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TEXT TALK, BODY TALK, TABLE TALK: A DESIGN OF RATIO AND PROPORTION AS CLASSROOM PARALLEL EVENTS

Dor Abrahamson

Northwestern University[1]

The paper describes the rationale and 10-day implementation in a 5th-grade classroom (n=19) of an experimental ratio-and-proportion instructional design. In this constructivist-phenomenological design, coming from our theoretical perspective, design research, and domain analysis, students: (1) link 'real-world' and 'mathematical' objects reciprocally through classroom enactment of word-problem situations vis-à-vis guided reading/writing of spatial-numeric inscriptions; (2) interpret and invent rate, ratio, and proportion texts as patterned cells in and from the multiplication table; (3) revisit and consolidate addition and multiplication as conceptual domain foundations. Students of diverse ethnicity, SES, and mathematical competence engaged successfully in discussing and solving complex problems, outperforming older students on comparison items.

INTRODUCTION

Substantial effort has been put into documenting and analyzing students' notoriously low performance in the domain of rational numbers (e.g., Behr, Harel, Post, & Lesh, 1993)—performance which has often been typified as evidencing 'additive reasoning' where 'multiplicative reasoning' may have been more effective. For instance, a student may reason that 3 : 4 is equal to 6 :7 because in both ratios there is a difference of 1 between the two numbers. It may be the case that students do not use multiplicative reasoning because they lack coherent cognitive models for understanding and applying multiplicativity in these cases. Plausibly, instruction should help students build such models on the basis of existing knowledge. Insightful design research has called for sensitive adaptation of instruction according to students' intuitive understandings (Confrey, 1998), and Vergnaud (1994) has suggested a broad view of the instructional domain of multiplicativity as encompassing not only schooled but also everyday knowledge. Some of this intuitive (Fischbein, Deri, Nello, & Marino, 1985), streetwise (Schliemann & Carraher, 1992), and folk (Urton, 1997) knowledge has been described in ways that illuminate prospective didactic solutions.

Our design-research principles have emanated from the perspective that students self-construct knowledge (Piaget, 1952) in response to their goals within social settings (Vygotsky, 1978) such as classrooms (Cobb & Bauersfeld, 1995). We thus strive to engineer an instructional design that affords: (1) opportunities for individual students to draw on their experience and intuitions as well as on their schooled knowledge in self-constructing understandings; and (2) engaging participatory activities that foster a supportive classroom climate (Fuson, De La Cruz, Smith, Lo Cicero, Hudson, Ron, & Steeby, 2000). When collaborating with teachers and students, we iteratively modify our design so as to maximize resonance with their difficulties, strategies, vocabulary, and emergent understandings.

We have found designs to be most effective when reading/writing/building activities allow students to 'connect' (Wilensky, 1993) the material to their previous understanding, where 'understanding' does not comprise static 'concepts' but "ways of acting and thinking" (von Glasersfeld, 1990, p. 37; see also Freudenthal, 1981). We are informed by Phenomenology (Heiddeger, 1962) in assuming that students' previous understandings—both the academic and non-academic components of their domain-specific 'conceptual field'—are implicit in their ways of acting and thinking, and become self-explicated through classroom problematizing that stimulates appropriation of mathematical artifacts. These mathematical-didactic views together with our domain analysis of ratio and proportion (Abrahamson & Fuson, 2003b) have brought us to conceptualize and design the domain in the following way:

In our design (see Figure 1): (1) '*Rate'* is an iteration of constant increments, e.g., "Every day Robin puts $3 in her kitty bank" ('+3 per day') or "Every day Tim puts $5 in his doggy bank" ('+5 per day') that produce successive cumulative totals, e.g., 3, 6, 9, 12, etc. or 5, 10, 15, 20, etc., which appear in this order in multiplication-table columns; (2) '*ratio'* is two linked rates occurring as a succession of <u>parallel</u> (contemporaneous paired) events, e.g., "Every day Robin puts $3 in her kitty bank and Tim puts $5 is his doggy bank," so '+3 & +5,' that produce successive rows in linked multiplication-table columns, e.g., 3 & 5, 6 & 10, 9 & 15, 12 & 20, etc.; (3) a '*proportion'* is two rows out of a ratio progression, e.g., 6 & 10 and 21 & 35 (forming a 'proportion quartet'); (4) the multiplicative relation between values in a proportion—whether 'scalar' (within column) or 'functional' (within row, Vergnaud, 1994)—emerges as a strategically useful property that shortcuts and can substitute for repeated addition in the solution of unknown-value proportion problems, e.g., 6 & 10 and 21 & '?' (see also Abrahamson & Cigan, in press).

METHOD

A class of 19 5th-grade students of mixed ethnicity and SES (34% free lunch) in an urban/suburban school participated in a 10-period experimental design that spanned two weeks. The class was co-taught by the author-designer and the teacher, who collaborated on preparing and debriefing each lesson. The author also held daily tutoring sessions with two students whom the teacher selected as having difficulty in mathematics. These sessions, conducted as semi-clinical interviews, served to help the students and probe for difficulties they were experiencing with the design. All lessons were audio- and video-taped and micro-analyzed daily towards modifying the design. The class had a large laminated multiplication table on the wall, and each student had two small multiplication tables, from one of which they cut columns. These columns could be put together to form rate tables or ratio tables, thus relating these forms to the multiplication table.

We pre- and post-tested the students on 13 items used in previous studies and gave the same pre-test in 5th-grade classes in two additional schools in the same district to check for the representativeness of the experimental class. These other students had the advantage that the test was conducted 2 months later in the school year.

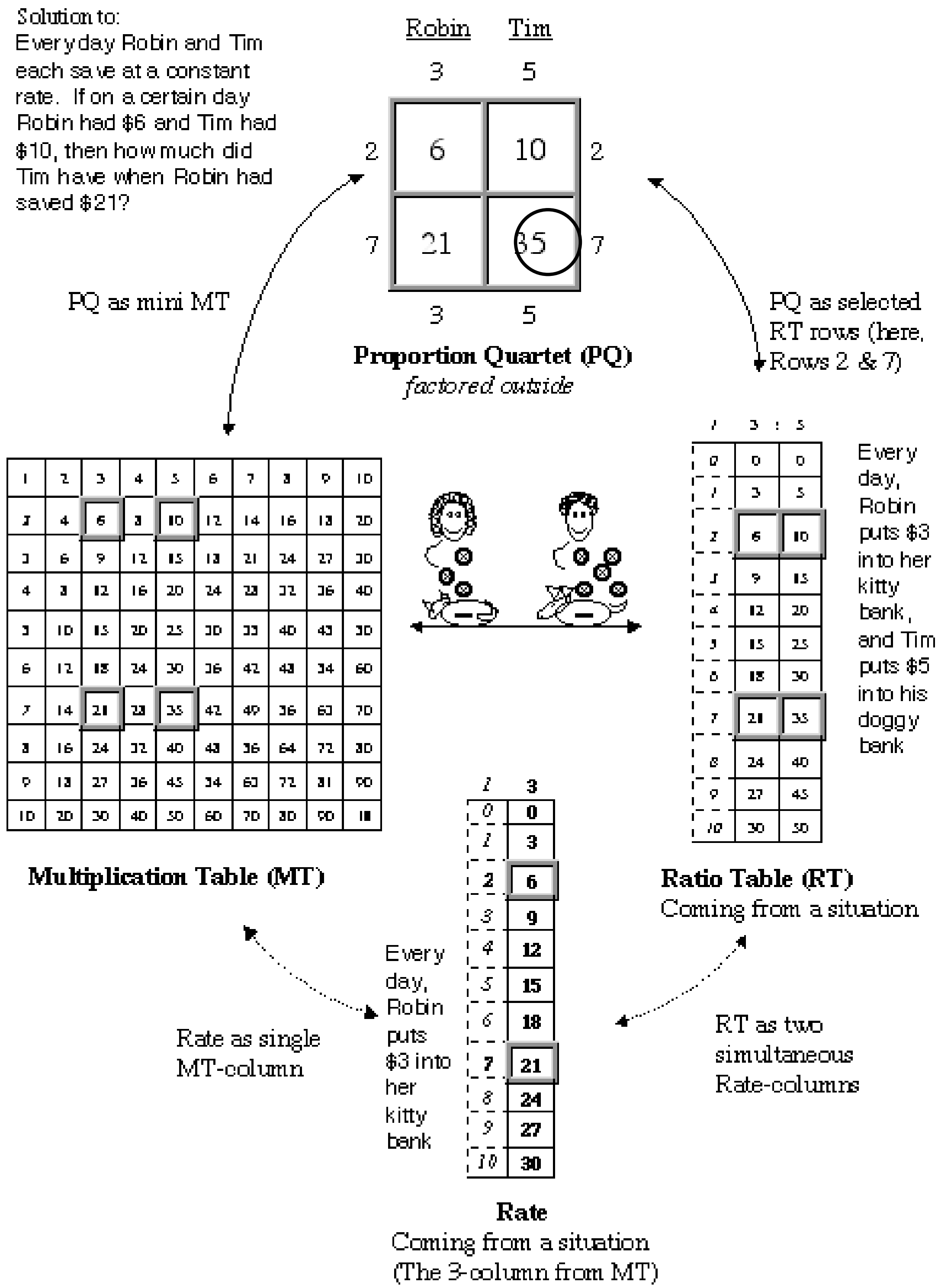


Figure 1. Design framework for conceptualizing ratio and proportion. From left and anticlockwise: (a) The multiplication table (MT); (b) rate table; (c) ratio table (RT); and (d) proportion quartet (PQ). Products and cells of a specific example problem (top left) are enhanced here for demonstration.

RESULTS

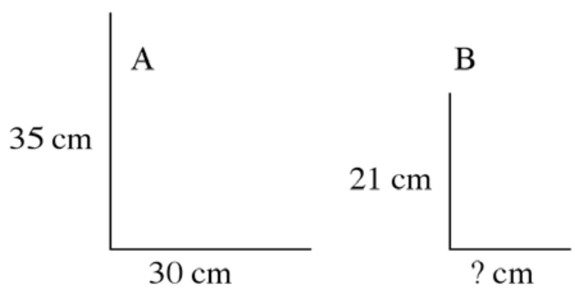
Items	Our Grade 5 (<u>n</u> = 19)	Comparative Studies
a. In a field trip, 12 people eat 16 boxes of food. How many boxes of food would 15 people eat?	72	45 Grade 6
b. To bake donuts, Jerome needs exactly 8 cups of flour to make 14 donuts. How many donuts can he make with 12 cups of flour?	83	19 Grade 6
c. The Boston Park Committee is building parks. They found out that 15 maple trees can shade 21 picnic tables when they built the Raymond Street Park. On Charles Street, they will make a bigger park and can afford to buy 50 maple trees. How many picnic tables can be shaded at the new park?	89	19 Grade 6
d. The two sides of Figure A are 35 cm high and 30 cm long. Figure B is the same shape but smaller. If one side of Figure B is 21 cm high, how long is the other side? 	78	05 Grade 6
e. If the ratio 7 to 13 is the same as the ratio *x* to 52, what is the value of *x*? Multiple choice: 7; 13; 28; 364	78	69 Grade 8
f. Fill in the missing number: 3:10 = __ : 100	94	03 Grade 5
g. After a long diet, Fat Cat weighed only 54 lbs. That was 90% of his normal weight. How much did Fat Cat weigh before the diet?	81	na

Table 1: Sample scores in % correct. Comparison items: a. Vanhille & Baroody (2002); b., c., & d. Kaput & West (1994); e. TIMSS (1995); f. Stigler, Lee, & Stevenson (1990)

Students' achievement on the posttest was higher than on the pretest (72% > 36%; $p < .0001$). In particular, in the 3 critical items, in which the multiplicative relation between the two ratios was non-integer (e.g., 6:10 & 21: 35; see items a, b, & c in Table 1), students' pre-and-post-test mean scores progressed from 10% to 81%, ($p < .005$). Also, students progressed on a fraction item even though fractions were not taught in the unit.

Pre-test scores from the 2 control classes in the same school district (36% and 29% correct) did not differ statistically from that of our study class (36%), indicating that this was not a class with special initial knowledge.

Students enjoyed solving numeric missing-value proportion problems given in the 'proportion quartet' format. In particular, students found these problems interesting in that different strategies—choice of factors and order of operations—led to the same solutions. Advanced students reported in feedback questionnaires and interviews that they had figured out how the 'proportion-quartet' solution format relates to their earlier understanding and strategies for addressing proportion situations—some of these students professed that they prefer the proportion-quartet system because its labeling reduces errors of misassigning values to variables, expedites their work, and facilitates interpretation of solution values back in terms of the situation variables and because they can confirm their solutions with the multiplication table.

Some unanticipated outcomes were that: (1) although many of the students were initially quite adept at reciting count-by sequences from multiplication-table columns and by heart, they had great difficulty in articulating these sequences in terms of addends and running totals or as multiplier-multiplicand relations; their use of the multiplication table—locating the cross product of two factors—appeared as acts of referencing a nominal chart that were devoid of any meaning of the multiplication operation; yet (2) using rate, ratio, and multiplication tables, students learned to alternatively interpret the tables' product structure as either additive or multiplicative and thus formed an additive-multiplicative model as a foundation of ratio and proportion; (3) the repeated-addition model of multiplication, e.g., '5 * 3 as +3, +3, +3, +3, +3' is more conducive to effective learning when expressed (uttered, indicated, inscribed) as a rate running total, e.g., as *0* + 3 = **3**, *3* + 3 = **6**, *6* + 3 = **9**, *9* + 3 = **12**, *12* + 3 = **15**, etc., wherein students keep a coordinated <u>parallel</u> double count—tracking the number of addends (running total of *3*'s, e.g., 1, 2, 3,4, 5) and the running cumulative total (e.g., 3, 6, 9, 12, 15); finally, (4) students became gradually quicker at multiplication "basic-facts" which they needed to solve the missing-value proportion quartets (using both multiplication and division).

In the weeks that followed our intervention, the teacher reported that students initiated using our multiplication-table poster and cutouts to find fraction equivalences, e.g., successive columns in the 2- and 5-row form the proportional progression 2/5, 4/10, etc.

DISCUSSION

Middle-and-high school students' well-documented additive errors in solving multiplicative problems are symptoms and not the sickness itself. Once students are no longer diagnosed as performing these symptomatic errors, one is still faced with the question of whether or not they are operating with understanding. Written explanations students offered in their daily worksheets, classroom participation, post-tests, and interviews to problems they had solved in the *absence* of the multiplication table focused on describing proportion as a spatial-numerical configuration in the multiplication table. That is, the multiplication table first shaped and then served as a criterion of proportionality. For instance, students used the multiplication table to reject the following problem as expressing proportion: “Bob and Joe are brothers who share the same birthday; When Bob was 3 years old, Joe was 5; Now Bob is 6. How old is Joe?” We view such a preliminary conceptualization of ratio and proportion as robust and not as narrow in that it builds directly on students’ understanding of addition, of multiplication as repeated addition, and of ratio and proportion as additive-multiplicative (Abrahamson, 2002a). Moreover, we suggest that our design addresses and fills two lacunae: (1) Generally, students’ execution of multiplication computation only ostensibly manifests an understanding of the operation—the running-total model connects addition and multiplication with understanding; (2) designs that do not incorporate any classroom enactment of ratio and proportion as a time-space phenomenon miss out on a crucial conceptual bridge between the parallel mathematical event-elements of text word problems and spatial-numerical inscriptions

Abrahamson & Fuson (2003a) interpret students’ developing cognition of ratio and proportion as following: Students who enact ratio-and-proportion word-problem situations embody verbally-kinesthetically and mobilize this text in the classroom time-space. These rhythmic gestures and utterances coincide with students successively tabulating the cumulative narrative-total, e.g., 3, 6, 9, 12, etc., in spatial-numerical structures such as rate tables that they build and fill in. Reciprocally, by interpreting inventively multiplication-table columns as rate and ratio stories, students return the spatial-numerical text to the verbal-kinesthetic classroom time-space and link inscribed forms with their enacted narratives. Thus, mathematizing narratives and narrativizing mathematics are reciprocal acts that, we believe, foster the development of interpretive schemas that are critical for sense making in the domain of ratio and proportion.

Our emphasis on the classroom temporal-spatial phenomenology of ratio-and-proportion commits us to a student-centered conceptualization of the domain: Whereas we appreciate the *mathematical* validity of referring to ratio as an ‘intensive’ quantity (Shwartz, 1988) and of qualifying the proportional relation as modeled on ‘splitting’ (Confrey, 1998), we locate the developing *cognition* of ratio in classroom acts of repeated linked adding of two quantities (a ratio) that plays out over classroom time-space. That is, values coming either from texts (word-problems) or spatial-numeric structures (tables) indicate particular cases of proportional relations and the computation of these values is facilitated by various inscriptions that students learn to execute, but the meaning of the proportional relation itself is individually mediated by verbal-kinesthetic embodiment.

To help students see anew the core spatial-numeric structure of our design, the very first task was an open-ended exploration that required students to find “interesting things” in the multiplication table. The surprisingly insightful patterns students discerned served as a “kickoff” for investigating the additive-multiplicative structure of the multiplication table, formed the basis of the classroom’s common mathematical vocabulary, and set the tone of this constructivist design. Many activities through which students embodied ratio and proportion involved moving back and forth between lexical and mathematical texts. For example, the teacher responded to a student’s additive error in the context of a discussion of the difference in the heights of two plants that grow at different rates. The teacher enacted a “slow-motion running competition” which the class followed gleefully as the distance between the two "runners"—the very slow runner (the teacher) and the slow runner (a student)—increased at every second. In a related activity, students placed their hands palms down on their desks, lifted their hands at different speeds (the right hand rising faster than the left hand), and watched how the gap between the hands grew.

Following the study reported in this paper, a modified version of the design was carried out in another 5th-grade classroom in the same school (Abrahamson & Fuson, 2003a), and later a further modified version was enacted in a different school. Also, we did extensions of this unit that included the study of percentage, geometrical similitude (Abrahamson, 2002b), and elementary coordinate graphing. The unit has been well received by mathematics teachers and supervisors in our district. Currently, we are working with several teachers in revising the design as a curricular unit of the Children's Math Worlds project. This unit reflects our current thinking on the imperative of initially grounding ratio and proportion in an explicit multiplicative spatial-numeric structure, the multiplication table.

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GENERALISING THE CONTEXT AND GENERALISING THE CALCULATION

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The Purposeful Algebraic Activity Project[1] *is concerned with the development of algebraic activity in pupils in the early years of secondary schooling. Here we report on the different ways in which we have observed children articulating generalisations during semi-structured interviews which are being used to gain snap-shots of this development. We describe how generalising the* context *appears to be distinct from generalizing the* calculation*, and discuss the implications of this for the design and implementation of our teaching programme.*

BACKGROUND

The importance of generalising as an algebraic activity is widely recognised within research on the learning and teaching of algebra. Kieran (1996) describes the first of three categories of algebraic activities as *generational activities* which involve

> *the generating of expressions and equations… for example equations involving an unknown that represent quantitative problem situations, expressions of generality from geometric patterns or numerical sequences and expressions of the rules governing numerical relationships.* (p. 272)

However there seems to have been relatively little research which looks in detail at what it is that novices actually do when they are encouraged to generalise, and how this differs from what more experienced mathematicians might do, or at what kinds of pedagogic tasks support the construction of meaning for generalising.

Mason *et al* (1985) describe three important stages in the process of generalising a pattern or relationship as *seeing*, *saying* and *recording*; that is, seeing or recognising the pattern, verbalising a description of it, and making a written recording. Several researchers have looked at the last part of this sequence, and particularly at the difficulties which students encounter in producing formal written expressions of generalizations (for example Macgregor and Stacey, 1993), and at the role of intermediate or idiosyncratic notations in the moving towards such formal expressions (for example, Bednarz, 2001, Ainley, 1999).

Radford (2001) offers a detailed linguistic analysis of pupils working on the problem of generalising a matchstick pattern, and identifies three levels of generalisation related to such geometric-numeric patterns. These are *factual* generalisations, which generalise numerical actions and enable students to tackle particular cases*, contextual* generalisations, which are performed on conceptual, spatio-temporal objects, and *symbolic* (algebraic) generalisations, which deal with algebraic objects which are

[1] The Purposeful Algebraic Activity Project is funded by the Economic and Social Research Council.

unsituated and atemporal. Whilst giving a detailed account of the linguistic indicators of the shifts between these levels of generalisation, Radford offers a description of the complexity experienced by students in constructing meanings for algebraic symbols, which he suggests are initially haunted by 'the phantom of the students' actions' (p. 88).

Within the Purposeful Algebraic Activity project we are studying the development of pupils' constructions of meanings for algebra during the early years of secondary schooling. We have based our data collection around the notion of *algebraic activity,* using the broad definition given by Meira (forthcoming), which implies that

> *The student who is unable to fully and competently use algebra as intended by experts may be engaged in algebraic activity …[when] he or she is producing meaning for algebra.*

The main strand of the project is the development of pedagogic tasks, based on the use of spreadsheets, designed using the principles of *purpose* and *utility* proposed by Ainley and Pratt (2002) (Ainley, Bills and Wilson, forthcoming). These tasks are being used in the introductory stages of teaching algebra to pupils in the first year of secondary schooling. The tasks are designed to support a range of algebraic activity, focussing on *generational* and *meta-level* activity (as described by Kieran, 1996) and making explicit links to *transformational* activity, which traditionally receives emphasis in the school curriculum.

Semi-structured interviews are being used to collect snap-shots of the development of pupils' algebraic activity. The interviews aim to explore different aspects of algebraic activity, and are to be repeated, with essentially the same set of questions, at regular intervals. Analysis of the first phase of interviews has given us some insight into the ways in which pupils with little experience of algebra generalise numerical relationships which differ in emphasis from, but may complement, Radford's (2001) categorisation.

DATA COLLECTION

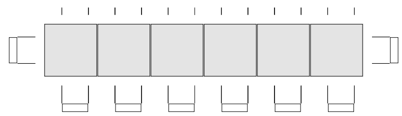
Two cohorts of pupils have been interviewed in the first phase of the project; cohort A at the end of their first year at secondary school (mostly aged 12), during which they had had some explicit teaching in algebra, and cohort B at the beginning of their first year (mostly aged 11). Twelve pairs of pupils from two schools were interviewed in cohort A, and fourteen pairs in cohort B. The pupils were selected by their teachers from those who were willing to take part, with the aim of making compatible pairs of similar ability. The pairs were distributed across the perceived ability range in the year group, and contained a balance of boys and girls.

The interviews were conducted by a researcher (the second named author), who presented the questions in written form, and also read them aloud. All interviews were video taped, and audio taped for transcription. Copies of any pupil writing were collected, and added to the transcripts, which were also annotated to include non-verbal behaviour observed in the video tapes. For the purpose of the analysis presented here, the contributions of the two individuals during the interview have generally been amalgamated, since our concern in this paper is less with the particular pupils than with the ways in which they are expressing their thinking.

WAYS OF GENERALISING

This paper focuses on responses to the question shown in Figure 1, which comes about two thirds of the way through the interview.

Tables and chairs are arranged like this in the school dining room:



Can you explain how to work out how many chairs are needed for each long table? Can you write it down?

Figure 1: The ‘tables and chairs’ question

This question was chosen so that we could see whether pupils were able to articulate a generalised relationship containing a variable (in Mason et al’s term *seeing and saying*). They were encouraged to write down (*recording*) so that we could see whether or not they would make use of an algebra-like notation. During the interviews, the researcher encouraged the children to articulate and share ideas, and, if they seemed stuck, sometimes suggested that they try particular numerical examples (e.g. *what would happen if there were 20 tables?*). The interviewer also prompted the pupils to try to write down an answer, but did not insist on this if they seemed reluctant to do so for any reason. Often the pupils preferred to say aloud what they would write.

Cohort A

All of the pairs in cohort A (those who had some experience of algebra), showed some evidence of algebraic activity, in that they were able to articulate a general relationship between the numbers of tables and chairs. Here are some examples of their statements.

1. *Two on each table except for the ends, which is three.*
2. *For each end one you would need three and … for each other table you would need two*
3. *For every table there’s two chairs plus the other two that are on the end*
4. *You would do, two for each table, two chairs for each table … and then you would need … two more for the sides*
5. *Well, every table you get, you need to double it and then … add ‘em on … the number of chairs must be double and add two.*
6. *Just however many tables double that, and then, plus two for the ends*
7. *So it could be times how many tables there are, times two … plus two*
8. *You take the tables and you times it by two and then plus two*

These examples vary in a number of ways. Clearly some are expressed more fluently and confidently than others. Statements 1 and 2 seem to see the arrangement in terms of two kinds of table (the ‘ends’ which have three chairs, and the ‘middles’ which have two) while the remaining statements seem see each table as having two chairs, and treat the chairs at the ends as extras. Although both of these are reasonable ways of describing the arrangement, the second is much easier to translate into a symbolic expression.

This range of statements could be analysed in a number of ways, but our initial analysis focused on a distinction between two different things which are being generalised: the context and the calculation. The first four examples appear to be descriptions of the *context;* the way in which the tables and chairs are arranged. We might see them as general instructions for how to place the furniture. In contrast, examples 5 – 8 describe the *calculation* required to find out how many chairs are needed. These statements are distinguished by the use of terms for operations, such as *double, times, plus, add*. They also include phrases that indicate a sense of a variable involved in the calculation: *every table you get, however many tables, how many tables, the* [number of] *tables.*

Within cohort A, four pairs of pupils gave general descriptions of the arrangements of furniture (the context) but did not describe the calculation, four gave general descriptions of the context and then of the calculation, and four gave descriptions of the calculation as their first response. We now look in more detail at these three groups of responses.

<u>Generalising the context only</u>

The four pairs who gave a general description of the arrangement of furniture, but did not describe the calculation, had been identified by their teachers as middle to low ability in mathematics. Three of these pairs initially gave a description which treated the end and middle tables separately (e.g. *if it's like a corner table you'd need three…but if it was one of the middle ones you'd only need two*). The calculation which arises from this description is more complex than the 'two chairs for each tables and two for the ends' image, and this may have been a factor which inhibited these pairs from being able to express a calculation. The researcher encouraged each of these pairs to try to write something down, but no pair in this group wrote or articulated a symbolic rule.

<u>Generalising the context then a calculation</u>

The four pairs who gave a description of the arrangement of furniture, and then later described a calculation, had been identified by their teachers as middle or high ability. They were generally more confident in approaching this question than those pairs who did not move beyond describing arrangements of furniture. In some cases the initial description of the arrangement of furniture was a dynamic one, focusing on what would happen as additional tables were added. For example, Pair 7 said:

> *'cause you just use this one again, it just goes to the end of the next table, but you need two more of those to put on to the side of it.*

As their discussion continued, Pair 7 described the calculation as follows:

> *so it could be times how many tables there are, times two … plus two*

Pairs 6, 7 and 8 managed to articulate algebra-like expressions for the calculation, though these were not always written down.

> Pair 6: *They could represent table as t and you could do two, I mean t two, no you could write*
> $\boxed{T_2 + 2}$

Pair 7: *But it could be, how many tables times by, n is tables, well, t is tables, times by two … t times two plus two could equal the chairs*

Pair 8: *So it would be like, um, c equals two t plus two*

Pair 5 gave a clear description of the calculation (*The number of chairs must be double and add two*), but then became confused when they decided that their first attempt at a written response $2\times x+2$ required some brackets.

Calculation as first response

Not surprisingly, the pairs who gave a description of the calculation as their first response were ones who had been identified as middle or high ability by their teachers. These pairs generally approached the question confidently, and with relatively little discussion. They were usually able to move easily to a written version of the calculation expressed in algebra-like notation, even though they sometimes struggled with syntax.

Pair 9: *double the number of tables add two,... t, you could, if you were doing it in algebra* $\boxed{T2+2}$

$\boxed{2x+2}$

Pair 10: *So we could try double it plus two? … So shall we say ...x, two x…plus two*

Pair 12: *However many tables, it would be n tables, times two… plus two*

In summary, this analysis suggests that although all the pairs in cohort A were able to generalise the relationship between the numbers of tables and chairs, those pairs who were successful in articulating a version of the rule in a recognisably algebra-like notation had also given a general description of the calculation necessary to work out the number of chairs. This led us to conjecture that being able to generalize the *context* was not sufficient to enable pupils to express the relationship in algebra-like notation, and that being able to generalise the *calculation* required was a significant 'bridge' which supported pupils in constructing meaning for a symbolic expression of the relationship.

Cohort B

To test our conjecture about the significance of a verbal description of the calculation as a pre-cursor of constructing meaning for an algebra-like expression, we looked at the transcripts from the interviews with cohort B. This group of pupils had just begun secondary school, and had had no real teaching of algebra prior to the interviews. Not surprisingly, fewer of the pairs in this cohort were able to use algebraic notation with any confidence, but nevertheless most showed evidence of algebraic activity which could be categorised in similar ways to that of cohort A.

However, three of the fourteen pairs were unable to give a general description of the relationship between tables and chairs. Their attempts included methods of counting the chairs, or specific numerical examples, though one pair felt the task was not possible 'unless it tells you how many tables you actually want'.

Generalising the context then a calculation

Four pairs initially gave generalised descriptions using the 'ends and middles' approach.

Pair 5: *You need three for the end tables and two for the rest of the tables*

The interviewer's normal prompt to explore pupils' thinking further was to ask if they could write something down. In response to this kind of prompt, these four pairs gave generalisations of calculations, but were not confident about writing these: for example,

Pair 5: *You could put end tables times three chairs and then middle tables times two chairs.*

Expressing algebra-like rules

The remaining seven pairs of pupils all managed to articulate a rule for calculating the number of chairs which was recognisably algebra-like, using a variety of non-standard notations. Only one pair produced a rule in 'standard' notation. One pair gave a description of the calculation as their first response, while the remaining six moved from general descriptions of the context to generalising a calculation, and then to a symbolic version of the rule (see examples in Figure 2.) Although their responses are less confident than those of cohort A, they display explicit attempts to work with a variable.

	Generalisation of context	Generalisation of calculation	Symbolic rule
Pair 9	*You have two on each end and then there's two on each side of the table*	*You could times how many tables there are by two and then add the extra two on the …(ends)*	How many tables x2 +2
Pair 10	*For one table you'd need two chairs or for an end one three*	*You'd like, you'd double it then add the two on the end*	A numb~~15~~ X 2 + 2 Originally they wrote '15' then crossed this out and wrote 'a number', saying "it's not necessarily gonna be fifteen"
Pair 12	*It starts off with three for the first one and then its carry on adding two and then when it gets to the end it's add three again.*	*Times two that amount … add two*	*Just write times the amount of squares* [i.e. tables] *by two then add two*

Figure 2: examples of responses from cohort B showing algebra-like rules.

The overall pattern from cohort B appears to support our conjecture. Those pairs who were able to articulate an algebra-like version of the rule had previously generalised the calculation required; generalizing the context did not seem to be sufficient to support pupils in moving to a symbolic version of the rule.

DISCUSSION

The Purposeful Algebraic Activity Project centers around a teaching programme in which we have designed spreadsheet-based tasks to support generational activity (Ainley, Bills & Wilson (forthcoming)). Typically, such tasks involve pupils in generalising relationships and expressing these as spreadsheet formulae in order to use the power of the technology to solve problems. The spreadsheet environment provides a purposeful context for the use of an algebra-like notation, but we see a key element of such tasks as the construction of meaning for such notation through the activity of generalising.

The analysis of the ‘tables and chairs’ question has provided an important insight into the ways in which such relationships may be generalised, and suggests that if pupils’ activity is focused only on generalising within the context (as in describing arrangements of the furniture), and does not move to generalising the required calculations, then an important link in the construction of meaning for the symbolic notation may be lost.

Although our study is similar to that used by Radford (2001) as the basis for his categorisation of levels of generalising, in terms of the type of problem and the age of the pupils, there are some differing features which make a direct comparison difficult. These differences lie in the questions on which the pupils were working. Although both questions are based on geometric-numeric patterns, in our study the pattern had been contextualised in an ‘everyday’ setting (tables and chairs), whilst Radford’s question is based on an arrangement of matchsticks. In designing our interview question we had deliberately set what could have been presented as an abstract spatial design in an ‘everyday’ context, in which the expression of a generalised relationship could be used in a purposeful way, which parallels the pedagogic tasks in our teaching programme. In Radford’s study, there is no contextual setting to which pupils might refer, and indeed in the extracts he quotes pupils refer only to the numerical (rather than the spatial) patterns.

Furthermore, in Radford’s study, the question is broken down into stages as pupils are asked to find the number of matchsticks used in the fifth, and then the twenty-fifth, term in the pattern sequence before being asked to generalise for any term. We had made a choice to present the problem with a single image rather than as a sequence in order to discourage term-by-term approaches which obscure the need for algebraic generalisation. These two features (problem context and the staging of the question) appear to have produced different kinds of responses from pupils. Although there seems to be much in common between our analysis and that of Radford, particularly in terms of the shift in the level of abstraction of the objects involved in the generalisation, the different features of the question seem to have altered the process of generalising for these relatively inexperienced pupils. Whilst we feel that the overall level of success pupils achieved in making some kind of generalisation of the relationship between the numbers of tables and chairs indicates that both the problem context and the one-stage question were effective in supporting the constructing meanings for generalising, we need to consider carefully what effect they may have on the construction of meaning for symbolic notation.

We see this as significant in relation to the design of pedagogic tasks. One way in which we might read pupils’ responses to the tables and chairs problem is that they focus too much on the context and fail to successfully ‘negotiate the boundary between the ‘mathematical’ and the ‘real’’ in the design of the question (Cooper and Dunne, 2000). They see the problems as essentially about arranging furniture, and do not appreciate the pedagogic intention to express a relationship or calculation. We did, indeed, find that some children became distracted by discussing other possible arrangements of the tables and chairs, thinking about how the chair had to moved as other tables were added, or asking how many children needed to sit down for lunch. Adding an element to the task which signals clearly the need to describe a calculation (such as ‘Could you tell the caretaker how to work out how many chairs she should get out of the storeroom?) might signal the pedagogic emphasis of the task to pupils in a meaningful way.

We also see the distinction we have observed as potentially significant in helping to focus teachers' interventions. We are working closely with a group of teachers in the development of the teaching programme, discussing teaching approaches as well as the content of the tasks. If teachers become aware of the different ways in which pupils may be generalising, this offers the opportunity for targeted interventions to support pupils' construction of meanings for formal notation by asking them to articulate and generalise their calculations.

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INTERVIEW DESIGN FOR RATIO COMPARISON TASKS

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In this article, which is part of an ongoing research, a classification is proposed for ratio comparison problems, according to their context, their quantity type, and their numerical structure. Deriving from this classification, an interview protocol was designed, and guidelines for the interpretation of answers into strategies were decided. A first case study was conducted and the results obtained are discussed, including comparisons of answers to problems of different contexts, quantity types, and number structures.

This paper reports part of an ongoing research on the strategies used by subjects of different ages and schoolings when faced to different kinds of ratio comparison tasks. A framework for the study of answers to ratio comparison probability tasks was presented at PME26 (Alatorre, 2002). The framework, which consists on two interrelated systems, one for interpreting answers as strategies and one for classifying the numerical structure of questions, was constructed within an investigation in which only adult subjects participated. One of the goals of this part of the research was to test its effectiveness with non-probability ratio comparison tasks and with younger subjects.

The partial investigations carried out so far show that the framework is indeed effective for a) the design of instruments aiming at studying the strategies used in different kinds of ratio comparison tasks, and b) the interpretation of the answers given by young subjects. This paper deals firstly with a classification of the kinds of problems; secondly, it presents a protocol design surging from this classification and in the frame of previous research; and thirdly, the actual results from a case study are discussed.

TYPES OF PROPORTIONALITY PROBLEMS

In the complex setting of proportional reasoning research, several ways have been put forward to classify the problems that can be proposed to subjects. They may be in turn grouped in classifications according to four issues that affect the subjects' responses: 1) the task, 2) the context, 3) the quantity type, and 4) the numerical structure.

The *task* that subjects have to accomplish was classified by Tourniaire and Pulos (1985) as "missing value problems" or "ratio comparison problems". To this basic classification other researchers, such as Lesh, Post and Behr (1988), later added more categories. In the research reported in this paper only ratio comparison problems are considered.

Among the classifications according to the *context*, Freudenthal (1983) distinguished couples of a) expositions, b) compositions, and c) □-constructs; Tourniaire and Pulos (1985) set apart d) physical, e) rate, f) mixture, and g) probability problems; and other authors, among which Lamon (1993), have distinguished h) well chunked measures, i) part-part-whole problems, j) associated sets, and k) stretchers and shrinkers. Although each of these classifications and categories corresponds to particular views and goals, the following considerations can be made. Categories a), d), e), h), and j) can all be considered as one and the same because they all involve two different quantities; the difference between d) and e) lies in the fact that the latter are word problems, and the difference between h) and j) lies in how familiar the subject finds them. Categories b) and i) can be considered as one, because they involve one quantity; f) and g) are in the same case and the difference among them may be considered important. Finally, categories c) and k), which are problems of a geometrical nature, can be considered as one. The left column of Table 1 displays the condensed classification of context resulting from these considerations. As examples, figure 1 shows a rate problem and figure 2 shows a part-part-whole mixture problem. This research does not deal with geometrical problems.

Rate problems: couples of expositions		Intensive quantity surging from two quantities: both discrete, both continuous, or one of each type
Part-part-whole problems: couples of compositions	Mixture	One quantity, discrete or continuous
	Probability	One quantity, discrete or continuous
Geometrical problems: couples of □-constructs		Two continuous quantities

Table 1: Problem classification according to context and quantity types

The most generally accepted classification of *quantity types* separates discrete from continuous quantities (Tourniaire and Pulos, 1985). The right column of Table 1 shows the possibilities in each context category. It may be noted that in rate problems two quantities are at stake, and thus the intensive quantity formed (Schwartz, 1988) can have three different origins. In figure 1, one of the quantities at stake is discrete (amount of plants) and one is continuous (soil area), whereas in figure 2 the quantity at stake (liquid amount) is continuous, although discretely handled through amounts of glasses.

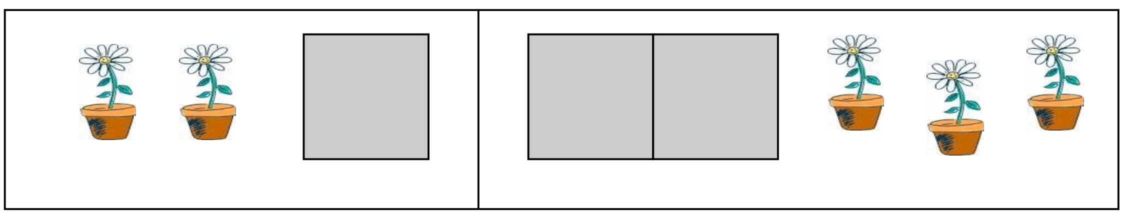

Figure 1. Garden problem (G). In which garden will the flowers be more cramped? (The squares stand for soil area where the flowers will be planted)

Among the classifications proposed for the *numerical structure*, two are considered here: Karplus, Pulos and Stage's (1983) and Alatorre's (2002). Before describing them, the notation used in this paper will be presented; it differs slightly from that of Alatorre (2002). In a ratio comparison there are always four numbers and two "objects" (1 and 2) involved. In each object there is an antecedent "a" and a consequent "c", and thus the four numbers may be written in an array, which is an expression of the form $(a_1,c_1)(a_2,c_2)$.

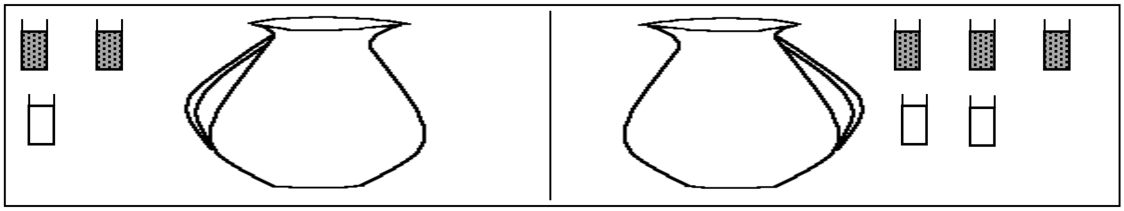

Figure 2. Juice problem (J). In which jar does the mixture taste stronger?
(The grey glasses contain concentrate and the clear ones contain water)

For instance, the numbers involved both in figures 1 and 2 are (2,1)(3,2). Also of interest may be the totals t=a+c ($t_1=3$ and $t_2=5$), the differences d=a–c ($d_1=1=d_2$), and the part-whole quotients p=a/c ($p_1=2/3$ and $p_2=3/5$). If in figure 2 there were 5 concentrate and 1 water glasses at the left side and 2 concentrate and 7 water glasses at the right side, the array would be (5,1)(2,7) and one would have $a_1=5$, $c_1=1$, $t_1=6$, $d_1=4$, $p_1=5/6$, $a_2=2$, $c_2=7$, $t_2=9$, $d_2=-5$, and $p_2=2/9$.

Karplus et al (1983) propose the categories W (integer ratio at least within one object), B (integer ratio between objects, in antecedents or in consequents), 1 (unit amount), and X (unequal ratios); they can be combined to form 5×2 categories: W, B, WB, WB1 and NIR (non integer ratio), all of which can occur in proportionality or non-proportionality (X) situations. For instance, the array (2,1)(3,2) of figures 1 and 2 is WB1X because the ratio 2:1 is integer both within (in garden 1 of figure 1, there are two plants for a square) and between (in figure 1 the area of garden 2 doubles that of garden 1), because one of the four numbers is 1, and because it is a non-proportionality situation.

Alatorre's (2002) proposition is a classification of all arrays in 86 different situations according to 17 different "combinations" –successions of results when an order relationship is established in the array between the pairs of numbers t, a, c, d, and p–, and 17 different "locations" –non-ordered pairs of the following alternatives for both quotients of the array: *u:* unit (p=1); *w:* win (1>p>_); *d:* draw (p=_); *l:* lose (_>p>0); and *n:* nothing (p=0)–. The array (2,1)(3,2) of figures 1 and 2 belongs to combination K11 (<<<=>) because $t_1=3<t_2=5$, $a_1=2<a_2=3$, $c_1=1<c_2=2$, $d_1=1=d_2$, and $p_1=2/3>p_2=3/5$; and to location *ww*, because both p_1 and p_2 are 1>p>_. The 86 situations can be grouped in six difficulty levels, labelled I to VI.

Both classifications may be used as complementary. Considering that six of the 17 combinations (K0 and K12 through K16) bear proportionality and the rest does not, Alatorre's 86 situations may be crossed with Karplus et al's basic categories W, B, WB, WB1 and NIR.

PROTOCOL DESIGN AND INTERPRETATION OF ANSWERS

One of the purposes of the research is to compare the strategies used by subjects when faced to different kinds of ratio comparison settings. As stated above, these may differ in their context, their quantity type(s), and their numerical structure. A protocol was designed, with ten different problems and fifteen different numerical questions.

Figures 1 to 8 show the eight main problems. The gardens, lemonade, notebooks, and blocks problems are rate problems. They vary in the types of the quantities at stake – one discrete and one continuous in the first two, two discrete ones in the third, two continuous ones in the fourth. The juice and the exams problems are part-part-whole mixture problems – the first one is continuous and the second one is discrete. The spinners and marbles problems are part-part-whole probability problems – respectively with a continuous and a discrete quantity. Two other problems were used as controls, one for the fractions algorithmisation and one for the concept of randomness.

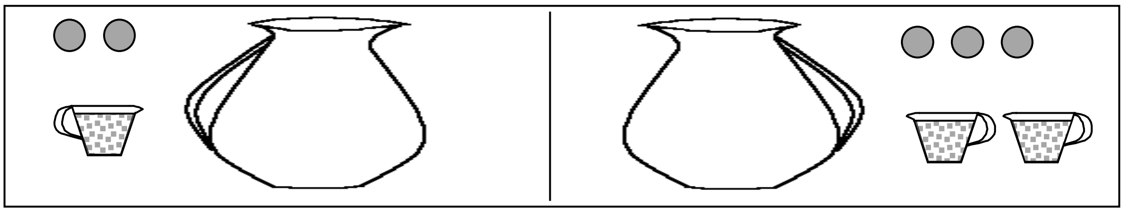

Figure 3. Lemonade problem (L). In which jar is the lemonade's taste stronger?
(The round figures stand for lemons and the cups contain sugared water)

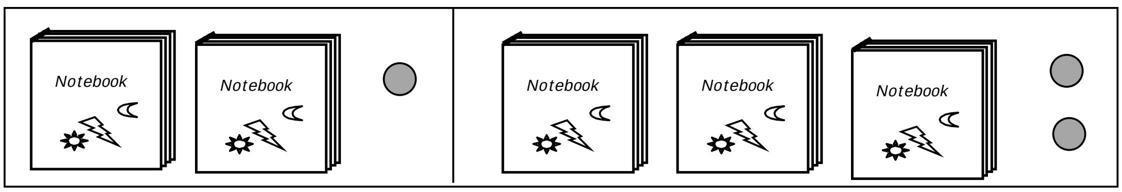

Figure 4. Notebooks problem (N). In which store are the notebooks cheaper?
(The round figures stand for coins)

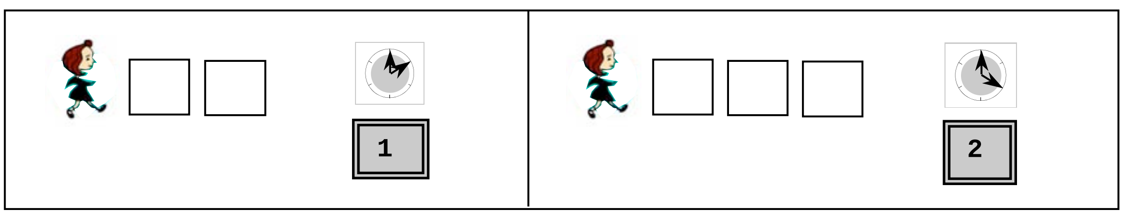

Figure 5. Blocks problem (B). Which of the girls walks faster?
(The squares stand for blocks, and the numbers represent minutes for walking them)

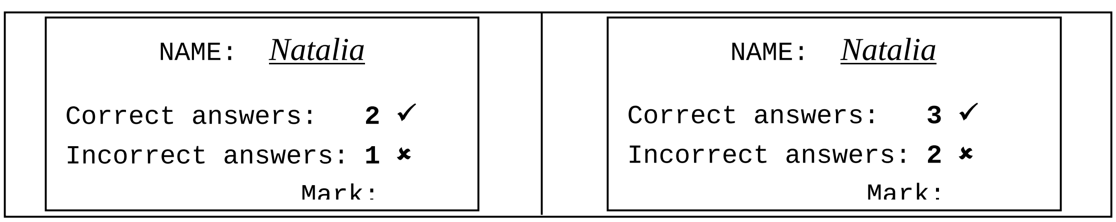

Figure 6. Exams problem (E). In which exam did Natalia do better?
(Additional question: what were her marks in both exams?)

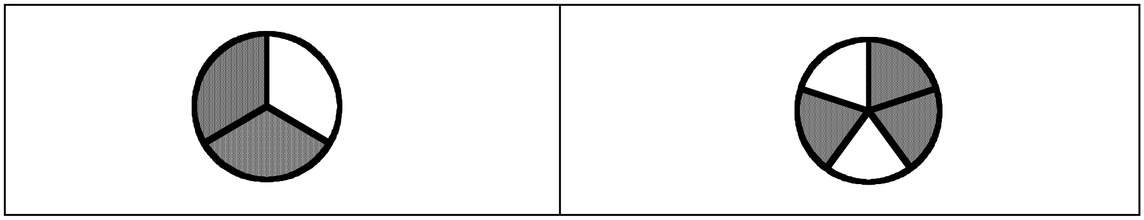

Figure 7. Spinner problem (S). In which spinner is a dark sector more likely to be marked?

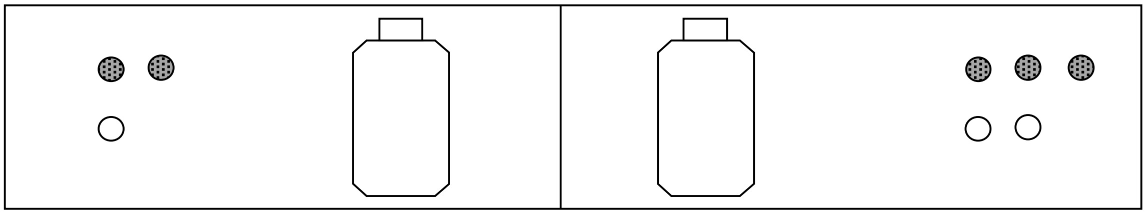

Figure 8. Marbles problem (M). In which bottle is a dark marble more likely to come out at the first try? (Bottles will be shaken with marbles inside)

Table 2 displays the 15 questions that were designed according to numerical structure. All the problems may be presented to subjects in each of 15 questions, except for question number 7, which does not have any sense in most rate problems. In figures 1 to 8 all the problems are shown in question number 4, which is the array (2,1)(3,2).

Question number	Left		Right		Karplus et al's	Alatorre's		
	a	c	a	c		Combination	Location	Difficulty level
1	2	3	2	3	B	K0	*l=l*	II
2	1	4	3	2	WB1X	K3	*wl*	III
3	2	3	2	2	WBX	K5	*wd*	III
4	2	1	3	2	WB1X	K11	*ww*	VI
5	3	3	1	1	WB1	K16	*d=d*	IV
6	2	2	3	2	WBX	K4	*wd*	III
7	3	3	2	0	WX	K7	*ud*	II
8	2	1	4	2	WB1	K15	*w=w*	IV
9	2	5	1	3	WB1X	K8	*ll*	VI
10	3	6	1	2	WB1	K14	*l=l*	IV
11	5	2	7	3	NIR	K9	*ww*	VI
12	4	6	2	3	B	K14	*l=l*	IV
13	3	2	5	3	NIR	K6	*ww*	V
14	2	4	3	5	W	K10	*ll*	VI
15	8	4	4	2	WB	K15	*w=w*	IV

Table 2. Antecedents (a), consequents (c), and classifications of the 15 questions

Since one of the purposes of the research is to observe the occurrence of different strategies, it is important that the protocol contains a variety of situations distributed among the 17 combinations and the 17 locations. Questions 11 to 15 are designed to be posed only to subjects capable of proportional reasoning, because of their higher difficulty (Karplus et al's categories, difficulty levels, and larger number sizes).

Alatorre's (2002) framework is to be used in the interpretation of answers. Because of space limitations, only a succinct summary will be offered here, with slight modifications in the notation. Simple strategies can be centrations or relations. Centrations can be on the totals CT, on the antecedents CA, or on the consequents CC. Relations, either "within" or "between", can be order relations RO (when an order relationship is established among a and c elements of each object and the results are compared), or subtractive relations RS (additive strategies), or proportionality relations RP. Composed strategies can be conjunctions *X&Y* (*X* and *Y* dominate), exclusions *X¬Y* (*X* dominates), compensations *X*Y* (*X* dominates), or counterweights *X*□*Y* (neither dominates). Strategies may be labelled as correct, sometimes depending on the situation (combination and location) in which they are used. Correct strategies are RP; RO in *wl*, *wd*, or *dl* locations; CA in locations with *n*; CC in locations with *u*; and some composed strategies.

Alatorre's (2002) framework was constructed for double urn probability problems (similar to problem M), where the antecedents are the favourable and the consequents are the unfavourable ones. Before any experimental evidence was obtained, it was anticipated that: 1) the strategies system would be applicable not only in probability and mixture part-part-whole problems (see Alatorre, 2000) but also in rate problems; 2) there would be no comparisons, additions or subtractions among a and c within objects in rate problems, because a and c belong to different quantities, and therefore no CT, RO, or "within" RS strategies were expected in rate problems; and 3) part-whole "within" proportionality strategies RP would only be observed in part-part-whole problems, and part-part "within" or "between" RP would be observable in all problems.

DISCUSSION OF THE RESULTS OBTAINED IN A CASE STUDY

A first interview has been conducted with Sofía, a ten-year-old fifth grader from a Mexico City private school. All the 8×15 questions were previously colour printed in 5"×8" cardboard cards. The interview, which lasted 57 minutes, was videotaped. Sofía was presented one problem at a time, for which question number 1 served in each case as an introduction to the context, and then followed one by one questions 2 to 10 (questions 11 to 15 were not posed to Sofía). The problems were posed in this order: marbles, juice, spinners, exams, notebooks, blocks, gardens, and lemonade. The question asked could always be answered by "left side" or "right side" or "it is the same"; Sofía was requested in each case to justify her answer. She was allowed to handle the cards at will.

The interview was analysed and an attempt was made to interpret Sofías's answers using Alatorre's (2002) framework. This was possible in all but two answers that will be commented later on. The results are shown in table 3.

