest obligation to complete the questionnaire, and would be more interested in doing so. We also offered the incentive of a full report of the findings to all who responded.

The questionnaires sent out in the two years were not identical due to the differing requirements in 2002 (e.g., questions on the use of the internet) but instead had a number of questions in common. However, on both occasions they provided valuable data on: the number of computers in each school; the level of access to the computers; available software; the pattern of use in mathematics teaching; and teachers' perceived obstacles to computer use (see Figure 1 for a selection of questions from the second survey). This data has enabled us to come to some conclusions about the changing nature of computer use in the learning of mathematics in New Zealand primary schools.

RESULTS

In 1995 39.1% of the primary teachers said that they did not use computers at all in their teaching, but this had fallen to 11.4% by 2002. Of those who use computers, 93% used them in their own classroom in the first survey, and this figure has remained constant through out the period. One change though is the growing number of teachers using ICT labs that have been set up in schools as numbers of computers have increased, with 71% now able to access computers in these labs, although all of these used them in their classroom as well.

In 1995 the primary schools had a reported mean of 7.0 computers per school, although teachers saw this as insufficient and were asking for more computers in their classroom. 2.9% of the schools claimed to own no computers at all, 22 schools (4.6%) had 20 or more and two schools had 40 and 41 computers respectively. In addition 8.3% of the schools had a computer room. By 2002 there had been a huge jump in these numbers, with an average of 46.4 computers per school and only two schools (0.02%) claiming not to have any computers at all.

	Computers in Primary Mathematics Questionnaire School: ______________________________ Decile: ______ Name: ______________________________ (Name of teacher who is filling in questionnaire on behalf of your school) Email: ______________________________ (Your personal or school email address)
Q1	How many of each of these computers does your school have? (Exclude computers for administration purposes only) PC _______ Mac ________ Other ____________________
Q3	Where do teachers have access to computers? **In their Classroom** ☐ **ICT Room** ☐ **Staffroom** ☐ **Library** ☐ **Office** ☐ **Nowhere** ☐ **Other**______________________________ If teachers do not have access to computers please go to Q13
Q7	Do any teachers use computer software programs with students in their Mathematics Programme? No ☐ Some ☐ Most ☐ All ☐ If no teachers use computer software with students in Mathematics go to Q11
Q8	How often do most teachers use computer software with students in their Mathematics Programme? **Everyday** ☐ **At least once a week** ☐ **At least once a month** ☐ **At least once a term** ☐ **At least once a year** ☐ **Never** ☐
Q9	Please rank these types of software in the order in which teachers most often use with their students in Mathematics. i.e. 1 for most often, 2 for next etc. Leave blank any you do not use. **Mathematics program** **Spreadsheet** **Database** **Data handling** **Logo** **Mathematics Game** **Other**____________
Q10	Please rank these mathematics strands in the order in which teachers most often use computer software in Mathematics. i.e. 1 for most often, 2 for next etc. Leave blank any you do not use. **Number** **Statistics** **Geometry** **Algebra** ____________ **Measurement**
Q16A Q16B	Would you like computers to be used more often in Mathematics at your school? If you answered yes to Q16A, what do you see as obstacles? Please tick. **Yes** ☐ **No** ☐ **Myself** **Lack of training** **School policy** **Government policy** **Availability of software** **Availability of Internet access** **Lack of technical support** **Availability of computers** Other(s)______________________________

Figure 1. Sample questions from the 2002 primary teacher computer use survey.

Now 68 schools (78.1%) have 20 or more computers with 27 (31.0%) having 50 or more and 4 schools have more than 100 computers. Also 71% of the schools now have a computer or ICT room. In one school there were 200 school computers and another 300 laptops belonging to students, who were all encouraged to obtain one. 10% of teachers themselves now have access to a laptop computer in the school. One reason for thus growth in the numbers of computers in primary schools is the support of the New Zealand government, which has pushed a considerable amount of funding into getting computers into schools. The question arises though as to whether these increased numbers of computers have changed the pattern of use in the teaching of mathematics, remembering that in most of these schools the teachers will be general teachers who teach all curriculum subjects.

COMPUTER USE IN MATHEMATICS TEACHING

Although access to computers is now relatively good for teachers and students, only 3% of schools said that all of their teachers use software with students in their mathematics programme and 15% said that most do. A large proportion of schools (70%), said that only some teachers at their school use software in mathematics, and 12% of schools did not use any software at all. When those who did use the computer were then asked how often they used it in their teaching the results were as shown in Table 1. From this we can see that there has been an increase in the frequency of computer use in mathematics. Now 3.4% claim to use the computer every day, and the number using them at least once a month has risen from 45.2% to 76.6%.

Table 1: Primary Teachers Using Computers in their Mathematics Teaching

Amount of Use	% of 1995 teachers (n=1500)	% of 2002 teachers (n=84)
Every day	---	3.4
At least once a week	31.9	43.7
At least once a month	13.3	29.5
At least once a term	11.5	11.4
At least once a year	3.2	6.9

The mathematics curriculum in New Zealand schools is divided up into strands, with the content being Number, Statistics, Geometry, Algebra and Measurement, along with a Processes strand. An emphasis on number work in the primary school is to be expected, and even in 1995 a sizeable proportion (83%) was using computers in the number strand content area in the schools, and this has continued with little change into 2002 (now 80%—see Table 2).

Table 2: Curriculum Areas where Primary Teachers are Using Computers

Area of Use	% of 1995 teachers (n=915)		% of 2002 teachers (n=84)	
	Some Use	Most Often Used	Some Use	Most Often Used
Number	83	59	80	62
Geometry	51	10	52	1
Statistics	45	11	64	22
Measurement	30	2	40	1
Algebra	25	2	44	0

Results based on replies to Q10 in Figure 1.

There has been, according to these figures, an increase in the use of computers for the learning of Statistics, Measurement and Algebra in the primary school over this period, with Statistics leading the way. This not surprising since there is a strong emphasis on Statistics in New Zealand schools (Statistics is seen as a separate subject from Mathematics in New Zealand), and it lends itself readily to an approach where the computer can be used to perform routine calculations such as means as well as investigational work.

To gain some idea of the variety of uses that computers are being put to in primary schools we asked the teachers to rank in order of regularity of use the types of software they employed in teaching mathematics (see Q9 in Figure 1). The results from each of the two surveys can be seen in Table 3.

In the first survey the most common use of software was mathematical programs. However, there was some doubt as to what type of programs these might be. This time we also included the option of mathematics games, and while many teachers (61%) also use other programs, it is these games that are the most popular choice, with 78% of teachers making some use of them.

Table 3: Types of Software Primary are using with Computers

Area of Use	% of 1995 teachers (n=915)		% of 2002 teachers (n=84)	
	Some Use	Most Often Used	Some Use	Most Often Used
Mathematical Programs	78	56	61	25
Word Processing	43	19	2*	1*
Graph Drawing Package	27	7	---	---
Database	23	5	32	1
Spreadsheet	22	6	59	14
Data Handling Package	12	1	20	0
Mathematics Games	---	---	78	45
Logo	---	---	10	2

*Likely to be an underestimate since it was not included in the options given.

This could be an interesting area for further research, since although it is possible to make good use of games in mathematics learning it is also very easy to use them simply in a drill and practice mode. As Noss and Hoyles (2000, p. 219) observe about computer games, they "typically cast children in the role of game-player, playing according to the rules programmed by someone else - a situation which however motivating sets strong boundaries around what might be learned." Research such as that of Moseley, Mearns and Tse (2001) supports this view and suggests that there are possible detrimental effects from the playing of computer games in primary schools. Noss and Hoyles' (2000) solution is to allow students to interact with games where they are in control, programming attributes and functions in Microworld-like games software.

It is also noteworthy that the use of the spreadsheet has also increased, with 59% (up from 22%) of teachers now making some use of them. The most surprising thing about this is that 41% of the teachers never use a spreadsheet in their mathematics teaching, even though the program is virtually always provided along with the computer. It seems that the spreadsheet is still very much under-appreciated as a freely available resource. The increase in the use of statistical data handling programs is not unexpected.

OBSTACLES TO COMPUTER USE

In the original survey 93% of the primary teachers responded that they would like to use computers more in their mathematics teaching, and in the latest survey 95% agreed with this sentiment. Since this must mean that a proportion of the teachers who are not currently using computers very much would like to, we are led to ask 'what factors do they perceive as preventing them from doing so, or are preventing them from making greater use?' The results from the two surveys on this aspect are shown in Table 4.

In 1995 there were two areas where the teachers wanted to see improvement in order to reach their goal of using computers more. They were the provision of resources, in terms of available hardware and software and the increasing of their confidence through satisfactory training.

Table 4: Obstacles Preventing Teachers Using Computers More in their Teaching

Obstacle	% of 1995 teachers (n=1500)	% of 2002 teachers (n=87)
Available Software	65	49
Available Computers	57	36
Lack of Training	56	41
Lack of Confidence	41	6
Government Policy	12	8
School Policy	9	0
Lack of technical support	---	25
Time, etc	---	13

In addition 8.3% of primary teachers in 1995 mentioned some other obstacle; in 2002 18% mentioned unavailability of the internet as an obstacle.

In 2002 more teachers see themselves as confident in computer use, which is possibly a natural consequence of the wider use of computers in society, but their concerns still surround the provision of resources and suitable training. While the percentages in each category have fallen over the seven years, 49% continue to perceive a lack of software as a barrier to increased use. It may be that through professional development along with better sharing of ideas and resources they could be directed to employ what others are using, such as spreadsheets. Certainly, with 41% citing it as a need, the provision of suitable in-service training for primary teachers is still a particular area needing to be addressed. In later discussion with a small group of the teachers it emerged that very little, if any, of all the professional development they had received had been targeted at teaching mathematics with computers. Instead the technology assistance had been too general to be of real help in mathematics.

It is somewhat surprising in view of the numbers of computers in the schools that 36% are still citing a lack of computers as a barrier to improving application of computers in their classroom. However, this may be a consequence of the increase of ICT rooms and may refer to an access problem rather than anything else. This time around 25% mentioned a lack of technical support and 13% mentioned a lack of time as factors preventing them using the computer more. The former may mean that computers are too often out of action, while the latter is a measure of the increasing pressure that teachers are under in today's schools.

GROWTH OF INTERNET USE

In 1995 use of the internet in primary schools was so rare (if it existed at all) that there were no questions asked in our survey about it. However, in the latest survey we did question schools about their access to the internet and the type of sites they made use of. 71% of the teachers have access to the internet in their own classroom, and 69% in an ICT room. Of these 20% had access only in the classroom and 18% only in an ICT room. Figures were very similar for student access and it seems access for teachers is also open to students. When it comes to using the internet in teaching mathematics though 44% never used it, and only 26% used it at least once a month. On closer inspection it appears that the teachers rarely used the internet as a tool for mathematics learning, but most used it as a resource base and as a support for their own professional development, citing: professional support and communication; preparing mathematics lessons and units; and preparing learning resources, such as accessing problem solving ideas. When the internet was used in the classroom it was nearly always for drill and practice games.

CONCLUSIONS

While the second survey is somewhat smaller in size than the first it does represent a larger number of teachers than the 87 schools suggests. However, some caution should still be exercised on the interpretation of the results. Over the period of seven years we believe we can say that there has been:

A huge increase in the numbers of computers in primary schools.
An increase in the number of teachers using computers and in the frequency of computer use.
A change from solely classroom use to a mixture of classroom and ICT room use.
A small growth in the use of laptops, with these now available to 10% of staff.

In spite of this progress in the hardware provision and use of computers there appear to have been relatively few changes in the curriculum areas where computers are used or in the type of programs used with the computer. However, the primary teachers are:

Still using mathematical games and programs, often in a drill and practice manner.
Increasing in the use of the spreadsheet, although 41% still never use them.
Using the computer more for algebra, statistics and measurement.

While most teachers (79%) believe that the use of computers can significantly enhance student learning in primary/intermediate mathematics a minority (15%) do not accept this. Although many felt that computers could enhance learning in primary mathematics, many did not really know how, or how to overcome the barriers to increased use. Apart from an increased personal confidence in the use of computers the following barriers remain, or have more recently appeared:

Lack of available software and hardware.
A lack of suitable training on using the computers in primary mathematics teaching.
Insufficient technical support.
A lack of time for learning and proper lesson preparation.

Hence, there is still a clear need to address the provision of primary teacher in-service training specifically aimed at the teaching of mathematics with computers. One method of training attempted in a study by Ainsworth, Grimshaw & Underwood (1999) was the creation by inexperienced primary teachers of Intelligent Tutoring Systems for use in their classrooms. The research showed that it is possible for teachers to construct systems matching the needs of their students even when they have had no previous experience in computer-based work. Certainly our research shows little progress in this are has been made in New Zealand over the past seven years, and if the training and resources teachers need, and are asking for, are not provided then it will lead to increased frustration and a loss of enthusiasm for technology on the part of teachers (Dunn & Ridgway, 1994).

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STUDENTS AND TEACHERS LISTENING TO THEMSELVES: LANGUAGE AWARENESS IN THE MATHEMATICS CLASSROOM

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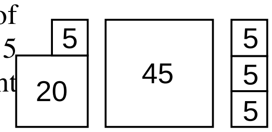
Taking seriously the recent call for critical language awareness in mathematics classrooms, I propose that students engage with their teachers in the analysis of the discourse in their classrooms. To explore the potential for increased mathematical understanding, I analyze transcripts of students and a teacher listening to and responding to their own dialogue in the context of student work on a mathematical investigation. The linguistic features analyzed include hedges, deixis and politeness.

BACKGROUND

A complex web of relationships among students, their teacher and powerful traditions emerges in every classroom, yet the discourse that forms the connections in this web is typically left unquestioned. Morgan (1998), in her call for language awareness in the mathematics classroom, promotes the direction of students' attention to language features of the genres in which they are beginning to participate. In this report, I consider what a language awareness program in a mathematics classroom might look like. How might students' mathematical learning benefit as they analyze the discourse in their classroom?

To frame this exploration, I analyze student and teacher interpretations of a particular classroom interaction involving grade 10 students (15 years old) in a pure mathematics class. I interviewed student groups and their teachers after audiotaping the students working on an investigation I had designed to relate to their curriculum (see below).

The 45 cm^2 square is the exact same height as the two stacks of squares beside it. The squares in the stack on the left have areas of 5 cm^2 and 20 cm^2. Each of the three squares in the stack on the right has an area of 5 cm^2.



a. Find stacks of squares that would be the exact same height as a square of area 72 cm^2.
b. Explain how to find the stacks that would match any other given square in height.

The transcripts in this report are drawn from two interviews, one with a group of three students and the other with their teacher, whom I call Mr. Penner. In each interview, I played an audio recording of an interaction between the students and their teacher during their work on the investigation one week earlier. This interaction, in which Mr. Penner happened upon the group some twelve minutes into their work, is represented in the following transcript. (Names of participants are also given as pseudonyms.)

A1 Natalie: Okay. I have a question. What we're trying to do right here, right? You find the area and all the lines, right. But instead of 45 we're finding 72, right?

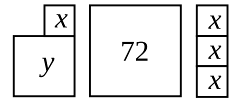
A2 Mr. P.: Well, sort of.

A3 Natalie: But, don't we need to know the ratio between 20 and 45, and then if this is 72 what would be the ratio then?

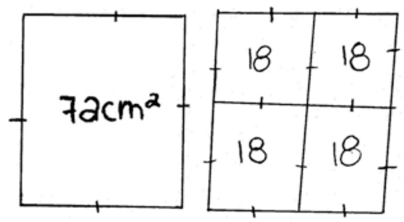
A4 Mr. P.: Let me show you one thing. If I wanted to find that. …

Natalie was thinking about ratios as a way of addressing the problem. She seemed to have something like the following calculation in mind:

$5 : 20 : 45 = x : y : 72$ $\Longrightarrow$



Mr. Penner was thinking about an approach that would use techniques he had taught this class one month earlier. He was thinking about square roots to express the heights of squares and addition sentences to represent stacks: $\sqrt{5}+\sqrt{20}=\sqrt{45}$ _ $\sqrt{8}+\sqrt{32}=\sqrt{72}$, for instance. The dialogue continued with Mr. Penner directing Natalie's group to look at heights using square roots. After this interaction with their teacher, Natalie and her group seemed to have forgotten about their ratio approach, which did not involve finding the heights. Though they lost the ratio idea, the group did find yet another approach that Mr. Penner had not considered before. They divided the 72-square up into smaller squares to construct stacks as required (see part of their work at right). The 72-square has the exact same height as two 18-squares stacked.



PARTICIPANTS LISTENING TO THEIR OWN DISCOURSE

I am interested in the benefits of having students and their teacher look back at their discourse. One result of exposing teachers and students to records of their classroom interaction is that it prompts them to reflect on their pedagogical and mathematical choices. It affords them the opportunity to consider what they might have done differently. There is also value in looking at language features in discourse.

What might Mr. Penner and his students have learned about mathematics learning if they were to look at their language practices? An analysis of the following interview transcripts uncovers potential directions that are worth considering for a language awareness program in which teachers and students analyze their own discourse. In the interest of self-similarity, I have chosen to analyze the interview transcripts in a manner that I foresee being used by language aware students and teachers. However, the objects of their analysis would be significantly different from mine here. Their analysis would focus on their mathematical dialogue. Here I analyze interviews to see what can happen when participants listen to their own discourse.

The first interview transcript is excerpted from my interview with Mr. Penner. In our interview, I introduced the audio recording saying, "I am going to play a bit of tape from Natalie's group." He began responding to the episode after listening for about ten seconds. He spoke while the tape was playing.

B1 Mr. P: I think, I think this is where I help them too much.
B2 I: Should I stop here? You know what you're …
B3 Mr. P: Yeah, I know what's going on here. I can picture it. Yeah. So, [*I stop the tape*]
B4 I: Yeah, so you helped them too much. Obviously you decided to help them.
B5 Mr. P: Yeah.
B6 I: So, just tell me how you feel about that …
B7 Mr. P: about helping?
B8 I: Yeah.

B9 Mr. P: I think that was after about half an hour. It was after a while anyway. I could see they really weren't getting the idea, so I gave them an example. There was an example on the page, but I gave them an example more directly, with the roots. So, I kind of opened up that whole subject to them, and then I think they kind of got a few after that point. So, I mean I really led them into it, but, [*laughs*] I mean, that group is a group that struggles in math. Out of the three people there's only one that's passing.

B10 I: I think I heard them say their highest mark was 52 in the group.

B11 Mr. P: Yeah, that's right. I guess I could have left them alone and they would never have got it. [*pause*] Possibly. With the roots. But, I think, was that the group that came up with some other creative method? I think they did actually [*referring to their work, which is partially reproduced above*], so maybe if I wouldn't have helped them along they would have come up with something.

Hedges: Mr. Penner's response to the audiotape began with a hedge: "I think, I think …" (turn B1). Rowland (2000) analyzes students' use of hedges when they are involved in mathematical conjecturing and reasoning. He describes hedges as "words which have the effect of blurring category boundaries…[and which] hedge the commitment of the speaker to that which s/he asserts" (p. 58), words such as 'sort of', 'I think', 'maybe', and 'perhaps'. Hedges are language devices that protect us from being proven wrong.

In this transcript, Mr. Penner hedged repeatedly, saying 'I think' five times, 'kind of' twice, 'maybe' and 'possibly', all in a short time span. By contrast, his students in the next transcript did not hedge at all. From whom was Mr. Penner hedging himself? Was he worried about my opinion (or judgement) of him? Indeed, we find ourselves in an uncommon predicament when confronted with records of our activity, not unlike a court where the defendant is confronted with artefacts or transcripts from prior testimony. Though I can imagine why Mr. Penner would be defensive in this provocative situation, his hedges may simply reflect his awareness that his recollections of the event were mere reconstructions. They may demonstrate his recognition of the complexity of reflection.

In a language aware mathematics classroom, teachers and students can be directed to become aware of linguistic hedges. In such a context, participants would be analyzing mathematical discourse instead of interview data. Their data would more resemble the transcripts in Rowland's work on hedges. Classroom participants' language awareness would afford them the opportunity to reflect on possible explanations for the hedging they find. For example, Natalie and Mr. Penner might wonder why he said "Well, sort of" (turn A2) instead of saying "No, you're wrong." Was his expressed uncertainty mathematically significant as well as pedagogically significant?

I suggest that there is no need to pinpoint the speaker's intentions in such a setting. Instead, the students and teacher might reflect on the importance of being unsure in mathematics and in learning. Though certainty is highly valued in mathematics, new ideas require space for investigation. Rowland (1997) names such a discourse space the 'zone of conjectural neutrality'. In various contexts, Rowland encourages teachers to provide such spaces for students, but students may also benefit from being aware themselves of the importance of uncertainty and vagueness in their mathematics practice and in their thinking about their practice. However, students need to be confronted with records of their mathematical activity in a way that minimizes the sense of confrontation.

Deixis: Following Mr. Penner's initial hedging words, he 'pointed' with the word 'this': "This is where I help them …" (turn B1). Such pointing with language is called deixis. Rowland and others (e.g. Pimm, 1987) have studied deixis in mathematics practice.

As mentioned above, Mr. Penner found himself in an odd position for this interview. How should he have pointed to himself? To what extent is/was the Mr. Penner on the audiotape the same person as the Mr. Penner in the interview? He initially pointed with the word 'this' which suggests a sense of proximity more than 'that' would suggest. He also spoke of the audio-taped episode in the present tense, though it had occurred a week earlier. The present tense suggests proximity as well.

In my interview with Mr. Penner, he switched from using present to past tense before turn B9. At the same time, he distanced himself by using the distal pointer 'that' instead of the proximal pointer 'this' to point to the event (turn B9). If he and I were working together in a context in which language awareness was part of the agenda, I could ask him why he might have switched tenses. The purpose of this question would not be to know what his intentions and feelings 'really' were during the interview. Those experiences are lost. Rather, the value in such a question would lie elsewhere. As we talked about possible reasons for him to distance himself from the event, we might, for example, learn more about how practice can be informed by revisiting past experiences.

The transcript below is from my interview with Mr. Penner's students who responded to the same audio segment, in which they were recorded interacting with Mr. Penner. The excerpt begins immediately after the point at which I stopped playing the audiotape.

C1 I: So what was he doing there?
C2 Janet: He was getting us to talk about it and then like …
C3 Natalie: Trying to solve it ourselves, like he's trying to give us hints
C4 Janet: Tell us if what we're doing, … if what we think is right.
C5 I: Did he, did he pay any attention to Natalie's question about whether ratios …
C6 Natalie: No.
C7 Janet: Not really [*all laugh*]
C8 I: You know what?
C9 Janet: | 'Cause that was wrong right? | [*simultaneous*]
C10 Natalie: | He didn't even hear me. |
C11 I: Noooo. In fact I thought of it when I was listening to it…I never thought of using ratios and I tested it in a whole bunch of ways and it is pretty interesting actually, it would work well. I thought it would work well.
C12 Natalie: Right!
C13 I: So, I played it for Mr. Penner too. So, you know, he had mixed feelings about it. So, you would have got something,
C14 Natalie: Yeah.
C15 I: something good anyway.
C16 Natalie: We needed more time on that project.

I am captivated by the deictic flow between these students and me, their interviewer. I began the interview with the distal pointer and the past tense: "What was he doing there?" (turn C1). Alternatively, I could have suggested proximity by starting with, "What *is* he doing *here*?". Janet replied using the past tense (turn C2), perhaps following my lead. Natalie followed Janet with a present tense utterance; perhaps resisting the kind of pointing I subconsciously chose to structure the conversation. Janet followed Natalie's proximal pointing, using the present tense (turn C4). I countered with the past tense (turn C5), and the rest of the conversation was in the past tense, at least where tense was clear. Natalie closed the conversation mirroring the distal pointing I started it with, using the past tense and referring to *that* project (turn C16).

In this conversation, I used distal pointing throughout, Janet's deixis resembled that of whomever she followed, and Natalie once tried proximal pointing, but seemed to acquiesce in my language structure. Mr. Penner's interview was similar in that he also appears to have followed my lead in terms of the proximity of our deixis. He started the conversation with proximal pointing and I turned it to distal pointing in turn B4.

As with the analysis of hedges, discussion of deixis in a language aware mathematics class would focus on mathematics discourse, not interview data. Spatial and temporal deixis present significant tensions in mathematics, similar to the tensions described above. How do students (or mathematicians) point to abstract, mathematical objects, to which they have no direct access? Surely students would benefit from discussing this difficulty. The alternative is for each student to just suffer through the difficulty, unaware that he or she is the only one struggling to communicate. Scholars are increasingly becoming aware of the significance of representation in mathematics and mathematics learning. (There has been a semiotics discussion group at the last two international PME conferences.) Students may also find such discussion important.

Tensions of temporal deixis are related to those of spatial deixis. Mathematicians have asked: Where is the mathematics? (e.g. Mason and Muller, 2001). We might also ask: When is the mathematics? Is a proof or geometric construction an artefact of a person's mathematical thinking or do such mathematical objects contain the mathematics in themselves? In other words, to what extent does mathematics exist independent from human agency? The way we talk about mathematics can tell us something about our answers to this question. Do we talk with the past tense about the mathematical choices made by a mathematician or a student, or do we talk in the present tense as if it does not matter who wrote or spoke the mathematics? Chris Bills (2002) has identified significant disparity between higher and lower achieving mathematics students in their use of pronouns and verb tenses when referring (pointing) to their mathematical thinking.

Politeness: Another feature of language studied by linguists is politeness. Liz Bills (2000) has drawn on this scholarship to investigate ways in which mathematics students and teachers save face and allow the others in their classroom to save face. Power and control are important aspects of politeness.

The above analysis of two transcripts uncovers features of politeness in the interview discourse in which I engaged my research participants. In terms of the proximity of deixis, both the students and the teacher followed my lead in the interviews. When I moved Mr. Penner's interview to the past tense, for example, it was like me saying, "You're wrong to talk about the past as if it were present." His transition to the past tense was like saying, "Okay, I'll use the past tense then." Though I did not foresee the extent to which the participants in my research would follow my lead, I am not surprised by it because they agreed in advance to cooperate with my research agenda. They could be expected to follow my structuring, not unlike students following their teacher.

My new-found awareness of interview politeness prompts me to be more critical when I read transcripts of interviews. To what extent do interviewees say what the interviewer wants to hear? School children are encultured to give their teachers what they want. They learn to say what adults in schools want to hear, but there are tensions when another adult is introduced into a setting of mathematics learning. In Bills' (2002) interviews, high achievers tended to follow their teachers' language structure and resisted the interviewer's language structure.

Besides informing the interpretation of interview-based research reports, this experience of interview politeness raises questions about any discourse. To what extent do participants merely follow the structuring of the person in power? In the mathematics investigation described and discussed here, for instance, Mr. Penner wanted to free his students to structure their own mathematics. They, however, were searching for the 'right' answer, the mathematics that would fit his expectations. In my interviews with Mr. Penner, he revealed his struggles with exercising intervention (or power), but his students seemed oblivious to the tensions their teacher felt in his pedagogical choices.

Why not open up this tension for classroom discussion? When students and their teacher revisit their dialogue, they can become aware of their politeness practices. Perhaps this awareness could free them to take more initiative in their mathematical practice.

DIRECTING AWARENESS IN THE MATHEMATICS CLASSROOM

In the above analysis of interview transcripts, I have pointed out ways in which awareness of language practices can inform mathematical learning. But, how can students and their teachers be directed to become language aware? Should they rely on outsiders' analysis of mathematics classroom discourse, or should they do analysis themselves? And, what particular texts would serve them best as exemplars for language awareness?

First, who should be involved in analysis? Current calls for language awareness assume that students ought to be the ones becoming more aware. These calls cast teachers as experts who can tell their students how to use language well in mathematics. However, the interrogation of stereotypical classroom roles, in which students are the only learners, has contributed to revelations of classroom realities that have allowed researchers and teachers to see the classroom in new ways (e.g. Matos, van Dormolen, Groves and Zan, 2002). Healthy language awareness programs need to identify all participants as learners who can both contribute unique insights and benefit from increased awareness.

My second question relates to the object of analysis. To uncover some of the language tensions that are at the heart of mathematics, I suggest that it would be most appropriate for participants to analyze their own discursive practice, not only that of others.

There is, however, awareness-building value in the analysis of texts generated outside the participants' first-hand experience. Video-taped mathematics classroom episodes have been played for teachers and students to research their perceptions (e.g. Ainley, 1988). Others have used videotapes to help mathematics teachers become aware of alternative possibilities for practice (e.g. Pimm, 1993; Sáenz-Ludlow and Perlwitz, 1994). In such research and teacher-development, viewers typically watch video episodes in which they themselves do not appear, and it is typically assumed that teachers and researchers, not students, have the potential to become more aware of mathematics classroom discourse.

To identify the 'practical rationality of mathematics teaching', Herbst and Chazan (in press) listened to teacher responses to videotaped mathematics classroom episodes. In their analysis of the teachers' responses to other teachers' practice, they note how a video is both record and artefact. As an artefact, the video-framed episode itself is analyzed by teachers who view it. As a record, the episode reminds the teachers of similar events and their contexts. This mix of subjective interpretation with objectification, which is inherent in any experience of artefacts, is an issue for any analysis of discourse, even if the interpreters are blind to the issue.

The conflict here relates to the nature of language. To the extent that words and speech acts have meaning in themselves, utterances can be analyzed without thought of personal experience. With this view, text extracts are mere artefacts. Though few current discourse scholars would espouse such a representationist view of language, it is difficult to hold a completely relativist view of language when we consider authentic texts as records of events that are situated in particular cultural contexts. Though a particular speech act in its context may mean many things, there are many more things that it cannot mean.

The classroom transcript above exemplifies the divergence of meaning different people can find in a particular discursive event. Mr. Penner appears not to have been listening to Natalie because her words bore little resemblance to his expectations. Instead of *listening to* her with an open mind, he was *listening for* his expected response. However, the nature of language makes it impossible for a teacher to listen with no preconceived notions.

Because of their intimate acquaintance with its context, participants in a language aware mathematics classroom ought to have an easier time of being aware of semiotic tensions in language when they analyze their own discourse. Such awareness can be healthy in mathematics classrooms, in which Sáenz-Ludlow (2001) notes a tendency to believe that meaning rests on symbols, independent of context. Awareness of this semiotic tension might help students understand how, as Duval (1999) puts it, "there is no direct access to mathematical objects but only to their representations" (p. 24). Such awareness would move beyond a mere increase in language power, which seems to be the aim of Morgan (2002): "Greater awareness … may help mathematics teachers and students to develop more purposeful and hence more effective use of language" (p. 7).

CONCLUSION

The analysis of language patterns in mathematics classrooms has provided researchers with insight into classroom culture and opened up opportunities for us to share this insight with teachers. I have described here how students may also benefit from the analysis of their own discursive practice.

Though there is pedagogical potential in directing mathematics students to language awareness, there is work to be done to find viable models for such cooperative analysis. I suggest that this would be best worked out in the context of a class committed to exploring the possibilities of language awareness structures in the classroom. There is even more research possibility here. We may gain new insights into mathematics learning if we listen to the insiders, students and their teachers, listening to themselves and analyzing their discourse. In my report, I will discuss these possibilities more fully.

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THE CONTEXT SENSITIVITY OF MATHEMATICAL GENERALIZATIONS

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Many theoretical explanations for knowledge transfer or generalization assume that such processes are rooted in the acquisition of abstract rules, principles, or schemata applicable in context-independent ways. This case study is part of a larger research program examining how what often appears to be generalized knowledge or performance is, in fact, supported by systems of context-sensitive knowledge. A microgenetic analysis of an undergraduate student's solution to two mathematically isomorphic probability problems demonstrates how he perceived the two problems as structurally and phenomenologically different despite his ability to apply a single, correct, schematic solution to both. The means by which he finally perceived the problems as structurally "alike" was influenced by both the relevant solution principle and the problem context.

INTRODUCTION

Many theoretical attempts to understand and explain what has been traditionally called the "transfer of knowledge" make a distinction between the underlying structure of a problem or situation and its "real world" context or surface features (see, for example, Anderson, Reder, & Simon, 1996; Gick & Holyoak, 1987; Reed, 1993; Singley & Anderson, 1989). From this perspective, the perception of two problem situations as supporting the same means of solution requires the problem solver to notice sufficiently the "objective structural similarity" of the problems, although the saliency of such similarities may depend on the individual's expertise (Gick & Holyoak, 1987). Thus, while problem contexts have long been understood to influence the types of structure perceived in a situation, successful transfer of an abstract mathematical solution or principle is said to depend on a presumably objective, context-independent structural similarity across problem situations.

The "surface features" of a problem are generally understood to refer to objects or object attributes non-essential to the problem's solution. Structural similarity across two situations depends on their sharing relational properties as well. One of the most widely cited theoretical frameworks for modeling the perception of structural similarity is Gentner's structure-mapping model (Gentner, 1983; Gentner & Markman, 1997), which posits the representation of situations as systems of object nodes, object attributes, and hierarchies of relations among nodes. The perception of a particular hierarchy of nodes and relations in one problem situation permits it to be mapped onto another problem of similar structure whose solution is known; or, perhaps, the problem's structure may be directly associated with an abstract solution method, principle, or schema for problems of such structure (Reed, 1993). Ultimately, abstract rules, principles, or schemata serve to explain how it is that knowledge is transferred and applied across contexts.

While numerous researchers have critiqued the very cognitive foundations of the traditional transfer paradigm from the perspective of situated cognition (see, for example, Bransford & Schwartz, 1999; Greeno, Moore, & Smith, 1993; Lave, 1988), recent attempts have been made to reconsider the problem of transfer, and the surface/structure distinction in particular, while maintaining a focus on the individual and cognitive dimensions of the phenomenon (Lobato & Siebert, 2002; Wagner, 2002). The case study analyzed in the present research demonstrates that the successful application of abstract rules can depend upon the acquisition of context-sensitive ways of perceiving structure within a problem situation. Most significantly, such structure is not located objectively in the problem situation itself; nor is it determined by the problem situation alone. Rather, it arises through the interaction of the problem solver's understanding of both the problem context and the relevant mathematical principle. From this perspective, the application of abstract rules or principles may depend on acquiring a network of supporting knowledge that is often sensitive to the variety of situations and contexts in which those rules or principles may apply.

THEORETICAL FRAMEWORK

The theoretical perspective of this analysis originates in diSessa's (1993) epistemological framework for the learning of concepts of science and mathematics. From this perspective, the acquisition of such concepts depends on the organization and systematization of prior knowledge that is often highly sensitive to context. Wagner (2002) offered the beginnings of how diSessa's epistemology could be used to address transfer phenomena. In his case-study analysis, descriptive and explanatory statistical ideas initially used by a student only in isolated situations coalesced into associations of knowledge more likely to be cued in a wider variety of contexts in which they had not been used before.

A second aspect of diSessa's work relevant to generalization and transfer is the theory of coordination classes (diSessa & Sherin, 1998). diSessa and Sherin argued that understanding at least some concepts (coordination classes) requires the acquisition of particular *readout* and *coordination* strategies, often dependent on context. In a prototypical example, the ability to perceive a single concept such as a force in different situations might require a student to learn to attend to ("read out") information about different features of the situation and coordinate this information in different ways in different contexts. While students may well be able to learn and state abstract principles or solution schemata, the use of those ideas across contexts depends on the development of increasingly sophisticated networks of context-sensitive knowledge. Such knowledge might include acquired means of perceiving different types of structure by attending to and coordinating different problem features in different situations.

The analysis presented here reveals how Philip, an undergraduate student, demonstrated himself capable of correctly applying normative probabilistic reasoning to two mathematically isomorphic problems while nevertheless *denying* that such reasoning applied in one of the two cases. Despite his ability to recognize and coordinate all the essential features of both problems to obtain a correct solution, the structure he perceived in the two problems differed substantially, and he resisted my extensive efforts to convince him of their normative isomorphism. Using fine-grained methods of microgenetic analysis (Schoenfeld, Smith, & Arcavi, 1993), I analyzed transcripts of Philip's interviews to reveal instances of useful, normative reasoning that he had used in other situations, but had not yet applied to the problem offering him difficulty. I will argue that I was able to use that analysis to construct a reformulation of the problem that convinced Philip of its normative solution by cueing his good reasoning strategies used in other contexts. Moreover, the means by which Philip learned to perceive new structure in the problem arose not from the problem situation itself, but through his understanding of both the relevant mathematical principle and the particular context. Philip's learned means of perceiving structure is thus shown to apply to *some but not all* situations in which the statistical principle applies. This behavior is not readily explained by those who posit that transfer takes place through the recognition of "objective" situational structure, or that problems supporting a particular abstract principle are understood to be alike because they all support a common structural interpretation.

METHODS

Philip was one of fourteen undergraduate students who participated in this research during the summer of 2001 or 2002. The students were enrolled in an introductory course in statistics at a large, public, university in the United States, and each agreed to meet with me for two hours each week in one-on-one sessions for the duration of their eight-week course. During the first half of each interview, I offered myself as a personal tutor, and the use of our time depended largely on the questions and concerns each student brought to the meeting. During the second half of each interview, students engaged in a variety of activities including think-aloud problem-solving sessions, computer simulations, experiments with simple randomization devices such as dice or spinners, and interviews and discussions about their understanding of probability and statistics. These interviews were audio- and video-taped, and salient portions of them were later transcribed for fine-grained examination. All work done during these interviews was directly related to the material the students were studying in their course, so most subjects indicated that they found their participation in this research both useful and motivational.

Among the problems given to the students to examine were collections of problems deemed mathematically similar or isomorphic because their solutions involved the same mathematical principle. The problems examined in the present analysis were presented to students after the mathematical principles relevant to their solution had been covered in their course. When students offered different or non-normative solutions to problems that I perceived to be similar, I asked the students to compare them and explain how they saw them as similar or different. I engaged in a deliberately instructive role only if students found such comparisons unhelpful in assisting their normative reasoning. The data permitted detailed analyses of how students succeeded or failed in offering normative solutions to problems deemed mathematically "alike," and how they came to identify problems as similar or isomorphic after failing to do so in their initial solutions to them.

THE CASE OF PHILIP

For reasons that will become clear as this section unfolds, the case of Philip will be told as a chronological narrative. The interviews relevant to this analysis took place during the final two weeks of Philip's eight-week course. Philip appeared to be highly engaged in all of our activities, and he showed himself to be successful in his coursework by earning an A in the course from his instructor (not the researcher). During the seventh of our interviews, I asked Philip to solve the following problem, which was accompanied by a picture of a circular spinner divided into ten sectors of equal size, seven of which were colored blue and three of which were colored green:

> Suppose someone spins the spinner at the right ten times in a row. Of the following possible outcomes, which is the most likely to occur?
>
> a) The spinner will land on blue five times and on green five times.
> b) The spinner will land on blue seven times and on green three times.
> c) The spinner will land on blue all ten times.
> d) All of the above are equally likely.
> e) It is not possible to answer this question.

A normative solution to this problem would require an understanding of a principle that might be stated as follows:

> The most likely (expected) result of a series of *n* observations of a binomial random variable with probability *p* of success on each observation is *np* successes.

For convenience, this principle will be referred to in this paper as the *binomial principle*. Neither Philip nor his classmates would have been expected to learn this principle in so specific a form. It is an idea that they would have been more or less expected to deduce from their study of the expected value of a random variable. It has, however, an intuitive appeal, since it predicts that the most likely outcome to occur from a series of ten spins of a spinner is the one that most closely resembles the proportion of blue and green sectors on the spinner, namely, seven blues and three greens.

Philip took very little time before offering a normative solution to the Spinner Problem:

> Philip: Well, it's still a small group of spins, so you could get all blue or about 50-50 green and blue, um, more often than if you were to spin it a hundred times. Because, obviously, I guess it would get closer to a 70-30 split. But, um, it's only ten spins, but I still think it would be close to about a 7-3 split, just because, it's kind of like a problem where, if you have an average of a box, and they say after 400 draws, what do you expect the average to be? You expect it to be right around the average of the box. And even for five draws I would expect it to be around the same. So, that's why I, I picked B. I mean, it's most likely it would be seven and three.

Not only was Philip's answer correct, his explanation was virtually picture-perfect as he correctly noted the role of sampling variability in samples of such a small size. He nevertheless recognized that the most likely outcome was a seven-to-three ratio, regardless of the size of the sample. In referring to "the average of the box," Philip made use of his classroom experience of using box models to simulate random draws. The average of the box was the expected value of the draws (in this case, 70% blue).

Immediately after solving the Spinner Problem, I presented Philip with the Box Problem. In this problem, he was asked to imagine a box containing 500 tickets, 350 labeled "B" for blue and 150 labeled "G" for green. If ten tickets were drawn at random from the box, what would be the most likely outcome? The five multiple-choice solutions offered to Philip corresponded precisely to those accompanying the Spinner Problem. From a mathematical perspective, the two problems are isomorphic, both asking for the most likely outcome of a series of ten draws of a binomial random variable with probability (of drawing blue) 0.7. Only a "surface feature," the means of making the random selection, has been changed. Philip, however, perceived something very different:

> Philip: If I were to keep the same logic, I'd say seven are B and three are G, and that would be the most likely. But the, the difference in this is there are so many, now, that are blue. There are a lot. I mean, in this case [indicates the Spinner Problem] there's only four more that are blue. In this case [indicates the Box Problem] there's, like, two hundred more that are blue, and, um, I'm pulling ten. OK, I'm gonna, I'm actually gonna s-, I'm gonna say C, that all ten of the slips say B, is, is most likely....

Philip's response is particularly striking because he demonstrated that he both remembered his reasoning from the Spinner Problem and could apply it to the Box Problem, but he nevertheless denied that such reasoning was appropriate. Thus, it cannot be argued that Philip simply did not recall the correct rules or that he did not learn them at a sufficiently abstract level to apply them. In extensive follow-up discussion, Philip maintained that draws from the box differed from spins of the spinner because the large number of blue tickets "overwhelmed" the green ones in a way that the few extra blue sectors of the spinner did not. In short, the structure Philip perceived in the problems that permitted him to recall and apply the binomial principle was not the same as the structure that would permit him to perceive both problems as appropriate instances of the principle.

Following Philip's initial solution to the two problems, he and I engaged in a 23-minute conversation during which I tried to no avail to convince Philip of the two problems' normative isomorphism. A detailed analysis of that conversation is beyond the scope of this paper, so I summarize here only particularly salient highlights. During our conversation, Philip repeatedly acknowledged the 70% proportion of blue in both problems, but found that correspondence insignificant compared to the difference in absolute numbers. While Philip's "problem" might be seen simply as a failure to realize that only the percentage of blues and not their absolute number was relevant, it became clear that for me as an instructor merely to *tell* him this was insufficient. Philip spontaneously acknowledged that the "mathematical" answer was "probably" the same for both problems, but he did not believe that such a mathematical solution had anything to do with what would really happen in the two situations. Even had he accepted the normative answer on my authority, it would not have changed his perception of the situations—the structure he saw—that led him to consider the problems differently.

Most of my attempts to convince Philip of the isomorphic nature of the two problems involved presenting him with reformulations of the spinner. At first, for example, I took the ten-sectored spinner and began sketching in extra lines as though to increase the number of sectors to 500, corresponding to the number of tickets in the box. Philip denied the relevance of this move, however, pointing out that by increasing the number sectors I did not increase the overall *amount* of blue present on the spinner. This stood in contrast to increasing a set of 10 tickets to 500 tickets, which would introduce an "overwhelming" amount of blue to the box not initially present. I countered by asking Philip to imagine my taking the 500 tickets and laying them on the spinner, as though to reconstruct the face of the 10-sectored spinner with the 500 tickets. Philip again denied the correspondence, noting that each sector of the spinner should correspond only to one ticket in the box. Thinking I finally had him, I suggested that we take all 500 tickets and lay them around the circumference of the spinner, with blue and green tickets lined up along the edge of their corresponding sectors. Then, I said, the needle would point to only one ticket at a time. Philip acknowledged that this "put a hole" in his argument, but he then surprised me by suggesting a reformulation of his own. In my suggested reconsiderations of the spinner, I never changed the placement of the original ten blue and green sectors, so even after subdividing them into a larger number of sectors, the actual distribution of color on the spinner's face remained unchanged. Philip asserted that if the spinner were subdivided into 500 sectors and the blue and green sectors were randomly scattered on the spinner's face, then the spinner and the box would match and the most likely outcome of ten spins of the spinner would be ten blues! He stated that the large number of blue sectors would overwhelm the intermingled green ones, and the needle would have a more difficult time picking out the green from among the blue.

In response to my questioning, Philip suggested that it would take a demonstration of draws from a box to convince him of the normative solution. Not only did I not have the requisite supplies to carry out such a demonstration, I was also skeptical of the wisdom of doing so. What if we had drawn ten blues on the first try? Even if a demonstration had supported my argument, would it have offered Philip any new understanding of the situation to convince him that the normative solution *should* be the correct one? After 23 minutes of debate, Philip and I agreed to take up the issue again during our next meeting.

In the five days before our next meeting, I closely examined Philip's reasoning about the Spinner and Box Problems, hoping to devise a means of convincing him of the problems' isomorphism. I looked for reasoning strategies Philip used successfully in contexts other than the Box Problem, and I noted two in particular. Philip twice defended his correct reasoning about the ten-sectored spinner by noting a one-to-one correspondence between the number of sectors on the spinner and the size of the sample: "There's only ten slots. And seven are blue and three are green. And you're only spinning it ten times, so you have to get one thing on, on each spin." Also, Philip reasoned normatively about spinner and box situations in which the number of blues and greens were equally likely. So, for example, he acknowledged the likelihood of five blues and five greens in ten spins of a spinner colored half blue and half green, as well as in ten draws from a box containing 250 blue and 250 green tickets. I predicted that a reformulation of the Box Problem designed to cue these reasoning strategies would enable Philip to reason normatively about draws from the box.

When we next met, Philip indicated that he remained firm in his non-normative expectations about draws from a box. In reply, I asked him to imagine a box of balls numbered 1 through 10, and a separate box of 500 balls, with 50 balls each numbered 1 through 10. Philip took a short time to acknowledge that there should be no discernable difference in drawing (with replacement) from either box, emphasizing his awareness that there were "the same number" of each class of ball in each box, and thus drawing on his good reasoning about equally likely outcomes. Finally, I asked him to imagine that the balls numbered 1 through 7 were colored blue and those numbered 8 through 10 were colored green. Philip immediately responded, "Yeah, that would convince me, then." After some probing on my part, Philip explained his change of reasoning:

Interviewer: You're, you're not just going along with me? [laughs]

Philip: No, it's a good way, it's a good way of looking at it. Because, I, I think about it with just the numbers on it, one through ten, and I'm thinking, well, now there's so many different choices in there, it's not just blue and green. It's, uh, it's any numbers. But, uh, it, but then when you throw the catch in, on the back, one through seven were blue, then it, it makes perfect sense, then. Because you're gonna pull some eights, and you're gonna pull some ones, and, the numbers between, you're gonna get seventy percent of the numbers one through seven. Yeah. And then, you throw the clause in that those are colored blue, it sheds light on it. [laughs] Yeah.

Philip's account that "it's a good way of looking at it" is precisely that: a learned way of looking at the situation and perceiving a new kind of structure in the problem. Whereas Philip had earlier attended only to the absolute number of blue, the numbering scheme encouraged him to attend to equal classes of balls, and his earlier reasoning about one-to-one correspondences between the spinner sectors and the sample now informed his reasoning about the box as he saw that "you're gonna get seventy percent of the numbers one through seven," thereby highlighting the proportional nature of the situation.

DISCUSSION

The means by which Philip arrived at a normative understanding of the Box Problem is of theoretical importance. First, the structure he came to perceive in the situation was by no means suggested merely by the situation itself. Rather, it was mutually influenced by the nature of the binomial principle and the problem context that permitted Philip to imagine the population under an imposed classification system. Thus, learning this "way of looking at it" is inherently tied to one's understanding of the binomial principle and should not be imagined as a structure that one *first* perceives in a situation that *then* reminds one of an appropriate rule or solution. Second, while Philip learned a powerful way of perceiving some situations, the means by which he perceived the normative solution to the Box Problem is not useful in all instances of the binomial principle. More specifically, Philip learned a way of perceiving structure in drawing from an available population, and such structure is not readily perceived in instances of the binomial principle for which no such population is available. Finally, the structure Philip learned to perceive in the Box Problem enabled him to reason directly that a draw of seven blues and three greens was the most likely outcome of ten draws; he did not need to "apply" an abstract rule or principle. While it is quite likely that students learn in the long run to apply such reasoning schematically (as Philip, in fact, initially did), this analysis suggests that the similarity of problems governed by a common mathematical principle lies not in a single structure shared by all, but in learned, context-sensitive ways of structuring them that enable problem solvers to perceive them as instances of that principle.

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HOW DOES FLEXIBLE MATHEMATICAL THINKING CONTRIBUTE TO THE GROWTH OF UNDERSTANDING?

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We examine aspects of flexible mathematical thinking in middle school students and its contribution to the growth of their mathematical understanding. We inductively analyzed data collected from video-episodes of an interactive, problem-solving based after-school mathematics class. Flexibility is essential in such classrooms since students must apply knowledge rather than simply retrieve it in its original context. The study provides links between flexibility and an existing theory of mathematical development to give researchers and teachers a better theoretical grasp of what happens in interactive classrooms.

IMPORTANCE OF FLEXIBLE KNOWLEDGE & THINKING

In the last decade, the focus in mathematics classrooms has changed from computational ability to higher-level problem-solving skills (National Research Council, 1989), as advocated in the National Council of Teachers of Mathematics Standards (2000). An important component of problem-solving is durability of knowledge, which is the ability to recall and use facts, skills, procedures and ideas. However, this does not imply its use in contexts other than those in which they were constructed. Flexibility is essential if students are to apply knowledge to new contexts, and is therefore a key ingredient in problem-solving. The goal of this study is to identify acts of flexible thinking, trace the events that surround these acts, analyze the mathematical knowledge that the students build, and understand how these acts, events and knowledge intertwine.

THEORETICAL FRAMEWORK

In psychology the term "flexibility" is often used without being defined (e.g. Jausovec, 1994), or is investigated via specific problems that are thought to require flexibility for their solution (e.g. Kaizer & Shore, 1995). Studies in mathematics education generally treat flexibility as the capacity to exhibit a variety of invented strategies or a large repertoire of strategies for solving problems (e.g. Heirdsfield, 2002). More specific uses of the term are offered in educational psychology and mathematics education. In particular, Spiro & Jehng (1990) define cognitive flexibility as the ability to spontaneously restructure one's knowledge, in adaptive response to radically changing situational demands. In mathematics education, Krutetskii characterizes flexible thinking as reversibility of thought (Krutetskii, 1969). Gray and Tall (1994) characterize flexible thinking in terms of an ability to move between interpreting notation as an instruction to do something (procedural) and as an object to think with and about (conceptual). Our conceptualization of flexible mathematical thought extends the literature reviewed above, based on the first author's classroom experience and inductive analysis of the data in the present study. It may be summarized by stating that a student thinks flexibly when they exhibit some or all of the following behaviors (Warner, Coppolo & Davis, 2002): interpretation of their own or someone else's idea (e.g. through questioning it and thus showing it to be valid or invalid; through using, reorganizing or building on it); use of the same idea in different contexts; sensible raising of hypothetical problem situations based on an existing problem: creating "What if…?" scenarios; use of multiple representations for the same idea; connecting representations.

METHODS/ANALYSIS

Ten sixth and seventh grade students, of mixed abilities, volunteered to participate for twelve weeks in an after-school mathematics class, in which students worked on combinatorial tasks (such as, Building Towers - finding the number of possible 4 tall towers of building blocks, choosing from 2 colors of block; Martino & Maher, 1999). Three to five students worked in each group and presented, discussed, and argued about, their solutions. Additional time was provided for students to further discuss, rewrite and reorganize their ideas. The teacher/researcher (first author) encouraged the students to discuss disagreements and differences, question and justify solutions, revisit ideas over time, generalize and extend ideas. A month later, individual and group interviews were conducted with half the students to examine the durability of their ideas.

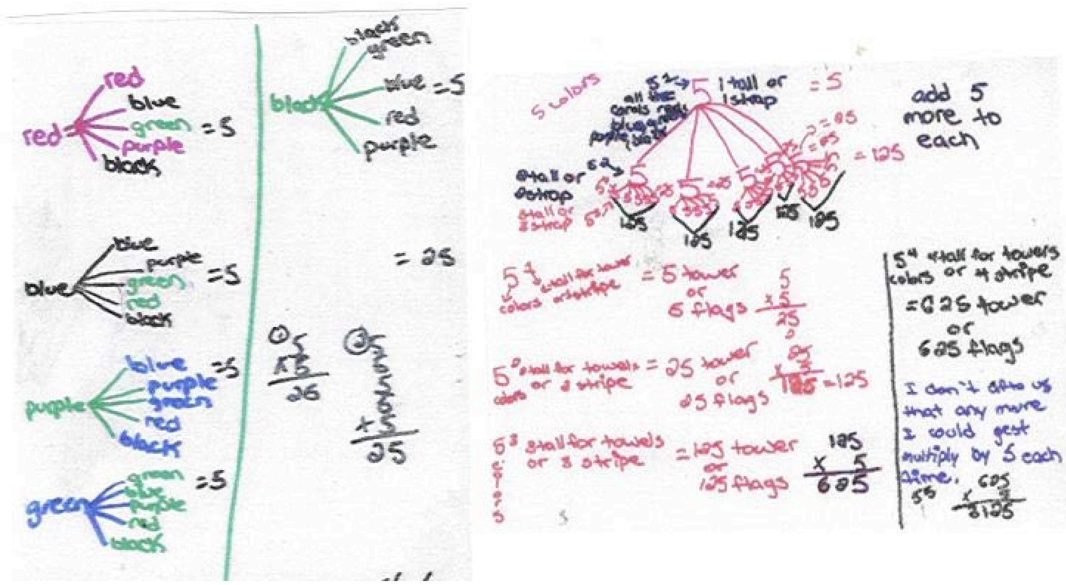
This study focuses on six of the hundred and fifty videotapes generated in this manner, as the students investigate a student-posed task, called the Castle problem. We coded transcripts to identify critical events (Powell, Francisco & Maher, 2001), determined by acts of flexibility. The analysis led beyond our characterization of flexibility (Warner, Coppolo & Davis, 2002) to a recognition of links between acts of flexibility and movement between layers in the Pirie/Kieren model for the growth of mathematical understanding (Pirie & Kieren, 1994). For example, questioning someone else's idea often resulted in a change to a new representation for the same idea, which moved the student from *image making* (doing something to get an initial idea of a concept) to *image having* (at which point the image is no longer tied to an action). In the following events, we examine the mathematical development of one student, Anna, by linking her acts of flexibility to movement through the layers of the Pirie/Kieren model. Anna is placed in a modified class based on her below average performance on standardized tests.

RESULTS

The students investigated six tasks, in depth, during the six month period of this study, two of which Anna connect to "The Castle Problem" (posed by Jessica): *If you're at the beach and there are 3 different kinds of buckets in front of you (small, medium, large). How many different ways can you build castles if there are a total of 10 castles?* The Flag problem (posed by Amanda) asks for all possible flags with two stripes, choosing from 5 colors. David extends it by also providing a choice of 5 shapes for the corner. Some of Anna's work for the Flag Problem, to which we refer later, is shown below.

The first day: understanding the task

In this first section, Anna uses a previous idea (see figure 1a below) in a new context while she is working in the *image making* layer. David begins by reading Jessica's written version of the Castle problem.



(a). (b).

Figure 1. Anna's work on the Flag problem

David	If you're at the beach, and there's three different kinds of buckets, small, medium and large, in front of you, how many different ways there are [sic] [*David laughs*] to make them if there are only a total of 10 buckets.
Anna	So, they're saying that there's only 10 of them.
T/R	So [looking at a student's written version of the problem] you have ten castles in total and you have a choice of a small, medium or large bucket.
Anna	This is hard.
T/R	So, basically you want to know how many possibilities there are.

Anna immediately begins writing, and although she says that this is a difficult problem, she immediately comes up with a correct strategy and begins to solve. (See figure 2a)

Anna	Small you can have…it's almost…small, medium…and large. [*drawing a tree with three branches coming out of each of the three sizes*]

When Anna uses the words "it's almost", we infer that she is *folding back* (Pirie & Kieren, 1994) to make a connection with a previous context. This facilitates her *image-making* in this context.

Ideas are questioned and a revision is offered

In this section, Anna uses a previous representation in another context. Being questioned by David deepens her understanding, which enables her to reorganize her ideas, flexibly move to a new representation, and move from *image making* to *image having* (Pirie & Kieren, 1994). Anna continues the next part of her solution (bottom of figure 2a). In the event below, David questions Anna's answer of nine possibilities and her second representation (See the bottom of figure 2a). He offers a suggestion and they continue drawing their trees using his idea.

David Can I ask you a question? How come you did small small, small small and medium? …because then there's three castles. [*David points to her trees.*]

Anna [*connecting the small small, medium medium and large large tree and writing a nine at the end of it]*…but, there are nine possibilities.

David No, there's more than nine. Hold on let me…this would be…let me get another sheet here. [*taking a new sheet*]

At this point Anna becomes excited - it appears that she recognizes her mistake. David begins to show her how to reorganize her idea by drawing the possibilities.

David You would have small small. Then you would have small medium and small large.

David Small, small and that branches three ways. [*He draws three branches coming out of small, small.*]

Anna Yes.

David So it's small small small. [*writing the word small coming out of a branch and drawing all possibilities for Small small, Small medium and Small Large…*]

Anna Medium…Now, would you do Medium, small?

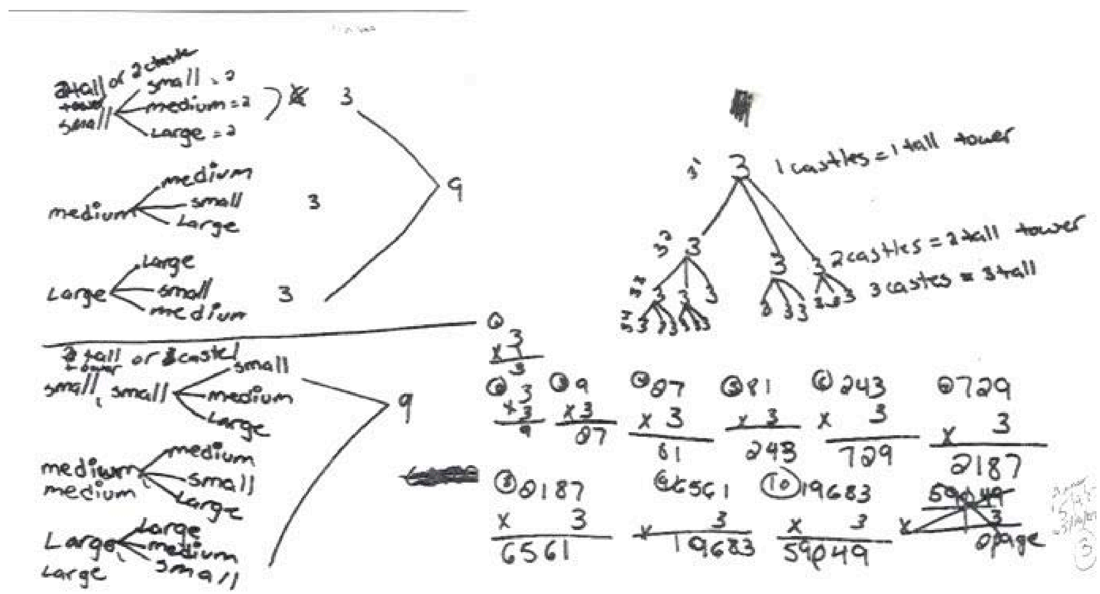
David Yeah. [*He continues with his tree.*]

David questions Anna's idea, points out what she missed and offers a way to revise it. Anna uses David's suggestion and reorganizes her tree. Anna now has a deeper understanding of her representation and how it connects to the task. David and Anna continue to draw their trees, separately, working in the *image making* layer. In the following event, Anna moves beyond this to *image having*. Immediately after David questions Anna again, she corrects her mistake, then flexibly uses an old idea (Figure 1b) to move to a new representation (Figure 2b).

David How come you wrote small small, medium medium, medium small, medium small?

Anna What do you mean? Small small, small medium, small large, medium small, medium small, oh. [*Anna fixes the second medium small and writes medium medium, instead.*]

After Anna corrects this mistake, she flexibly changes her representation to a "three tree" (Figure 2b), which moved her from *image making* to *image having*. She no longer needs to write small, medium and large: she has an image of the three sizes and writes a "*3*" to represent them. It appears that being questioned by David pushed her to understand her mistakes which moved her to a deeper level of understanding.



(a) (b)

Figure 2. Anna's changes of representation

Questioning someone's idea and showing that it is invalid

In this section Anna questions David's assertion, shows that it is invalid, and explains why. This necessitates *property noticing* (Pirie & Kieren, 1994). David begins by writing a nine with nine branches coming out of it. Note that Anna questions David's idea and shows it is invalid by providing a reason for using the three in terms of the properties noticed about her images.

T/R Well, what does the nine represent?

Anna Wait, no, first you should do three.

David Wait, let me explain. Nine, sort of like, this would be…are you saying...maybe nine is not the right number?

Anna Yeah, I think three.

David Right. I'm sorry it is three. [*draws a line under his 9 with 9 branches*]

T/R Why three?

Anna Because there's three, there's small, medium and large.

David Right.

Anna So, out of that three you get three more and it's…you get three from small, three from medium and three from large. [*points to the 3's in the 2nd layer of her tree*] Then from that you get three from small, three from medium, three from large. [*points to the 3's in the third layer of her tree*]

Noticing properties by setting up hypothetical situations

In this section, Anna connects the sizes in the Castle problem to the colors and shapes in the Flag problem, by re-explaining her hypothetical situation she created for Flags. She *folds back* (Pirie & Kieren, 1994), then builds on the idea by creating a new hypothetical situation for Castles. When Anna tells us exactly what the tree would look like without drawing it, she also has an image of the castles, choosing from 4 sizes. She builds on her old hypothetical situation by adding another size (XXL), choosing from five sizes, and notices even more properties about the images she has.

T/R So what would this first three represent? [*pointing to Anna's previous representation in which the numeral 3 appeared*]

Anna One castle? [*writing 'one castle' next to the first three*] …and then this one would be two castles [*writing '2 castles' next to the second layer*] …and David's problem would be the same thing except we're talking about size and he's talking about shapes.

T/R Hmm, so how is David's problem the same?

Anna Ok, like you see, I moved on and you can go on with this problem too…like in David's problem I said what if you add another color or add another shape? …In this one, you can say what if you add another size.

T/R Maybe you can work on that, too, what if you did add another size?

Anna It would be four [*tapping her pen next to the three branch*], then four comes from that, [*drawing the four branches with the pen in the air*], that would be four, eight, twelve, sixteen [*tapping her pen on each imaginary spot that the four branches would be in on her paper*]…and then out of those four another four come out. It's the same thing. [*finding her previous work for flags, figure 2b, and pointing to it*] I say five over here [*pointing to the top five*] and then five come out [*pointing to the five branches with her pen*] and for each five you add five more.

T/R Hmmm. So what would that five represent in Castles?

Anna You would need…small, medium, large, extra large, extra extra large.

Anna sets up new hypothetical situations for the existing problem, *folds back* to a previous hypothetical situation she set up for another context, and connects representations for the two problems. In the next line, Anna appears to be saying that "moving on", or setting up hypothetical situations, allows her to move to other representations and discover new things about the problem. Notice that she moves to another representation [*multiplying by 5*] while saying this.

Anna Can I tell you something? Well, when I moved on, I said you times by five each time. If you do that, lets say another person says I want you to do small, medium, large, extra large, so you already have the answer.

T/R Maybe we can solve that after we are done solving this one. Maybe we could extend the problem. Do you want to do that when we're done solving this one?

Anna It's the same thing except you're adding another size or you're adding another shape or color.

T/R Hmmm, interesting. I'd like you to label them that way if you could.

Anna This one? [*pointing to the 3 labeled 'one castle'*]

T/R How else would you label that besides one castle?

Anna One tall or... (*now connecting Building Towers - See Background section*)

T/R: Go ahead. [*Anna writes '2 castles = 2 tall' next to the second layer of the tree and '3 castles = 3 tall' next to the third layer of the tree*.]

Her talk about "moving on" advanced Anna towards using new representations, namely, multiplying by five and labeling each level with the number tall. Her *property noticing* is now on the level of sophistication at which the properties amount to identifying an isomorphism between the three problems (Castles, Flags and Towers).

Moving from formalizing to observing

We now pass over Anna's move from *property noticing* to *formalizing* (generalizing to create a "for all" statement) in order to highlight her moving beyond this layer to *observing* - noticing properties about "for all" statements. She does this by setting up hypothetical situations for the existing problem (for example, "If I add another size or color to 3 to the n, I get 4 to the n"), by connecting representations (using exponents to label each layer of the tree) and by connecting contexts (labeling each "n" and base with the information that it represents in Castles, Towers and Flags). All of these labels can be seen in Figure 3, which were written by Anna after moving back and forth between the layers over the course of three days.

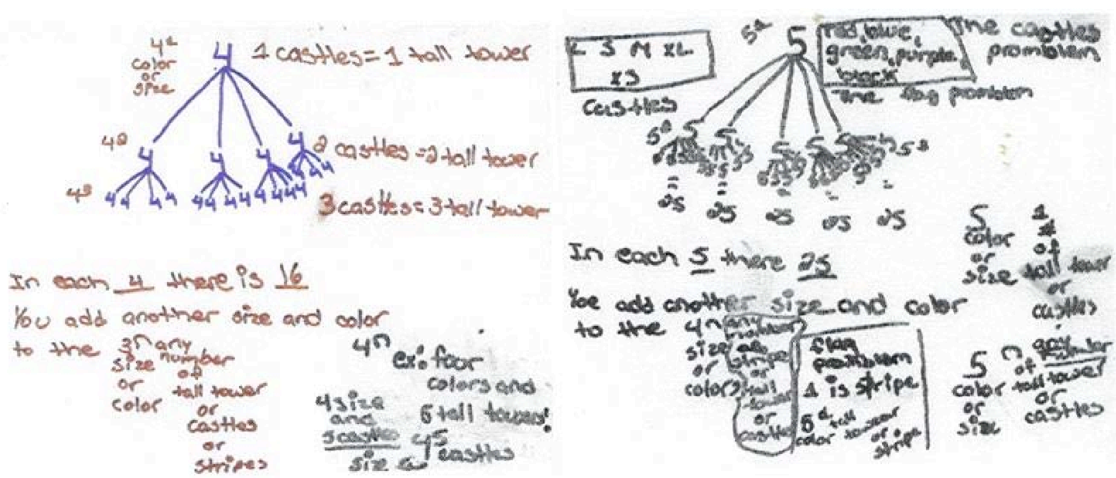


Figure 3. Anna's labeling of diagrams, written at the end of the 3rd day

CONCLUSION

Through a detailed analysis of one student's work on combinatorial problems we have shown that the transitions from one to the next layer in the Pirie/Kieren model of understanding occurred in association with acts of flexible mathematical thinking. We hypothesize that it is the acts of flexible thinking that act as catalysts for transitions in the Pirie/Kieren model for growth of mathematical understanding. We also hypothesize that the success of interactive classrooms is partially due to the opportunities and encouragement they provide for students to engage in flexible thinking. We have also shown that even students who are considered to be of below average ability can develop sophisticated mathematical understanding in an environment in which questioning, debate and discussion are encouraged.