These results can be analysed in several ways, and some conclusions may be drawn. The first and very important one is that Alatorre's (2002) framework's strategies system can be successfully used with ratio comparison problems different from the double urn task for which it was originally constructed. The two only answers for which the system does not have any category were given in problem S. Both of them display the same kind of reasoning, which can best be exemplified in the answer to question S-5 (see figure 9):

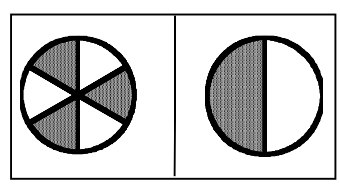

Figure 9. Question S-5

Sofía: The left side, because if you join the three parts they are bigger and besides they are more spread apart.

This reasoning seems to be based on the size and the distribution of the spinners' sectors. Both factors are not related to the ratio comparison task but to the concept of randomness and to the visual perception of the sectors.

An overall analysis of the strategies used by Sofía was performed. Centrations were slightly more used than relations and than composed strategies, which is also the case with adult subjects in double urn tasks (Alatorre, 2002). Among centrations, CA and CC were by far the most frequently used, and it is noticeable how they were often competing against each other, alternatively excluding each other or in counterweight. Sofía seemed to be equally attracted by the side with the largest antecedent or with the smallest consequent. All relations were "within", except for a "between" RS strategy in question N-4 ("it's fair that for one notebook more they charge one peso more", see figure 4). RS and RP relations were only used when they led to the answer "it is the same" (question 4 for RS and questions 5, 8, and 10 for RP). Also noticeable is that once a correct strategy is found, it may be thrown down by an incorrect one, as in questions J-5 and N-5.

As expected, CT was only observed in a part-part-whole problem (S-4). However, RO and RS did not respond as expected. Although RO appeared more in part-part-whole problems, it did come out four times in rate problems: G-3, L-2, G-6 and L-6 ("right side, because there are more lemons than cups"). Also, RS did appear twice in a "within" form in rate problems: L-4 and G-4 ("it is the same, because there's one more flower than squares in each" – see figure 1), contrary to what was expected. This apparently shows that Sofía faced no dilemma in putting together different quantities for comparisons.

	Marbles (M)	Juice (J)	Spin-ners (S)	Exams (E)	Notebooks (N)	Blocks (B)	Gardens (G)	Lemo nade (L)
2	CA	CA	***RO***	***CA & CC***	***CA & CC***	***CA & CC***	***CA & CC***	***RO***
3	CC	CC	CA	CC	***CC * CA***	***CC * CA***	***RO***	CC
4	CA RS	CA □ CC CA	CT CT	CA CA □ CC	CA ¬ CC RS	CC ¬ CA CA □ CC	RS	RS
5	CA * RO	***RP*** CA	Other	***RP***	***RP*** CA	CC ***RP***	***RP***	***RP***
6	CA	RO	CA		***CA * CC***	CA	RO	RO
7	***CC***	***CC***	***CC***	***CC***	***CC***			***CC***
8	RO	CA * RO	Other	RO	CA ¬ CC	CC ¬ CA	***RP***	***RP***
9	CA ¬ CC CC ¬ CA	CC ¬ CA		CC RO		CA ¬ CC CA □ CC		CC
10	RO	CC ¬ CA		RO			***RP***	CC

Table 3. Strategies used by Sofía. Correct strategies appear in bold italics. (Two successive answers were given to some of the questions 4, 5, and 9).

Among the RP strategies, only one was a part-whole "within" strategy: E-5 ("it is the same, because in both exams Natalia had one half correct and one half incorrect answers"), and all other RP's were part-part "within" strategies. This behaviour follows the predictions. Most RP's appeared in problems with at least one continuous quantity.

The results in table 3 were also analysed column-wise and row-wise. A column-wise analysis allows comparing the eight different contexts and the part-part-whole (first four) vs. the rate (last four) problems. Rate problems were apparently easier for Sofía to answer correctly than part-part-whole ones, which goes in agreement with Tourniaire and Pulos' (1985) opinion in the sense that it is easier to handle two different quantities than only one. Among the rate problems, G was the one with more correct solutions and B was the one with the least. Among the part-part-whole problems, E was the easiest and M the most difficult. These results could also be due to the order in which the problems were administered; further interviews should vary this order.

A row-wise analysis of table 3 allows comparing the results obtained in questions with different numerical structure. The results were consistent with the levels proposed by Alatorre (2002): The easiest question was number 7 (level II, see table 2), which was always answered with a correct strategy, followed by level III questions 2 and 3, and the most difficult questions were level VI questions 4 and 9, in which incorrect strategies were always used. Among the level IV proportionality questions, the correct RP strategy was more frequently used in question 5, where the ratio was 1:1, than in questions 8 or 10, where it was 1:2 or 2:1. Proportionality was easier for Sofía to distinguish in 1:1 ratios than in other ratios.

CONCLUSION

Alatorre's (2002) framework has proved effective for the design of interview protocols as well as for the interpretation of answers to ratio comparison tasks of different kinds, that is different contexts (rate and part-part-whole, mixture and probability) and different quantity types. It also proved effective with a young subject. The only answers that were not interpretable in terms of the framework could be attributed to flaws in the concept of randomness or related to the visualisation.

The framework also allowed a comparison of the subject's answers among problems of different contexts and quantity types, and among questions of different number structure. It is to be expected that it also permits to compare the behaviour of different subjects towards ratio comparison tasks.

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MODELING OUTCOMES FROM PROBABILITY TASKS: SIXTH GRADERS REASONING TOGETHER

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This report considers the reasoning of sixth grade students as they explore problem tasks concerning the fairness of dice games. The particular focus is the students' interactions, verbal and non-verbal, as they build and justify representations that extend their basic understanding of number combinations in order to model the outcome set of a probabilistic situation. Several models are posed and students discuss and debate their differences and similarities. Analyses of student-to-student and student-to-researcher dialogue reveal an emphasis on sense making and collaboration in the quest for meaning. This resulted in richer and deeper understanding of the conditions of the problem and the varying interpretations developed by the students.

INTRODUCTION

The 15 children in the study, 7 boys and 8 girls, were students in a K-8 elementary school in a small, working-class community in the United States. During the students' elementary and secondary grades, the researchers periodically facilitated problem-solving sessions during extended classroom sessions[1]. The problem-solving episodes described here document the discourse and work of the children as they developed solutions for a series of combinatoric and probability tasks in March of their 6th grade year. Our intent was to study in detail how the children dealt with the complexity of the task. In particular, how did they negotiate differences and become convinced of the reasonableness of ideas that were posed?

THEORETICAL FRAMEWORK

The basic tenet of the theoretical perspective guiding this research is that meaningful and useful mathematical knowledge is built by the learner through active engagement in challenging problematic situations over extended periods with opportunities to reconsider and modify ideas over time (Alston, Davis, Maher & Martino, 1994; Davis & Maher, 1997; Kiczek & Maher, 1998; Davis & Maher, 1990). This development is often cyclical, involving the construction and reconstruction of representations from and with which to develop, compare and justify solutions individually and in discourse with others (Davis & Maher, 1997, Maher, 1998). Considerable experience working with what may initially appear as messy data enables learners to gain rich and deep experience in posing theories, testing them, and carry out ideas. Our view is that engaging with complex tasks offers opportunity to work through the confusion and make strategic choices. Immersion in

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[1] This research, funded in part by NSF grants MDR9053597 (directed by Davis and Maher) 981486 (directed by Maher) is within a longitudinal study of children's mathematical thinking from grades 1 -12, with individual task-based interviews for some of the students continuing to their junior college year.

complexity requires the making of judgments about what features are essential in reaching the goal (Maher, 2002). Making strategic choices and decisions about mathematical ideas that have appeared strong, well-grounded, and effective, when tested within novel situations require the learner to fold back, extend and rebuild more efficient and flexible representations in order to deal with the constraints of the new experience (Pirie, Martin & Kieren, 1996).

METHODOLOGY

During two consecutive 80-minute sessions, the children worked on 5 tasks. Each group was videotaped throughout the sessions. These videotapes were analyzed for the identification of critical events in the students' mathematical thinking by pairs of researchers. Justification of each critical event and its transcript were verified by at least one objective observer. These transcripts, the researchers' analyses, and the students' work, are the data for this report (Maher, 1995, 1998).

The first two tasks required the students to generate all possible two digit numbers from the four digits: 1, 2, 3 and 4, first with no constraints and second if there was only one of each digit to use in any given 2-digit number. For each of Tasks 3 and 4, the students were presented a Dice Game. They were asked upon a first reading to predict whether the game appeared fair, and then, after playing the game several times, to decide whether or not it was fair, to justify their conclusions, and to modify the game, if necessary, to make it fair. Following are the game instructions:

> Game One (A Game for two players): Roll one die. If the die lands on 1, 2, 3 or 4, Player A gets a point (and player B gets 0). If the die lands on 5 or 6, Player B gets a point (and Player A gets 0). Continue rolling the die. The first player to get 10 points is the winner.

> Game Two (Another Game for two players): Roll two dice. If the sum of the two is 2, 3, 4, 10, 11 or 12, Player A gets a point (and player B gets 0). If the sum is 5, 6, 7, 8 or 9, Player B gets a point (and Player A gets 0). Continue rolling the dice. The first player to get 10 points is the winner.

During Session One, the three groups developed solutions to each of the first three tasks sequentially, discussed their ideas informally and shared their conclusions in whole-group discussion. The second Dice Game was presented to the groups shortly before the end of this session. The students made their initial predictions within their small groups and began playing the game. As the session ended, the students were asked to continue thinking about this task at home and to bring back any data and conclusions to share at the next session.□Because of the space limitations for this paper, the present analysis focuses on the development of probabilistic ideas through the specific lens of one of the three groups during the second session. This group, sitting at Table 3, included 5 children, 2 boys, Matt and Mike, and 3 girls, Michelle, Amy and Magda. Interactions with other students in the class, the researchers and references to the first session are included as they influence or provide context for the mathematical activity of the five children.

RESULTS

In the final minutes of Session One, the students were asked to share their predictions as to whether the game was fair. Almost all agreed that the game was not fair with Player A having the advantage for receiving points for 6 sum scores (2, 3, 4, 10, 11 and 12) as compared to 5 scores for Player B. A few students noted that Player B had numbers that were "easier to roll" which might make the game fairer. The following discourse occurred among several of the children during this whole-group discussion and provides a context for the debate in Session Two.

Michelle: *(Table 3, predicting that player A had an advantage in the game): ...because Player A has .... um ...*

Amy: *(Table 3): More numbers ...*

Michelle: *Yeah - They actually have 6.*

Amy: *That's what I said.*

Ankur *(Table 1): but 7 is the most commonly rolled number.*

Amy: *It's 6 to 5 (numbers for the two players).*

Ankur: *7 is the most commonly rolled number. Look at all the combinations it has ....6 and 1, 5 and 2, and 3 and 4 - and they can, they can be reversed because you have two dice. There's like 6 possible ways.*

As the students entered the classroom for Session Two, they immediately began discussing the task. Some of the students shared ideas from what they had done at home, while others began to play the game. As pairs of students began playing, the researchers encouraged them to record their actual scores, to show not only the sums, but also the numbers on each die for each roll. The ideas and conclusions within each of the three groups were quite different. The students at Table1 returned with data from home that supported Ankur's earlier statements. Stephanie and Milin had each played numerous games with their fathers and had developed charts to represent what they had found. The four students at Table 2 and the five students at Table 3 appeared not to have thought as deeply about the game and were still uncertain about which player had the advantage.

The 5 children at Table 3 immediately separated into two smaller groups (the boys and the girls) and spent the first half-hour of the session playing the game, carefully recording not only the sum for each roll, but the number on each of the dice. Michelle first stood quietly as an observer behind Amy and Magda, but was encouraged by one of the researchers to develop her own record, playing "against herself". As they shared their results, the five agreed that their original prediction about Player A's advantage was not correct and began to develop strategies to modify the game. The researcher provided overhead transparencies and asked students to prepare their ideas to present to the whole group. The following transcripts of critical episodes during the session document the students' continuing debate about the meaning of the number combinations that were rolled as a means of modeling the outcome space for the game.

Episode One, Negotiating Meaning

Stephanie (Table 1) joined the group at Table 3 to compare solutions.

Stephanie: *Matt, explain your theory to me. I have to find out if they're the same. OK ….Show me.*

Matt: *All right. Since B has the most chances. B has 13 chances. A has only 9. So we figure the way to make it at least a little bit fair is to give them 12 and give them 8, or 7, or which ever one you want to give them.*

Stephanie: *But that doesn't ensure a chance of a win.*

Matt: *Well why don't we try and do it that way? Neither does yours.*

Stephanie: *I know. ………… I didn't say it did. Okay. What's with this? (pointing to Matt's overhead transparency) These are the probabilities? Right?*

Matt: *That's what's with that.*

Stephanie: *So you switched the 12 and the 8?*

Matt: *Yeah. Basically. Yeah.*

Stephanie: *Okay, but…….…B still has a better chance. Because look… what I'm saying is…I'm not saying that this is going to make a difference because for everybody's, no matter what you do, there's never a sure chance of winning. So I'm not saying this is bad. What I'm saying is…let me write on this.*

Stephanie takes a sheet of blank paper and wrote as she was speaking. She copied the two sets of outcomes, drew a line across the page under each set and then wrote the frequency under the first four outcome numbers.

Stephanie: *We have 5, 6, 7, 12 and 9 against 2, 3, 4, 11 and 8. Okay. Now, the probabilities. 5 is 4, 6 is 5, 7 is 6, 12 is 1.*

Matt: *7 is not 6.*

Stephanie: *Yeah. Here let me show you mine. I'll be right back.*

*Matt (*speaking to his group while Stephanie is gone)***:*** *How did she get 5 for 6?*

Magda: *How does she …….* (takes Stephanie's paper and begins to write on it).

Stephanie went back to her seat at Table1 and returned with a matrix chart that she had prepared. The chart headings were the numerals 1 through 6 across the top and down the side. In each cell was the resulting sum of the two digits. Referring to this chart, she continued writing numbers and frequencies on another paper for Matt while she talked (See Figure 1 below.)

Stephanie: *Here, look, Matt. . 2 has 1. There's one way to roll 2. … 3, There's 2 ways to roll 3. You get a 2 on one die and a 1 on the other. Or you can get a 1 on one die and a 2 on the other die. 4 has 3. 2 and 2,….3 and 1,*

Matt:	*So you're doubling everything? …..*
Mike:	*Will you explain this, I don't understand? (Figure 1 below.)*
Stephanie:	*I will in a minute, Mike. All right. 2 has 1 probability. 3 has 2.*
Magda:	*No, it has one!*
Stephanie:	*No it has 2 … because 3 can be rolled with a 2 on one die and 1 on another die or with a 1 on one die and a 2 on the other die.*
Amy:	*But if you don't look at the graph, then it's the same thing. (referring to Stephanie's matrix shown in Figure 1 below.)*
Stephanie:	*It's the same thing but it's used on different dice.*
Mike:	*What is the difference if it says 2 on this side? … What's the difference if it's this or this?* (manipulating the dice as he speaks) *They're both the same thing!*
Amy:	*But if you look on the graph…. It shows different … 1 and 2 and …whatever.*
Magda:	*Can you see the difference? See? (holding the dice) Can you see the difference? That's the …...*
Stephanie:	*That's not the point. See? I can't tell the difference either.*
Mike:	*Exactly. You don't need that. Okay. What is 1 and 2 and 2 and 1? They're the same answer. You're trying to see the difference – the answer.*
Stephanie:	*No. You're trying to show the probability……*
Matt:	*Then why do you have 6 probabilities if there's only 3?*
Amy:	*Because she's changing them. She's changing.*
Michelle:	*She's changing the order of the numbers.*
Amy:	*She is going, 1 and 2, 2 and 1 (writing the two combination for Matt as he manipulates the dice) Stephanie? Are you looking at the total? What are you trying to go at, - other than the probability?"*

At this point, one of the researchers joined the group, listened to the discussion that was going on, and provided dice of two colors, one red and one white. She encouraged the children to investigate further and think about whether the two outcomes were indeed the same. Stephanie returned to her group to complete the preparation of the transparencies.

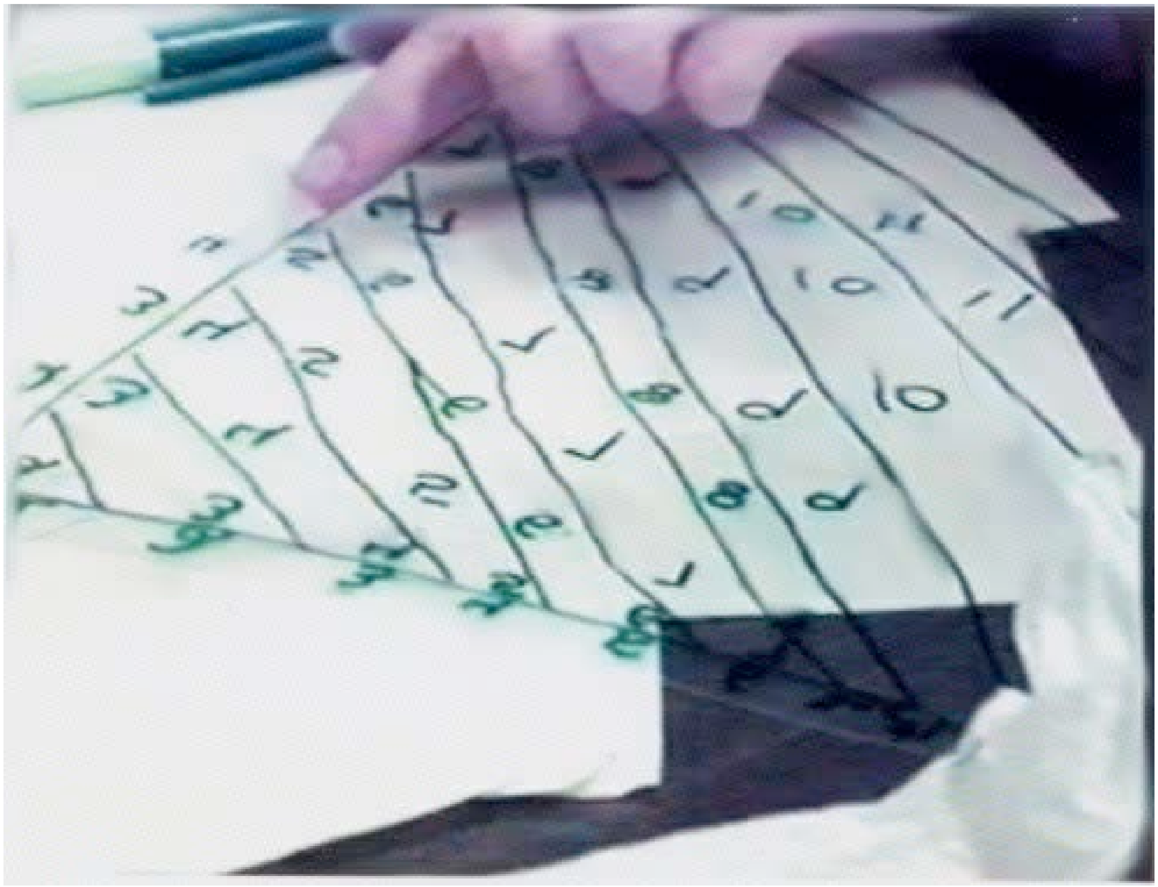

Figure 1: Stephanie's matrix representation of the outcome space

Episode Two: Continued debate

The children at Table 3 continued experimenting with the different dice, discussing the game and challenging each other to predict the sum of a particular roll for some 20 minutes. The following vignette documents that the debate was still unresolved.

Matt:	*What do you want? 6? (rolling the dice).*
Mike:	*There's your 6! (manipulating the dice to form a sum of 6).*
Amy:	*This is the same as ……(reaching for the dice to make a different sum).*
Matt:	*Say you get a 5 and a 1 here, right? It's the same.*
Amy:	*All you have to do is add them up and then …. This is 6 and that's 6. 6 equals 6. It's exactly the same. (writing as she speaks) 6 = 6, 7 = 7, 8 = 8.*
Matt:	*Which one is the one you had before? See, you can't tell .... My point is over!*

Episode Three: Confronting the differences

At this point, 60 minutes into the session, the researcher called the groups together. All the students agreed that Player B had the advantage, despite their earlier predictions. When asked to describe the advantage, Amy asserted that there is one way to roll a 3, with a 2 and a 1. Ankur disagreed.

Ankur: *For the 3, there's 2 and 1 and 1 and 2. 2 is on one die and 2 is on the other die. and 1 is on one die and 1 is also on the other die. It's a different combination.*

The researcher asked for someone to clarify what appeared to be the disagreement.

Shelly (Table 2): He's saying that if you have 1 on one die and 2 on the other, but you could also have 2 on one die and 1 on the other. But it's the same thing. We're working with what it equals up to, not the numbers that are on the dice. We're working with what it equals.

Jeff (Table 2): *Unfortunately, he makes so much sense – because actually you do have to do that to get them. See – look – because if you roll – this die might show a 1 and this die might show a 2 – but the next time you roll it might be the other way around. ……and that makes it 2 chances to get that….even though it's the same number, there are 2 different ways to get it …..*

Stephanie: *Therefore there are 2 different ways to get 3!*

Matt: *For some reason that makes sense. I don't know why, but it does.*

Stephanie then shared her matrix chart, now prepared on a transparency, with the class.

Episode Four: Some resolution

After the discussion and presentations, the researcher presented a final task to the whole group. Their responses indicated considerable resolution about the initial debate, but raised a final issue.

Researcher: *Okay. I have another problem. Roll two dice. If the sum of the two is 7, Player A wins the game. If the sum is 8, Player B wins. Continue rolling until you have a winner. Suppose you have a choice of being Player A or Player B, which would you choose and why?*

Chorus: *Player A.*

Angela (Table 1): Because 7 has the most possibilities…….

Matt: *7 has the same amount as 8 does. . 7 has the same amount as 8 does….*

Several Other Students: No.

Matt: *It does, it does.*

Stephanie: *It doesn't. In this chart, 7 has 6 different ways , 8 has 5. Even though there's really no difference and again, it's a game of luck… I'd rather be Player A because Player A has -maybe - 1 more chance than Player B.*

Matt: *Oh, wait! 7 does! … She's right because you can't switch the 4 and 4 around. It's the same two numbers. It's like impossible. You can't switch the 4 and 4 around because it's the same two numbers, no matter which way you put it. It's a 4 on one die and a 4 on the other."*

Michelle: *That's why she (referring to Stephanie) said there's only 5."*

Matt: *I know.*

CONCLUSIONS

The children, on their own initiative extended the focus of the debate, moving from group to group as they compared and negotiated shared meaning among themselves, displaying powerful evidence of both individual and collaborative ownership of the ideas. They concurred that 1,2 and 2,1 represented different outcomes. They used the concrete representation provided by manipulating the dice to understand and articulate that a pair of "doubles", even when the dice are moved around, still models one outcome in the set.

Early strong understanding about numbers, multiple representations involving composing and decomposing: "6 is the same as 5 + 1, 4 + 2, etc." is basic to children's understanding of arithmetic. Extending this understanding so that the same number combinations can model more complex situations takes time for exploration and conversation. Through sharing, questioning, and debate, children put forth hypotheses, and explore their reasonableness. In so doing, they cycle back to earlier ideas, test these ideas, and develop a deeper and more flexible understanding.

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A WEB-BASED SURVEY TO ASSESS PROSPECTIVE ELEMENTARY SCHOOL TEACHERS' BELIEFS ABOUT MATHEMATICS AND MATHEMATICS LEARNING: AN ALTERNATIVE TO LIKERT SCALES[1]

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Because of the importance of beliefs, mathematics educators need to consider ways to assess beliefs and belief change. Beliefs, because they must be inferred, can be difficult to measure, particularly with a common metric that enables one to compare individuals. Some limitations of Likert scales are identified and overcome in a newly developed instrument in which prospective teachers are provided scenarios to interpret. The instrument captures qualitative data that are quantified for purposes of comparison. Results from two administrations of the instrument demonstrate that it is an effective tool for assessing belief change.

Imagine that prospective elementary school teachers (PSTs) are entering a mathematics course and that you could change one of their beliefs to increase the likelihood that the PSTs would be poised to benefit from what the instructor had to offer? What belief would you change? None of several mathematicians and mathematics educators to whom we have posed this question has questioned its implicit underlying assumption, that beliefs can make such a difference. We take these responses as confirming evidence of the important role beliefs play in mathematics teaching and learning. In considering this question ourselves, we identified seven beliefs we would like to cultivate in PSTs; three are listed in Figure 1.

Belief 1 (About mathematics). Mathematics, including school mathematics, is a web of interrelated concepts and procedures.

Belief 3 (About knowing or learning mathematics or both). Understanding mathematical concepts is more powerful and more generative than remembering mathematical procedures.

Belief 7 (About children [students] doing and learning mathematics). During interactions related to the learning of mathematics, the teacher should allow the children to do as much of the thinking as possible.

Figure 1. Beliefs of interest.

Given the importance of beliefs, assessing beliefs and belief change would benefit mathematics educators. As part of our large-scale research project, Integrating Mathematics and Pedagogy (IMAP), we needed an instrument to assess the belief change

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that we expected as the result of treatments presented concurrently with prospective teachers' first mathematics course for teachers. We first explain why we considered Likert-scale instruments insufficient for assessing the belief change of interest. We then describe the web-based belief instrument we developed and implemented and provide a rationale for its benefits over Likert-scale instruments. Finally, we provide data that indicate that our survey is general enough to capture a range of positions on beliefs but sensitive enough to capture change.

The term *belief* is so common in education literature today that many who write about beliefs do so without defining the term and instead assume that researchers know what beliefs are (Thompson, 1992). We identify four components of beliefs that are important for the way we attempt to measure beliefs. First, beliefs influence perception (Pajares, 1992). That is, beliefs serve to filter some complexity of a situation to make it comprehensible, and, therefore, when inferring beliefs, one must determine to what one attends in a situation. Second, beliefs are not all-or-nothing entities; they are, instead, held with different intensities (Pajares, citing Rokeach, 1968); thus, when measuring beliefs, we consider tasks that offer multiple interpretation points. Third, beliefs tend to be context specific (Cooney, Shealey, & Arvold, 1998), and, hence, we situate belief items in particular contexts and infer a respondent's belief on the basis of his or her interpretation of the context. Fourth, beliefs might be thought of as dispositions toward action (Cooney et al.,1998; Rokeach, 1968); therefore, we infer one's belief on the basis of how the person might act in a particular situation.

Why Likert Surveys Are Insufficient for Our Purposes

However one chooses to define (or not define) beliefs, "for the purposes of investigation, they must be inferred" (Pajares, 1992, p.315). Our work required us to identify a belief instrument that might be administered to large numbers of prospective elementary school teachers years before they were in the classroom.

We identified three problems with Likert Scales and attempted to overcome them with our instrument. (Figure 2 lists two items drawn from Likert-scale belief instruments.) First, knowing how a respondent interprets the words used in items is difficult. For example, for Item 2 (see Figure 2), one needs to know whether the respondent distinguishes among situations in which the child listens: when teachers demonstrate procedures, when teachers present problem situations, and when students share unusual thinking. For Likert items, respondents are asked to agree or disagree with statements, whereas in our survey, respondents use their own words to react to, or answer questions about, learning situations. Although this format does not remove the need to draw inferences, it reduces it.

Second, we think that beliefs can be inferred by determining to what one attends in a complex situation, and Likert scales seldom provide contexts. For example, in Item 1 (see Figure 2), would whether one imagined a kindergarten class or an 8^{th}-grade algebra class be relevant? In our instrument, each item is embedded in a context, so one can better determine to what the respondent's attention was drawn.

Item 1. In mathematics, perhaps more than in other fields, one can find set routines and procedures. (Collier, 1972)

Item 2. It is important for a child to be a good listener in order to learn how to do mathematics. (Fennema, Carpenter, Loef, 1990)

Figure 2. Two Likert items.

Third, Likert items do not carry with them good ways for determining how important the issue is to the respondent. One may respond in a way that may indicate the existence of a belief that is not central to the respondent. McGuire (1969) stated, "When asked, people are usually willing to give an opinion even on matters about which they have never previously thought" (p. 151). For example, a respondent may agree strongly with Item 2 (see Figure 2) but not believe that listening matters as much as speaking or other activities in mathematics. We addressed this issue in our belief survey by drawing inferences from that to which respondents attended in learning episodes and when they attended to certain issues.

Our Belief Instrument

We set out to create an instrument to assess beliefs that might affect PSTs' subsequent learning of mathematics: beliefs about mathematics and mathematics understanding and learning. We wanted an instrument that would provide a common metric for measuring change in individuals and for comparing individuals to one another. We also wanted the instrument to provide qualitative data that could be used for more holisitic analysis. To avoid the limitations of Likert scales, outlined above, we developed an instrument in which PSTs construct responses instead of choosing from options provided. We later quantified these constructed responses using rubrics. We designed items to measure only the seven beliefs we had identified (three of the seven are listed in Figure 1).

Development of Instrument

This instrument and the accompanying scoring rubrics were developed over a 2-year period by the authors with support from other staff members. We used a recursive cycle of development that included piloting segments of the instrument, analyzing PSTs' responses to the segments, revising the segments, and piloting them again.

The instrument contains seven segments. Each segment includes several questions about a particular situation. Four segments are in the domain of whole number, two are in the domain of fractions, and one is a more general teaching segment. The chosen domains were the domains of focus for our experimental treatments and were important topics in the mathematics-for-teachers course in which the PSTs were enrolled. Two segments included video clips of individual children doing mathematics problems with an interviewer. Each segment is associated with two or three beliefs, and each belief is assessed using a separate rubric for each of two or three segments. Overall, we developed 17 rubrics for the instrument.

To illustrate how we assigned scores for each belief, we describe one of the segments, the rubric used to assign scores to prospective teachers' responses to that segment, and the scoring system used to combine scores on individual rubrics to determine an overall score for the belief. The scores on segments and on beliefs reflect the amount of evidence a respondent provided related to the belief. This scoring is in keeping with the idea that beliefs can be held with different intensities and are more or less central (Rokeach, 1968).

In the segment, prospective teachers watch a video clip of a teacher presenting a story problem to a 6-year-old child in a one-on-one setting: "There are 20 kids going on a field trip. Four children fit in each car. How many cars do we need to take all 20 kids on the field trip?" After a long pause, the child stated 10 as his answer. He confirmed that he had guessed when the teacher asked. The teacher then directed the child to show her the kids by counting out 20 cubes. She then reminded him that 4 kids fit in one car and asked him to show her 4 kids in one car. She directed him to make another group of 4 for the next car, and he followed her directions. She continued in this fashion until he had made five groups of 4 cubes. She then reminded him that each group stood for a car and prompted him to count each of the cars. She counted along with him.

Our interpretation of this clip was that the teacher was overly directive and focused the child's attention on counting cubes instead of on understanding the relationships among the quantities in the situation. She could have provided prompts that were less specific, to see whether the child could solve the problem with less help. For example, she might have invited him to try to use the cubes to represent the situation and then waited to see what he would do before providing him with additional help. The clip featured a familiar real-world context and manipulatives. The teacher was encouraging in her tone of voice and in providing the child with praise. These positive features of the clip were quite attractive to some respondents, leading them to focus on these aspects instead of on the excessive guidance offered by the teacher, as is evident in the following response:

> *I thought it was good that she let him try and answer the problem first and then she showed him how to figure it out using the blocks .... They need to test things out themselves and then see the different ways to approach a problem.… I think the strengths of this video were allowing the child to think on his own and solve the problem.… I didn't see any weaknesses in this video clip. I really liked it.*

In addition to being impressed with the teacher's use of blocks, this respondent wrote about the importance of letting the child figure out the problem for himself. She used the rhetoric that we would like PSTs to employ, but in this case she applied the rhetoric in a context in which we believe it was inappropriate. Responses like this one reminded us of the critical role that context played in inferring beliefs from responses. Without knowing the context to which these comments were directed, one might interpret this response as providing strong evidence of Belief 7.

The complete rubric used to score this segment (for Belief 7) is provided in Figure 3. We were particularly concerned about whether the respondents noted that the teacher did too much leading and when they noted that fact. Those that noted excessive guidance in their response to the first prompt, "Please write your reaction to the video clip. Did anything stand out for you?" provided strong evidence of this belief because the issue mattered so much to these respondents that it shaped their interpretation of the episode. For subsequent prompts, "Identify the strengths of the teaching in the episode" and "Identify the weaknesses of the teaching in the episode," some respondents noted that the teacher might have provided too much guidance after the third prompt. In this case, we determined that the respondents provided some evidence of the belief. It was not strong evidence because the issue did not matter enough to them to shape their initial interpretation of the clip. Because the survey was web-based, respondents could not change earlier responses. Open-ended questions allowed us to discern which issues mattered enough to respondents to affect their interpretations.

<table>
<tr><td>No Evidence
Overall satisfaction with guidance provided by teacher
No teaching weaknesses identified</td><td>No Evidence
Thought the teacher should explain more.</td></tr>
<tr><td>Weak Evidence
In initial response does not mention teacher’s excessive guidance. In second prompt expresses satisfaction with the guidance offered by the teacher. In third prompt, points out that child may not have needed so much help.</td><td>Weak Evidence
In initial response suggests that this problem was too hard and inappropriate for this child to solve.</td></tr>
<tr><td colspan="2">Evidence
In initial response does not mention that the teacher did too much leading. In second prompt identifies cubes or story problem or positive reinforcement as strengths but does not talk about teacher’s guidance as strength. In third response, critiques teaching for being too leading.</td></tr>
<tr><td colspan="2">Strong Evidence
In initial response notes that the teacher was too leading. In third prompt criticizes teaching for being too leading.</td></tr>
</table>

Figure 3. Scoring rubric for Belief 7/Segment 7.

Each belief was measured by more than one segment to give a valid measure. In the case of Belief 7, the second segment used to measure the belief included the only general (not context specific) segment: Respondents were asked, first, whether they would ever ask children to solve problems without first showing them how and, second, to explain their answer. Their explanations were assigned scores on the basis of the amount of autonomy they planned to provide for children and their rationale for doing so. Respondents who suggested that children understand more mathematics when they devise their own solution strategies were given a *Strong Evidence* score.

To determine a final score for each belief, we combined individual scores from each rubric. Because of the ordinal nature of the scores, summing scores and discussing means for each belief were inappropriate. We developed a rubric-of-rubrics system that could be applied to each belief. In this system, we accounted for the differing strengths of beliefs by having a range of scores for the rubrics and the belief scores.

Because the instrument did not employ Likert scales, traditional tests typically performed on surveys were not appropriate. We confirmed the validity and reliability of the instrument by administering it to 18 PSTs and conducting follow-up interviews. We also administered the instrument to five mathematics educators experienced at teaching and researching PSTs’ beliefs. We had extensive follow-up conversations with them; they confirmed that the questions on the instrument and the rubrics used to score responses were valid measures of beliefs. When coders used the rubrics, 20% of the responses were coded by two coders, and we achieved, on average, 84% reliability on all 17 rubrics.

HOW EFFECTIVE IS THE SURVEY?

We were interested in determining whether the belief survey was sensitive enough to measure a range on each of the seven beliefs and whether the instrument would measure belief change. For 159 PSTs enrolled in the first of four mathematics courses for prospective elementary school teachers, we administered the assessment as a pretest (at the beginning of the course) and as a posttest (at the end of the course). (In our IMAP study we assigned these 159 PSTs to treatments, but presentation of those data go beyond the scope of this paper.) Pretest results indicate that most of the PSTs initially showed *no evidence* of holding each belief (see Table 1), and nearly all the PSTs (90%, 77%, and 96% for Beliefs 1, 3, and 7, respectively) fell into one of two categories, showing either no evidence or weak evidence. We found variation in scores in the pretest, showing that the instrument captured individual differences.

	No evidence	Weak evidence	Evidence	Strong evidence
B1 Pretest	60% (95)	30% (47)	9% (15)	1% (2)
B1 Posttest	19% (30)	40% (64)	25% (39)	16% (26)
B3 Pretest	64% (102)	13% (20)	18% (29)	5% (8)
B3 Posttest	28% (44)	11% (18)	26% (41)	35% (56)
B7 Pretest	71% (113)	25% (39)	4% (7)	0% (0)
B7 Posttest	40% (64)	36% (58)	19% (30)	4% (7)

Table 1. Pretest and Posttest Scores for Beliefs 1, 3, and 7 (n = 159).

Posttest results indicate that many PSTs' beliefs changed over the semester, with far fewer *No Evidence* scores and a greater number of *Strong Evidence scores* on the posttest (See Table 1). Many PSTs' responses were still coded as indicating no evidence of the beliefs. We interpret these results as indicating that our belief survey was not simply a measure of information that could be easily learned or parroted back to us by PSTs over the course of a semester in which they took a mathematics course for PSTs, as might be the case in a test of knowledge. The range of interpretations possible for each segment allowed PSTs' beliefs to emerge. Clearly, some had changed and some had not.

CONCLUSIONS

A major strength of our instrument is that it uses video clips and learning episodes to create contexts to which users respond in their own words rather than choose from one of several options. This format provides qualitative data that can be used for a variety of purposes. It also provides detailed information about the respondents' interpretations of the questions they are asked. This strength comes with a cost in terms of time required for coders to learn to use the rubrics and translate the constructed responses into quantified responses. Whether this "price" will be too high for those seeking a belief instrument is an important question left to be answered.

The mathematics of the instrument was whole number place value and rational number. We suspect that the instrument would have been different were it intended for different content, say geometry, or for a different population, say preservice secondary school teachers or practicing elementary school teachers. For example, we have given little thought to whether different approaches are needed for investigating the relationship between concepts and procedures in geometry and arithmetic. Our pilot work using the instrument with in-service teachers has been encouraging, because we found that we continued to capture a range of scores. We make no claim about the efficacy of this instrument with secondary school teachers.

Although our instrument measured change between the beginning and end of the treatment, it seemed neither "too easy" nor "too difficult"; that is, it measured neither a floor effect nor a ceiling effect. Although the high scores on the pretest were few, we did measure variation, and on the posttest we found low- and high-scoring PSTs.

Beliefs are inferred by someone who holds beliefs. The most those inferring the beliefs can do is to be clear about what those beliefs are and how those beliefs were operationalized so that others considering using the instrument can decide whether they value those beliefs and whether they agree with how those beliefs were measured. We would be presumptuous to claim that we have created an instrument to measure beliefs about mathematics and mathematics learning, so we will state only that we have created an instrument that measures seven specific beliefs about elementary school contexts. We think that we have developed a belief instrument that can effectively measure quantitative differences while still capturing the individual voice of respondents that, in the past, has been captured only through qualitative approaches.

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'SENSING': SUPPORTING STUDENT UNDERSTANDING OF DECIMAL KNOWLEDGE

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Informed by theory and research in inquiry-based classrooms, this paper examines how classroom practices support students' understanding of decimals. Data from a six-month teaching experiment, based on the work of Moss and Case's (1999) use of percentages and metric measure as visible representations for students' emerging understanding of decimals, indicated that understanding was significantly influenced by a classroom climate that supported sense making. Sensing, established by a shared expectation, was used and extended in the form of sociomathematical norms associated with mathematical argument and authority.

INTRODUCTION

Construction of robust decimal concepts is dependent upon students engaging in an active learning process in order to integrate prior whole and fractional number thinking and build multi-levelled and multi-connected decimal concepts (Boufi & Skaftourou, 2002). Within New Zealand, decimal fractions are formally introduced to students aged nine or ten years. For students within this age group, constructing conceptual understanding of decimal fractions is traditionally difficult (Moss & Case, 1999). Such understanding requires reconstruction of prior whole and fractional number concepts and integration of place value concepts using base 10 notation to represent the fractional quantities (Irwin, 1999). The process is lengthy with regular recurring misconceptions and partial understandings occurring as students integrate their prior knowledge with new learning along the path to sense making (Condon & Hilton, 1999).

While many research studies on decimals illuminate the nature of student misconceptions (Stacey & Steinle, 1998), or the role of specific teaching activities (Helme & Stacey, 2000; Hiebert & Wearne, 1986), increasingly classroom studies focus on the role of students' informal understandings (Boufi & Skaftourou, 2002). This study reports on a teaching experiment based initially on the work of Moss and Case (1999) that promoted the use of percentages and metric measure as visible representations for students' emerging understanding of decimals. Fundamental to the success of the teaching experiment was a mathematical learning environment that supported students' sense making—an environment in which student reasoning was foremost, with an expectation that they explain, explore alternative ideas, and validate or reorganise their thinking.

This paper explores the defining features of the classroom climate which supported the extension of the making of explanations from beyond a classroom social norm to a sociomathematical norm and the way in which this promoted students' development of rich understandings of decimals. Our theoretical framework is derived from Cobb and Yackel's (1996) emergent perspective. From this perspective the construction and reconstruction of decimal concepts is assumed to be both an individual and social activity. In such a learning environment, challenges to thinking patterns occur during discussion and debate as students actively engage in making sense of other's explanations, to elaborate, argue, and justify mathematically their own current thinking (Kazemi & Stipek, 2001; McClain & Cobb, 2001; Wood, 2002). Thus, learning involves reflexivity between individual student activity and participation in classroom practices that support collective activity.

RESEARCH DESIGN

The study involved a 6-month classroom teaching experiment conducted at a large inner city primary school. Students came from predominantly higher socio-economic levels and represented a range of ethic backgrounds.

A collaborative partnership between the researchers and the teacher supported the development of a hypothetical learning trajectory and an instructional sequence, which through on-going discourse and data analysis was revised and modified as required (Cobb, 2000). Following individual interviews, four students were selected as case studies to represent the range of decimal misconceptions common to students within the age group. Data were collected from case study student interviews, 15 lesson observations, and classroom artefacts.

Analysis of data used a grounded approach identifying codes, categories, patterns, and themes. These were used in conjunction with participant dialogue in order to give voice to the students as they participated in classroom practices designed to support the construction of decimal concepts.

The research classroom was first and foremost distinguished by the extent and quality of productive discourse that supported students' reflective reconstruction of decimal knowledge. What characterised this classroom socially and intellectually was a shared expectation that all students actively engage in examination, analysis, and validation of their decimal understandings through reasoned discourse (Wood, 2002). Such expectations, while not unique to this classroom, are fundamentally different from that experienced by many primary aged New Zealand students (Walls, 2002).

RESULTS AND DISCUSSION

Establishing classroom norms that support students' understanding requires both in-depth knowledge of the mathematics involved and the students' mathematical thinking (Peterson, Fennema, Carpenter, & Loef, 1989), and the ability of the teacher to advance students' mathematical thinking (Fraivillig, Murphy, & Fuson, 1999). The focus of this paper is to articulate the teacher's development and support of students' understanding of decimal through her expectation, use, and extension of 'sensing'— a norm, which represented not only active listening and making sense of explanations, but also questioning, clarifying meaning, making predictions and justifying conjectures.

Expecting 'sensing'

The establishment and maintenance of this expectation of sensing was achieved though a variety of pedagogical strategies. Clearly evident in the classroom participation structure was an expectation by the teacher that all students would actively engage, not only physically, but also mentally in all mathematical activity. This included an expectation that individual students were responsible for active listening and making sense of explanations in their collaborative groups and in the large group sharing sessions which occurred at the completion of each teaching/learning episode. The teacher regularly reinforced this notion of ‘sensing’ through her directives to the class:

> Now while you are listening to the explanations I want you to turn your sensing on, ask questions at any time and search for answers. Listen carefully so that you can predict what might be said next.

Moreover, students consistently maintained the expectation that during any mathematical discussion all explanations should make sense. They recognised that it was their personal responsibility to actively question further in order to understand or clarify an explanation. This is illustrated in the following extract when one student has expressed confusion and another student, assuming collective responsibility, instructs the first student:

> Well just listen to him and see if he can make it clearer otherwise think of some other questions we need to ask him to make him explain it better.

The effect of individual students asking questions not only probed all group members' current understanding of decimal concepts, it supported thinking at a deeper conceptual level, fostered through the exchange of ideas. This is illustrated in the following episode in which three students discuss the ordering of 0.9, 0.9015, and 0.90146.

Sara:	That's biggest [pointing at .9] and that's smallest [pointing at .90146].
Fay:	But why? Why do you think .9 is the biggest?
Sara:	I think that is the biggest because that is the tenths, like that's only tenths so like that's a larger number. This [.90146] goes into something like thousandths and they are smaller bits, so it makes it smaller.
Fay:	Yeah but this could be .90 and this [.90146] is just a teeny bit more than .9.
Sara:	So is the .9 bigger than .90146?
Fay:	[Using decimal arrow cards] But if you see that .9 means 9 tenths but this [.90146] has 9 tenths, one thousandth, 4 lots of tenths of thousandth, and 6 lots of hundredths of thousandths more, so it is the biggest.
Jane:	Can you explain that again?
Sara:	What she is saying is that one is only nine tenths, but that also has ones of thousandths, tenths of thousandths and hundredths of thousandths, so it's the biggest actually, not the smallest.

Through such discussion a model was developed which represented the group’s collective strategy and culminated in a problem solution.

Exploration of problems involving decimal operations that were counter intuitive to whole number thinking supported reconstruction of understandings through cognitive conflict as the students argued a path through to group consensus. This is illustrated in the following episode involving three students’ joint construction of a notation scheme to represent the subtraction of 0.7 from 2.30. In response to Eric’s recording of 1.60, Brenda using 'whole number thinking' says:

Brenda:	So it will be 1.23?
Eric:	No it will be 1.60.
Brenda:	But that's not .70 it's 7 [pointing at the .7].

Eric : It's .70, seven tenths is 70% [pointing at approximately 70% on an enlarged number line on the floor]

Philip: [Extending the explanation] Oh yes, it's .70 because if that was a 7 there would be a zero and it would be .07.

The listening audience used other students' questions and answers to reformulate their own thinking about decimals. This is illustrated in a written reflection recorded following a problem-solving activity involving a prolonged collective discussion about decimal numbers below 0.1 and the role of zero as a placeholder.

> I lurnt [sic] from Jane's mistake. I had ideas about it and when Helen commented on it I thought back over my thinking, sort of recapping and found out I was write [sic].

In doing so the students are demonstrating a shift from participating in making an explanation—to making the explanation itself an object of reflection.

Extending 'sensing'

Making explanations of mathematical thinking is common practice in many classrooms. However, a fundamental focus of reform-based mathematical classrooms is the way in which explanations extend beyond procedural description to mathematical reasoning based on justification of specific problem-solving strategies. The teacher regularly reinforced the need for explanations to be extended as she instructed the students:

> Don't forget to work together so that you can explain how and why the strategy you used worked for you all. Don't look for a quick solution; prove it this way and then that way then check back to your benchmarks that you use. If you are having problems go back to what you know so that you are all able to go forward again.

During small group problem solving episodes, student questioning and argumentation extended explanations to include justification. The students recognised the power of supporting verbal mathematical explanations or arguments with a demonstration, usually in the form of an invented notational scheme, graphical representation, or physical re-enactment. This is illustrated as the students collaboratively constructed a notational scheme in order to add 34.07 and .005

Jane: So what did you do? Can you show me again?

Sara: We added them together 34.07 and .005 and that gave a score of 34.075 because…

Jane: Why?

Eric: [Pointing at the notation] Those are five thousandth [records 5/1000 on the sheet] and those are seven hundredths [records 7/100 on the sheet].

Fay: But I thought those were a thousandth. I thought the seven was a thousandth.

Eric: [Points at his earlier recording] No those are the five thousandth and those are the seven hundredths, see, and you need ten of those to make one of those but you haven't got ten of those so it's not one of those.

Sara: Yeah so you just leave it like that 34.075

Jane: I still don't get how you got that .075 because plus .07 and .005 it just doesn't make sense.

Eric: Yeah it does because look… [Eric illustrated his point with a concrete model].

Multiple strategies were often recorded in small groups with the most 'sophisticated' strategy presented to the larger class group. Selection of the most sophisticated strategy, while contingent on understanding by all group members, also included 'taken as shared' knowledge of the most efficient or 'tidy' steps. The students took for granted their right to question or challenge explanations, and in response, receive an explanation based on mathematical reasoning. This is illustrated in the following example in which three students solve a problem involving the subtraction of 0.37 from 3:

Stefan: I think it is .63 because if he has got 3 metres and he only wants 2 metres and 37 centimetres then if you add .6 onto that and .03 more it makes 3 metres.

Sara: Yeah but does that work and how are we going to show it?

Stefan: Well on a number line so start at 2.37 and add .6 oh no put the .03 first.

Georgia: But why?

Stefan: I am making a tidy number, like sort of rounding, so everyone can see, and then you just have to put 6 tenths.

In addition, the students understood that collectively and individually they needed to be able to give clear, logical explanations of a group strategy to the larger sharing group. This influenced their behaviour in collaborative groups where they discussed ways to explain and validate their mathematical thinking. The expectation that other students would also be expected to 'make sense' of one's explanation led to extended discussion and translation across representations as the students sought explanations that they deemed would be understood and acceptable to their peers:

Brenda: What's like maybe an easier way of explaining it?

Eric: Oh yeah because remember that Miss Smith said that we all have to understand how to do it so can you guys do that if you use my way? You could do a number line what about that?

Brenda: Yeah cos I go with number lines. I find them easier to understand and so does everybody I reckon.

Eric: Well you do know that you just write down the same strategies but it looks a bit easier to understand.

As students engaged in these mathematical conversations about decimals, the prolonged discourse, which was a feature of many such interactions, frequently led to the reconstruction of students' erroneous thinking patterns.

Using sensing

The intellectual climate established in the classroom not only held all students responsible to participate in 'mathematical conversations'—it maintained an expectation that all members of the listening audience would consider seriously other student ideas proposed during discussion. During group presentations the teacher used the notion of 'wait time' to allow other students the opportunity to make sense of the mathematical concepts being explained and also to support engagement in reflective analysis of similarities and differences of solution strategies with their own group strategies. This is illustrated in the following episode in which the student making an explanation and recording each step numerically is paused by the teacher and asked by another student to justify his thinking:

Adam: Can you explain where you got the .033 from? Oh but I see…

Eric: We're trying to make a tidy number. See three thousandths and seven thousandths is another hundredth then three hundredths and seven hundredth is another tenth. Nine tenth plus one tenth is another whole so that is seven.

Adam: Yeah that's an efficient step.

In this way, the sharing of notational schemes increased students' awareness of more conceptually advanced mathematical thinking:

> Next time I will do William's way because it was easy to do and easy to track and cut down some of our steps. But then again doing it our way I knew exactly what we were doing and as well I was sort of using tidy numbers to make the steps quicker. [Student journal entry]

Verification of mathematical thinking was also an integral part of the process as the students focused their attention on similarities and differences in the mathematical concepts and not merely procedures. In order to maintain focus on the mathematics in explanations, there was an accepted norm that small group solutions were only shared with the large group if the group could verify that their explanation differed mathematically from those already shared. This is illustrated in the following explanation of a strategy to add 1.13 and 1.28.

Brenda: Start from 1.13 and add on .07. Do you know why?

Jane: So it gets you to .2, a tidy number.

Brenda: Then plus 1 metre and 20% of another metre, so 2.4, add .01or 1% so it's 2.41. That's a different way because I was using some percentages as well as some tidy maths and this way is quicker. The last group used fractions and they took lots more steps in that bit…

During large group discussions, listening students also reflected on their own solutions monitoring the differences and identifying possible errors in their group solution:

Jane: That's different from ours.

Fay: Yeah that is where we are wrong…oh yes see where we added the wrong tenth and hundredth to the wrong ones.

The acceptance of errors as a basis for mathematical inquiry provided ongoing opportunities for all students to engage in mathematical analysis and re-conceptualisation of their developing understandings. Often the teacher would place two group's record of their solutions alongside each other, reinforcing the right of each group to warrant their own explanation:

> These children are checking their thinking, let's give them some thinking time, and then they can explain what they did differently and justify their reasons.

Such 'teachable moments' provided a potential source of cognitive conflict—an essential element in the construction and reconstruction of robust decimal fraction concepts.

In this climate the growth of intellectual autonomy was evident; students demonstrated confidence in their own ability and that of their peers to make mathematical decisions and warrant their own solutions. Rather than appeal to the teacher for help or authority students regularly explored alternative strategies and validated their solutions to decimal fraction problems through alternative representations of percentages, metrics, fractions and decimals.

CONCLUSIONS

The teaching experiment was designed to build on students' informal understandings and strategies. However, descriptions of the learning environment presented in this paper clearly recognise that the teacher needed more than an awareness of the array of possible student misconceptions and current conceptions about decimals; the teacher needed to provide a learning environment that challenged the traditional classroom participation structure for all of the participants. In the classroom involved in this study, a participation structure of collaborative interaction and discourse was central to students' mathematical development, both in its role as supporting individual construction and transformation of decimal concepts and as a social act within the mathematical community.

The participating teacher spontaneously developed the metaphor of 'sensing' to negotiate with her students the shared expectation of sense making as central to the learning process. This strategy overtly positioned the teacher as her own author of an inquiry-based pedagogy. Through her establishment and ongoing support of 'sensing', the teacher provided a classroom environment where all members interactively constituted the social and sociomathematical norms. Their experiences at examining, discussing, and reflecting on their mathematical constructions assisted their on-going development of decimal understandings.