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YOUNG CHILDREN'S UNDERSTANDING OF EQUALS: A LONGITUDINAL STUDY

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This paper examines the change in young children's understanding of equals as equivalence over a three-year period and their ability to express this understanding in real world problems. Seventy-six children participated in the longitudinal study. The results indicate that approximately one third of the sample had an understanding of equals as equivalence and the difficulties and misunderstandings they experienced in Year 3 persisted through to the end of Year 5. Many of those who appeared to have an understanding of equivalence could not express this understanding as real world problems.

INTRODUCTION

In spite of the wealth of research in the algebraic domain, including student learning, teaching strategies, and the use of technology, it still remains a barrier for participation in high levels of mathematics (National Council of Teachers of Mathematics [NCTM], 2000). Many students at both the high school and tertiary level continue to experience difficulties. In an answer to this continuing problem current research has turned to the elementary years with a particular focus on arithmetic as a key access to algebra (Carpenter & Levi, 2000; Carraher, Schielmann, & Brizuela, 2001; Kaput & Blanton, 2001; Warren & Cooper, 2001). It is believed that one of the most pressing factors for algebraic reform is to develop in elementary students the arithmetic underpinnings of algebra (Warren & Cooper, 2001), and to extend these to the beginnings of algebraic reasoning (Carpenter & Franke, 2001). The aim of this reform is to allow elementary school students access to powerful schemes of thinking about mathematics (Carpenter & Franke, 2001) that assist them in participating in algebra at later grades. The arithmetic underpinnings of algebra include understanding of operations, arithmetic properties and equals, the focus of this paper (Boulton-Lewis, Cooper, Atweh, Pillay & Wills, 1998).

EQUALS AND EQUIVALENCE

Freundenthal (1983) delineated a number of roles for the equal sign. These included: (a) indicating a task or a question (e.g., 3+4 = ?: 3+? = 7). In this instance the = sign suggests that an answer needs to be found; (b) representing equivalent situations (quantitative sameness), the symmetric quality of the equal sign where the left and right of the sign mean the same thing (e.g., 18 ÷ 3=24 ÷ 4); (c) stating something is true for all values (e.g., a+b=b+a) and, (d) introducing a new variable (e.g., a+b=c). In the elementary school the primary focus seems to be on the first of these roles as many children interpret equals as a sign to do something (Behr, Elwanger & Nicols, 1980; Carpenter & Levi, 2001). Most children do not have an understanding of equals as representing quantitative sameness (Carpenter & Levi, 2001), or stating something is true for all values, in particular recognising examples of the commutative law as being true. Warren (2001) found that when presented with examples such as 2+3=3+2 a significant number of children stated that *it is not true because the equal sign is meant to go last and the plus first,* or offered 2+3=5+2=7. The second instance is an example of linearity of thinking where the children simply work from left to right (Saenz-Ludlow & Walgamuth, 1998). It was conjectured that for these children the role of the equal sign as indicating a question and an answer needed to be found, was so strong that it interfered with other understandings of the equal sign. Past research with young children has also tended to focus on the symbolic representations of equivalent situations (e.g., 4+5=6+3). Few studies have explored young children's ability to apply this understanding to real world contexts.

The dichotomy between representing mathematical procedure or applying the knowledge of arithmetic is referred to as contrasting solving by purely formal processes according to formal rules with the application of 'real knowledge' to solve real world problems. The first is characterised as task and performance, where the child is given a task and is simply asked to perform. In this instance the 'fixed words are well-shaped utterances of arithmetical language, .… and are simply automatic linguistic utterances" (Freundenthal, 1983, p464). For example, in the case of 7-4 saying '"takeaway four from seven" is an instance of using arithmetic language to echo the process. This is seen as the most primitive relation. The second requires recognising the underlying structure of the relationships between the quantities (MacGregor & Stacey, 1998), and applying this understanding to create problem situations, changing between the symbolic register to natural language register. Duval (2002) referred to this as a mathematical transformation involving a conversion, the most difficult transformation in mathematics. In this instance language is not simply the formal language of mathematics but entails the coordination of natural language with correct mathematical language, a difficult process. It is suggested in the literature that incorporation of 'real world' language adds to the difficulty of the task. Impacting on this complexity is the belief held by Pririe and Martin (1997) that language only serves problems where the equal sign appears just before the answer, thus reinforcing the equals indicating an action.

The aim of paper is not only to examine how young children's understanding of equals as equivalence changes over a three year period but also to ascertain whether their understanding simply represents a mathematical procedure or can they express this understanding in real world contexts.

METHODS

Sample

The sample was comprised of 76 children from four elementary schools in low to medium socio-economic areas in Australia. The children were all participants in a three-year longitudinal study investigating early literacy and numeracy development. By the conclusion of the study the children had completed Year 3, Year 4, and Year 5 of their elementary schooling. The average age of the sample at the beginning of the study was 8 years and 6 months and at the conclusion of the study was 10 years and 6 months. Prior to commencing the study, all had completed the first three years of formal education.

Instruments

At the end of each year all children completed a written test. After completion of the written test, each child was also interviewed. These interviews served to illuminate the responses on the written tests. A number of items were also common across the three years.

At the end of year 3 children were asked in the interview if the number sentences on the following cards were true or not true and to explain why they were true or not true.

2 + 3 = 3 + 2	**31 + 16 = 16 + 31**
2 - 3 = 3 – 2	**31 - 16 = 16 - 31**

Figure 1 Cards used for Task 2

The results of this activity indicated that children's understanding of "=" impacted on the responses given. Many of the responses in category 3 stated that *2+3 doesn't equal 3* and offered either 2+3=5 or 2+3=5+2=7, confirming the claim that many children in elementary grades generally think that the equal sign means that they should carry out the calculation that proceeds it and the number following the equal sign is the answer to the calculation (Warren 2001).

In order to further probe this misconception, the following task was developed.

Find the missing number : $7 + 8 = \square + 9$

Figure 2 Written question for the Year 4 and Year 5 test.

At the end of Year 4 and Year 5 the children were asked to solve the above question on a written test. In the interview that followed the completion of the written test, they were given their responses and asked to explain how they obtained their answer and to give, where possible, a word story for $7 + 8 = \square + 9$. They were all familiar with 'word stories' as this is a significant component of the Queensland syllabus. The Year 4 results indicated that most could provide a story for the problem $\square - 15 = 41$. A typical response to this problem was *I had some lollies and gave away 15. Now I have 41 left.*

RESULTS

The results for this task fell into four broad categories. The categories and frequency of responses for each are presented in Table 1.

Table 1 Frequency of responses for $7 + 8 = \square + 9$

Interpretations for equals	Year 4	Year 5
1. Correctly identified the unknown ($\square$ =6)	20 (26%)	28 (37%)
2. Interpreted equals as indicating the answer ($\square$ = 15)	23 (30%)	20 (26%)
3. Applied linear thinking to the number sentence ($\square$ = 24)	19 (25%)	18 (24%)
4. Other	14 (19%)	10 (13%)

An analysis of the data using a Chi Squared ($\square^{2}_{3} = 2.23$) test indicated that there was no significant difference between the responses across the two years. Over half the children persisted in seeing the role of = as indicating an action, in finding the answer. They either simply focused on the 7 + 8 = $\square$ component of the question, providing 15 as the unknown, or calculated 7+8=15+9=24, exhibiting linearity of thinking. This misconception persisted throughout Year 4 and Year 5, indicating a certain robustness to this understanding over an extended period of time. In most instances, where children were capable of provide a word story to illustrate the problem, these stories reflected their thinking. For example, for those who gave the answer 15 tended to give word stories such as *I had 7 lollies and someone gave me 8 more. How many do I have altogether?* The following section examines the responses of the students who correctly identified the unknown (Category 1).

Correct responses

Thirteen of the 20 children who correctly responded to this question in Year 4 also provided correct responses in Year 5. Of the other 7, in year 5 two believed that it should be 15 and two responded with 24, indicating that they had changed their perception of equal to one of indicating the answer. The other three respectively responded with 7, 8 and 13.

All of the children when in Year 4 could describe the procedure they used to find the unknown, procedural mathematical language. An example of this is utterances such as "7 plus 8 is 15 and 9 plus something is 15 so it must be 6". One during this process actually changed his mind and now believed that he was wrong on the test and the answer should be 24. Eleven could not provide an everyday story using natural language for the equation (No story). Of the remaining nine, one provided an inappropriate story using all the numbers (e.g., *I had 8 apples on a tree and 7 more grew that made 15 and then 9 died*) (Inappropriate story). Four gave a story for each side of equation (e.g., *I had 7 lollies. Mark gave me 8 and my friend had 6 lollies and Mark gave him 9)* (Story for each side). Two others, when telling their stories insisted on finding the answer for the whole equations, indicating a tendency towards linearity of thinking (e.g., *There were 7 monkeys in a tree and 8 more came and then there were 15 there. Then another 6 monkeys came and another 9 monkeys came and altogether there were 30 monkeys in the tree)* (Story for each side with closure). Only one child gave a story that represented the problem (Appropriate story). James said *I had 7 lollies and then I got 8 and that equalled 15 and then my best friend got 6 lollies and then he got 9 lollies and he got the same as me*. In nearly all instances the children needed to calculate the unknown ($\square$ = 6) before they could provide a story.

Similar trends existed in the Year 5 results. Table 2 presents the frequency of responses for each of the above trends for Year 4 and Year 5.

The results indicate that as children moved from Year 4 to Year 5 many more found the correct answer for the unknown and many more could also give appropriate stories for the equality. One of the Year 5 stories started with the two groups of 15 and then broke them into their components.

Table 2 Frequency of responses for word stories

Types of Stories	Year 4	Year 5

1. No Story	12	11
2. Inappropriate Story	1	5
3. Story for each side of the equation only	4	1
4. Story for each side with closure (e.g., *that makes 30)*	2	2
5. Appropriate story	1	10
Total	20	29

He said,

> There was two lots of 15 animals in two paddocks and the farmer wanted to split them up but he couldn't use the same amount 'cause he wanted to split them up into four separate paddocks but he couldn't do that because – well he could but he had to have different numbers because they were all small. Well one was small and then it goes a bit bigger, medium, like that. And they had to be at least one or two apart the numbers, or three. So what ones would he have? 7 plus 8 or 6 plus 9.

With regard to the 13 children who correctly responded in Year 4 and Year 5, 7 were incapable of providing any story for the problem in their Year 4 interview. Of these 7, by the end of year 5, 4 could still not provide an appropriate story of the equality. The remaining three attempted to provide a story. Of these three, one provided a story for each side of the equation and two provided an appropriate story. Of the remaining 6 children who correctly identified the unknown in Year 4, 4 provided an appropriate story in Year 5 and the remaining 2 provided inappropriate stories. The following figure summarises how their stories changed over the intervening period.

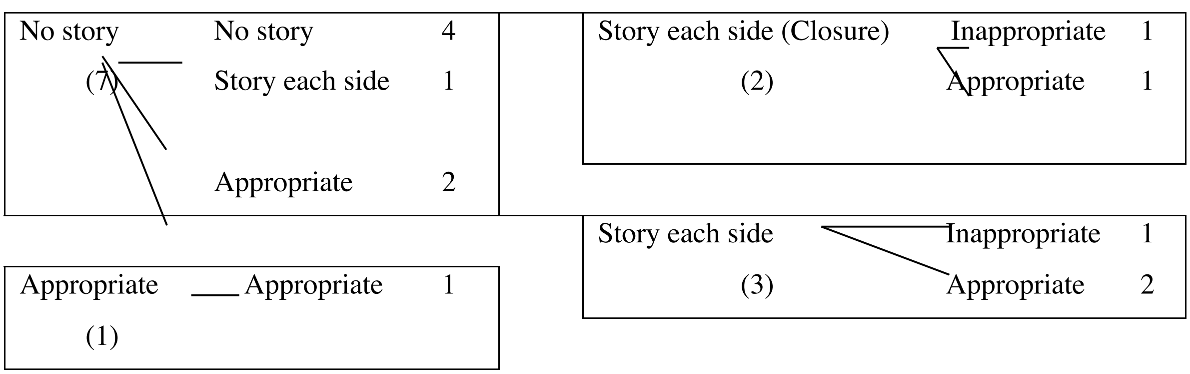

Figure 1 Frequency of thirteen children's story from Year 4 to Year 5

In both instances, the stories considered as inappropriate were such that the answer to them was the unknown (6). For example, Jan said *there were 15 seats in the circus and 9 were taken how many seats were left*. From the above results it seems that children can arrive at the ability to tell appropriate stories from different pathways. Even though the categories delineated in Table 2 suggest levels of development (from no story to story for each side to appropriate story) it seems that children can simply go from no story to appropriate story without passing through the other levels of story telling. Interestingly none of these thirteen children provided stories with closure, and the child who could tell an appropriate story in Year 4 provided an appropriate story in Year 5.

DISCUSSION AND CONCLUSION

First, from the responses given in Table 1, it can be seen that many reflect common misconceptions identified in the literature (e.g., equal as indicating the answer and computing from left to right to find an answer). This research adds to the literature in that it seems that once these misconceptions are formed they seem to remain fairly stable. This could reflect the types of problems elementary children are commonly presented with where the answer occurs after the equal sign. Further research needs to occur to ascertain (i) the robustness of these misconceptions impact on understanding equal as equivalence in the later years, and (ii) the types of activities that would assist young children to challenge these misconceptions.

Second, many of the children were capable of language utterances that reflected the processes they used to find the answer. Even when they were incorrect their utterances matched their misconceptions. The majority found difficulties in expressing the symbolic representation in a real world language context. Although in many instances their real world representations mirrored their misconceptions. For example, with the answer 15 some children gave examples such as, *I had 7 pencils and my friend gave me 8 more. How many do I have altogether [15].* As young children are negotiating mathematical understanding what roles do mathematical utterances and real world language play in the negotiation of meaning? How is their interpretations of symbols influenced by their ability to represent the symbols in real world terms. In other words, do they interpret the above expression as 7+8=15 simply because that is the expression they can express in real world language. Duval (2002) suggests that mathematical utterances of the mathematical processes entail staying in the one mathematical register. This is considered as the easiest mathematical activity, and thus is commonly the activity that occurs in many classrooms. On the other hand expressing symbols in real world language entails changing mathematical registers. Duval (2002) refers to this as a conversion, one of the most difficult mathematical activities. It involves a mapping from one register to another as compared to a mapping within the same register. Comprehension in mathematics commonly involves the coordination of at least two registers of semiotic representations. The results of this research support Duval's (2002) claim that such coordination of registers does not come naturally, but it is in this coordination that mathematical thinking occurs.

Third, while there is no evidence that supports a sequence in growth of ability to use language to provide real world stories, it seems that a significant number of children (10) could pose relevant word problems for this context by the end of Year 5. Also, many of those who were unsuccessful on the task provided language problems that mirrored their responses. In most of these instances the problems provided were structured such that the answer appeared just after the equal sign, the type of word problems commonly privileged in many classrooms (Pririe and Martin, 1997). Further research needs to be carried out in order to explore the role of 'real world' language on the interpretation of symbols, does it indeed favour certain interpretations.

This research indicates that some young children are not only capable of correctly interpreting equivalent situations but also can recognise the underlying structure and express this in an appropriate real world context. But in most instances they needed to assign a value to the unknown before they could create a problem. There was also a reluctance to use words such as 'some' for the unknown. Most past research has tended to focus on how young children can represent word problems in symbols and not the reverse process. The influence of this capability and the need to find the unknown before being able to pose a problem on their ability to represent word problems in symbols needs further investigation. The research also indicates that children's narrow conception of the equal sign not only occurs on early in their development but also persists as they progress through their elementary years.

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STATISTICAL VARIATION IN A CHANCE SETTING

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A series of 13 survey questions based on a 50-50 spinner is used to explore school students' understanding of statistical variation in a chance setting. Five questions set the context by assessing understanding of theoretical expectation and representation of repeated trials in a stacked dot (line) plot. Four questions provide opportunity to display appreciation of variation from point expectation and four questions address variation from distributional expectation. A total of 707 students in grades 3 to 9 answered some or all of these questions. A subset of 334 students then took part in a unit of study on chance and data emphasising variation. These students answered a post-test including the same items. Analysis showed a progression across the years of schooling, plateauing at grade 7 and improvement for all grades after instruction. Implications are considered.

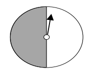
INTRODUCTION

Although research into school students' understanding of variation has not progressed as rapidly as research into other topics in the chance and data curriculum, interest in developing tasks that will allow students to display their appreciation of variation is growing. Zawojewski and Shaughnessy (2000) became frustrated with a national survey test item based on drawing 10 objects from a container with 50% red that required students to predict a specific number of reds. The item did not encourage a range of answers and Shaughnessy, Watson, Moritz, and Reading (1999) revised the item, employing various formats to allow students to describe not only the most likely outcome but also what variation might occur.

In a chance setting there are two aspects of variation that effect the distribution of the outcomes that appear over repeated trials. First is the variation from the theoretical value expected, the most likely outcome; for example, from a container with 50% red objects sometimes one might draw 4 red out of 10 or 6 red out of 10. This is the variation that produces the probability distributions, such as the binomial, which are studied in the senior secondary years. In practice an experimental distribution does not match exactly the theoretical distribution: there is also variation from the ideal shape of many outcomes. The development of tasks to reflect student understanding should give students opportunity to show appreciation of expectation (theory), of the likelihood of some variation from that expectation, of a distribution of expected outcomes, and of potential unrealistic variation from the distribution of expected variation.

Working in a straight-forward chance setting, as done by Shaughnessy et al. (1999), would appear to be important in order to focus on aspects of variation rather than details associated with working out probabilities or searching for causes of change. To explore appreciation of expected variation in a chance setting, students can be asked to predict a number of individual outcomes or to draw a distribution of outcomes. Both of these alternatives were tried with an interview protocol developed from the work of Shaughessy et al. (1999) by Kelly and Watson (2002). Students had more difficulty with drawing a distribution than with predicting a list of possible outcomes. Even for students who could create a graph, the degree of variation displayed was usually greater than reasonable. The limited practical experience of students, however, suggested that it was unrealistic to expect a good fit to a theoretical model for a large number of trials.

Set 1. A class used this spinner.



Q1. If you were to spin it once, what is the chance that it will land on the shaded part?

Q2. Out of 10 (50) spins, how many times do you think the spinner will land on the shaded part? Why do you think this?

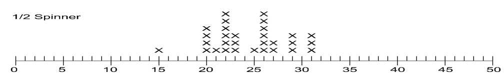
Q3. If you were to spin it 10 (50) times again, would you expect to get the same number out of 10 (50) to land on the shaded part next time? Why do you think this?

Q4. How many times out of 10 (50) spins, landing on the shaded part would surprise you?

Q5. Suppose that you were to do 6 sets of 10 (50) spins. Write a list that would describe what might happen for the number of times the spinner would land on the shaded part?

______, ______, ______, ______, ______, ______

Set 2. A class did 50 spins of the above spinner many times and the results for the number of times it landed on the shaded part are recorded below.



Q6. What is the lowest value?

Q7. What is the highest value?

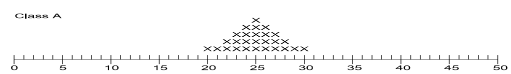
Q8. What is the range?

Q9. What is the mode?

Q10. How would you describe the shape of the graph?

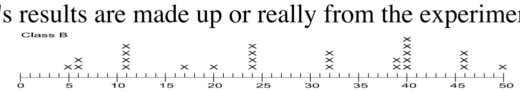
Set 3. Imagine that three other classes produced graphs for the spinner. In some cases, the results were just made up without actually doing the experiment.

Q11. Do you think class A's results are made up or really from the experiment?



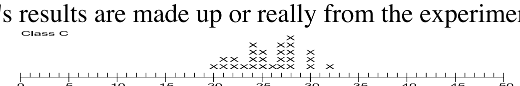
☐ Made up ☐ Real from experiment - Explain why you think this.

Q12. Do you think class B's results are made up or really from the experiment?



☐ Made up ☐ Real from experiment - Explain why you think this.

Q13. Do you think class C's results are made up or really from the experiment?



☐ Made up ☐ Real from experiment - Explain why you think this.

Figure 1: Spinner questions used on survey

An alternative to asking students to produce their own distribution of outcomes is to present them with completed graphs representing repeated outcomes and ask which are reasonable and which are not. To do this requires a simple presentation, which like the basic chance setting, does not complicate the issue at hand. The stacked dot (or line) plot is well-suited to this task and has been found useful in various contexts for allowing students to focus on variation (Konold & Higgins, 2002), and on telling a story with data (Watson & Kelly, 2002c). The plot directly displays frequencies vertically as dots or Xs along a baseline with the scaled data values labeled.

This study is based on a series of 13 questions, shown in Figure 1. The scenario is the repeated spinning of a 50-50 spinner with the first set of five questions introducing the spinner and the potential for variation in outcomes in repeated trials. The second set of five questions introduces the stacked dot plot and explores student familiarity with the basic characteristics of the graph of a distribution of outcomes in sets of repeated trials. The third setting offers three stacked dot plots, two with unlikely variation, e.g., too perfect or too much variation, and one with likely variation, for evaluation. The tasks are based on classroom activities developed by Torok (2000).

The research questions for this study are based on the usefulness of the tasks to assess student appreciation of variation in a chance setting and change after instruction. For a subscale of five Prerequisite questions (PRE), Q1, Q6, Q7, Q8, and Q9: Do students appreciate the theoretical chance involved and can they interpret a representation of a distribution? Is there a trend over grades and a difference after instruction? For a Point Estimate Variation subscale of four questions (PEVar), Q2, Q3, Q4, Q5: Do students' acknowledge the role of variation when predicting outcomes for a single and/or repeated trial of a spinner? Is there a trend over grades and a difference after instruction? For a Distributional Variation subscale of four questions (DisVar), Q10, Q11, Q12, Q13: How do students describe appropriate variation and can they accurately identify appropriate and inappropriate variation in an established distribution? Is there a trend over grades and a difference after instruction.

METHODOLOGY

Sample. Sample 1 consisted of 707 students from grades 3, 5, 7, and 9 in nine public schools in the Australian state of Tasmania who were surveyed as part of a larger study on school students' understanding of statistical variation. Sample 2 was a subset of 334 students from Sample 1 who received instruction on chance and data focusing on variation in five of the schools. These students were given a post-test to measure the effect of instruction on change. Sample sizes in the four grades are given in Table 1.

	Grade 3	Grade 5	Grade 7	Grade 9	Total
Sample 1	150	181	184	192	707
Sample 2	72	82	91	89	334

Table 1: Number of students per sample in each grade

Procedure. The questions in Figure 1 were part of a larger survey investigating students' understanding of statistical variation (Watson & Kelly, 2002a, 2002b). Students in grades 3 and 5 were presented with the Q1 to Q5 only and were asked Q2 to Q5 out of 10 spins. Students in grades 7 and 9 were given all thirteen questions, and were asked Q2 to Q5 out of 50 spins. All students were administered the surveys in class time. For grades 3 and 5 the researchers showed a spinner to the class before commencing the survey demonstrating the purpose of the spinner and how it worked in case there were any classes that may not have used spinners yet in their studies.

Students in Sample 2 were in the experimental group undertaking a unit of work on chance and data. Grades 3 and 5 were taught by the same teacher (provided by the research team), whereas grades 7 and 9 were taught by their usual mathematics teacher. Details of the lessons taught can be found in Watson and Kelly (2002a) for grades 3 and 5, and Watson and Kelly (2002b) for grades 7 and 9. During the intervention, all classes in Sample 2 were exposed to lessons using spinners and all grade 7 and 9 classes had graphing activities, most involving stacked dot plots. The students in Sample 2 were re-administered the same survey approximately 6 weeks after the completion of the lessons, in the same way it was administered for Sample 1.

Analysis. For this study the prerequisite questions, Q1, Q6, Q7, Q8, and Q9 were coded on a correct-incorrect basis. For these questions a code of 1 was given to responses of "50,""Half," "50/50" or the equivalent for Q1, "15" for Q6, "31" for Q7, "15-31" or "16" for Q8, and "22 and/or 26" for Q9. Responses to all other questions were categorized and coded by the authors to reflect an increasingly sophisticated appreciation of variation. In particular, the criterion for determining the appropriateness of the variation displayed in responses to Q5 was based on a simulation of 1000 outcomes using an EXCEL spreadsheet. The standard deviation for each simulation was calculated and then plotted, and appropriate variation was determined by values within the middle 90% of the distribution (0.6 – 2.3 for 10 spins; 1.3 – 5.0 for 50 spins). Examples of responses for codes for Q2 to Q5 are given in Table 4 and examples for Q10 to Q13 are given in Table 5 of the Results section. The scoring rubrics developed for these items were devised specifically to reward an appreciation of variation. It is acknowledged that others might devise different rubrics for different purposes, for example for Q10.

Although F-tests were performed for the PEVar scale across grades, t-tests are reported here as appropriate to describe observed differences for pairs of grades. T-tests were used to compare grades 7 and 9 on the PRE and DisVar scales also. Paired t-tests were used for Sample 2 with respect to pre- and post-test scores. The results will be discussed in the order of the research questions, with respect to the PRE questions, the PEVar questions, and the DisVar questions.

RESULTS

Prerequisite Ideas

Questions Q1, Q6 to Q9, in Figure 1 do not address variation but are necessary or potentially relevant to the other questions. Table 2 contains the percent correct on each item for grades 7 and 9 in Sample 1. For Q1, the grade 3 percent correct was 22.0% and for grade 5 it was 62.4%.

	Q1	Q6	Q7	Q8	Q9
G7	80.4	76.1	34.2	27.7	2.2
G9	85.4	71.9	33.9	31.8	16.1

Table 2: Percent correct for PRE questions in Sample 1

For Sample 2, Table 3 contains the pre- and post- performance on the PRE questions. With the exception for Q6, there was an improvement on all questions for both grades 7 and 9. For grade 3 the corresponding change for Q1 was from 23.6% to 40.3%, and for grade 5 it was 62.2% to 68.3%. Overall instruction assisted the basic ideas.

		Q1	Q6	Q7	Q8	Q9
G7	Pre	83.5	81.3	34.1	31.9	2.2
	Post	89.0	80.2	50.5	50.5	28.6
G9	Pre	86.5	76.4	42.7	34.8	24.7
	Post	94.4	77.5	53.9	41.6	31.5

Table 3: Percent correct for PRE questions for Sample 2 before and after instruction

Point Estimate Variation

Table 4 contains examples of responses to the PEVar questions. Without acknowledging variation, students were more likely to give code 2 responses to Q2. Only 5.3% of grade 3 students acknowledged variation in Q2, increasing to 7.2% in grade 5 and 19.6% in grade 7, but dropping to 12.5% in grade 9. For Q3, again appreciation of variation increased slightly with grade.

Code	Summary	Examples
Q2. Out of 10 (50) spins, how many times do you think the spinner will land on the shaded part? Why do you think this?		
3	Variation	24 because it is close to half; About 5, it's equal
2	Theoretically correct; Anything can happen	25 because there's an equal chance for shaded and white 25, divide by 2; 50/50, you can't estimate
1	Inappropriate reasoning	5 out of 10 because you have a good chance of getting this one; Very good … it might land on the white side
0	NR	Don't know
Q3. If you were to spin it 10 (50) times again, would you expect to get the same number out of 10 (50) to land on the shaded part next time? Why do you think this?		
3	Variation	Not exactly because the spinner would vary slightly
2	Anything can happen; Chance	No, it's the luck of the spin; Yes, same odds
1	Non- chance theories	No, because nothing has changed at all Yes, because you do the same as you did the first time
0	NR/no reason	Yes, just guessing
Q4. How many times out of 10 (50) spins, landing on the shaded part, would surprise you?		
1	Reasonable responses	Grade 3 & 5 – 0,1,2,8,9,10; Grade 7 & 9 – <20, >30
0	Inappropriate responses	Grade 3 & 5 – 3,4,5,6,7, Grade 7 & 9 –20 to 30 Ambiguous (e.g., Not many), misinterpretation
Q5. Suppose that you were to do 6 sets of 10 (50) spins. Write a list that would describe what might happen for the number of times the spinner would land on the shaded part?		
3	Appropriate variation	5, 2, 6, 7, 8, 9 (SD = 0.6-2.3 – 10 spins) 30, 20, 32, 26, 24, 18 (SD = 1.3-5.0 – 50 spins)
2	Too small, too wide or no variation (strict probability)	5, 8, 10, 1, 3, 2, 1 (SD <0.6 >2.3 – 10 spins), 5, 30, 26, 18, 45, 50 (SD <1.3 >5.0 – 50 spins)
1	Lop-sided, All <5 or >5 (10), All <21 or >30 (50)	2, 1, 3, 0, 4, 0 (10 spins) 1, 3, 7, 8, 9, 11 (50 spins)
0	Inappropriate response	25, 50, 75, 25, 50, 75 (out of range)

Table 4: Codes and examples for Q2 to Q5 completed by grades 3, 5, 7, and 9

Suggestion of appropriately surprising outcomes in Q4 increased from 42.7% of grade 3 students to 76.6% of grade 9 students. There was, however, a monotonic decline over the grades in the ability to provide reasonable variation in the six suggested outcomes of trials in Q5, with 36.0% in grade 3, 32.6% in grade 5, 22.3% in grade 7 and 18.2% in grade 9 doing so. In fact across the grades 1.3% of grade 3, 15.5% of grade 5, 14.1% of grade 7, and 17.2% of grade 9 suggested outcomes in strict accordance with theoretical probability, e.g., 5, 5, 5, 5, 5, 5 (10 spins) or 25, 25, 25, 25, 25, 25 (50 spins). At the same time the percent predicting too much variation fluctuated with 20.7% in grade 3, 14.4% in grade 5, 31.5% in grade 7, and 26.0% in grade 9 doing so.

Table 5 contains the means and standard errors for all grades for Sample 1 on the PEVar subscale. There was a steady rise in performance from grades 3 to 7, with a plateau evident between grades 7 and 9. There were significant differences between grades 3 and 5 ($p<.001$) and between grades 5 and 7 ($p<.01$) on the PEVar subscale.

	PEVar				DisVar	
	G3	G5	G7	G9	G7	G9
Mean	4.15	5.28	5.94	5.81	6.18	6.29
Std Error	0.203	0.172	0.154	0.153	0.203	0.201

Table 5: Means and standard errors for Sample 1 on the variation subscales

As can be seen in Table 6 all grades in Sample 2 showed a significant improvement on the PEVar subscale from the pre- to the post-test ($p<.001$ – grades 3, 5, and 7; $p<.01$ – grade 9). Again a plateau effect is evident from grades 7 to 9 on the PEVar subscale for the post-test after a rise in performance from grades 3 to 7 on these items.

		PEVar				DisVar	
		G3	G5	G7	G9	G7	G9
Pre	Mean	3.89	4.93	5.81	6.12	5.93	6.30
	Std Error	0.336	0.269	0.224	0.217	0.296	0.280
Post	Mean	5.19	6.24	6.97	6.79	8.80	7.20
	Std Error	0.282	0.192	0.226	0.191	0.290	0.271

Table 6: Mean and standard errors for Sample 2 on the variation subscales

Code	Summary	Examples
Q10. How would you describe the shape of the graph?		
3	Acknowledges variation	Most of it land landed on 20; Jaggedy; Up and down
2	Reasonable shape description	Pyramid; City; Hill; Melbourne (Physical objects) Triangle; Rectangular; Circle (Geometry)
1	Focuses on graph and axes	Line graph; Column graph (Graph types) Straight; Flat; Even; A line (Axes)
0	Unreasonable responses	Small; Strange; Different; Big (Illogical); Don’t know (NR)
Q11-Q13. Imagine that three other classes produced graphs for the spinner. In some cases, the results were just made up without actually doing the experiment.		
3	Right choice with appropriate reason	Q11: Made up – Shape of a triangle Q12: Made up – The range is too big Q13: Real – Not even but around 25
2	Right choice with vague reason	Q11: Made up – It would be impossible Q12: Made up – Doesn’t look real Q13: Real – More stable graph
1	Right choice with no reason Wrong choice with data based reason	**Q11: Made up – ? / Real – Around 25, the average** Q12: Made up – They are! / Real – All over the place Q13: Real – Don’t know / Made up – All in one spot
0	Wrong choice with no reason, illogical or vague reason	Q11: Real – They would not lie Q12: Real – It’s random Q13: Made up – It looks wrong

Table 7: Codes and examples for Q10 to Q13 completed by grades 7 and 9

Distributional Variation

Table 7 contains examples of responses to the DisVar subscale answered by students in grades 7 and 9. Neither grade showed very much imagination (code 1 or above), in answering Q10, with 13% of grade 7 and 16.7% of grade 9 mentioning something to do with variation. For the three questions asking about real or made up distributions, 73.1% of Sample 1 students judged appropriately with either vague or appropriate reasoning, for Q11 (codes 2 or 3). For Q12, this percent dropped to 40.2% indicating more difficulty in appreciating too much rather than too little variation. For the reasonable distribution in Q13, 64.1% of students gave vague or appropriate reasons in supporting this view. For Sample 2, the percents before and after instruction for Q11 were 73.9% and 86.7%, for Q12 were 36.1% and 57.2%, and for Q13 were 63.3% and 77.2%. This indicates that after instruction students were more easily able to identify appropriate and inappropriate distributions by giving data-based reasons in their arguments.

Table 5 for Sample 1 shows that there is no significant difference between grades 7 and 9 on the DisVar subscale. Table 6 reveals that for Sample 2 both grades improved significantly after instruction (G7, $p<.001$; G9, $p<.01$). Although grade 9 students improved significantly on the DisVar subscale, the grade 7 students not only improved significantly, but also performed significantly better than the grade 9 students ($p<.001$) on this subscale after instruction.

DISCUSSION

Three aspects will be covered in relation to the implications of the results: outcomes of the study, limitations of the study, and educational issues for the understanding statistical variation.

The three subscales based on Figure 1 allowed the presentation of outcomes to answer the research questions in relation to prerequisite knowledge, point estimate variation, and distributional variation. All three aspects will be important in future research and classroom planning. Teachers need to be aware, for example, of initial unfamiliarity with "chance" by young children and the different degrees of recognition of lowest and highest values in a graph by middle school students; of the lack of spontaneous acknowledgement of variation without prompting; and of the apparent greater difficulty of recognizing too much variation as inappropriate compared to too little variation. Tasks such as Q5 and Q11 to Q13 should provide useful diagnostic information for teachers about students' beliefs regarding the extent of reasonable variation. The instances where students predict no variation in six repeats of many trials in Q5 (e.g., 12.6% overall in Sample 1) or claim that the distribution in Q11 is likely to be real rather than made up (e.g., 13.6% overall in Sample 1) suggest a lesson in the teaching of probability: Expectation must be balanced by variation. On the other hand, the instances of providing too much variation in six repeats of many trials in Q5 (e.g., 23.3% overall) or not realizing that the distribution in Q12 was made up rather than real (e.g., 22.3% overall) point to another issue for the classroom: Students need many hands-on experiences to develop an appreciation for "how much" variation is reasonable. This will not happen in a single lesson.

Limitations of the study from a measurement perspective are associated with using 10 repeated spins for grades 3 and 5 in Q2 to Q5, and 50 spins for grades 7 and 9. This may account for the fluctuating performance across grades for Q5. Having little control over the teaching arrangements in grades 7 and 9 may have contributed to greater improvement in performance on the PEVar and DisVar subscales for grade 7 than grade 9. The plateau of performance at grade 9 may also reflect the classes chosen for participation by the schools or an overall lack of continuing interest in chance and data by schools over the middle years.

Variation is everywhere and has different manifestations depending on where it occurs and what causes it. The choice of scenario found in this paper was intended to simplify some aspects of variation in order to concentrate on the statistical aspects of variation from a point estimate and variation from the theoretical distribution of random outcomes related to the point estimate. Children as young as six have little trouble appreciating that variation occurs in chance settings (Kelly & Watson, 2002), however, what is not easy is finding tasks that can determine to what degree students imagine variation occurring. Asking about repeated trials is a starting point. It would appear, however, that some type of visual presentation, probably graphical, is required before the more complex issue of appreciation of variation from distributions can be explored. It will be interesting to follow future research as other tasks are developed to investigate student understanding.

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A PROCEDURAL ROUTE TOWARD UNDERSTANDING THE CONCEPT OF PROOF

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In this paper, I describe how undergraduates can develop their understanding of the concept of proof by viewing the act of proving as a procedure. Such undergraduates first understand proof as an ***algorithm****- or a step-by-step mechanical prescription for proving certain types of statements. The students can then condense this algorithm into a* ***process****- or a shorter list of global, qualitative steps. By reflecting on the process, successful students can view proof as an* ***argument****- something that establishes the veracity of a mathematical statement. I illustrate a student in real analysis who followed this learning progression to learn the concept of proof by induction. Finally, I note that many students who view proof as a process fail to ever view proof as an argument and I discuss the consequences of these students' narrow view of proof.*

INTRODUCTION

Undergraduates can successfully come to understand advanced mathematical concepts in qualitatively different ways (Pinto and Tall, 1998). Some students develop their understanding of a concept by focusing on the definition of the concept and its logical entailments; others use their intuitive understanding of a concept to give meaning to the concept's definition (Pinto and Tall 1998, 2002). Still other students can learn about a mathematical concept by first viewing it as a process and only later viewing the concept as a static mathematical object. (cf. Sfard, 1991; Tall, Thomas, Davis, Gray, and Simpson, 2000).

The purpose of this paper is to describe one way that undergraduates may come to understand the concept of proof. Proof is an intricate mathematical concept with rigorous, intuitive, procedural, and social dimensions (e.g. Weber, 2002a). As such, Simpson (1995) suggests that there may be more than one way that a student can develop their understanding of this concept. Simpson describes how some students can learn proof logically as rules for manipulation, while others may learn proof as reasoning, with an emphasis on argumentation. In this paper, I suggest another learning progression that students may traverse to understand the concept of proof. Students may develop their understanding of proof by viewing common proof techniques as mechanical procedures. Only later will successful students reason about why they are applying these proof techniques and extract standards for what constitutes a valid proof.

RESEARCH CONTEXT

The data from this study was collected within the context of a first course in real analysis taught at a regional comprehensive university in the southern United States. This course was taught in a traditional manner by a professor unaffiliated with this study. Six students in the course volunteered to participate in this study. Of these six students, three were taking a proof-oriented mathematics course for the first time. The other three had completed a course entitled "Mathematical Reasoning", a bridge course introducing students to the concept of proof. However, at the beginning of the real analysis course, these three students all claimed to have little understanding of the concept of proof, and their performance on their first assignments and their early interviews with the author of this paper support these students' self-assessments. Hence all six students in the real analysis course were still developing their conception of proof.

Each students participating in this study met with the author of this paper every other week. There were eight such meetings during the fifteen-week course. During these meetings, students discussed a wide variety of topics with the interviewer, including their attitudes about proof and advanced mathematics. Students also answered questions designed to probe their formal and intuitive understandings of the concepts and proof techniques in the course. Finally, students were asked to construct basic proofs. All interviews were recorded and transcribed.

A PROCEDURAL APPROACH TOWARD UNDERSTANDING PROOF

In this section, I describe how some undergraduates developed their understanding of proof by first understanding proof as a procedure. Before doing so, let me state that the learning progression I describe below was not the only way that the undergraduates in this study learned the concept of proof. Some undergraduates in this study primarily viewed proofs as the formalization of their intuition (cf. Weber, 2002b). These students used their conception of mathematical proof to produce arguments and used the feedback that they received on assignments and examinations to refine their conception of a mathematical proof. At times, other students viewed constructing proofs as finding strategies for manipulating logical sentences (cf. Weber, 2001). These students' performances will be the subject of future reports.

In this paper, I will concentrate on students who learned about the concept of proof via a procedural route. Such students, when successful, first understood proof as an *algorithm*, then as a *process*, and finally as an *argument*. At this stage, it should be noted that there is much in common between this learning trajectory and the process-object theories of concept acquisition (e.g. Tall et. al., 2000; Dubinsky and McDonald, 2001). I will describe each stage of my hypothesized learning trajectory in more detail below. In the following sub-section, I will present a case study of one students' successful learning of proof by induction within the context of this trajectory.

Proof as algorithm- Many of the undergraduates in this study first learned a proof technique (e.g. proof by induction or proving that a sequence converges to a specific value) as a step-by-step prescription for proving certain types of statements. Each of the steps in this algorithm was relatively mechanical and highly tied to a specific type of problem. At this stage, the undergraduates were generally unaware of the overall nature of the procedure that they were incorporating. As a result, undergraduates with an algorithmic understanding of a proof technique often could only apply this technique toward a very specific type of problem. For instance, after many undergraduates first learned how to prove that linear functions were continuous at a particular point, they could not prove similar statements for other types of functions (e.g. quadratic and rational functions). In short, the undergraduates were able to mimic a technique that the professor had demonstrated for them, but they did not really understand what they were doing, could not generalize the technique, and could not justify why this technique established the veracity of a statement.

Proof as process- After the undergraduates applied a proof technique multiple times and in different contexts, some were able to *interiorize* the algorithm into a mathematical process. The algorithm previously viewed as a set of mechanical actions was condensed into a shorter list of global, qualitative steps. These global steps were not highly specified manipulations on a specific type of statement (divide by the leading coefficient of this polynomial), but rather involved accomplishing a general goal (solve an arbitrary inequality, find a delta that satisfies a particular condition). For this reason, the undergraduates who viewed a proof technique as a process were able to divorce the technique from the specific statements it was originally used to prove. This allowed the undergraduates to generalize their proof technique to a larger number of cases, including cases that they may not have seen before. It appeared that seeing the same proof technique applied within different contexts helped undergraduates view a proof technique as a process, as they seemed to extract common global steps from the ostensibly different proof procedures.

Students with a process understanding of a proof method often can flexibly apply this method over a wide range of cases, yet their epistemological understanding of the concept of proof may still be naïve. Many of the undergraduates in this study could apply a proof technique in a large number of cases, but they did not view proving as convincing or explaining, nor did they view a proof as a convincing explanation. Rather such students viewed the act of proving as a process that one applies to receive credit on examinations, not unlike computing a derivative in a first calculus course (cf. Moore, 1994). To these undergraduates, the proof itself is the successful completion of that process. The proofs these undergraduates submitted were not so much arguments, but rather chronological accounts of their work (cf. Dreyfus, 1999). Potential consequences of this narrow view of proof are described in the concluding section.

Proof as argument- By reflecting upon the proof procedure that they were applying, successful undergraduates began to think about why they were asked to apply that procedure and what that procedure was designed to accomplish. In this way, some undergraduates were able to extract standards for what constituted a valid proof from the procedure itself. Just as encapsulating processes into mathematical objects is inherently difficult (e.g. Tall et. al., 2000; Dubinsky and McDonald, 2001), it appears that viewing proof as an argument rather than a process was a very challenging transition for the undergraduates to make. Most undergraduates did not even attempt to make sense of the procedures that they were applying.

A CASE STUDY OF A STUDENT LEARNING PROOF BY INDUCTION

I illustrate the procedural route toward understanding proofs by describing David's progression toward learning proof by induction. The real analysis course was the first proof-oriented course that David enrolled in. Throughout the course, David expressed a strong desire to make sense of his mathematical work and he generally viewed proving as formalizing his intuition. He tried to understand how the concepts of the course fit together, frequently described and reasoned about concepts with descriptive diagrams, and borrowed analysis texts from the library so the he could obtain different viewpoints on the course topics. In this way, he was not unlike the gifted and successful student described in Pinto and Tall (2002). However, the set theoretic proofs that David was introduced to in the beginning of the semester were so alien to him that he resorted to learning how to write such proofs by rote.

David first learned how to use proof by induction as a linear sequence of steps that he could perform to verify identities involving summations. Five weeks into the course, I asked David to use induction to verify an inequality. When this occurred, he was unable to make much progress beyond establishing the basis case. After I gave David several suggestive hints, he produced what appeared to be a thoughtful proof by induction. The following excerpt reveals that he did not understand what he had just accomplished.

> I: Good.
>
> David: OK, so what do we do now?
>
> I: Well, we uh… do we need to do anything now?
>
> David: Are we done?
>
> I: You tell me. Have we shown the basis case?
>
> David: Yes.
>
> I: Did you show that P(k) implies P(k+1)?
>
> David: Um… I think so.
>
> I: Do we need to do anything more in a proof by induction?
>
> David: I guess not… [pause]…
>
> I: Why did you think that we weren't done before? Do you know why what you wrote down is a correct proof?
>
> David: Well because this proof doesn't look like the other proofs we did [by induction]. The other ones had a lot of equations and this one has sort of just one string. I mean I know it's correct. I see why it is proved for the P(k+2) and P(k+3) case. This one just looked different.

David clearly did not understand the essence of proof by induction. Further, his scheme for validating proofs was at least partially dependent upon how superficially similar his new proof was to previously observed or constructed proofs. On his mid-term examination, David was asked to verify an equality involving products (i.e. Prove $(1 + 1/1)(1+1/2)\ldots(1+1/n) = n+1$). Again, he was unable to make meaningful progress despite claiming to spend 20 minutes on the question. In our interview after the examination, he indicated that he found the question to be unfair, because he'd "done induction using summation, but not induction using multiplication."

The following excerpt taken from our sixth interview twelve weeks into the course illustrates that David had acquired a more flexible understanding of mathematical induction.

> I: Could you describe how you would use induction to prove that $2^{n-1} \leq n!$?
>
> David: Well the basis case just gives …. 1 is equal to 1. To solve the inductive step, I would have to see how (n+1)! related to n! and how 2^{n+1} related to 2^n. I think that I would approach it

in some way of handling the factorial. If I can expand (n+1)! In some way, I can see how it relates to n. If I can see how they are related, I can use my inductive hypothesis.

Although David had trouble articulating his ideas, it appears that he had acquired powerful general strategies for approaching proof by induction. David's use of these global strategies allows him to approach induction problems involving factorials and exponents, even though he had not encountered such problems before. However, as the following excerpt taken from the same interview reveals, David still does not understand why proof by induction is a legitimate proof technique:

David: And I prove something and I look at it, and I thought, well, you know, it's been proved, but I still don't know that I even agree with it [laughs]. I'm not convinced by my own proof!

By the seventh interview, David had acquired a strong understanding of proof by induction, as the following excerpt demonstrates:

I: How would you go about proving this by induction [presents the statement: 2^n □ n! for $n \geq 4$]? Note that this isn't like regular induction. I'm not asking you to prove it for all natural numbers, just n is bigger than or equal to four.

David: Wouldn't I, wouldn't I just prove it for four and then prove the inductive step?

I: Yes. Yes you would. Could you explain to me why your proof would work?

David: Well because of the domain, I guess, for lack of a better word. You know we start at 4, it works on that first element, then it's going to… if it works there, then it works for the k+1 th element, then it's going to work for five, then six, then seven, and all the bigger elements than that.

In our final interview, I asked David to reflect on how he came to understand induction as a proof technique. By his account, for most of the semester, he understood induction as mimicking a procedure, and he was unsure why the technique showed something was true for all natural numbers. When I asked how he achieved the greater understanding that he had at the end of the course, he responded:

"I wouldn't say it was any one thing, I guess, practice with it, thinking about what I'm doing, you know talking about it with you and [the professor], just being exposed more to it… I don't know. I think it is a combination of things".

In summary, through the first half of the semester, David understood proof by induction as an algorithm that he could apply to prove identities involving summations. He was not able to apply proof by induction to other statements, such as inequalities or identities involving products; indeed, he thought such questions were unfair. Later, David understood proof by induction as a process. He was able to skillfully approach problems that he had not seen before, but he still did not see why an inductive proof was a convincing mathematical argument. Toward the end of the semester, through an ill-defined combination of exposure, discussion, and reflection, David was able to extract meaning from the process that he was applying to understand why proof by induction was a valid proof technique.

DISCUSSION AND CONCLUSIONS

The undergraduates in this study often did not view the act of proving as establishing the mathematical certainty of a statement, but rather as a process that one executes to receive credit on assignments and examinations. In this section, I describe how this narrow view of proof caused these students much difficulty and confusion as the course progressed. When validating their own proofs, these undergraduates would often compare how similar their proof was in form to previous arguments that they had observed or constructed. When validating others' proofs, their judgments were influenced by how similar the observed proof was to what they would produce (cf. Selden and Selden, in press). These students also tended to view the rigor and precision that are required of a formal proof to be purely academic, or in accordance with a mathematical ritual (cf. Harel and Sowder, 1998). Throughout the course, there were proofs in which the order of the formal presentation differed significantly from the process of creating the proof. "Delta-epsilon" proofs and the verification of identities via induction are two common examples. The undergraduates who viewed proof as process could not comprehend why one should present a formal argument in a different order than by which it was produced; they were genuinely baffled by what they perceived to be a bizarre mathematical convention.

As Simpson (1995) suggests, there may be multiple ways that one can learn the concept of proof. In this paper, I have described one learning trajectory that students in a real analysis course followed to make sense of proof techniques. When a student first learns to construct proofs by following a set of mechanical steps, it is tempting to minimize that student's efforts as "rote learning". Hence it is important to note that students like David can use a procedural understanding of a proof technique as a basis for developing a sophisticated understanding of the concept of proof. However, it is equally important to note that most students who attempted to learn proof procedurally were unable to view a proof as an argument by the end of the course. In some respects, undergraduates who were only able to understand proof as a process performed adequately in real analysis; they were able to construct a wide variety of proofs and some went on to earn a high grade in the course. However, these undergraduates' limited view of proof also caused much confusion about how to formally present a mathematical argument, left them unable to effectively validate their own work, and lead them to acquire misleading beliefs about advanced mathematics.

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TEACHING ANGLES BY ABSTRACTION FROM PHYSICAL ACTIVITIES WITH CONCRETE MATERIALS

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An angles teaching sequence was designed, in which students were guided to abstract a general concept from physical activities with concrete materials. The three design principles used were familiarity, similarity recognition, and reification. The resulting teaching sequence was tested in a field study involving 25 teachers of Grades 3-4. The data collected demonstrate the significance of the three design principles, and the results from pre and post assessment interviews provide evidence for the overall effectiveness of the lesson sequence. Also earlier findings that angle is a multifaceted concept which is difficult to learn are supported.

School students have great difficulty learning the angle concept (Clements & Battista, 1992). Douek (1998), for example, described how difficult it was for students in Grades 3 and 4 to interpret the inclination of the sun in terms of angles. The problem seems to be that angle is such a multifaceted concept. Close (1982) and Krainer (1989) have discussed the variety of angle definitions used in mathematics, and many authors have noted the difference between dynamic (movement) and static (configurational) aspects of the concept (Close, 1982; Kieran, 1986).

Mitchelmore and White (2000a) proposed a theory of learning angle concepts by successive abstraction and generalization. According to this theory, an abstract concept is "the end-product of ... an activity by which we become aware of similarities ... among our experiences" (Skemp, 1986, p. 21) and generalisation is the process which extends the meaning of such a concept to include further experiences. Mitchelmore and White proposed that young children initially recognize superficial similarities between physical situations and abstract everyday concepts such as corner, junction, door, and roof. Gradually, they recognize deeper similarities between these objects and form restricted angle concepts such as corner, turn, and slope. A general abstract angle concept emerges as children recognize the even deeper similarity between these restricted angle concepts. At this point, students can interpret a wide variety of situations in terms of angles because they recognize them as consisting of two linear parts that meet at a point and realize that the relative inclination of these two parts has a crucial significance.

Mitchelmore and White (2000a) showed empirically that the major difficulty in learning to identify a physical angle situation lies in identifying the two linear parts of the angle. They found that children can identify so-called 2-line angles (e.g., corners of a room, road intersections, pairs of scissors), where both arms of the angle are visible, as early as Grade 2. On the other hand, 1-line angles (e.g., doors, windscreen wipers, ramps) are more difficult to understand, and even by Grade 6 many students cannot interpret 0-line angles (e.g., doorknobs, pirouettes, wheels) in terms of angles. For 1- and 0-line angles, one or both linear parts have to be imagined or remembered.

Mitchelmore and White (2000b), incorporating the ideas of Sfard (1991), advanced a method of teaching angles called Teaching for Abstraction, based on the following three principles:

> Familiarity. Students should first become familiar with a variety of angle situations.
>
> Similarity. Teaching should then focus on helping students recognize the similarities between these situations.
>
> Reification. Activities should be undertaken whereby the recognized similarity becomes abstracted to an angle concept that can be operated on in its own right.

Following exploratory research on this model using one-to-one and small-group teaching (White & Mitchelmore, 2001), a sequence of lessons was designed and tested in the field during 2001 and 2002. This paper reports and analyses the results of this study.

METHOD

The authors wrote a sequence of 15 lessons which initially explored 2-line angles (corners, scissors, and body joints) and then moved on to 1-line angles (doors, clock hands, and slopes). The lessons followed the principles of Teaching for Abstraction as follows.

> Familiarity. Students explored the angle situations separately to learn about their crucial, angle-related features.
>
> Similarity. Lessons involved frequent direct matching, indirect matching, and other forms of attention to angular similarities:
>
>> In direct matching, the angle in one situation was physically superimposed upon the angle in the other situation. For example, the corner of a pattern block was fitted into the corner of a window frame.
>>
>> In indirect matching, an intermediate angle-like object was used to indicate how the angles in the two situations were similar to each other. For example, a bent straw was used to show that two angles were “the same”.
>>
>> In selective attention, the arms, vertex, and opening of an angle were identified.
>
> Reification. Several activities were aimed towards abstracting the angle concept from the concrete situations:
>
>> Students made abstract drawings of angles with approximately the correct size and orientation.
>>
>> Acute, obtuse, and right angles were defined.
>>
>> Students described the angle concept in their own words.

The angles teaching sequence was tested in a total of 25 Grades 3 and 4 classrooms. All the teachers (20 female and 5 male) attended two one-day workshops, one before and one after teaching about 8 lessons selected from the sequence. Grade 3 teachers taught mainly lessons on 2-line angles, whereas Grade 4 teachers taught both 2-line and 1-line angles. Each teacher identified a target group of 8 students in her or his class and administered them a pre and post assessment interview designed by the researchers.

Data were collected from four sources: (1) teachers’ written comments and comments made in focus group discussions at the second workshop, (2) the researchers’ notes taken during the workshop discussions, (3) work samples from 130 students, and (4) pre and post assessments from about 200 students.

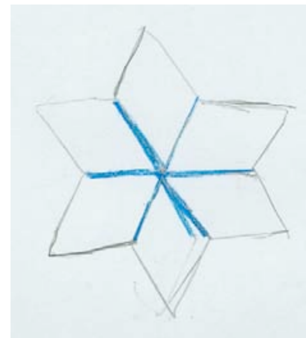
RESULTS AND DISCUSSION

In this section, we present examples illustrating the significance of familiarity, similarity, and reification in the learning of angles. We also present evidence that students learnt to recognize angular similarities and did, to some extent, reify the concept of angle.

Familiarity

There were several instances where the importance of students' familiarity with physical angle situations became clear.

Students were allowed a period of free play with pattern blocks prior to exploring how their angles fitted together (see student's drawing on the right). The pilot testing had shown that, without this familiarization process, students were distracted by making creative patterns and concentrated on the whole shapes instead of the angles at their corners.



A lesson in the sequence dealt with angles made by opening a pair of scissors. Students were clearly unfamiliar with the function of the pivot, but quickly grasped its significance.

Without this first step, it would not have been possible to discuss angles of opening at all. Similar, essential familiarization processes took place when students learnt in other lessons about how the bones of the body are jointed and how doors are hinged.

A negative example occurred in a lesson devoted to angles of slope. Although students easily recognised the angle of slope when a ruler was placed on a table, they had difficulty when this prop was removed and many drew arbitrary second lines (see student's drawing at right). Many students were clearly not sufficiently familiar with the idea of the horizontal, and drew arbitrary second lines.

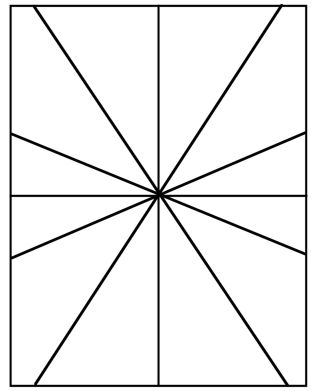
This lesson should have been preceded by lessons devoted to the concepts of vertical and horizontal.

A more positive example occurred when measuring angles. The approach taken by the researchers was to use the 30o corner of one of the pattern blocks as a unit angle. We felt that the introduction of degrees at this point would be unnecessarily complicated, so suggested that teachers invent a name for this unit such as a "Kevin." However, several teachers argued that, since students were familiar with the idea that a right angle is 90o they could easily work out that each unit was 30o. This proved to be correct.

Similarity

Students focused on angular similarities in different situations by matching the angles or identifying the arms and vertex in each situation (selective attention).

Matching. There were several situations where the use of matching assisted development of an abstract angle concept. In one such example, students were challenged to find whether the angles at the centre of a "windmill" (see figure on right) were equal. Most students initially thought that the angles at the top and bottom were larger than those at the sides. When they directly matched the 30o corner of a pattern block with each of the angles at the centre of the windmill, they were surprised to find that all the angles were equal. This exercise focused students' attention on angle as an amount of opening and demonstrated dramatically that the length of the arms is irrelevant.



Another instance occurred with the opening of a door. Students were introduced to this angle situation through examples such as their classroom door, which provided a visible line for the closed position of the door. Students then looked at a "floating door" (one made from a piece of card where the line of the closed position was not visible). A bent straw was used to compare the opening in the two types of door and to help students identify the invisible line of the angle in the second case.

In the assessment interviews, students were given three pairs of angle situations and asked to check that the angles were equal— which they did either by direct matching (placing one angle on the other) or indirect matching (placing an intermediary object on both angles). Figure 1 shows the percentage of the sample that could correctly match the pairs of angles before and after the lesson sequence.

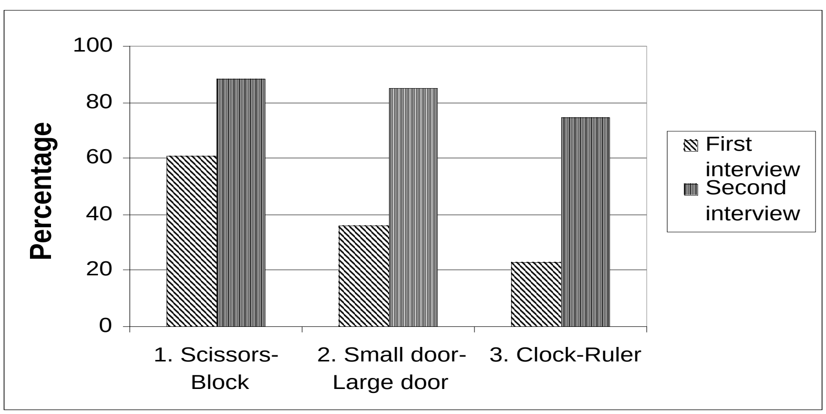
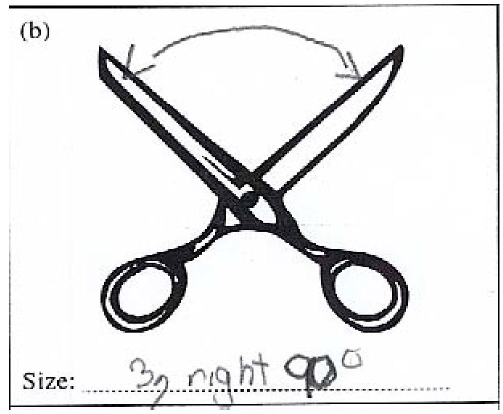


Figure 3: Percentage of sample correctly matching angles.

As expected, the first task, which dealt with 2-line angles, was rather easier than the other two, which dealt with 1-line angles. But the data clearly show that, in all cases, students' ability to match the angles improved substantially over the teaching period.

Selective attention. Many students initially had difficulty identifying the parts of the angle (even in 2-line situations). For example, in the example shown on the right, the student clearly had some idea about the angle of opening, but could not identify the arms or the vertex. This common problem was addressed in the teaching sequence by having students draw lines from the pivot along the blades of the scissors. A similar difficulty occurred with angles made by limbs of the body. Some teachers marked the joint with a dot and the arms with felt pen to make the angles clearer.



Others built models out of pipe cleaners to help students identify the angles better.

In the assessment interviews, students were asked to identify the arms and vertex of various angles. Figure 2 shows the results. Again, students recognized 2-line angles more easily than 1-line angles but in all cases recognition increased dramatically as a result of the teaching. Another task not reported here showed that it was substantially easier for students to identify the arms of an angle than its vertex.

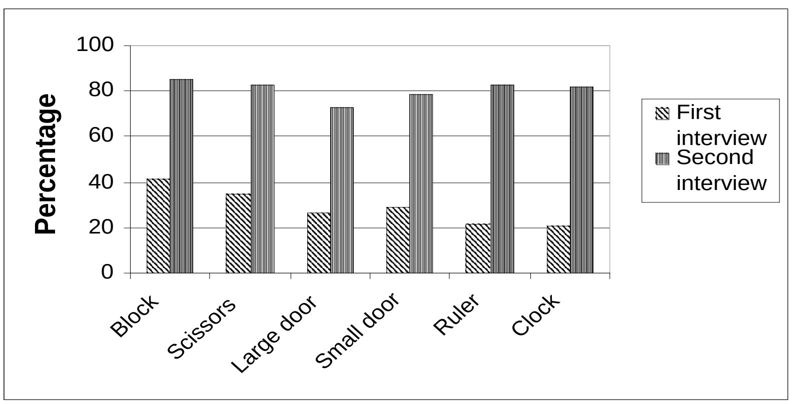


Figure 2: Percentage of sample correctly identifying arms and vertex of each angle.

Reification

In one interview task, students were asked to match an angle in a physical situation with a given abstract angle size. Sample tasks were to move one hand of a clock through a right angle from 2 o'clock and through half a right angle from 7 o'clock, to open a door (represented by a straw on a diagram of a door opening) through an obtuse angle, and to draw an acute angle.. Correct identification increased from 45% to 81% for right angles, from 29% to 82% for obtuse angles, and from 42% to 75% for half a right angle. The large percentage knowing about right angles is not surprising and the large percentage able to draw an acute angle could have been artificial because most angles encountered were acute and angle diagrams are usually drawn acute. However, the dramatic increase in half right and obtuse (in particular) indicates that after the teaching, most students could relate abstract angle sizes with physical counterparts and the technical terms right and obtuse had for most students become quite general.

The last question of the assessment interviews asked students to define an angle. As expected, there was a wide range of responses and it was often difficult to decide exactly what a student was trying to communicate. Eventually, the following categories were determined:

> Two arms, vertex, and opening: responses indicating that an angle consisted of two lines meeting at a point and making some attempt to express an angular relation between the lines (using words such as opening, turn, space, area, gap, distance, size and measurement. Typical responses were:
>
> > Two lines that meet at a point. Size between the two arms near the point, not the length of the arms.
> >
> > Two straight lines that come together at one point vertex. The angle is the opening between the two arms
> >
> > It measures the turn from one line to another.
> >
> > An angle is like two sides with apex at the top [uses hands to show] at different degreasers [sic].
>
> Two arms and a vertex: responses indicating that an angle consisted simply of two lines meeting at a point. Typical responses were:
>
> > It's a thing that has a vortex [sic] and two arms.
> >
> > A pivot which has two arms coming out of it
> >
> > Two arms: responses that merely mention the presence of two lines, but suggest a general situation. Typical responses were:
> >
> > Two diagonal lines like a mountain
> >
> > Two bits of thing which come together
>
> Vague responses named a particular type of angle, referred to degrees or to a part of a specific object, or were vague to the point of being incomprehensible. "Don't know" responses were also included in this category. Typical responses were:
>
> > Something that is 90o, or more or less than 90o
> >
> > An angle is the amount of degrees between two points.
> >
> > It's like a triangle. It's used to measure the space.
> >
> > A thing that bends in a certain way

Figure 3 shows the results for each of the four categories in this item before and after the teaching.

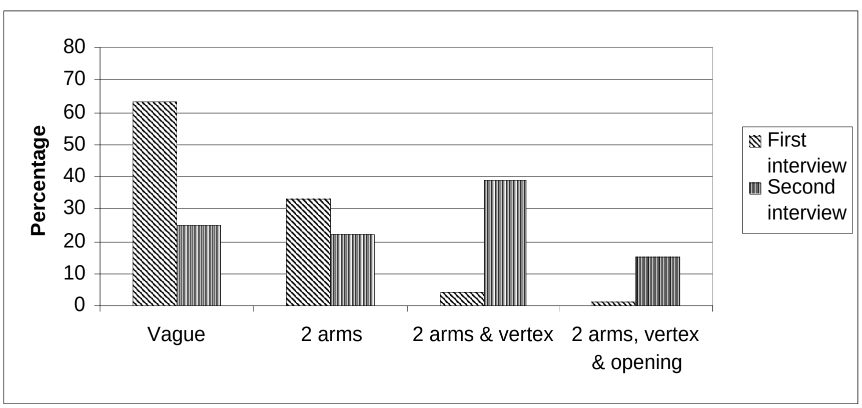


Figure 3: Percentage of sample giving various definitions of an angle.

As a result of the teaching the percentage of responses indicating awareness of at least the two arms and vertex of an angle increased from 5% to 54%, the percentage of vague responses falling off dramatically. The least salient feature of the angle concept was clearly the opening aspect.

CONCLUSIONS

The results of this field study confirm the significance of familiarity, similarity, and reification in learning the angle concept. The importance of familiarity in establishing essential pre-requisite knowledge or necessary understanding of angle situations has been shown to be fundamental to the learning process. The data also show that similarity recognition can be taught and that the three types of matching (direct, indirect, and selective attention) are powerful tools in building an abstract angle concept. There is ample evidence that the emphasis on similarity recognition and reification in the teaching model resulted in many students developing a quite sophisticated concept of angle as early as age 10.

However, our data also emphasize earlier findings that angle is a multifaceted concept which is difficult to learn. The opening aspect of an angle and the significance of the vertex are particularly difficult and would seem to need more emphasis in the teaching. Many of the students in this study were still some way from reifying angle as "an amount of turning between two lines joined at a point."

Finally, the fact that the Teaching for Abstraction model was so successful in designing an angle teaching sequence suggests that the method could be applied to other mathematical concepts. The processes of familiarity, similarity, and reification are clearly worth further study.