Overall, the classroom environment portrayed a vision of mathematics learning—neither wholly individual nor wholly social—which enabled connections to be made between the person, the cultural and the social. Students appeared to 'learn' from their participation in the cultural milieu of their classroom rather than from other students or the teacher per se (Yackel & Cobb, 1996). The 'sensing' metaphor encapsulated the learning agenda advocated by Kirshner (2002) in which specific mathematical dispositions are targeted for instruction through the development of the classroom microculture. Sensing, as understood by the participants in this study, appeared to go some way to mediate the conceptual and dispositional goals for student learning.

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NON-EXAMPLES AND PROOF BY CONTRADICTION

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Researches in Mathematics Education about proof by contradiction revealed some difficulties of the students but also that this kind of argumentation comes spontaneously in certain situations. In this paper we shall show some processes that might lead the student to produce a proof by contradiction. In particular, we shall point out a deep link between a certain kind of examples (which we call non-examples) generated during the stage of conjecture production, and the structure of the argumentation with which the conjecture is justified.

INTRODUCTION

There are few researches in Mathematics Education which have specifically dealt with proof by contradiction. An analysis of these studies points out that the proof by contradiction, from cognitive and didactical points of view, seems to have the form of a paradox.

First, it is well known that proving by contradiction is a complex activity for the students of various scholastic levels. See, for example, Bernardi (2002), Antonini (2001), Reid (1998), Epp (1998), Thompson (1996), Barbin (1988), Leron (1985).

On the other side, some studies describe proof by contradiction as an argumentation that students spontaneously produce. For example, Freudenthal says:

"The indirect proof is a very common activity ('Peter is at home since otherwise the door would not be locked'). A child who is left to himself with a problem, starts to reason spontaneously '... if it were not so, it would happen that...' " (Freudenthal, 1973, p. 629).

Freudenthal concludes that "*Before the indirect proof is exhibited, it should have been experienced by the pupil*" (Freudenthal, 1973, p.629). Also Thompson writes:

"If such indirect proofs are encouraged and handled informally, then when students study the topic more formally, teachers will be in a position to develop links between this informal language and the more formal indirect-proof structure."(Thompson, 1996, p.480).

Following Freudenthal, we take as working definition of *indirect argumentation* an argumentation like "*…if it were not so, it would happen that…*". A more articulate definition has been developed in [Antonini, 2003] but the one just given is enough for the topics treated in this paper.

Assuming that students spontaneously produce indirect argumentations, it is very interesting to study favourable conditions that can help the generation of such argumentations. The goal of this paper is to investigate the processes that lead to the construction of an indirect argumentation. In particular, we shall focus on some factors that favour the rise of indirect argumentation.

THEORETICAL FRAMEWORK

A suitable theoretical framework is the Cognitive Unity which has been introduced and developed in [Garuti et al.,1996a,1996b,1998; Mariotti et al.,1997; Pedemonte,2002]. The framework of Cognitive Unity is based on the following:

"during the production of the conjecture, the student progressively works out his/her statement through an intensive argumentative activity functionally intermingled with the justification of the plausibility of his/her choices; during the subsequent statement proving stage, the student links up with this process in a coherent way, organising some of the justifications ("arguments") produced during the construction of the statements according to a logical chain." (Garuti et al. 1996b, p.113).

Cognitive Unity regards the relations between processes of argumentation and of proof construction. In particular, Pedemonte (2002) analyses and compares the structure of the argumentation and of the proof produced by students. In this theoretical framework, we basically pose two questions. First, which are the factors that can favour the production of an indirect argumentation. Then, which factors can induce or block the transition from this argumentation to a proof by contradiction. In this paper we focus on the first question.

We begin by remarking that the analysis of the processes of the construction of the conjecture has shown how frequent it is for the students to produce one or more examples during the solving of an open-ended mathematical problem. We divide these examples in two classes: the ones that verify the conditions given by the problem and the ones that do not. In particular, a student facing a task like "*given an hypothesis A what can you deduce?*" can produce examples which verify A and examples which do not verify A. The latter will be called *non-examples*.

Non-examples are particularly interesting because, with the construction of a non-example, students seem to produce indirect argumentations like "*…if it were not so, it would happen that…*". Ones a non-example has been generated, the student seems to be led to a formulation of a conjecture in negative terms, like "*given A, it is not possibile for B to be true*".

Moreover, sometimes the students links the argumentation of the conjecture with the argumentation given to justify the fact that the example generated is a non-example: "*it is not possible for B to be true because, otherwise, A would not be true (as seen in the non-example)*".

The hypothesis we formulate is the following:

In task like "given A what can you deduce?" the conjecture can be produced via the analysis of a non-example. The argumentation that justifies the fact that the generated example is a non-example can be re-elaborated and become part of the argumentation of the conjecture. In this case, the argumentation takes an indirect form.

METHODOLOGY

The research that we are going to explain here is part of a wider work on proof by contradiction which is the subject of [Antonini, 2003]. To develop the research for this paper we interviewed 9 pairs of students: 7 pairs from the last three years of secondary school (grade 11 to 13) and 2 pairs from the second year of the degree course in biology at the University. The students have been given an open-ended problem and they have been asked to formulate a conjecture and to prove it. We decided to interviewed pairs of students to favour the rise of argumentative processes. This kind of methodology has already been successfully used by other researchers (for example see Pedemonte, 2002).

The choice of the problem for the interviews has been guided by the hypothesis made on the relations between the production of a non-examples and the construction of an indirect argumentation. The problem is the following:

Two lines r and s on a plane have the following property: each line t intersecting r, intersect s too. Is there anything you can say about the reciprocal position of r and s? Why?

Let be A the property "*each line t intersecting r, intersect s too*". The problem asks to determine the reciprocal position of two lines verifying A. Hence the text is of the form: "*given A, what can you deduce?*". Moreover, since the position of the two lines is not known, indeed it is what has to be found out, the students have no initial configuration to refer to and that they can use to start the study of the problem. We think that this lack of a starting point can lead to the production of examples and to the analysis of various cases. In particular we expect that the students will consider two cases: two parallel lines and two crossing lines. Property A is verified only in the first case. Indeed, when r and s intersect each other, it is possible to find a line intersecting r and not intersecting s (just take any line parallel to s). We expect the students to produce non-examples, i.e. examples not verifying A, and, moreover, we expect them to propose an indirect argumentation for the conjecture formulated in this way.

Valerio and Cristina: non-examples and indirect argumentation

In what follows we give some excerpts from an interview in which it is evident how the production of a non-example determined the indirect structure of the argumentation. The two students, Valerio and Cristina (grade 13) are very well considered by their teacher.

18. V,C: .....
19. V: Oh, they [r and s] cannot be perpendicular.
20. C: No, that's not possible.
21. V: They cannot be perpendicular because otherwise it [the line t] could be parallel to one of the two and do not intersect the other one (he makes a drawing, see figure 1)

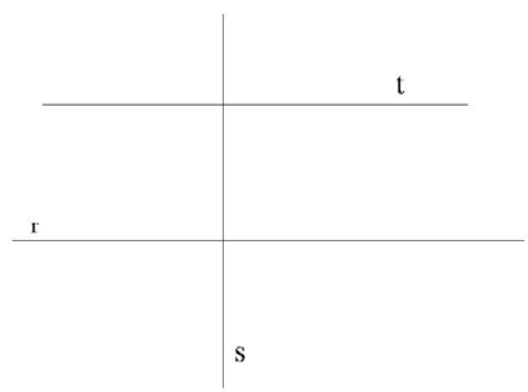

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Figure 1

Valerio produce a non-example made by two perpendicular lines. The conjecture is formulated in negative terms ("*r and s cannot be perpendicular*") and it is justified indirectly: “*they cannot be perpendicular because otherwise*” A would not be true. Later, during the interview, Valerio generates a second non-example which recalls the first one.

31. V: Well, [the line t] cannot be parallel to any of the two lines because, if we have two crossing lines, even if they are not perpendicular (he makes a drawing, see figure 2), if [t] is parallel to one of the two, it intersect just one of them.
32. C: Yes, it’s the same situation of the two perpendicular lines.

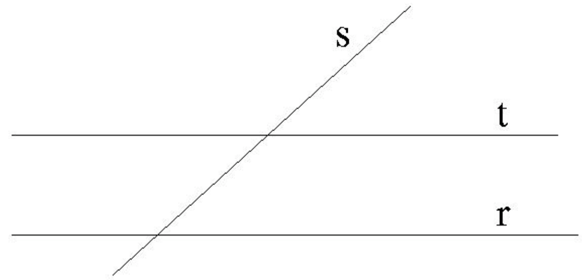

Figure 2

The second non-example is made by two crossing lines which do not verify the property A. The argumentation is indirect: the two lines do not intersect each other because, “*if we have two crossing lines*” they do not verify A. The second non-example is a generalisation of the first one: the condition of being perpendicular lines is generalised to the condition of being crossing lines. It is interesting to notice that the two students are aware of the fact that the second non-example is a generalisation of the first one. Indeed, Valerio says “*even if they are not perpendicular*” and Cristina adds “*Yes, it’s the same situation of the two perpendicular lines*”. Moreover, the argumentations of the fact that they are non-examples have the same structure, and finally, the argumentation of the conjecture is an indirect argumentation. Later on Valerio and Cristina go on as follows:

40. V: Oh yes, then they [r and s] definitely have to be parallel.
41. C: Parallel.
42. I: Why?
43. V: Because, they would never intersect each other if they are parallel.
44. C: Because…
45. V: They would never intersect each other and then there could not be a situation like this (he points at figure 2), in which, since they [r and s] cross, the line t is parallel to r or to s and then it [t] does not intersect any of them.
46. C: The line…
47. V: Am I making myself clear? If they [r and s] are not parallel there will be always a point in which they intersect, there can always be a situation in which there is a parallel to one line only, which [the parallel] then intersect just one line.

The conjecture is now expressed in affirmative terms: *the two lines are parallel*. The argumentation of the conjecture recalls the argumentation of the fact the ones produced are non-examples. Hence such argumentation is indirect: *if they are not parallel, it exists, like in the two non-examples, a line which intersects only one of the given lines r and s*. Later on, during the proof production stage, Valerio says:

87. Hence, it is enough to prove that, if we menage to find a line parallel to one of the two which does not cross both, we have proved that they [r and s] definitely have to be parallel… by contradiction, yes.

The argumentation is still the same. We remark two important aspects. First, we are dealing with a case of *cognitive unity* between argumentation and proof construction: Valerio affirms that to prove the conjecture we need to do exactly what we have done before during the construction of the non-example. Second, what Valerio says expresses a continuity between the production of the non-example and the proof which is going to be by contradiction: the argumentation is based on the second non-example, which is a generalisation of the first one.

DISCUSSION AND CONCLUSIONS

The presented protocol is very enlightening.

It is confirmed that indirect argumentation can arise spontaneously: in the protocol, we can observe many indirect argumentation spontaneously produced. Moreover, the hypothesis we formulated on the production of non-examples and on the role they can have in the production of indirect argumentation seems to be confirmed.

Valerio and Cristina start by noticing that if r and s are perpendicular there is a line t intersecting r but not intersecting s. Hence the configuration of the two perpendicular lines is a non-example. The generalisation of this configuration leads to the following conjecture: if the lines r and s intersect each other then the property A is not verify. The same argumentation that justifies the fact that the lines could not be perpendicular has been applied to the case of the crossing lines, leading to an indirect argumentation: the two lines are parallel because *if it were not so, it would exist* a line t intersecting r but not intersecting s. The conjecture (r and s are parallel) is true because *if it were not so, we would be* in a situation like the non-example.

We have interviewed eighteen students, almost all of them produced non-examples, but sometimes the generation of non-examples did not lead to an indirect argumentation. This happened when the students, instead of focusing on the analysis of a non-example (as Valerio does), produced a set of situations in which each example and non-example remain isolated.

We believe that the process followed by Valerio to produce a conjecture is a special case of one of the processes of statements production like "*if A then B*" described in [Boero et al. 1999]; in particular the one classified as PGC 4 and described as follows:

"the regularity found in a particular generated case can put into action "expansive" research of a "general rule" whose particular starting case was an example; during research, new cases can be generated"(Boero et al., 1999).

It seems that the other students we have interviewed produced the conjecture according to the model classified as PGC3 in [Boero et al., 1999]. This model is described as a "*synthesis and generalisation process starting with an exploration of a meaningful sample of conveniently generated examples*". In our opinion, the models suggested in [Boero et al., 1999] can be further analysed and their classification can be refined according to our distinction between example and non-example. A study in this direction requires a deeper research that we consider of great interest from both the cognitive and the didactical point of view.

As regards the transition from the argumentation to the proof, the case of Valerio can be considered similar to the transition from example to proof through what Balacheff (1987) calls *generic example*. As the *generic example* of Balacheff, we can formulate the hypothesis that the *non-example*, in some cases, can be developed to a certain level of generality (becoming a *generic non-example*), so that the argumentation of the fact that the example is a non-example is used to support the conjecture: this argumentation takes an indirect structure.

The non-example one is not the only process that can lead to a proof by contradiction. Further researches are needed to point out some others. The non-example process seems to be linked to a certain kind of task ("*given A what can you deduce*?"). It is reasonable to think that different kinds of task can give rise to different processes and that the kind of task given to the student is one of the fundamental variables for the production of argumentation with different structures.

The educational implications of this research are very interesting.

The studies of Cognitive Unity have been showing that the processes of conjecture production are extremely important for the student in order to construct the proof. From the didactical point of view, a meaningful approach to theorems seems to be the one of producing them entirely, starting from the production of the conjecture, going through the argumentation and, finally, reaching the proof. In this theoretical framework, we think that a way to help the student overcome the difficulties of proof by contradiction can be to favour argumentations that can become proofs by contradiction.

The relevance of this research relies on the fact that it offers to the teacher tools to implement didactical activities in order to introduce pupils to proof by contradiction.

Indeed, we have pointed out a tool which can lead students to indirect argumentation. From the didactical point of view, is very important guiding the students to the awareness of the structure of their argumentations, in order to offer them the opportunity to construct a proof by contradiction starting from the indirect argumentation produced.

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THEMATIZATION OF THE CALCULUS GRAPHING SCHEMA

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This article is the result of an investigation of students' conceptualizations of calculus graphing techniques after they had completed at least two semesters of calculus. The work and responses of 27 students to a series of questions that solicit information about the graphical implications of the first derivative, second derivative, continuity, the value of limits, and the inter-relationships among these concepts was analyzed from their interviews. A double triad was developed to describe students' schema as a framework for the analysis. The study centered on the way students coordinated the various elements of each question, their strategies and difficulties. It was found that coordinating concepts to solve complex problems in a graphical setting is a difficult process. Only two students were considered to have thematized the schema.

Many research studies have been carried out to find the difficulties students have with specific calculus concepts. There is not that much research about the way students coordinate knowledge to solve complex calculus problems, so this study contributes to this area. This study extends a previous study (Baker, Cooley & Trigueros, 1999, 2000) in examining students' abstractions of calculus graphing techniques and the coordination of the elements of a calculus graphing schema. It enlarges the previous work in probing students' understanding to determine if there is evidence of thematization of this schema.

ANTECEDENTS

Many researchers have used the concept of schema to describe cognitive structures of individuals. Piaget studied the concept of schemes in many of his works. Whatever is repeatable and generalizable in an action is what Piaget refers to as a scheme. Any single scheme in itself does not have a logical component, but schemes are coordinated with each other, and this fact results in the general coordination of actions.

Robert Davis (1984) wrote about "frames", a concept which has some similarities to schemas. He wrote that each person holds in memory a vast collection of knowledge "representation structures" (RS), or "frames". The collection continues to grow as one learns new things. The growth manifests itself as new, more complex frames which are built on previously held frames. He relates these frames to explicit observable behaviors, and includes properties that frames have in common.

Skemp (1987) explains schema as a hierarchy of concepts, each level building from the previous ones. A schema for him is a conceptual structure that includes not only the complexities of mathematics, but also the relatively simple structures which coordinate sensori-motor activity. Doerfler (1989) discusses protocols and these have some similarities with Davis' representation frames and Skemp's schema. The basis for his work is the genetic epistemology as developed by Piaget and, in particular, the interpretation of Piaget's work by von Glasersfeld.

Piaget and Garcia (1983, 1989) wrote in detail about schemas and their development. In their book Psychogenesis and the History of Science (1983/1989), They discuss the development of a schema progressing through the three stages called the triad. These stages are referred to as the intra-, inter- and trans- stages. At the intra- stage of a schema, particular events or objects are analyzed in terms of their properties. Explanations at this level are local and particular. An object in the intra level is not recognized by the learner as necessary, and its form is similar to the form of a simple generalization. The student's use of, comparison of, and reflection upon isolated ideas leads her or him to the construction of relations and transformation in the inter level. In the inter- stage, the student is aware of the relationships present and can deduce from an initial operation, once it is understood, other operations that are implied by it or can coordinate it with similar operations. When a student reflects upon these coordinations and relations, new mathematical structures evolve. Through synthesis of the inter level transformations, the student is reconstructing an awareness of the completeness in the schema and in the trans-level, can perceive new global properties that were inaccessible at the other levels.

The Action Process Object Schema theory (APOS) is based on Piaget's work and has been illustrated and elaborated in a number of papers (see e.g., Asiala et al., 1996; Clark et al., 1997; McDonald, Mathews, and Strobel, 2000). In this theory, a schema is a dynamic collection of action conceptions, process conceptions, object conceptions and other schema and the relationships between them. The most distinguishing characteristic between the three stages is the ability of the person to demonstrate an understanding of the relationships between the actions, processes, objects and other schema and the development of a schema is not necessarily a linear progression through the three levels.

According to Piaget and Garcia, and in APOS theory, at the trans stage, a person may thematize the schema. A schema is considered thematized when it becomes a reality for the individual, has reached a conscious level, and may be treated as a new interesting object. When a schema has been thematized it can be unpacked so that its various parts can be used and repacked as needed. When presented with a new situation, the subject knows which parts of the schema are applicable and which are not. The structures of the schema provide true comprehension of what would be understood before. A new level of intelligibility, where the understanding is now fundamental is reached.

The theory of a triad of stages in schema development has since been utilized in several studies of student understanding in mathematics. In their study of student understanding of the chain rule, Clark et al. (1997) found that the APOS theory involving actions, processes, and objects was not adequate for analyzing their data on student understanding but that the triad of Piaget and Garcia (1983, 1989) was useful in interpreting the levels of understanding. McDonald et al. (2001), also utilizing APOS theory and the triad, studied the development of the sequence schema.

In both the calculus study by Baker, Cooley & Trigueros (1999, 2000) and in the study on systems of differential equations by Trigueros (2001), the notion of schema was explored and expanded. The idea of schema interaction was introduced in the first of these papers, and a framework consisting of the intermingling of two other schemas was described. In this work it was shown that the triad proved to be a useful tool in describing the interaction of two specific schema, referred to as the property schema and the interval schema, that the researchers observed the students using in trying to solve the non-routine calculus problem that was posed. The study indicated that students retain and use certain calculus concepts while disregarding others. In particular, the two predominant areas of concern were coordinating overlapping information across contiguous intervals on the domain and coordinating properties that were explicitly stated rather than derived from a formula. Based on these observations, a model of schema interaction was described for the first time. As a consequence of this study several further research questions were raised. One of them was whether other students, who addressed the same graphing problem, could also be described by the same model. Another issue was if the behavior of students changes when presented with multiple questions of a similar nature instead of only one task. Finally, an important research question was to study thematization of the schema and its characteristics. This current paper addresses these questions.

THEORETICAL FRAMEWORK

The theoretical framework used to design and analyze this research project is a two-dimensional schema referred to as a Calculus Graphing Schema, which was presented in a previous paper. This model was developed on the basis of a genetic decomposition of the processes and objects contained in the schema as well as their relationships.The main objects in the schema are the first derivative, second derivative and continuity. These were then also identified in relationship to the graph of the function with its intervals over the real numbers and the possible relationships between all of these aspects.

The property schema has two important aspects: understanding the graphical property associated with analytical conditions concerning the main objects of the schema and the coordination of all the properties pertaining to a given interval in the domain of the function. At the intra-property level, a student can interpret only one analytical condition at a time in terms of its graphical feature. A student at this level typically utilizes solely the first derivative condition and is often aware of other properties but cannot coordinate them to produce a graph. If two properties overlap, the student describes the behavior of the graph using only one property. If he or she tries to use more than one property, the student cannot complete her or his description and resorts to using only one property.

At the level of inter-property, the student begins to coordinate two or more conditions simultaneously. This coordination, however, is not applied throughout all overlapping graphical properties given for an interval. The student is considered to be at the trans-property level if he or she can demonstrate coordination of all the analytic conditions to the graphical properties of the function on an interval. At this point, the student expresses or demonstrates a coherence of the schema. That is, the student clearly recognizes what behaviors of a function can be included in the graph and what cannot.

The main aspects of the interval schema are understanding the interval notation, connecting contiguous intervals, and coordinating the overlap of the intervals. The interval schema involves coordinating conditions across contiguous intervals as well as different conditions on the same interval. Distinguishing different sections of the domain is a typical problem in introductory calculus. At the intra level of the interval schema, a student works on isolated intervals and is not able to coordinate information across intervals. The overlap of intervals or connection of contiguous intervals causes confusion. At the inter-interval level, the student begins to coordinate two or more contiguous intervals simultaneously. This coordination, however, was not applied throughout all connected intervals or across the entire domain. The student is considered to be at the trans-interval level if he or she is able to describe the coordination of the intervals across the whole domain. He or she is able to overlap intervals and connect contiguous intervals. The student also demonstrates coherence for the schema by describing which manifestations in the graph are allowed by the overlap and connection of the intervals and which are not.

Students who were determined to be reconstructing their knowledge at the trans-property, trans-interval level were considered candidates for thematization. A student is considered to have thematized the schema if he or she is able to continue to coordinate all of the given properties across all intervals as singular properties are withdrawn. If this is demonstrated, then the student is able to unpack the schema and repack it without the removed property, coordinating the remaining property. The student is demonstrating conservation in how the remaining conditions interact, which behaviors of the graph now remain and which have been removed. The mathematical structures of the Calculus Graphing Schema have become a fundamental part of their understanding and may be used in total as an object. Another important aspect of thematization is consistency, in this case the student consistently is able to pack and unpack the properties of the function throughout the questions, which can be solved by means of the same coordinations.

METHODOLOGY

The purpose of this study was two-fold: to determine if the double triad would work as a tool in another similar situation and to extend the interview to include questions that would allow a student to demonstrate the thematization of the schema. Therefore, the researchers decided to interview those students found to be successful enough in calculus that there could be a possibility of thematization. These students represented a broad mix. Fifteen students were from a private university in Iowa, while the remaining twelve were from a private university in Mexico City. The majors of these students included mathematics, engineering, mathematics education, economics, and natural sciences.

The reliability of the model derived from the observations of an interaction between the property schema and the interval schema was tested with a series of related questions, as well as the non-routine calculus question used in the previous study by Baker, Cooley & Trigueros (2000). The series of questions was developed by the researchers with the goal of having more in-depth data of student understanding, examining student responses and techniques from various perspectives, in order to provide evidence of conservation in their understanding and if thematization of the schema had occurred. The students were asked to respond to these questions in extensive, detailed, audio-taped interviews. The interviewers were mathematics professors with a background in mathematics education research. The students answered the questions, explained their thought processes, asked questions, and explained the methods they used to assemble their graphs and other work. Their written work was also kept as part of the interview data. The researchers reviewed all the written and oral work. Each student interview was analyzed independently by at least two researchers. After the analysis, the two researchers discussed their observations and conclusions. There were no discrepancies in the final categorization of the students. Those students who were determined to have reached the trans-property, trans-interval level were reviewed again by all three researchers to determine if there was evidence of thematization. The three researchers discussed this analysis and agreed on the students who had thematized the schema.

The questions asked of the students required an understanding of what it means for the first derivative to exist, its representation as the slope of the tangent line and as rate of change, and the relationship between derivative, the limit, continuity, and shape of the curve in different intervals of the domain of the function. They also required an understanding of the significance of the second derivative, of its existence and representation as the concavity of the curve, its graphical meaning and the significance of inflection points. The relationship between first derivative, second derivative, continuity and limit needed to be used in all the problems and must be coordinated on different intervals of the domain. Students needed to decipher which conditions and properties are dependent upon continuity. The questions were posed in different representational forms and were more demanding as the interview progressed in terms of asking for coordination of more objects ending with the same problem used in the previously mentioned paper.

In order to examine possible thematization of the Calculus Graphing Schema, several of the interview questions remove certain conditions (continuity, first derivative and second derivative) while the student is asked to explain what characteristics the graph of the function retains. The ability to describe the relationships among the remaining characteristics, to synthesize the properties and separate them, packing and unpacking them as needed, was considered evidence of thematization.

DISCUSSION

Our analysis of the student interviews showed that students were demonstrating similar behaviors and responses as the students in the previous study. We observed students having the same difficulties at the same points in the final question. Students were found to rely heavily or exclusively on the first derivative and struggle with the limit condition on the first derivative at a point ($x = 0$ in this case). Coordination of differentiability and continuity in cusp points proved to be very difficult for students. The series of questions helped students stay focused on the issues under study and to demonstrate their knowledge in different contexts, but still the students exhibited the difficulties mentioned above consistently throughout all of the problems.

It was interesting to observe students who were able to construct the graph in the last problem, but who demonstrated an unstable conceptualization when trying to answer other questions. We also found that removing the continuity, first derivative and second derivative in the last question provided a good tool to observe the strength and coherence of the students' schema, as well as the ability to unpack it and repack it, thus demonstrating thematization. Indeed, these last questions were very revealing in what the students were able to consistently coordinate and what was a weaker conceptualization.

There were only two students who were consistently able to discuss the relationships between the different processes and objects in the schema, and who were also able to unpack the schema in the final questions related to removing certain conditions while holding others constant and give consistent answers. This was a surprising result considering that all of the students in the sample had successfully completed their calculus courses and were considered above average by their teachers. In contrast with the other students, those with thematized calculus graphing schemas use their knowledge in a flexible way. We show here an example of one of these students when solving the removing conditions parts of the last problem.

(a) Sketch the graph of a function that satisfies the following conditions:

h is continuous

$h(0) = 2,\ h'(-2) = h'(3) = 0,$ and $\lim_{x \to 0} h'(x) = \infty$

$h'(x) > 0$ when $-4 < x < -2$ and when $-2 < x < 0$ and when $0 < x < 3$

$h'(x) < 0$ when $x < -4$ and when $x > 3$

$h''(x) < 0$ when $x < -4$, when $-4 < x < -2$, and when $0 < x < 5$

$h''(x) > 0$ when $-2 < x < 0$ and when $x > 5$

$\lim_{x \to -\infty} h(x) = \infty$ and $\lim_{x \to \infty} h(x) = -2$

(b) Do there exist other graphs besides the one you drew that satisfy the same conditions? Justify your response.

(c) If we remove the continuity condition, and the other conditions remain, does the graph change? In what way? Do other possible graphs exist? If other graphs exist, could you sketch one example?

(d) If we remove all of the first derivative conditions, and the other conditions remain, does the graph change? In what way? Do other possible graphs exist? If other graphs exist, could you sketch one example?

(e) If we remove all of the second derivative conditions, and the other conditions remain, does the graph change? In what way? Do other possible graphs exist? If other graphs exist, could you sketch one example?

S: Now b, what did it say? Can any other graphs, apart from the one you drew, satisfy the same conditions? No, like before, since the conditions that are established aren't too specific. They only say that the point is (0,2). So it isn't unique. You can move it to a different place, make it taller or shorter and lower the peak or the maximum. Other than the (0,2) and the asymptote, the rest can be in any other place, and, no, it's not unique.

I: The last part, what happens if you leave all the conditions the same but take away the condition of continuity? Can other graphs exist?

S: If the graph changes, for example at negative 4, it could be broken because there are open intervals and they don't say anything about the negative 4 so that it could break there. Also at 0 there could be a little hole. It could be something like this example in the drawing.

I: Are those the only points that could change?

S: I think so because those are the points where the derivative isn't defined. At the others it is defined. So even if they don't tell you that the function is continuous, you know that it is because if it is derivable, it's continuous, like in the other question. It's the same thing. So only at the 0 and the negative 4.

I: Does the fact that they tell you that it goes through the point (0,2) have any affect?

S: Oh yes, I forgot about that. Then there can't be a hole. There has to be a point. There could be an asymptote and the point there in the axis. No, because it grows and grows, but then it could come out from underneath and it would grow from the left and grow by the right…and the point, but no because then the limit wouldn't exist. What a mess! And then, no, only the negative 4 would change, I think. Yes, only the negative four.

In contrast, another student, also at the trans-property, trans-interval level, was not able to succeed with the removal of the conditions:

I: And if you take off the condition of continuity?

S: If the condition of continuity is removed the graph would change, because now it can be broken in any of the points I marked on the x-axis. It can become something like this, like the one I am drawing here.

I: Are you sure that it can break in all those points?

S: Aha, yes, in all of them there is a change in the properties and if it doesn't have to be continuous well, you can draw little holes there.

CONCLUSIONS

The results of this study show that the double triad model presented above is a good tool to study the coordination of different concepts used to solve calculus graphing problems in different representational contexts. It also shows that the integration of knowledge is a slow process and that even good students finish their courses without having a strong understanding of the relationship between them. Even if the students in this study struggled less with issues that were shown to be difficult before, they still demonstrated the same kind of difficulties as the students had in the earlier study. Coordination of properties on different intervals remained a difficult task for most of the students. This study shows as well that students' knowledge is not stable. What can be considered a slight change in the conditions of a problem causes some students to reconsider their answers and to exhibit difficulties that they didn't show while solving the original problem.

This is, as far as we know, the first study that focuses on thematization of schema. It shows that it is a difficult process and that it takes a lot of time. Even very good students seem to finish their calculus courses with what we can consider weak and unstable schemas.

Further research focused in widening our understanding of thematization and schema development is needed to find ways in which students can be helped through this difficult process and to design didactical strategies that can foster schema development.

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ATTENTION TO MATHEMATICAL STRUCTURE DURING PARTICIPATION IN A MATHEMATICS CLASSROOM TASK BY LEARNERS OF ENGLISH AS AN ADDITIONAL LANGUAGE (EAL)

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How do students who are learning English as an additional language (EAL) participate in classroom mathematics? To investigate this question, I collected recordings of primary school EAL students as they worked with peers on a mathematics classroom task. Drawing on theoretical and methodological work in discursive psychology (Edwards, 1997), I analysed students' interaction and identified several patterns of attention, including a pattern of attention to mathematical structure. In this paper I report how the students in the study used attention to mathematical structure as part of their joint work on the task. My analysis shows how EAL students are able to participate successfully in mathematics classroom interaction. My analysis also highlights the social nature of this interaction.

INTRODUCTION

The aim of the study reported in this paper was to investigate how students learning English as an additional language (EAL) [1] participate in classroom mathematics in the UK. A key starting point was the observation that such students *are* able to take part in mathematics lessons. Rather than see multilingualism as a problem, therefore, I wanted to explore *how* such students participate in doing mathematics.

There are approximately 500,000 EAL students in the UK, most of whom participate in mainstream mathematics lessons. There has, however, been no sustained research in the UK into the nature of this participation and its relation to the teaching and learning of mathematics and little research elsewhere in the world. The research that has been conducted (e.g. Adler, 1995, 2001; Khisty, 1995; Khisty *et al.*, 1990; Moschkovich, 1996, 2002; Setati, 2002) largely focuses on the teacher's perspective and is based in classrooms in which more than one language is used during mathematics lessons. In the UK, the home languages of multilingual students are *not* normally heard in mathematics lessons. This situation raises the question: How do EAL students participate in their mathematics lessons? This paper reports results from a recent study designed to address this question. As there has been little previous research in this area, I was not attempting to identify features of participation in mathematics classroom interaction that were *specific* to EAL students. The aim was to counter deficit model assumptions about EAL learners (described by Moschkovich, 1996) by exploring the nature of such students' participation.

THEORETICAL FRAMEWORK

The participation of students from a diverse range of cultural and linguistic backgrounds presents a challenge for the analysis of classroom interaction. In particular, it is difficult for me, as a researcher from one cultural and linguistic background, to make assumptions about what students from different backgrounds mean by what they say. Although we may use the same words, what we mean by those words may be quite different. To address this issue, I have developed a theoretical and methodological approach (Barwell, 2001) drawing on discursive psychology (Edwards, 1997; Edwards & Potter, 1992) and conversation analysis (Sacks, 1992). This approach centres around the notion of *attention*.

For discursive psychologists, language is conceptualised as primarily "a medium of *social action* rather than a code for representing thoughts and ideas" (Edwards, 1997, p. 84, orig. emph.). Social action is foregrounded as the primary function of language, which is seen as having evolved through social interaction, and therefore as being structured both by and for social interaction. Thus 'psychological' notions, such as thinking, meaning or attending are examined discursively, to understand, for example, how participants in interaction construct thinking, and what different ways of constructing thinking might achieve. Taking this perspective therefore avoids attempting to say what EAL students mean by their words, in preference for analysing *how* they use language to construct mathematical thinking or meaning.

In the study reported in this paper, I focused particularly on the notion of attention. By attention I do not mean an internal aspect of focusing the mind on some aspect of the outside world (see, for example, Mason & Davis, 1988). Instead I treat attention discursively, drawing particularly on what conversation analysts call 'participants' attention' (Sacks *et al.*, 1974). This view of attention arises from the social organisation of talk, which includes features such as taking *turns* to speak, so that successive turns form an unfolding sequence of interaction in which each turn builds on what has gone before. For this to be possible, participants must indicate through their words, what of the preceding interaction is relevant to their current contribution. Through their words, therefore, participants display *explicit* attention to some aspect of the immediate past. Since this attention is explicit, however, it is also available to analysts. A first level of analysis therefore consists of identifying *patterns* in students' attention.

Language, of course, is highly flexible. There are many possible ways of explicitly attending to relevant aspects of preceding interaction. A second level of analysis therefore examines *how* attention is brought about, and what particular ways of attending achieve in terms of the on-going socially organised talk. Before outlining some of the results of this form of analysis, I will outline the nature of the data collected.

DATA COLLECTION

The study focused on two Year 5 (9-10 years) classes taught by the same teacher in consecutive years. Over the two years, 10 EAL students participated in the study. Since interaction involving the teacher was heavily cued by the teacher's contributions, I recorded students working together without a teacher present. The primary data therefore consisted of transcripts of pairs of students working together on a mathematics classroom task. Both the task and the pairings selected were based on ethnographic observations of mathematics lessons over several months. The task required students to work together to *write arithmetic word problems* with some general focus, such as addition or division. Students were paired in a variety of ways including: 2 EAL students sharing a home language; 2 EAL students from different language backgrounds; 1 EAL student with one monolingual student; 2 monolingual students. Altogether I collected and transcribed 20 recordings of students working on the word problem task. I also collected a variety of other data, including: copies of students' work, information about students' attainment; interviews with teachers; classroom observation notes and video-recordings of mathematics lessons.

ATTENTION TO MATHEMATICAL STRUCTURE

My discursive analysis of EAL students' interaction revealed four patterns of attention [2]. In this paper I focus on one: students' attention to the mathematical structure of their emerging word problem. For a definition of the structure of a word problem I drew on Verschaffel, *et al*. (2000):

> the nature of the given and unknown quantities involved in the problem, as well as the kind of mathematical operations(s) by which the unknown quantities can be derived from the givens...[and] the way in which an interpretation of the text points to particular mathematical relationships (Verschaffel, Greer & de Corte, 2000, p. x).

This definition does not concern any particular operation which may be used (or expected to be used) to solve a word problem, but is concerned with the mathematical relationship between the various quantities which occur in such problems. In examining EAL students' attention to the mathematical structure of their word problems, I am not attempting to categorise the structure of their problems for myself. My task is to analyse how the students themselves attend to aspects of the problem structure, as *they* see it.

I will illustrate the pattern of attention to mathematical structure, before setting out some of the ways in which this attention was used by students. In the following extract, Tahira (EAL) and Verity (non-EAL) are starting to work on writing a new word problem. I have asked them to write problems 'about' division (for transcription notation, see [3]).

Tahira and Verity

36	T	I can think of times one
37	V	mm
38	T	if/[if there were um
39	V	[y-
40	T	if you had um/ twelve sweets
41	V	twelve/ sweets/
42	T	and you had um/
43	V	you had/ how many people?/ six people
44	T	six people
45	V	yeah 'cause half of twelve is six/ if you/ had/
46	T	six people
47	V	six people/ how many sweets/ how many sweets would they get each/ ac-
48		no/ actually/

49 T twelve divide six

50 V yeah twelve divided by six/

51 T do times

52 V no we're not allowed to do times/

In this extract, the two participants attend to mathematical structure on several occasions. Tahira offers to 'think of times one' (line 36) as the opening move in writing the problem. I cannot say whether she actually has a problem in mind; only that she explicitly attends to the structure of the future problem. The two students then propose and negotiate various generic aspects of the problem (see Gerofsky, 1996), including a scenario about a quantity of sweets. Having considered 'six people' (lines 43-44), Verity attends to mathematical structure again, this time in the form of an arithmetic relationship between 12 and 6, ''cause half of twelve is six' (line 45). Following further discussion of the wording of the problem, Tahira attends to the same relationship using different words, 'twelve divide six' (line 49). This attention to structure is maintained by Verity, who reformulates Tahira's statement (line 50). Tahira then continues the attention to structure, saying 'do times' (line 51), a request which Verity rejects. Thus at several points in the above sequence the two students attend to mathematical structure, attention which is seen as co-constructed by both participants. Throughout the data collected in this study, EAL students and their peers attended to mathematical structure as part of the process of creating word problems, as in the above extract.

SOME USES OF ATTENTION TO MATHEMATICAL STRUCTURE

Analysis of *how* the students who took part in the study used attention to mathematical structure as they created word problems together revealed a number of uses, including:

- justifying generic details,
- managing the social relationship between the participants,
- critiquing the emerging word problem,
- modifying word problem structure.

The first three of these uses are illustrated by the sequence involving Tahira and Verity. For the fourth, I will use another extract from the data. I will address each use in turn.

Justifying generic details

Tahira and Verity discuss a number of generic details for their problem, including the number of sweets and the number of people. One of the features of the word problem genre is that such details are essentially arbitrary (Gerofsky, 1996). In generic terms, it makes little difference how many sweets or people there are. In the above sequence, mathematical structure is used as a way of *justifying* the choice of particular numbers:

40 T if you had um/ twelve sweets

41 V twelve/ sweets/

42 T and you had um/

43 V you had/ how many people?/ six people

44 T six people

45 V yeah 'cause half of twelve is six/ if you/ had/

Verity uses attention to structure, 'cause half of twelve is six' (line 45), to justify her suggestion of 'six people' (line 43). Indeed it is difficult to see "half of twelve is six" acting in any other way. Consider some alternative ways this exchange could have been. Verity could have said 'you had/ six people/ half of twelve is six.' The recourse to structure seems to act as a justification, even without the causal language of "'cause". Even if the structure is introduced first, it still appears to act as a justification. Verity could have said, for example 'half of twelve is six/ so let's do if you had six people.'

Managing the social relationship between the participants

What, socially, is achieved by using attention to structure in the way described above? Other forms of justification are possible. Verity could have said 'trust me', or 'I know best', or 'obviously', all persuasive devices rather than accounts for her ideas. She could have invoked previous experience of word problems, remembering a particular problem or citing the teacher, for example. All these possibilities, however, rely on Verity and therefore have implications for the social nature of the discussion. In particular such approaches require Tahira to accept either Verity's judgement or her memory. Attending to mathematical structure, by contrast, distances Verity from her account, so making it both more authoritative and less personal (Edwards and Potter, 1992, p. 162). By attending to mathematical structure, Verity draws on the objective rhetoric of mathematical reasoning. In mathematical discourses, the arithmetic relationship between numbers is not seen as a personal matter. By drawing on the objective authority of mathematical discourse, Verity avoids forcing Tahira to submit to Verity's personal authority. Thus attention to mathematical structure is also used to manage the *social relationship* between the two participants.

Critiquing the emerging word problem

At the start of the above sequence, Tahira offers to 'think of times one' (line 36). Towards the end of the negotiations, she brings attention to the operation required to solve the emerging word problem. She formulates this attention in terms of division:

47 V six people/ how many sweets/ how many sweets would they get each/ ac-
48 no/ actually/
49 T twelve divide six
50 V yeah twelve divided by six/
51 T do times

Tahira is characterising the problem as one of division, a proposal with which Verity agrees. Tahira then argues for a different kind of problem, 'do times' (line 51) [4]. The form of Tahira's initial attention to structure (line 49) sets up 'do times' as a contrast, so that her words act as a *critique* of the problem which has been created. This critique ties in with Tahira's opening offer to 'think of times one', a task which has not been completed. A frequent use of attention to mathematical structure in the data collected in this study is to critique word problems in this way.

Modifying word problem structure

The above uses of attention to mathematical structure can be seen in the sequence shown below, which also illustrates an additional use: modifying word problem structure. The sequence features Cynthia (EAL) and Helena (non-EAL) and is taken from the last few lines of the transcript of the students' work on their first problem. They have been asked to write problems 'about' addition. Prior to this sequence, Cynthia and Helena have agreed on and Helena has written down: "Daniel has a job he gets pay £415 in a month". They are now negotiating the question which will conclude the problem.

Cynthia and Helena

100	C	how many in a week/ no oh yeah/ how many in a week
101	H	(...) okay then/ how many/ how many/ how much money does he get/ in
102		a **year**/
103	C	in a week
104	H	a week?
105	C	no that's (...)
106	H	no cause/ you said in a **month**/
107	C	yeah/ no/ I said/ [no/ I said/ Daniel has a job he **gets paid** four &
108	H	[how many
109	C	& hundred and fifteen pound in a **month**/ how many in a **week**
110	H	how much he gets
111	C	yeah/ how-how much he get/ on one week
112	H	that's **dividing** innit
113	C	oh yes that's divide/
114	H	that's sort of like dividing cause there's four/ four weeks in a month so
115		that's **four** divided by (three) I mean four hundred and fifteen
116	C	I'll just do/ how many in a year//

In this sequence, the two students disagree over whether to conclude with a question about 'how many in a year' or 'how many in a week'. In generic terms, either would be acceptable. Their negotiations consist of each student restating their preferred question until Helena shifts attention to mathematical structure, characterising the use of week as 'that's dividing innit' (line 112). Cynthia agrees and Helena then elaborates through continued attention to the structure of the problem. This attention to structure then provides a basis for Cynthia to accept Helena's preference for the questions 'how many in a year'. Thus for Cynthia, a division problem is transformed into what is potentially an additive problem. Attention to mathematical structure is therefore implicated in *modifying* the word problem structure.

CONCLUSION

The analysis reported in this paper shows how the EAL students in this study are able to participate in the use of attention to mathematical structure to conduct a number of social actions, including: justifying generic details, managing the social relationship between participants, critiquing emerging word problems, and modifying word problem structure.

These findings highlight the close inter-relation between social and mathematical concerns. Attention to mathematical structure is an intrinsic part of the mathematical nature of the word problem task. At the same time, this attention is used to conduct social actions and to manage the on-going relationship between the students, as for example, when the objective nature of mathematics is used to render a point of view more authoritative and less personal. Finally, these findings show some of the ways in which EAL students *can* participate in mathematics classroom interaction, at least in the context of the word problem task.

NOTES

1. English additional language (EAL) refers to any learner in an English medium environment for whom English is not the first language and for whom English is not developed to native speaker level. Native English speakers are described simply as monolingual.

2. The 4 patterns of attention are: attention to the word problem genre, attention to narrative experience, attention to mathematical structure and attention to written form.

3. Transcription conventions: Bold indicates emphasis. / is a pause < 2 secs. // is a pause > 2 secs. (...) indicates untranscribable. ? is for question intonation. ( ) for where transcription is uncertain. [for concurrent speech. & for utterances which continue on a later line.

4. The interpretation ‘do times’ is supported by Tahira’s intonation.

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LEVELS OF SOPHISTICATION IN ELEMENTARY STUDENTS' REASONING ABOUT LENGTH

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Because cognition is the core substance of understanding and sense making, cognition-based assessment is essential for understanding and monitoring students' development of powerful mathematical thinking. The Cognition-Based Assessment System (CBAS) project is applying the results, theories, and methods of modern research in mathematics education to create an assessment system that can be used to assess in detail the cognitive underpinnings of the progress students make in developing understanding and mastery of core mathematical ideas in elementary school mathematics. In this report, I briefly describe initial CBAS research on the development of students' reasoning about length.

THEORETICAL PERSPECTIVE

An important finding of modern research in the psychology of mathematics learning is that for particular mathematical topics and within particular age ranges, students' development of conceptualizations and reasoning can be characterized in terms of levels of sophistication. These levels start with the informal, pre-instructional reasoning typically possessed by students in the age range; the levels end with the formal mathematical concepts targeted by instruction. The levels describe not only what students can and cannot do, but their conceptualizations and reasoning, cognitive obstacles that obstruct learning progress, and mental processes needed both for functioning at a level and for progressing to higher levels. The pedagogical importance of these levels is that instruction that produces conceptual understanding and powerful reasoning for a mathematical topic must be firmly guided by detailed, research-based knowledge of the development of students' cognition for the topic (Carpenter & Fennema, 1991; Cobb, Wood, & Yackel, 1990).

METHOD OF ANALYSIS

To investigate the sophistication of students' thinking about length, the CBAS project examined how students reasoned about the lengths of a variety of straight and non-straight paths. Based on a review of the research literature on length, 19 assessment tasks were developed and administered to students in grades 2-5 (ages 7-11) in one-on-one, videotaped interviews. Levels of sophistication in students' reasoning about length were synthesized from analysis of videotapes, summaries, field notes, and transcriptions of students' work. I first summarize the levels, then the cognitive processes underlying these levels.

FINDINGS: LEVELS OF SOPHISTICATION FOR STUDENTS' REASONING ABOUT LENGTH

Measuring length involves determining how many unit lengths are contained in a given length; it therefore involves the use of number to make judgments about length. However, before students acquire the concept of length *measurement*, they often reason about length using non-measurement techniques. And, although non-measurement strategies appear before measurement strategies, non-measurement strategies continue to develop in sophistication even after measurement strategies develop. In fact, sophisticated non-measurement strategies are essential for understanding length-based geometric properties of shapes.

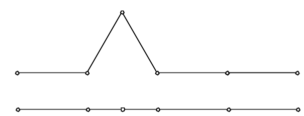
NON-MEASUREMENT (NM) REASONING

For non-measurement reasoning about length, students do not use numbers. Instead, they reason using vague judgments, direct or indirect comparisons, transformations, or geometric properties.

NM Level 1. Informal Holistic or Movement-Based Comparison

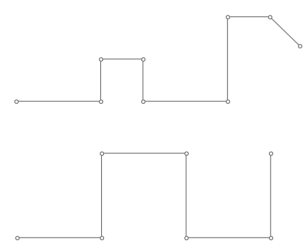
Students compare lengths holistically or using informal strategies that are imprecise and often vague. For instance, students might judge path lengths based on the amount of time or effort they imagine would be required to traverse the paths. Students have not yet separated length from the physical contexts in which they have experienced it. Elementary students may bring several perspectives to their reasoning, often with vestiges of the reasoning Piaget described for younger children. For instance, students might compare two non-straight paths by looking only at their endpoints, not what occurs between the endpoints.

Task L4. If an ant had to crawl along these paths, which path would be longer for the ant, or would they be the same? JAK drew segments joining the left endpoints and the right endpoints of the two paths and said, "I think they are pretty much the same."



Task L4. SAS said the top path was longer because "it wastes its time going up that way" [motioning to the two diagonal segments in the top path]. She said that the bottom path "just goes straight." When asked what she meant by "wastes its time," SAS said, "It is a lot easier if you just go straight and if you were carrying something it would be harder." When asked what makes it harder, she responded, "It is harder to hold something and when you turn it is harder."

Task L5. If an ant had to crawl along these paths, which path would be longer for the ant, or would they be the same? HT said that the second path would be longer, "cause [tracing the path with a finger]...like when it gets to these parts here [tracing along the square indentation in the bottom path] it has bigger squares, so it would take longer to get through."

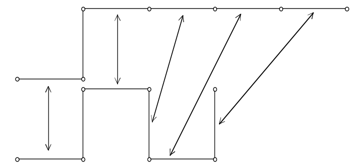


NM Level 2. Componential Comparison

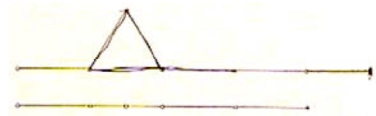
Instead of reasoning about paths holistically, students systematically operate on components of paths. They compare paths segment-by-segment using visual, often transformation-based, strategies.

Task L1. If these were wires and I straightened them, which would be longer, or would they be the same? Each segment between dots is the same. *AR claimed that the two wires were the same.*

She matched segments in the bottom wire with segments in the top wire (as shown in the figure), saying, for each match, "This [segment] is the same as this [segment]."

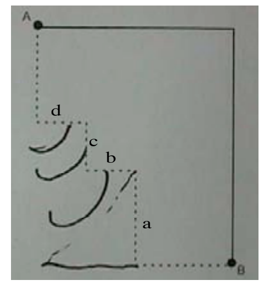


Task L4. On the top path, MB drew one segment at the right end and another under the triangular indentation, saying, "One of them will make this [pointing to the



added segment on the right] and one of them will be at the bottom [pointing under the indentation]. And it [the top path] will be a little bit bigger by one line [pointing to the added segment on the right]."

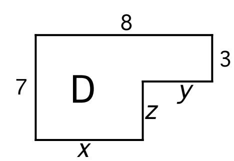
Task L12. Which path from A to B is shortest? JOK concluded that both paths are the same length: "If you put this here [dotted segment *a* onto the horizontal base] and this here [dotted segments *b, c, d* onto the left side]…they match up and they make a straight line." [Although this reasoning is not far off in this case, it is a strictly visual componential comparison that is logically flawed.]



NM Level 3. Property-Based Comparison

Students compare lengths using concepts such as perpendicularity, parallelism, and geometric properties of shapes. Students' use of these properties, however, may be informal and not stated explicitly.

In determining the perimeter for Shape D, AK said, "Side X plus side Y equals 8 because these sides are across from the top which is 8. Side Z is 4 because it and the side of length 3 are across from the side of length 7."



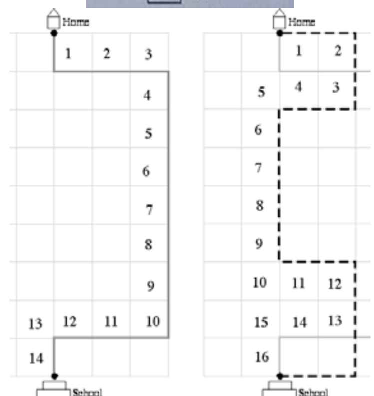
MEASUREMENT (M) REASONING

M Level 0: Pseudo-Measurement

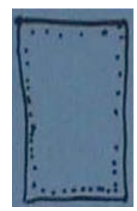
The numbers students use to reason about length do not properly represent the iteration and enumeration of unit lengths. Students fail to properly connect counting acts to unit lengths. For instance, they might recite numbers as they continuously move their fingers along a segment. Or they might count dots or squares in ways that are inappropriate for length measurement.

Task L4. CG counted the number of dots on both shapes and said the paths would be the same length because they each have 6 dots. He said that if you straightened the top shape, it would be "perfectly in line" with the bottom shape.

Task L3. Which path from home to school is shortest, the gray path, or the dotted path? SA said, "You can count the squares and whoever [sic] has the less is the shortest." As shown in the following figure, SA counted squares along the gray path 1-14, then along the dotted path 1-16 [but mistakenly switched back to the gray path at the very end, 13-16]. SA said, "So the gray line is shorter because it has less squares."



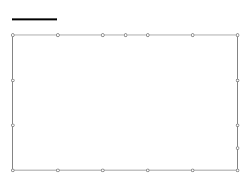
Task L16 Supplemental. The interviewer asked JAK if she could draw a rectangle with a distance around of 40. When asked how she knew the rectangle she drew was 40, JAK drew dots along its inside edge, stopping when she counted 40. (There are actually 38 dots on her paper.)



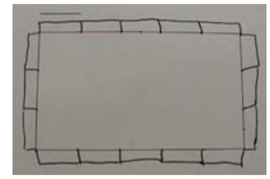
M Level 1: Enactive/Figurative Unit Length Iteration

Students use 2d or 3d shapes to represent unit lengths, but unlike in M Level 0, these 2d/3d shapes have a 1-1 correspondence to properly located unit lengths (though there may be gaps, overlaps, and variations in length). Although students have not disembedded unit lengths from physical manifestations such as rods or squares, they have abstracted the linear extent of 2d/3d units sufficiently to use them as representations of length units. Indeed, because students are attending to linear extent—not, for example, to squares per se—there is no consistent double counting around corners.

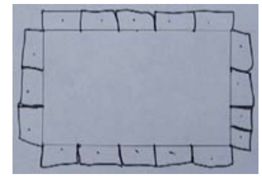
Task L9. How many black rods does it take to cover around the gray rectangle? SAS drew rectangles around the outside perimeter of the rectangle, then answered 16. Each rectangle she drew had endpoints that matched the given dots on the rectangle, and so were equal in length to the given rod. When asked how she got 16, she responded, "I thought of this one, [motioning across the given rod, then the rod she drew], and I tried to measure it as much as that one was."



Task L9. SA drew 2 vertical segments from the endpoints of the black rod down to the gray rectangle. She then created 4 additional, same-size rectangular figures on the top side of the gray rectangle. On the right side of the gray rectangle, she drew 2 rectangular shapes corresponding to the given dots, then, using the given dots to guide her work, she created2 square-like shapes.



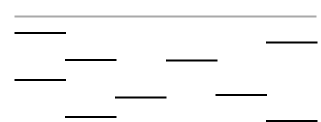
SA continued along the bottom of the gray rectangle, making 5 rectangular shapes, and on the left side, creating 3 more rectangular shapes. SA then counted the rectangular units she made, getting 17.

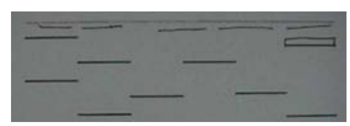


M Level 2: Measurement by Iterating Unit LENGTHS: Unit Length NOT Properly Maintained or Located During Iteration

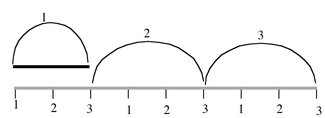
Because students have abstracted unit lengths to the level necessary to disembed them from physical manifestations, they specifically count length units. However, because students do not properly coordinate unit lengths with either each other or the whole object being measured, gaps and overlaps occur. There is an inability to maintain the unit. Also, because there is a lack of proper structuring of the *set* of iterated unit lengths, students often lose track of their counting.

Task L7. How many black rods does it take to cover the gray rod? JAK drew a rectangle on the far right black rod, copied that length above the rod and under the gray rod, then continued to draw black rods under the gray rod. She gave an answer of 5 rods, explaining, "I measured this one [pointing to the far right black rod] with 2 little short lines and then a long line and it gave me a clue for how long it was and then you just draw how long the lines are and that gives you how many."



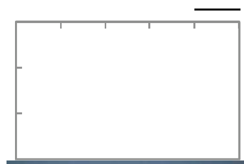


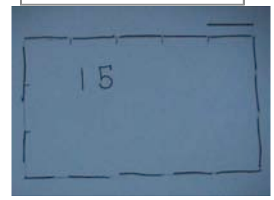
Task L8. How many black rods does it take to cover the gray rod? SA said that she knew that the black rod takes 3 hash marks on the gray rod. She drew a vertical segment from the right end of the black rod to the third hash mark on the gray rod. SA then counted the fourth, fifth, and sixth hash marks, "1, 2, 3" and marked the sixth hash mark and said, "have one."



She counted, "1, 2, 3" on the seventh, eighth, and ninth hash marks and said "have one." She returned to the beginning of the gray rod, pointed to each section she created, and counted "1, 2, 3."

Task L10. How many black rods does it take to cover around the gray rectangle? JAK drew segments all the way around the rectangle's perimeter, rotating the paper so that she could draw horizontally. She made and counted 15 segments. When asked to explain, JAK said, "These little squiggly lines [given hash marks on top and left sides] helped me measure….This side [pointing to right side of the rectangle] and on the bottom…I was picturing in my mind that there were squiggles."



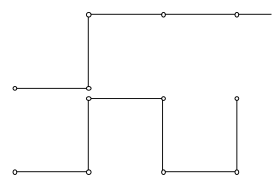


Task L15. [Given a 5-by-7 rectangle and a "broken" ruler that starts just before 1 and ends just after 9.] Use the broken ruler to measure the distance around this rectangle. JOK lined up the 1 on the ruler with the left endpoint of the base, then moved the ruler to the height, again lining up the 1 on the ruler. He said the distance around the rectangle was 28. When asked for the dimensions of the rectangle, JOK replied, "6 by 8."

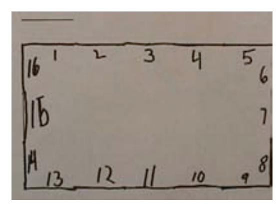
M Level 3: Measurement by Iterating Unit LENGTHS: Unit Length Properly Maintained

Students are able to operate on their abstractions of unit lengths. They can use the coordination operation to properly relate the position of each unit with the position of the unit that precedes it so that gaps and overlaps are eliminated. Some students at this level can also coordinate and integrate unit lengths with the whole—so the whole is clearly seen as iterations of the unit. Some students can also iterate composites of unit lengths.

Task L1. If these were wires and I straightened them, which would be longer, or would they be the same? SS said, "Just by looking at it I can tell that these are like the same size between two dots, and so I would count by twos." SS counted the top wire by twos as she pointed to the segments to get 6, then counted the bottom wire by twos to also get 6. SS said, "And so I would know that they were the same."



Task L9. HW said, "This [black rod] is about as long as between these two [dots] here." She then drew a black path around the rectangle, one segment between two dots at a time. For the third segment on the right side, she ignored the extra dot. HW counted each segment as she drew it, writing the corresponding numerals inside the rectangle. She got 16.



M Level 4: Abstract/Applied Measurement: Reasoning about Length without Iterating Units; Using Rulers Meaningfully

Students can operate on lengths numerically *without iterating unit lengths*. Iterable units have reached the symbol level.

Task L10. BW counted the spaces between the hash marks on the top side of the rectangle, getting 5 rods. He said that since the bottom was the same as the top, it would also be 5. He then counted 3 rods on the left side (which also has hash marks). He said that the left side was equal to the right side, so the right side also would be 3. BW then said, "3+3=6 and 5+5=10. So it takes 16 black rods."

Task L16. Give the lengths of the sides of three different rectangles that have a total distance around of 200 units. SS said, "If you want all the sides the same, it would be 50 for all sides." SS next said, "60+40 and that would be 100, since you need two." For a third rectangle, SS said, "It could be 70+30=100, since you have two of the 70+30's, so 200."

COGNITIVE PROCESSES

General Processes

Among the cognitive processes necessary for mathematical reasoning, abstraction is critical. *Abstraction*, which has several levels, is the process by which the mind selects, coordinates, unifies, and registers in memory some aspect of the attentional field (Battista, 1999). At its most basic or *perceptual level*, abstraction isolates something in the experiential flow and grasps it as an item. When material has been sufficiently abstracted so that it can be re-presented in the absence of perceptual input (visualized), it has reached the *internalized level*. Material has reached the *interiorized level* when it has been disembedded from its original perceptual context and it can be freely operated on in imagination, including "projecting" it into other perceptual material and utilizing it in novel situations. At the second level of interiorization, one can operate on symbols (in von Glasersfeld's "pointer" sense) as substitutes for abstractions.

Three additional processes that are fundamental to understanding students' reasoning about length are spatial structuring, coordination, and use of mental models. *Spatial structuring*, a type of abstraction, is the mental act of constructing an organization or form for an object or set of objects (Battista, 1999). It determines an object's nature or shape by identifying its spatial components, combining components into spatial composites, and establishing interrelationships between and among components and composites. *Coordination* arranges abstracted items in proper position relative to each other and relative to the wholes to which they belong. *Mental models* are nonverbal recall-of-experience-like mental versions of situations; they have structures isomorphic to the perceived structures of the situations they represent (Battista, 1999). Mental models consist of integrated sets of abstractions that are activated to interpret and reason about situations that one is dealing with in action or thought. In particular, they permit visual reasoning.

PROCESSES APPLIED TO LENGTH

The levels-of-sophistication example episodes described earlier illustrate that in reasoning about length, level of abstraction is important because it determines the sophistication of the abstractions and operations students can apply in reasoning about length. For instance, to use the non-measurement strategy of componential comparison, students must interiorize the paths so that they can decompose them into parts and establish a one-to-one correspondence between the parts. For measurement strategies, once students have interiorized a length unit, they can employ the *units-locating* process to locate unit lengths by coordinating their positions with each other and (later) with the whole. The sophistication of this coordination is a major factor in determining the level of students' reasoning about length measurement. At first, students exhibit no coordination of unit lengths. Then, as they attempt to iterate a unit length, they coordinate each successive unit with the unit that precedes it. Next, students see a particular unit length in relation to the sequence of unit lengths; for instance, they see a unit as the third unit from one end, enabling them to understand the location of this unit. Finally, students see the whole length as a composite of unit lengths. In the latter two instances, the units-locating process is integrated with the process of *structuring* all or a portion of the whole length. Indeed, interiorization enables students to integrate successive iterations of abstracted length units into an operationalized, structured system that allows students to (a) properly interrelate iterated units to avoid gaps and overlaps, (b) relate iterations to the whole object so that the whole can be conceptualized as a composite of units, and (c) maintain, via the generalized and systematized unit, the invariance of the unit length in multidimensional contexts. A second level of interiorization enables students to operate on measurements as symbols—that is, it enables students to meaningfully reason about measurement numbers without having to iterate unit lengths.

The example episodes also illustrate that the exact substance of what is abstracted is critical to reasoning properly about length. For instance, for holistic comparison of paths, do students abstract distances between endpoints or motions along the paths? Or, with measurement strategies, students often use two-dimensional shapes such as squares or rectangles as their length units. They have not yet abstracted length from these units, even though length is embedded in them. Indeed, most perceptual manifestations of length used by students in classrooms—rods, squares, cubes, and so on—possess multiple length dimensions (as well as area and volume), so students often have great difficulty properly abstracting linear extent. (This difficulty suggests that more thought and research should be directed to the types of concrete materials that are used instructionally for measuring length.)

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AUSTRALIAN INDIGENOUS STUDENTS' KNOWLEDGE OF TWO-DIGIT NUMERATION: ADDING ONE TEN

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With consistently low levels of academic performance and high rates of absenteeism (Bourke & Rigby, 2000), the students in this study are the most educationally disadvantaged group in Australia with respect to mathematics. This paper reports on a study undertaken with Years 5-7 students in a Queensland aboriginal community to determine their baseline knowledge of whole numbers. The results of a numeration test were analysed to identify major misconceptions, and selected students were interviewed to establish whether misconceptions were language-, context-, or mathematics-based, and then to "peel back" to culture- specific language and activities if language and concept proved to be the main sources of misconceptions. The interviews revealed that misconceptions were generally related to language and mathematics schema.

Research context. The students in this study live in a dispossessed multi-language indigenous community that was created through Government policy that had a penal objective. The history of this community has been one of exploitation, dependency and suppression of culture. Yet, although maintaining contacts with their language homelands, members strongly view their current community as their home. As a consequence, a culture has evolved that combines aspects of their original language group cultures and Western culture, a community language (called Aboriginal English) that is based on, but different from, standard English. The local political system is built on shifting divisions and language-group loyalties which impede developing consensus with respect to local pragmatic activities that may benefit the inhabitants' life chances, and this includes a community approach to education that would improve attendance and learning. The community has very low adult employment necessitating a reliance on welfare, leading to most of the problems described by Fitzgerald (2002), namely, alcohol and substance abuse, family violence, low health, transient population, discrimination and exclusion.

The community is an 8-9 hour drive from the University so that research visits (one week, four times a year) were set up in advance and consequently the actual research activities were highly contingent on the realities of the research context. For example, community, parent and student permission to undertake the research was obtained by the school's Community Liaison Officer in advance of the research activity but the students were often not available during the research visits.

This reported research project was undertaken at the local primary school. The teachers were generally in their first or second year of teaching and the school has a history of high turnover of staff. Within the year of this research project, there were five principals, four of whom were in Acting positions. Each teacher has an indigenous aide but, generally, the aides appear to be there for behaviour management rather than pedagogical purposes.

Mathematics context. Place value is one of the basic mathematics concepts with which indigenous students in this community have difficulty, with many students attending the secondary school unable to understand two- and three-digit numbers. In this they are not alone; research (Baturo, 1997, 2000, 2002; Jones et al., 1994) has produced a plethora of evidence that students have great difficulties in acquiring an understanding of place value. Baturo's numeration model (1997; 2000) gives some indication of the complexity of place value in that she indicates that there are three hierarchical levels which take account of Halford's (1995) complexity model. In particular, several researchers have pointed to the difficulty students have with: (a) *grouping/unitising* which involves quantifying sets of objects by grouping by 10 (in a base-10 system), treating the groups as units (Steffe, 1988), and using the structure of the notation to capture the information about the groupings (Hiebert & Wearne, 1992; Ross, 1990); and (b) *counting* principles (Baturo, 2002) such as *odometer* which require an understanding that a place is "full" when it has 9 units (which could be ones, tens, tenths, etc), that recording the next number requires a new position to the left of the place under consideration, and that numbers increase in value as they "move" to the left (and, conversely, decrease in value as they "move" to the right).

Modern teaching of mathematics to indigenous students is focusing on integrating mathematics into indigenous culture and experience so that the power of meaningful contexts can be harnessed in learning (Roberts, 1999). This recognition and valuing of the distinct cultural differences between indigenous and non-indigenous Australian cultures is a recent policy (Department of Education, Training and Youth Affairs, 2000); in the past, the dominant assumption in relation to the indigenous cultures of Australia was that both the peoples and their cultures would become assimilated into main stream Anglo Saxon culture (MacGregor, 1999). This concern for recognising and valuing indigenous culture reflects the emerging *ethnomathematics* position that "the accumulated experiences of the individual and one's ancestors are responsible for enlarging natural reality through the incorporation of mindfacts [ideas, particularly mathematics facts]" (D'Ambrosio, 1997, p. 16). It is particularly supported by Day (1996), who concluded that successful educational performance was closely linked to a healthy sense of indigenous identity.

In this study, primary indigenous students' understanding of place value for three-digit numbers was tested and difficulties associated with adding one and adding ten were probed with interviews. The interviews focused on whether mathematical tasks not understood in Standard English could be understood and completed if placed in an everyday out-of-school context familiar to the indigenous students.

METHOD

The methodology was predominantly qualitative using semistructured individual interviews.

Subjects. Eighteen Years 5-7 students who had undertaken the Diagnostic Mathematical Tasks (DMT) test (Australian Council of educational Research, 1994) participated in the semistructured individual interviews. The test was selected and administered by the school's Learning Support Teacher. The DMT comprise seven sets designed to be administered from pre-school to Year 6 in Victoria and Years 1-7 in Queensland. However, at the project school, students were administered the test that was two years lower than their current year level. That is Year 5 students were administered DMT Level 3, Year 6 students DMT Level 4, and Year 7 students Level 5. However, poor performance by the Years 6 and 7 necessitated administering yet a lower level.

Instruments. There were two main instruments: (1) the DMT test results; and (2) a research-developed interview schedule comprising "first-level" tasks which focused on basic three-digit numeration (i.e., reading numbers when presented in pictorial and symbolic forms, place value to hundreds, and adding 1 and 10 to a given number). The schedule also included "peel-back" activities should the students be unable to answer the first-level tasks.

The task under discussion in this paper is shown in Figure 1. For the purposes of this paper it is categorised as *place value counting*.

276
Write the number that is 1 more. __________
Write the number that is 10 more. __________
Contingent on performance with adding:
Write the number that is 1 less. __________
Write the number that is 10 less. ___________

Figure 1. Place value adding interview task.

For all first-level tasks, the first peel-back was to write a dollar sign in front of the number to determine whether this would trigger a real-world context for the symbolic representation. If the dollar sign did not elicit a response, then play money was used to represent $276. The second peel-back was to reduce the 3-digit number to 2 digits (i.e., 276 to 76). If this failed, the number 76 was proved in a money context as described for 276 and real money was used. To check the robustness of the student's response at this level of peeling back, an isomorphic number (e.g., 37) was embedded in either a card game or a sporting game context, depending on the student's out-of-school interests. (An earlier interview had elicited that many of the students had a passion for card games, football or netball.) For example, four playing cards comprising 3 tens and 1 seven was shown to the student who was asked to say what the score for this hand was and then say what it would be when another card with 10 was placed with the original 4 cards. If the student was more interested in sport, then the following scenario was given: *Your team scored 37 points in a football/netball match; the other team scored 10 more points. What was their score*?

Procedure. The test was administered and marked by the school's Learning Support Teacher in conjunction with the class teachers. We entered and analysed the results and provided these to the school along with a list of students (comprising a mixture of high, medium, and low-performing students) that we wanted to interview on an individual basis. However, the high absenteeism of the students resulted in our interviewing any available student from Years 5-7.

Analysis. The students' test results were recorded and then individual items were combined according to the numeration concepts of number identification, place value, counting, grouping and regrouping, comparing, ordering, and estimating. However, for the purposes of this paper only those related to adding 10 or 100 (for Year 7) are reported (see Figure 1 for the items and Table 1 for the results). The particular items on which the results were obtained are provided in

RESULTS

DMT results

Figure 2 tabulates the DMT items which led to the interview tasks whilst Table 1 provides the class means per item.

Place value adding

DMT3, Item 319. *Write the number 1 more than 299.* (Administered to Years 5 and 6.)

DMT3, Item 320. *Write the number 10 more than 163.* (Administered to Years 5 and 6.)

DMT5, Item 5024. *Add 100 to 20305.* (Administered to Year 7.)

Place value counting

Finish these counting patterns:

DMT3, Item 328. ___, ___, 73, 83, 93, ___.
(Administered to Years 5 and 6.)

DMT4, Item 4027. ___, ___, 1135, 1235,
1335, ___. (Administered to Year 7.)

DMT5, Item 5039. ___, 7974, 7984, 7994,
___, ___. (Administered to Year 7.)

Figure 2. DMT items on which the interview task was based.

	DMT means	
Year	Adding 1/10/100*	Counting sequence
Year 5 (*n* = 13)		
_ +1	Item 319: 07.7%	Nil example
_ +10	Item 320: 23.1%	Item 328: 00.0%
Year 6 (*n* = 8)		
_ +1	Item 319: 75.0%	Nil example
_ +10	Item 320: 50.0%	Item 328: 37.5%
Year 7 (*n* = 20)	Item 5024: 45.0%	Item 5039: 10.0%
		Item 4027: 50.0%

*Adding 100 was applicable to Year 7 only. For Year 7, there were DMT5 items related to adding 1 or 10.

Table 1 Indigenous Years 5-7 Students' DMT Means in Relation to Place Value Counting

Place value adding. Year 4 performed better on adding 10 than adding 1, a behaviour attributed to the artifact of the number provided (299) as the students needed to invoke the odometer principle twice. It is interesting to note, however, the improved performance on this item that was exhibited by Year 6. Furthermore, the Year 6 students performed much better on adding 1 than on adding 10, suggesting that the counting procedures may be more developed than place value.

Place value counting. The results show that the students performed better on the place value adding item than on the place value counting activity within each test level. This behaviour could be the result of having first to identify the adding 10 difference in the counting activity, having to count both forwards and backwards, and/or having to invoke the odometer principle. With respect to Year 7, the higher performance on the Level 4 counting (Item 4027) may have been the result of not having to invoke the odometer principle.

Interview results

Overall, 18 students were interviewed _ 5 from Year 5, 6 from Year 6, and 7 from Year 7. Their results are provided in Table 3. With respect to adding 1 to 276 (see Figure 1 for the interview task reported on in this paper), all the Year 7 students were able to do this successfully. Of the 6 Year 6 students, 4 were successful and the remaining two students were able to do it when the number was reduced to two digits, that is, 76. Of the 5 Year 5 students, two students only could add 1 to 276 but the remaining three students could add 1 to 56. Table 2 provides the results for the interview tasks and the various peel-back tasks.

	Interview task		
Task	Year 5 (*n* = 5)	Year 6 (*n* = 6)	Year 7 (*n* = 7)
Add 1 to 276	2	3	5
_ with $	0	0	0
_ with play money	1	0	0
_ 2-digit number	2	3	2
_ 2-digit with $	N/A	N/A	N/A
_ 2-digit number/play money	N/A	N/A	N/A
_ 2-digit number with cards	N/A	N/A	N/A
_ 2-digit number sport score	N/A	N/A	N/A
Add 10 to 276	0	2	2
_ with $	0	0	0
_ with play money	0	2	0
_ 2-digit number	2	1	2
_ 2-digit with $	0	1	0
_ 2-digit number/play money	2	N/A	2
_ 2-digit number with cards	1	N/A	1
_ 2-digit number sport score	N/A	N/A	N/A

*Adding 100 was applicable to Year 7 only. For Year 7, there were DMT5 items related to adding 1 or 10.

Table 2. Indigenous Years 5-7 Students' Interview Results in Relation to Place Value Counting

Because of the poor performance with respect to adding 1 and 10, the contingent tasks related to subtraction 1 and 10 (see Figure 1) were not included in the interview. The following protocols encapsulate the students' responses to the place value adding tasks.

Robert (Year 5)

Interviewer: What would 1 more than this number (pointing to 276) be?

Robert: 376.

Interviewer: What would be 10 more (pointing to 276)?

Robert: Ten more? Two thousand and seventy-six.

Peeling back to money and a 2-digit number didn't help Robert who was yawning throughout the interview and didn't appear interested in participating.

Interviewer: Do you play card games?

Robert: Yes. I earn a lot of money in card games. Do you play black and red?

Interviewer: Yes. You good at that?

Robert: Yep. Seven Down?

Interviewer: What's Seven Down?

Robert: You have to go, like, seven downwards. You know Two Card?

Interviewer: No. How do you play Two Card? [Robert explained the game.]. I'm going to do Coo'nCan _ If your score was already 45 points and you scored 10 more points in the next game, how many points would you have?

Robert: 60.

Interviewer: You've got 10, 20, 30 40 and 5 (showing with real cards) and I give you another 10 (showing card), what have you got?

Robert: 55.

Interviewer: How did you know that?

Robert: Because I added it up.

Interviewer: But when I said here "10 more than 45", you were having difficulty there. Is it easier with the cards?

Robert: Mmm.

Janice (Year 7) was considered a high performer within her class.

Interviewer: What's one more than 276?

Janice: 277.

Interviewer: Ten more?

Janice: 300?

Interviewer: What if I cross out the two, what number is there now? [76] What's one more than that? [77] What's ten more?

Janice: 80.

Interviewer: Let's put out $76 with this play money. What if I gave you one more $10 – how much would you have then?

Janice: 86

Interviewer If your team scored 35 points at netball (Janice liked netball) and the other team beat you by 10 points, what would they score?

Janice: 40…what they have?

Interviewer: They beat you by 10. You scored 35.

Janice: 45.

Interviewer: Well done.

DISCUSSION AND CONCLUSIONS

The results of all of these elementary place value/seriation tasks are of concern considering their importance in the development of the number system. Without adequate foundational knowledge, students cannot develop the facility for numbers (i.e., number sense) that is the focus of current mathematics syllabi (e.g., NCTM, 2000).

The seriation items evoked low performances suggesting that adding 1 or 10 to a 2- or 3-digit number may have been a nonprototypic task for these students. Baturo (2002) undertook a similar study with middle-high socioeconomic Western students being required to add 1 tenth or 1 hundredth to decimal numbers. That study produced similarly poor results suggesting that there is an inherent mathematical difficulty embedded in place value adding tasks.

There was some evidence that "peel-back" activities, namely using contexts such as money, cards, sports or reducing the number of places that need to be considered (e.g., 3- to 2-digit numbers). The results of both of these studies indicate the fragile nature of students' understanding of place value and seriation. Students cannot seriate without good place value knowledge. Whilst the contextual peel-back activities appeared to be effective, it is suspected that they were "of the moment". Therefore, the challenge for teachers is to find ways of abstracting the mathematics embedded in these activities.

With respect to teaching numeration processes, Ross (1990) claimed that children learn to represent numbers with concrete manipulatives (as practised in Queensland schools) through following the teacher's directions rather than from thinking about what they have constructed. The results of this study suggest that teachers need to be more creative in the types and levels of examples they provide to ensure that students have the robustness and flexibility of knowledge that is required for number sense.

Teaching indigenous students is fraught with difficulties associated with language, socialisation and cultural problems which are exacerbated by poor attendance. Teaching "good mathematics" in these circumstances would challenge experienced teachers but the teachers in this community have, historically, been very young teachers in their first or second ears of teaching. They should be applauded for the mathematics concepts they have been able to establish in such a difficult situation.

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USING INSTRUCTIONAL REPRESENTATIONS OF RATIO AS AN ASSESSMENT TOOL OF SUBJECT MATTER KNOWLEDGE[1]

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This study posits that instructional representations are viable assessment tools of subject matter knowledge within the context of lesson study. Evidence is provided and validated, suggesting that representations provide valuable insights into the depth and accuracy of the knowledge prospective teachers bring to instructional settings. It is conjectured that those with weak subject matter knowledge may ask too many open-ended questions, and over rely on their students to "remember." Those growing in their understanding may be reluctant to cede ownership to their students while those with strong understanding include many representations in their lessons.

PROBLEM, FOCUS, AND FRAMEWORKS

The development and growth of teachers' subject matter knowledge of ratio is the research problem addressed in this study. The link between teachers' knowledge of mathematics and the quality of classroom instruction was established in a number of studies (Ball, Lubienski, & Mewborn, 2002). Without a deep understanding of mathematics (Baturo & Nason, 1996), it is difficult for teachers to change their instructional practices to incorporate curricular reforms. While much has been learned about proportional reasoning over the last 20 years (Lamon, 1995) assessing teachers' content knowledge of ratio and proportion is difficult from an educative perspective. Teachers' knowledge, in the traditional sense of paper/pencil testing, does not correlate with traditional measures of student achievement. Wilson, Schulman, and Richert (1987) suggest that this failure is attributable to the narrow definitions given to teacher knowledge. They posit that an assessment include what is known about the subject matter and how to present that subject matter to others. We aim to study what prospective teachers know about ratio from the instructional representations they select in a lesson plan. Instructional representations are viewed as a link between content and pedagogy by a number of researchers (Wilson, Shulman, & Richert, 1987; Ball, Lubienski, & Mewborn, 2002). Words, pictures, graphs, objects, numbers, symbols, and contexts (including examples, metaphors, analogies) constitute the instructional representations that convey mathematical ideas. They serve as powerful connections between what teachers know about mathematics and what they know about teaching mathematics. The focus of this study is to determine if the initial instructional representations of prospective teachers are viable assessment tools of their subject matter knowledge. The study reported here provides base line information to inform a larger study on the growth of understanding of ratio among prospective teachers.

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Two theories conceptually guide this research: a) a model of the knowledge base of teaching proposed by Schulman (1986), and b) the theories of Pirie and Kieren (1994) that describe the growth of mathematical understanding. Of interest are two components of the knowledge base, subject matter knowledge and pedagogical content knowledge. Subject matter knowledge is defined as the procedural and conceptual knowledge of mathematics, as well as, the connections and relationships within ideas. Shulman defines pedagogical content knowledge as the subject-specific instructional strategies, instructional representations, and teachers' knowledge of students' understanding. The model of the growth of mathematical understanding, developed by Pirie and Kieren (1994), gives perspective to the emerging ideas expressed by the prospective teachers' selections of instructional representations of ratio. The Pirie-Kieren theoretical model is one of actions and interactions, tracing the back and forth movement of the learner's ideas between and among eight levels of understanding activities where the learner builds, searches, and/or collects ideas (Pirie & Martin, 2000). The movement toward inner layers is termed *folding back* and serves an important role in the growth of understanding as it may signal a fundamental shift in the learner's understanding. For this study we focus on the first five activities of understanding. The innermost level is that of *primitive knowing* consisting of all of one's previous knowledge and serves as the reservoir from which to build subsequent understanding. Moving outward within the model, *image making* and *image having* are learner activities that involve making a new image or revising an existing image, and then abstractly manipulating that image. It is these two levels of activities that play a prominent role in our analysis. Other levels of activities used here are *property noticing*, and *formalising*. Identifying properties of the constructed image defines property noticing and perhaps the images are formalised when a method, rule, or property is generalised (Pirie & Kieren, 1994). To begin the process, the researcher presents a problem or task to the learner so as to observe, record, and interpret the growth of understanding within a mathematical context.

OBTAINING AND ANALYSING THE EVIDENCE

Prospective elementary teachers at a large, Australian university volunteered to participate in a semester-long teaching experiment. It is noted here that instruction and terms associated with teaching ratio and proportion differ across countries. US students are introduced to fractional notation of ratios very early in instruction while most Australian students never use fractional notation. Here we report on the cases of three undergraduates beginning their third year of study. Stephanie and Maria were in their mid-30s, returning to the university after previous career and family experiences. Abbie was in her early 20s and had gone directly from high school into college. Lesson study was chosen for the teaching experiment of the larger study and here we focus on the first three tasks of lesson study. Described by Lewis (2002), lesson study is a model of Japanese professional development where communities of teachers come together as researchers to recursively develop, discuss, and teach a single lesson over an extended period of time. Teachers are cast in the role of researchers as they examine, test, and modify the lesson. Several researchers adapted these techniques to study the growth of understanding of slope, similarity, and right triangle trigonometry among prospective US high school mathematics teachers (Berenson, 2003). Findings suggest that prospective teachers come to the lesson study tasks with a variety of mathematical understanding, and as they engage in individual and group lesson study activities, grow in their understanding. Evidence was collected individually in the two videotaped interviews, and included the artifacts developed during the three activities of lesson study. In the initial interviews the prospective elementary teachers were asked to recall what they learned about ratio and proportion in school, what their teachers did, and what their textbooks showed. The interviewer assessed the undergraduates' ideas during this initial interview, and if necessary taught the missing subject matter. Then the subjects were asked to plan a lesson to introduce the concept of ratio to an average class of seventh graders. The initial, 15-minute interview was followed by 45 minutes of planning time where the undergraduates had access to textbooks, materials, and manipulatives. No particular format was requested for the plan although subjects were encouraged to write down their ideas to present in the 30-minute follow-up interview. Data were analyzed using categorical aggregation to find patterns. The conjectures or assertions drawn from the patterns of the data summarize the findings.

DIVERSITY OF REPRESENTATIONS AND UNDERSTANDING

Part – Part Comparisons	Simplifying	**Missing Value**	**Notation**	**Applications**
Table to show number of boys to girls* (T) G | B 15 | 10	How can we group girls and boys so that each group looks the same? (T)	If 15 G to 10 B is equal to 5G* to 2B, then how would we work out the number of girls to boys if there are 200 boys? (T)	3 : 2 = 15 :10 Or 3 girls = 2 boys 15 girls 10 boys (T)	If we know proportion of one thing to another we can predict for larger pop. (T)
Use counters or bears to represent class (S)	Use bear counters to find equal groups. (S) Make a table of trials: G., B., & Leftover. (S)	Ratio Table. G. 5 10 15 _ _ 125 500* B. 2 4 10 _ _ 50 200 (T)		Ask where seen e.g. fuel consumption, recipes (T & S)
	Use student models to show not equivalent e.g. 6 g & 4b leaves 3 g & 2b left over. Draw Table: G B Leftover 15 10 0 6 4 5 7 5 1 3 2 0 Arrows relate equivalence* (T)			Give real world problem to groups. If space shuttle uses 1000 litres fuel to travel to Mars which is 10,000 km from earth, how many litres?* (T & S)

S = student task; T = teacher task; * = accuracy/appropriateness questioned

Table 1: Maria's instructional representations

Tables 1-3 characterise the subjects' instructional representations from which are drawn the individual assessment of each teacher's subject matter knowledge of ratio. Images of the instructional representations are categorized according to the subject matter included in each individual plan. Maria's plan includes part – part comparisons, simplifying ratios, missing value problems, ratio notation, and applications (See Table 1 above). Data from Stephanie's plan is sorted into part – part comparisons, simplifying ratios, building up strategies, and notation (See Table 2). Images in Abbie's plan are rates, missing value problems, notation, and applications. Under each of these subject matter categories are the instructional representations included in each plan. For example, Maria selects an instructional representation of a data table of the number of girls and boys in the class and then moves to using teddy bear counters to represent part – part comparisons. Within each category, an order is implied from top to bottom but not across subject matter categories. Areas of understanding are identified in the narrative, and incomplete, questionable, or missing ideas are highlighted with an asterisk in the tables. The initial "S" or "T" in each cell denotes who will use the representation, students or teachers.

Maria appears to have a very good understanding of part – part ratios, using tables and manipulatives to represent the boy/girl comparisons of the classroom. Her initial choice of words, comparing boys to girls, and then the notation (15 g to 10 b) calls for monitoring a potential problem of order in writing ratios. She moves beyond an algorithmic understanding of simplifying ratios toward a conceptual understanding with the use of manipulatives to model possible equivalent ratios of 15 : 10. This understanding is incomplete as she was not aware that the bear representation of *6 green – 4 yellow – 5 left over* combination was equivalent to 15 : 10 and 3 : 2. It suggests that simplifying ratios means "lowest terms" rather than equivalence.

Her second use of ratio tables to find missing values provides information that Maria was able to construct multiple sets of equivalent ratios. Additional assessment is needed to determine if the misrepresentation of 15 : 10 = 5 : 2 and subsequently in the table of missing values is a computational problem or an indication of lack of planning time. The notation that Maria chooses to present to her students includes the colon and fractional notation. Her use of labels in the fractional notation provides confirming evidence that she understands the importance of order in writing ratios. In the representation of *the real world* space problem, we question the accuracy of the fuel and distance quantities. Overall, Maria's plan has more instructional representations than the other plans, and we assess her understanding of ratio subject matter to be deeper than the other prospective teachers. Maria will grow in her understanding of ratio and proportion as she collects more instructional representations, noticing the properties of the ideas embedded within the representation to formalize deeper understanding of the subject matter.

Two essential differences between Stephanie (see Table 2 below) and Maria's plans are the number of representations and who plans to use the representations. Maria has more instructional representations, yet gives students some of these representations to build understanding of difficult concepts. Stephanie uses fewer representations in her plan with little student ownership of the representations. Her representation of part – part summing to the whole is an example of property noticing of this type of ratio, and indicates a growth in her understanding of part – part ratios.

Part – Part Comparisons	**Simplifying**	**Adding Up Strategy**	**Notation**
Student taste test of 2 differ items intended to bake for par tea (S)	Reduce to low number. Hopefu (they) will divide 5.* (S)	How many of each it do we need to cook 100 people? (T&S)	Ratio – explain ter comparing two quantiti 10 : 15 (T)
Make a chart of preferences (T Item 1 | Item 2 10 | 15	So for every 2 peo who like item 1, 3 l item 2.	2/3 adds up to 5 10/15 adds up to 25 So you need someth that adds up to 1((T&S)	
Explain: Out of 25 people in class 10 prefer item 1, 15 pre item 2. Item 2 is the m popular. (T)	Use colored counte Shows 10 yellow a 15 red. Then shows yellow and 3 red. *(T	Finding equivalent ra means same equivalent fractio Use fraction sticks remind them. (T)	
Both together add up to 25 total or whole. (T)			

S = student task; T = teacher task; * = accuracy/appropriateness questioned

Table 2. Stephanie's instructional representations

She chooses to teach simplifying ratios but "hopefully they will know to divide by 5" reflects her lack of knowledge of simplifying ratios. Representations of equivalent ratios using fraction bars may flag a weak understanding of proportion. Her building up strategy appears to indicate an image that is non-traditional in terms of introducing ratio problems but rooted in her own experience of cooking for 100 people. Overall, Stephanie's understanding of the subject matter is emerging as noted by the number of representations in her plan. Stephanie can grow in her understanding by collecting more instructional representations of ratios, simplifying ratios, equivalent ratios, and different types of problems and strategies.

Rate Comparisons	**Missing Value**	**Notation**	**Applications**
Introduce water bill classroom tap drippi The tap drips 5 tin every 2 minutes.* (T)	I left my office at 5pm yesterday an was going to measure the water loss putting a bucket under the leaking t but I forgot. How could I work out h much water was lost between the ho of …….? * (S)	What would be an ea way of presenting t data? Teacher introdu of concept of rate ra and how it is written. (	Can you think of other ways that we could use this idea of ratio? (T&S)*
Show chart: Drips Time ooooo 2,00 5 2 (T)	Student groups share their finding w class. Class questions findings. (S)		

S = student task; T = teacher task; * =

Table 3: Abbie's instructional representations

At first glance Abbie's plan (Table 3) appears to be exactly what we want beginning teachers to adopt, open-ended and student-centered instructional approaches. While the pedagogy may be sound, there are very few instructional representations of the subject matter. Abbie poses a problem that is computationally very challenging (5 drops : 2 min), and requires unit conversions of minutes to hours. The students are asked to solve the problem, explain and question the solution strategies, and come up with applications of ratios. The assessment that Abbie lacks the subject matter knowledge of ratio and proportion is based on what representations are missing rather than what she included. To grow in her understanding, Abbie needs to collect many more images of ratio, noticing properties to formalise her understanding.

VALIDATING THE ASSESSMENT

We conclude from the analysis of their instructional representations that Maria understands most of the ideas of her plan deeply, Stephanie understands some initial ideas, and Abbie may have very little idea understanding of the subject matter. We looked back to our initial interviews with each undergraduate and the extent to which they remembered or were able to communicate their ideas in the first interview. Maria's transcript speaks to very good understanding of ratio, proportion, and missing value problems. Her initial image of ratio was *how two different factors relate to each other* and her example was to use a rate comparison of dollars to lollies. When asked how she would explain ratios to students she replied, *You have to know the context. Suppose you have $2, then you can buy 3 lollies. If you have $4, how many can you buy?* To explain her definition of proportion to the interviewer, Maria wrote 2 : 3 = 4 : 6. Though unsure of her ideas, Stephanie was able to create images of ratios relating to her own experiences in cooking and entertaining, growing in her understanding by folding back to her primitive knowledge. In response to the interviewer's request to define "ratio," Stephanie said it was a *proportion like boys to girls. If there were 119 boys in a football club and 1 girls then we write this as 119:1.* When asked later in the interview what the term proportion meant to her, Stephanie said, *My expenditures are not in proportion to my income.* Stephanie was able to solve missing value problems without any additional instruction. While highly motivated to be an excellent eacher, Abbie's primitive knowledge of ratios and fractions was very thin. She recalled that math was her worst subject in school and that she did not enjoy math class. Her teachers were very strict and punished frequently. The only thing she remembered about ratios was how to represent them with a colon. It was necessary for the interviewer to teach Abbie some of the comparison ideas of ratios and fractions so that she was able to continue with the next lesson study activity, the lesson plan. During this instruction Abbie was able to grow in her understanding of ratio to solve several missing value problems. Triangulating the results with the initial interviews, we conclude that the assessments emanating from the instructional representations are valid.

IMPLICATIONS

Conjecture 1. *If lessons do not balance the number of open-ended questions with other instructional representations it may indicate weak subject matter knowledge.* Among some beginning teachers there is the belief that all teachers need to do is to ask students an open-ended question to "inform" the other students. What they fail to realize is that students are not capable of "spontaneously generating" knowledge without instructional assistance. An open-ended approach requires deep subject matter knowledge in order to design a variety of instructional representations that lead to student understanding.

Conjecture 2. *If the teacher "owns" all of the instructional representations then it may indicate a level of uncertainty with the subject matter*. Teachers own almost none of the representations in the first conjecture, but it is more common to find teachers who own most of the representations. Teachers who are the primary users of the representation in the lesson appear to be collecting images of the subject matter before releasing them to the students. Teachers may be unsure of the subject matter and feels the need to "try out" these new images. They may even fold back to primitive knowledge to make new images during the process of teaching.

Conjecture 3. *If the teacher expects students to "remember" an algorithm or "guess" an answer correctly for a major concept in the plan, it may indicate the teacher's lack of deep understanding*. The primitive knowledge of many prospective teachers includes an algorithmic understanding of the subject matter. They are able to use procedures to solve problems and calculate answers. It is only when they begin to plan their lessons that they realize a deeper understanding of concepts, rules and definitions is necessary.

Conjecture 4. *If a teacher uses many accurate instructional representations, then it indicates a deep understanding of the subject matter*. It requires deep mathematical understanding for teachers to generate multiple representations of the subject matter. When teachers employ multiple representations of ideas and relationships that are accurate, the assessment of strong subject matter knowledge is made. The converse of this conjecture may also be true. If there are few accurate representations of the subject matter, then additional assessment and instruction are warranted.

Many teacher educators assign lesson plans to assess prospective teachers' use of pedagogy. A number of researchers advocate for preparation that moves beyond pedagogy as a primary focus. Programs are advised to incorporate subject matter knowledge along with pedagogy so as to provide prospective teachers with experiences in what and how to teach (Ball, Lubienski, & Mewborn, 2002; Sowder et al. 1998). Here we propose an assessment alternative of subject matter knowledge that moves beyond paper and pencil tests and finding the "correct" answer. These assessments are embedded within teaching tasks of lesson planning that are common to most preparation programs.

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A SOCIAL EXTENSION OF A PSYCHOLOGICAL INTEREST THEORY

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Based on an individual interest theory as a sensitising theory, empirical data are used to gain social interest concepts, as there are situated collective interest and interest-dense situation. These concepts serve as a basis for a social extension of a psychological interest theory. Its construction combines social interactions, the dynamic of epistemic processes and mathematical valency of situations in maths lessons. However, this paper is restricted to the presentation of results concerning social interactions. The construction process of theoretical interaction types is outlined and leads to a typology of interactions which provides the theoretical background for interaction analyses of interest-dense situations and their genesis processes.

INTRODUCTION

Researchers agree that interest is an outcome of social processes. But there is a lack of empirical studies investigating social situations, especially social interactions in maths classes in order to find out interest supporting conditions (Bikner-Ahsbahs 2001, 2002). The main reason for this seems to be that it is difficult to link an individual to a social perspective within one empirical study. In my study the psychological interest theory and its implications were used in a sensitising way (Brandt/ Krummheuer 2001, p. 11) to gain the concept of *interest-dense situations* which describes situations in maths classes with a high potential for the support of interest development. The investigation of these situations led to the construction of a contextual theory (Brandt/Krummheuer 2001, p. 199) about the genesis of *situated collective interest* and its impact on the development of interest.[1]

MATHEMATICAL IDENTITY AND INTEREST IN MATHS

A concept of personal interest is a concept of intrinsic motivation. Nevertheless, it cannot be restricted to it. Interest research is always connected with research dealing with self determination theory and personal development (Deci 1992, 1998) which includes research of the development of mathematical identity. Mathematical learning practice always produces relationships between mathematics and the students through the discursive processes and the production processes of mathematical ideas, no matter the participants are actively participating or not. The way students advocate mathematics in discursive processes forms the quality of mathematical identity (Klein 2002, Boaler 2002). Whether mathematical identity includes interest in maths or not, is due to the experience of competence and autonomy within the production processes of mathematical ideas, the amount to which students engage themselves within these

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[1] The theoretical background of interest research concerned with the individual view of interest development was presented at PME25 (Bikner-Ahsbahs 2001).

processes and the experience of valuable and sense-making mathematical contents connected with positive emotions (Krapp 2002, Bikner-Ahsbahs 2001, Deci 1992, 1998).

Now I am going to propose a concept of mathematical identity which assumes the necessity of interaction. This concept explains the emergence of interest within the construction process of mathematical identity.

AN INTERACTIONIST VIEW ON IDENTITY AND INTEREST

Krappmann proposes a concept of identity which is balanced between an individual and a social view of identity based on the necessity of social interaction (Krappmann 1968). He assumes that interaction only continues, if the participants of the discursive processes take up the interlocutors' contributions and if they simultaneously work out and express their individual views. This means that a person with a balanced identity shares views of others in a discourse, while simultaneously he or she develops and expresses his or her individual view on and preferences for special aspects and methods. Mathematical (learner) identity can be understood as a balanced concept of identity within mathematical learning practices, as it is constantly (re-)constructed by adopting, refusing or (re-)constructing mathematical ideas, mathematical methods, and other ways of interacting or not-interacting with mathematical tools and material in discursive processes. Hence, mathematical identity can be regarded as a construct which describes the relationship of a person with mathematics. This relationship becomes evident through one person's behaviour that is basically dependent on - and stimulated by - experienced mathematical learning practice. In a similar way interest development can be regarded as a constantly balanced process between individual and social relatedness to mathematical learning practices, in which a person (re-)constructs an epistemic relationship with experienced mathematical contents, expressing his or her valuing and emotional relationship to it.

Usually interest is an individual concept which requires an appropriate approach that is psychologically focusing on individuals. Analyses intended to investigate social and epistemic conditions in classes require different approaches concerning social interaction patterns, social practices and epistemic actions. A theoretical approach intended to describe social conditions which foster or hinder interest development in maths lessons has to include views of individual interest. This is the idea for the construction of the basic concept of "interest-dense situations".

THE CONCEPT OF INTEREST-DENSE SITUATIONS

Using the psychological interest theory (Krapp 2002, Bikner-Ahsbahs 2001) as it is developed so far as a sensitising theory (Brandt/Krummheuer 2001) and the concept of balanced interest development as a sensitising concept, I constructed the concept of *interest-dense situations* based on the collected data on the one hand and the empirical results of interest research on the other hand. In brief, an *interest-dense situation* is a *situation during a maths lesson which initiates interest activities*, that is the emergence of *situated collective interest.*

What does *situated collective interest* mean?

In a maths camp young people come together because they are collectively interested in maths. In classes this is usually not the case. But sometimes a kind of *situated collective interest* emerges. That is a construct which *describes a relationship between the active participants in the class and the mathematical content. This relationship can be observed through interaction processes showing high amount of student involvement in the activity, student constructions of further-going meanings, and mathematical valency of the situation* (Bikner-Ahsbahs 2002). That means that the students all together construct further-going meanings turn by turn, one after another is getting involved in the activity and the value status of the situation is tied up with its mathematics. Situations in which situated collective interest emerges are called *interest-dense,* and *interest density* is used as a synonym for situated collective interest.

During an interest-dense situation the active participants do not have to be individually interested in the topic area, in the sense that they are aware of their interest. Since they act as if they were interested, they at least begin to build up a kind of situated interest as a balanced, epistemic, positively valued relationship to the mathematical content. Therefore, active participation in interest-dense situations is likely to foster the development of individual or situational interest as components of mathematical identity.

The concept of interest density now leads to the basic research question for the data analyses: How do social interactions, the dynamics of the epistemic processes, and the constructions of mathematical valencies have an effect on the genesis and stabilisation of interest-dense situations? Although the theory is already worked out, I will restrict my presentation to social interactions and the construction of theoretical types of interaction structures, which can help teachers in fostering the development of interest and help them to avoid enhancing the development of disinterest.

METHODOLOGICAL FRAMEWORK

The design of data collection was already presented at PME25 (Bikner-Ahsbahs 2001, 2002). In this paper I will focus on data analysis concerning social interaction practices. The analysis was done by using video recordings of all lessons of one class from half a school year, except the lessons which were involved with test taking.

The data show two different kinds of interest-dense situations, as there are *ad-hoc-interest-dense situations* and *generative-interest-dense situations*. Ad-hoc-interest-dense-situations are initiated by the students asking deep questions or contributing far going ideas. Generative-interest-dense situations are initiated by the teacher based on mathematical tasks, problems or questions the teacher begins with and the way these situations are organised. In ad-hoc-interest-dense situations situated collective interest emerges spontaneously. However, through generative-interest-dense situations the hole genesis process of interest density is observable, hence reconstructable.

My analysis of interest-dense situations uses an interpretive approach reconstructing structures of meanings by interpreting the interactions at three levels: the level of information (locutional), the level of generating meaning through acting (illocutional), and the level of intention and effect (perlocutional) (Beck/ Maier 1994). The method of analysis follows a recursive structure enhancing the theoretical content cycle by cycle. Every cycle of analysis comprises the comparison of an ad-hoc-interest-dense scene with a scene which begins in a similar way but in which interest density ceases. Through analysing these contrasting scenes in a comparative way it was possible to construct two theoretical types (Kluge 1999) of social interaction structures which foster or hinder the emergence or stabilisation of interest density. Based on these types I gained a marking space made of two dimensions, the teacher and the student behaviour, with two features each. A crossing table gives an overview about possible theoretical types of interactions (fig.1). These theoretical types are seen as theoretical descriptions being helpful to describe, analyse and diagnose real situations.

An analysis of the data which did not show situated collective interest led to a more precise description of all possible fields in the crossing table which creates a typology (Kluge 1999) of interactions. This typology was the theoretical background for the analysis of the genesis processes of generative-interest-dense situations from the perspective of social interactions. Applications of this typology to real situations have to include, that a typology is not a classification of reality. It describes real situations more or less and transitions between the types cannot always clearly be fixed in reality.

A TYPOLOGY OF INTERACTIONS

student behaviour / teacher behaviour	**expectation dependent**	**expectation independent** ((re-)constructing own r which is)	
	(anticipating the teacher's expectations)	part of the e pectation space	not part of the exp. S space pace
expectation controlled (expecting concrete students' answers)	expectation-dominant interaction structure	F lowing	confligating
situation controlled (re-)constructing the students' meanings	misunderstanding	expectation-recessive interaction structure	

Figure 1: A typology of interactions (expectations are meant as the teacher's content specific expectations)

Each of the two dimensions in the crossing table show two features: expectation-controlled and situation-controlled teacher behaviour; expectation-dependent and expectation-independent student behaviour. This leads to four different situations. However, the reconstruction of different kinds of interactions through the data distinguishes five different situations:

- two situations which are balanced by interaction structures;
- two situations with inherent conflicts showing incompatible teacher and student behaviour (misunderstanding and conflicting interactions) and

- one situation which shows a flow of interactions in which the students' utterances seem to be part of the expectation space of the teacher. The students act concerning their own constructions whereas the teacher handles the utterances like a reconstruction of his own expectations.

If the teacher and the students focus on the teacher's content specific expectations, a stable balance of social interactions emerges: the *expectation-dominant interaction structure*. The function of this interaction structure is to reproduce teacher expectations. If the teacher and the students focus on the students constructions of meaning, another but labile balance of social interactions may emerge: the *expectation-recessive interaction structure*. The function of this interaction structure is to enable students to (re-)produce mathematical meanings.

The comparison of generative-interest-dense situations with ad-hoc-interest-dense situations shows that ad-hoc-interest-dense situations immediately begin with an expectation-recessive interaction structure and maintain interest density until this pattern ceases or the task is finished. Unlike ad-hoc-interest-dense situations, the process of genesis of generative-interest-dense situations begins with an expectation-controlled teacher behaviour. In this case situated collective interest is generating more slowly so that the starting point of situated collective interest usually cannot precisely be fixed. Based on the crossing table we find a wide range of different generating processes which all have in common, that as soon as interest density emerges, we find an expectation-recessive interaction structure. Therefore, an expectation-recessive interaction structure is necessary for the genesis of an interest-dense situation, but not sufficient.

I will now present a short summary of the analysis of a scene, in order to give you an impression of the way how these interaction structures are constructed.

CONSTRUCTION OF THE EXPECTATION-RECESSIVE INTERACTION STRUCTURE

The presented scene shows a prototype of an interaction structure that fosters the emergence and stabilisation of interest density in ad-hoc-interest-dense situations. Anji refers to a group activity in which a group of three boys had to divide four pieces of liquorice into three parts while each length of the four pieces was not divisible by three. The group made a long piece out of all pieces by putting one after the other. The whole length was divisible by three then. Since they had to find more then one way of dividing their sweets into three equal parts they invented a way of dividing the pieces lengthwise by dividing the round cross-section into three. As the class had not measured angles before this group had to find out how to divide a circle into three equal parts.

1 Anji: I've a question. they've divided it from the top downwards sure but how do they know then what 120 DEGREES means.

2 T: I see' you now want to go back to the set square once again. won't you'

3 /Anji: no (.) yeah but if they ,that's such a small piece and how do they know that because ,they can't do that with the set square

4 /S: yeah that's round of course.

5 /Ernst: but that's round of course

6 S: that is round of course

7 T: Tom yes. that's such a practical problem isn't it' how do I do it if that's such a very small one and not such a BIG circle.

8 Tom: you must put zero in the centre and then it will work anyway I think (.) then you must only keep the lines in mind going from 120 until you can draw them.

9 T: well I see ,there must be additional ,we're going to practice that sometime

10 /S: yes but how'

11 /Rahel: Mr Kramer I have another silly question ,how do we get the centre OUT ,how have they got it OUT because that is so small

12 S: that's of course

13 T: that's a problem too ,exactly. that's a practical problem (...) well ,how do you get the centre of such a small circle anyhow (.) ,exactly. these are questions'

14 Anji: with a small compass'

15 T: yeah you can get it out with a compass ,only if you draw a circle first' then you'll have the centre but if you already have got a circle'

16 Rahel: yes

17 T: that's exactly what Geometry deals with

18 Andy: I KNOW that

19 T: there are possibilities to get that out a-n-d you may puzzle on it at home probably somebody might find a possibility'

20 /Rahel: yes I know-

21 T: well at home after all ,we'll just use that as a part of the homework' you draw a circle ,but you'll erase (.) the centre and when you have got the circle. you'll try how can you find the centre

22 /S: but you do put that thing in there

23 /S: you stick it in

24 T: you stick it in

25 /S: wha'

26 S: um

27 T: yeah but you can act as if you didn't have it. how can you find it then. probably there is a possibility probably you'll find something and then you try try once again to divide into three (..) well at first drawing a circle' then doing like you didn't have the centre' you can't find it anymore and how can you find it again when you have it how can you divide it into three then

TRANSCRIPTION KEY

S(s), T	student(s), teacher
EXECT	emphasized or with a loud voice
e-x-a-c-t	prolonged
exact.	dropping the voice
exact´	raising the voice
,exact	with a new onset
(.),(..)...	1, 2 ... sec pause

The teacher tries to understand what the students mean. He does not force the students to answer in a special way and he does not show his own content specific expectations. We can assume that he has some, but they do not seem to be important. Instead, he tries to reconstruct the students' goals and he tries to reflect and understand the way the students act. His behaviour is focused on the children’s contributions and not on his own ideas. He is (re-)constructing the students meanings, acts in a *situation-controlled* and not in an *expectation-controlled* way concerning his own content specific expectations.