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UNDERGRADUATE STUDENTS' MENTAL OPERATIONS IN SYSTEMS OF DIFFERENTIAL EQUATIONS

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Purdue University Calumet

This paper reports on research conducted to understand undergraduate students' ways of reasoning about systems of differential equations (SDEs). As part of a semester long classroom teaching experiment in a first course in differential equations, we conducted task-based interviews with six students after their study of first order differential equations and prior to instruction on SDEs in order to obtain baseline data on their conceptual resources for SDEs. Interpretative analysis of the interview data generated the following three themes pertaining to student reasoning in situations dealing with SDEs. First, students used their conception of rate as a reasoning tool. Second, students used quantification as a mental operation, and third, students enacted what we call a function-variable scheme in their efforts.

INTRODUCTION

Undergraduate mathematics education research is a growing area of interest and current areas of concern include, but are not limited to, calculus, proof, abstract algebra, number theory and differential equations. Research in differential equations has primarily focused on student understandings of and difficulties with single differential equations (e.g., Rasmussen, 2000; Artigue, 1992; Habre, 2000; Zandieh & McDonald, 1999) while much less research has focused on students' understandings of SDEs (for some brief reports in this area see Rasmussen, 2000 & Trigueros, 2000). The purpose of this paper is to begin to fill this gap by offering a theoretical account of students' mental resources and ways of reasoning when thinking about and solving SDEs.

THEORETICAL BACKGROUND

Since differential equations are expressions of rate, we build upon previous work regarding student conceptions of rate in our analysis of student thinking about SDEs. Piaget (1970) documented that young children's concept of speed exists before their concept of time. According to Piaget, children initially understand speed in a relational sense. That is, if one person "passes" another then the person passing has a greater speed. Children first develop a concept of speed as a "quantified motion" and later conceptualize speed as the coordination of distance and time in proportion to each other. Thompson (1994a) extends Piaget's seminal research to older children's understanding of rate and ratio, finding that students initially understand rate (speed) as a single quantity that is a value attached to motion and not as a ratio of distance and time. Developing a conceptual understanding of rate involves integrating distance and time together as a ratio. Complementarily, Confrey and Smith (1994) suggested that rate is a unit per unit comparison and focus on the idea of "per" and the mental construction of units for both distance and time.

Thompson & Thompson (1994) presented levels that they suggest are needed to construct a sophisticated understanding of speed as rate. To summarize this framework, students initially understand speed as a unit of motion. For example, going 25 mph means going

25 miles in one hour (a unit they call speed length). For students, this conception evolves into an understanding of the relationship of distance, speed, and time. This relationship can be briefly explained in the following way: as time changes, the distance changes proportionally, the ratio $\Delta d/\Delta t$ remains constant, and that ratio is the speed (rate).

More recently, Carlson, Jacobs, Coe, Larsen, and Hsu (2002) suggested that we might understand students' progression of rate and covariational reasoning in terms of progressively more complex mental actions. These researchers suggested five levels of mental actions as students mature in their understanding of covariational reasoning. To give an example of this framework, the first level is identified when a student understands there is a relationship between two variables. Level 5 is identified when students understand instantaneous rate in terms of instantaneous values for a function.

In their work, Rasmussen & Whitehead (2002) posit that an interconnected but somewhat hierarchical set of stages may also frame students' understanding and use of rate in differential equations. These stages depict students' reasoning as it becomes increasingly more sophisticated. The stages range from using rate as a ratio of two discrete values to dynamically using rate as a function to infer changes to the structure of the space of solutions for single differential equations. All three of the above frameworks offer ways to think about students' understanding that we can build on in SDEs.

Adapting Piaget's notions of image, mental operations, and schema to rate, Thompson (1994a) aimed at "capturing the multiple reconstitutions that take place in individuals as they progress toward the construction of mathematical objects such as ratio and rate" (p. 180). In this report we aim to capture and describe analogous progress students make as they construct increasingly sophisticated understandings in SDEs and in our analysis we make use of these same constructs. According to Thompson (1994b), an image is "the kind of knowledge that enables one to walk into a room full of old friends and expect to know how events will unfold" (p. 125). It is more than a mental picture; it is a dynamic construct originating in a student's experience, both mathematically and otherwise. A mental operation, on the other hand, "is a system of coordinated actions that can be implemented symbolically, independent of images in which the operation's actions originated" (Thompson, 1994a, p. 182). For example, when a student puts two values in order, she uses a mental operation. A scheme is an organization of actions that is repeatable and generalizable (Piaget, 1970). Finally, images develop through students' intentions to meet goals at any given time with the mental resources, operations, and schema they enact in that situation, but mental operations and schema exist within images as well. The images of rate and solution that students develop when learning SDEs can be framed using these constructs.

METHODOLOGY

We conducted videotaped, task-based, semi-structured individual interviews with six students as part of a semester long classroom teaching experiment (Cobb, 2000) in differential equations for engineers at a midsize university in the United States. The theoretical purposes of the teaching experiment were to characterize both individual and collective reasoning about differential equations (from a dynamical systems point of view) and to contribute to the theoretical body of knowledge about undergraduate students' mathematical thinking as it evolves in the complexity of classroom life.

Instruction sought to develop a participation structure (Erickson, 1986) where students routinely explained their thinking, provided reasons for their conclusions and interpretations, and attempted to make sense of other students' ideas and approaches (Yackel, Rasmussen, & King, 2000). As a result, students approached the interview sessions with an appreciation that the research team was genuinely interested in understanding their thinking and reasoning.

In both the day to day and retrospective analysis of classroom events, we used the interpretive framework developed by Cobb and Yackel (1996) that strives to coordinate psychological and sociological perspectives. In the analysis reported here, we foreground a psychological perspective while a sociological point of view remains in the background. Analysis was informed by a grounded theory approach as described by Glaser and Strauss (1967) and proceeded in the following manner: Transcriptions of the interviews were summarized per student per task. Analyses of these summaries were discussed with both authors in order to develop a shared sense of interpretation and to minimize inappropriate interpretations. We often returned to the original data as we compared interpretations and sought confirming and/or disconfirming evidence for conjectures. Systematic review of these summaries then led to the identification of three themes that cut across most all of the problems dealing with SDEs.

RESULTS

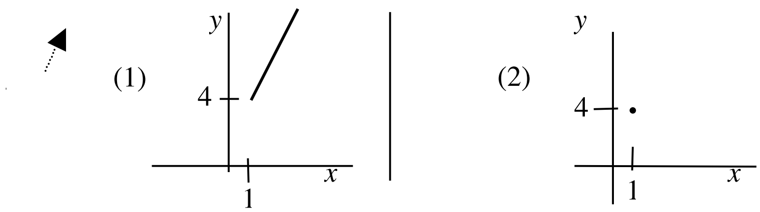
The three interview tasks dealing with SDEs were designed to afford us insight into ways of reasoning that were available to students before instruction on SDEs. Students had not solved any problems like these in the past, but they had completed approximately seven weeks of instruction dealing with single differential equations. We next review the three tasks, provide telegraphic results of student responses, and then discuss our analysis and interpretation in terms of three cross cutting themes.

TASK 1: In this task, we look at systems of rate of change equations designed to inform us about the future populations for two species that are either competitive (that is, both species are harmed by interaction), or cooperative (that is, both species benefit from interaction, for example bees and flowers). Which system of rate of change equations describes competing species and which system describes cooperative species? Explain your reasoning.

$$\text{(A)}\quad \begin{aligned} dx/dt &= -5x + 2xy \\ dy/dt &= -4y + 3xy \end{aligned} \qquad \text{(B)}\quad \begin{aligned} dx/dt &= 3x - 2xy \\ dy/dt &= y - 4xy \end{aligned}$$

TELEGRAPHIC RESULTS: All students were able to correctly predict that system A was cooperating and B competitive. A typical strategy used the fact that the second term in each DE in (A) would cause the rate of change of one variable to increase when the other variable increased and the second term in each DE in (B) would cause the rate of change of one variable to decrease when the other variable increased. Another strategy used involved considering what would happen if x or y was 0. Students also used their understanding of the situation to reason about the rate quantities, using specific values to reason about the behavior of the size of the populations. For two students, however, we could not tell if they were thinking about a rate of change quantity or about an x or y quantity.

TASK 2: Imagine the following: You are up in a hot air balloon looking down at a skateboarder. He has a piece of chalk on the bottom of his board that is drawing a line on the ground as he moves. Here is what you see on two different rides.

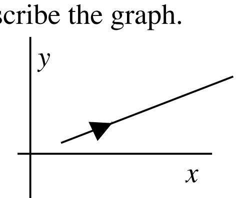


Sketch possible x(t) and y(t) graphs for each ride.

Follow-up A: Students were given concave up graphs for x-t and y-t and asked to trace the line in ride (1) that would produce the given x-t and y-t graphs.
Follow-up B: Repeat for a concave up graph for x-t and a concave down graph for y-t.

TELEGRAPHIC RESULTS: All students in the interviews sketched appropriate x-t and y-t graphs for both of the skateboard rides. For (1), they sketched linear graphs; their approach involved imagining the skateboarder's trace as it developed over time and then sketching the horizontal and vertical displacement of the line on the x-t and y-t axes. Two students also suggested concave up and increasing x-t and y-t graphs if the skateboarder speed was not constant. For (2), all students sketched horizontal lines for the x-t and y-t graphs. The typical reasoning students used to justify their graphs was that time "goes on" and the skateboard does not move, so a horizontal lines are the best representation. In Follow-up A, four students discussed the increasing speed of the skateboarder, using the changing slope as justification. Two also reasoned that two concave down graphs would also be acceptable and explained that the skateboarder would be slowing down. In Follow-up B, four students claimed it would not be possible to produce such graphs while two said that it would be possible.

TASK 3: The following is a sketch of x vs. y that is produced over time. Look at each of the following suggested systems of differential equations and decide if any of the choices could describe the graph.



(1) $\frac{dx}{dt} = x + y$, $\frac{dy}{dt} = -x + y$ (2) $\frac{dx}{dt} = x + y$, $\frac{dy}{dt} = x + y$ (3) $\frac{dx}{dt} = 2x - 2y$, $\frac{dy}{dt} = x - y$

TELEGRAPHIC RESULTS: Four of the students correctly identified (3) as a possibility by imagining a parameterization of x and y as functions of t and using their image of the graph to predict what the rate of change equations could be. A typical strategy involved the fact that the line has a slope less than one and so x would be greater than y, and since the line has positive slope, both x and y would be increasing. They coordinated x and y as dynamic quantities (as variables and functions). One student showed surprise when he realized that the one x-y graph could represent both rate of change equations and then went on to correctly solve the problem. The other two students made no progress.

DISCUSSION AND ANALYSIS

We developed the following three themes to characterize students' mathematical understandings of systems of differential equations: rate use, quantification as a mental operation, and function-variable scheme. These themes are intended to be complementary rather than disjoint. For example, we often saw students using the mental operation of quantification to construct a rate quantity, which they then used to reason about the values of the variables/functions x and y. This overlapping or simultaneity of themes occurred frequently, but for discussion purposes we illustrate and clarify each theme separately.

Rate Use. Other researchers have offered ways to describe how students' understanding of rate develops. We build on and extend this work to detail students' use of rate as a tool for building other mathematical images such as population predictions (Task 1) and function (Tasks 1, 2 & 3). Several of the interviewed students used their understanding of rate to reason and reach conclusions. For example, they typically used the idea of positive and negative rates as being increasing and decreasing, which connects with Carlson et al.'s (2002) framework for mental action (level 4/5). They also used rate as a ratio of displacement and time (Thompson & Thompson, 1994), discussing changes in x and y compared to a change in t by coordinating the values discretely (Task 2). Finally, they used rate to make predictions about how functions change over time in Task 1,2, and 3.

As an illustration, consider the following transcript with one of the students, Arthur, who used his understanding of rate to reason that the given graphs in Follow–up B to problem 2 were not possible.

> Arthur: I can't do this. Um. Um. As time elapses, it goes faster in the y direction. And it slows down in the x direction. On this one [indicates problem 2 Follow-up B] I don't know how to move my pen. Because, on that one, [indicates problem 2 Follow-up A], they were both, they both increased. On this one, they both looked about the same. On this one, um, [Pause] I'm lost.
>
> Karen: OK. Could you say what's bothering you about it?
>
> Arthur: Well. This one's increasing and this one's [Refers problem 2 Follow-up B]. This one's this one's increasing, too. But the rate of the rate of this one is increasing. The rate of increase on this one is decreasing. So, I've got a problem ...

Arthur used rate as a tool to reason about the possible x-t and y-t graphs in this problem. In the last statement, he says that the rate is increasing for the x-t graph and decreasing for the y-t graph. This is a conflict for him because he looks at the original skateboarder's graph in the x-y plane and reasons that the rate (slope) of it requires that the x-t and y-t graphs need to have similar rates (note his first statement) for the constant slope in the x-y plane to be maintained. Rate is then a tool for him, and slope, rate, and speed all integrate together to be used in reasoning about the skateboarder's rate in the parameterized curves representing movement in the horizontal and vertical direction.

For us, rate use means that the students have an image of rate that provides them powerful ways to reason in tasks involving motion and/or change, as exemplified in Arthur's excerpt. Situated within a problem setting, students' images involve a conception of rate as a quantity that determines the dynamic behavior of a function, such as if it is increasing and decreasing, and rate as a comparison of quantities as they change dynamically.

Quantification as a mental operation. Thompson (1994a) posits that quantification is an important mental operation and our analysis finds support for this position. He defines a quantitative operation as "a mental operation by which one conceives a new quantity in relation to one or more conceived quantities" (p. 185). An example of a quantitative operation is combining two quantities additively. Quantification begins with a mental action and creates a new quantity (not a specific number, but an image of a value) by operating on original quantities. In each of the tasks, quantification was an important part of students' sense making and reasoning.

For example, on Task 1, x and y represent the populations of two different species. Students mentally operated on the two different population quantities to conceive a new quantity: the rate of change of each of the populations. By mentally acting on the notion of "how big is the population," students conceived of rate as related to quantified values for population. Most of them did this by assigning values to x and y and reasoning with the results. The following is an example from Jere's response on Task 1.

> Jere: This one here would be cooperating because they're both helping to the, the general equation, they're both adding to it. This one, they're both, I guess, pulling away from the, from the function. From the differential equation. And, only one is putting in.
>
> Karen: I'm just trying to understand your, your thinking. When you say both are putting in.
>
> Jere: Yeah. Both species. If you have bees and flowers, then, you know, your bees are helping your flowers.
>
> Karen: Oh. So. When you say add to the equation can you say a bit more about?
>
> Jere: They're increasing. Here, you only one species that is negative. So. If you would add. Oh. For a cooperative species, you'd want a positive rate of change. Right? You'd wantthem to help each other… and a competing species, one of them, one of the, one of these equations over time is going to drop, and one is going to go up.

Jere was mentally operating on the x and y quantities to create new quantities (rates of change of x and y). For example, he said, "they're both adding to it…the differential equation." Rate use then affords him a means to mentally act on these DEs to expand and quantify the two systems in relationship to each other and to the actual situation.

In Tasks 2 and 3, students also used quantification as a mental operation. They first quantified the rate of change in x and y that is present in (A) (in the x-y plane) and coordinated the variation of x, y and t (time). They operated on the graph to conceptualize tri-variation of the three quantities and produced two new quantities, the co-variation relationship of x vs. t and y vs. t. Then they moved to representing these as graphs of x-t and y-t that displayed similar co-variation with respect to time.

The mental operation of quantification in these students' mathematical reasoning is complex. The quantification of rate from the variables x and y is intertwined with the quantification of the functions x(t) and y(t) using rate and both are grounded in the students' understanding of the situations portrayed in the tasks.

Function-variable scheme. In earlier studies of single differential equations, Rasmussen (2000) found that one of the most difficult images students need for schemes of mental operation is that x and y need to simultaneously be both variable and function for a robust conception of differential equations and solutions to differential equations. Students who have developed ways to think about these "things" as both variables and functions then create a mental scheme that puts together images and understandings from both function and variable to create new ways to reason. Students who were able to reason appropriately about the third question showed that they might be at different depths of development for this scheme. In fact, three students showed a particularly strong understanding of this function-variable relationship. We illustrate this third theme with a portion of transcript from Jeff's reasoning on Task 3.

Jeff: The change in y is positive. The change in x is positive. Both increasing. For all t that we see, it's moving slightly faster in the x direction that it is in the y direction. They look almost equal but not quite. [Pause] So. OK. dx/dt is greater than dy/dt. And dx/dt and dy/dt are both greater than zero. They're both positive. That's positive x. That's a positive change in x. This says x is greater than y. Which means this is a negative value. So, I would say that this [system (1)] is invalid... [pause]

Karen: Tell me what you're thinking.

Jeff: Although initially they could be equal, my value of x is greater than my value of y. This is greater than this y. Which means that this is a negative slope for y. Which it is not.

Jeff: This [system (2)] is telling me that change in x is equal to the change in y. Seems close but that's not quite right. That would be a little bit steeper. [About system (3)] So 2x minus 2y is going to be positive because x is greater than y. So this is positive. And x still greater than y. So, this is positive. But this is still greater than this. So, I would have to go with choice three.

We were actually quite surprised at the depth of reasoning that Jeff used before having ever studied SDEs. Jeff was able to reason about this because, in our view, he conceptualized x and y as both function and variable operating simultaneously in this situation. When he needed to see x and y as variables to substitute numerical values for and use to determine something about dx/dt and dy/dt, he did so. When he needed to see them as functions of time that are changing continuously and operating "underneath" the situation, he did that. He seemed to move effortlessly between the two mathematical objects enacting a scheme that was easy for him to use and us to observe. His mental operations coordinated into a robust scheme to reason in this situation. Other students also showed development of this mental scheme, although with different depths of understanding. We posit that this function-variable scheme developed for our students in their study of first order differential equations and served them well in reasoning about mathematical ideas in SDEs.

CONCLUDING REMARKS

This research offers a snapshot of students' reasoning and sense making about SDEs prior to formal instruction. Its significance lies in the fact that it builds on theoretical constructs developed for younger children, laying a foundation for understanding student thinking about SDEs. In addition, although not discussed here, this theoretical analysis is proving to be pragmatically useful in our efforts to refine and revise instructional materials. We are currently analyzing data from the classroom teaching experiment (including classroom videorecordings and end of the semester interviews) as students' progress in their study of SDEs. We anticipate that the theoretical ideas developed here will be useful in this retrospective analysis of student thinking and instructional design and may extend to other areas as well.

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EMPIRICAL GENERALISATION AS AN INADEQUATE COGNITIVE SCAFFOLD TO THEORETICAL GENERALISATION OF A MORE COMPLEX CONCEPT[1]

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The impact of prior learning on new learning is highlighted by the case of Dean, a Year 8 student who developed his own method to find the sum of the interior angles of a polygon without knowing why his method worked. Enriched transcripts and visual displays of the cognitive, social (Dreyfus, Hershkowitz, & Schwarz, 2001) and affective elements (Williams, 2002) of Dean's interrupted abstraction process informed the identification of factors that inhibited Dean's constructing process. It was found Dean possessed an empirical, not theoretical, generalization (Davydov, 1990) about sums of interior angles of triangles that was an inadequate cognitive artifact for constructing the new more complex theoretical generalization. The study suggests use of tasks designed with the opportunity develop assumed knowledge in conjunction with new concepts.

This case forms part of a broader study of factors that promote or inhibit the process of student-initiated and student-directed abstraction of mathematical concepts, without mathematical input from an external source (like the teacher, text-book, or students external to the group) during the abstraction process. Dean's case illustrates an interesting phenomena; his inability to utilize the procedure specified and demonstrated by the teacher appeared to trigger his development of an alternative more complex strategy. Whilst the class as a whole developed the empirical pattern 'you add a hundred and eighty degrees each time', Dean drew upon prior knowledge that he considered to be part of a different topic 'angles in triangles add to a hundred and eighty degrees' to develop his new method 'you add another hundred and eighty degrees for each triangle in the polygon'. Dean was unsure his method was correct (even though he had checked it with specific examples) because he had been unable to 'see' the mathematical essence behind the prior knowledge he utilized; Dean did not know what angles were or where they were positioned in triangles and polygons. Dean had developed an 'empirical generalization' using a less complex empirical generalisation (Davydov, 1990) to scaffold his thinking. Dean did not develop a theoretical generalization (Davydov, 1990) or abstract a new mathematical insight (Dreyfus, et al., 2001). This report examines why a student who had demonstrated the capacity to recognize the relevance of mathematical ideas, that he considered external to the lesson focus, did not also gain mathematical insight into the method he developed.

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LITERATURE REVIEW AND THEORETICAL FRAMEWORK

Student-initiated and student-directed abstraction (discovered complexity) has been found to be associated with high positive affect and optimization of learning conditions (Barnes, 2000; Csikszentmihalyi & Csikszentmihalyi, 1992; Williams, 2002). Discovered complexity occurs when students spontaneously formulate a question related to a newly found mathematical complexity and work with unfamiliar mathematical ideas to explore this further (Williams, 2002). Discovered complexity is a subset of 'abstraction'; an activity of vertical reorganisation of ‘previously constructed mathematical knowledge into a new structure’ (Dreyfus, et al., 2001, p. 377). 'Vertical' refers to a new mathematical structure as opposed to a strengthened connection between a mathematical structure and a context (‘horizontal’). Dreyfus, et al. (2001) identified observable cognitive elements of the process of abstraction: (a) ‘recognising’—seeing a previously known mathematical structure within a new context or realising a previously known mathematical structure fits a new context; (b) ‘building-with’—using a combination of previously generated abstracted entities in a new context; and (c) ‘constructing’—using assembled resources to vertically reorganise a mathematical structure. In the present study, to facilitate examination of the inhibited process of abstraction, 'building-with' is taken to include use of a rule where the mathematical essence behind the rule is not known; conceptual ideas that enable the justification of a rule or pattern are not present.

The six categories of dialectic social interaction (control, elaboration, explanation, query, agreement and attention) (Dreyfus, et al., 2001) can assist in determining whether ideas are student-initiated and/or student-directed. For example, a student-directed interaction that was not student-initiated would be controlled initially by an external source but once the process of abstraction had commenced, all elaboration, explanation, and agreement would emanate from an internal source. The group would decide whether to attend to any query or attention from an external source. To reduce the difficulties associated with identifying cognitive artifacts assembled by students for use during the process of abstraction, post-lesson video-stimulated reconstructive student interviews can be used to 'make visible' additional cognitive activity (Clarke, 2001). Nisbett and Wilson (1977) have shown people can produce accurate reports of their own cognitive activity if salient stimuli (like video-stimulation) are provided.

Williams (2002) used Csikszentmihalyi and Csikszentmihalyi's (1992) concept of flow to develop indicators of a student's affective state. The state of flow exemplifies the enhanced quality of the learning experience that can occur where task involvement is associated with a high level of positive affect. The indicators of task involvement are: eyes on the task; pens on the task page and/or bodies leaning in towards the task; and, participating in the interaction. A more intense task involvement can be inferred where these previous indicators occur in conjunction with the following indicators: lack of awareness of the world around; building on each other's ideas (latching comments); and exclamations of pleasure. More detail about flow, body language and positive affect can be found in Williams (2002).

In this study, the cognitive and social elements of the process of abstraction are examined in conjunction with the affective indicators 'visible' on the video and 'audible' in the interview to answer the question: Why was the process of abstraction inhibited for Dean?

RESEARCH DESIGN

The Year 8 lesson studied was the 12th lesson in a sequence of 16 lessons in a government school in a lower middle-class area in Australia. The teacher was seen by his school community to display 'good teaching practice'. Three video cameras operated simultaneously in the classroom to capture the actions of the class as a whole, the teacher, and a pair of focus students. Data included videotape of the lessons, post-lesson video-stimulated student interviews, and photocopies of student work and lesson tasks (Clarke, 2001). The video-stimulated interviews were intended to reconstruct the learners' perspective. A student was given the remote control to a mixed video image with the focus students at center screen and the teacher as an insert in the corner. The student was asked to identify and discuss the parts of the lesson that were important to that student.

RESULTS AND ANALYSIS

The aims and outcomes of Lesson 12 for Dean and for class members in general are now described. The teacher intended students to learn the algebraic rule for finding the sum of the interior angles of a polygon. Class members used a table of student-generated results to find the rule 'add a hundred and eighty degrees each time'. Dean did not attend to the method the class developed; he developed his own method 'add a hundred and eighty degrees for each triangle'. The teacher then used student pattern recognition to formulate '(n-2) x 180° (where n is the number of sides of the polygon)'. Dean demonstrated some proficiency in applying this rule. The evidence now reported is drawn predominantly from: (a) the mixed image video of the lesson (V); and (b) the video stimulated, post-lesson reconstructive student interview (I). These sources are identified in the text.

For the purpose of analysis, Lesson 12 was divided into episodes or intervals in time during which Dean focused on a particular idea. The first 15 of the 21 episodes in the lesson have been included in Table 1. Episodes 16-21 have not been included and will not be discussed because they relate to the development of the algebraic rule and Dean did not link this rule to his ideas developed earlier in the lesson. In Table 1, episodes are numbered consecutively in Column 1 according to the time at which the episode occurred (see Column 2). The context of the episode (Column 3), and a description of the episode (Column 4) are also included. Column 5 displays the cognitive elements of the process of abstraction observable for Dean (V & I). The 15 episodes included 5 off-task and 10 on-task episodes. Three of the off-task episodes (1, 10 & 14) were instigated by Dean who engaged in across-class whispered conversations with Cam in relation to her hat that had been confiscated from Dean. Of the 10 episodes where Dean focused on mathematics, he struggled to understand what was expected in Episodes 4, 7, 8, 9, and 11 (I & V). Dean demonstrated observable cognitive elements of the process of abstraction in Episodes 2, 3, 12, and 15, and there was insufficient evidence to detect Dean's cognition during Episode 6. As can be seen from Table 1, the critical intervals in time occurred in Episodes 12 and 15 where Dean began constructing new ideas. The earlier episodes are now briefly described in preparation for a more detailed analysis of Episodes 12 and 15. As a variety of activities occurred simultaneously during whole class talk and organized pair-work, indicators of involvement were used to determine the focus of Dean's attention. The types of activities that occurred simultaneously in this classroom included: (a) groups of students engaged in their own on-task or off-task talk during whole class-talk; or (b) students engaged in talk in pairs or in larger groups whilst the talk of adjacent groups and the teacher (assisting other groups) was also audible.

No	*Time*	*Context of Episode*	*Episode description*	*RBC*
1	9:16-9:32	Across Class Off-Task Talk	The confiscated hat (i)	
2	11:16-13:52	Task Instruction	No. of sides of polygon	R
3	14:14-16:30	Task Instruction; Procedure	Make triangles by joining vertices	R
4	16:30-17.19	Investigation	No. of triangles in Polygons	
5	17:42-18:12	Small Group Off-Task Talk	Put Popeye Away	
6	18:53–20:40	Student responses to Col. 3.	No. Triangles in Polygons (ii)	
7	20:02-22:40	Task Instruction; Procedure	Sum of interior angles of polygon	
8	22:54-25:10	Simultaneous Work	Dean sorted work; Ted cut out triangles	
9	25:10-25:30	Simultaneous Work	Class discussed results; Dean struggled with procedure (Ep. 7)	
10	25:56-28:20	Across Class Off-Task Talk	The confiscated hat (ii)	
11	28:20-32:26	Small Group Instruction	Teacher demonstrated procedure (Episode 7) to Dean and Ted.	
12	32:26-33:48	Simultaneous Work	Dean attempted procedure; Ted found pattern; teacher assisted	R B C?
13	34:15-34:20	Small Group Off-Task Talk	The pot plant	
14	34:25-35:22	Across Class Off-Task Talk	The confiscated hat (iii)	
15	34:20-37:30	Whole Class Summary	Dean develops novel method	R B C?

Key: Cognitive elements: R, recognizing; B, building-with; C?, constructing interrupted

Table 1: The first 15 of the 21 Episodes in Lesson 12 from Dean's Perspective

In Episodes 2, 3, 4, and 6, the teacher explained the task by providing sheets: (a) a table with headings 'Name of Polygon', 'Number of Sides', 'Number of Triangles', and 'Sum of Angles'; and (b) named polygons with 3-10 sides. He demonstrated the procedure for sectioning each polygon into triangles—join all vertices of the polygon to a particular vertex. Students were required to fill in the columns 'Number of sides' and 'Number of triangles' which was Year 7 work for Dean (I). Dean was unsure how to fill out the third column of the table in Episode 4 as evidenced by his comment 'Oh I get it' when he heard an adjacent group explaining late in the episode (V). The teacher introduced the procedure for finding the sums of interior angles of polygons in Episode 7 by reminding students they knew the answer for the first polygon (a triangle). In an earlier lesson, the teacher had torn off the 'corners of the triangle' (teacher's wording) and licked them and placed them together on the board commenting: 'see they make a hundred and eighty'. In Episode 7, the teacher demonstrated a similar procedure with a quadrilateral—cut out the triangles, tear off their 'corners', place them together, and find the total angle. When asked why they were doing this (by a student), the teacher replied: 'to find the answer'. Dean's comments in class and his interview-reconstruction of Episodes 9, 11, and 12 demonstrated he did not know how to implement the teacher's procedure. Dean knew he had to rip off corners but did not know what to do after that as illustrated by an excerpt from Dean's interview [Key: '{…}' dialogue omitted; '[ ]' researcher's comments]:

Dean I always put [pause] I didn't know [pause] where the corners went [pause] in it {…} I was doing it all different- I was facing them *out* … and *up* {…}

Dean did not know to face the vertices of the angles towards the center and juxtapose them to see the total rotation. He had the angle vertices pointing many different ways. Dean's lack of understanding of the purpose of the teacher's procedure was further illustrated by Dean's response to a question from Simon who asked Dean why you tear not cut the corners. Dean shrugged initially, then when he overheard the teacher explain to an adjacent group: 'cut not tear so you know which are the corners', Dean turned to Simon and laughed: 'You have got to find which is the corner'. In Episode 11 Dean made an unsuccessful attempt to find the sum of the interior angles of a quadrilateral. The teacher then demonstrated the procedure for Dean: 'Right, tear, tear, tear, we get the pointy ends in together, and that gave us a hundred and eighty, straight line.'

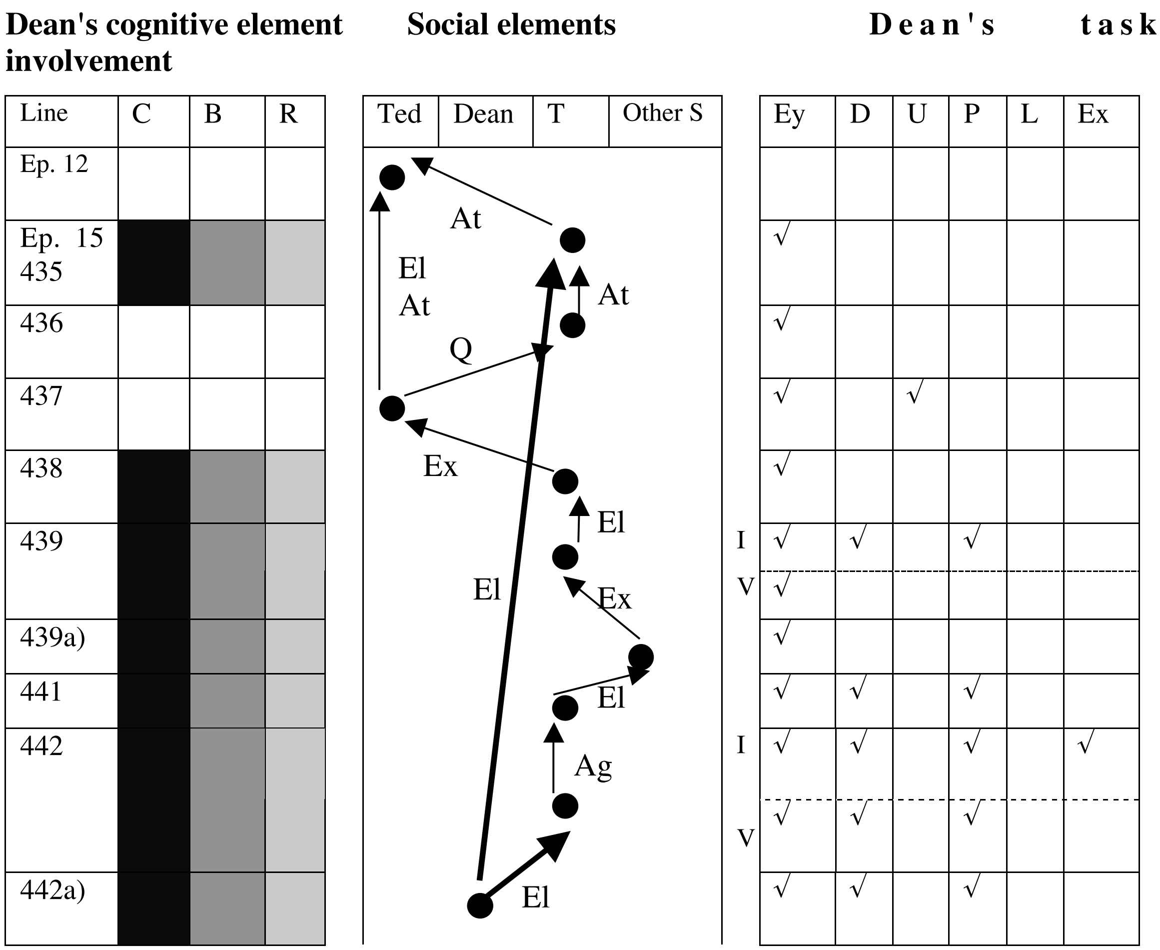
Dean controlled the start of Episode 12 as he attempted the teacher's procedure with a pentagon. The teacher progressively took control by repositioning the 'corners' Dean had placed. The actions of the teacher assisted Dean to see that the teacher's procedure required the 'corners to face in' (V), but did provide Dean with reasons for why this was so (I). The teacher engaged in a parallel but separate dialogue with Dean's partner Ted who developed a numerical pattern: 'add a 180 each time'. Dean paid no overt attention to this parallel dialogue until the teacher asked Ted 'what's 360 plus 180?'. The teacher then used the 'corners' he had juxtaposed with Dean to evaluate Ted's response of 540^{o} and exclaim: 'Aha it is'. Dean then looked from Ted's table, to the juxtaposed angles, to his own page. Dean did not attempt the teacher's procedure again after Episode 12.

Enriched transcripts and visual displays of Dean's abstracting process were generated for Episodes 12 and 15 but due to space constraints have only been included for Episode 15. Enriched transcripts include descriptions of Dean's body language and interview comments beside the relevant lines of transcript (Table 1). Episode 15 is now summarized and supported with the enriched transcript (Table 2) and a visual display (Figure 1) that differs in several aspects to the displays developed by Dreyfus, et al. (2001). The display used in this paper contains evidence of task involvement (see key to Figure 1), and every line of transcript is related to Dean's cognition rather than to the cognition of the speaker. In Figure 1, line numbers are listed down the left hand side with the inclusion of lower case letters after line numbers to indicate dialogue captured on the student (but not the teacher) microphone. The three groups of columns display Dean's cognitive activity (left), social elements of Episode 15 (center) and Dean's task involvement (right). In Lines 439 and 442, the 'I' and 'V' beside the task involvement columns show indicators drawn from interview and video data. Due to space limitations, task involvement indicators for interviews have only been shown for the lines where Dean's video-stimulated reconstructive interview data provided evidence not available from Dean's body language in the classroom video (Lines 439 and 442).

Line. Time	**Dean's new method 34:53-37:30**	**Relevant Excerpts from Post-Lesson Video Stimulated reconstructive Interview**
435. 35:47	T: // [wrote table headings on board] Those people who managed to work it out for the four-sided shape found it was 360. Around about this point Ted had an idea of what was going on. [Dean listened, wrote, looked at board].	**Dean:** {…} on the board it said [pause] two triangles [pause] which is [pause] 360º [pause] and then I just thought [pause] you keep each triangle you had [pause] 180º and then [pause] *so on* but … yeah.
436/7 35:58	T: What did you think was going on Ted? Ted: One eighty you plus one eighty. [Dean watched board]	
438. 36:03 439. 36:04 439a)	T: Every time you add one eighty. So we did the five-sided shape and we got five hundred and forty. Which was one time around plus another half. [Dean watched the board] Tessa: Oh so you always go up by one eighty.	**Dean:** {…} so a 180 plus 180 is 360- plus another hundred and eighty [pause] be 540. **Interviewer**: How did you know that? **Dean**: He said [pause] in a previous lesson [pause] each triangle adds up to 180- that was when we {…} the protractor [pause] and we did all those lines {…} once you get a triangle {…}. In a previous lesson [pause] hard to explain. I may have *known* that like in a *previous* lesson but [pause] I probably didn't think of it for [pause] because it's *another topic* [pause] like we were just doing *that* [pause] *then* {…} finding out the shapes and triangles {…} **Interviewer:** What does … 'adds up to 180' mean? **Dean:** Um … I'm not actually sure.
441. 36:17	T: // Every time you go up by one … you keep adding up 180. [Dean looked at the developing table then wrote]	**Dean:** But there's another way you can do it as well {…} each triangle you get [pause] in the shape [pause] which is 180º [pause] so that would be 180, 180, 180 {…} 360, plus another hundred and eighty {…} 540.
442. 36:22	T: [to all] Sally had a six-sided shape // and she said I've got two three sixties. Which is 720. [Dean wrote in his book. When he finished his calculation he leant slightly towards Ted and Ted's page and pointed to Ted's page]	**Dean:** I think *that's* where I got it from actually [fast confident voice] {…} Hold on![Rewinds video] {…} [slow pensive] *yeah hold on* [fast pace] [pause] see how that's got [pause] 1, and then 2 and then 3 and then 4 and then 5? [pause] Yeah wait on! [fast pace] {…} yeah I think that's where I got it from. See it goes up by 180 each time.
442a. 36:23	Dean: // That'll be ten eighty. [Dean looked at Ted]	

Key: '[ ]' researcher comments; '//' simultaneous event; '{…}' dialogue omitted; 'italics' student emphasis. Numerical values rather than words have been used where a number was correctly worded.

Table 2. Enriched transcript for Episode 15: Dean's new method



Key. ***Cognitive:*** C, constructing; B, building-with; R, recognizing. ***Social:*** C, control; El, elaboration; Ex, explanation; Q, query; Ag, agreement; At, attention
Involvement (video, interview): Ey (eyes, respond to questions); D (point body or pen to task, try to answer fully); U (unaware, cut to add ideas); P (participate, few prompts needed); L (latch to each other's comments, cut in eagerness to answer); Ex (exclaim, pace and emphasis). 'Cut': interrupt interviewer's question. '▬': aspects of the social interaction contributing to Dean's idea.

Figure 1. Visual display of Cognitive, Social, and Involvement for Episode 15

During Episode 15, Dean spoke once (Line 442) and attended selectively to classroom discussion. Where the class focused on Ted's pattern, Dean focused on the number of triangles and adding a new 180º for each triangle (Line 436, interview comments during Lines 435, 439, 441 & 442). Towards the end of the episode Dean softly told Ted his result from building-with his new method (Line 442a in Table 2 & Figure 1). During most of Episode 15, Dean paid attention to the table on the blackboard. He sometimes leaned over his work and wrote calculations (Lines 435-441). As the table on the board was progressively completed, Dean reflected about the number of triangles in polygons with 3-6 sides (I, Line 441). Dean built-with his newly developed sequence of procedures to find the sum of the interior angles in a polygon with more than 6 sides (Line 442a).

DISCUSSION AND CONCLUSIONS

Dean self-instigated pursuit of a discovered complexity: 'the corners of each new triangle make another 180°' when the teacher placed the 'corners' of triangles in Episode 12. This led to self-directed exploration of links with the idea: 'angles in a triangle add to a hundred and 180°' (self-selected from another topic (Line 439, Table 2)). He horizontally reorganized a mathematical structure by using the number of triangles in each polygon (I: Line 435, Table 2) and successively applying the rule 'angles in triangles add to a hundred and eighty degrees' for each triangle (I: Line 441, Table 2). Dean's focus of attention on Column 3 on the board was self-directed; the class focused on numerical patterns in Column 4. Dean identified video around Line 442 (Table 2; 6-sided shape) as when his new idea crystalised (empirical generalization). Dean's perception of angles, that he was unable to identify in diagrams, as amorphous entities associated with polygons provided an inadequate cognitive artifact to scaffold the integration of ideas and gain insight from the relative positions of the interior angles of the triangles and the polygon (theoretical generalization). Nevertheless, Dean's ideas were more mathematical than those used by the class. Implications for practice include: (a) knowledge assumed to be simple and understood by all (like the idea of angle) may not be understood and could inhibit the development of a new more complex concept; and (b) students could benefit from tasks that provide opportunities to develop assumed knowledge and more complex concepts simultaneously. The interesting theoretical question for further study is: What are consequences of the subsequent development of an inadequate cognitive artifact (from an empirical generalization to a theoretical generalization) in relation to any related more complex empirical generalization the student may have already developed?

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COMPARING COMPETENCE IN TRANSFORMATIONAL AND GENERATIONAL ALGEBRAIC ACTIVITIES

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The Purposeful Algebraic Activity Project[1] is a longitudinal study of the development of pupils' algebraic activity in the early years of their secondary schooling. Here, we report on our empirical findings from the initial semi-structured interviews. We analyse the responses of three pairs of 12-year-old pupils to a range of algebra questions. In our analysis, we identify broad similarities in the 'answers' pupils gave to transformational questions and quite significant differences in the pupils' responses to generational questions. We consider the implications for assessment, and discuss the potential of spreadsheets for developing pupils' appreciation of the need for an algebra-like notation.

BACKGROUND

Kieran (1996) describes three kinds of activities within the scope of school algebra:

- *generational activities*: generating expressions and equations that are the objects of algebra, expressions of generality from geometric patterns or numerical sequences, and expressions of the rules governing numerical relationships
- *transformational activities*: rule-based activities including collecting like terms, factoring, expanding, substituting, simplifying expressions and solving equations
- *global, meta-level activities*: such as problem solving, modelling, finding structure, justifying, proving and predicting (p. 272)

In the project Purposeful Algebraic Activity, we use Kieran's classification above, together with a broad notion of *algebraic activity* (Meira, forthcoming).

Generational activity, particularly the translation of a verbal representation of a problem into an algebraic one, has been identified as a major obstacle for pupils (Kieran, 1997). MacGregor and Stacey (1997) interpret pupils' early misrepresentations, not as indicative of low levels of cognitive development, but as 'thoughtful attempts to make sense of a new notation' (p. 15). They identify the sources of these errors as analogies with other symbol systems, intuitive assumptions and pragmatic reasoning about an unfamiliar notation system. Ainley (1999) found that 11-year-old pupils were comfortable with talking about, representing and operating mathematically with unknown quantities. Their written representations reflected their lack of experience of the conventions of notation rather than their difficulties with algebraic activity. Research has also shown that technology can be highly successful in helping students give meaning to algebraic expressions (Thomas and Tall, 2001; Sutherland and Rojano, 1993).

Transformational activity has traditionally been given significant attention both in schools and in research (e.g. Küchemann, 1981). Within the literature, some researchers use the transformational activity of solving equations in order to define a boundary between arithmetic and algebra (Filloy and Rojano, 1989; Herscovics and Linchevski, 1994). Rather than establishing levels of pupils' competence or defining boundaries, we

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focus our attention on the *meaning* of *algebraic activity*. Whilst we acknowledge that detachment of meaning is powerful in transformational activity, we recognise that while pupils may engage in routines of action, they may not appreciate the value of doing so:

> 'Even those students who manage to handle algebraic techniques successfully, often fail to see algebra as a tool for understanding, expressing and communicating generalisations, for revealing structure, and for establishing connections and formulating mathematical arguments' (Arcavi, 1994, p. 24)

Although there appears to be relatively little research comparing *generational* and *transformational activity*, the literature on assessment raises two issues of particular relevance. It is widely acknowledged that school tests tend to assess quantifiable skills, and that where particular aspects of mathematics are assessed, there is an incentive to 'teach to the test'. We suggest that while transformational activities are relatively easy to measure, assessing generational activity and global, meta-level activity is potentially more problematic. As well as the *kind* of activity assessed, Cooper and Dunne (2000) discuss the *context* of the question. They refer to pupils' responses to two algebra questions used to assess whether pupils can express a simple function symbolically (from a U.K. national test taken by 13-14 year olds). One item is *realistic* with the imagined context of planting trees in an orchard; the other is *esoteric*, requiring pupils to work out the perimeter of labelled figures. Cooper and Dunne observe that pupils are significantly more successful at answering the *esoteric* question, and have difficulties constructing the intended goal of the problem as algebraic for the realistic item.

DATA COLLECTION

As part of our longitudinal study we are conducting semi-structured interviews at regular intervals to trace the development of pupils' algebraic activity during the first three years of secondary schooling. The interview questions cover a number of themes, and include *generational*, *transformational* and *global, meta-level activity*. We report here on the initial set of interviews with twelve pairs of pupils at the end of their first year of secondary school (mostly aged 12), during which they have experienced some formal algebra teaching. Their teachers were asked to identify compatible pairs of pupils across the perceived ability range from those who were willing to take part. The first named author conducted the interviews, with each question presented in written form and read aloud. Throughout the interviews, pupils were encouraged to articulate and discuss their responses, all of which were video taped and audio taped. Calculators, paper and pens were available for the interviews, but most pupils chose to say aloud their responses. The transcripts were annotated to include non-verbal behaviour, and any written work.

Here, we report on the responses of three pairs of pupils to a selection of *generational* and *transformational* questions from the initial interviews. The three pairs were chosen to illustrate typical responses from each of the ability groups:

Megan and Thomas	High ability	School M
Natasha and Holly	Middle ability	School M
Mollie and Grace	Low ability	School N

In our analysis we discuss the contributions of each pair, but where appropriate (where they disagree, for instance) we focus on the contributions made by specific individuals.

TRANSFORMATIONAL QUESTIONS

During their first year of secondary school, pupils learn to simplify expressions and construct and solve simple linear equations. We wanted to see whether pupils could *do* these things, and how they talked about doing these things. The following table summarises the pupils' spoken responses to the transformational questions. The first four questions involve simplifying expressions (from Küchemann, 1981); the last four involve solving equations (first two of which are from Herscovics and Linchevski, 1994).

Question	Megan and Thomas (High ability)	Natasha and Holly (Middle ability)	Mollie and Grace (Low ability)
2a+5a	**Seven a**	**Seven a**	**Seven a**
2a+5b+a	**Three a plus five b**	**Three a plus five b**	G: **Three a add five b** … the answer would be eight … two a add five b … M: Three a equals five b
3a-b+a	**Four a minus b**	**Four a minus b** Three a minus b Three a plus one b	**Four a take away the b**
(a-b)+b	a minus two b a plus b	H: a minus two b or a plus two b … N: I don't think you can simplify	[Evaluates for particular numbers]
14+n=43	**Twenty-nine**	**Twenty-nine**	**Twenty-nine**
23=37-n	M: Sixty … T: **Fourteen** [Megan agrees]	N: What minus thirty seven equals twenty three? H: **Fourteen** [Natasha agrees]	G: You'd do n take away thirty seven equals twenty three M: **Fourteen** [Grace agrees]
3t=t+3	T: (frowns) t is one M: … Nought or something … T: **One and a half**	H: That just means the same thing … N: Three times t plus t plus three	Four t add three
7+4u=70-3u	[Unable to solve]	[Unable to solve]	[Unable to solve]

Table 1: Extracts from pupils' spoken responses to transformational questions, correct answers highlighted in bold

Using the distinction between *answers* and *strategies*, (van den Heuvel-Panhuizen and Fosnot, 2001) we observe that there are broad similarities between the answers given by the three pairs. With the exception of one equation (3t=t+3), the three pairs were either all correct or all incorrect in their response.

Megan and Thomas (High ability) were noteworthy in terms of the immediacy and confidence of their responses. Thomas was the only pupil of the three pairs who was able to solve the equation 3t=t+3. Clearly, solving an equation had some meaning for Thomas, for when asked what he thought algebra was useful for, he responded 'If you, like, don't know a number and something then, but you know the rest of the thing you can, like, call it x and try and work out what it is.' He was able to use trial and improvement, going beyond the taught method of using inverses.

Natasha and Holly (Middle ability), although giving the same answers, engaged in significantly more discussion, evidenced by the lengthy transcripts. Their talk is an honest account of some of their dilemmas. For instance, when trying to simplify 3a-b+a:

Holly: You can't minus the b. These are the ones what I don't understand really, because it's really confusing, I think, because you don't know whether to plus a or take away a …

Natasha: I think you do, I think it's four a minus b [writes 4a-b] I think that's how you write it, 'cause you can't minus a b from an a because they're different letters

Holly: Yeah, but then, it doesn't exactly say we have to add an a does it? 'Cause it could mean take away an a. No it doesn't.

Importantly here, we observe that they gave the correct *answer* in the end. Natasha and Holly said that they had done a lot of algebra in class, and particularly referred to the transformational activities of simplifying expressions and solving equations.

Mollie and Grace (Low ability). In terms of the *answers* they gave, Mollie and Grace's achievements were comparable with those pairs identified by their teachers as high and middle ability. They correctly simplified three out of the four expressions. However, when asked to explain their method of simplifying 2a+5a, it became clear that Mollie and Grace were using what has become known as *fruit salad algebra*: 'anything can be an a, so you can put two apples add five apples, so then you can, like, add the five and the two … seven, and then you can put seven a equals seven apples.' Whilst they showed some competence in transformational activity, their skills were limited to simple transformations. This relative 'success' may be attributed to previous experience of such tasks. Their language indicates awareness of the *rules* of algebra: 'put the one a on the two a' and 'carry it over.' But with the more difficult equations, unlike the other pairs, Mollie and Grace could not make sense of the *meaning* of the equations; instead focusing on trying to identify which part was the 'answer.'

Analysis of the pupils' responses to transformational questions has shown that there is not a great deal of difference between the three pairs. Whilst all pupils could participate in the discourse of simple transformational algebra, the interview situation revealed some differences in the *meaning* that they constructed for what they were doing.

GENERATIONAL QUESTIONS

One aspect of our research involves tracing the longitudinal development of pupils' generational activity and their attempts to use notation. In the interviews, we wanted to identify whether pupils can generate expressions and equations, and whether they used particular conventions for notation. We report here on the pupils' responses to five such questions.

4. Jack is three years older than Chloe. What can you write for Jack's age?
5. George's big brother gets twice as much pocket money as George.
   What can you write for how much pocket money George's big brother gets?
6. David is 10 cm taller than Con. Con is h cm tall.
   What can you write for David's height?
8. Explain what you think the rule is in the table.
   Write the rule in terms of the letters in the top row.

a	b
1	2
2	3
3	4
4	5

Write the rule in any other way that you can.

11. Can you solve 5x99 in your head?
    Can you write down how you could multiply *any* number by 99?

Figure 1: Generational questions called *Jack's age, Pocket money, David's height* (from MacGregor and Stacey, 1997), *Rule in table* and *Multiply 99*

Unlike the transformational questions, analysis of pupils' responses to the generational questions shows quite significant *differences* between how the pairs responded:

Question	Megan and Thomas (High ability)	Natasha and Holly (Middle ability)	Mollie and Grace (Low ability)
4. Jack's age	**c plus three equals j**	H: Eight and Chloe would be five … N: **Three plus c** … H: Yeah, it would be three c … N: Three minus c	G: Jack could be nine and Chloe could be M: Six … they could be any age but as long as they're three years, Jack's three years older than Chloe
5. Pocket money	**Two g equals b**	**g times two** **g two**	G: George could get two pounds and George's big brother could get six pounds … M: However much George gets, then his brother has to get two pounds more
6. David's height	**h plus ten equals d**	**h plus ten**	G: Con could be one centimetre and David could be eleven centimetres … M: The h can be any number (gestures, palms upwards) and, um, as long as David is ten centimetres taller
8. Rule in table	**b equals a plus one**	**a plus one** a plus one equals b [with prompting]	Add the one [with prompting] [Unable to express using letters]
11.Multiply 99	Round ninety-nine to a hundred then times it by the number and take away the number	5 ×100 (− 5) 495	[Unable to solve 5x99; unable to understand method]

Table 2: Extracts from pupils' spoken responses to generational questions, correct answers highlighted in bold

Megan and Thomas (High ability) generated expressions and equations fluently. Their answers indicate their familiarity and confidence with using conventional notation. For example 'two g equals b' in Pocket money shows awareness of the convention of putting the letter term before the number, and 'b equals a plus one' in Rule in table, shows awareness of the convention of putting the dependent variable first. Although in Multiply 99, Megan and Thomas were unable to write anything down when asked, the question does not make it explicit that algebra should be used, and Megan's explanation 'Round ninety-nine to a hundred then times it by the number and take away the number' is entirely appropriate. Over a range of questions, a pattern emerges of pupils who are confident in generating their own expressions and equations, and for whom algebra has some meaning.

Natasha and Holly (Middle ability) Natasha and Holly successfully answered almost all of the questions. They discussed each question carefully but were not always secure in their responses. For instance in Jack's age, they first suggested ages for Jack and Chloe, then generated a number of different expressions: 'three plus c', 'three c' and 'three minus c'. Interestingly, they tended to generate expressions rather than equations, which was problematic for Rule in table, since the rule needed to include both a and b. Natasha initially suggested 'a plus one,' but needed prompting to include 'equals b' in her response. The expressions that they generated do not show the same awareness of conventional notation as in Megan and Thomas' responses. In Multiply 99, Natasha and Holly used the number 5 to illustrate the method; they did not use a letter to represent how to multiply any number by 99.

Mollie and Grace (Low ability). While we observed subtle differences between the high and middle ability pairs, we observed quite significant differences between both of these pairs and Mollie and Grace. In essence, their answers to the generational questions do not include letters or symbols. A pattern emerges whereby Grace sought to evaluate the unknowns, choosing suitable (or unsuitable) values for Jack's age, George's pocket money and David's height. Her responses suggest that she did not recognise the questions as algebra problems, but as arithmetic problems, to which a numerical answer was required. Mollie's language and gestures seemed to indicate that she was comfortable with unknowns and most of the relationships. When asked David's height, Mollie concluded:

> 'Well, h, yeah, again, the h can be any number (pointing to the h and when she says *any number* she gestures with both hands, palms facing upwards) and, um, as long as David is ten centimetres taller than any of that, any number that h is'

Drawing upon Cooper and Dunne's (2000) interpretation of pupils' responses to *realistic* algebra questions, we acknowledge that questions such as *David's height* are problematic in that they are presented in the everyday language of arithmetic and they do not explicitly ask pupils to write an expression. Mollie's response clearly demonstrates an understanding of the *relationship* between the heights, although she did not use notation. We recognise that the use of notation is an implicit expectation, but equally recognise that other pairs of pupils were aware of this convention. Mollie and Grace's difficulties extended over a broad range of generational questions including the *esoteric* questions (*Rule in table, Multiply 99*) where they were unable to generate equations.

Analysis of the pupils' responses to *generational questions* has shown important differences between the pairs. The pairs identified as high or middle ability successfully used algebraic notation to generate expressions and equations. However, those identified as low ability did not use algebraic notation to generate expressions and equations.

DISCUSSION

Analysis of data from the three interviews leads us to tell three different stories about the pairs of pupils. Megan and Thomas, identified by their teachers as high ability, had some sense of what letters are used for in algebraic notation. They solved generational and transformational questions with a confident grasp of notation conventions. Natasha and Holly, identified as middle ability, were also successful over a broad range of questions, particularly when prompted in an interview situation. However, they were less confident with the syntax of algebraic notation. Mollie and Grace, identified as low ability pupils, also engaged in algebraic activity. They could clearly solve simple transformational questions, but had difficulties with generational algebra. In summary, we found broad *similarities* in the pupils' answers to *transformational* questions, and *differences* in their responses to *generational* questions. The interview data gives only a snapshot of pupils' development, and our focus has been on three pairs of pupils, but this finding is representative of what we have found across pairs in the set of interviews. We conjecture that the reason for this finding many lie in the teaching emphasis on transformational activities (which may be particularly prominent in School N), and the nature of pupils' learning experiences of generational activities.

The picture that emerges from our analysis has important implications for assessment and for teaching. If school tests focus on *transformational* activities, they may be giving a misleading impression of how much pupils understand. Competent performance on transformational activities may disguise pupils' difficulties with generational activities, and with constructing meaning for transformational algebra. Hence, whilst we recognise the importance of the assessment of transformational activity, we feel that greater consideration should be given to assessing generational activity (and global-meta level activity). Our initial analysis of pupils' responses to school test items on algebra suggests that our concern about assessment implications is valid. We observe on one test used in School M, transformational items outnumber the generational, and that there are broad similarities between pupils' responses to transformational items, and differences in their responses to generational items, reflecting the findings from our interview data.

We believe that pupils need more opportunities to engage in generational activities, and particularly in activities where they can appreciate why generating expressions is a useful thing to do. In the Purposeful Algebraic Activity project, we have designed a teaching programme using spreadsheet-based tasks (Ainley, Bills and Wilson, forthcoming). The tasks are designed to engage pupils in solving purposeful problems, and incorporate *generational* and *global, meta-level activity*. We feel that generating expressions and attending to the process of denoting is an important foundation for understanding. The spreadsheet provides a context for pupils to construct formulae *themselves*. Importantly, there is a *purpose* in doing so, for example to generate more data in order to solve a problem. We have also built in to the teaching programme reasons to move away from the spreadsheet and engage in *transformational* activity so that 'the non-letter-symbolic representations and their transformations can be used to make contact with or give meaning to the letter-symbolic representations that are traditionally involved in algebraic activities' (Kieran, 1996, p. 275). We see spreadsheet algebra as important in developing pupils' confidence *with* and appreciation of the need *for* an algebra-like notation.

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COMPLEXITY IN TEACHING AND CHILDREN'S MATHEMATICAL THINKING [1]

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In the research conducted, the relationship between teaching complexity and children's mathematical thinking was investigated in 4 'reform' classes and 1 conventional elementary class (7-8 years). Forty lessons were analyzed for the type of teaching and children's mathematical thinking revealed during class discussion. The results indicate increased complexity in teaching and level of children's thinking was highly related to the kinds of interaction that distinguished three class cultures. These findings complete a previously proposed theoretical framework that integrates teaching and learning by detailing acts of teaching in relation to complexity of children's thinking.

It is well documented that the concerted effort in the U. S. to change conventional mathematics teaching to forms of pedagogy that coincide with learning for conceptual understanding is more difficult than initially anticipated. One reason may be, in part, due to the fact that this requires the development of far more complex and sophisticated pedagogy than was understood or even known at the onset of the effort (Wood, Nelson & Warfield, 2001). Although research on learning over the past century has influenced our knowledge of learning, similar transformation in our understanding of the teaching practices has yet to occur. Educators, such as Darling-Hammond (1996) believe the challenge for education in this century is the advancement of "knowledge for a different kind of teaching . . . that goes far beyond dispensing information, giving a test, and giving a grade" (p. 7). From recent studies such as Askew et al. (1999) and Franke et al. (1998) we are beginning to understand what characterizes the complexity in new forms of teaching and how this relates to student learning. However, it still remains that "only a few studies exist which empirically examine teaching in these classes with the same detail and attention to theory building as found in the investigations of learning" (Wood, 1998, p. 193).

In previous research we have examined class cultures for differences in pedagogy and found that teaching for conceptual understanding does not consist of a singular practice, but rather varies on two dimensions—expectations for class members' participation and the breadth of pupils' thought (Wood & Turner-Vorbeck, 2001). While these two dimensions differentiate the nature of teaching in 'reform-oriented' class cultures, the relationship of teaching to children's mathematical thinking was only theoretically conjectured. Therefore, the purpose of this research report is to present the results of an investigation into the relationship between teaching and children's mathematical thinking

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in order to account empirically and theoretically for differences in the complexity of teaching. In the next section, we discuss revision of the previously proposed theoretical framework that includes the addition of results from empirical analysis of children's mathematical thinking.

THEORETICAL FRAMEWORK

In recent research, we examined the nature of children's mathematical thinking and found substantial differences existed not only between conventional and reform-oriented classes, but also among reform-oriented classes in terms of the mathematical thinking and reasoning students' revealed during class discussion (Wood, 2002; Wood, Williams & McNeal, in review). Specifically, these differences could be attributed to variations in the nature of student participation and the level of students' mathematical thinking. Although, these two dimensions, participation and thinking, continue to form the basis of the theoretical framework, the recent analysis resulted in a reconceptualization of the prior three class cultures into two, *strategy reporting* and *inquiry/argument* as shown in Figure 1.

Thinking Dimension

The *thinking dimension* of the theoretical framework represents children's increasing responsibility to engage in and reveal higher levels of thought and reasoning during class discussion. In creating this dimension, it is hypothesized that children's increasing responsibility to engage in higher levels of thinking is connected to the type of culture that exists in the class. The vertical black arrow in Figure 1 represents the axis, *Responsibility for Thinking*, which relates to students' mathematical thinking and highlights differences between reform class cultures.

Cognitive theories following constructivism but drawing specifically from work in mathematics by Krutetskii (1976) Hershkowitz, Schwartz, and Dreyfus (2001) and elaborated by Williams (2001) were used to connect the thinking dimension to types of children's mathematical thinking described in the fourth column of Figure 1. Krutetskii's descriptions of mathematical reasoning describe the mental mathematical activity that underlie the observable epistemic actions described by Dreyfus, Hershkowitz and Schwarz (2001) that occur in the process of mathematical abstraction and generalization. These levels of thinking not only indicate a deepening in thought processes but also represent a means to particular kinds of knowledge outcome that represents the hypothesized development of increasingly integrated knowledge networks or structures that differentiate the three class cultures (reform and conventional).

Conventional Class Culture

Discussion Context	Reporters	Listeners		Mathematical Thinking
	Student	Teacher	Student	
Conventional	tell right answers tell prescribed procedures	evaluate ask test questions	pay attention check answers and procedures	*Recalling* recalling answers and prescribed procedures

Reform Class Culture

Discussion Context	Explainers	Active Listeners		Mathematical Thinking
	Student(s)	Teacher	Students	
Strategy Reporting	tell different strategy or method clarify solutions	accept solutions elaborate solutions	listening to decide if own strategy is different	*Recognizing* comprehending applying *Building with* analyzing
Inquiry/ Argument	give reasons justify defend solutions	ask questions provide reasons ask for justification make challenges	ask questions for understanding & clarification disagree and give reasons make challenges & justify	*Building with* synthetic-analyzing evaluative-analyzing *Constructing* synthesizing evaluating

Responsibility for Thinking

Responsibility for Participation

Figure 1

Participation Dimension

The *participation dimension* of the theoretical framework consists of the extent to which it is possible for all students to participate actively in the class social interaction and discourse. The horizontal black arrow in Figure 1 represents the axis, *Responsibility for Participation,* which illustrates the increase in students' interaction as they take more responsibility to participate in the ongoing discussion. From these empirical findings, we linked student participation to the norms constituted in a class and the social interaction patterns that evolve. The results indicate that teachers in each culture establish different social norms for children's participation (Wood, McNeal, & Williams, in preparation).

The increasing demand on student thinking and the expectation for social interaction is the basis upon which meaning making, individual and collective, is accomplished in the class and represents a link between student cognition and classroom social processes. Individuals, in this perspective, are seen as constructing knowledge in the interpersonal activity of negotiation of meaning that leads to the attainment of mathematics that is held as common or taken-as-shared knowledge. The assumption that underlies the Responsibility for Participation axis is that the increase in responsibility to participate in the class culture is consonant with increases in student autonomy in learning (Bruner, 1981).

As shown in Figure 1, the two types of reform class cultures are considered to be 'deepening' in terms of responsibility for student thinking and for participation and, therefore, are hierarchical and nested. The main focus in a strategy-reporting class is on children's presentation of different strategies for the problem-centered tasks. Children presenting their solutions may be asked to provide more information about how they solved the problem by the teacher and sometimes by other student listeners. Classes classified as inquiry are those in which children offer different solution methods, as in the strategy reporting classes, but they provide reasons for their thinking in order to make sense to others. Student listeners and teachers in these classes may ask questions for further clarification and understanding. Argument classes contain the features of strategy reporting and inquiry but, in addition, a challenge or disagreement from student listeners or teachers initiates an exchange that prompts the thinking of justification in support of student ideas. Although inquiry and argument are distinguished by differences in interaction patterns and knowledge construction, they are better characterized as a single class culture, because the lines between reasoning and justification are somewhat blurred.

The revised theoretical framework, shown in Figure 1, presents thinking in conventional class culture along with the mathematical thinking that is revealed in reform class cultures. The categories, *recognizing*, *building with*, and *constructing,* and the types of mental activity are ordered on Figure 1 to represent increasing complexity in the thinking children revealed.

This addition of empirical results about the nature of children's mathematical thinking is a key contribution to the revision of the theoretical framework. Nevertheless with this information at hand, the relationship among teaching processes and student thinking still needs to be defined. Therefore, the purpose of this research report is to describe in greater detail the manner in which teachers promoted mathematical thinking.

METHODOLOGY AND ANALYSIS

Data Source

The data consisted of videotapes selected from 2 classes identified by their culture as strategy reporting, and 2 consisting of both inquiry and argument and 1 class, previously identified as conventional that was selected as a contrast case. This particular class was selected because it consisted of conventional textbook lessons and problem-solving lessons.

Methodology

The methodology and analysis followed a qualitative research paradigm and procedures similar to those of Strauss & Corbin (1990) in which categories were developed from the data, examined for confirming and disconfirming evidence and revised. For each of the classes, 30 lessons were selected for analysis. From each of the reform-oriented classes, a subset of 8 lessons selected as representative of each class culture was selected from the second semester for more intensive analysis of the interaction patterns that structured the class discussions. These 8 lessons (16 for each culture, strategy reporting and inquiry/argument) were the data source for the analysis of the ways teachers prompt students' collective thinking. From the conventional class, 8 lessons were selected; 2 consisted of textbook only instruction, which focused on place value and 2-digit addition and subtraction. The remaining 6 lessons consisted of both open-ended problem solving and textbook instruction. This combination provided the unique opportunity to examine textbook and problem solving lessons within conventional instruction.

Analysis

The coding scheme previously developed for investigating interaction patterns in reform-oriented mathematics class discussions was extended for the current study to include additional codes developed for conventional mathematics class instruction. Following the coding and identification of structure, the interaction patterns were further analyzed to identify consistent and repeatable patterns across the lessons, and then categorized and labeled. The transcripts of class discussion in each lesson were also used to analyze teachers' prompting questions in relation to children's thinking. Using coding categories developed for examining teacher's questions and statements each line in the transcript was coded. Finally, the analysis of the interaction patterns, teacher prompts, and children's thinking were recombined to recreate the discussion in the order the interaction patterns occurred and reanalyzed.

RESULTS AND DISCUSSION

The results of the analysis of the interaction patterns define and delineate the specific nature of each class discussion. The analysis reveals that not only does the frequency of interaction between teachers and children increase progressively across the four cultures but also the nature of interaction changes indicating increasing opportunities for student discourse and participation. The analysis also reveals that the types of interaction patterns vary across the classroom cultures with the textbook culture consisting of the fewest kinds of interaction patterns and inquiry/argument the most.