On the other hand the children ask their own questions in a self assured way. They even refuse the teacher's information about the topic of Geometry in general. The students are involved in the problem which they want to get solved. They construct their own sense-making mathematical meanings and they are not concerned with reproducing teacher expectations. They act in an *expectation-independent* way. Teacher behaviour matches with student behaviour. That stabilises the interaction and supports the production of mathematical ideas by the students.

This scene shows that an *expectation-recessive interaction structure* gives the students access to the construction of their own sense-making mathematical meanings and to experience themselves as competent and autonomous participators within the discursive practices. The teacher himself is not passive. He focuses on the students’ constructions, shows interest in their constructed meanings and tries to understand their behaviour.

Further analyses show that content specific expectations of the teacher which dominate the teacher’s behaviour, hinders a successful emergence of interest density, on the other hand if the teacher abstains from his content specific expectations, he will be able to focus on the constructions of the students' meanings. Then the teacher will be able to support the emergence of interest actions and interest density. However, the support of interest only works, if the students act in a adequate way: They have to concentrate on their own thinking. This is not usual, because often the students try to reconstruct the teacher’s expectations, even though the teacher does not really show any. The students may interpret the teacher’s behaviour as a hint for being on the wrong track.

CONCLUSIONS

The constructed types are theoretical types which provide basic concepts for the constructed theory. Theoretical types cannot be observed empirically, but data give access to prototypes corresponding as far as possible but not in all perspectives to them. The interrelations of the theoretical types are used to generate the theory as a contextual theory (Brandt/Krummheuer 2001, p. 199) with a limited scope and with deep insight in the processes concerning the genesis of interest-dense situations in maths classes during the learning of fractions at the age of about 11 in a German gymnasium. Using such local theories it will probably be easier to change teacher-student-relations towards the support of interest development in every day maths classes. However, innovative practice cannot mean implementing theoretical types. Practice has to deal with unexpected situations. These theoretical types can serve as an orientation which may help teachers diagnosing real situations, making decisions and developing and implementing suitable prototypes depending on the contextual environment, on their view of mathematics and the mathematical learning process and on the behaviour of the students. However, this theory does not include the impact on interest development of individuals yet, because the analyses of the individual data are not finished. But there is growing evidence in the data that active participation in interest-dense situations do support the development of interest.[2]

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PARTICULAR AND GENERAL IN EARLY SYMBOLIC MANIPULATION

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The teaching and learning of early algebra draw heavily on arithmetic and the relationship between these two forms of activity is much debated. Drawing on interviews with 12 year old pupils we consider the ways in which some pupils used substitution of numbers for letters to extend their ability to manipulate algebraic expressions. We argue that teaching programmes need to emphasise 'seeing the particular in the general' alongside the manipulation of general expressions.

INTRODUCTION

Studies of the early stages of algebraic understanding have often been premised on the idea that algebra is a natural extension of arithmetic. This premise gives rise to a host of studies which have treated as crucial the move from arithmetic to algebraic thinking (e.g. Filloy and Rojano, 1984). Though the debate about where arithmetic ends and algebra begins has never quite been resolved, researchers have seemed to focus on finding where algebra becomes difficult, and how teaching interventions might be designed to enable pupils to overcome cognitive obstacles, bridge cognitive gaps or cross the didactic cut (Herscovics and Linchevski 1994, Filloy and Rojano 1989).

More recent work has argued that the late introduction of algebraic thinking in the school curriculum is at least in part responsible for pupils' subsequent difficulties (Carraher, Schliemann and Brizuela, 2001). Carraher et al describe how children aged 8 and 9 years began to use algebraic notation, though whether they were operating on the unknown was disputed by the research forum respondents (Linchevski, 2001, Tall, 2001). This study, and others looking at algebraic activity amongst young children (for example Blanton & Kaput, 2002; Slavitt, 1999), set the context for a widespread conviction that the teaching of arithmetic in elementary schools must be seen as providing the foundations from which algebraic thought can develop. This imperative is now enshrined in policy documents in both the US (NCTM, 2000) and the UK (DfEE, 1999).

Two kinds of activities which might be seen as drawing directly on pupils' arithmetic experience are the simplification of expressions through symbol manipulation (which tends to encourage 'letter as object' thinking) and the evaluation of expressions through substitution (which tend to encourage thinking of the letter as standing for a number). The ways in which algebra is typically introduced in schools tend to treat these two activities separately, rather than making explicit links between them. In this paper we look at some evidence of the way in which 12 year old pupils make use of their arithmetic understanding to work on problems which require manipulation of algebraic symbols. We make the assumption that when a child sees the usefulness of making a substitution of a particular value for an algebraic symbol, then this is one instance of what

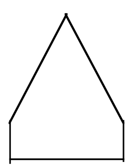
Mason (1993) calls ‘seeing the particular in the general’. It is a demonstration that the child sees the generality, and not just the symbol.

BACKGROUND

The data reported on here was collected as part of the Purposeful Algebraic Activity Project[1], a three year longitudinal study. In this study we are designing and using a series of algebraic tasks based on the use of spreadsheets with 11-13 year olds in two local comprehensive schools. One of the key features of these tasks is that they make use of the spreadsheet as both a context and a motive for generalising arithmetic processes (see Ainley, Bills and Wilson (forthcoming)).

As part of the project we are conducting interviews with two cohorts of pupils at different stages during the early years of secondary schooling. Through these interviews we intend to track pupils’ developing understanding and algebraic proficiency. The interviews on which this paper is based were conducted with cohort A, when they were aged 12, at the end of their first year in secondary school.

The interview questions were chosen to explore a variety of aspects of pupils’ algebraic capabilities. The three questions on which we report here were selected to examine pupils’ ability, at a simple level, to use the normal rules of arithmetic to simplify expressions that contain letters standing for numbers. They were:

<table>
<tr>
<td>Question 3:What is the distance around t
</td>
<td>Question 12:
Simplify the following:
i. $2a + 5a =$
ii. $2a + 5b + a =$
iii. $3a - b + a =$
iv. $(a - b) + b =$</td>
<td>Question 13: Are these statements true or false?
i. $\dfrac{6+6+6}{3} = 6$
ii. $\dfrac{x+x+x}{3} = x$</td>
</tr>
</table>

Questions 3 and 12 are drawn from the CSMS survey (Küchemann, 1981). Correct answers to question 3 and the first two parts of question 12 could be given using only a ‘letter as object’ type of reasoning (Küchemann, 1981). Question 3 gave a context for the interpretation of the value of the letter, whereas question 12 gave none. Parts (iii) and (iv) of question 12 introduced negative signs or subtraction operations. Whereas in part (iii) the term $-b$ could be treated independently, and therefore as an object, in part (iv) an operation had to be performed on $-b$ in that it had to be combined with $+b$. We therefore expected the four parts of question 12 to have decreasing facility, as was found by Küchemann.

Question 13 was adapted from a question used in a national test set for 14 year olds. Part (ii) demanded that the letter be interpreted as a generalised number, and the presence of

[1] The Purposeful Algebraic Activity Project is funded by the Economic and Social Research Council

question 13(i), along with, in some cases, the encouragement of the interviewer, supported pupils in considering particular values as substitutes for the letter x.

Our concern in this paper is to consider the evidence that we have from our interviews about pupils' ability to use two kinds of reasoning – the 'letter as object' reasoning and the 'letter as generalised number' reasoning described by Küchemann. The former we expect to be characterised by reification ('the x') and/or reference to moving the letter ('put the a and the $3a$ together'). The latter we expect to recognise by reference to operations on the value ('divide $3x$ by 3') or by substitution of particular values into the algebraic expressions.

METHOD

Twenty four pupils from two local comprehensive schools were interviewed in pairs in the first cohort. The pupils were selected by their teachers from those who were willing to take part, with the aim of making compatible pairs of similar attainment. The pairs were distributed across the perceived attainment range in the year group, and contained a balance of boys and girls.

The interviews were conducted by a researcher (the third named author), who presented the questions in written form, and also read them aloud. All interviews were video taped, and audio taped for transcription. Copies of any pupil writing were collected, and added to the transcripts, which were also annotated to include non-verbal behaviour observed in the video tapes. In the analysis presented in this paper we have not generally tried to separate out the contributions of the two individuals to the interview, except where there was an obvious difference of approach.

FORMS OF REASONING

Question 3

Ten of the twelve pairs gave a correct answer to this question, most of them doing so quickly and without very much discussion. The other two pairs tried to answer the question by giving a value to x. One of these pairs, Mollie and Grace (low attainers), gave an answer of 50 and when asked to explain said

> "For x centimetres that's 15, and then we added 15, 15 and 15, which is 45, and then we did 5 add 5, which makes 50".

The other, Nathan and William (high attainers), debated whether x should be taken as 10 centimetres or 15 centimetres and came to no conclusion.

Of the ten pairs who answered correctly, a number gave indications of 'letter as object' reasoning, for example:

> "There's three xs up there and you add them up together. Cause x isn't a number and you can't add the x all together" (Sophie and Lauren, middle attainers)

> "There's three xs so you'd say three x" (Natasha and Holly, middle attainers)

Others gave indications of interpreting x as a number, for example:

> "Cause there's three sides that says x centimetres, so it's $3x$ centimetres" (Kieran and Emily, high attainers)

> "There's three xs so it would be three times x" (Adam and Connor, middle attainers)

<u>Question 12</u>

The first two parts of question 12 were answered correctly by all pairs and presented little difficulty for most.

One pair gave a very clear indication of using 'letter as object' reasoning:

> "anything can be an a, so you can put two apples and five apples, so then you can, like, add the five and the two, which is seven, and then you can put seven a equals seven apples" (Mollie and Grace, low attainers)

Others were less clear:

> "there's two a there and an a there and they're exactly the same, well, not in number wise, but they're both as, so you can add those two, which is three a" (Olivia and Lucy, low attainers)

Question 12(iii) proved more challenging. Three pairs failed to reach the correct answer, whilst a fourth pair did not agree on their answer. One of the three pairs that failed to reach an answer explained their difficulties as follows:

> "No, because it's three a minus b and you can't, normally like this when you can put the a with that one, but you can't put the a with the three a because three a, that a is being added to the answer that you get from three a minus b" (Amy and Georgia, high attainers)

Of the successful pairs, few gave any explanation of their answers, but one pair gave some insight into their reasoning:

> "It would be, say, forget about the take away b for the moment, we'll put three a add the other a, so that would be four a, take away the b" (Mollie and Grace, low attainers)

Question 12(iv) caused difficulties for nearly all the pairs. Four pairs achieved some degree of success, and these all involved some substitution in their reasoning.

Kieran and Emily (high attainers) disagreed at the outset, Emily interpreting the bracket as a signal to multiply, apparently because they had been expanding brackets recently in lessons. Kieran began by saying

> "I think it would just be plain a. Cause it's a minus b, but there's a plus b as well, so the b, both bs would cancel out"

A discussion ensued in which Emily, encouraged by the interviewer, substituted a equals 2 and b equals 1 into the expression, but failed to see the significance of the result. Kieran responded with

> "If you don't get it in algebra, why don't you just change it to numbers and then do it in numbers and change it back into algebra and then you've got the answer, like you just did then".

Emily remained unconvinced.

Adam and Connor (middle attainment) had already used substitution in 12(i) to establish that the expression should be simplified to $7a$ and not $7a^2$. For part (iv) Connor was at first interested in whether a is bigger than b, and said:

> "So if a is bigger it would stay the same, because it's take away b and then plus b"

In order to address what happens when a is smaller, they both choose more values to substitute and become convinced that, whatever the values of a and b, the expression is equivalent to a.

Amy and Georgia (high attainers) disagreed at first. Georgia suggested "*a* plus *b*?" and Amy responded, "No, I'd say *a* on its own would be that".

A discussion followed in which Georgia became persuaded of Amy's view. During this Amy wrote:

$a = \cancel{2}5 \quad b = \cancel{8}3$

$5 - 3 + 3$

Mollie and Grace (low attainers), having acknowledged the meaning of the brackets, said:

> "So you'd say *a* take away *b*, um, they can be any numbers, so it doesn't matter what numbers they are"

They went on to collaborate over the substitution ($a = 2$, quickly changed to $a = 4$, $b = 3$) as follows:

Grace	So it could be two
Molly	Take away three. No.
Grace	No, four
Molly	~ yeh. No. Four take away three, so, and then
Grace	The answer could be one
Molly	One, and then you add b
Grace	Add b
Molly	Which could be, hold on, yeh, three, then it would be, what, um
Grace	Or you could do four take away three
Molly	Three, is one. And then because the *b* is a three, so it would be four, so that's four

They do not reach a final statement that the expression is equivalent to *a*, but it is not clear whether this is because they do not see that this is the case or because they are encouraged to go on to the next question.

Of the other eight pairs several mentioned the difficulty created by the presence of brackets in the question, and $a + 2b$ and $a – 2b$ were popular answers.

<u>Question 13</u>

All the pairs were successful with 13(i). Some pairs explicitly calculated before announcing a decision, whilst others announced that it was true without obviously calculating. Some of these made reference to a calculation when asked for a justification.

Ten pairs were agreed that the statement in 13(ii) was true, whilst two agreed it was false. Of these ten, six pairs did not make use of any particular values in justifying the truth of the statement. Typical responses were:

> "cause you've got a number, you times three, then you're dividing it by three, so it's the same" (Amy and Georgia, high attainers)
>
> "yeah, that's the same, because there's three of them, and so divide by three would just take you back to *x* and that would be *x*" (Adam and Connor, middle attainers)

A further three of the successful pairs came to their conclusion having been prompted by the interviewer to consider particular values for *x*. For example, Natasha and Holly (middle attainers) worked as follows:

Holly	But it probably will be true then. Um, because I think that, If you do, if you do six; six plus six plus six is eighteen, then you have to do eighteen divided by three which is six
Interviewer	uh-huh
Natasha	and you do ten as well, it would be thirty, divided by three is ten, and then
Holly	Yeh
Natasha	Seven, twenty one divided by three is seven
Holly	Yeh, it's true.

The final successful pair, Mollie and Grace (low attainers), arrive at a similar conclusion after spontaneously deciding to try 'any numbers', though with considerably more hesitation over the calculations required.

The two pairs who agreed that the statement was false made no attempts to substitute particular values and seemed confused by the syntax.

DISCUSSION

After a year or more of algebra instruction these pupils showed a good deal of competence with basic 'letter as object' reasoning, not only in simple cases of questions 3 and 12(i), (ii), but also in the more demanding context of 12(iii). Question 12(iv) proved very challenging, even for high attainers. Two features of this question seem to have caused difficulty, i.e. the brackets and the negative sign. Difficulties with negative signs have already been documented, and the literature is well summarised in Vlassis (2002), which also sheds some new light on possible reasons for the difficulties. Our interest here is in the strategies used by the pupils who were successful in 12(iv) which was a difficult and non-routine problem. The use of substitution was part of this successful strategy in each case, and the successful students came from across the attainment range.

Similarly, in question 13 we observed some pupils working competently with a 'letter as object' approach, and others addressing this non-routine problem successfully using substitution.

Bazzini et al (2001) use Frege's model of sign/sense/denotation to describe pupils' engagement with algebraic symbolism. They describe how for some students the connection between sign, sense and denotation is lost as they follow a procedure, whereas for others different senses of the same sign can be activated simultaneously so that non-routine problems can be approached. The appropriate use of substitution in the problems we set seems to us to be evidence of an activation of a sense for the algebraic expression which is different from a 'letter as object' understanding.

Cerulli and Mariotti (2001) describe the work of Francesca who was able to transform the expression $(a + b)(a - b)$ into $a^2 – b^2$, but was unsure whether (10 + 13)(25 –3) was the same as 10□ 25 - 10□3 + 13□25 - 13□3. For Francesca it seems that in the algebraic context the sense being activated relates to symbols which can be manipulated according to certain rules. There is evidence that she does not recognise a numerical analogue of the expression as being subject to the same rules of manipulation. When presented with a similar expression containing values rather than algebraic symbols she does not immediately see it as a particular example of a general equivalence. The implication is that she was not seeing $(a + b)(a - b)$ as a generalisation of particulars, but merely as a symbolic expression. In Mason's terms she has not 'seen the particular in the general'.

In our experience of teaching programmes in the UK, the activity of substitution is practised in the context of evaluation of algebraic expressions, but it is not often linked with developing thinking about symbolic manipulation. Indeed substitution may be widely regarded as not part of a scheme of algebraic development (See, for example, Ursini and Trigeros, 2001).

Bazzini et al (2001) recognize the importance of flexibility, or the activation of different senses, in the solution of non-routine problems. Seeing the particular in the general, as evidenced by a purposeful use of substitution, as well as operating at a general symbolic level, offers one form of flexibility. We have evidence that continuing to see the particular in the general can help pupils to 'activate different senses' and solve the non-routine problem.

Within our teaching programme we exploit opportunities to support pupils in understanding the 'particular and general' nature of algebraic symbols. The spreadsheet environment contains a powerful ambiguity: when a formula is entered in a column, it can be 'filled down' to operate not just on a single cell, but on a range of cells in a column. The symbol used as a cell reference can then be seen as both particular (the number I am going to enter in this cell) and general (all the values I may enter in this column). In future interviews we plan to look for evidence of how working in the spreadsheet environment has supported the development of flexibility in pupils' algebraic thinking.

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THE NATURE OF SCAFFOLDING IN UNDERGRADUATE STUDENTS' TRANSITION TO MATHEMATICAL PROOF

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This paper explores the role of instructional scaffolding in the development of undergraduate students' understanding of mathematical proof during a one-year discrete mathematics course. We describe here the framework adapted for the analysis of whole-class discussion and examine how the teacher scaffolded students' thinking. Results suggest that students who engage in whole-class discussions that include metacognitive acts as well as transactive discussions about metacognitive acts make gains in their ability to construct proof. Moreover, students' capacity to engage in these types of discussions is a habit of mind that can be scaffolded through the teacher's transactive prompts and facilitative utterances.

BACKGROUND FOR THE STUDY

Exploring Sociocultural Aspects of Students' Understanding of Proof

> "The concept of proof is one which not only pervades work in mathematics but is also involved in all situations where conclusions are to be reached and decisions to be made. Mathematics has a unique contribution to make in the development of this concept, and […] this concept may well serve to unify the mathematical experiences of the pupil" Harold P. Fawcett (1938)

Since the statement above was written, the assumptions about proof as a logical argument that one makes to justify a claim and to convince oneself and others, and its role in mathematics, have not changed. Mathematicians and mathematics educators unanimously agree on the importance of proof in mathematics and the necessity for students to develop both the understanding of concepts related to proof and the skills to read and write proofs. However, the ability to read and do proofs in mathematics is a complex one that depends on a wide expanse of beliefs, knowledge, and cognitive skills and that is uniquely shaped by the social context in which learning occurs.

Research on students' understanding of mathematical proof has focused on cognitive issues, including the development of students' proof schemes (Harel and Sowder, 1998) and students' misconceptions and difficulties with proof (e.g., Balacheff, 1988; Chazan, 1993; Porteous, 1990; Senk, 1985). However, the effect of sociocultural factors on students' transition to mathematical proof, particularly in undergraduate settings, remains a virtually unexplored domain. Thus, we are engaged in a study of the role of the social in how students in a one-year, undergraduate mathematics course come to understand proof.

We see an emphasis on the social character of proof as situated within the broader theoretical perspective that development cannot be understood apart form the *social* context in which it occurs (Vygotsky, 1962/1934). In particular, Vygotsky maintained that "higher voluntary forms of human behavior have their roots in social interaction, in the individual's participation in social behaviors that are mediated by speech" (Minick, 1996, p. 33), and that students' development of self-regulatory thinking occurs through a process of internalizing events that originate on the social plane. As part of this, he postulated the notion of a zone of proximal development (ZPD) as a way to conceptualize learning.

Scaffolding and the Zone of Proximal Development

The ZPD is defined as the space characterizing one's potential for development through the assistance of a more knowing other (Vygotsky, 1962/1934; Litowitz, 1993). As a diagnostic, the ZPD intends to assess not only those cognitive functions that one possesses, but also those that are in the process of development by virtue of the learner's *interaction* with more knowing others, cultural tools, and so forth (Kozulin, 1998). Since learning is viewed as a product of interaction, it follows that one's development within the ZPD is affected by the intellectual quality and developmental appropriateness of these interactions (Diaz, Neal, & Amaya-Williams, 1999). In other words, the extent of one's development within the ZPD is predicated in part upon how the more knowing other organizes, or *scaffolds*, the task at hand. Thus, if we intend to understand development within the ZPD, we must think about if and how tasks can be scaffolded to extend one's learning.

As a construct inseparable from the ZPD, *instructional scaffolding* is a mechanism for observing the process by which the learner is helped to effect his or her potential learning (Stone, 1993). Practically speaking, it refers to the "provision of guidance and support which is increased or withdrawn in response to the developing competence of the learner" (Mercer, 1995, p. 75), and it is based on the appropriation, not simple transfer, of ideas between teacher and student. However, understanding the subtleties by which this occurs is a complex process that requires sensitivity to the learner's goals as these goals emerge in the course of activity (Wells, 1999). Thus, within the classroom, scaffolding presupposes that the teacher is continuously attending to students' thinking in order to access their individual (and communal) ZPD. For example, knowing how to give hints that focus and challenge a student's thinking requires a deep knowledge of students' individual learning capacities with respect to the task at hand. The complexity increases for the teacher because hints are often given in large group settings that necessarily conceal individual differences and thus diminish the teacher's capacity to attend to them.

From this perspective, we came to view the nature of scaffolding and when and how one's learning is scaffolded as a critical part of understanding how students learn to construct mathematical proofs. Thus, within the broader purpose of exploring sociocultural factors in undergraduate students' transition to mathematical proof, we focus here on instructional scaffolding and how it supported the development of students' capacity to write and express rigorous mathematical proofs. In particular, we share our findings on the following specific questions:

h. What is the nature and meaning of instructional scaffolding in the classroom in the development of students' proof ability?

i. How do different types of scaffolding prompts from the teacher affect students' self-regulatory thinking about proofs?

METHODOLOGY

Participants, Data, and Setting

Participants for the study were two cohorts of undergraduate mathematics students, with 50 students per cohort, enrolled in a one-year discrete mathematics course that emphasized mathematical argumentation and proof. Classroom instruction was videotaped and selected small group discussions were audiotaped. Whole class and small-group episodes were selected for transcription and analysis. Additionally, students were given pre- and post-assessments which were analyzed to identify the generality, form and competency of students' arguments and which we took as evidence for shifts in students' capacity for self-regulatory thinking (see Blanton & Stylianou, 2002). Finally, students' individual written proof constructions were collected biweekly. The study reported here focuses on data collected during whole-class and small-group discussions that occurred in the first semester of the course.

The instructor (the same for both cohorts) worked to establish expectations that students explain their reasoning and make sense of and challenge each other's explanations and justifications. Students submitted regular assignments in which they wrote proofs and reflected about their thinking. Classroom activity focused on group problem solving and included alternative forms of assessment (e.g., group exams, reflective writings).

RESULTS

A Framework for Analyzing Instructional Scaffolding

We begin here by describing how our focus narrowed to instructional scaffolding and the subsequent framework we adapted for its analysis. Our previous work provided both a general description of how students evolved in their capacity for argumentation and written proof and quantitative results that students were learning to construct increasingly rigorous proofs (see Blanton & Stylianou, 2002). However, we wanted to more carefully detail the mechanisms of classroom interaction that mediated the collaborative, or public, development of students' proof ability. Consequently, our focus shifted to analyzing the discourse structure in whole class discussions, using each speaker's turn as the unit of analysis. For purposes of analysis, we found it useful to distinguish between *public* and *private* cognition, where we take public cognition to mean mathematical knowledge that is publicly owned and constructed. As we analyzed discourse data, it became apparent that the teacher's utterances, because of their intent coupled with her function as a more knowing other, were fundamentally different than those of students. Thus, we could not analyze these data as a group discussion such as that among peers, but had to attend to the dynamic created by the different purposes of the speakers. This redirected our attention to the teacher's utterances in order to detail the nature of instructional scaffolding and how it extended students' development within the ZPD.

We based our framework for analysis on the work of Kruger (1993) and Goos, Galbraith, and Renshaw (2002). In particular, we found Kruger's (1993) framework for the analysis of transactive discussion helpful in identifying each person's contribution to the collaborative structure of the whole class interaction. Transactive discussion is characterized by clarification, elaboration, justification, and critique (of one's on or one's partner's reasoning). Moreover, transactive discussion refers to the ways that people publicly engage with metacognitive utterances. Thus, we drew from the work of Goos *et al*, itself an extension of Kruger's framework, to analyze metacognitive utterances that functioned as "New Idea" or "Assessment". However, since the work of Kruger (1993) and Goos, *et al* (2002) is based on peer group analysis, we needed to extend their frameworks by analyzing the intent of the teacher's utterances as well. Thus, in our framework for analysis of whole-class discussion, utterances were cross-coded in terms of metacognitive acts (New Idea; Assessment), transactive utterances, and the nature of scaffolding in the teacher's utterances.

The Nature of Instructional Scaffolding in Students' Proof Construction

From our analysis, we found that the teacher's utterances consisted of *transactive prompts* and *facilitative utterances*. By facilitative utterances, we mean instances of revoicing or confirmation. We define transactive prompts to be a form of scaffolding in which the teacher's questions promote transactive discussion among students. In particular, the teacher's utterances consisted primarily of *requests for* clarification, elaboration, justification, and critique, all of which formed the basis for a complex, interconnected dialogue by which students engaged in metacognitive acts, transactive discussion, and transactive discussion about metacognitive acts. Goos, *et al*, (2002) found transactive discussion of metacognitive acts to be a significant factor in successful (small group) collaborative problem solving. We conjectured a similar effect on whole-class discussion and we argue that, to the extent that the teacher's transactive prompts were able to facilitate transactive discussion in whole-class dialogue, she was able to scaffold students' thinking in publicly constructing mathematical proofs.

To support this claim, we share here the coding and analysis of an excerpt from a 60-minute classroom episode that occurred during Week 4, where the task was to construct a proof that $\sqrt{2}$ is irrational. Codes of utterances are italicized in the protocol. The analysis focused on characterizing the structure of dialogue surrounding transactive prompts and facilitative utterances in the whole class discussion in order to understand how the teacher was able to scaffold student thinking and what this suggested about student development within the ZPD. Student names are pseudonyms.

Teacher: Why is that true ('$2q^2 = p^2$' fails for odd numbers)? (*request for justification)*

Anthony: We could prove that an odd times an odd is an odd. (*new idea)*

Teacher: Yeah. We could do something like that. That would certainly work. That would be a more general case in fact, instead of a particular case. *(revoice and confirm)*

Degan: We already know that 2 times any integer is going to be an even number anyway. *(new idea)*

Jarrod: That's what I was going to say. The left side ($2 p^2$) is always even. *(elaboration)*

Teacher: OK, So here's something ($2p^2$) that's always going to be even, so you're saying that if p is odd, [then] p^2 is odd, so you'd have an odd number equal to an even number? *(clarification)*

Jarrod: Yeah.

Teacher: True. So if p is odd, it fails. (*revoice and confirm)* Are we done? *(request assessment of proof status)*

In the above episode the teacher aims to scaffold the students towards the construction of a particular proof. The classroom had agreed the previous day that a proof by contradiction would be an appropriate strategy to use, and the teacher initiated the discussion by re-stating the agreed upon plan. The teacher restrains her comments in only three types: (a) requesting clarification, elaboration, justification, or assessment, (b) revoicing and/or confirming a student statement, and (c) elaborating on a student-originated idea. While the teacher herself avoids engaging in transactive discussion (except in the one case where she elaborates), her goal is to encourage her students to do so as they gradually progress in their proof construction.

The scaffolding here takes two forms. The obvious form of scaffolding is the teacher's confirmation of students' ideas. By revoicing and confirming student-originated ideas, the teacher lends authority and confidence to students, as the "more knowing other", to proceed along the student-suggested path. The second form of scaffolding is the teacher's repeated requests for students to engage in transactive discussions. And while by the first form of scaffolding the teacher shares responsibility for the proposed action (through a tacit approval), the second is a transfer of responsibility for a construction of a proof from the teacher to the students (through her requests for assessment and critique). A second difference is with respect to the overall goal of each form of scaffolding. The former involves utterances specific to a given mathematical problem. The latter is a theme that permeates the entire semester; it is about the development of the habit of mind of being inquisitive and engaging in metacognitive acts.

The question that arises is whether the teacher, through the two forms of scaffolding, accesses students' ZPD. Our coding and analysis suggest a tentative hypothesis: Students' proposal of "new ideas" and their subsequent elaboration and justification of these ideas in a way that furthered the construction of a proof indicates their development within the ZPD. Pre-test results (Blanton & Stylianou, 2002) suggested that prior to instruction students were not able to construct this proof. However, the teacher's transfer of the proof responsibility through transactive prompts supported students in making significant contributions to the proof. With respect to our first research question, we claim that while both types of scaffolding prompts impact student proof ability, it is likely that prompts that encourage transactive discussion are the most crucial in the development of students' proof ability. Further study is needed to understand whether the two types of scaffolding prompts impact students' reasoning differently at different stages of the proof construction.

We were further interested in examining possible patterns of transfer of the teacher's scaffolding prompts in students' small group discussions. We conjectured that transactive patterns in whole class dialogue, led initially by the teacher, eventually would be internalized by students in their acquisition of self-regulatory thinking. Our subsequent coding of students' small group discussions provided evidence that students assumed the role of scaffolding each other with the same transactive prompts their instructor earlier urged them to use to scaffold their own thinking about proof. In this sense, we argue that the forms of argumentation essential for proof-building were becoming a habit of mind for students *independent of the teacher's participation in the dialogue*. The following excerpt, which occurred during Week 5, is a small-group discussion for which the task is to prove that for any even integer, n, n^2+1 is odd.

Mike: Does this show this… this is true? *(request for clarification)*

Justin: Say, assume n^2+1 is even so then you can throw out… (*clarification)*

Mike: Right…. *(confirm)*

Justin: [voicing his algebraic work] n^2+4l^2+1 … n^2+4l^2…. *(elaboration)*

Mike: So that's where I show n is odd down here. *(confirm)*

Steve: Ummm….yeah *(confirm)*

Mike: But aren't you trying to show n is odd? *(request for clarification)*

Justin: I did. *(clarification)*

Mike: I don't know…I don't really think you… I don't think you proved it yet, but that could be close. Because you're trying to show its odd and all you proved is n^2+1 is even. *(critique)*

Steve: Alright, what are we trying to show? *(request for assessment of proof status)*

Justin: I know but I showed its even when I say $n=2l$. *(clarification)*

Steve: We showed its odd… n^2 we showed is…. *(clarification)*

The discussion in the small group is fundamentally different than the whole-class discussions. While in both episodes the main objective is to produce a correct proof to a given problem, in the small group discussion there is no instructional intent to scaffold student thinking. The "more knowing other" becomes the "more capable peer" in a discussion among equal partners. This difference is reflected in the type of utterances in the small group discussion that fall into two categories: (a) requests for clarification/elaboration/justification (transactive), and (b) responses to these requests. Justin appears to be the more inquisitive partner, but his requests do not imply that he assumes the role of the scaffolding instructor. His requests are the expression of his attempt to follow his partners' reasoning and negotiate meaning with them, not to get his partners to engage in transactive discussion for its own sake. However, *indirectly* and unintentionally, his requests have an impact similar to the teacher's earlier requests: The other two students are forced to clarify their reasoning and, subsequently, advance their own understanding and thinking. With respect to our second research question, we make an initial claim that students appropriated the structure of whole-class dialogue, scaffolded by the teacher's transactive prompts and facilitative utterances, and used this to advance their own and their classmates' proof construction.

DISCUSSION

Concerning the notion of scaffolding, Stone (1993) notes that little attention has been paid to the mechanism by which the transfer from mentor to student is accomplished. Indeed, little, if any, research has focused on instructional scaffolding in tertiary mathematics settings. As such, this study was intended to provide insights into how students appropriate strategies for advanced mathematical reasoning and how instructional scaffolding supports this. Our results suggest that students who engage in whole-class discussions that include metacognitive acts as well as transactive discussions about metacognitive acts make gains in their ability to construct mathematical proofs. Moreover, students' capacity to engage in these types of discussions is a habit of mind that can be scaffolded through the teacher's transactive prompts and facilitative utterances. This has serious implications for the nature of whole-class discourse that occurs in advanced mathematical settings, at least for those that deal conceptually with mathematical proof. In effect, it suggests that students can internalize public argumentation in ways that facilitate private proof construction if instructional scaffolding is appropriately designed to support this.

More work is needed, however, to further detail the nature of instructional scaffolding and its longitudinal effect on students' capacity for small-group and individual proof construction. For example, analyses that would establish the increasing use of transactive prompts in students' private proof construction subsequent to whole-class discussion would further our understanding of the development of students' proof conception and would provide a critical link between instructional acts and the development of one's transition to mathematical proof.

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CHILDREN'S CONCEPTIONS OF INFINITY OF NUMBERS IN A FIFTH GRADE CLASSROOM DISCUSSION CONTEXT

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"How many are the numbers?", "How many numbers are there between 1 and 2?". These tasks were set in three fifth grade classes in an educational environment characterised by an alternation of classroom discussions orchestrated by the teacher and students' production of individual reports. They allowed us to analyse both the short term evolution of students' conceptions about infinity of numbers through social interaction, and their relationships with the cultural environment. Moreover, during classroom discussions we observed an autonomous shift from the original tasks to a new question "Is it possible that an infinity of numbers exists?". A number of related naive epistemological positions (that seem to depend at least partly on the students' cultural environment) were detected.

INTRODUCTION

Preceding research studies suggest that what children say about infinity depends on both task and context. Both the way of formulating the task (textual aspects) and the mathematical context (e.g. geometrical or numerical) seem to be influential (see Monagham, 2001, p. 240 ; 254). Moreover, students' conceptions of infinity are open to changes and shifts in the short term within a suitable computer learning environment (see Sacristan Rock, 2001). However, we can observe that most research on children's conceptions of infinity takes into account only their individual performances, often in a situation of interaction with an adult (usually, the interviewer); only few research studies deal with classroom social interaction situations. In particular, Bartolini Bussi (1989) describes a teaching experiment on infinity in grade IV based on classroom discussion orchestrated by the teacher. In that situation, some interesting shifts in students' positions were observed and many arguments coming from students' school and everyday life culture entered the debate. The history of mathematics provides some evidence about strong relationships existing in some crucial periods (in ancient Greece and, especially, in the second half of XIXth century) between specific epistemological conflicting positions and theoretical constructions concerning infinity, and general philosophical assumptions depending on the culture of the time (see Jahnke, 2001, pp. 189-192). All this legitimates the following questions:

A) how can classroom discussions (orchestrated by the teacher) influence the short term evolution of students' conceptions about the infinity of numbers?

B) is there a counterpart for young students of what emerges in the history of mathematics, in terms of the relationships between their conceptions about infinity of numbers and their cultural environment?

The aim of the research reported in this paper is to study what fifth students say about the infinity of numbers, taking into account both the educational context (in particular, what happens during classroom discussions orchestrated by the teacher) and the cultural context (in particular, with reference to religion, available technological tools, and school learning about numbers).

THEORETICAL FRAMEWORK

A preliminary methodological problem, highlighted by preceding studies (cf. Monagham, 2001, p. 240), concerns the way of considering what students say in relation with what they think about infinity (and related subjects) within an educational context where relevant hints come from the formulation of tasks and from schoolmates' and teacher's interventions. We decided to use the expression "conception of infinity of numbers" to designate what the student says in a given moment, being aware that in many cases no firm, conscious and clear acquisition, independent of the communication context, underlies his/her utterances. Different analyses of students' conceptions of infinity have been made in preceding studies (see Monagham, 2001 for a survey). We share the point of view expressed by D. Tall (see Tall&Tirosh, 2001, p. 133) about the epistemological distinction (and cognitive tension) between "*'natural' concepts of infinity*," possibly bearing contradictory aspects, *"that arise through extending finite experiences to the infinite case, and 'formal' concepts of infinity framed in modern axiomatic approaches"*. According to the axiomatic method, *"selected finite properties can be formulated to give corresponding axiomatic theories"*. In particular, we will try to detect (in what students say) early traces of a "sequential" point of view (based on the idea of moving from each number to the next one, ... and so forth), which could be related to Peano's axiomatisation of numbers, as well as of a "cardinal" (or "set") point of view, based on the consideration of "quantity of elements", which could be related to a systematisation of numbers based on set theory.

Focus on the relationships between students' conceptions and their cultural environment drew our attention on the need of considering arguments that enter a discussion about infinity of numbers and come from non-numeric domains. Different tools were available; we have considered "Mental models" (see Fischbein, 2001) and "Conceptual metaphors" (see Nunez, 2000). In particular, Nunez provides a frame to describe mental operations performed through the use of conceptual metaphors (defined as *"cross-domain mappings that project the inferential structure of a source domain onto a target domain")*. It allows also to distinguish between different kinds of conceptual metaphors that intervene in our study: mainly grounding metaphors (the source domain is in everyday life experience); and linking metaphors (the source domain is mathematical but different from the domain of numbers under scrutiny).

Considering the evolution of students' conceptions from a social construction point of view, the need for analysing such evolution in relationship with the educational setting (classroom discussions, etc.) and the cultural environment led us to adopt a Vygotskian perspective. This choice was made in order to frame and interpret in a coherent way some specific phenomena related to social interaction and cultural belonging (in particular, the internalisation process; the shift from content conceptions to meta-theoretical considerations) – cf Vygotskij (1990)

We conclude this Section with two assumptions concerning terminology taken from Boero, Douek & Ferrari (2002): *argumentation*: "*both the process that produces a logically connected (but not necessarily deductive) discourse about a subject, as well as the text produced by that process*"(the context will allow to choose the appropriate meaning*). Argument*: "*a reason or reasons offered for or against a proposition, opinion or measure* (Webster Dictionary) *– it may include linguistic arguments, numerical data, drawings, and so forth*".

THE TEACHING EXPERIMENT

Three fifth-grade classes of 16, 21, 22 students were involved in the study in March 2002. Teachers belong to the Genoa Research Group in Mathematics Education.

The sequence of tasks and the educational context

According to the usual sequence of activities in the Genoa Group Project, classroom activities concerning infinity of numbers were organised as follows:

FIRST PHASE: First classroom discussion (20'-25'): *"Sometimes in the last months we have met the problem of how many are the numbers –whether finite or infinite. Now it is the moment to start dealing with this problem in depth"*. The discussion was followed by the production of an individual text concerning *"What do you think about the problem that we have discussed?"*. The following day, 3 or 4 (according to the class) individual texts, representing different positions, were selected by the teacher, photocopied and distributed to all students, with the task of identifying both analogies and differences with personal productions. The teacher helped some students (through 1-1 interaction) to perform this task. The aim of this phase was to create a shared baggage of ideas and references for the subsequent phase.

SECOND PHASE (1-2 days later): Second classroom discussion (35'-40') *"In order to go in depth into the problem that we have discussed in the last few days, it is useful to try to answer the following question: 'How many numbers are there between 1 and 2?'. Try to do your best to answer this question!"*. Then students were asked to produce an individual, written *"detailed report about your position and its motivations"*. The choice of setting a task different from *"How many are the numbers?"* was meant to avoid mere repetitions of arguments and positions that had been expressed in the preceding phase, and engage students (who had not produced some arguments) to appropriate them in order to tackle a different question.

Concerning the mathematical background, most students were able to deal with finite-decimal numbers. They were able: to perform arithmetic operations with them (both using paper-and-pencil methods, and pocket calculators); and to use them in measuring activities, in particular to represent/read numbers like 2.5 or 0.82 on the ruler. They had already experienced the fact that the division 1:3 produces the result 0.3333...

Criteria for analysing students' productions

We have tried to analyse the evolution of students' conceptions during the discussions, and in comparison with their individual texts, by considering the "sequential" and "cardinal" points of view (cf our theoretical framework). The progressive maturation and clarification of such conceptions were also analysed. In order to study the relationships between students' conceptions of the infinity of numbers and their cultural environment, we have considered both the nature and the sources (technology, school subjects, religion, and personal preceding elaboration) of the arguments brought by students (specially metaphors). The origin (when, why and how) and the distribution and possible evolution of the most frequent arguments in the three classrooms were considered too, in order to detect some effects of classroom discussions on the students' argumentation.

SOME OUTCOMES

As expected according to preceding studies (cf. Monaghan, 2001 for a survey) different positions concerning the answers to the main tasks *("how many are the numbers"? "How many numbers are there between 1 and 2 ?")* were expressed initially; both the "sequential" and the "cardinal" conceptions emerged in each class in the motivations for each position reported in the written texts. Some language ambiguities were detected (cf Monaghan, 2001, pp. 240, 241) : in particular the sentence*: « I cannot count all the numbers »* might mean *"I have not enough time to count all the numbers, they are too many"* or *"I cannot reach the last number".* Another ambiguity depended on the use of *"infinito" (in Italian)* as a noun *("infinity")* and an adjective *("infinite").* And the adjective *"infinito"* was applied both to a number like 1.1111... and to the sequence of natural numbers: 1, 2, 3...

Consistently with the aim of this study, we will analyse: how positions (and conceptions related to them) evolved in the educational setting described in Subsection 3.1; how language ambiguities were dealt with; and what links with the cultural environment emerged.

The evolution of students' conceptions

21 students out of 59 clearly moved from one position to another (16 students moved from the "finite" position to the "infinite" position - see Debora below; 5 in the opposite direction – see Sabrina, in the last Subsection), by taking into account different positions - and the related arguments - brought into the debate by their schoolmates. For instance, Debora initially said that only ten numbers exist between 1 and 2 *("2.1; 2.2; ...2.9")*, then (exploiting a hint from a schoolmate) she considered also 2.01; 2.02; etc. up to 2.99 *("still a finite number"*), then she said that *"in any case, they are finite: let they be 9 or 99 or 999, it is the same! A finite number").* A critical point was reached when Ivan proposed the sequence 1.1; 1.11; 1.111; etc. (Debora): *"now I understand: I cannot get the last number, I can go on in an endless counting"*.

As concerns students' conceptions, in some cases we have observed that students moved from a "cardinal" view to a "sequential" view (see Debora), in other cases, they moved in the opposite direction. In general, classroom discussions allowed to develop more and more precise and sophisticated positions (through selection and integration of arguments brought by peers, and/or refinement of arguments, under the pressure of contrasting peers: see Debora again). An interesting aspect was the repeated crossing of lines of argumentation where the "sequential" view and the "cardinal" view became more and more sophisticated as a consequence of the need to consider the others' views (see Paolo's interventions in the following excerpt) :

> (Paolo): *"There is no last number, if I make 1.9; 1.99; 1.999 I get nearer and nearer to 2 but there is always another number in between, like 1.9999"*

(Beatrice) *"You say that they are so many, but all of them cannot stay there, they are too many!"*

(Paolo) *"but they are as many as… as if I count 1, 2, 3, 4, as many as 9s, but with whole numbers I go on with steps of 1, and I carry on without finding one last whole number, while here I go on with a smaller and smaller quantity*

Some students' positions were refined through an appropriate treatment of language ambiguities during discussions. The ambiguity inherent in *"the impossibility to count all numbers"* was clarified in the three classes through the teachers' hint of considering the numbers of the pocket calculator. Students realized that *"there is a last number ; they are finite, but we cannot count all of them because we have not enough time, while in the case of integer numbers there is no last number"*.

Classroom discussions were a source for social construction of knowledge and subsequent internalisation. Here is an example where some arguments produced by peers were integrated (in the subsequent individual text) as components of an inner dialogue, echoing the classroom debate, that brought to a substantial evolution of the student's position: (still Debora, second text) *"If I think that numbers between 1 and 2 are finite, be they 9, like 1.1 up to 1.9; or 99, like 1.01 up to 1.99, I cannot cope with the example brought by Ivan. Indeed the sequence 1.1; 1.11; 1.111; 1.1111 is endless, and those numbers do stay between 1 and 2. To answer the remark made by Sabrina, I can observe that if she would say that one point one million times 1 is the longest number for her mind, I could say: one point one million and one times1, and I am sure that she could think about it as well! So, the sequence is really endless"*.

Metaphors

Students' conceptions about infinity of numbers between 1 and 2 were related by them, during the discussions, to arguments belonging to different cultural domains. We can distinguish between three kinds of metaphors:

- metaphors where the source domain was mathematical - but in a different domain of mathematics, generally geometry: (Federica): *"The points on the line between 1 and 2 are finite, because they cover only a short line, and the same should happen for the numbers between 1 and 2"*

- metaphors related to ordinary life experience, like in the following excerpt:

(Valentina) *What does it mean to say that infinite numbers exist, if we cannot count them because we must die?"*

(Stefano) *"I agree, man is not everlasting but life is everlasting"*

(Valeria)*"The woman's body ends, but she creates another woman, and so life goes on to infinity"*

(Emanuele) *"Numbers create other numbers, to infinity, by multiplying. Each number is finite, but an infinite list is produced"*.

- metaphors directly or indirectly related to religious ideas (the eternity of God and/or soul, the unboundedness of Universe as the realm of God): (Emi) *"Numbers are infinite, because the time needed to count them would be infinite, like the life of our soul. Infinite numbers do exist, because it is like our soul: we cannot reach its end, because we die before, but this fact does not mean that soul gets to an end"*.

What is the role of these metaphors in classroom discussions? In few cases their use seems to depend on the mere need of communication in order to overcome the lack of technical terms. In some cases, they are produced (or used by someone who did not produce them) as arguments for the plausibility of a hypothesis (see the example of Federica). In other cases, they support the shift to meta-theoretical positions, like in the preceding case of Emi.

The problem of existence of infinity

Preceding considerations bring us to the most interesting (in our opinion) result coming from this study: the emergence in the three classes of an "existence problem" for infinity as a relevant issue for students to deal with when they must choose between accepting or rejecting the idea of an infinity of numbers. The existence problem emerged (in different modalities) as a result of the transition from the debate about *"how many numbers..."* to the question *"can infinite numbers exist?..."*. Here is an example about how this transition occurred through the intermediate problem of the existence of numbers that cannot be reached by counting:

> (Clelia) *"Numbers are infinite, because we can imagine numbers that grow bigger and bigger, with no limit",*
> (Stella) *"And also smaller and smaller, like 0,1; 0,01; 0,001"*
> (Enrico) *"We can be sure that a number exists only if we can reach it by counting*"
> (Amelia) *"I cannot count up to one million, but one million exists: we used it for liras"*
> (Ezio) *"I agree with you for one million and for one billion too, but for numbers that we do not know... That we do not use... Do they exist?"*
> (Sabrina) *"If we cannot touch or see something, we are not sure that it exists"*
> (Clelia) *"But we think that God exists! And God is eternal! Like the time necessary to count all numbers"*.

Students dealt with the existence problem under three different perspectives:

- existence considered as the possibility of "experiencing it": this test usually brought students who chose it to the exclusion of the possibility of infinity of numbers. In some cases this position took the flavour of a proto-philosophical assumption, in other cases it heavily relied on pragmatic considerations (related either to paper and pencil ordinary uses of numbers, or to the availability of calculators). The case of Sabrina is interesting: she moved from a mature expression of potential infinity existence *("if I write 1, 1,1, I get an infinity of numbers, because I can add how many 1s I want. There is not a last number, I could increase it by adding 1 to the right!"*, to a doubt preparing rejection, and related to accessibility to experience and pragmatism *("I cannot say if there exists a last number: it may be that we cannot think about a longer number, it could not enter our mind. And in any case a longer number would be a useless number, if we cannot think about it."* But Fadel considered the impossibility of experiencing infinity as a necessary condition for its existence: *"We cannot arrive to the last number, so numbers are beyond our possibility of knowing. Infinity cannot be known, because if we could get to know it, it would not be infinite."*

- existence thought of as an inner consequence of the structure of the number system: this was frequently related to the "sequential" point of view about infinity *('by adding 1 we always get a bigger number , but we cannot reach the last number'*, *'by writing 1 on the left we still get a number between 1 and 2, different from the preceding one: 1; 1.1; 1.11; and so on'*). But in one case the same consideration brought to the exclusion of the possibility of infinity (due the impossibility of reaching 2 on the line through the points 1.9; 1.99; 1.999 - two conflicting tacit models related to infinity seem to intervene: cf Fischbein, 2001).

- existence as the possibility of an independent, non-accessible reality frequently related to religious transcendence and/or space and time unboundedness: *"Numbers do exist, and always existed, and always will exist even if we do not think about them. They are endless. They are like God, who already existed before the creation of man, and man was not there to think about Him"*

Comparing the three classes, we have observed that in one of them the discussion about the "existence problem" did not develop very well because of frequent interventions of some students who refused to consider the infinity of space, or the eternity of God, as pertinent arguments in a discussion about the infinity of numbers.

DISCUSSION

Our study suggests that the complexity of the problem of children's conceptions of the infinity of numbers in school is perhaps even bigger than preceding studies had revealed. We must deal with this problem not only in individual, developmental terms (see Fischbein, 1979; Monagham; 2001) and in the social construction perspective (see Bartolini Bussi, 1989) ; we also need to consider the cultural environment and the didactical contract (specially as concerns arguments that are legitimate for students).

As concerns the "existence problem", we can ask ourselves what were the conditions that enabled students to pose it and deal with it.

The classroom discussion context seems to be a convenient environment for this because each position forces the supporters of the other positions to move to a meta-theoretical consideration, in order to defend their own position. This happened in the three classes and fits what was observed in some teaching experiments concerning other subjects and conceived in a Vygotskian perspective (see Bartolini Bussi&al, 1989: shift from empirical to theoretical, and then to meta-theoretical considerations about geometrical entities). The analysis of the data collected in our teaching experiment (see the previous Subsection) suggests that, in a classroom discussion context, the emergence of the "existence problem" and the possibility of a passionate debate about it might depend on:

- Preceding classroom experience of exploration of the number domain (specially as concerns the generation of decimal numbers like 0.33333...); and preceding discussions about the "numbers of the pocket calculator".

- Frequent questioning about what is beyond the immediate reach of our experience, nurtured in other classroom activities (production of interpretations, of predictions, etc.: see Douek, 1999 for some examples).

- The familiarity with an “existence problem” related to transcendent realities (like the soul, or God) due to parallel Catechism. This could explain the frequent shifts to the consideration of arguments taken from religion.

As concerns educational implications, the emergence of an “existence problem” for mathematical entities related to the question “How many numbers ...” suggests to consider the problematique of the infinity of numbers as an extremely interesting opportunity to develop the students’ sensibility and (later) their awareness about the nature of mathematical entities. Concerning this issue, we must recognise that the students’ potential of epistemological thinking is surprisingly high in grade V!

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INVESTIGATING THE MATHEMATICS INCORPORATED IN THE REAL WORLD AS A STARTING POINT FOR MATHEMATICS CLASSROOM ACTIVITIES

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In this report we present preliminary results of a study on the relationship between informal out-of-school and formal in-school mathematics and the ways each can inform the other in the development of new mathematical knowledge, in this case concerning computation in base 12, 24 or 60. This study is based on a teaching experiment involving a sequence of classroom activities in upper elementary school based on the use of cultural artifacts, interactive teaching methods and the introduction of new socio-math norms, in an attempt to create a teaching/learning environment focused on fostering a mindful approach toward realistic mathematical modeling. In this way we wish to create a new tension between school mathematics and everyday-life experience in which cultural artifacts, incorporating mathematics, can play a fundamental role in bringing students' out-of-school reasoning experiences into play.

INTRODUCTION

The critical problem of how to manage the relationship between informal out-of-school and formal in-school mathematics has been the subject of our studies for some years. Although mathematics learning and practice in and out of school differ significantly, (Resnick, 1987; Nunes, 1993) it seems that the conditions which often make extra-school learning more effective can, and must be re-created, at least partially, in classroom activities. While some differences between the two contexts may be inherent, many can be narrowed if classroom learning processes that are closer to those occurring in out-of-school mathematics practice can be created and promoted. This can be implemented in the classroom, for example, by encouraging children to analyze some '*mathematical facts*' (Bonotto, 2001a) embedded in appropriate '*cultural artifacts*' (Saxe, 1991).

The study presented in this report is part of an ongoing research project aimed at showing how the use of suitable cultural artifacts can play a fundamental role in bringing students' out-of-school reasoning experiences into play, by creating a new tension between school mathematics and everyday-life knowledge with its incorporated mathematics. In particular, the artifacts can be used as motivating stepping-stones to launch, at a first stage, new mathematical knowledge.