The results of the analysis revealed that the frequency and level of teacher prompting through questioning or statements varied considerably across these class cultures. Moreover, the frequency and complexity of teacher prompts for mathematical thinking progressively increased across the types of class cultures, with the conventional environment, both textbook and problem solving, being predominately situations of prompts for recall for children. Instances of teacher prompts for mathematical thinking in the conventional textbook class culture were infrequent (N=14) and consisted of recognizing (comprehending and applying) mathematical ideas as shown in Figure 2. In the strategy reporting class culture teacher prompts for mathematical thinking dominated (N=86) with a majority of questions consisting of recognizing (comprehending and applying) and building with (analysis). In inquiry/argument classes, teacher prompts for mathematical thinking were most frequent (N=177) with more questions focused on building with (synthesis analysis and evaluative analysis) see Figure 2.

Other analysis revealed that questioning in conventional teaching was directed at prompting children to give teacher expected information, while strategy reporting and inquiry/argument emphasized student exploration of methods and justification of student ideas. Comparison of strategy reporting and inquiry/argument revealed teachers' differed in the frequency of prompting for mathematical thinking during inquiry interactions and situations involving proof and justification. The resolution of differences in students' answers was dealt with differently by teachers in the two reform cultures. Teachers in strategy reporting class cultures emphasized proof of a correct answer through the use of concrete objects, while teachers in inquiry/argument relied on children's explanation and justification to resolve differences in reasoning.

Conventional Class Culture

Class Discussion	Mathematical Thinking Revealed	Teacher Prompts
Conventional	*Recalling* recalling answers and prescribed procedures	What is the answer? Two plus 3 is_____?

Reform Class Culture

Class Discussion	Mathematical Thinking Revealed	Teacher Prompts for Mathematical Thinking
Strategy Reporting	*Recognizing* comprehending applying *Build with* analyzing	I'm confused. Would you tell us again what you thought? Does this make sense (do you understand)? How did you decide this? Why would that tell you to subtract? Any comments on the answer/method? Why? Why would you do that? What is happening? Are there patterns? Is there a different way you can do this?
Inquiry Argument	*Building with* synthetic-analyzing evaluative-analyzing (warrant) *Constructing* synthesizing evaluating	How are the 2 things the same? What is the same about each method? Does this make sense (is the method reasonable)? Why not? How do you know that? Why do you think that? Can you link all the ideas you found in some overall way? Does it always work? Is it always true? Why does this happen?

Responsibility for Participation

Figure 2

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STUDENTS' UNDERSTANDING OF PROOF BY CONTRADICTION

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Two hundreds and two students of 17~20 years old were surveyed on their understanding of proof by contradiction. Five abilities were identified for interpreting their understanding. A two-streamed model of understanding proof by contradiction was constructed statistically. To analyze the negating of a statement with quantifier 'only have one', interviews were conducted to reveal the relationship among the language used, Chinese or English, in their thinking process and their logical judgment.

INTRODUCTION

Understanding proof by contradiction shall mean to have both the procedural and conceptual knowledge of proof by contradiction. The procedure knowledge is: negating the conclusion q, and then inferring a mathematical fact or assertion that is contradicted to p. The conceptual knowledge is: "if ~q then ~p" implies "if p then q". This step, which is the principle of proof by contradiction, is based on the law of contrapositive. Historically, proof by contradiction is a method necessary in constructing mathematics systems. It was realized by today's mathematicians that mathematics systems might not be constructed if proof by contradiction has not been used in mathematics.

Young children may already have experiences informally of using reasoning with contradiction in their playing. It was found that 7~8 years old reasoned using contradiction in game playing and in checking conjectures (Reid & Dobbin, 1998.) However, the following three issues about the learning difficulties of this indirect proof method were notified in the related studies. These issues have motivated us to carry out this study.

The first is "When to use proof by contradiction?" Analyzed the interviews with six mathematicians about when they would think of using proof by contradiction, two criterions were mentioned: (1) the given conditions are not able or not easy to be manipulated; (2) The negation of conclusion reveal an obvious representation within a familiar system. These criterions are trivial to mathematicians, but are not familiar to senior high school students. For proving the irrationality of $\sqrt{2}$ is just their first experience of using proof by contradiction. Barnard and Tall (1997) studied the difficulties experienced by students of 16~19 years old on proving $\sqrt{2}$ is irrational. They highlighted six themes to show the difficulties. The initial one is the overall notion of proof by contradiction. This overall notion might include the explanations of 'what is' and 'when to use' proof by contradiction. Reid and Dobbin(1998) suggested that the difficulties students have with standard proof by contradiction in mathematics may arise from issues of the need from which their reasoning arises. When a student is asked to read the proof of $\sqrt{2}$ is irrational, what needs drives that proving? They argued that it is very rare that a need to verify comes into play. To feel a need to verify one must be uncertain of the result. In the case of $\sqrt{2}$, it is unlikely that there is any uncertainty at all.

Ability of Negating a Statement

Procedurally, negating a statement is the first task on processing proof by contradiction. Mathematical statements very often contain certain quantifiers, such as: all, only one, some of, don't exist, etc. Abilities of negating a statements with/without quantifier shall be one focus of this study.

Abilities of Reasoning Contrapositively

Zepp; Monin and Lei(1987) developed a test using implicational and disjunctive sentences for studying common logical errors in English and Chinese. They found that 54% of their university subjects were able to use the law of contrapositive in their reasoning on content-free tasks. Regarding to this facility, they argued that if the learning of logic principles depends on one's experience, it is possible that different logical principles may be learned for different situations. The logic in the mathematics classroom may be different from that in daily life environment. Thus, this study intended to develop items in both mathematical and daily life contexts for the investigation. In summary, this study aims for investigating abilities of negating a statement and their thinking process, recognizing the law of contrapositive, recognizing the procedure of proof by contradiction, and for constructing a model of understanding proof by contradiction.

METHODOLOGY

This study were conducted as following: (1) paper and pencil questionnaire; (2) individual and focused groups interviews; (3) field testing: In order to find a suitable model, items were analyzed. Percentile for each item was used to analyze the performance level of the samples. Factor analysis was used to analyze the abilities needed in the process of doing a proof by contradiction method. (4) proposing the model of proof by contradiction; (5) teaching experiment: After the model was proposed, a three hours teaching experiment were conducted to verify the model (Lin, F. & Cheng, Y. 1997. pp.557-591). The teaching experiment is not described in detail in this paper.

Item Development

Items in the questionnaire are grouped into three categories. Each category includes items in the context of daily life and the context of formal mathematics. Category 1: negating a statement. This is considered as a prior ability for proof by contradiction. Eight items belong to this category with different contexts and different quantifiers. Category 2: recognizing the procedure of proof by contradiction. Two items belong to this category. One asked students to recognize the procedure of proof by contradiction, the other asked students to state the procedure. Category 3: recognizing the law of contrapositive. Four items belong to this category, three are in the context of daily life, and one is in the pure mathematics context.

Interviews

Two kinds of interviews were conducted in this study. Individual interviews were conducted for testing the clarity of the items and exploring students' understanding. Interviews with focused groups were conducted to understand the thinking process of students and teachers related to negating a statement in daily life. During the interviewing process, we have observed that in the thinking process the language obstacles, Chinese or English, affected their performance. In order to clarify the relationship, we conducted some interviews in the ordinary class hours. Statement with the quantifier “only one” was used to interview a class of university students and a class of junior high school mathematics teachers in a summer program, and graduate students.

Subjects

One hundred and forty 11th grade students and sixty-two college students major in mathematics or mathematics education participated in the questionnaire survey. According to the college entrance examination data (Lin, 1991), the distribution of these students’ scores was similar to the distribution of the scores of the whole population of senior high school graduates in Taiwan. Seventy-one samples, mathematics majors or junior high school in-service teachers, participate in the group interviews.

RESULTS

Negating a Statement

Table 1 gives the correct frequency of two items and the classification of each question according to the quantifier terms and the context.

Table 1. Facilities on Items of Negating a Statement

Item No.	1-1	4-1	1-4	4-4	1-3	4-2	1-2	4-3
Quantifier	none	none	some	some	all	all	only one	only one
Context	Real	Pure	Real	Pure	Real	Pure	Real	Pure
Correct	93.0%	84.6%	82.1%	73.2%	49.5%	53.4%	19.8%	16.8%

The types of the quantifier affect the performances. About 90% were able to negate statements without quantifier; On the other hand, about 80% were not able to negate the quantifier “only one”, which is the hardest question in this test. The reason for this will be elaborated later. The followings are some statements with their most frequent errors we found in this study. For example, 40% students negated the statement “all people are my friends” as “no one is my friend”, and 16% students thought the statement “all three angle of the triangle ABC are acute” should be negated as “no angle of Triangle ABC is acute”. There are 55% students thought the negation of the statement “Engle has only one brother” was “Engle has more than one brothers”, and 54% students thought the negation of the statement “the graph of function f(x) intersected x-axis at only one point” was “the graph of the function f(x) intersected x-axis at more than one points”.

The effect of context is revealed when the items are easier. For the statements with the quantifier “some” and the statements without the quantifier, the correct frequency of items in the daily life context were about 10% higher than items in pure mathematics context. However, no difference was found for harder items, such as the statements with the quantifier “all” and “only one”.

Recognizing and Stating the Procedure

Procedural knowledge of proof by contradiction includes two steps: negating the conclusion, and inferring a result that is contradicted to the assumption or a known fact. Table 2 gives students’ response to the related items.

Table 2. Facilities on items of recognizing the procedure

Response type	Recognized proof by contradiction	Given correct explanation	Correctly stated the procedure

	(item 6-3)	(item 6-3)	(item 8)
Correct frequency	83.7%	36.6%	53.9%

About half of the students were able to write the procedure of proof by contradiction. Although 83.7% students could recognize a proof by contradiction, but only 36.6% students were able to give a correct explanation. There is about 20% students who were able to state the procedure but not able to apply it when it was needed. In addition, there were 6.9% students made the mistakes of writing the procedure of proof by method of exhaustion instead of proof by contradiction.

Recognizing and Using the Law of Contrapositive

The conceptual knowledge of proof by contradiction is the law of contrapositive. Students need to recognize that "IF P THEN Q" and its contrapositive "IF ~Q THEN~P" are equivalent. They need also to realize that "IF ~P THEN ~Q" and "IF QTHEN P" are not equivalent.

Table 3. Facilities of Items on recognizing the law of contrapositive

Item No	2	3-a	5-a	3-c	5-c	3-b	5-b	3	5	7
Context	Real	Real	Pure	Real	Pure	Real	Pure	Real	Pure	Real
Types of statement	~p=>~q	~p=>~q	~p=>~q	~q=>~p	~q=>~p	q=>p	q=>p			
Correct frequency	58.4%	49.5%	66.3%	53.5%	48.5%	73.3%	65.8%	24.7%	30.1%	23.7%

Table 3 gives the correct frequency of the four items and the classification of each question according to the type of the statement and the context. More than 70% did not have the conceptual understanding of proof by contradiction. They could not correctly recognize that "If P then Q" is only equivalent to "If ~Q then ~P", but not equivalent to "If P then Q" or "If ~Q then ~P". About 49% thought that each pair of the statements in the items was not the same. Only about half students had the necessary conceptual knowledge, thinking that "If P then Q" is equivalent to "If ~Q then ~P". This result is similar to the result in Zepp, Monin, & Lei (1987). Students with the misconception that "If P then Q" are the same as "If Q then P" is fewer than the above misconception. However, there are still 30% had this misconception. Table 3 also shows that reasoning contrapositively in the context of daily life was a little harder than in the context of pure mathematics.

Language, Thinking, and Logical Reasoning: -- example of negating "only one"

According to Nakamura's study about the thinking of Eastern species (__, ___ 1992), it was found that Chinese language emphasizes on individual, practical objects, and believe in the sense and intuition, but lacking the understanding of the general rules. Therefore, in Chinese language usually the facts were stated, and less attention was paid on the logical reasoning method and skills. On the other hand, western mathematics curriculum emphasizes on the logical reasoning and formal proofs. This difference might cause the obstacles in the thinking process for Chinese students in mathematics learning.

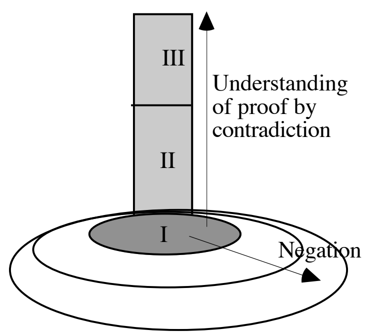
Analyzing students' responses to negating the term "only one", it was found that less than 20% were able to negate it. More than 70% used the term "none" or "more than one" to negate “only one”. In order to understand this more deeply, this study investigates the relationship between the Chinese language and the logical thinking. The sentence “I have only one brother” was used in the classroom interview. We found that the thinking models shows varieties affected by the words leading thinking or the thinking leading words. For those belong to “Words leading thinking”, they negated the statements by way of speaking or reading, and then translated the conclusion back to the spoken or read language. In this way, the concept of negating is first negating the Chinese words, and then explaining the meaning of the negated words in English or Chinese way. They might be thinking in English. For example, the negation of “I have” is “I don’t have”. So, the statement was negated as “I don’t have one brother”, which is “I have no brother”.Or they might be thinking in Chinese. For example, in Chinese, the words for “have only one” were in the order “only-have-one”. First added “not” to the statement with the term “only-have-one” was negated into “not-only-have-one”. However, the words “not-only” in Chinese means “more than one”, so the conclusion followed as “I have more than one brother”. For those belong to “Thinking leading words”, they negated according to the semantic of the term “have only one”, and then followed the meaning of the negated statement to form the conclusion. They might think in semantic way. In Chinese the semantic meaning of “have only” is “few”. Therefore, negating “have only” would follow as “more” and getting the conclusion “I have more than one brother”. Or they might think in Pseudo-semantic way. For example: (1) “have only one” means “have”, so the negation is “not have”. Students who followed this type of thinking got the conclusion “I have no brother”; (2) Under the assumption of “have”, some students negated the amount “one” and got the conclusion “I have more than one brother”; (3) “have only one” means “have just one”. Therefore, the conclusion is “I have no brother or have more than one brother”. Some student thought that “have only one” is related to a logical term. The Chinese term “have only one” is “exactly one”. So the conclusion is “I have no brother or I have more than two brothers”.

LEARNING FACTORS AND UNDERSTANDING MODEL

SAS pro factor was used to conduct the factor analysis. Only those factors with egenvalues higher than 1 were considered. It was found that there were five major factors that affect the understanding of proof by contradiction. The characteristics of each factors were analyzed and named them accordingly. **F1**: able to negate statements with simplest quantifier. Four items assessed the ability of “negating the statements without quantifiers or with the quantifier some”, in pure mathematics or real world situation. **F2**: able to recognize the law of contrapositive. Four items are related to “recognizing the law of contrapositive”, in pure mathematics or real world situation. Students who gave correct answers were not only able to recognize the equivalence relation of “p=>q” and “~q=>~p”, they were also able to recognize that “p=>q” were neither equivalent to “q=>p” nor to “~p=>~q”. **F3**: able to negate the statements with the quantifier ‘ only one’. Two items are related to “negating statement with quantifier only one”, in pure mathematics or real world situation. Students who correctly answered theses items 1-2 and 4-3 could overcome the obstacles of Chinese language and correctly negate the statements with the quantifier “exist only one”. **F4**: able to describe the procedural knowledge. Two items are related to “recognizing and stating the procedural knowledge of proof by contradiction". **F5**: able to negate the statements with the quantifier "all". Two items are related to this factor.

In order to establish the development model of understanding proof by contradiction, the method used by the research programe "Concepts in Secondary Mathematics and Science"(CSMS) is adopted (Hart(ed), 1998.) According to the item difficulties, items in this study can be classified into five groups, each corresponding to the five factors discussed in previous section. It was found that items with the item difficulty value higher than 70% were those items related to factor F1; items with the difficulty between 30% and 20% were related to factor F2; items with the difficulty below 20% were related to factor F3. Those items between 55%and 35% can be grouped into two according to the item characteristics and were related to factor F4 and F5 respectively.

This study combines the two dimensions M1 and M2 as a model of the development model of understanding proof by contradiction. Following figure gives a simple description of the model. M1 describes the development of the knowledge of proof by contradiction. F1-F4-F2 describes three levels of M1 from the bottom to the top. M2, which describes the development of the ability of negating statement, is considered as the basic ability of proof by contradiction. F1-F5-F3 described three levels of the abilities of M2.



CONCLUSION

From the result of our study, it might be reasonable to make conclusion about how students learn proof by contradiction. In summary, according to the field test, interview results, and teaching experiments, this study proposed a development model of the understanding of proof by contradiction. The first step of proof by contradiction is to negate the conclusion. After a student is able to negate a basic statement, he/she can begin to learn the procedure knowledge of proof by contradiction. However, only until a student understands the law of contrapositive, he/she will know why the procedure is finished. The ability of negating a statement might be developed unrelated to the understanding of the procedural knowledge of proof by contradiction. In order to verify our conjecture, a teaching experiment has conducted to check the understanding model of proof by contradiction. The result of that experiment shows that it is possible to help student to understand the method and apply it through the real world situations.

This study also found that the difficulty levels of students' negating a statement can be ordered decreasingly as negating statements without quantifier, negating "some", negating "all", and negating "only one". Besides, according to the result of the interviews, Chinese language might cause addition difficulty in negating statements with some quantifiers. In Chinese language, students negated statements either according to the words or according to the thought. It was found that students might have mistakes in negating a statement because the universal set (whole set) they used were different from the universal set usually used in mathematic. Since this study did not investigate the negating of “or” and “and”, it is unable to know whether the negating of statements with these two terms will form another level or include in one of the levels found in this study.

According to the field test, most the students in this study recognized the procedural of proof by contradiction, but only half of them can describe the process of proof by contradiction. Among those who described proof by contradiction method correctly, about one fifth of them could not apply it. More instruction design can be investigate based on this model to help students in understanding proof by contradiction and applying it.

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WHAT DOES 'POSITIVE' ATTITUDE REALLY MEAN?

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In this communication we analyze a dichotomy that pervades research on attitude in mathematics education: the classification of attitude as positive / negative. After highlighting the ambiguity of the term 'positive', and some problems related to it, we suggest that the positive / negative dichotomy is too simplistic to take into account the deep interaction between affect and cognition that characterises mathematics learning.

INTRODUCTION

In the field of mathematics education, research on attitude has always been motivated by the 'belief', common to mathematicians and mathematics educators, that '*something called "attitude" plays a crucial role in learning mathematics'* (Neale, 1969, p.631). More recently *attitude* is being considered together with *beliefs* and *emotion* to be one of the constructs that constitute the affective domain (Mc Leod, 1992). De Bellis & Goldin (1999) also include *values* in this domain, and view affect as the most fundamental, and most unrecognized in importance, of the internal representational systems. Even if the meaning of the various terms is not always agreed upon, or even made explicit (Hart, 1989; Pajares, 1992) there is consensus on the fact that emotions and beliefs deeply interact: on one hand, most emotions have a cognitive component (Mandler, 1984; Ortony et al., 1988), on the other, beliefs can have an emotional counterpart (this is the case for example of beliefs about self, such as self-efficacy beliefs, as underlined by Bandura, 1986). As regards attitude, an emotional component is generally explicitly recognised in this construct, often together with a cognitive component, mainly identified with beliefs.

Most researchers have underlined the need of some theory for research on affect, in order to make clearer the connections among the various components, and their interaction with cognitive factors in mathematics education (Mc Leod, 1992). Research on attitude has been judged to be particularly contradictory and confusing[1], due to the fact that it has given more emphasis to the creation of measurement instruments rather than to the elaboration of a theoretical framework (Kulm, 1980; Ruffell, Mason & Allen, 1998). In a previous work (Di Martino & Zan, 2001) we suggested that one of the reasons for this confusion is that a large portion of studies show a lack of a clear definition of attitude. Attitude tends rather to be defined implicitly and a posteriori through the instruments used to measure it (Leder, 1985; Daskalogianni & Simpson, 2000).

The problem of a lack of a clear and agreed upon definition is in the same way common to social psychology. The most recent theories agree on the multidimensionality of the construct, and make reference to a *tripartite model*, according to which attitude has a

[1] The work providing coherent information focused on the description of the differences between groups of people, usually males and females (Fennema, 1989).

cognitive, an affective, and a behavioural component (Eagly & Chaiken, 1998). Within the field of mathematics education many explicit definitions of attitude towards mathematics refer to this tripartite model (Leder, 1992; Ruffell et al., 1998), even if it is possible to find some explicit definitions according to which attitude is simply a general emotional disposition (Haladyna et al., 1983). The idea of attitude that emerges from this latter definition, which we called 'simple' (Di Martino & Zan, 2001) is not considered very significant by many mathematics education researchers, who underline the importance of linking a positive emotional disposition with an epistemologically correct view of the discipline (Ernest, 1988). Kulm (1980) suggests that '*It is probably not possible to offer a definition of attitude toward mathematics that would be suitable for all situations, and even if one were agreed on, it would probably be too general to be useful*' (p. 358). In this way, the definition of attitude assumes the role of a '*working definition*' (Daskalogianni & Simpson, 2000). This position sees the construct of attitude functional to the problems the researcher poses himself: we consider it to be useful in these terms in the context of mathematics education as long as it is not simply borrowed by the context in which it appears, i.e. the social psychology, but is characterised as an instrument capable of taking into account problems typical of mathematics education. This is in line with the position of Ruffell et al. (1998), who see attitude as an observer's construct.

Regardless of the presence of an explicit definition and its nature, the instruments used to assess attitude often make implicit reference to the tripartite model, in that they take into consideration beliefs and behavior as well as emotions. The assessment of each component opens significant problems: for instance the limitations of questionnaires (the most common assessment instruments) have been underlined. As regards beliefs, the mismatch between beliefs expoused and beliefs in action is well known (Schoenfeld, 1989). Many researchers have also underlined the necessity of taking into account not only single beliefs, but also how an individual's beliefs are structured and held, i.e. his/her belief systems (Pajares, 1992; Di Martino & Zan, 2001). Regarding emotions however, researchers have underlined the limits of instruments such as questionnaires and interviews in capturing emotional reactions that are not conscious (Schlöglmann, 2002).

But as well as these, the assessment of the interaction between the various components poses other problems, above all when this assessment takes the form of measurement, as happens in most studies both in mathematics education and in social psychology. Even if the instruments used appear increasingly sophisticated, the measurement generally results in a reduction to the favorable / unfavorable bipolarity, possibly obtained by summing points relating to the three dimensions: cognitive, affective and behavioral. While some scholars play down this operation, observing that 'the correlation among measures of the three components, although leaving room for some unique variance, are typically of considerable magnitude' (Ajzen, 1988, p.22), others consider this reduction as contradicting the recognised complexity of the tripartite model (s. Eagly & Chaiken, 1998).

THE POSITIVE / NEGATIVE DICHOTOMY

Synthesizing the evaluation of attitude in the favorable / unfavorable dichotomy recalls the dichotomy of positive/ negative attitude that pervades both implicitly and explicitly mathematics education research. For example, classic studies performed on the relationship between attitude and achievement (see for example the meta-analysis of Ma & Kishor, 1997) in fact investigate the correlation between *positive* attitude and success. In this way, studies aiming to change attitude end up in reality setting the objective of transforming a 'negative' attitude in a 'positive' one.

But is the notion of 'positive' attitude in mathematics education capable of taking into account the deep interaction among beliefs / emotions / behaviour that is the basis of the tripartite model?

In mathematics education just like in social psychology, the characterisation of attitude as positive is for the most part linked to a measurement and therefore score: in general (for example when instruments such as the Thurstone or Likert attitude-scaling techniques or the semantic differential technique are used) the score is obtained by summing points relating to the single items. The choice of scores to be assigned to the items naturally leads to a positive/negative type evaluation of each one. This characterisation is not so frequent in qualitative type studies, and when it is present, it is generally accompanied by a description of the factors (behavior, beliefs, emotions) from which it is obtained. In any case, the evaluation of a positive attitude brings us back to a positive evaluation of at least one of the components: emotions, beliefs, behaviour.

But even a partial analysis of literature highlights that the adjective 'positive' is used with different meanings, not just in different studies, but even within the same study. More precisely, this meaning varies depending whether 'positive' refers to emotions, beliefs, or behavior:

1) When it refers to an emotion, 'positive' normally means 'perceived as pleasurable'. So anxiety when confronting a problem is seen as 'negative', while the pleasure in doing mathematics is evaluated as 'positive'.

2) When it refers to beliefs, 'positive' is generally used with the meaning 'shared by the experts'.

3) When it refers to a behavior, 'positive' generally means 'successful'. In the school context, a successful behaviour is generally identified with high achievement: this naturally poses the problem of how to assess achievement (Middleton & Spanias, 1999).

But in reality the three meanings overlap. For example, in the case of beliefs, sometimes 'positive' means that it is supposed to elicit a 'positive' emotion. A typical case is represented by the belief 'Mathematics is useful', which is also used in questionnaires aimed at measuring just the emotional dimension of attitude (i.e. the 'simple' definition of attitude: s, Haladyna et al., 1983). But often 'positive' referred to a belief means that it is supposed to be related to a 'positive' behavior, i.e. to a successful behavior. Sometimes the latter meaning is also used for emotions, implicitly admitting that a 'positive' emotion toward mathematics, being pleasurable, is necessarily associated with a 'positive' behavior in mathematics. In reality various studies (Evans, 2000) suggest the possibility that for certain subjects, an optimal level of anxiety exists, above which, but also below which, performance is reduced. The problem is that generally the difference between the various meanings is rarely made explicit: in this way, an a priori assumption is often made as to what should in effect be the result of the research, for example, that a belief which is 'positive' because it is shared by the experts is associated with a 'positive' behaviour in that it is successful.

Depending on the criteria used to evaluate an attitude, it is therefore possible that there are different results: for example, an attitude can be evaluated 'positive' as regards the emotional dimension, but 'negative' regarding the cognitive dimension, or viceversa. This is what Hannula (2002) observes, describing the evolution of the attitude of Rita, a lower secondary school student: he underlines that, using the term 'attitude' in a traditional manner, *'in the beginning Rita had an 'attitude' that was negative and positive at the same time'* (p.42). The problem is only apparently overcome when the algebraic sum of the two components results in a single evaluation. Furthermore, as we have observed, beliefs are often used to assess the significance of the emotional dimension, or evaluated according to their 'behavioural' consequence, and this increases ambiguity.

SOME PROBLEMS RELATED TO THE POSITIVE / NEGATIVE DICHOTOMY

If the researcher does not make explicit his/her choices, it becomes problematic to interpret results obtained within a study, and to perform comparisons with different studies: we are not surprised therefore at the difficulty encountered when coordinating and interpreting widely different results collected in order to study the correlation between 'positive' attitude and achievement.

But even if this ambiguity is overcome by making explicit the choices made, in our opinion other problems remain.

The first is that the separate observation of beliefs and emotions does not allow researchers to pick up the interaction between the two constructs, in particular, the emotional dimension of beliefs and the cognitive dimension of emotions.

But trying to consider this interaction (as apparently is the case when a belief is evaluated according to the emotion that it elicits or to the associated behaviour) poses another problem: in evaluating a belief as 'positive' or 'negative' according to its emotional or behavioral component, we assume not only that a certain belief has an emotional component, but also the significance of that emotional component; not just that it is linked to behaviour, but also the type of behavior. In the end, we accept a cause/effect model according to which the same belief 'causes' the same emotion or the same behavior in all individuals.

The inadequacy of this model has emerged from a recent study (Di Martino & Zan, 2002) in which we set the problem of investigating the interaction between beliefs / emotions.

In this study, we considered some beliefs typically used to assess attitude toward mathematics, and considered 'positive' in that they are beliefs shared by experts.

Here we refer only to one of the two different questionnaires that we prepared, and to the first belief, i.e.: 'In mathematics there is always a reason for every thing'.

In questionnaire 1 we proposed the following formulation:

Choose the option you most agree with:
☐ In mathematics there is always a reason for every thing
☐ It is not true that in mathematics there is always a reason for every thing
And:
☐ I like ☐ I don't like ☐ I find indifferent
…this characteristic of mathematics.

If we call B the agreement with the belief ‘In mathematics there is always a reason for every thing’, and non B the agreement with the belief ‘It is not true that in mathematics there is always a reason for every thing’, we may combine these two possibilities[2] with the three options:

☐ I like ☐ I don’t like ☐ I find indifferent

By labelling these three cases respectively with +, — , and 0, we have the following theoretical possibilities:

(B, +); (B, -); (B, 0); (nonB, +); (nonB, -); (nonB, 0).

Our study highlights that these possibilities are not only theoretical. Questionnaire 1 was administered to 211 high school students (aged between 14-18), and we found the following data:

(B, +)	(B, -)	(nonB, +)	(nonB, -)	(nonB, 0)	(B, 0)	TOT
95	27	17	14	20	38	211

Therefore only 109 students (i.e. 51.7% of the sample) associated belief B with a positive emotion, and belief nonB with a negative one.

The two combinations (B, -) and (nonB, +) appear particularly interesting in this context, since there is a discrepancy between the emotional and the cognitive component of the belief B in these pairs, i.e. the emotional and the cognitive components are ‘opposite’. This discrepancy highlights the limits of observing these components *separately*, without taking into account *the interaction* between beliefs and emotions. In particular, as regards the (B, -) pair, traditional assessment of attitudes would interpret the agreement with a belief that is shared by experts as *positive*, even if it would interpret an emotion such as ‘I dislike it’ as *negative*. Vice versa the case (nonB, +) would be considered ‘positive’ according to the emotional component, but negative according to the cognitive one[3].

But above all, the real presence of all the possible pairs underlines that a certain belief can elicit different emotions in different individuals: this renders it impractical to evaluate a priori the emotional component of a belief as positive or negative.

CONCLUSIONS

The reduction to a positive / negative dichotomy in the characterization of attitude toward mathematics brings up many types of problems both on a theoretical and didactical level.

We have highlighted three problems at a theoretical level, linked on one hand to the fact

[2] There is also the possibility that the subject has neither belief B nor belief nonB. Of course, this possibility is not easily detectable using questionnaires.

[3] These considerations refer only to the process of assessment; regarding interpretation, we suggested several possibilities for such a discrepancy, for example that the emotional component + or – may be directly linked to belief B, but also indirectly, through interaction with other beliefs, such as beliefs about self (for example ‘I am not able to understand these reasons’). From a didactical point of view this difference is important, since it suggests different kinds of intervention.

that the adjective ‘positive’ is used with different meanings, and on the other to the subjective nature of the interaction between emotions / beliefs:

- The fact that there can be different meanings to the word ‘positive’ relating to attitude poses a problem of ambiguity. This underlines the necessity that the researcher makes explicit his/her choices (Zan & Di Martino, 2003) to permit communication, particularly in the interpretation of results and confrontation with other research.
- The fact that the criteria used to evaluate the emotional component and the cognitive component are different, and can even result in contradictory evaluations (see the study cited above in the cases (B, -) and (nonB, +)), highlights the problem of managing to consider both of the components in a single evaluation. It also highlights the limits of a positive/ negative evaluation which makes reference to a single component. This type of evaluation can only be viable in our opinion in the case of particularly simple problems such as those encountered when predicting the choice of mathematics courses, for which it can be sufficient to refer simply to the emotional dimension.
- But above all, the subjective and complex nature of the cognition / emotion interaction highlights the limits of a *normative* model for attitude, which searches for universal laws and rules of behaviour, subject to a cause and effect explanation (s. Cohen & Manion, 1994).

This brings us to the other problem: the inadequacy of the positive / negative dichotomy from a didactical point of view. This dichotomy hides extremely different situations and needs under the same label, when they in fact require different interventions. For example, the negative emotional disposition of a pupil who sees mathematics as made of nonsensical rules requires a completely different intervention compared to the negative emotional disposition of a pupil who has an ‘epistemologically correct’ view of the discipline, but a low sense of self-efficacy.

In conclusion, the positive / negative dichotomy appears inadequate to confront the complexity that characterises the interaction between emotions/beliefs, and therefore also the more significant problems that the context of mathematics education presents. In order to account for this complexity, we consider that it is necessary to overcome this reduction, using a multiple approach to assess attitude, privileging the description in context, as performed by some researchers (Leder, 1992; Ruffell et al., 1998; Daskalogianni & Simpson, 2000; Evans, 2000; Karsenty & Vinner, 2000; Hannula, 2002).

According to this interpretive approach, attitude becomes ‘a construct of an observer’s desire to formulate a story to account for observations’, rather than ‘a quality of an individual’ (Ruffell et al., 1998, p. 1): a construct useful to understand motives of intentional actions, rather than to explain the causes of behaviour.

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TRANSLATION OF A FUNCTION: COPING WITH PERCEIVED INCONSISTENCY

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A horizontal translation of a function is the focus of this study. We examine the explanations provided by secondary school students and secondary school teachers to a translation of a function, focusing on the example of parabola $y=(x-3)^2$ *and its relationship with* $y=x^2$*. The participants' explanations focused on attending to patterns, location of zero of the function and pointwise calculation of the function values. The results confirm that the horizontal shift of the parabola is, at least initially, inconsistent with expectations and counterintuitive to most participants. We articulate possible sources of this perceived inconsistency and describe a pedagogical approach aimed at resolving it.*

Imagine the graphs of (a) $y=x^2$ and (b) $y=(x-3)^2$. Check your image with that of a graphing device. If you're not surprised, it's probably because you've already explored the relationship between the two parabolas in the past. Most people conjecture that graph (b) will appear 3 units to the left of graph (a). The surprising result is that (b) is actually 3 units to the right of (a).

As students, we have memorized this result as one of the counterintuitive facts of mathematics. As teachers, we have attempted to explain this phenomenon to our students, often creating a conflict between their intuition and our explanations. As researchers, we became interested in how teachers and students deal with the phenomenon and what explanations they provide to themselves and to others. Moreover, we wished to understand possible sources of students' "wrong" intuition and sought pedagogical solutions.