This quasi-experimental study involves a teaching experiment based on a sequence of classroom activities in upper elementary school aimed at developing new mathematical knowledge concerning computation in base 12, 24 or 60. In this case the artifact used is a weekly TV guide issued as a supplement of a well-known daily paper. These activities are also based on the use of interactive teaching methods and the introduction of new socio-mathematical norms, as outlined by Yackel & Cobb (1996), in an attempt to create a substantially modified teaching/learning environment. This environment is focused on fostering a mindful approach toward realistic mathematical modeling, i.e. both real-world based and quantitatively constrained sense-making (Reusser & Stebler, 1997). In our approach, informal out-of-school and formal in-school mathematics, despite their specific differences, are not seen as two disjunctive and independent entities. Instead, the aim is a process of gradual growth, in which formal mathematics comes to the fore as a natural extension of the student's experiential reality, as in Gravemeijer (1999).

THEORETICAL AND EMPIRICAL BACKGROUND

In common teaching practice the habit of connecting mathematics classroom activities with everyday-life experience is still substantially delegated to word problems. However, besides representing the interplay between in- and out-of-school contexts, word problems are often the only example providing students with a basic sense experience in mathematization, especially mathematical modeling. During the past decades, a growing body of empirical research (e.g. Freudenthal, Schoenfeld, Verschaffel, De Corte) has documented that the practice of word problem solving in school mathematics promotes in students the exclusion of realistic considerations and a "suspension" of sense-making, and rarely reaches the idea of mathematical modeling and mathematization (see Verschaffel, Greer, & De Corte, 2000, for a review of these studies). Furthermore, it has been noted that the use of stereotyped problems and the accompanying classroom climate relate to teachers' beliefs about the goals of mathematics education (Verschaffel, De Corte, & Borghart, 1997; Asman & Markovits, 2001).

If we wish to establish situations of realistic mathematical modeling in problem-solving activities, changes must be made. In particular, the type of activity used to create an interplay between mathematics classroom activities must be replaced with more realistic and less stereotyped problem situations. These should be more closely related to children's experiential world and meaningful. The extensive use of suitable cultural artifacts could be a useful instrument in creating a new link between school mathematics and everyday-life with its incorporated mathematics (Bonotto, 2001b).

The artifacts we introduced into classroom activities (e.g. supermarket bills, bottle and can labels, rulers, the cover of a ring binder, see Bonotto, 2001a; Bonotto & Basso, 2001; Bonotto 2003) are concrete materials that are meaningful to children as they are part of their real life experience, offering significant references to concrete situations. This enables children to keep their reasoning processes meaningful and to monitor their inferences. As a consequence, they can off-load their cognitive space and free cognitive resources to develop more knowledge (Arcavi, 1994). We can thus make use of children's familiarity with the chosen artifacts and allow them to express their intuitions and produce their own anticipations, in the sense of "*prospective learning*", Freudenthal (1991). These anticipations precede, and may be functional to, any systematic learning process. Furthermore, the double nature of the artifacts, that is belonging to the world of everyday life and to the world of symbols, to use Freudenthal's expression, makes it possible to move from the situations in which it is usually utilized to the underlying mathematical structure and vice versa, in agreement with '*horizontal mathematization*' (Treffers, 1987).

THE STUDY

In this study we decided to use the TV guide from a well-known weekly magazine in order i) to extend students' capacity to calculate from base 10 to b ase 12, 24 or 60, ii) to develop the conceptof equivalence between time intervals expressed in different ways (days, hours, minutes), iii) to introduce informally the concept of fractions.

The children in the classes involved did not know how to carry out computation with hours and minutes, however they all knew how to add and subtract in base 10, and remembered from the previous scholastic year that an hour is made up of 60 minutes.

To check students' familiarity with TV program guides, the experience was preceded by a phase in which children were asked to bring to class magazines and daily papers they usually use to choose TV programs. Magazines read and used only by parents, that led however to discussion within the family, were also accepted. It was found that the timetable of television programs, directly or indirectly, is part of the experiential reality of the children involved in the experience. All said that they knew the starting time and duration of their preferred programs, and that they were able to regulate TV viewing with their daily activities.

Participants

The study was carried out in two third-grade classes (children 8-9 years of age) in a suburb of the city of Padua by the official logic-mathematics teacher, in the presence of a research-teacher. The first class consisted of 20 pupils (10 girls and 10 boys), the second class of 21 (10 girls and 11 boys). In each class there were three children with learning difficulties, and two in the first class and one in the second who displayed demotivated behavior towards school activities.

As a control, two third-grade classes (children 8-9 years old) were chosen from another area of Padua, in keeping with the following criteria: i) congruence of socio-cultural background, ii) homogeneous level of performance with the two classes involved in the teaching experiment, (as confirmed by the outcome of the pretest), iii) use by teachers of a traditional teaching method.

Materials

After time to collect, read and comment on the various TV guides gathered by the children, it was decided that all children should work on the same TV guide in order to be able to manage and organize the classes better. This guide also has a section, on the two following pages, dedicated to a review of the films to be televised, where the starting time, duration, but not the finishing time, can be found. Among the details presented is the date of production from which it is possible to calculate the age of the film.

Procedure

The teaching experiment took place from February to April 2001, for a total of 12 hours a week of class activities, divided into 10 sessions. The first 5 sessions were dedicated to familiarization with the artifact, classification of the various programs according to typology (news, cartoons, films, etc) and to discovering the mathematical facts included, selecting from the many found. The remaining 5 sessions concerned 2 experiences involving two different opportunities offered by the artifact chosen.

In the first experience, using the table of television programs, the children were asked to organize their day, and then the week, keeping in mind their activities and commitments, and not exceeding an hour and a half of television a day.

The second experience, which took place in 2 two hour sessions, was aimed at reading and interpreting the numerical data in the artifact used - this time the reviews of the two films. The aim also included calculating the duration of the two films in minutes and converting them to hours, and finally establishing a strategy to find the finishing time of the film (see fig.1 for the requirements of the second experience). The children were then left free to discover other spontaneous scientific dilemmas, for example the age of the film.

Each session of these two experiences was divided into three phases. In the first, each pupil was given an assignment to carry out individually. The children were asked to answer all the questions in writing, individually. In the second phase, the results obtained through personal reflection and elaboration were discussed collectively, sometimes corrected, and then systematized and re-elaborated. The third was aimed at the elaboration of a collective written text comprising the clearer and more convincing explanations emerging from the whole-class discussion.

Questions asked: Make an evaluation of the information presented in the film review, in particular the time the film ends. Write down the procedure you used.	**Film** ***Courage*** *** Rete 4 / time: 16.00 Producer... With... Review… Comedy Italy 1956 Duration 95' O
Questions asked: Make an evaluation of the information presented in the film review, in particular the time the film ends. Write down the procedure you used.	**Film** *The Secret of the Old Forest* ** Channel 5 / time: 0.30 Producer... With... Review… Fairytale Italy 1993 Duration 134' O

As far as the control classes were concerned, the class teachers dedicated, within the same time period, exactly 12 hours to class activities regarding reading and calculation of time duration measured in hours and minutes, according to the modality and techniques normally used in elementary school.

DATA

The research method was both qualitative and quantitative. The qualitative data consisted of students' written work, audio recordings and fields notes of classroom observations and audio recordings of mini-interviews with students. The quantitative date was collected by means of pre- and post-tests, administered to the two experimental classes as well as the other two control classes. The two tests were constructed by the official class teachers, not the research-teacher, by taking some items normally used in the bimonthly tests utilized by the same teachers.

Both the pre- and post-tests were organized in such a way as to evaluate the effects of learning on time duration and fractions. The structure of items remained basically the same in the pre- and post-test, although post-test items included more difficult data or figures.

Research questions and hypotheses

In terms of learning processes, it was decided to continue gathering information on the way a particular artifact could be utilized as a motivating stepping-stone to launch new concepts or algorithms.

The first general hypothesis was that the children in the teaching experiment class, thanks to the opportunity they had to refer to a concrete reality (the cultural artifact), explore their strategies and compare them with those of their schoolmates, were able to grasp the calculation of hours and minutes more effectively, and the equivalence between time intervals expressed in different forms (days, hours, minutes) compared with the control class, who received a more traditional teaching method.

It was also hypothesized that using the clock face, which is divided into half and quarter hours, would allow participants to work out the concepts related to fractions according to "*prospective learning*" (hypothesis II).

Furthermore, we hypothesized (hypothesis III) that, contrary to the practice of word-problem solving documented in the literature, children in this teaching experiment would not ignore the relevant, plausible and familiar aspects of reality, nor would they exclude real-world knowledge from their observations and reasoning. Finally children would also exhibit flexibility in their reasoning, by exploring different strategies, often sensitive to the context and quantities involved, in a way that was meaningful and consistent with a sense-making disposition and closer to the procedures emerging from out-of-school mathematics practice.

SOME RESULTS

Some early results from the second experience are reported. From the first film review, all the children except one, were able to elaborate in their own words the information regarding starting time, channel, year of production etc. Of 41 children, 28 noted and commented on the judgment of the review (for example "*as it has 3 stars, it means it is good*"). Some noted that the film had a green symbol (children's viewing) and 16 children calculated the age of the film, even if they were not explicitly asked to do so, therefore activating a problem-posing procedure. 26 children worked out the equivalence "*95 minutes = 1 hour and 35 minutes*" and 29 mentioned the time the film finished. We note the case of a child, who we will call Emanuele, a repeating student with serious scholastic demotivation and learning difficulties. At the end of the first phase, the written report, he handed in a blank sheet. It was therefore decided to test his knowledge and thought processes by individual interview. We discovered that he knew how to read the data in the artifact and how to correctly work out the equivalence by referring to his preferred interest, football. In fact he knew that the duration of a football match is 90 minutes, and that it corresponds to an hour and a half because he always watches sports programs with his father, the most well-known of which is called "*novantesimo minuto*" ("*ninetieth minute*"). Therefore, 95 minutes for him was equivalent to an hour and half plus 5 minutes. The case of Emanuele therefore confirmed our third hypothesis.

Regarding the task for the second review, we found that 15 children calculated the age of the film, 20 correctly interpreted the time 0.30 as "*half past midnight*", 27 children worked out the equivalence between minutes and hours correctly, and 21 worked out the time the film would have finished. During the whole-class discussion the entire class participated with great interest in the problem raised by a child that 0.30 and 24.30 may not be the same times.

Some significant extracts are presented from written work regarding the finishing time of the film. These show the activation of strategies sensitive to the context and quantities involved and also the emergence of problem posing activities.

Claudia: *"I pretended that the film started at exactly 0. I put the 30 minutes to one side. I added 2 hours and that makes 2. I added the 30 minutes and so I got 2 and 30. I added the 14 minutes and so arrived at 2 hours and 44 minutes."*

Claudia tried to simplify the data as much as possible to be able to calculate with greater surety. The explanation was extremely clear, expressed in the language and terminology normally used by children, and for these reasons during the class discussion it led to curiosity, attention, understanding and participation by classmates who were unable to find the finishing time.

Gregorio's protocol included the following:

"1) I found 2 hours and 14 minutes in this way: 60+60=120+14=134. 2) To arrive at 2.44 it was 0.30+2 ore=2.30+14=2.44. 3) To get 8 years we worked out 1993 to arrive at 2001 makes 8 years. We can see 51 minutes of the film "The executors".

It can be seen that in the end that Gregorio faced a spontaneous dilemma with a film whose review was next to the one assigned and whose viewing time partially overlapped. He posed the question *"Once the film "The Old Forest" is finished, how much of the film "The Executors" can I watch?"*. This shows how the use of an artifact may evoke situations that are in fact experienced, activating the ability to pose and resolve problems.

On the basis of the qualitative results we can say that this experience has reinforced knowledge of the hours in the day and led to calculation in base 60 by means of an informal, non conventional, procedure on the basis of intuition linked to the context or the quantities involved. Among the children's protocols, attempts at formalizing calculation in rows and columns also appeared.

As far as the outcomes of the pre- and post-tests are concerned, the errors in the experimental group diminished by 46%, while those of the control group remained more or less stationary. The two parts of each test are outlined, that is the first part testing reading ability, calculation of hours and the equivalence between time intervals expressed in different forms, and the second part regarding knowledge of the concept of fractions. It emerged that the greater improvement in the experimental group's performance is relative to the abilities tested in the second part of the test, where the concept of fractions was evaluated. There was in fact a 63% reduction in errors in the case of the experimental group, while errors increased for the control group. This is also documented by the statistical analysis that we are elaborating.

CONCLUSION AND OPEN PROBLEMS

From the results it appears that the teaching experiment had a significant positive effect on achieving learning goals, in particular enhancing and understanding the calculation of hours and minutes and the equivalences between time intervals expressed in different forms, and enhancing a first approach to the concept of fractions in a way that is meaningful and consistent with a sense-making disposition. This was not the case in the control group where an increase in errors was found in the second part of the test. It could be supposed that the control group, who received a more traditional type of teaching, may have acquired general algorithmic procedures and formal rules, but these were not well mastered and therefore did not improve performance.[1] The first two research hypotheses were therefore confirmed.

It was also confirmed by the qualitative results that using the TV guide did not activate rigid and general algorithmic procedures but rather specific heuristics, that have an inner consistency and value. The strategies were flexible, local and sensitive to number sizes (hypothesis III), and were such that children often made reference to parts of the hour (half and quarter hours) to be able to manage calculations better. This aspect made them more sensitive to the concept of fractions according to *prospective learning* and therefore led to the distinct improvement (63%) by the experimental classes in the second part of the post-test.

Furthermore by asking children i) to select other cultural artifacts from their everyday life, ii) to identify the embedded mathematical facts, iii) to look for analogies and differences, iv) to generate problems (e.g. discover relationships between quantities), the children were encouraged to recognize a great variety of situations as mathematical situations, or more precisely "mathematizable" situations.[2] In this way, children are offered numerous opportunities to become acquainted with mathematics and to change their attitude towards mathematics.

For the real possibility to implement this kind of classroom activity, there needs to be a radical change on the part of teachers as well. They must try to i) modify their attitude to mathematics; ii) revise their beliefs about the role of everyday knowledge in mathematical problem solving; iii) see mathematics incorporated in the real world as a starting point for mathematical activities in the classroom, thus revising their current classroom practice. Only in this way can a different classroom culture be attained. On the basis of the experience of this and our other studies, we entirely agree with Freudenthal (1991), that the main problem regarding rich contexts is implementation requiring a fundamental change in teaching attitudes.[3] As in other studies (Verschaffel et al., 1999), the effective establishment of a learning environment like the one described here makes very high demands on the teacher, and therefore requires revision and change in teacher training, both initially and through in-service programs.

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[1] Many studies have pointed out that local strategies developed in practice are more effective than arithmetic algorithms which are usually taught in school to give students powerful general procedures, but which are, in fact, often useless in out-of-school contexts (Schliemann, 1995).

[2] We think that the development of a problem-posing program integrating the promising outcomes of recent studies (English, 1998; Silver et al., 1996) should also permeate the entire curriculum.

[3] Cultural artifacts used, and the way they have been used, can be considered as " *contexts*" or " *rich materials*" in the meaning given by Freudenthal, who underlines their qualities through a comparison with structured material.

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FOURTH GRADERS SOLVING EQUATIONS[1]

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We explore how fourth grade (9 to 10 year olds) students can come to understand and use the syntactic rules of algebra on the basis of their understanding about how quantities are interrelated. Our classroom data comes from a longitudinal study with students who participated in weekly Early Algebra activities from grades 2 through 4. We describe the results of our work with the students during the second semester of their fourth grade academic year, during which equations became the focus of our instruction.

INTRODUCTION

Successful implementation of algebraic activities in elementary school are described by Bodanskii (1991), Brizuela, Carraher, and Schliemann (2000), Brizuela and Lara-Roth (2001), Carpenter and Franke (2001), Carpenter and Levi (2000), Carraher, Brizuela, and Earnest (2001), Carraher, Schliemann, and Brizuela (2000, 2001, 2003), Davis (1985), Kaput and Blanton (2001), Schifter (1999), Schliemann, Carraher, and Brizuela (2001), Schliemann and Carraher (2002), and Schliemann, Goodrow, and Lara-Roth (2001). Our own work has shown that third graders can learn to think of arithmetical operations as functions rather than merely as computations on particular numbers, that they can operate on unknowns, and work with mapping notation, such as $n \rightarrow 2n - 1$. We have also found that graphs of linear functions are within reach of fourth graders.

These demonstrations, however, may not have convinced some mathematics educators that young children can learn algebra. Previous research has highlighted students' *difficulty* in solving equations when unknown quantities appear on both sides of the equality (e.g., Filloy & Rojano, 1989; Herscovics & Linchevski, 1994). Many attributed such findings to developmental constraints and the inherent abstractness of algebra, concluding that even adolescents were not ready to learn algebra (Collis, 1975; Filloy & Rojano, 1989; Herscovics & Linchevski, 1994; Linchevski, 2001; MacGregor, 2001; Sfard & Linchevski, 1994). Further, some have claimed that students are engaging in algebra only if they can understand and use the syntax of algebra and solve equations with variables on both sides of the equals sign (see Filloy & Rojano, 1989).

It is our belief that, as previously stressed by Booth (1988), Bodanskii (1992), Kaput (1995), and Schliemann and Carraher (2002), among others, the difficulties middle and high school students have with algebra result from their previous experiences with a mathematics curriculum that focuses exclusively on arithmetic procedures and computation rules. With the classroom data we will describe, we will show that, if children are given the opportunity to discuss algebraic relations and to develop algebra notations, even fourth graders will be able to solve algebra equations.

OUR APPROACH TO ALGEBRAIC NOTATION

[1] This paper is part of a larger longitudinal study sponsored by the National Science Foundation (Grant #9909591, awarded to D. Carraher and A.D. Schliemann).

Algebraic-symbolic notation is one of several basic representational systems of mathematics. In a *narrow* sense, algebraic reasoning concerns only algebraic-symbolic notation. In the *broad* sense we adopt in our research and in this paper, algebraic reasoning is *associated with* and *embedded in* many different representational systems. Although some educators argue against any and all uses of algebraic-symbolic notation in the early grades, we feel it is better to frame the issue in a *broad* context. By *broad context* we mean to ask more generally how written notations relate to mathematical reasoning and algebraic concepts in particular.

In a previous study, we found that children can use mathematical notations not only to register what they understand, but also to structure their thinking; that is, notations can help further children's thinking (Brizuela, Carraher, & Schliemann, 2000), allowing them to make inferences they might otherwise not have made. Conventional notations help extend thinking (Cobb, 2000; Lerner & Sadovsky, 1994; Vygotsky, 1978), but if they are introduced without understanding, students may display premature formalization (Piaget, 1964). For these reasons, students need to be introduced to mathematical representations in ways that make sense to them. Much of our research has focused on introducing mathematical symbols in meaningful ways. Our approach relies on introducing new notations as variations on students' spontaneous representations (Brizuela & Lara-Roth, 2001; Carraher, Schliemann, & Brizuela, 2000). Our classroom intervention data have shown that young students can meaningfully learn to use *algebraic-symbolic notation* to express generalizations they have reached while exploring problems in open-ended rich contexts. Our next step was to investigate whether elementary school children could also deal with written algebra equations and with the syntactic rules of algebra.

In interview studies, Brito Lima and da Rocha Falcão (1997), Schliemann, Brito-Lima, and Santiago (1992), and Schliemann, Carraher, Pendexter, and Brizuela (1998) have shown that seven year-olds can understand the basic logic of equations, and that third graders can develop representations for algebraic problems and, with help from the interviewer, solve linear equation problems using different solution strategies, including the syntactic rules of algebra. Furthermore, Bodanskii (1992) found that fourth graders introduced to algebra notation and equations from grade 1, could solve algebra problems and equations, performing better than sixth and seventh graders who received five years of arithmetic instruction starting algebra in grade six only. Other promising results come from Lins Lessa (1995) who found that, after only one individual teaching session, fifth graders could solve verbal problems or situations presented on a balance scale that involved equations as complex as $x + y + 70 = 2x + y + 20$ or $2x + 2y + 50 = 4x + 2y + 10$. She also shows that, in the post-test, the children's solutions were based on the development of written equations and in more than 60% of the cases they used algebra syntactic rules for solving equations.

In the longitudinal study we partially report here, we introduced children to equations as an extension of their work on functions and on graphs of linear functions. In this paper we will report on the final results from one of the classrooms we worked with.

The Classroom Intervention and its Results

We worked with 70 students in four classrooms, from grade 2 to 4. Students were from a multiethnic community (75% Latino) in Greater Boston. Each semester, from the beginning of their second semester in second grade to the end of their fourth grade, we implemented and documented six to eight Early Algebra activities in their classrooms, each one lasting about 90 minutes. The activities related to arithmetic operations, fractions, ratio, proportion, and negative numbers. Our goal was to examine how, as they participated in the activities, the students would work with variables, functions, positive and negative numbers, algebraic notation, function tables, graphs, and equations.

The last six lessons we taught in fourth grade focused on equations and algebraic notation. Each lesson focused on a problem that had unknown amounts in it and that could be represented with equations, as in the following example:

> Mike and Robin each have some money. Mike has \$8 in his hand and the rest of his money is in his wallet. Robin has altogether exactly three times as much money as Mike has in his wallet.
>
> Which phone plan is better? Plan #1: You pay \$0.10 per minute for all calls. Plan #2: You pay \$0.60 per month plus \$0.05 per minute for calls.

When presented with the problems, children were not asked to find a "right" answer, but to consider all possibilities, to draw the graphs of two functions, and to consider an answer only after they had gone through these steps. During the weeks leading up to the lesson we will focus on in this paper, the children felt fairly comfortable dealing with unknown amounts and some of the children were able to gradually use N to represent the unknown amounts, although some of them still used iconic representations. During the last lesson in fourth grade, the following problem was presented to the class:

> Two students have the same amount of candies. Briana has one box, two tubes, and 7 loose candies. Susan has one box, one tube, and 20 loose candies. If each box has the same amount and each tube has the same amount, can you figure out how much each tube holds? Each box?

A box, two tubes, and 7 candies in a transparent bag are put on Briana's table; a bag, a tube, and 20 candies in a transparent bag are put on Susan's table.

The students start by discussing the problem and Aarielle recalls that it is similar to the "wallet problem" (see above) they had solved six weeks before. Kauthaumy states that Susan has 13 more candies in her bag than Briana does, and Albert observes that Briana has an extra tube of candy. When the teacher of this lesson (David Carraher) asks if they could figure out how many candies there are in a tube or in a box, most of the students answer that they couldn't. However, less than 14 minutes into the class, Albert explains that Briana's tubes have to have 13 candies in them so that the tube plus the 7 loose candies could be equal to Susan's 20 candies. Briana agrees with Albert and Cristian notes that it doesn't matter how many candies are in the boxes.

Mariah asks Albert to explain why he thinks there are 13 candies in each tube. He answers that the amount in a tube plus the 7 loose candies would be equal to Susan's 20 candies. Mariah asks about Briana's second tube and Albert assures her that it still works because Susan also has one tube. Carissa further explains that the candies in Susan's bag make up for the extra tube that Briana has. David (the instructor) asks how many candies in the bag make up for the tube and Albert replies 13, which would leave 7. A few minutes later, David asks, "How do we know that the tubes have 13 and that the girls are holding the same amount if we haven't peeked in the tubes yet?" Cristian replies that this is called algebra and Briana and Mariah explain that they used algebra to subtract and make educated guesses. David challenges the children to prove that there are indeed 13 candies in a tube. Cristian explains that we can use N to stand for a tube, and the class as a whole agrees that a different letter should be used to stand for the boxes. When the students sit down to work on ways of representing the problem in writing, Anne, a member of the research team, asks Carissa and Susan to explain the problem for her. Carissa explains that Briana and Susan have the same amount and thus Susan's bag, that had 20 candies, is really like 13 plus 7. So Susan has 13 extra candies, so that has to be the amount in Briana's extra tube.

Each of the students in the class produce their written account of the problem. Although most of the children in this class of 18 students made iconic representations for the problem (78%), one third of them included an equation in their representation and more than one third (39%) included a letter in their representation, to stand for one or more of the unknown values.

Figure 1. Nancy

Nancy's written work (see Figure 1) is an example of an iconic representation. She first works with the amounts given for the loose candies (20 and 7) and correctly uses the difference of 13 between these two amounts as the value for what is inside the tubes, showing one tube on Susan's table and both of Briana's tubes as having 13 in them. Although Nancy acknowledges that Susan starts out with 20 loose candies, on the table she shows her as having 7 and 13—just like Briana. One interesting feature of Nancy's work is the question mark that she places on the two boxes—the amount of candies in the boxes is unknown (hence the question marks) and will and can remain unknown to the very end. In our longitudinal study, this was not the first time that we had observed children using question marks to represent unknowns.Ramón's written work (see Figure 2) is also interesting in the way in which he is able to integrate both an iconic representation and algebraic notation (N+7). He consistently uses the N to show what is in the tubes of candies. In addition, he uses his notation to solve the problem and show his solution to the problem. He represents Susan's (1) and Briana's (2) candies iconically, then matches what they have and crosses out the matching amounts on both sides. He does not assign a value for the boxes and appears to have no problem crossing them out since there is one on each side. Through his matching, he arrives at the conclusion that, in order to have equal amounts of candy, Susan must be left with 7 and 13 (20) and Briana must be left with N+7, making N be 13 candies.

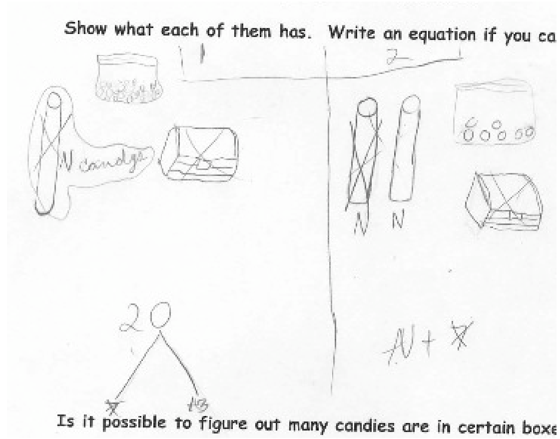

Figure 2. Ramón

Figure 3 shows that Albert uses an equation to represent the problem. He uses both N and Z as the unknowns. He starts by using N to represent the amounts in the tubes and in the boxes but soon uses Z for the amount of candy in the tubes. After matching the equal amounts, he appears to have used the letters interchangeably as he finally reaches the equation 20=N+7.

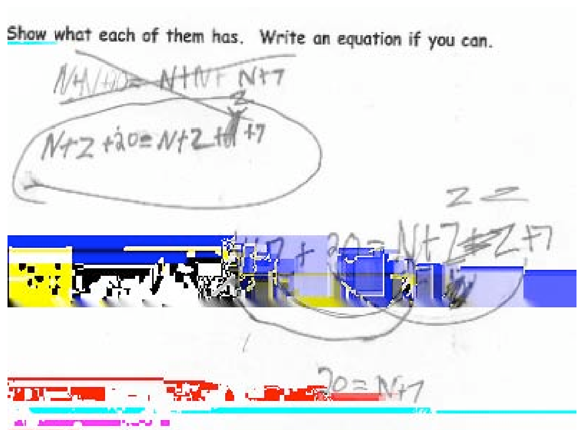

Figure 3. Albert

Figure 4 shows a very sophisticated representation by Cristian. Although similar to Albert's, Cristian's notations are of added relevance given the explanation written out at the side of the equations that he matches up. Cristian has set Susan's and Briana's amounts equal and matched the elements in the two sides of the equation.

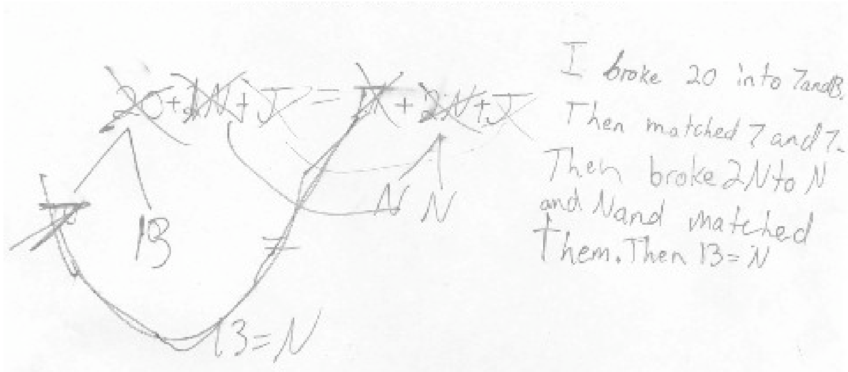

Figure 4. Cristian

Once the group as a whole meets to discuss the problem and the students' written representations, David writes on the board that T is the amount in each tube and B is the amount in each box. He asks the class, "How much did Briana have?" The class calls out "2T + B + 7". He then asks, "How much did Susan have?" and the class calls out, "T + B + 20". When David asks whether these expressions can be simplified, Albert suggests matching up the Bs. David does so and crosses them out: "Now we have 2T + 7 and on the other side we have T + 20. How could we simplify them further?" Carissa suggests putting 7 in Susan's bag and leaving 13 out in a pretend tube. David does so and Aarielle writes on the board, breaking up the 20 into 13 and 7 and matching up the two sevens. David erases the 7 from the board to leave 2T = T + 13. Cristian suggests matching two tubes to leave T = 13. David does this with the actual tubes and records it on the board: "So T has to be 13. Let's count the tube candies." Subsequently, Kauthaumy counts the candies and finds 13. The children shout out, "Hooray", expressing their excitement at their accurate calculations.

INTERVIEW RESULTS

At the end of the school year, 1 to 4 weeks after the last class, we individually interviewed the children on a series of problems. In the last part of the interview, children were asked to represent in writing and to solve the following problem: "Harold has some money. Sally has four times as much money as Harold. Harold earns $18.00 more dollars. Now he has the same amount as Sally. Can you figure out how much money Harold has altogether? What about Sally?" Of the 18 children from this class who were interviewed, 10 represented Harold's initial amount as N, X, or H and Sally's amount as Nx4. For Harold's amount after earning 18 more dollars, eight children wrote N + 18. Four children wrote the full equation N + 18 = N x 4 and eight children correctly solved the problem. However, only one systematically used the algebra method to simplify the equation. Another child, when prompted, correctly explained the algebra method. Apparently, as the children worked in their written representations, they easily inferred that Harold's starting amount was 6. As Albert stated, "I thought about six because it just popped in my head."

DISCUSSION

The kinds of activities we developed over the last six weeks of our longitudinal study were not simple or easy for the students. Nevertheless, they were able to deal with the challenges we proposed and, at the end of only six meetings on equations, many were able to represent and meaningfully discuss and analyze problems involving unknown amounts on both sides of an equality. In the classroom, at least a third of the students in this class could represent the problem as an equation, solve the equation, and meaningfully explain why they could manipulate the elements in the equation. In the interviews, more than half of the children correctly represented the amounts in the problem using letters to stand for unknown amounts. Our results suggest that dealing with equations is not beyond fourth graders' mathematical understanding and that much more can be achieved if the same kind of activities become part of the daily mathematics classes offered to elementary school children.

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AN EXAMINATION OF HOW PEOPLE WITH DIVERSE BACKGROUND TALK ABOUT MATHEMATICS TEACHING AND LEARNING BOTH FACE-TO-FACE AND ON-LINE

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The purpose of this study is to examine how the diverse members of the mathematics education community (in-service secondary mathematics teachers, in-service secondary mathematics teachers, university mathematicians and university mathematics educators) talk about a multimedia case study on mathematics teaching and learning when they interact both face-to-face and on-line. Also, the researchers compare on-line and face-to-face discussions. Initial findings indicate that both on-line and face-to-face discussions of the case study promoted questioning, answering and informing among the participants.

INTRODUCTION

The purpose of this study is to examine how the diverse members of the mathematics education community (in-service secondary mathematics teachers, in-service secondary mathematics teachers, university mathematicians and university mathematics educators) talk about a multimedia case study, Making Weighty Decisions (Bowers, Doerr, Masingila & McClain, 2000), when they interact both face-to-face and on-line. In addition, we will examine how the computer-mediated discussion differs from the face-to-face discussion by using computer mediated discourse analysis (CMDA), an area of study in the field of computer-mediated communication that employs the tools of discourse analysis to understand issues related to language and language use in computer-mediated settings. CMDA is applied to face-to-face communication as well as on-line communication (Herring, 2001).

The present study is an extension of a previous one (Koc, Herring & Brown, 2002) that examined how pre-service secondary mathematics teachers, in-service secondary mathematics teachers, mathematicians, and mathematics teacher educators communicate about the same multimedia case study through on-line communication. This research focuses on face-to-face communication as well as on-line communication, and their comparison.

LITERATURE REVIEW

Teachers frequently experience a sense of isolation in their professional life at schools (DuFour, 1999; Rogers & Babinski, 1999; NCTM, 2000). This routine of school life prevents teachers from engaging in collegial relationships with other teachers; for example, sharing, discussing, reflecting and planning with colleagues (Cochran-Smith & Lytle, 1993; Hyde, Ormiston & Hyde, 1994; Little, 1987). Efforts are being made to transform this common culture of teacher professional from one of isolation and individualism to one of professional community. The new social reality brings its own norms, including collegiality and trust among teachers (Lieberman & Miller, 1999). Collegiality breaks the isolation of the classroom; in particular, teachers begin to participate "in a professional community that discusses new teacher materials and strategies and that supports the risk taking and struggle entailed in transforming practice" (McLaughlin & Talbert, 1993, p.15). In addition, building effective relationships among teachers may entail a common or shared professional language of teaching (Brandes & Erickson, 1998). Talking about teaching and learning in a discourse community is an invaluable opportunity for teacher educators as well as in-service and pre-service teachers to reflect on personal and colleagues' teaching experiences.These current reform ideas are not only applicable in face-to-face interactions, but also in computer-mediated contexts (Levin & Waugh, 1997).

Teacher conversation is proposed as a medium of learning to teach. In particular, talking about teaching and learning provides opportunities for teachers to construct knowledge about their practice (Richert 1992). Conversing with other teachers creates a venue for teachers "to examine their beliefs and experiences (p.190)," so it promotes being reflective about teaching and learning. In this study, we examine what teachers, teacher educators and mathematicians are doing when they talk about mathematics teaching and learning.

The literature indicates that educators possess a large amount of interest in online environments for teaching. As yet, however, little systematic research is available that compares online communication with face-to-face communication in educational settings. Since Internet use has become widespread in teaching and learning contexts, it is essential to examine how computer mediated communication (CMC) influences human learning with respect to face-to-face communication. Key questions that need to be examined include, "Is CMC an effective tool in teaching and learning?", "Does CMC affect equality of participation among individuals?" and "Does CMC, in as much as it provides more time for reflection, enhance the sophistication of language used in educational settings?". This paper illustrates some applications of Computer-Mediated Communication (CMC) in a university-school collaboration project, the Collaboration for Enhancement of Mathematics Instruction (CEMI) project.

In the proposed paper, we will report how the computer-mediated discussion differed from the face-to-face discussion by using computer mediated discourse analysis (CMDA). Computer-mediated discourse analysis (CMDA) is a discourse analysis method to identify patterns of structure and meaning in language use in computer-mediated communication (CMC) (Herring, 1996). CMDA utilizes various data analysis tools, including rate of participation and speech acts. Rate of participation includes counting the number of messages, sentences, and words said/posted by the individuals. Speech act is an utterance conceived as an act by which the speaker does something. Some of the frequently used speech acts are informative, inquire, neutral proposal, conclusion, confirm, qualify, directive, comment, and prompt (Francis & Hunston, 1992).

RESEARCH QUESTIONS

Specifically, this paper reports answers to the following questions:

- What are diverse members of the mathematics education community (in-service secondary mathematics teachers, in-service secondary mathematics teachers, university mathematicians and university mathematics educators) doing as they talk about mathematics teaching and learning?
- How does computer-mediated communication (CMC) differ from face-to-face communication with respect to the first research question?

METHODS

This study examines the discussions about the multimedia case study *Making Weighty Decisions*. Six discussion groups were formed with each group consisting of a high school mathematics teacher, a university mathematician or instructor (a graduate level mathematics student), a mathematics educator, and several pre-service teachers (two to four per discussion group). Each discussion group member was then asked to view the multimedia case individually and then engage in face-to-face and online discussions. Online discussion prompts were provided initially to encourage discussion group members to reflect on the teacher's role in planning for and facilitating classroom activities, the mathematical content of the lesson, and the level of student thinking throughout the lesson. Members were also encouraged to raise their own issues. Online discussion proceeded for approximately five weeks. Discussion groups met face-to-face to discuss the case study twice during those five weeks; all face-to-face discussions were audiotaped.

The goal of the discussions was to encourage discussion among the participants that was focused on mathematics teaching and learning. Since they had diverse backgrounds in teaching and learning mathematics, development of a way of talking about mathematics education among them was necessary. It was hypothesized that talking about teaching and learning would be an effective means of building a common language among the participants with diverse backgrounds-- mathematics teachers, pre-service mathematics teachers, university mathematicians and mathematics educators. Also, it was thought that on-line discussion opportunities could produce more discussions and have more participants be engaged in the discussions. In addition, on-line communication would attract more participation from the group members who do not participate well in face-to-face communication. On-line discussions took place within the Inquiry Learning Forum (ILF) (http://ilf.crlt.indiana.edu). The ILF, hosted at Indiana University, "is a web-based professional development tool designed to support a community of in-service and pre-service mathematics and science teachers creating, sharing, and improving inquiry-based pedagogical practices" (Barab, Makinster, Moore, Cunningham & the ILF Design Team, 2001, p. 3).

DATA SOURCES

Each of six discussion groups engaged in two online discussions. The first discussion lasted two weeks (from 09/11/2000 to 09/25/2000) and then members met face-to-face to discuss the CD for approximately one hour. Next, another three-week online discussion took place (from 09/25/2000 to 10/16/2000). Finally, a second face-to-face meeting provided an opportunity for the participants to share their ideas. The transcripts of on-line and face-to-face discussions of the multimedia case study by six discussion groups are our data sources. Totally, we analyzed postings and transcripts of 38 people.

DATA ANALYSIS

We used the exchange structure of Francis and Hunston (1992), originally developed by Sinclair and Coulthard (1975) for the analysis of classroom discourse, to analyze the transcripts of the discussions. Exchange structures were sequences of speech acts (agree, inquire, inform, react, etc.) produced when individuals are engaging in conversation. The model was developed for face-to-face conversation, but has been applied to educational CMC by Herring and Nix (1997). The goal of the present analysis was to understand what kind of speech acts takes place in face-to-face and on-line communications in discussing the multimedia CD. Additionally, we compared the speech act usage of groups within the groups.

RESULTS

What follows are some examples of the initial results. Although we have many interesting results, we cannot report all of them due to space limitations.

The average number of messages per individual for discussion1 and discussion2 is 3.81 and 2.50, respectively. Interestingly, in discussion1, the average number of messages per mathematician (5.00 messages) is significantly higher than the average number for the entire population in discussion1 (3.81 messages). Because the mathematics educators posted only two messages during the first discussion (average= .67 messages), the average number of messages in that discussion session was decreased.

Overall, the participants were engaged in 678 speech acts throughout the online discussions. The speech act analyses indicate that generally the participants are informing each other, sharing their observations, inquiring and commenting on their own statements as they discuss the multimedia case study. These results are consistent with the findings of Herring and Nix (1997) for a distance education course. However, unlike in Herring and Nix's study, the participants use very few directive speech acts, suggesting a relatively polite and egalitarian environment.

It was also revealed that that males share their observations (45 times) more than females (31 times), while females asked more open-ended questions. Pre-service teachers shared their observations mostly (47 out 76 times). This may be because as students they regularly prepare assignments including reflections, descriptions and observations.

Basically, the on-line and face-to-face discussions of the case study promoted questioning, answering and informing among the participants. The initial findings include only qualitative representations of the data. Quantitative findings will accompany qualitative reports. Also, we compare and contrast on-line and face-to-face discussions, so it is helpful to understand the benefits of on-line professional development tools.

The big benefits of the present study are twofold: 1) Teacher educators will understand how people with diverse backgrounds in mathematics education talk about mathematics teaching and learning, and 2) Effects of an on-line professional development tool will be observed.

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MATHEMATICAL IDENTITY IN INITIAL TEACHER TRAINING

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This paper addresses issues of identity among trainee teachers as they progress through college in to their first year of teaching mathematics in primary schools. We examine how we might conceive of the trainees confronting mathematics in the context of government policy instruments. We suggest that teacher identity be produced at the intersection of the trainee's personal aspirations of what it is to be a teacher and the external demands they encounter en route to formal accreditation. We also suggest that participation in the institutions of teaching results in the production of discourses that serve to conceal difficulties encountered in reconciling these demands with each other.

RECONCILING PERSONAL ASPIRATIONS WITH EXTERNAL DEMANDS

Teaching is about empowering young learners and as such can be seen as a very worthy profession, around which a new teacher can harness more personal aspirations, such as feeling that one has social worth and a clear identifiable professional purpose. However, in adopting a broader perspective on how social improvement might be achieved, the role of individual teachers often takes second place to the wider social agenda. Individual teachers become participants in a collective programme where their personal aspirations need to be filtered through a set of socially defined demands. Such demands get to be meshed with the requirements for accreditation as a teacher and the regulations governing everyday practice as a teacher in schools. Trainee teachers, in a UK study to be discussed here, were, for example not keen on having their individual practices as teachers and mathematicians gauged against the externally defined definitions of what it is to be a teacher, as for example, in government sponsored inspections carried out by the Office for Standards in Education (OfSTED).

> "It feels as if they're checking up on you all the time, … they're not leaving it to your own professionalism …but the university have to cover their own backs don't they, with OfSTED (inspectors) coming."

The study coincided with the introduction of the National Numeracy Strategy, a high profile government initiative defining the content and conduct of mathematics lessons in great detail. Whilst most students regarded the Strategy and its daily "numeracy hour" as "very useful", it resulted in nearly all schools and individual teachers in the sample abandoning their own more personalised schemes of work. And it was not uncommon for some teachers to find the Strategy a little over-prescriptive:

> "The numeracy hour, it's so prescriptive as to what you have to do, when you have to do it and how long you do it for, so it shapes the whole numeracy hour of every day of every week of the school year."

But, for many trainees interviewed their personal aspirations were disrupted more by an unwelcome component of the overall job description of a primary school teacher, namely, the actual need to teach mathematics in the first place. Most students in our sample (60% plus) had experienced significant emotional turmoil in their own experience of mathematics whilst pupils at school:

> "It was just a case of doing the sums but you didn't realise why you were doing the sums. I think the teacher's role played a big part in it as well because the atmosphere she created, it wasn't a very, it was just a case of if you can't do it, you should be able to do it now. It wasn't very helpful or you didn't feel like, she wasn't very approachable, you didn't feel like you could go to her and say I'm having trouble with this and I need some help, it was just a case of don't even bother going to a teacher, just very much a case of you have to meet the standard and if you don't then you're a failure. So I didn't really enjoy maths at all."

Attitudes such as those expressed here were very common in the study and worked against a clear passage to feeling comfortable about producing a conceptualisation of teaching through which their personal aspirations could be achieved.

But in analysing such data there seemed to be a need to adopt a certain amount of caution (cf. Convery, 1999). What is concealed in such a story? Surely this interviewee did not have just one teacher, introduced here as "she". The trainee appears to be personifying his entire experience of many teachers in just one teacher who is required to carry the weight of this individual's perceived suffering at school. We may wonder as to which narrative devices individuals employ when they are requested to recount experiences that happened some ten to twenty years earlier. For what reasons do they construct such images of themselves and what present demands are concealed in these images? How do teachers tell the story of their lives to rationalise their current motivations? Freud might suggest that a repetition of such a story may be a form of resistance, an insertion of a fixed image, that blocks off the possibility of building memories in a more creative way (cf. Ricoeur, 1981, p. 249). The reworking of memory into a story is not the memory "as it was" but rather a probing that creates something new; a present day building of the past, shaped by current motives, but perhaps also distorted by things the student would rather not confront. Hence we examine what we might learn from such teacher accounts with particular reference to their current practice.

THE EMPIRICAL STUDY

This paper draws on two studies funded by the UK Economic and Social Research Council. The first study focused on the four years of B.Ed. training (Brown, McNamara, Hanley and Jones, 1999). The second study focused on the transition from the fourth year of training to the first year of teaching. The cumulative report has recently been completed (Brown and McNamara, under review). The particular aims of the second study upon which we focus in this chapter are:

1) To examine how the students'/teachers' conceptions of school mathematics and its teaching are derived.
2) To examine the impact government policy initiatives relating to mathematics and ITT, as manifest in college and school practices, have on the construction of the identities of the primary student and first year teachers.

The studies were situated in the B.Ed. (Primary) programme at the Manchester Metropolitan University in the UK. The empirical material produced provided a cumulative account of student transition from the first year of training to the end of the first year of teaching. The first study spanned one academic year and interviewed seven/eight students from each year of a four-year initial training course from a total cohort of some 200 students. Each student was interviewed three times at strategic points during the academic year, at the beginning of the year, whilst on school experience, and at the end of the year. The study took the form of a collaborative inquiry between researcher and student/teacher generating narrative accounts within the evolving students'/teachers' understanding of mathematics and pedagogy in the context of their past, present and future lives. The second study, which followed a similar format, spanned two academic years. In the first year of the study a sample (n=37) of 4th year students was identified. Each student was interviewed three times during this year. The sample included seven students involved in the earlier project, five of whom were tracked for a total of four years. In the second year of the study a small number of these students (n=11) were tracked into their first teaching appointment. Each of these students was interviewed on a further two occasions. These interviews monitored how aspects of their induction to the profession through initial training manifested itself in their practice as new teachers. A particular focus was on how aspects of the college training continue to influence the new teacher's practice in school, with an emphasis on mathematics teaching practice.

Specifically, the body of students that the research focused on were those who were training to be primary teachers and who, as part of their professional brief, would have to teach mathematics. Significantly, whilst all the students that were interviewed held a GCSE (16+) mathematics qualification as required for entry to college, none had pursued mathematics beyond this. Nor had any of the students elected to study mathematics as either a first or second subject as part of their university course. The research set out to investigate the ways in which such non-specialist students conceptualise mathematics and its teaching and how their views evolve as they progress through an initial course.

IDENTITY

> Identity should not be seen as a stable entity- something that people *have*- but as something that they *use*, to justify, explain and make sense of themselves in relation to other people, and to the contexts in which they operate. In other words, identity is a form of argument. (MacLure, 1993, p. 287, author's own emphasis).

The notion that "identity" is something people *use* became a significant research theme. So, those ways that the "self" perceived the world, including certain worries concerned with the learning and teaching of mathematics, became in our view central to how mathematics was constituted. Taking note of the figurative language that was used by students when talking about themselves, particularly in relation to mathematics, provided glimpses into some of their beliefs and orientations about learning and teaching (Munby, 1986; Schon, 1979). After all, mathematics as such does not exist in any tangible sense but nevertheless produces tangible effects as though it does exist. Mathematics does not impact on our lives as mathematics *per se* but rather through the social practices that take up mathematics into their forms (Brown, 2001). Such social practices cannot be separated from personal engagements in them and the affective products of such engagements. Mathematics itself is thus necessarily shaped through the often emotionally charged activity that gives it a form. As an example, trainee teachers observed often presented a fairly clipped "didactic" version of mathematics, nervous as they were about opening it up as a field of more creative enquiry. A key focus which emerged from our readings of the transcripts was how in describing their past mathematical experiences, it seemed that negative perceptions of self were resituated as positive traits in accounts of their present teaching.

NEGOTIATING A SOCIALISED MATHEMATICAL IDENTITY

More broadly within the UK, mathematics curriculum materials have become high profile and rigorously enforced. Nevertheless, there are many accounts of mathematics, ranging from those built within the discourse of such government-sponsored materials to others generated more by the trainees themselves. Meanwhile, training institutions, schools, mathematicians, employers and parents all have some say in what constitutes school mathematics. For the trainee teachers interviewed, it seems impossible to appreciate fully and then reconcile all of the alternative discourses acting through them. In confronting the disparity between these alternatives, we have argued elsewhere (Brown and McNamara, under review) that the trainees produce an image of themselves as functioning professionals, in which the failure to reconcile perspectives is swept under the carpet. The individual trainee may, for example, buy into official story lines and see their "own" actions in those terms. This does not have to be seen as a problem. But it may mean that the trainees subscribe to intellectual package deals laid on for them rather than see the development of their own professional practice in terms of further intellectual and emotional work to do with resolving the contradictory messages encountered. As one teacher commented in carrying out research for a higher degree: "Why do we need to do research to find out what good teaching is when the government is telling us what it is?"

Any supposed resolution then of the conflicting demands cannot be achieved without some compromises. Certain desires will always be left out. The teacher however may nevertheless feel obliged to attempt such a reconciliation and to have some account of her success or otherwise. As an example: for so many of the trainees interviewed, mathematics was a subject that filled them with horror in their own schooling. Yet such anxieties seemed less pervasive once the trainee had reached "Qualified Teacher Status". How had this been achieved? It would seem that those who so often had ambivalence towards the subject of mathematics did not continue to present themselves as mathematical failures. Rather, they told a story in which their perceived qualities had a positive role to play. For example: "I like to give as much support as possible in maths because I found it hard, I try to give the tasks and we have different groups and I try to make sure each group has activities which are at their level. Because of my own experience." (Yr. 4). Another student comments: "The first one that springs to mind which I believe that I've got and which I think's very important particularly in maths as well, would be patience" (Yr. 4). A new teacher is more expansive. "Well I'm sensitive towards children who might have difficulty with maths because I know how it might feel and I don't want children to not feel confident with maths. ... I use an encouraging and positive approach with them. ... Because I think if you're struggling in maths the last thing you want is your confidence being knocked in". Such happy resolutions to the skills required to teach mathematics can provide effective masks to the continuing anxieties relating to the students' own mathematical abilities. The evidence in our interviews pointed to such anxieties being sidestepped rather than removed since they were still apparent in relation to more explicitly mathematical aspects of our enquiry.

CONCLUDING THOUGHTS

How then might we better understand the teachers' task of their own professional development? Professional development in the UK has it seems come to be seen in terms better achieving curriculum objectives as framed within the National Numeracy Strategy. The new teachers seemed very comfortable with this Strategy as an approach to organising practice, even if many did find it very prescriptive. The Strategy does seem to have provided a language that can be learnt and spoken by most new teachers interviewed. In this sense the official language spanning the National Numeracy Strategy and the inspectorial regulation of this seemed to be a huge success. This does however point to a need to find ways of adopting a critical attitude in relation to the parameters of this discourse in that certain difficult issues are being suppressed rather than removed. For example, when confronted with mathematics from the school curriculum of a more sophisticated nature the new teachers remained anxious. The National Numeracy Strategy and college training however had between them provided an effective language for administering mathematics in the classroom in which confrontation with more challenging aspects of mathematics could be avoided. If true this points to certain limits in the teachers' capacity to engage creatively with the children's own mathematical constructions. And perhaps further professional development in mathematics education for such teachers might be conceptualised in terms of renegotiating these limits.

Surely policy initiatives must promote improved practice that transcends the conceptualisations embedded within specific initiatives. It seems essential to keep alive debates that negotiate the boundaries of mathematical activity in the classroom and how those boundaries might reshape in response to broader evolving social demands. It would be unfortunate if the prevailing conception of teacher development reached further towards the preference of providing sets of rules with the teacher seeing their own professional development in terms of following those rules more effectively.

Trainees and teachers seem to be increasingly *interpellated* by multiple discourses and risk ending up speaking as if they were a ventriloquist's dummy. Immersed as they are in socially acceptable ways of describing their own practice, the obligation to identify with these can generate resistance to the desire (rather than ability) to produce an identity of their own. Mathematics seems to have a habit of deflecting people from creative engagement into rule governed behaviour as a way of dampening the emotional difficulties engagements with it can provoke. It seems important that further professional development is seen in terms of teachers seeking to recover and then develop some sense of their own voice towards participating more fully in their own professional rationalisations. Effective implementation of the National Numeracy Strategy is one thing. But we do need to guard against this restricting the teachers' need and desire to reconceptualise and develop their practice in their own terms. Very often research focused on mathematics education is seen from the external perspective of mathematics experts detecting the formation of mathematics in classrooms or from the perspective of government officials concerned with administering schools and the standards they achieve. In a professional environment increasingly governed through ever more visible surveillance instruments, such as high profile school inspections there is a sense of needing to be what one imagines the Other wants you to be. Freud's concept of the super ego seems to be ever more reified in an environment of supposed or intended control technology. By focusing more on the perspective of the emotionally charged individual teacher at the centre of the classroom and what they have to say, development within classroom practices can perhaps be conceptualised more by those within the classrooms. Surely this is a worthy aspiration.