BACKGROUND

Teaching, learning, or understanding of functions has been an important focus in mathematics education research in the past decades. However, within a variety of research reports focusing on functions, only minor attention has been given to the transformations of functions. A common treatment of transformations of functions in pre-calculus courses involves a consideration of a graph of a function f(x) on a Cartesian plane. Functions f(ax), f($-x$), f(x)+k and f(x+k) correspond to a dilation, reflection, vertical translation and horizontal translation of f(x) respectively.

Eisenberg and Dreifus (1994) conducted an extensive exploration on students' understanding of function transformations, focusing on visualization. They recognized the difficulty in visualizing a horizontal translation in comparison to a vertical one, suggesting that "there is much more involved in visually processing the transformation of f to f(x+k) than in processing the transformation of f to f(x)+k " (p.58). Baker, Hemenway and Trigueros (2000) have investigated understanding of transformations of functions and confirmed that vertical translations appear easier for the students than horizontal translations. Further, students' difficulty with function transformation was attributed, at least in part, to their incomplete understanding of the concept of function.

Though the difficulty presented by a horizontal transformation of a function has been acknowledged in prior research, there has been no attempt to investigate how students and teachers cope with this difficulty and how it is possible to overcome or at least to reduce this difficulty; these are our goals. We examine the understanding of horizontal translation of functions in general, and of parabolas in particular. We explore how participants cope with the difficulty presented by a horizontal translation and suggest a pedagogical approach to transformations of functions that addresses the "mystery" of graphs "moving in unexpected directions".

METHODOLOGY

Participants in this study were preservice secondary teachers (n=15), practicing secondary teachers (n=16) and grade 11/12 students (n=10). The teachers were volunteers attending courses at the University in which this study was conducted. The students were volunteers referred by the participating teachers, all classified as having "above-average" ability. All the participants were asked to predict, check and explain the relationship between the graph of $y = x^2$ and the graph of $y = (x - 3)^2$. In addition, if the issue did not come up naturally in the conversation, the participants were asked to discuss the graph of $y = x^2 - 3$ and compare it to $y = (x - 3)^2$. The analysis of responses attended to (1) common trends in explanations, and (2) attitudes towards perceived inconsistency.

RESULTS

The fact that the shape of $y = (x - 3)^2$ is a parabola that is congruent to the canonical parabola y=x2 was taken for granted by teachers and students alike. The fact that the graph of $y = (x - 3)^2$ is a horizontal translation of $y = x^2$ was also mentioned as requiring no further explanation. Therefore, we turn our focus now to the direction of this horizontal translation.

1. Students' responses

The participating students had learned about transformations of quadratics (conic sections) before this study was conducted. Nevertheless, half of the students predicted the spatial location of $y = (x - 3)^2$ incorrectly. There was no sincere attempt on students' part to explain the phenomenon, though all admitted that the observed location of the parabola was counterintuitive. The following conversation with Mitch is rather representative of students' responses. The interview took place after Mitch incorrectly sketched the graph of $y = (x - 3)^2$.

Int: Would you please check this.

Mitch: No problem. (pause) Yes problem. I should have remembered this. It moves in the wrong direction.

Int: What do you mean, wrong direction?

Mitch: It's minus 3, so you sort of expect it to move left. But it moves the other way. It moves right.

Int: Oh, and why is that?

Mitch: That's the way it is. It's always like that. I really knew that. Just remember to do the opposite. But not always. ... Only if your number is in the brackets.

Int: Please explain.

Mitch: OK. When you do something to x in the brackets, like x–3 or x+2 your graph moves the opposite way. For x+2 it moves left, 2 times. For x–3, the one you wanted, it moves right, 3 times. And once you remember that, it works for all the graphs.

Int: You mentioned something about brackets.

Mitch: Yes, it moves the opposite way only when there are brackets. Without brackets, say for $y = x^2 + 3$, written this way, it moves the way you want it to move. This one will move up, and with negative-3 it will move down. This one is doing exactly what it should.

Int: So the other one is doing something it shouldn't.

Mitch Not really, just not what one would guess, but the other way around.

Int: Does it bother you?

Mitch: Not really. You just see on the calculator what it's doing and you know it will do it all the time.

Int: I wonder why the graph [of $y = (x - 3)^2$] moves the way it moves. It's not what I expected, it's not what you expected, at least initially. Can you help me understand this?

Mitch: (pause). Maybe try small steps. Try $y = (x - 1)^2$. You see, it goes right one step. And try $y = (x - 2)^2$. It will go right 2 steps, you see them together here [demonstrates with graphing calculator]. So after seeing this you do not expect (x–3) to do something different, do you?

Int: Is this how your teacher explained it to you?

Mitch: I'm not sure he explained this at all. But graph it once and you know how it works.

In examining Mitch's response we note the initial confusion, which is rapidly corrected based on the feedback from the graphing calculator. The claim "I should have remembered" demonstrates that the phenomenon is not new to Mitch, he has encountered this behavior of graphs earlier and expects to base his knowledge on this recollection. The claim "It moves in the wrong direction" and further clarification on what is meant by "wrong direction" presents a conflict between the result and the expectations; however, these expectations have been adjusted based on experience. Further, Mitch generalizes his experience of observing translation of the parabola ("once you remember that, it works for all the graphs") and also limits the scope of this observation ("only if your number is in the brackets"). Mitch explains that while the vertical translation matches his expectations, as "it moves the way you want it to move," the case of the horizontal translation in his approach is "just remember to do the opposite". However, Mitch is not troubled with the apparently acknowledged counterintuitive behavior of $y = (x - 3)^2$. His main concern appears to be with remembering this behavior, rather than with understanding it. Responding to the interviewer's quest to understand the location of the function, Mitch suggests the consideration of a pattern. Observing graphs of $y = (x - 1)^2$ and $y = (x - 2)^2$ implies the location of the graph of $y = (x - 3)^2$. This explanation is consistent with his strong belief in internal consistency of this pattern – "graph it once and you know how it works".

Remembering "to do the opposite" was a common strategy for coping with perceived inconsistency. There was no desire on the part of student-interviewees to understand or explain the phenomenon. When an explanation was requested, it relied on rules to be memorized or on generalizations from attending to patterns.

2. Teachers' responses

All the teachers participating in this study have sketched the graph of $y = (x - 3)^2$ correctly. However, for the practicing teachers it was an immediate and effortless recall from memory, the way one would recall, rather than derive, a multiplication fact 7x9. It was evident that for some preservice teachers the horizontal translation of a parabola was not in their immediate repertoire of knowledge, but the location of the graph was derived correctly and without major effort.

Compared to uniformity in students' tendency to rely on memorized rules, there was more variety in teachers' responses to the interviewer's request to explain the move of the parabola. We summarize below several themes that emerged in teacher's responses.

Citing rules. About one half of the teachers interviewed referred to the "rule of horizontal translation". According to this "rule", $y = (x - 3)^2$ has the same shape as $y = x^2$ but is located 3 units to the right. Thus, teacher's reliance on rules is indicative of students' reliance on memorizing these rules.

Pointwise approach. Plugging numbers into the equation and then plotting the points seemed more convincing for teachers than simply accepting what the computer or graphing calculator was showing. Some teachers explained that they saw advantage in using the point-by-point creation of $y = (x - 3)^2$ as an explanatory tool for their students. Interestingly though, a pointwise approach was not mentioned by any of the students. It appears that the utility and availability of graphing calculators and the lack of extensive experience with creating graphs manually, point by point, influences students' perception of graphs and suggests that the convincing power of teachers' explanations needs to be reexamined. Moreover, the influence of graphing calculators and graphing software on participants' explanations deserves the attention of further research.

Attending to zero and "making up". Another common explanation was to find the zero (x=3) of the parabola and imply that the rest of the points are "symmetrically determined" around it. Those teachers were prompted to explain in what way the location of zero would determine the location of the rest of the points. In most cases preservation of shape and symmetry were put forward as justifications. However, several teachers presented variations on the following explanation: "to get to a particular y-value, the x-value must be 3 greater because 3 is being subtracted". This was usually an extension of attending to the zero of the function and an attempt to explain "what happens to other points on a graph is the same as what happens to zero", without an explicit calculation. It was noted that explanations similar to these can help in justifying what happens after the result of translation is presented, rather than help in predicting the result.

A search for consistency. It has been acknowledged that the direction of translation of parabola is inconsistent with intuitive expectations. This creates a dissonance for both students and teachers. This dissonance is further aggravated by the fact that the vertical translation operates "as expected", that is, the graph of the function + 3 is a vertical upward translation by 3 units of the graph of $y = x^2$. Therefore, the learners face not only a counterintuitive behavior of functions, but also inconsistency in the fact that some translations act according to their expectations while others do not.

In summary, teachers provided a variety of explanations, most common of which were citing the rules, considering the function pointwise and attending to the zero of the function. Most teachers were not completely satisfied with their explanations, but claimed to "have never seen a better one". Preservice teacher's responses differed from responses of practicing teachers on two accounts. The first is the ease of recall, acknowledged early in this section. Automatic and fluent retrieval is considered to be one of the indicators of expert knowledge (Bransford, Brown & Cocking, 2000), and therefore it is not surprising that practicing teachers exhibited better expertise in the subject matter they taught than preservice teachers. Second, as expected by assuming expertise not only in content but also in a pedagogical content knowledge, practicing teachers, had a better understanding of the perceived inconsistency and of the problematics that a horizontal translation presents to a learner. However, there was no significant difference between the attempts of practicing teachers to explain the translation and the attempts of preservice teachers.

ANALYSIS: COGNITIVE AND PEDAGOGICAL OBSTACLES

The results indicate that the ability to determine correctly the direction of horizontal translation does not imply understanding of this function transformation and the ability to justify it. From the perspective of a learner, the behavior of functions is counterintuitive. From the perspective of an observer, the learners' intuition is misleading and it presents an obstacle to their understanding. In order to understand the sources of this intuition we turn our attention first to the notion of an obstacle, cognitive and epistemological.

The notion of an epistemological obstacle was introduced by the French philosopher and scientist Gaston Bachelard. Bachelard described the development of scientific knowledge as constrained by intrinsic, and at times unrecognized, factors associated with the process of understanding, rather than by the complexities of the phenomena. Herscovics (1989) reserved the term "epistemological obstacles" for obstacles that are encountered in the development of knowledge in a discipline, while denoting the obstacles in the conceptual development of an individual learner as "cognitive obstacles". Sierpinska (1994) considered a broader interpretation of epistemological obstacles and described them as obstacles to some change in the frame of mind, that can be attributed to the historical development of knowledge in a discipline, as well as to the knowledge development of an individual learner. One of the constraints identified by Bachelard as a potential epistemological obstacle is possibly deceptive intuitions (Norman & Prichard, 1994).

Turning to translation of functions, learners' experiences with numbers suggest that adding 3 results in moving in the positive direction (right), while subtracting 3 (or adding negative 3) results in moving in the negative direction (left). On a surface, such a view is an overgeneralization of prior experience of adding and subtracting numbers on the number line. Examining the issue closer, we believe that a possible source of an obstacle is in trying to see (b) $y = (x - 3)^2$ as a transformation of (a) $y = x^2$, and in doing so creating confusion between the function defining parabola and the function of transformation. Students' focus is on the algebraic representation of functions. Comparing the algebraic representation $y = x^2$ and $y = (x - 3)^2$, students notice, correctly, that (b) is derived from (a) by substituting (x–3) in place of x. As mentioned earlier this interpretation is extended, mostly intuitively, to the view of the number line: for any number x, x–3 is located 3 units to the left of x. We suggest that the main source of difficulty here is in seeing this algebraic replacement as a transformation (x moves to x–3) and trying to imply the geometric transformation, the movement of the graph, from the algebraic substitution. That is to say that the transformation $f(x) \mapsto f(x-3)$ is simplified to be viewed as $x \mapsto (x-3)$. Such a view is in accordance with what Hazzan (1996) described as reducing abstraction, which is a strategy used by learners to cope with complexity. In our example students' attention is on an object of a variable (x) rather than on a more abstract object of a function f(x).

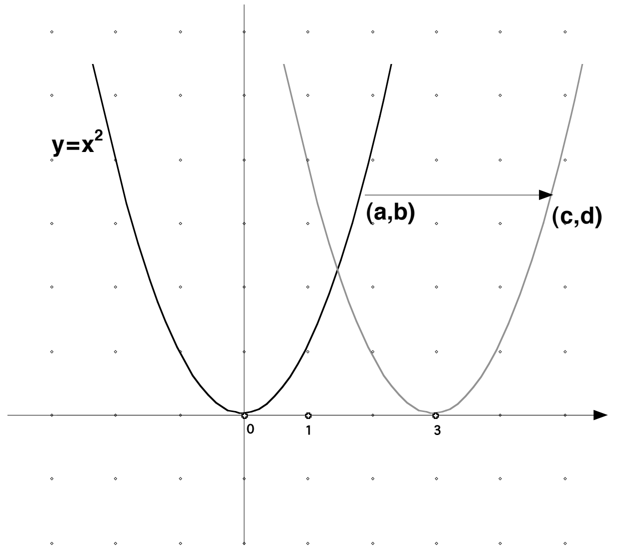
Though the studies of Eisenberg and Dreifus (1994) and Baker et. al. (2000) agree on the difficulty presented by horizontal translation, they differ in a way of explaining this difficulty. Eisenberg and Dreifus see the difficulty in visual processing of information. Baker et. al. attribute the difficulty to the complexity of mental construction needed to process a horizontal transformation. We described above how students' difficulty can be attributed to a cognitive-epistemological obstacle and to human tendency to reduce the abstraction level when facing complexity.

However, there is another possible source of the difficulty, and it is in the instructional sequence in which the discussion of transformation of functions takes place. Students' and teachers' focus on algebraic representation is not their fault and may not be their choice. This choice is prescribed by a traditional curricular approach to function transformations. This approach embeds the discussion of transformations of parabolas, quadratic relations, or polynomial functions in general in the context of functions, rather than in the context of transformations. This standard pedagogical approach could be reinforcing the obstacle, rather than trying to overcome it. Our belief is that the problem can be avoided, or at least minimized, by a pedagogical approach that situates transformations of functions in the context of transformations. We outline this approach below.

PEDAGOGICAL APPROACH

In situating transformations of function in the context of transformations, rather than functions, we focus on translations. Translation on a plane is defined by a vector (or directed segment) that specifies the direction of the motion and the distance. It is our experience that students tend to identify translations by breaking this motion into its horizontal and vertical components. This view leads to the natural introduction of the formal notation for translation $(x, y) \mapsto (x+a, y+b)$ or $T((x,y)) = (x+a, y+b)$, where a and b are horizontal and vertical components of the motion respectively. This function notation for a transformation is often referred to as a mapping rule.

Connecting mapping rules and resulting translations, students will identify positive and negative values of 'a' with motion to the right or to the left respectively; positive and negative values of 'b' with motion up and down respectively. Furthermore, they will associate the strict horizontal motion with b=0 and strict vertical motion with a=0. Since any set of points can be translated according to the mapping rule, this set of points can be a parabola. In particular, applying $T((x,y)) = (x+3,\ y)$ on a canonical parabola $y = x^2$ represents a horizontal translation by 3 units to the right.



$T((x,y)) = (x+3,\ y)$ applied to graph of $y = x^2$

The task now becomes connecting the translated image to its algebraic representation. Recall that the set of points of the source is described by $y = x^2$. Without loss of generality, focus on a point (a,b) of the source set that was translated to the point (c, d) of the image set. According to the specific translation performed, $d=b$ and $c=a+3$. We wish to connect c and d in an equation. Relating c to d we obtain the following: $d=b;\ c=a+3$ which implies $a=c–3$. However, $b = a^2$, as (a,b) is a point on the source parabola. Substitution leads to $d = (c - 3)^2$. Since the above is true for every point of the image set, the image of the translation is described by the equation $y = (x - 3)^2$. (Of course, one can work directly with x's and y's, but switching to a's b's and c's can be helpful for students). This explains the "unexpected" appearance of "–3" in the horizontal translation to the right.

Consideration of a geometric transformation as a starting point is not limited to translations. We envision an approach in which students explore a variety of transformations, making a connection between the conventional definition of a transformation and its effect on specific sets of points on the plane. Such sets could be simple geometric shapes in the initial stages of exploration and graphs of specific functions or relationships in later stages.

The goal of our pedagogical approach is not to change students' minds about the direction of horizontal translation. This is done successfully with the use of graphing devices, either electronic or manual. Our goal is to balance the dissonance presented by the discord between initial intuition and graphical image.

Future research will determine whether the intervention or instructional sequence presented above is helpful for students' understanding of the translation of parabola in specific and of any function in general. So far, we explored such intervention on a smaller scale, presenting it to teachers as an alternative pedagogical approach. The participating teachers referred to this view of transformations as "an eye opening clarification" or a "pedagogical AHA!".

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TEACHERS' CHARACTERISTICS: INTRODUCING INVESTIGATIONS TO YOUNG CHILDREN

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This study explored the thinking exhibited by two teachers as they implemented a mathematical activity that they had designed for the purpose of introducing young children to investigations. Video data of these activities were analyzed for the distinguishing teaching characteristics of individuals and for the quality of specific teaching characteristics. The analysis revealed substantial differences in individuals' distinguishing teaching characteristics. Furthermore, the use of a lesson evaluation instrument also demonstrated differences in the quality of their specific teaching characteristics. These differences suggest that the teachers' implementations created considerable variation in the learning opportunities for their classes. Further avenues for studying the teaching of mathematics to young children are also proposed.

Mathematical investigations have been advocated for children because they provide opportunities for them to develop thinking skills and content knowledge (Baroody & Coslick, 1998). However, there is scant knowledge about young children's learning from investigations or the ways that teachers can support their learning. The implementation of inquiry-based approaches results in a radically different curriculum (Taber, 1998). Hence, my colleagues and I have explored some of the key issues in implementing mathematical investigations with children of seven to eight years. These issues include: the types of tasks that promote investigation, how children learn from investigations and the difficulties they encounter, and how teachers can support or inhibit students' reasoning in investigations (Diezmann, Watters, & English, 2001a, 2001b, 2002). The finding that teachers can inhibit students' reasoning highlights the need to understand teachers' thinking about the implementation of investigations. Investigations require teachers to reconceptualize the nature of mathematics and to teach mathematics in new and different ways (Taber, 1998). Teacher's thinking about how to create learning opportunities for students can be explored by studying teaching characteristics (Doerr, 2002; Doerr & Lesh, 2002). Thus, the purpose of this paper is to explore the characteristics of teachers as they implement an investigatory activity with young children. This should provide some insight into teachers' thinking about this unfamiliar aspect of teaching.

THE LEARNING POTENTIAL OF INVESTIGATIONS

Mathematical investigations should be challenging and motivating (Greenes, 1996):

> Investigations present curiosity provoking situations, problems, and questions that are intriguing and captivate students' interest and attention. (p. 37)

Thus, investigations support the development of thinking in two ways. First, they provide students with the opportunity to learn mathematics in context, which has cognitive and motivational advantages (Brown, Collins, & Duguid, 1989). Second, investigations enculturate students into mathematics practices, which empower them to discover, invent and use mathematics to understand the world (Lappan & Briars, 1995).

learn from reading (Brown & Campione, 1991). While children are learning how Patterns and relationships can be explored through problem solving (Romberg, 1994), representations (Goldin, 1998), physical and thought experiments (Simon, 1996), and reasoning (Russell, 1999). As context and culture are significant factors in the teaching of thinking (Sternberg, 1994), investigations are ideal for nurturing thinking.

Teachers can support students' thinking in investigations through their selection and implementation of mathematically challenging tasks. A fundamental goal of an effective task is to "stretch" all children's thinking irrespective of their current capabilities (e.g., Diezmann, Thornton, & Watters, in press). The challenge of a task is not fixed but can be moderated at any point in its "life" from its selection, through to its announcement by the teacher, its implementation, and finally, to the products accepted by the teacher (Henningsen & Stein, 1997). Thus, the learning potential from a task is influenced by a teacher's ability to maintain adequate challenge throughout its life.

The implementation of investigations with young children needs to accommodate their limited content knowledge, and generally, their unfamiliarity with investigations. Children's content knowledge is typically supported through the use of stories to contextualize mathematical situations (Whitin, 1994) and manipulatives, which provide concrete referents for abstract mathematical ideas and relationships (Hartshorn & Boren, 1990). Although some guidance is provided in implementing investigations (e.g., Baroody & Coslick, 1998), there are no well-established ways to support children's learning from investigations. One of the difficulties that young children confront when beginning investigations is their lack of understanding of <u>what</u> the problem to be investigated is and <u>how</u> to explore that problem (Diezmann et al., 2001a). Thus, before students can learn from investigations, they need to learn how to investigate _ just as students need to learn how to read before they can to investigate, their teachers are learning how to support them to investigate.

STUDYING CLASSROOM TEACHING

The assessment of teaching characteristics within a lesson is problematic. Brown et al. (2001) developed a classroom observation instrument based on previous work by Saxe (1991) and their own work in a large five-year Numeracy study. This instrument focuses on four key teaching characteristics of effective mathematics lessons, namely Tasks, Talk, Tools, and Relationships and Norms. Each of these characteristics is comprised of various components. For example, Tasks comprises (1) mathematical challenge, (2) the integrity and significance of the mathematical tasks, and (3) children's interest in the task (See Table 1). Brown et al.'s instrument describes four levels for evaluating the quality of these components. However, they concluded that this instrument had shortcomings due to its low predictive power for student attainment scores and argued that the instrument failed to account for the human factor _ the teacher-class relationship. However, a further factor that seems likely to impact on predictive power is the validity of test scores as a measure of the learning that occurs when teachers emphasize Tasks, Talk, Tools, and Relationships and Norms. This issue is beyond the scope of this paper. However, Tasks, Talk, Tools, and Relationships and Norms are variously argued to be important in creating rich and supportive learning environments for investigations (e.g., Baroody & Coslick, 1998; Greenes, 1996). Thus, evaluation of these specific teaching characteristics should provide some insight into the "learning potential" of an environment, which aims to promote learning through mathematical investigations.

Table 1: Specific teaching characteristics and their components.

Tasks
1. Mathematical challenge for all pupils
2. Integrity and significance of the mathematical tasks in the lesson
3. Engage interest in the mathematics of the lesson
Talk
1. Teacher talk that focuses on mathematical meanings and co-construction of knowledge
2. Teacher-pupil talk about mathematics
3. Pupil talk that focuses on reasoning and mathematical understanding
4. Management of talk to encourage pupils to talk about mathematics
Tools
1. Range of modes of expression including oral, visual, and kinaesthetic
2. Types of models used to represent mathematics ideas
Relationships and Norms
1. Community of learners comprising teacher and pupils
2. Empathy towards pupils' responses

(summarised from Brown et al., 2001, p. 14)

Doerr (2002) has also highlighted the importance of the human factor in the understanding of teaching characteristics. She argues that the identification and understanding of the distinguishing teaching characteristics of an individual implementing a particular lesson with a specific class within a realistic context can provide insight into that teacher's thinking. Thus, Doerr's (2002) approach can accommodate individual (teacher), contextual (class/school), relational (teacher-class), and situational variability (lesson). Brown et al.'s (2001) instrument provides the means to examine the situational variability of teaching in greater depth.

The analysis of a single individual's teaching characteristics is not ideal due to "cultural blindness" to these characteristics. The term "cultural blindness" is used here to describe the situation noted by Hiebert and Stigler (2000) where aspects of teaching are so common that they are "invisible" to members of the culture including teachers themselves. Doerr (2002) accounted for this blindness to some extent through her comparison of the characteristics of more than one teacher. As this blindness results from high familiarity with the tasks of teaching, novel tasks should make the teaching characteristics of an individual more visible both to that individual and to observers. Thus, teaching characteristics in the implementation of an investigatory activity should be most apparent when more than one teacher is implementing a novel activity. Additionally, the use of Brown et al.'s instrument may provide particular insight into the quality of specific teaching characteristics that are important in investigations.

DESIGN AND METHODS

This research is part of a case study on teachers' implementation of mathematical investigations with young children (Diezmann et al., 2001a, 2001b, 2002). The study reported here explores teachers' characteristics in implementing a story-reading activity, which was used as an introduction to a ten-week program of mathematical investigations. This particular activity was chosen for study because (1) teaching characteristics are overt in a story-reading activity, and (2) children's literature is promoted as a means of engaging students in meaningful mathematical thinking (Whitin, 1994). The participants were two teachers, who taught in the same large outer metropolitan school. Ms I and Ms U each had in excess of ten years teaching experience and taught comparative mixed ability classes with 25 and 26 students respectively. Each week the teachers implemented a 90-minute session of mathematical investigations. The teachers and I met regularly to discuss, plan, and debrief the investigations program. Thus, these sessions were consistent in many ways with "lesson study" (Hiebert & Stigler, 2000). The data reported here are observational lesson data collected by three strategically focused video cameras monitoring the teacher, the whole class, and salient events. These data were analysed for emergent themes of distinguishing teaching characteristics (Doerr, 2002), and subjected to an evaluation of specific teacher characteristics (Brown et al., 2001). Only limited data are presented here due to space limitations. However, teacher interviews, and teacher and researcher notes supported the interpretation of video data.

Prior to the first lesson, both teachers participated in a planning session and decided to use "The Doorbell Rang" (Hutchins, 1986) to introduce their students to investigations. Both teachers agreed that this story provided an ideal context for an introductory investigation and was mathematically relevant for their classes. This story commences with a small number of children sharing out a batch of 12 cookies. The doorbell then rings announcing the arrival of more children and a subsequent need to re-share the cookies. This story line repeats until there are 12 children present to share 12 cookies. The climax of the story occurs when the doorbell rings again but this time it is Grandma with another batch of 12 cookies. The mathematics in this story includes division, multiples of 12, doubling, and inverse relationships (children and cookies). At the culmination of the planning session, it appeared that both teachers would implement this activity similarly with children acting out the story using real cookies.

RESULTS AND DISCUSSION

This section presents a brief description of how teachers implemented the activity and the three most distinguishing characteristics of their implementation. Only three characteristics are discussed due to space limitations. The evaluation of the participants' specific teaching characteristics is then presented.

Teachers' Implementations and Distinguishing Characteristics

<u>Ms I</u> selected children for the characters in the story and directed them to a small group of tables where the cookies were to be shared. The remainder of the class was seated nearby on the carpet. Ms I provided an overview of the story and directed "mother", to collect the tray of cookies. She paused each time a new group of children "arrived" in the story and waited until the nominated children acted out their arrival and shared the cookies. The actors had a good view of events, however, the children who were seated on the floor had difficulty seeing. This activity culminated with Ms I asking whether <u>all</u> children in the class had a share of the cookies, instructing some children to share their

cookies with those who did not have one, and encouraging the class to eat their cookies. Ms I also advised children that they had participated in a mathematical investigation.

The three most distinguishing characteristics of Ms I's activity implementation follows.
A. The goals of the activity were implemented at a superficial level. Ms I's focus from commencement to conclusion was to faithfully act out the story. For example, she provided the child chosen as "mother" with a broom for sweeping. After the story concluded, Ms I checked whether all children had cookies and instructed some children to share their cookies with those who had none.

Ms M: Have we all shared the cookies?"

Class: Yeeeeeees.

Ms M: We haven't shared yet! … Lily doesn't have one and Ed doesn't have one.

B. The mathematics was self-evident. During the planning session prior to this activity, Ms I endorsed the selection of this book for its wealth of relevant mathematics for her class. However, she did not capitalize on the substantive mathematics in the storyline. The only mathematics in which students engaged were simple counts of the number of children, and one-to-one correspondence between children and cookies.

C. Investigations were straightforward, and fun. Ms I's assumed the role of the director and narrator in the acting out of this story. She organized the props, selected the actors, and cued them when to enter. Ms I's scripted approach implied that investigations were straightforward rather than ill-defined and complex. The eating of cookies at the end of the story created a sense of fun. However, this physical enjoyment was unlike the mental satisfaction that results from the successful completion of a challenging task.

<u>Ms U</u> organized the class to sit in a large circle, asked for volunteers to act out the story, and directed the actors to a nearby location from which to enter at the appropriate times. She questioned the remaining students about "how many" people were seated at the table in the story and laid a square tablecloth inside the circle of children. She then read parts of the story frequently pausing to ask a range of questions. These questions included general recall _ "What did Ma say?" _ and prediction _"What do you think will happen next?" Ms U also incorporated many mathematically-oriented questions throughout the story reading that ranged from simple questions _ "Have they got the same amount?" to more complex questions _ "How did that (when there were four children to share the cookies) compare to when Zeb and Cia were by themselves?"

Children had opportunities to engage in thought experiments and physical experiments by predicting the outcomes of the sharing and by acting out the story. Prior to the conclusion of the story, there was a groan when "the doorbell rang" again and the 12 children each only had one cookie remaining. Ms U concluded the activity with a discussion about the mathematical situations throughout the story. She then posed the question of whether what they had done was a math investigation. Children recorded their responses, which were discussed later. At lunch, the children ate the cookies.

The three most distinguishing characteristics of Ms U's activity implementation follows.
A. The role of the teacher was to stimulate and support children's thinking. Ms U cued the children to think about the mathematical situations. For example, her mathematical

questions built on from each other — "How many do you each have?", "How did they share this time?", and "What did they do, because two extra people came?".

B. Highly structured representations support mathematical understanding. Ms U used the square tablecloth on the carpet to position the actors to sit and share the cookies. She sat the first two children opposite each other with their share of six cookies on a plate. When the next two children arrived, they sat on the vacant sides of the tablecloth. Each of the first two children then shared their cookies with a newcomer. This arrangement made the act of giving the newcomers half their cookies very explicit. Ms U later explained that she had organized this visual layout to emphasize the mathematics.

C. A community orientation was supportive and focuses students on a shared goal. Ms U fostered a supportive community orientation in many ways including ensuring that everyone could easily see the story being acted out. This support seemed related to the common goal of understanding the various mathematical situations throughout the story. For example, towards the end of the story when the actors only had one cookie left, Ms U's response indicates an appreciation of the children's concerns when the doorbell rang.

Ms U:	As the doorbell rang (Story text).
Actors:	(Muffled raised voices). I don't want to share.
Ms U:	Oh, now we've got a bit of concern, what are we concerned about? (The actors started putting their hands over their cookies showing concern and groaning.) I wonder why? (A rhetorical question)

There were substantial differences between Ms I and Ms U's distinguishing teaching characteristics. This difference, for example, was reflected in how the cookies were used in the story. Ms U capitalized on the cookies as manipulatives to support thinking, whereas Ms I mainly used the cookies as story props. In summary, while Ms U seemed highly focused on mathematics learning, Ms I seemed to lack a similar focus.

Specific Teaching Characteristics of Instruction

Figure 1 presents an analysis of the teachers' implementations of this activity using Brown et al.'s (2001) instrument for the evaluation of mathematics lessons. The numbers on Figure 1 correspond to the codes for the components of these teaching characteristics (See Table 1). For example, in the Task section, "1" refers to "mathematical challenge". No hierarchical relationship should be implied from these numbers. Brown et al.'s instrument has four levels, which describe the quality of components of specific teaching characteristics. "Nil" here indicates that there was no evidence of that component or that it was ineffective. "Low", "medium" and "high" indicate increasing levels of quality.

The teachers' implementations of the lesson differed in three ways (See Figure 1). First, each component of the specific teaching characteristics was observed in Ms U's implementation to at least a low level, whereas there was no evidence of components related to effective use of Tools in Ms I's implementation. Second, there was variety in the components of Ms U's teaching characteristics; whereas only one component of each of Ms I's demonstrated teaching characteristics was evident. Third, the majority of the components in Ms U's implementation were coded at the highest level. In contrast, those components demonstrated by Ms I were coded as non-existent or ineffective. Thus, overall, there were substantial differences in the quality and extent of demonstrated specific teaching characteristics by Ms I and Ms U.

	Tasks			**Talk**				**Tools**		**Relationships & Norms**	
Levels	**1**	**2**	**3**	**1**	**2**	**3**	**4**	**1**	**2**	**1**	**2**
High	U	U	U	U			U		U	U	
Medium					U	U					U
Low			I				I				I
Nil	I	I		I	I	I		I U	I	I	

Key: I= Ms I; U= Ms U

Figure 1: Levels of Effective teaching characteristics.

CONCLUSION

This study focused on exploring the mathematical learning potential of a story-reading activity that was designed to introduce young children to mathematical investigations. The two analyses of this activity revealed substantial but consistent differences between the teachers' implementation of the activities. Ms U's implementation reflected the principles of effective mathematics instruction (e.g., Brown et al., 2001) and investigatory approaches (e.g., Baroody & Coslick, 1998), and consequently, had high learning potential for children. In contrast, Ms I's implementation was inconsistent with these instructional principles, and hence, had low learning potential. This study also revealed three potentially fruitful avenues for investigation. First, why would an experienced teacher fail to capitalize on the mathematics in the story despite participating in the joint planning of this activity? Second, to what extent might "story reading" activities result in limited mathematical learning opportunities? Story reading may cue off-task teacher behavior through "seductive detail" _ the interesting but unimportant information in text (e.g., Alexander, Kulikowich, & Jetton, 1994). Finally, what other typical practices may camouflage a lack of mathematical learning opportunities? For example, manipulatives can sometimes be an impediment to reasoning rather than an aid (Marojam, 1992). Thus, the study of teacher characteristics has provided considerable insight into teachers' thinking about how young children learn mathematics and the role of the teacher in an introductory investigation. Additionally, this study has suggested three further avenues for exploring the teaching of mathematics to young children.

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