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USING RESEARCH TO INFORM PRACTICE: CHILDREN MAKE SENSE OF DIVISION OF FRACTIONS

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The purpose of this paper is to share the strategies that children invented to solve problems involving division of fractions in the absence of algorithmic instruction. The data come from two sources, the first being a teaching experiment and the second being the regular classroom practice of this teacher/researcher. It was found that when the conditions of the teaching experiment were replicated, the three methods that students constructed to solve the identical problems involving division of fractions were the same as those seen in the original group. The three methods involved reasoning concerning natural numbers, measurement and fraction knowledge, all of which relate to counting.

INTRODUCTION AND THEORETICAL FRAMEWORK

In the 1993 – 1994 school year, a research study was conducted in the form of a teaching intervention into a regular classroom, in a 4th grade suburban elementary school in NJ.[1] The topics addressed included several related to fractions. The premise of the study was that students could be provided with opportunities to develop an understanding of fraction concepts before the formal introduction of algorithms. The selection of a 4th grade is significant because it is the year prior to the one in which students are formally introduced to most algorithms involving fractions.[2] Several research papers (including Maher, Martino, R. B. Davis, 1994; Alston, R. B. Davis, Maher, Martino, 1994; Steencken, Maher, 1998; Martino, Maher, 1999; c.f. Steencken, 2001; c.f. Bulgar, 2002; Bulgar, Schorr, Maher, 2002) have traced and documented the emergence and development of powerful mathematical ideas about fractions, and the conditions that were in place in an attempt to better understand how children construct knowledge about fractions in the absence of algorithmic instruction. Other researchers have also investigated the development mathematical ideas prior to the introduction of more formalized procedures. Kamii and Dominick (1997) compared students who were taught algorithmically with a group that was not. They found that the non-algorithmic group was more successful and that even when they made errors, the errors resulted in more reasonable responses. They concluded that teaching algorithmically may impede number

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2 New Jersey's Core Curriculum Mathematics Content Standards have been designed to be philosophically consistent with the NCTM Standards. The current version represents revisions made in 2002.

sense as it adversely affects what was taught about place value and forces children to give up their own thinking.

> When we try to teach children to make relationships between numbers (logico-mathematical knowledge) by teaching them algorithms (social-conventional knowledge), we redirect their attention from trying to make sense of numbers to remembering procedures. (Kamii & Dominick, 1997 p. 59).

Detailed studies of several of the sessions in the above-mentioned teaching intervention have been undertaken (Steencken, 2001; Bulgar, 2002). During the four sessions dealing with division of fractions, it was observed that children used one of three successful means, all related to counting, to solve the problems.

In an attempt to achieve similar outcomes in a regular classroom setting, the conditions of the teaching experiment were identified, studied and then replicated by this teacher / researcher, as part of regular classroom practice. This teacher / researcher had spent years studying the development of children's mathematical ideas and thought deeply about what was observed. Three important elements were noted and replicated. First was the establishment of a classroom community, described below in greater detail, that is consistent with recommendations set forth by the National Council of Teachers of Mathematics (NCTM 2000) and conducive to the development of an inquiry-based classroom. The other significant elements are the selection of appropriate tasks and teacher interventions designed not to interfere with the natural trajectories of children's thinking.

This particular paper seeks to resolve the following issues. How can we use what has been gleaned from the teaching experiment to make a positive impact on classroom teaching? And perhaps more importantly: Are the strategies that children constructed so robust that under similar conditions comparable favorable outcomes can be achieved?

METHODS AND PROCEDURES

This study involved replication for a multiple-case research methodology. Further, the design is considered to be a literal replication since each case was expected to yield similar outcomes as a result of similar conditions being in place (Yin 1994).

BACKGROUND, SETTING AND SUBJECTS

The teaching experiment took place in a small suburban NJ district, over the course of a year. The focus of this intervention was to investigate the development of children's mathematical ideas about fractions. This particular 4^{th} grade class consisted of 25 students. The four sessions involving division of fractions took place in December 1993.

One of the goals of this teaching experiment was to create a classroom community in which student inquiry and discovery were of paramount importance. Students were not told that their work was correct or incorrect. Instead, they were questioned and encouraged to justify their solutions, taking personal responsibility for the accuracy and completeness of their work. The overarching perspective was that if students were invited to work together and conduct thoughtful investigations with appropriate materials, they would be able to build mathematical ideas relating to fractions (Maher, Martino, Davis, 1994). Throughout this experiment, the teacher/researchers worked to promote a classroom culture that supported children as they explained, explored, and reflected upon mathematical ideas. Children were always invited to talk about their thinking, and they were challenged to defend and justify their ideas. The children were encouraged to build models of their solutions and share them. In discussing their solutions, children listened to each other and developed convincing arguments to support their ideas. Discourse among students was encouraged and considered a significant component of the classroom community. Cobb, Boufi, McClain & Whitenack (1997) indicated that when students engage in discourse with peers, there is growth and development of mathematical ideas. In this study, the researchers used responsive questioning to elicit explanations, to help students develop appropriate justifications and to redirect them when they were engaged in faulty reasoning. Justification of ideas was considered to be an integral component of the discourse. Cobb, Wood, Yackel & McNeal (1993) clarify the difference between justification and explanation. They say that we are expected to justify our reasoning when our thinking is understood and challenged. In contrast, we are expected to explain when our reasoning is not understood and a clarification is requested.

The students being studied in their regular classroom environment attended a small parochial school that attracts children from a several surrounding communities. For this part of the study, the work of a 5th grade class during the school year 2000-2001 was examined. This academically heterogeneous class consisted of 13 girls. In May 2001, they were assigned the identical task as the students in the teaching experiment.

These students had experienced a very traditional[3] classroom environment prior to the 5th grade. They were used to being told whether or not their answers were correct and being shown procedures for doing mathematics. In contrast, upon entering 5th grade, they were encouraged to take responsibility for convincing others that their solutions were correct and were expected to write about what they were thinking on a regular basis. Discourse was of paramount importance. Responsive questioning took place to encourage mathematical thinking by attempting to elicit verbalization of mathematical thought. The classroom community was one in which students' ideas were always respected. Alternate strategies were encouraged, shared and discussed. They were invited to discuss their thinking and to submit ideas in writing. Students were not taught algorithms. When they recognized patterns and could justify that these patterns were valid, they created generalizations, which they could apply to future problems. In general, the classroom community was modeled on that of the teaching experiment described above.

DATA AND CODING

All of the research sessions were videotaped using two or three cameras. Students' original written work and teacher/researchers' notes were also carefully collected. Transcriptions and detailed narratives of the data were recorded and a coding scheme was designed to flag elements for study. The four classifications of codes used were intended to record teacher interventions, ideas expressed, representations used by students, and justification and reasoning by students.

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[3] Traditional, in this case refers to a more didactic environment of the type described in Cuban, 1993.

Just as the students in the research group had done, the 5th grade students wrote about their mathematical experiences. The original written work that they completed while engaged in the identical activity as the 4th graders form a second set of data, which were examined in this study. In addition, field notes from

two graduate students visiting from Rutgers University and the observations of this researcher, the regular mathematics teacher of the class, provide a triangulation of sources for these data. Audio recordings also exist, but these were made primarily to monitor the nature of teacher interventions.

THE TASK

Both populations worked on the task[4] entitled "Holiday Bows" [5] which was designed to provide students with a meaningful context for understanding division of a natural number by a fraction. The task involved finding out how many bows of several fractional lengths could be made from various sizes of ribbon. For example, one of the questions was how many bows, each one-third meter in length, could be made from a piece of ribbon that is six meters in length. Students in both groups had access to actual ribbons, pre-cut to the specified sizes, meter sticks, string and scissors.

RESULTS AND DISCUSSION

What emerged from study of the research intervention was that the students who were part of the 4th grade class used one of three specific means as their basis for solving problems involving division of a natural number by a fraction, and that all three of these solution strategies involve counting. These solution strategies have been identified as:

Reasoning involving natural numbers

Reasoning involving measurement

Reasoning involving fractions.

Students who solved the problem using reasoning involving natural numbers converted the meter lengths of ribbon to centimeters so that the division made use of whole numbers. In many cases, these students also used reasoning involving fractions and measurement, but because they performed the division using natural numbers, this is seen to be the primary strategy. For example, when finding out how many bows, each 1/3 meter in length, would be made from one meter, they took the 100 centimeters and divided it into three equal parts, showing an understanding that one-third of a meter involves dividing by three. Many were not able to come up with the exact number, thirty-three and 1/3, but they were able to estimate what 1/3 of a meter would be and then

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4 Students worked on many tasks, however this report focuses on one that provides the context for analysis.

5 This task was developed by Alice Alston to be used with a 5th grade class and was then modified to be more "open-ended" for the 4th grade class. Carol Bellisio (1999) reported on the 5th grade version.

mark off approximately where the 33 centimeters would fall. Then, they counted their markings to find that there would be three such bows.

Students who solved the problem using reasoning involving measurement as their primary strategy created a unit of measurement equal in length to the desired bow, placed it along the given amount of ribbon, and then counted how many times it fit. For example, to find how many bows, each 1/3 meter in length, could be made from two meters of ribbon, they cut a piece of string equal to 1/3 meter and then counted how many times this measuring tool could be placed along the two meters of ribbon. In many cases the measuring tool was constructed by taking a one-meter piece of string and dividing it equally into the required number of parts. That is, if a 1/3 meter measuring tool were needed, a one-meter piece of string would be folded into thirds and then cut.

When students used their knowledge of fractions as their primary strategy to solve the problem, they recognized that each meter contains an equal number of fractional pieces and then they multiplied this number by the number of meters. For example, to find out how many bows, each 1/3 meter in length, could be made from two meters of ribbon, a student would recognize that in each meter there would be three one-thirds, so there would be three bows. They would then take that number of bows, three, and multiply it by two (or use repeated addition) since there were two meters. The counting is evident in the multiplication, which involves the iterated number of thirds.

A significant portion of the activity consisted of division of a natural number by a unit fraction. Students had more difficulty with division involving a non-unit fraction divisor. Those who reasoned using natural numbers, had to work to recalculate what two-thirds of a meter would be in terms of centimeters. Those who used measurement as the primary strategy, had to create a larger measurement tool and did so based upon the measuring tool they used for the unit fractions. For example, they had to construct a one-third-meter piece of string before they could construct a two-third meter measuring tool. Those who reasoned using fractions had great difficulty because they struggled to give meaning to the piece that was "left over". That is, it was not clear how many two-thirds there were in one. In spite of these obstacles, students were able to solve the problems involving the division of a natural number, by a non-unit fraction divisor. In some cases, they had to make adjustments to the schemes they had built to solve the other problems.

When the same task was implemented under replicated conditions, as part of regular classroom practice, the identical three solution strategies were observed.

Nicole used reasoning involving fractions to solve the problems. She writes the following about her solutions.

> What I did for the 1 meter ribbons are: If you have 1 meter and the length of a bow is 1/2 a meter then obviously it's 2. I did the same thing for 1/3 = 3, 1/4 = 4, 1/5 =5.
>
> What I did for 2 meters is: Since 1 is 1/2 of 2 I add the #'s for 1. For example: for the 1/2 meter [bow] and 1 meter [ribbon] I got 2. So I added 2 + 2 = 4. So 4 is my answer.
>
> What I did for 3m is: I added what I got for 3m. 1m + 2m. For example 1/2 of 1 meter was 2 1/2 of 2 meters is 4 so I added the [sic] together 2 + 4 = 6.

Sarah used reasoning involving measurement. She states the following.

I figured all the answers out by putting the string next to the ruler and finding the "Ribbon Length of Bow" and seeing how many strings I could get to fit to that length.

Being accustomed to justifying her solutions, she continues by attempting to substantiate her work by writing the following.

Explanation: I think my method works because when you measure the string to the right length and see how many strings you can measure it [the pre-cut ribbon length] to, you get an answer.

An interesting difference in the outcomes observed in the two populations is that the students in the 5^{th} grade group rarely used the solution method of reasoning involving natural numbers, while it was a very commonly used method in the 4^{th} grade group. Perhaps this is a function of the difference in grades. In 4^{th} grade, there is a great deal of emphasis on division of natural numbers, while in 5^{th} grade, the curriculum contains more introductory fraction study. This would seem to be consistent with Vygotsky's Zones of Proximal Development (Vygotsky, 1978). Olivia was the only 5^{th} grader to refer to reasoning involving natural numbers and she did not come up with this type of solution initially. At the conclusion of her written work she says the following.

I figured out a shorter way to explain this & it makes more sense. It works as follows: 1 meter = 100 centimeters. You could change the amount of meters you have into centimeters. Thus, let's say you have to make bows each 1/2 of a meter. Figure out how many centimeters = 1/2 of a m. 50 centimeters = 1/2 of a m because half of 100 is 50. Then see how many times 50 goes into 100. However many times 50 goes into 100 is how many bows you can make with each bow 1/2 of a m. & with 1 m. You can also do this with 1/3 of a m. or 1/4… as long as you change 1/3 or 1/4 of a meter into a # amount of centimeters. You can also do this with 2 or 3m… of string as long as you change 2 or 3m… into centimeters. I think this works because you have to figure out how many 1/3rds or $1/4^{th}$ s of a m. go into 1 m. That is saying the same thing as a certain # of centimeters go into 100 or 200 or 300. Or you could do $1 \div 1/4$ and you would get 4. That is the same thing as $100 \div 25 = 4$. They both = the same thing which proves they both work.

As these students moved from the unit fractions to the non-unit fractions, they also had to adjust their strategies. Linda solved the problems that required division by a unit fraction using reasoning involving fractions. She assumed this method was no longer valid when faced with a non-unit fraction divisor and therefore employed the strategy of reasoning involving measurement. The following appeared in the field notes of one researcher who was present.

When she got to the question of 6m ribbon and 2/3m bow, she started measuring. I asked her why she didn't just use her multiplying method, she replied, "cause there's a 2 there not a 1, so you can't do it, you can only do it when there's a 1, so I have to measure it if there's another number there." It's ironic how she understands that the 2 in the numerator makes her method invalid, but she doesn't understand why. (C. Hayworth, unpublished notes, May 24, 2001)

CONCLUSIONS

The initial research study was undertaken to learn more about how children constructed knowledge about fractions. While there is much research showing that many children have experienced great difficulty in solving problems that involve fractions (for example: Tzur, 1999; Davis, Hunting & Pearn, 1993; Davis, Alston & Maher, 1991; Steffe, von Glasersfeld, Richards & Cobb, 1983; Steffe, Cobb and von Glasersfeld, 1988), these students demonstrated mathematical understanding of division of fractions, thought to be the most complex aspect of the elementary mathematics curriculum (Ma, 1999).

The ability of the students in the teaching experiment to successfully construct means by which to solve problems involving division of fractions prompted an examination of conditions in place during the teaching experiment. The three primary issues which stood out in the investigation are development of a classroom community, choice of appropriate problems and teacher interventions.

The teaching experiment described here became a model for classroom instruction of division of fractions. Specific strategies used by the students in the research group to construct methods of solving problems involving division of fractions were observed. These robust methods were also used by students completing the identical task in their regular classroom setting. These students were also given the time and the opportunity to explore mathematical ideas deeply, in a supportive environment where their ideas are respected, and they became empowered to think like mathematicians. They collaborated, experimented, hypothesized, tested their hypotheses, built concepts and took great pride in their accomplishments.

In this situation, research did inform classroom practice. Teachers need to be provided with more opportunities to study and learn from research. In addition, more research experiments need to be undertaken to provide information to classroom teachers that will lead to a situation wherein all children will have an equitable opportunity to build powerful mathematical ideas and to think like mathematicians.

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DEVELOPING AND CONNECTING CALCULUS STUDENTS' NOTIONS OF RATE-OF-CHANGE AND ACCUMULATION: THE FUNDAMENTAL THEORM OF CALCULUS

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An overview of the conceptual underpinnings, reasoning abilities and notational issues related to learning the Fundamental Theorem of Calculus is provided. Using this theoretical framework, curricular materials were developed to promote these understandings and reasoning abilities in students. Results from a study that investigated the effectiveness of these materials on first semester calculus students' understandings of the FTC revealed significant advances in their understandings of accumulation and the FTC. Some specific difficulties that were observed in select students provided insights for further refinement of the theoretical framework and for revision of the FTC activities.

INTRODUCTION AND BACKGROUND

The Fundamental Theorem of Calculus has been described as one of the intellectual hallmarks in the development of the calculus (Boyer, 1959). However, studies have documented that most first semester calculus students do not emerge from the course with an understanding of this concept; nor do they appear to be developing the foundational reasoning abilities needed to understand and use the FTC in applied settings (Bezuidenhout & Olivier, 2000; Kaput, 1994). Student difficulties with the Fundamental Theorem of Calculus have been attributed primarily to their impoverished view of function (Carlson, 1998; Thompson, 1994) and rate-of-change (Thompson, 1994). However, little research is available articulating what is involved in knowing and learning this concept. The purpose of this paper is to provide additional clarity about the understandings and reasoning abilities involved in learning and using the FTC. It also reports the results of a study that investigated the effectiveness of curricular materials for first semester calculus students that were developed using this framework as a guide.

Reasoning about and with the Fundamental Theorem of Calculus involves mental actions of coordinating the accumulation of rate-of-change with the accumulation of the independent variable of the function. The accumulating quantity can be imagined to be made of infinitesimal accruals in the quantities, which when thought of multiplicatively, make up the accruals in the accumulating quantity. Both a process view of function and *covariational reasoning* have been shown to be foundational for coordinating these accumulations (Thompson, 1994).

Covariational reasoning refers to the coordination of an image of two varying quantities, while attending to how they change in relation to each other (Carlson, Jacobs, Coe, Larsen and Hsu, 2002). A more detailed characterization of covariational reasoning has been articulated in a Covariation Framework that characterizes covariational reasoning in terms of the mental images that support the mental actions of coordinating: i) changes in one variable with change in the other variable; ii) the direction of change in one variable with changes in the other variable; iii) the amount of change of one variable with changes in the other variable; iv) the average rate-of-change of a function with changes of the independent variable; and v) the instantaneous rate-of-change of the function with continuous changes in the input variable. The mental image that supports all five mental actions has been classified as Level V covariational reasoning. As this body of literature suggests, it seems reasonable that students should develop both a process view of function and Level V covariational reasoning abilities prior to their study of the Fundamental Theorem of Calculus.

Thompson (1994) has described the Fundamental Theorem of Calculus as a means of expressing the relationship between the accumulation of a quantity and the rate-of-change of the accumulation. He advocates that an understanding of the FTC involves coordinating images of respective accruals in relation to the total accumulation. According to Thompson, this is the idea that motivated Newton's development of the Fundamental Theorem. Newton first determined the average rate-of-change of an area and determined that the total area could be computed by multiplying the rate-of-change by the accumulation of the independent variable. This led to his observation that the rate of change of the accumulated quantity is equal to the immediate accrual. This line of thinking emphasizes the importance of understanding that the accrual is a multiplicative relationship and that the total accumulation is made of infinitesimal (multiplicatively composed) accruals of the quantities (e.g., accruals of lines compose area and accruals of area compose volume). It is these understandings that enable the relationship expressed by $\frac{d}{dx}\left(\int_a^x f(t)dt\right) = f(x)$ to be appreciated and understood. A more careful articulation of the reasoning abilities and understandings related to accumulation and the Fundamental Theorem of Calculus are provided in the theoretical framework for this study.

THEORETICAL PERSPECTIVE:

The FTC Framework

This framework contains four dimensions that describe the foundational reasoning abilities and understandings of the Fundamental Theorem of Calculus.

Part A: Foundational understandings and reasoning abilities

(FR1) Ability to view a function as an entity that accepts input and produces output.

(FR2) Ability to coordinate the instantaneous rate-of-change of a function with continuous changes in the input variable (Level V covariational reasoning).

(FU1) Understanding that the average change of a function (on an interval) = the average rate-of- change (multiplied by) the amount of change in the independent variable.

(FU3) Understanding that the multiplicative relationship that represents the accrual of change on an interval can be represented by area.

Part B: Covariational reasoning with accumulating quantities.

The Mental actions of the Fundamental Theorem of Calculus (The function refers to the rate-of-change function, *f*).

(MA1) Coordinating the accumulation of discrete changes in a function's input variable with the accumulation of the average rate-of-change of the function on fixed intervals of the function's domain.

(MA2) Coordinating the accumulation of smaller and smaller intervals of a function's input variable with the accumulation of the average rate-of-change on each interval.

(MA3) Coordinating the accumulation of a function's input variable with the accumulation of instantaneous rate-of-change of the function from some fixed starting value to some specified value.

Part C: Notational aspects of accumulation

Notation	Meaning
$F(x) = \int f(x)dx$	i) The antiderivative of f is F ii) f is the function that describes the rate-of-change of F.
$F(x) = \int_a^x f(t)dt$	i) The value of $F(x)$ represents the accumulated area under the curve of f from a to x; ii) The value of $F(x)$ represents the total change in F from a to x.

Part D: The statements and relationships of the FTC

$\int_a^b f(x)dx = F(b) - F(a)$	i) The accumulated area under the curve of f from a to b is equal to the total change in F from a to b.
$\frac{d}{dx}\left(\int_a^x f(t)dt\right) = f(x)$	i) The instantaneous rate-of-change of the accrual function at x is equal to the value of the rate-of-change function at x

METHODS

The subjects were 24 beginning calculus students enrolled in the same section of first semester calculus at a large university in the United States. A Pre-Calculus Concept Assessment Instrument (focused primarily on assessing the reasoning abilities and understandings described in Part A of the FTC Framework) was administered to the students at the beginning of the semester, and a post-instruction written assessment instrument was administered at the end of the course. The mean score and number of correct responses for each item were compiled. Four students who were somewhat representative of the diverse understandings of the class (based on their performance on the Pre-Calculus Concept Assessment Instrument) were invited to participate in eight (~75 minute) clinical interviews. The interviews were conducted in pairs and were designed to gain information about students' ability to understand and reason using the major concepts of the course (covariation, limit, derivative, accumulation, the FTC). During each interview the students were asked to complete a collection of thought revealing tasks (Lesh, 2002) that paralleled the conceptual focus of instruction during the previous two weeks of the course. Each pair verbalized their thinking while responding to the written problems and questions. The role of the interviewer was to promote discussion among the pair of interviewees and to gain insight into the understandings and reasoning of the individuals. Digital videos of the sessions were transcribed, coded and analyzed, using the FTC framework.

THE COURSE

The text for the course was *Calculus Early Transcendentals* (Stuart, 1999). However, about half of the instruction was delivered using the *Conceptual Calculus Modules* currently under development by the first author. Each module contains a collection of in-class and take-home activities designed to promote the development of students' conceptual connections and reasoning abilities relative to the central concept of the module. Carefully sequenced prompts and tasks (situated in context whenever possible) were included to promote students' articulation of their thinking.

The *Precalculus Concept Assessment* instrument was administered to the students at the beginning of the semester. Instruction during the first two weeks of the semester included a strong focus on the foundational reasoning and understandings described in Part A of the FTC framework. Post-instruction assessment of these understandings suggested that most students emerged from this instruction with these reasoning abilities and understandings. Instruction leading up to the FTC module included a balanced focus on concept development, acquisition of notational understanding, facts and procedures, and the development of students' mathematical practices and problem solving behaviors. Students were expected to be regular participants in the classroom. Whole class discussion, group work and lecture were the primary modes of instruction.

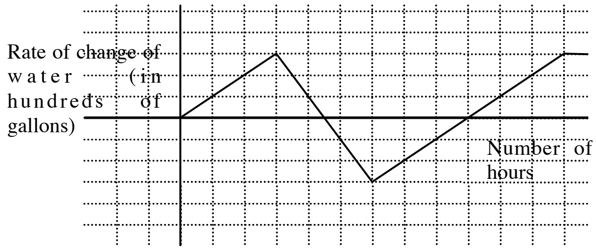
RESULTS

Select data from administering a post-instruction written assessment of students' understandings related to accumulation and the FTC are reported. The presentation of results provides a statement of the item, the number of students who provided a correct response (out of the 24 who completed the course), and the mean score (out of 3) on each part of each item.

The collection of responses on Item 1 suggests that the beginning calculus students in this study were proficient in applying covariational reasoning with accumulation tasks. Over 70% of the students completing the course provided a completely correct response to parts d, e and f (Item 1), suggesting proficiency in coordinating the accumulation of a function's input variable with the accumulation of instantaneous rate-of-change of the function from some fixed starting value to some specified value (MA3). Over 90% of these same students also provided a correct response to the prompts that assessed students' understanding of the notational aspects of the FTC (parts b and c).

Item 1: The Water Problem

Let f represent the rate at which the amount of water in Phoenix's water tank changed in (100's of gallons per hour) in a 12 hour period from 6 am to 6 pm last Saturday (Assume that the tank was empty at 6 am (t=0)). Use the graph of f, given below, to answer the following.



	Number Correct (out of 24)	Mean Score (out of 3)
a. How much water was in the tank at noon?	21	2.8
b. What is the meaning of $g(x) = \int_0^x f(t)dt$	24	3.0
c. What is the value of $g(9)$?	22	2.7
d. During what intervals of time was the water level decreasing?	22	2.7
e. At what time was the tank the fullest?	17	2.3
f. Using the graph of f given above, construct a rough sketch of the graph of g and explain how the graphs are related.	17	2.4

Student responses on Item 2 also suggest that this collection of students possessed both a strong understanding of notational aspects of the FTC (parts a, b) and proficiency in applying covariational reasoning with accumulation tasks (parts f, g, and h). Responses on parts c, e and i indicate moderate proficiency in understanding the statement of the FTC, with only about 60% of these students providing correct responses on this collection of questions.

Item 2: The Circle Problem

Consider a circle that expands in size from $r = 0$ to $r = x$. Let A be a function that represents the accumulation of the rate-of-change of the circle as it increases in size from $r = 0$ to $r = x$.

	Number Correct (out of 24)	Mean Score (out of 3)
a. Define $A(x)$ as an accumulation function.	18	2.3
b. Construct a circle and illustrate what $\int_2^4 2\pi r\, dr$ represents.	22	2.8
c. Describe what $\frac{d}{dx}\int_a^x 2\pi r\, dr$ represents relative to the circle.	14	1.8
d. Construct the graph of f (the rate of change of the area of a circle), on the axes on the left and the graph of A (as defined above) on the graph on the right. Label your axes.	20	2.6
e. Explain how the two graphs are related.	16	2.2
f. Construct the graph of A. Estimate the area under the graph of A from $r = 1$ to $r = 5$ using eight approximating rectangles and	19	2.5

right endpoints		
g. Given that n represents the number of subdivisions on the interval from r = 1 to r = 5, explain what is involved in letting the $\lim_{n \to \infty}$ for this interval.	20	2.6
h.What is the result of this evaluation?	19	2.7
i. What does $A'(x) = f(x)$ mean in the context of this situation?	16	2.1

Results for item 3 reveal that most of these students recognized this question as an application of the FTC. They were also successful in translating the situation to symbols (Part D of the FTC framework). However, most students had difficulty recognizing that they needed to sum the distance traveled in both the positive and negative directions.

Item 3: The Distance Problem

A particle moves along a line so that its velocity at time t is $v(t) = t^2 - t - 6$ (measured in feet per second). Find the distance traveled during the time period from $t = 1$ to $t = 4$. Show Work!

Number Correct (out of 24): 12 Mean Score (out of 10): 7.2

Number of students who set up the integral correctly: 23

Common Error: Computed position from start instead of total distance traveled.

The collection of student-responses on these items suggests that most of the students completing the course emerged with proficiency in using and understanding notational aspects of the FTC (Item 1, parts b and c; Item 2, parts a, b). These results also suggest that these students were able to apply covariational reasoning with accumulation tasks. Their understanding of the statements and relationships of the FTC tasks were weaker, with only a little over half of the students completing the course providing correct responses to the collection of questions assessing this ability.

Interview Results

The four interview subjects were Lisa, Harold, Chad and Katie. Lisa received a C in the course, Harold and Chad received B's, and Katie received an A. Analysis of the four interviews revealed that: i) all four students were able to apply covariational reasoning with accumulation tasks; ii) all four students possessed a strong understanding of most of the notational aspects of the FTC; and iii) Chad, Katie and Harold possessed a strong understanding of the statements and relationships of the FTC. Select interview excerpts from the interviews in which these four students explained the reasoning they used to respond to the above items follow.

When responding to Item 1, Chad demonstrated that he was able to coordinate the accumulation of time and accumulation of rate (MA3). He also demonstrated that he understood the role of the input variable to g (i.e., determining the upper limit of the integral); however, further probing suggests that he had some confusion about the role of x and the rationale for labeling the independent variable of f using another variable.

I: So what did you notice about the relationship?

Chad: One figure is always twice the area of the other.

I: Explain the meaning of $g(x)$ (see item 1c above).

Chad: I see g as giving the amount of area under the graph of f.

I: What does the input variable *x* represent?

Chad: This tells you how far out on the right on the graph of *f* you want to go (student sweeps his hand across the graph.)

I: Can you explain this in the context of the question.

Chad: Um…since *f* is the rate of water flowing into the tank and *g* is the integral of *f* from 0 to *x*, when you find $g(x)$ you are finding how much water came in or went out of the tank from the starting time, up until the time that you want….that is the time *x* (MA3).

I: So, how do you think about evaluating $g(9)$?

Chad: I see that as finding the time that passes from 0 to 9 and thinking about how much area gets added under the curve as I move along. I see that water is coming into the tank, first at an increasing rate, then at a decreasing rate. Then after 4_ hours, water starts to go out of the tank (MA3). As you add up the area under the curve you see that the same amount of water comes in between 0 and 4 _ that goes out between time 4_ and 9….so, the result is that there is no water in the tank after 9 hours have passed.

I: How are *g* and *f* related?

Chad: The derivative of *g* gives the graph of *f*. What I don't get is why *t* is the variable that is used in *f*. I never really understood this on some of the other problems we did either.

The interview responses to item 2 (parts d and e) revealed that three of the interview subjects (Harold, Chad and Katie) held a strong understanding of the statement of the FTC for the circle problem. These three students were proficient in constructing the graphs of *f* and *A*. They were also able to provide a clear articulation of how the two graphs are related. Harold set up a table that computed the accumulation of the area under the graph of *f* from 0 to various values of *x*. He continued to explain that he viewed the accumulation of the area from 0 to specific values of the input to *f* as producing a value that provided the total area. He went on to explain that he also viewed the accumulation of area as the output of *A*. Later in the interview, he expressed that he viewed the accumulation of rate-of-change of the circle as adding up "infinitely many infinitesimally small" circumferences. When probed to explain how to use the graphs to compute $\int_2^5 2\pi r\, dr$, he responded that it could be computed in several ways. He continued by subtracting the two areas under the graph of *f*; he then drew a picture of the circle and shaded the area represented by this definite integral. He went on to explain that what he was actually finding was $A(5) - A(2)$ and expressed that this value was just the difference in the heights of the graph of *A* between $r = 5$ and $r = 2$.

The interview with Lisa revealed some weaknesses in her understanding of the statements and relationships of the FTC. More specifically, she was unable to articulate what $\frac{d}{dx}\int_a^x 2\pi r\, dr = 2\pi x$ expressed about the relationship between accumulation and accrual. Her response suggested that she did not view $\int_a^x 2\pi r\, dr$ as a representation of the accrual of $2\pi r$ from some specific value *a*, to some specified value for *x*. Her utterances suggested that she did not view this as an object that she was able to differentiate.

CONCLUSIONS AND DISCUSSION

The quantitative and qualitative data suggest that most of the first semester calculus students in this study completed the course with a strong understanding of notational aspects of accumulation. They also demonstrated an ability to coordinate the accumulation of a function's input variable with the accumulation of instantaneous rate-of-change, from some fixed starting value to some specified value, for various contextualized situations. Although some weaknesses were observed in some students' understandings of the statements and relationships expressed in the FTC, the performance of this collection of students relative to the attributes of accumulation and the FTC expressed in this framework were relatively good, especially if one compares this with what has been reported of secondary teachers and graduate students (Thompson, 1994).

The framework for this study served as a useful tool for analyzing students' reasoning abilities and understandings relative to both conceptual and notational aspects of the FTC. The results of this study suggest that further refinement of Part D of the framework is needed. In particular, the weaknesses that were observed suggest that the framework needs to include a more careful articulation of the mental actions involved understanding and applying the statements and relationships expressed by the Fundamental Theorem of Calculus. This refinement should also lead to the development of additional ideas for curricular tasks and prompts to better assist students in developing these understandings and reasoning abilities.

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GUESS MY RULE REVISITED

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We present classroom research[1] on a variant of the guess-my-rule game, in which nine-year old students make up linear functions and challenge classmates to determine their secret rule. We focus on issues students and their teacher confronted in inferring underlying rules and in deciding whether the conjectured rule matched the rule of the creators. We relate the findings to the tension between semantically and syntactically driven algebraic reasoning.

FROM SEMANTICS TO SYNTAX

There are diverse approaches to algebra depending upon the relative mix of modeling, generalized arithmetic, mathematical structures, functions, and other considerations. Clearly, not all approaches will be equally appropriate for the young learner. It stands to reason, for example, that if algebra is introduced to elementary students as the syntactically-guided manipulation of formalisms (Kaput, 1995), many young learners are going to be left behind. We find compelling the evidence that children's early mathematical learning benefits from reasoning about rich contexts, from thinking about relations between quantities, from trying to solve word problems. This general approach to mathematics is shared by many schools of thought and research traditions (Vergnaud, 1985; Schwartz, 1996; Davydov, 1991; Smith,)[2]. It has led us to highlight modeling and mathematization in developing learning tasks during our three year longitudinal investigation of Early Algebra learning among 70 second to third grade students in greater Boston. The mere names of tasks we developed—the Heights Problem, the Piggy Bank Problem, the Best Deal, Phone Calling Plans—reveal our bias in grounding early algebra activities in rich situations about which students had considerable intuition and prior experience.

But early on we came to realize that certain representational tools—tables, number lines, graphs, and algebraic-symbolic notation, for example—were going to assume increasingly important roles in our students' mathematical lives. And the students clearly were not going to invent these notational systems on their own. This led us to elaborate activities in which special mathematical representations would become the object of direct discussion and reflection over the course of many lessons. It has been encouraging

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[2] Carpenter and Franke (2001) have built a program of Early Algebra instruction and research based on the premise that open sentences can constitute the main point of departure for introducing algebra into the early mathematics curriculum. Although we take a different view, we follow their work with interest, and find it noteworthy that our work in theirs draws inspiration from the groundbreaking work of Davis (1966-67), among others.

to see students incorporate tables, number lines, and graphs into their repertoire of "spontaneous" representations.

The case for algebraic-symbolic notation, however, has been somewhat different. We found that our students generally used algebraic notation to describe functions that they had come to identify through reflecting upon rich situations. However, we saw little evidence that such notation was exerting an influence on the course of their thinking. They seemed to be merely using the notation to register what they had concluded.

We realize that, as many have noted, at some point students will have to be able to reason directly upon and with the written notation. When should this transition occur? Will the prior emphasis upon richly contextualized reasoning militate against this progression towards notation and syntactically driven reasoning? What sorts of situations are likely to promote this new form of reasoning? What issues must students contend with along the way?

The Guess My Rule game suggested itself as a promising context for getting students to focus on written algebraic notation as an object of discussion because students would have to compare expressions written by the rule makers with those posed by the rule guessers and decide whether certain variations express the same or different underlying rules.

The Guess-my-Rule game has often been used in mathematics education at different grade levels as a way to introduce children to linear functions (see, for example, Davis, 1967, 1985; Carraher, Schliemann, & Brizuela, 2001, 2003; Schliemann, Carraher, & Brizuela, 2003). This activity essentially provides students with values from a function's domain (input) and the corresponding value from the range (output); based on the data, students try to infer the function. In order to play the Guess My Rule game, students must accept that: 1) each input must result in a single output; and 2) a function is consistently used for all values of the solution set, that is, the rule cannot change. Children in younger grades enjoy participating in this guesswork, even though they may not be fully aware of the teacher's role in moving the discussion towards the recognition of linear functions. Davis (1967) refers to children's experience in Guess My Rule as "readiness-building for functions" and makes a case for initially allowing students to figure out how to solve functions through their own means.

OVERVIEW OF CLASS AND THE RULES STUDENTS CHOSE

The following data come from a 90 minute classroom of 18 students in third grade; data from three other classes will be analyzed elsewhere. The students were already somewhat familiar with algebraic notation for functions. However, they had partaken in limited discussions about equivalent expressions for functions or about operating on functions. After several examples, students broke into 'groups' of one to four to make up their own rule. They made considerable efforts to ensure that other groups did not eavesdrop and discover their choice of rule as they were choosing it. After they chose their rule, they completed a table with input values of their choosing and the respective output values.

As students formulated their rules in groups, they discussed which combinations of operations and addends would result in a tricky rule yet one they could manage. Further, they openly discussed the importance of choosing a single rule that would account for all the data. All the groups applied their rule to inputs of 1 through 10, and several included higher, landmark numbers, such as 50 or 100.

In the following section, we will summarize the discussions from one of our four classrooms using Table 1 to provide an overview.

Group 1's rule: $N \times 7 - 3$

Table 1 (above) summarizes[3] the results of the activity during which each group of students generated a number of examples of input and output (with the input often coming from the guessing students) and challenged their classmates to guess their rule. For example, group 1 decided to use as their secret function, $n \times 7 - 3$. When given 4 as an input number, they correctly told the class that 25 would be the output. One of the classmates conjectured that the rule was $n \times 5 + 5$. This was a reasonable candidate; it is a *local solution*, that is, it matches the data for that particular instance. When the next input-output pair is given, a student among the guessers suggests that the rule is $n \times 5 + 7$. This is a local solution that does not accord with the first data pair, (4, 25). Likewise a local solution, $n \times 6$, is proposed for the ordered pair (3, 18). The final guess, n+7, at first perplexed us. After David wrote the input of 3 above the input 4, the list of inputs was ordered as counting numbers. Joey noticed that each successive output increased by 7. N+7 may not correctly describe the function; however, it captures something about the pattern of outputs: 18, 25, 32, 39.

The class does not solve the function, which is revealed by the rule makers before passing the floor to Group 2.

Group 2: What Counts As a Solution?

Group 2 secretly chose as their rule "$N \times 5 \times 4 + 1$". One of the guessing students suggests one million as input; the rule makers correctly reveal that the output will be twenty million and one. Cristian immediately raises his hand and says "I know it! I know it!" His conjecture is “N times twenty plus one”. A dialog ensues regarding whether Cristian has discovered the rule used. One of the creators of the rule, Joseph, sounds out a rasping buzzer noise[4] that signals the student has given a wrong answer.

The teacher notes that Cristian's answer happens to be consistent with the data pair. At this point Joseph maintains that Cristian got it wrong but "got it right in a different way".

Group	Functions	Examples	Guesses	Local solution

[3] For simplicity we represent here students’ conjectures through standard mathematical notation even though they were spoken; many of the spoken conjectures were annotated by the teacher on large paper; rule makers also wrote down their rules.

[4] The allusion is to a television game-show contestant being informed that his answer is incorrect.

1 Melissa, Alanna, Nancy, & Maria	N□7-3	4→ 25 5→32 3→18 6→39	n□5 +5 n□5 +7 n□6 n+7	✓ ✓ ✓ ✗
2 Joey, Joseph, & Adam	N□5□4+1	1,000,000→ 20,000,001 1→21 2→41 …→61 …→81 …→101	 n□20+1 n□20+1	 ✓* ✓*
3 Omar, Kevin, & Anthony	K□2-2	5→8 100→198 0→-2 2→2 3→4 4→6 6→10 7→12 15→28	[n+] 3 [n]+3 KK+2 [K] □2-2 K□2-2 K+K-2	✓ ✗ ✓ ✓ ✓*
4 Matthew	[N] +50-20	92→122 0→30 1→31	 N + 8 + 22 N+30	 ✓* ✓*
5 Cristian	C□3+2-4+5	-1000→-2997 0→3 1→6 2→9 3→12 4→15	N□2+997 N□3 N□4+1 Y□3+3	✗ ✗ ✗ ✓*

*expression is not identical to the expression of the rule makers.

Table 1. Summary of secret functions, examples presented in class, and conjectures made by students.

After David leads the students through an additional data pair, Joseph continues to insist that Cristian was wrong. Joey now concedes that Cristian has solved the problem. By the end of the discussion, Joey and Joseph seem to believe that there are two ways to look at the issue. And indeed there are: Cristian's expression is not identical to theirs; nonetheless, it seems to work.

David writes the rule-as-created, N □5□ 4 + 1, on chart paper. To encourage them to see the rules as equivalent, he asks the students to find another way to express the part, □5□4;

they respond correctly, “times twenty”, and David writes □20 under the factors. In the end, Joseph and Joey, rule makers, agree that Cristian’s rule works but take different stances on the issue of correctness. (Adam, the third rule-maker has not expressed his view.)

> Joseph: He got it wrong, but he got it right in a different way.
>
> Joey: He solved it right, but he did it in a different way.

Group 3: Is K + K the same as K □ 2?

Group 3 choose as their secret rule K □ 2 –2. The first input-output pair, (5, 8), is written in mapping notation as 5→8 on the chart paper. Joseph, now in the role of a conjecturer, says emphatically “three” while Joey says “plus three”. Briana raises her hand and answers that the rule is “plus three”. We transcribed their answers in Table 1 as “[n+] 3” and “[n]+3”, employing brackets to indicate which parts were editorially inserted by us. David realizes that Briana’s rule makes no explicit reference to the variable. So he introduces the letter B (students often prefer to work with the initial letters from their own name) as a means of completing the expression of the rule:

> David: Who thinks they know the rule? Briana…
>
> Briana (hand raised): Plus three?
>
> David: So if you start out with B (writes “B→” on chart paper), B becomes what? What’s the rule, that you think it is? What should we do to the B? [Another student quietly says, “plus three] …According to your rule, Briana,
>
> Briana: Three
>
> David: No…Is three always the answer? So…we’ve got to do something with the B. What do we do to the B? You said it’s plus three, right? [Briana nods in agreement.] So actually, your rule, Briana, is *B plus three* (as he completes writing B → B +3). So that’s Briana’s rule. Let’s see if this works.

When going through the next input, 100, David asks what Briana’s rule would predict the output to be. Several students answer, “103, ” and David agrees and Briana confirms by nodding her head. When the rule-makers reveal that their answer is 198, Briana, turns with perplexed surprise to the student sitting beside her. David himself is surprised and he asks to peek at the rule written on the makers’ sheet; he confirms that they have correctly given the output. When Briana acknowledges that her rule did not work, David clarifies that “it only worked for the first one.” In this way he calls attention to the fact that a conjecture might work locally (case 1) without working globally.

> David (summarizing): So we can actually say that this was a good guess but it is wrong, because it doesn’t work for both of them.

The rule-makers proceed to supply the class with additional information, namely the outputs for inputs of 0, 2, 3, 4, 6, and 7. Joey notes with interest, the pattern of the output: “It’s going in a pattern: 2, 4, 6, 8.”

Someone says aloud, “How do you count by twos?”, apparently meaning to say “what rule would yield a pattern that increases by twos?”

Maria conjectures that the rule is "K K plus two", which David transcribes as "K K +2" and asks, "What is KK? K times K or K plus K?" While Maria is thinking, Cristian suggests "times two, minus two", a correct answer, although it leaves the variable implicit. We represented his answer in Table 1 as: [K] □2-2. In the classroom, David writes K → on the board while asking "K becomes….?". A couple students [possibly Joseph and Joey, once again] respond: "…K times 2 minus 2".

Going back to Maria's answer "K K plus 2", David pursues the issue of the identity of "K + K" and "K □ 2". David now realizes that it may not be clear to the students that K□2 and K+K are interchangeable. He pursues the issue a bit, using N as a variable, but there is no convincing evidence that the students truly accept the identity, n□2 = n+n, in written or spoken form.

Group 4 (Mathew): [N] + 50 - 20

On the basis of the input output data (see Table 1) Cristian conjectures that Mathew's rule is N + 8 + 22. Others take the rule to be N +30. Matthew states his rule as "+50 –20" yet seems comfortable with the mapping, formulation, "N → N +50 –20", encouraged by David. Once again, there is a discrepancy between the maker's and conjecturers' expressions. David tries to argue that the various formulations express the same underlying rule because they can all be simplified to N +30. Students may not be fully convinced by his points, but in a sense they are being encouraged, through this and other examples, to accept the general notion that expressions that look different may be interchangeable.

Group 5 (Cristian): C□3+2-4+5

Cristian, wishing as he typically does, to provide the class with a very challenging problem, suggests using –1000 as the initial input, yielding –2997 as output. Only a couple of students are following the discussion at this point; it is late in the class and a negative input is a bit strange for them. After going through a number of input-output pairs, students suggest N□3, N□4+1, and Y□3+3 as possible answers. The final conjecture is consistent with Cristian's rule, although, once again, it is expressed in a different form. Since this discussion is a bit rushed (class is ending and David wants two remaining groups to at least state their rules), it is not fully clear whether some students believe that discovering the rule requires using the same letter adopted by the rule-makers.

The precise letter chosen *was* an issue in other classrooms, as the following dialogue from another class shows, after a discussion in which a student conjectured that the rule was A → A□5 –3.

(Ruler-makers come up to the front and write on the chart paper: K□5-3)

Teacher: Okay. They're saying that their rule is K□5-3. Is that the same thing [as A□5-3]?

Student: That's what I said!

[Erica points at the letter "A" in Albert's iteration of the rule]

Erica: Yeah, but the letter!

Student: The letter’s different!

Teacher: Does it matter if you start with a K?

Assorted students: No! No!

Teacher: And the K becomes K times 5 minus 3? Is that the same as doing this? (pointing to Albert’s rule)

Students: Yes!

Teacher: So they’re really the same rule. (To Erica) So I would say that Albert solved it, right?

Erica: Yeah. Paul got it right too, he said the same thing.

By the end of this interaction, Erica accepts the teacher's statement that Albert got it right, and adds that Paul, another student who concurred with Albert, is right as well. At issue was the idea that letters in algebraic expressions are arbitrary placeholders.

DISCUSSION

There was some evidence that third grade students from an urban public school with a prior background in early algebra activities based on functions and modeling could begin the transition from semantically driven to syntactically driven algebraic reasoning. We would hope to see students taking part in more prolonged and in depth debates about equivalent functions and identities. Furthermore, although students may initially find persuasive the fact that two rules produce the same output from a set of input values, eventually they need to abandon this approach and move towards *proving* the functions are equivalent. The discussion about □4 □5 simplifying to □20 exemplifies this sort of shift. But graphing the data may prove useful in lending meaning to the rules and (dis)proving their equivalence. Ultimately, we want students to be able to operate on equations in ways that preserve the solution set without having to resort to thinking about the original situations that gave rise to the equations. This does not mean that students should abandon, once and for all, semantically driven reasoning, for it may prove useful in other contexts, even some that entail the use of advanced mathematical reasoning. We look forward to encountering additional research that explores the transition and tension between semantically driven and syntactically driven mathematical reasoning.

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BUILDING THEORIES: WORKING IN A MICROWORLD AND WRITING THE MATHEMATICAL NOTEBOOK

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In the framework of a long term teaching experiment we present an Educational approach based on the use of a dynamic geometry software and a symbolic manipulator. Here we present the general ideas of the followed approach focusing on how meanings can originate from phenomenological experience and evolve under the guidance of the teacher. In particular we will focus on meanings related to the ideas of theory, axiom and theorem.

INTRODUCTION

The research project, this paper will report on, started some years ago in the framework of a long term teaching experiment, which is to be considered a "research for innovation": action in the classroom is both a means and a result of the evolution of research analysis (Bartolini Bussi, 1996 p. 1). One of the main objective was to investigate the feasibility of a teaching approach centred on the use of the microworlds (Cabri-Géomètre and L'Algebrista), and aimed at developing theoretical thinking in both Geometry and Algebra (Mariotti, 2001, 200; Cerulli & Mariotti, 2002). Despite the differences between Algebra and Geometry teaching, a common educational approach was used, on which we are going to discuss: some aspects will be considered and some examples will be presented,

MICROWORLDS AND SEMIOTIC MEDIATION

The teaching experiment was carried a out and is still in progress at the 9th and 10th grade, level; it has been designed and developed within the vygotskian theoretical framework with particular reference to the *notion of semiotic mediation*. Given an artefact it can be used by the teacher to exploit communication strategies aimed at guiding the evolution of meanings within the class community ; this may be also the case of the computer which can be used by the teacher in order to direct the learner in the construction of meanings that are mathematically consistent (Mariotti 2002).

Our approach is based on the following general hypothesis: "*Meanings are rooted in the phenomenological experience (actions of the user and feedback of the environment, of which the artefact is a component), but their evolution is achieved by means of social construction in the classroom, under the guidance of the teacher*" (Mariotti 2002). Thus an artefact can be a source for the construction of *meanings* by its users, but consistency with Mathematics is not a priori guaranteed and needs to be built under the guidance of the teacher. As a consequence, activities within a microworld need to be interlaced with other social activities guided by the teacher in order to reach the construction of the mathematical meanings she is aiming to.

Based on these assumptions our approach is organised in the following cycle of activities:

1. Problem solving activities within and outside the microworlds: this is the field of phenomenological experience where we assumed meanings to be rooted.

2. Production of reports (written or oral) concerning problem solving activities: students' experience is fixed into signs on which collective discussion will be based.
3. Collective discussions, i.e. Mathematical Discussions according to the definition given by Bartolini Bussi (1998): starting from the produced reports the teacher tries to guide the class in the construction of socially shared meanings, consistent with didactical aims.
4. Production of reports concerning collective discussions: the results achieved in collective discussions become part of the class culture, and as such are expressed and fixed into written text, that may serve as a basis for future activities.

This cycle describes only the general structure of the teaching sequence and focuses on the main aspects we want to discuss in this paper. In particular the articulation between experiences centred on activities within the microworld and experiences centred on semiotic activities, consisting both in producing and interpreting texts.

WORKING IN A MICROWORLD AND WRITING A NOTEBOOK

The two microworlds share interesting features which according to the shared Vygotskian framework, are similarly exploited both the Geometry and the Algebra teaching experiments.

1. objects and commands can be thought as external signs of the fundamental elements of a corresponding mathematical theory (Geometry or Algebra).

For instance, basic *tools* are signs of *axioms* and *definitions of a Theory*; *new tools* may be introduced using a specific command (Macro construction in Cabri, Il Teorematore – i.e. Theorem Maker in L'algebrista); such new commands become signs of *theorems*;

2. actions within the microworld correspond to fundamental metatheoretical actions, concerning the construction of a theory.

For instance, *adding new buttons* to those already available corresponds to the meta-theoretical operation of *adding new theorems* to a theory. In the case of Cabri it is possible to create macros that synthesise geometrical constructions and that can be used at any moment. In the case of L'Algebrista it is possible to create new buttons representing equivalence relationships between algebraic expressions and that can be used at any moment by the user in order to transform an expression into another one.

Due to the described feature (for more details see Cerulli & Mariotti, 2002 Mariotti, 2001) L'Algebrista and Cabri result to be good potential environments for phenomenological experiences concerning the production and the use of theorems. Furthermore, they offer the possibility to experience the act of adding commands to the software. In other terms, once a semiotic link with mathematics is built (see Cerulli,2003), the two microworlds make it possible to directly experience the development of mathematical theories by proving and adding theorems, through the effective operations of creating and adding new commands.

Together with Cabri and L'algebrista, another specific tool characterises our experimentation: the *notebook* (ital. "quaderno di classe"). Each pupil is asked to edit a *notebook* where any result, discussed and socially accepted in the class, will be reported. In particular, it contains the list of the axioms and theorems (either in algebra or geometry) of the theory the class is working with, and when a new theorem is produced it is added to the list. Thus the notebook may be considered a representation of the culture and the history of the class, where the elements of the theory are fixed into ordered sequences, so that both the elements and their logic relationships are represented.

Collective discussions, edition of the *notebook* and writing reports, are different kinds of verbalisation activities. In the limits of this paper, we cannot carry out a detailed analysis of the dynamics between such different activities, in particular, taking into account the different registers (Duval,1995); in the following we may refer to all of them using the generic term "verbalisation activity" in order to distinguish them from “practical” activities taking place within Cabri and L'Algebrista. The reason why, within our approach, we use both activities in the microworlds and verbalisations is to be found in our basic theoretical hypotheses. In fact, within both the microworlds it is possible to realise phenomenological experiences concerning some aspects of mathematical activity, which are not so easily experienced in other environments. For instance, the fact that commands are signs of theorems and axioms makes it possible to use them as instruments. Such an instrumental approach to theorems gives proving activities a practical flavour, that it is impossible to be obtained otherwise (Cerulli 2002). In other terms, these two microworlds may offer a very rich phenomenological experience to pupils, in order to build specific mathematical meanings.

On the other hand microworlds put strong constrains (on purpose!) on the actions the user can perform, and thus also on the way he/she can express him/herself. Thus it seems important to have other environments with less constrains and more familiar to students. Furthermore, within Cabri and L'Algebrista, communication occurs between user and the machine, and is characterised by a rigid set of signs: the production of a new sign might then be inhibited. This could be an obstacle from a Vygotskian perspective, where the production of new signs is assumed to play a key role in the production and evolution of meanings, as it permits communication and involvement of new meanings into discourses.

For these reasons, we based our teaching experiment also on verbalisation activities. In fact, on the one hand they guarantee more expressiveness, on the other hand they facilitate the production of new signs to be used and shared in the social discourse, leading to production and evolution of meanings.

Once a practice is verbally expressed, it is possible to talk about it, and once the culture of the class is fixed in a notebook, it is possible to talk about it and eventually to compare it with what is written in the mathematics textbooks.

STRATEGY TO GUIDE THE EVOLUTION OF MEANINGS

According to our hypotheses, the meanings, arising from phenomenological experiences within microworlds, have to evolve, under the guidance of the teacher, towards the mathematical meanings the teaching/learning activity aims at. In our teaching experiments, the main structure of class activities can be schematised as in figure 1. Meanings originated in the phenomenological experience are shared within a collective discussion, fixed in the sets of command of Cabri and L'Algebrista and then reported in the personal *notebook*. Starting from this general idea we may consider the two different cases of axioms and theorems.

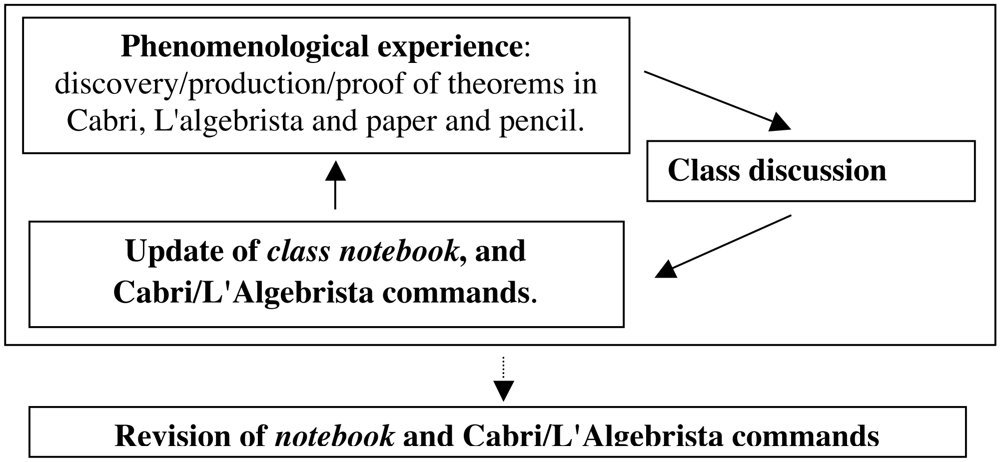

Figure 1 The main structure of the class activities

The case of axioms.

One possible way to introduce axioms is to begin working within a microworld: when it is firstly approached by the student, it presents a ready made set of commands. Such commands are *given* and can be used to work within the microworld. Thus the student *is faced with* a *given* set of commands that *are the only means of action* within the microworld, and that are actually *used to accomplish specific tasks*. Such an experience, under the guidance of the teacher, is then verbalised and socialised through a collective discussion, aiming at the formulation and the acceptance of a set of axioms, directly related to the given set of commands. Finally, each axiom is fixed into a statement on the *notebook*. Thus, at the end of this cycle, one obtains a set of commands in the microworld, a set of axioms in the culture of the class, and a set of statements in the notebook; furthermore, the fact that axioms are generated from commands, and statements from axioms, constitutes per se a link between them and may foster the idea that commands, and statements, are both signs representing axioms.

The case of a theorem.

Despite their differences, Cabri and L'Algebrista both allow the user to *create a new command*, *using given commands*. Starting from the new command and the sequence of actions producing it, a new *theorem*, with its *proof*, may be introduced in the culture of the class through verbalisation and socialisation, via collective discussion. The theorem is then fixed in the *notebook* as a statement together with a sequence of signs representing its proof.

Once introduced, theorems and axioms (and corresponding commands) can be used to accomplish new tasks, but their status in the culture of the class is different, as the processes generating them. Axioms originate from ready made commands, whilst theorems originate from command built on the commands already available. The dependence relationship, stated between new commands and the commands used to create them provides an operational referent to a logical structure in the organisation of the theory, as it is collectively built by the class.

When axioms and theorems are reported on the *notebook*, and new commands inserted in the microworlds, they may result to be ordered chronologically, however their different status, and the dependence relationships, may not always be evident. The activity of revising the notebooks gives the opportunity of reflecting and organising the set of axioms and theorems following their logical relationships.

The notebook (personal, but based on shared productions) together with the sets of commands of the microworlds, represent the "culture" and the history of the class. As a consequence, updating and revising them means to update the class culture. This is certainly a meta theoretical activity, corresponding to the construction of a mathematical theory; it may be interpreted as a phenomenological experience that raises meanings, that are then to be developed under the guidance of the teacher. The revision of the *notebook* (and updating of the corresponding set of commands) has to be interpreted from this point of view: it involves class discussions and writing of reports, and focuses on analysing the culture that has been produced along the history of the class.

In the following section we are going to discuss three example showing traces of the internalisation of the previous basic aspects, as a consequence of social activities.

THE STATUS OF AXIOMS AND THEOREMS: SOME EXAMPLES

The following examples are drawn form the data of our teaching experiments; the first concerns the proof provided by a pupil, the second an episode during a collective discussion and the last an excerpt from a report on a collective discussion.

Sum between monomials. After an activity within L'Algebrista and a collective discussion the theorem of the sum of monomials was introduced. Pupils are required to prove, in the paper and pencil environment, that "13*m+m*17=30*m". Elena, although not explicitly asked, gives two different proofs of the statement (fig. 2),. The first proof is produced using only axioms ("proprietà"), while the second one is produced using also the mentioned theorem, that she calls "Teorema 2". At each step the pupil indicates what axiom or theorem has been used to transform the expression: "com" stands for commutative property; "dist" stands for distributive property; "bottone di calcolo" stands for "button of computation", a command of L'Algebrista that executes only sums between numbers. Signs such as "bottone di calcolo" and the practice of underlying expressions show how symbolic manipulation, and proof of equivalencies between expressions are rooted in phenomenological experiences that take place in L'Algebrista (Mariotti &

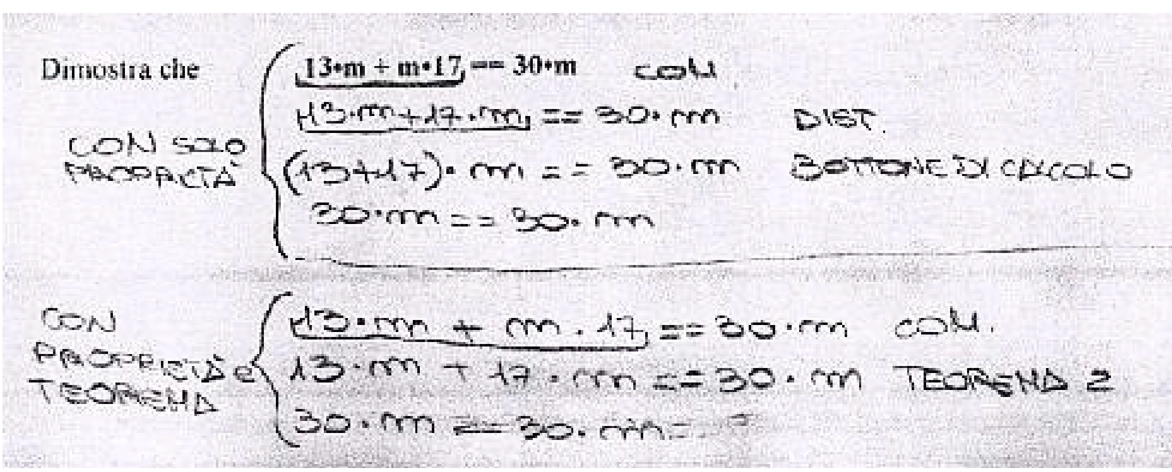

Figure 2 Elena's proofs

Cerulli, 2001). Finally, the fact that Elena writes "Teorema 2" originates in the social practice of the class of giving names to theorems and ordering them chronologically.

The Axiom/Theorem. In this episode the teacher (T) begins the lesson by asking pupils to recall what they said, 3 months earlier, about equations. At that time, they discussed the statement "**A=B ←→ A–B=0**", now she wants to start from this point in order to introduce the standard principles to solve equations.

Excerpt1

1. T: So, the first question is, do you remember what we have been doing at the end of last year? What did we focus on?
2. Tcl: The axiom theorem *(ita.: assioma teorema)*
3. Cri: axiom theorem one

[…]

6. T: What is it?
7. Tcl: if A is equivalent to B then A minus B is equivalent to zero.
8. T: come to write it *(on the blackboard)* and then explain why we called it axiom theorem

[…]

12. *Tcl writes on the blackboard:* a == b <=> a – b

[…]

14. T: do you remember why did we call it axiom theorem? Is it normal to call something "axiom theorem"?

[…]

20. Bzc: we didn't **know if…it was proved**, we took it as an axiom, last year, but **if later we are able to prove it** … we left it undecided.

In this activity, the history of the class becomes the source a new discussion. There is an element of the theory, which the class community decided to acquire and use even if its status it not clear, they "left it undecided" [20]; for the moment they take it as it is, but they know that in future they may go back to discuss its status: "but if later we are able to prove it…" [20].

The theorem of the bisector. After a first sequence of activities , the teacher sets up a collective discussion with the aim of revising the pupils' personal *notebooks*. From the comparison of the pupils' notebooks the teacher guides a mathematical discussion: the objective is that of ordering the sequence of the theoretical elements, as they are reported in the notebooks, and at giving them the right status: are they axioms, theorems or definitions? After the discussion each pupils is asked to write a report on such activity. During the discussion some time was devoted to the construction of angle bisector and the proof of the corresponding theorem called "*bisector theorem*" . In particular, different proofs were proposed, based on the application of different theorems. Trace of this part of the discussion can be found in the report of a pupil, and witnesses that the pupil Stefano writes:

*We then switched to examine the proof of the bisector theorem, one of my classmates stated that the bisector theorem could be proved also with the isosceles triangle, but to do that we would have needed to have the last theorem concerning the perpendicular. If I say that, even having the theorem, we couldn't use it, **it doesn't mean that we are fool** but simply that when we began [the proof] **we didn't have it, and our means for proving were in minor quantity**.*

CONCLUSIONS

A description of the general principles of our educational approach was given focusing on how some features of Cabri, L'Algebrista, and paper and pencil, may be used in order to foster the ideas of theorem, proof and theory. The phenomenological experience originated in the described environments may be exploited to produce mathematically consistent meanings; the revision of the history of the class may be used to build meanings related to the logical structures of mathematical theories.

The given examples show how pupils reached a good control in managing theoretical and meta theoretical aspects. Our basic hypothesis that this result is due to:

- the key role played by the class notebook as a store of the "class culture" and as input for class discussions
- the control of logical and chronological organisation of class findings obtained through the revision of the notebook

A better formulation, and a verification of such hypothesis is one of the focuses of our present research.

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TEACHER INVESTIGATIONS OF STUDENTS' WORK: MEETING THE CHALLENGE OF ATTENDING TO STUDENTS' THINKING

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The purpose of this research report is to describe some of the findings from a study of teacher investigations of students' work. The intent of the teacher investigations was for teachers to interpret their students' thinking as revealed on non-routine, thought-revealing mathematical tasks, known as Case Studies for Kids (Lesh, Hoover, Hole, Kelly, & Post, 2000)[1]. This research report focuses on instances during the investigations when the teachers engaged in what the author has termed 'mini-inquiries', discussions during which the teachers addressed why their students thought about the associated case studies as they did or the teachers addressed the underlying mathematical complexities associated with the case studies. During these mini-inquiries, the teachers met some of the challenges of attending to students' thinking that Confrey (1993) and Schifter (2001) have identified.

RATIONALE FOR THE STUDY

The latest reform efforts in mathematics education in the United States stress the importance of teachers attending to and understanding their students' mathematical thinking. The National Council of Teachers of Mathematics (NCTM, 2000) states, "Effective teaching involves observing students, listening carefully to their ideas and explanations, having mathematical goals, and using the information to make instructional decisions" (p. 19). This view of effective teaching has specific grounding in several research projects, including Cognitively Guided Instruction (Fennema, et al., 1996; Franke & Kazemi, 2001), the Purdue Problem-Centered Mathematics Project (Cobb, et al., 1991), SummerMath (Simon & Schifter, 1991), the Kenilworth Project (Maher, Davis, & Alston, 1992; Maher & Martino, 1992), the Mathematics Case Methods Project (Barnett, 1998), and the work of Putnam and Reineke (1993). Collectively, these research projects have found that when teachers attend to their students' mathematical thinking, potential benefits include:

- The ability on the part of teachers to construct or select appropriate, worthwhile mathematical tasks;
- A shift from teacher-centered didactical instruction to student-centered problem-solving instruction;
- Higher levels of conceptual understandings by students without compromises in their computational performances; and
- More positive beliefs of teachers and students toward mathematics.

Despite these benefits that may occur when teachers attend to their students' thinking, Ball (1997a; 2001), Confrey (1993), and Schifter (2001) point out that focusing on

[1] Case Studies for Kids are also known as Model-Eliciting Activities.

students' thinking can prove challenging for several reasons. First, students do not always express their thinking in ways that are logical to adults. Students often present unconventional and multiple representations for thinking about a given mathematical problem. Second, although students may appear not to understand a particular mathematical concept, there may be sense in their thinking. Teachers have to de-center from their own perspective and imagine what the view of the students might be like. Third, identifying the conceptual issue that a student is currently trying to understand can be difficult when the students' thinking is illogical. Teachers need to identify the concept in order to help move the student forward in his or her understanding. Finally, teachers often tend to focus on the pedagogical aspects of a learning situation rather than focusing on the mathematics expressed by students. Paying attention to a myriad of pedagogical aspects often causes teachers to lose sight of the mathematical ideas that their students are expressing.

To address these difficulties, Ball (1997a) describes three approaches with "promise for equipping teachers with the intellectual resources likely to be helpful in navigating the uncertainties of interpreting student thinking" (p. 808). One of these approaches is investigating artifacts of teaching and learning, such as students' written work. Several other sources support Ball's suggestion for teachers to investigate students' work (Allen, 1998; Blythe, Allen, & Powell, 1999; Driscoll & Moyer, 2001; NCTM, 2001), and several teacher development projects (Katims & Tolbert, 1998; Kelemanik, Janssen, Miller & Ransick, 1997; Saxe, Gearhart, & Nasir, 2001; Schorr & Lesh, in press) have found that when teachers engage in investigations of students' work, they have the potential to gain several benefits including:

- An expanding conception of what students are able to do mathematically;
- The realization that although students' methods may appear different from a teacher's approach, students' methods may still be valid; and
- The development of abilities to interpret students' thinking in class and to make appropriate future instructional decisions.

Despite these results, what is still missing is an in-depth investigation of how the examination of students' written work influences teachers' interpretations of students' thinking. Specifically, how the individual activity of examining student work coupled with the collective interpretation of this work influences teachers' development. Therefore, this study examined a particular instantiation of teachers investigating students' work. In particular, the purpose of the study was to closely examine (a) the teachers' collective interpretations of their students' thinking and (b) the social processes (patterns of interaction and norms for interaction) that occurred during the investigations.

THEORETICAL FRAMEWORK (ORIENTATION)

The theoretical perspective for the study was micro-sociology, which focuses on the face-to-face interaction of individuals and how these individuals act in relation with one another in everyday life (Blau, 1987; Charon, 1999; Gerstein, 1987). Micro-sociologists believe the social structure for these social interactions is composed of normative interaction and discourse patterns (Berger, 1963; Cicourel, 1974; Goffman, 1967; Gumperz, 1983). The research tradition for the study was ethnography of communication

(Hymes, 1986; Saville-Troike, 1989). Ethnographers of communication strive to describe the many different ways of communicating which exist within a community.

METHODOLOGY AND ANALYSIS

Seven middle grade teachers (of students aged 12-14 years) participated in the study by engaging in five investigations of their students' work, which occurred during the 2001-2002 school year. The purpose of each investigation was to interpret students' mathematical thinking as revealed in their students' work on a Case Study for Kids.

Mathematics Tasks

Case Studies for Kids are explicitly designed to help middle school students develop conceptual foundations for deeper and higher order ideas in pre-college mathematics (Lesh, et al., 2000). The tasks are non-routine because each task asks students to mathematically interpret a complex real-world situation and requires them to formulate a mathematical description, procedure, or method for the purpose of making a decision for a realistic client. Because groups of students are producing a description, procedure, or method, students' solutions to the task reveal explicitly how they are thinking about the given situation (Lesh, Cramer, Doerr, Post, & Zawojewski, in press).

Procedure

For each of the five teacher investigations, the teachers attended two teacher workshops. At the Introductory Workshop, the teachers completed the Case Study for Kids and discussed the mathematics inherent in the activity, expected students' responses, and implementation issues. Then, the teachers implemented the case study within their own classrooms. After implementation, the teachers attended the Follow-Up Workshop, where they discussed their interpretations of their students' mathematical thinking and ultimately developed a Consensus Students' Thinking Sheet. The sheet synthesized the students' ways of thinking into three or four primary solution strategies, included examples of students' work, described the mathematics that the teachers believed the students used while invoking the solution strategies, and outlined the teachers' perception of the efficiency and the effectiveness of each of the solution strategies. Requiring the teachers to create Consensus Students' Thinking Sheets provided the opportunity to study the teachers' *collective* interpretations of their students' thinking and the *social processes* that occurred.

Analysis

To capture the teachers' collective interpretations and social processes, the data sources consisted of transcripts from the videotapes recorded during the teacher workshops and the teachers' synthesis of their students' solution strategies recorded in the Consensus Students' Thinking Sheets. For the analysis of the data, a 'grounded theory' approach was used, as described by Strauss and Corbin (1998). Specifically, the procedures used were open coding, the process of naming concepts in the data, defining categories, and developing categories in terms of their properties and dimensions, and axial coding, the process of relating categories by identifying which categories are subcategories of other categories. Initial analyses have been conducted on the data from the first, third, and fifth teacher investigations. Analyses on the data from the second and fourth teacher investigations are in progress.

PRELIMINARY RESULTS: MEETING THE CHALLENGE OF INTERPRETING STUDENTS' THINKING

Throughout the first, third, and fifth teacher investigations, the teachers engaged in 17 patterns of interaction considered to be *mini-inquiries*. Specifically, the teachers engaged in inquiry discussions during which they addressed rationales for why their students thought about or interpreted the case studies as they did or they addressed the underlying mathematical complexities of the case studies. For 16 of these 17 mini-inquiries, the teachers met some of the challenges of attending to students' thinking, as identified by Confrey (1993) and Schifter (2001). Specifically, for seven of the teachers' mini-inquiries, the teachers de-centered from their own perspective of the case study and considered how their students viewed the case study. For three of these seven instances, the teachers not only de-centered from their own perspective and considered the view of the students; they also closely looked for sense in their students' thinking when the students' thinking did not appear entirely logical. Finally, for nine of the teachers' other mini-inquiries, the teachers identified the mathematical conceptual issue with which the students were struggling or were using to approach the case study.

The following excerpt illustrates the pattern of interaction and discourse that occurred during one of the teachers' mini-inquiries. This occurred during the first investigation when the teachers were examining their students' work from the Summer Jobs Case Study for Kids. For this task, students are to develop a procedure that will enable a concessions vendor to rehire the six most productive employees from last year's nine employees. Students are provided with data for each employee about the hours worked and the money made during the months of June, July, and August for the busy times at the park, the steady times, and the slow times. During this interaction, the teachers are observed to de-center from their own perspective to consider why their students chose to average some of the data provided with the case study[2].

69 Author: Okay, any other? [The teachers are discussing the mathematics associated with one of the students' solution strategies. This question is asking the teachers if they feel there are any other mathematical skills or concepts associated with this particular solution strategy.]

70 Lauren: I think when the kids, I hate to bring this up, but I think when the kids found the averages, it seemed more realistic to them because the numbers were more, the numbers were smaller and they seemed just realistic, but while that's not a gigantic amount of money, it's just a monthly, you know amount of money, it seemed

71 Author: So, you're . . .

72 Jim: Well, I think it shortened the categories

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[2] Pseudonyms are used for the three of the seven teachers that participated in this particular interaction: Lauren, Jim, and Tom. 'Author' refers to the author of this proposal, who served as the facilitator of the teacher investigations and as the researcher.

73 Lauren: Right

74 Jim: All of a sudden you've got all these different groups of hours, different groups of money, but when you have an average, all of a sudden you've got one number.

75 Lauren: Right.

76 Jim: I think it made the data more manageable.

77 Author: Well, Lauren is touching upon something cause like on my own spreadsheet when I played with it, I just added up all the money for say Maria and all the hours for Maria and then I divided the two, so what I had was dollars per hour kind of in the sense of across the whole summer; whereas if the kids found the average per month, they'd be finding the dollars per hour per month and maybe it's easier for them to think about having dollars per hour within a month?

78 Tom: A shorter amount of time (he's shaking his head to agree).

79 Author: You know, instead of the overall amount? Maybe that's what makes it [easier]. Okay, did we get most of the math?

As illustrated by this excerpt, for most of these mini-inquiries, the teachers engaged in a particular pattern of interaction. The pattern began whenever a teacher offered an insightful comment about (a) the mathematical skills needed for a particular solution strategy, (b) an error some of the students made while using a particular solution strategy, (c) why the students thought about the case study as they did or used a particular computation (as in line 70 above), or (d) how the students interpreted information provided in the problem statement. Most of these comments were prompted by one of two things. One type of prompt (for 5 out of the 16 mini-inquiries) was when I asked the teachers if they had anything further to add to our discussions, such as whether they felt there were any remaining solution strategies or mathematics associated with the solution strategies (as in line 69 above). The other prompt (for 10 out of the 16 mini-inquiries) was the topic under discussion. Frequently, the topic under discussion reminded the teachers of something that they had observed and thereby led them to share their observation with the group.

Once the pattern began, the initial sharing of the insightful comment was typically followed by additional comments or sharing from the other teachers. Sometimes the additional comments simply provided support for the original comment, and the teachers moved on to discussing another topic. However, more commonly, the teachers made several additional comments and therefore contributed to the original comment (as in lines 72, 74, 76, and 78 above). Thus, the pattern of interaction typically consisted of successive comments by the teachers in which they built on the original comment about how the students thought about the associated case study or about the underlying complexities of the case study, frequently providing more insight into the students' thinking. The pattern usually ended when either I encouraged the teachers to return to the task of creating the Consensus Students' Thinking Sheet (as in line 79 above), when a teacher made a comment about a new idea, or when a teacher offered a comment that resolved an issue under discussion about what the students did to solve the case study or how they thought about the case study.

During this pattern of interaction, a powerful norm for interaction appeared to guide the teachers' behavior. Specifically, an expectation seemed to exist that the teachers should consider rationales for their students' thinking. In other words, the teachers appeared guided by a norm that moved them beyond explaining how the students solved the associated case study to considering possible rationales for why the students solved the case studies as they did. Additional information and detail will be provided during the research session about the teachers' mini-inquiries and findings from the analyses in progress of the second and fourth investigations.

CONCLUSIONS AND IMPLICATIONS FOR EDUCATION

The mini-inquiries allowed the teachers to address why their students thought about the associated case studies as they did and therefore to address underlying mathematical complexities associated with the case studies. Thus, the mini-inquiries engaged the teachers in a deeper level of analysis of their students' thinking than simply reporting their students' solution strategies. In addition, while engaging in the mini-inquiries, the teachers met three of the challenges of interpreting students' thinking. First, they were able to de-center from their own perspective and to consider their students' view of the case studies. Second, they were able to seek sense in their students' thinking, even when the students' thinking was not entirely logical. Third, they were able to identify the conceptual issues with which the students were struggling or were using to solve the associated case study. The findings from this study provide initial support that engaging teachers in investigations of students' work holds promise for assisting teachers with the challenges of attending to students' thinking.

However, the teachers only engaged in these mini-inquiries intermittently throughout the teacher investigations and as the facilitator of these investigations, I did not recognize these mini-inquiries during 'real-time', thereby missing the opportunity to take advantage of these occasions. Thus, some questions remain about how to increase these occasions and their power for assisting teachers with interpreting students' thinking. For example, can teacher educators facilitate teacher investigations in such a way that teachers will be more likely to engage in mini-inquiries? If so, how? What pedagogical content knowledge is necessary for teacher educators to recognize powerful interactions such as these and to use them as starting points to further teachers' insightful examinations of students' work?

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TEACHERS' CONCEPTIONS OF MATHEMATICAL WORD PROBLEMS: A BASIS FOR PROFESSIONAL DEVELOPMENT

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This paper reports on a study of mathematics teachers' thinking in the teaching of contextual or word problems [WP] with particular focus on teachers' conceptions of WP and the relationship to teaching. The 20 participants included Grades 1-12 preservice and inservice teachers. Data consisted of interviews and classroom observations. The findings indicated 8 ways in which the teachers conceptualized WP, e.g., WP as object and experience, and a model of WP as a nesting of mathematics and social contexts. These conceptions played a significant role in framing their teaching of WP in terms of 4 teaching perspectives, including a paradigmatic and a phenomenological approach. Implications for teacher development based on the findings are also discussed.

Recent reform recommendations in mathematics education (e.g., NCTM 1989, 2000) assign a significant role to problem contexts in developing meaning for mathematics and to a problem-solving perspective of teaching and learning mathematics. Implementing such recommendations suggests an increase in importance in the use of a range of contextual problems or word problems [WP], "routine" or "non-routine", in the classroom. This paper considers the teacher as a basis for understanding the teaching of WP. The paper is based on a 3-year project that investigated teacher thinking in the teaching of WP. In particular, it reports on inservice teachers' conceptions of WP, the relationship to their teaching and implications for professional development.

BACKGROUND AND THEORETICAL PERSPECTIVE

It would seem that, given their long history, WP should be easily recognizable and easily defined. But the definition/conceptualization of WP is not always clear-cut. For example, some consider WP as including problems normally in symbolic mode expressed in words, e.g., What is the sum of 8 and 5? Others consider them to be only those that are "story problems". This is reflected in the different ways WP are described in the literature, although the latter case tends to be the preferred view.

Leacock (1910), in his discussion of "the human elements in mathematical word problems", described WP as "short stories of adventure and industry with the end omitted" (p. 118). More recently, Verschaffel et al. (2000) offered the following:

> Word problems can be defined as verbal descriptions of problem situations wherein one or more questions are raised the answer to which can be obtained by the application of mathematical operations to numerical data available in the problem statement [p. ix].

Gerofsky (2000) in her study of WP from the perspective of genre theory concluded:

> Literary analysis of word problems suggests that they are *like* religious or philosophical parables in their non-deictic, "glancing" referential relationship to our experienced lives, and in the fact that the concrete images they invoke are interchangeable with other images without changing the essential nature of the word problem or parable [p. 131].

WP have also been described in terms of the structural components that characterize them. Verschaffel et al. (2000, pp. x-xi) summarized these components as:

> The mathematical structure: i.e., the nature of the given and unknown quantities involved in the problem, as well as the kind of mathematical operation(s) by which the unknown quantities can be derived from the givens. The semantic structure: i.e., the way in which an interpretation of the text points to particular mathematical relationships … The context: What the problem is about … The format: i.e., how the problem is formulated and presented.

When WP are viewed as "genuine problems", they are also associated with the position and state of problem solver required to solve them, e.g., the problem solver wants something but does not know immediately how to get it.

Studies on WP have traditionally focused on the learner to study cognitive and affective factors that aid or hinder his/her performance as a problem solver. For example, studies on arithmetic WP (Carey, 1991; Cummings, 1991; Debout, 1990; Reed, 1999; Verschaffel et al., 2000) have looked at the mathematical and linguistic structure of these problems in relation to the children's performance; factors that affect the difficulty of the problem for children; strategies and methods children use; the errors children make in their solutions, and children's suspension of sense making in doing WP. A similar situation has existed for studies on high school algebraic WP where, for e.g., the focus has been on students' errors and methods in the translation process (Crowley et al., 1994; MacGregor & Stacey, 1993; Reed, 1999). Given this focus on the learner, studies on WP have generally ignored the classroom teacher. Thus, while these studies have enhanced our understanding of important issues associated with the learning of WP, they offer very little on the teacher and her/his role in teaching WP. This is likely to be a significant limitation in our understanding and means of improving the teaching of WP. The study in this paper is intended to make explicit aspects of teachers' thinking and classroom behaviors that frame their teaching of WP.

The study, then, is framed in the context of WP and in the theoretical perspective of teacher thinking in which teachers are viewed as creating their own meaning to make sense of their teaching, i.e., a constructivist orientation of knowledge construction. The importance of researching the teacher is associated with the view that teachers are the determining factor of how the curriculum, mathematics in this case, is taught. This validates the importance to learn from teachers what they do and how they make sense of what they do in the classroom. This is reflected by the increased focus on researching the mathematics teacher in recent years. There is a growing body of literature on mathematics teachers' content knowledge, beliefs, conceptions, classroom practices, learning, professional development and change (e.g., Lampert & Ball, 1998; Chapman, 1997; Fennema & Nelson, 1997; Leder et al., 2003; Schifter, 1998; Thompson, 1992; Tzur et al, 2001). These studies have provided us with insights on, for example, the relationship between beliefs and teaching, deficiencies in teachers' content knowledge, and the challenges of teacher education and change. However, ongoing research to understand the teacher's perspective of specific topics like the teaching of WP is important as we try to reform the teaching of mathematics.

RESEARCH PROCESS

The 3-year study on teacher thinking in teaching WP followed a humanistic research approach (Chapman, 1999) framed in phenomenology (Creswell, 1998), i.e., the focus is on the participants' meaning, what they value, and how they make sense of their experiences. The participants were 20 teachers from different local schools. They included 6 pre-service and 14 in-service teachers at elementary, junior high and senior high school levels. The main criterion for selecting the teachers was willingness to participate. However, a subset of the Grades 7-12 inservice teachers were included because in addition to satisfying this criterion, they were considered to be excellent teachers in their school systems. All of the participants were articulate and open about their thinking and experiences with WP.

The main sources of data for the study were open-ended interviews, classroom observations and role-play (Chapman, 1999). Role-play scenarios allowed the teachers to act out, instead of talk about, a situation, e.g., presenting a WP to the class. Interview questions were framed in a phenomenological context to allow the teachers to share *their* way of thinking and to describe their behaviors as lived experiences (i.e., stories of actual events). The interviews examined the participants' thinking/experiences with WP in three contexts: (i) past experiences, as both students and teachers, focusing on teacher and student presage characteristics, task features, classroom processes and contextual conditions, (ii) current practice with particular emphasis on classroom processes, planning and intentions, and (iii) future practice, i.e., expectations. The interview did not suggest particular attributes of WP to talk about. Questions were often in the form of open situations, e.g., telling stories of memorable, liked and disliked classes involving WP that they taught, giving a presentation on WP at a teacher conference, and having a conversation with a preservice teacher about WP. Classroom observations focused on the teachers' actual instructional behaviors during lessons involving/related to WP. Special attention was given to what the teachers and students did during instruction and how their actions interacted. Complete teaching units over a 2-week period were observed for each teacher. Post-observation discussions with them focused on clarifying their thinking in relation to their actions.

The data (audio-taped transcripts and field notes) were thoroughly reviewed by the researcher and two research assistants working independently to identify attributes of the teachers' thinking and actions that were characteristic of their perspective of teaching WP. In particular, conceptions about WP were deduced from the data based on significant statements and actions that reflected judgements, intentions, expectations, and values of the teachers regarding WP that occurred on several occasions and in different contexts. These attributes were grouped into themes and validated by comparison of the findings by the three reviewers and triangulation of the findings from interviews, classroom observations and role-play.

FINDINGS

The findings are presented only for the inservice teachers. There were three dominant features of all of the teachers' thinking that played a significant role in framing their teaching of WP – their conceptions of WP, their perceptions of students in relation to doing WP, and their conceptions and intentions of the WP-teacher-student relationships. The focus here is on their conceptions of WP. The following eight ways of characterizing WP emerged from the teachers' thinking and classroom behaviors. All quotes in the following sections are the teachers' thinking taken from the data.

Conceptions of WP

(I) *WP as computation/algorithm*: This view is associated with the simplicity of the WP based on their transparent semantic structure, i.e., they have language that explicitly suggests the solution to the situation, e.g., “take away”, “putting together”. One high school teacher explained, “They're extra, they're not necessary, they're trivial and they do little, most of the time I think to enhance a topic.”

(II) *WP as problem*: This is viewed in three ways. (a) The relationship between student and problem: E.g., “All word problems are real problems if students have not encountered them before. … I don’t think there’s anything in the problem that makes it necessarily routine or non-routine. … No problem is routine if you’ve never seen it before.” (b) The nature of problem/solution: This is viewed in terms of two situations. First, there are “problems for which students must deduce a structure to determine a solution.” Second, there are “problems for which students must impose a structure on problem to create a solution.” (c) The teacher’s intent: This relates to when and how a WP is introduced to students by the teacher, for example, a teacher could take a potentially routine WP and problematize it. So, “If they are given to students at the right stages as something beyond their level of experience at this time… [they] could be used to practice their problem solving skills.” “If you want it to be a problem solving type of question, it’s all in how you present the question.”

(III) *WP as Enigma*: This view is associated with WP students cannot relate to contextually and/or mathematically. Such WP are “intimidating”, “threatening”, and can erode “students’ confidence”. One high school teacher explained, “You are fearful of those problems because you don’t understand where they’re coming from. ... So it really becomes a problem because you can’t make sense out of the wording.”

(IV) *WP as object*: This view treats WP as consisting of universal properties independent of the student. The key idea is that a WP “is more of a declarative statement” and has a unique or pre-determined interpretation of the mathematics and semiotic structures/ contexts established by the author or equivalent authority (eg. the teacher) of the WP. The goal of the student is to uncover/identify the designated meaning and solution of the WP.

(V) *WP as contextualized mathematics*: This view treats WP as a way to frame mathematics and “not seen as a separate topic”. The WP “should be done with every topic” and “not [the] end of a unit” or “not [as] a separate unit”. They should form the “basis for presenting each concept” and be intertwined with other concepts. “You always introduce a new concept or idea in a context of a WP.” “I come to realize that everything about mathematics is framed within word problems.” “It should be something that is integrated throughout the year and throughout each of the lessons.”

(VI) *WP as experience*: This view considers WP in terms of a phenomenological relationship between WP and student – e.g., a lived experience, real, dependent on student, linked to intention/interest/value. The meaning of the WP is personally determined and justified, i.e., it is dependent on the student and not the author of the WP. The meaning is what it calls forth in the mind of the student, the particular association or images it excites. In order for students to accomplish this in the context of a positive experience, the WP should: “Capture their attention.” “Invite them, intrigue them and prod them to want to solve it.” “[Be] the students’ story.”

(VII) *WP as tool*: This view deals with the utility of WP for students' learning. In this context, there are two levels of WP – (a) those that are: "A means to apply concepts or practice a skill they have seen most recently in class." (b) Those that are a means of exploring new mathematics and fostering mathematical thinking, e.g., "to get the kids to handle a new situation where it does not seem like anything that we've done before."

(VIII) *WP as text*: This view considers WP as conveyors of knowledge. For example, "[A way] to transfer information to somebody else." "A way to share mathematical experience with another." "Stories from which you can extract mathematics."

In addition to these 8 ways of viewing WP, figure 1 is a schematic model of a WP that emerged from the teachers' thinking and classroom behaviors. This model considers the WP in terms of its mathematics context and perceived social contexts. The mathematics context of the WP is situated in the problem situation, which is situated in a social context, which is (or should be) situated in the student's experiences, knowledge and ability. The problem situation is a specific case of the social context of the problem and acts as a bridge between personal and impersonal aspects of the problem, i.e., together they make the mathematics context meaningful. To illustrate the model, consider the example:

> The perimeter of a pool table is about 7.8 m. Four times the length equals nine times the width. What are the dimensions of the table, in meters?

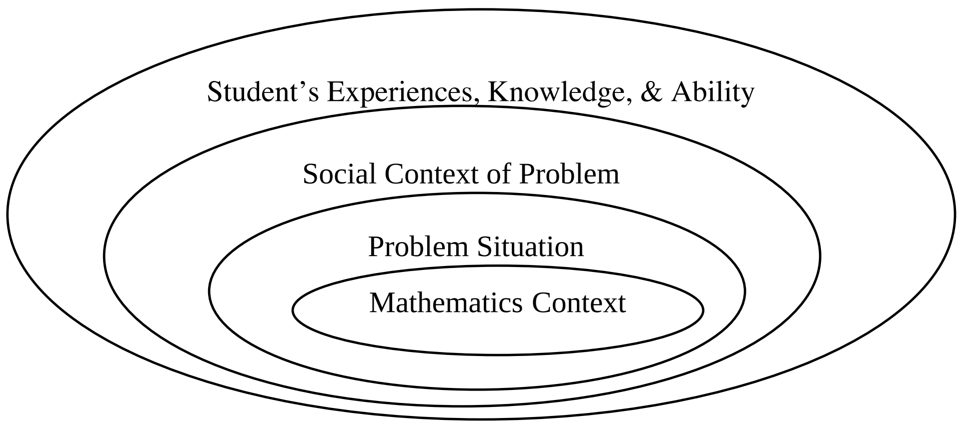

Figure 1

The social context could be, for example, playing pool, sport equipment, sports, and/or games. The problem situation is the pool table, i.e., a particular case of the social context. The mathematics context consists of two parts, "the math information" – perimeter, arithmetic/algebraic relationship and "the goal" – finding the dimensions of the table in meters. The students' experiences determine the aspects of the social context of the problem that emerge.

Relationship to Teaching

Each of the preceding conceptions of WP had an impact on teaching in terms of how it was or was not enacted in the teachers' teaching. Most of the conceptions were present in all of the teachers' thinking, but they were emphasized to different degrees in each teacher's teaching of WP. This contributed to the uniqueness of each teacher. While a description of each case is important to understand the relationship between the conceptions and teaching, it cannot be adequately provided here, thus the focus will be on those features that can be generalized to groups of the teachers.

Paradigmatic [Pa]	Paradigmatic-Narrative [PaN]	Phenomenological [Ph]	Paradigmatic-Phenomenological [PaPh]
- Universal interpretation - Mathematical solution - Social context of WP may or may not match personal experience - Suppresses students' resonance in social context of WP	- Universal interpretation - Mathematical solution - Context of WP reflects personal experience - Allows resonance in social context of WP to share personal stories; to "socialize"	- Subjective interpretation - Mathematical and social solutions of WP - Social context of WP is direct match to experience - Allows resonance in social context of WP as basis for solution to WP	- Subjective interpretation - Mathematical solution - Social context reflects personal experience - Allows resonance in WP social context to critique/ revise context, examine assumptions, rule out social solutions of WP

Figure 2. Teaching Perspectives of WP [*All items in table refer to WP*.]

There were 4 perspectives of teaching WP in which the teachers could be grouped to show the relationship of the more dominant features of their conceptions of WP on their teaching, i.e., *paradigmatic, paradigmatic-narrative, phenomenological, paradigmatic-phenomenological* [figure 2]. These perspectives are characterized by the degree to which the teachers viewed WP as object versus experience, which also influenced how the other conceptions of WP were enacted in their teaching. Figure 2 summarizes some of the key attributes of each perspective that relate to how the teachers presented WP and/or allowed students to interact with the WP in their teaching. The Pa approach emphasizes WP as object, i.e., the WP has a universal interpretation, a particular mathematical solution and is not influenced by personal social context. Thus Pa suppresses students' interaction with the social context of the WP. The Ph approach emphasizes WP as experience, i.e., what is meaningful from the student's perspective, thus multiple interpretations and both mathematical and social solutions of the WP are recognized. Ph allows students to interact with the social context of the WP.

The PaN and PaPh approaches integrate aspects of Pa and Ph. PaN is unique in that it allows the social context of the WP to provide a basis for students to share real-life stories triggered by it during teacher-led discussion of the WP, but with no link to the problem solution. PaPh allows students to critique the social context of the WP and to revise it, if necessary, to make it more meaningful to them as part of the solution process. In relation to the WP model (figure 1), Pa recognizes the mathematics context, problem situation, and student's knowledge while the three others recognize all of the components of the WP model, but in different ways. The Grades 1-2 teachers were more Ph, the Grades 3-6 teachers were more PaN, the Grades 7-12 teachers who were recommended as excellent teachers were more PaPh while the others were more Pa. Most of the PaPh teachers started as Pa and grew into the other based on their perceptions of the students, in particular what motivated them and helped them to learn. The Pa and PaN teachers taught the WP the way they solved them, while the Ph and PaPh teachers did not, but focused on the way the students were interacting with them. In general, the Ph and PaPh teachers were also more flexible, student-centered and inquiry oriented in their teaching than the others.

IMPLICATIONS FOR PROFESSIONAL DEVELOPMENT

The findings indicate that teachers' conceptions of WP are not limited to a simple definition based on structural features, but have scope and depth in a pedagogical context. The findings also offer a possible range of ways of thinking about WP that teachers could hold and suggest that there is an important relationship between the teachers' conceptions of WP and their teaching that could limit or enhance how WP are perceived, experienced and learnt by students. This has implications for teacher development in the teaching of WP and problem solving, e.g., explicit consideration of the teachers' conceptions of WP in professional development activities may be necessary particularly when fundamental changes in teaching are the desired goal.

The findings offer information of ways of thinking and teaching that could be used to enhance how we work with teachers on two levels. First, although these ways are not intended to state how things should be but how they are and could be, they could form a basis for helping teachers to broaden their perspectives of WP. Second, and more importantly, the findings offer particular structures against which other teachers could examine their own perspectives and assumptions, either through reaction against or resonance with them, to gain understanding of their thinking and teaching.

Some activities that could facilitate such use of the findings are: (1) Ask teachers to individually make up WP and to reflect on and describe what they thought about to do so. They then share and reflect on their thinking in small groups. (2) Given a WP, ask teachers to make up one with a similar structure/context and a different structure/context, then reflecting on and discussing their thinking. (3) Given a set of WP that reflects the structure of figure 1 in different ways, e.g., different mathematics context and WP situation, ask teachers to determine and discuss the nature of each WP without actually solving them, then discussing if, when, and how they will use them. In these activities, the 8 conceptions of WP and figures 1 and 2 can provide a basis for interpreting what the teachers do, to pose questions to facilitate depth in their reflection and to allow them to become aware of alternative ways of thinking of and teaching WP.

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TESTING A COMPREHENSIVE MODEL FOR MEASURING PROBLEM SOLVING AND PROBLEM POSING SKILLS OF PRIMARY PUPILS

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The study reported in this paper is an attempt to develop a comprehensive model of measuring problem solving and posing (PSP) skills based on Marshall's schema theory (ST). A battery of tests on PSP skills was administered to 5th and 6th grade Cypriot students (n=2519). The Rasch model was used and a scale was created for the battery of tests and analyzed for reliability, fit to the model, meaning and validity. The analysis revealed that the battery of tests has satisfactory psychometric properties. The identified scale verifies previous findings suggesting that a number of variables are interwoven in the problem solving process. Yet, problem representation possesses a critical role in the process. The scale also suggests that achievement in posing problems is affected by the type of given information. The findings are discussed with reference to intended uses of teaching mathematics and suggestions for further research are drawn.

INTRODUCTION

Though problem solving (PS) has always consisted an integral part in mathematics education, it was only after the evolutionary work of George Polya that researches and mathematics educators realized the importance of elaborating on the process of solving problems. As a consequence, a number of models have been proposed to describe the cognitive elements involved in that process (i.e., Anderson, 1993; Mayer & Hegarty, 1996; Verschaffel, Greer & De Corte, 2000). Most of the aforementioned models provide for general approaches and strategies for PS, irrespective of the problem type. On the contrary, ST proposed by Mashall (1995), elaborates on routine problems presenting a comprehensive PS approach. ST aims to provide solvers with a number of cognitive schemata that can be used as guides during the PS process. It also employs the idea of using simple external representations (diagrams) which act as learning aids in retrieving and enhancing cognitive schemata (Goldin, 1998; Diezmann & English, 2001).

ST focuses mainly on the structure of the problems, providing five distinct problem structures (change, group, compare, restate and vary) that capture most routine problems that are usually presented to primary students. The former three problem structures can be used to solve additive problems, while the last two are mainly used for solving multiplicative structure problems. For each situation, Marshall (1995) proposed an appropriate diagram, which is expected to help students recognize the problem situation and solve the problem. Combinations of the above-mentioned structures could be helpful in solving more complex problems (two or three step problems).

Marshall (1995) also, identified four main elements (types of knowledge) involved in the PS process: identification, elaboration, planning and execution knowledge. The first type of knowledge refers to identifying the structure of a problem, and thus, can be considered as the most important part for schema activation. The second type of knowledge refers to recognizing the details that are distinct to each schema. Selecting the appropriate diagram, placing data in it and drawing equations from it can be considered as elements of this type of knowledge. The planning knowledge refers to setting a solution plan for solving a given problem and it is usually conceived as unifying all needed decisions in order to arrive at a solution (thus, it includes elements of the two aforementioned types of knowledge). This type of knowledge is more prevalent in solving multiple step problems. Finally, the last type of knowledge includes executing algorithms.

The model described above was first introduced in upper elementary grades (4th to 6th) in Cyprus in 1998, with minor amendments. Specifically, only four problem structures were introduced, given that restate problems were embodied in comparison problems. Problem-posing (PP) activities were also included, since the significance of PP is nowadays well accepted (Silver & Cai, 1996). The present study builds on a previous study that investigated whether the first two types of knowledge mentioned in the model in relation to additive problem structures might help us form a developmental model measuring PS skills based on ST (Kyriakides, Philippou & Charalambous, 2002). In this paper, we report on testing a more comprehensive model including: (a) all problem structures, (b) one-step and multiple step problems (2 and 3 step problems), and (c) the former three types of knowledge, since execution knowledge refers mainly to executing algorithms. In this context, the main aims of this study were: (a) to develop a comprehensive model for measuring pupils' skills in problem solving and posing (PSP) one-step and multiple step problems, and (b) to collect empirical data in order to examine its validity.

THE DEVELOPMENT OF THE BATTERY OF TESTS ON PS

To answer our research questions, a battery of 48 tests on PSP was constructed guided by existing research and theory on assessment of PSP skills in Mathematics and by taking into account ST. Furthermore, a key requirement in designing the tests was its alignment with the mathematics curriculum that was operative in Cyprus. Thus, items were mainly based on ideas presented in ST as well as on activities included in the curriculum of Cyprus primary schools.

The specification table of the tests (Table 1) included fourteen levels of PSP skills related to three types of knowledge. Levels 1-3 referred to identification knowledge. Specifically, the first two levels included tasks examining the verbal identification of the schema needed for solving a problem (i.e., students were requested to identify the structure of a given problem or select a problem representing a given structure). The third level included tasks examining students' ability to select information and pose questions in order to produce problems of a given structure. The following four levels (levels 4-7) included tasks related to elaboration knowledge, which is linked to the use of diagrams. Namely items included choosing the correct diagram representing the structure of a given problem or selecting a problem that could be represented by a given diagram (4th level), placing the data and the unknown quantity of a problem in the correct position of a given diagram (5th level), setting equations for given diagrams (6th level), and posing problems based on specified diagrams (7th level). Items related to planning knowledge (levels 8-14) were similar to the above described, although they mainly referred to multiple step problems. Specifically, the items of the 8th level were similar to those of the 1st level (thus, these items included elements of the identification knowledge).

Types of knowledge	Levels	Items of the batte
Identification knowledge	*1. Verbal recognition of problems**	*1-12*
	*2. Selection of problems based on a given structure**	*13-24*
	*3. Posing problems of a given structure**	*25-40*
Elaboration knowledge	*4a. Diagrammatical recognition of problems**	*41-52*
	*4b. Selection of problems based on given diagrams**	*53-64*
	*5. Filling in data and unknown in given diagrams**	*65-100*
	*6. Setting equations based on given diagrams**	*101-127*
	*7. Posing problems based on given diagrams**	*128-151*
Planning knowledge	*8. Verbal recognition of problems** (I)*	*164-183*
	*9a. Diagrammatical recognition of problems** (E)*	*184-213*
	*9b. Selection of problems based on given combin structures** (E)*	*214-223*
	*10. Filling in data and unknown in given diagrams** (E)*	*224-263*
	*11. Setting equations based on given diagrams** (E)*	*264-338*
	*12. Posing multiple step problems** (E)*	*339-378*
	*13. Recognizing, representing and solving problems**	*152-163*
	*14. Recognizing, representing and solving problems***	*379-398*

* one-step problems, ** multiple step problems,

(I)=identification, (E)= elaboration knowledge is also prevalent

Table 1: Specification table of the tests on PS based on ST

Similarly, levels 9-12 were analogous to levels 4-7 (thus, these items included elements of the elaboration knowledge). The remaining two levels referred to setting and carrying out all needed actions to solve either one-step problems (13th level) or multiple step problems (14th level). The specification table guided the construction of a battery of tests with 398 items, representing all levels. Levels 1-7 and 13 included tasks of all four problem structures, while levels 8-12 and 14 included combinations of the four problem structures.

METHODS

The items in the final version of the battery of tests were content validated by four experienced primary teachers, two mathematics textbooks writers, and two university tutors of Mathematics Education. The "judges" of the tests were asked to mark-up, make marginal notes or comments on or even rewrite the items. Based on their comments, amendments were made, particularly where terminology used was considered as unfamiliar to primary pupils. The final version of the battery of tests (available on request) was administered to all 5th grade (1184) and 6th grade (1335) pupils from 27 primary schools selected by stratified sampling (1298 of the subjects were boys and 1221 were girls). The Extended Logistic Model of Rasch (Andrich, 1988; Rasch, 1980) was used and the data were analyzed by using the Quest program (Adams & Khoo, 1996). The data were initially analyzed with the whole sample (n=2519) for all items together. The analysis was repeated with each of the four groups (grade 5, grade 6, boys and girls) of the sample, to investigate whether the battery of tests was consistently used by each group of the sample.

FINDINGS

Table 2 provides a summary of the scale statistics for the whole sample and for each of the four groups of the sample. We can observe that for the whole sample and for each group the indices of cases and item separation are equal or higher than 0.85 indicating that the separability of the scale is satisfactory (Wright, 1985). We can also see that the infit mean squares and the outfit mean squares are close to 1 and that the values of the infit t-scores and the outfit t-scores are approximately zero. It can be claimed that there is a good fit to the model. The comparatively high value of outfit t-scores for persons can be seen as an indication of the relatively low separability of the persons scale and this can be attributed to the fact that the test was administered to children of a limited age span (only children of the two upper grades were included in the survey) and thereby the variation among their abilities was relatively low.

STATISTICS	**Whole (n=2519)**	**Boys (n=1298)**	**Girls (n=1221)**	**5^{th} grade (n=1184)**	**6^{th} grade (n=1335)**
Means (items)	0.00	0.00	0.00	0.00	0.00
(persons)	0.13	0.05	0.23	0.07	0.19
Standard deviation (items)	0.98	0.96	1.01	0.99	0.98
(persons)	0.97	1.00	0.95	0.95	1.01
Seperability* (items)	0.94	0.88	0.88	0.88	0.89
(persons)	0.86	0.86	0.85	0.85	0.86
Mean Infit mean square (items)	1.00	1.00	1.00	1.00	1.00
(persons)	1.00	0.98	1.00	0.98	1.00
Mean Outfit mean square (items)	1.02	1.02	1.03	1.03	1.02
(persons)	1.02	1.02	1.03	1.03	1.02
Infit t (items)	-0.05	-0.03	-0.02	-0.03	-0.03
(persons)	0.02	0.00	0.04	0.00	0.03
Outfit t (items)	0.03	0.04	0.05	0.05	0.04
(persons)	0.11	0.11	0.11	0.11	0.11

Separability* (reliability) represents the proportion of observed variance considered to be true.

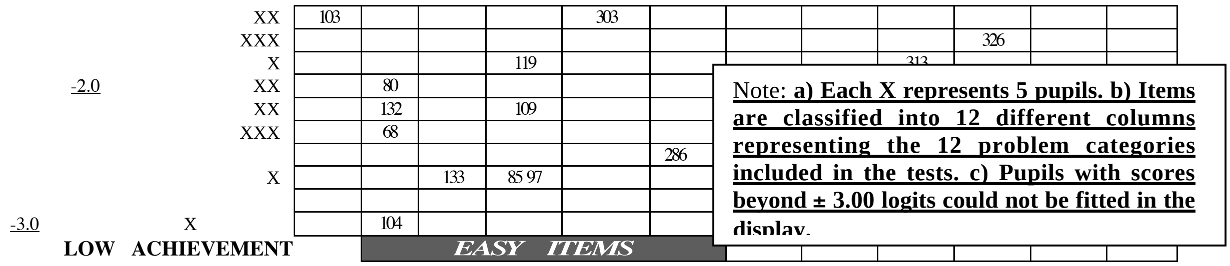
Table 2: Statistics relating to the scale for the whole sample and the four groups

Figure 1 illustrates the scale for the 398 test items with item difficulties and the whole group of pupils' measures calibrated on the same scale. The items appear in twelve columns. The first four represent the four problem structures (1=change, 2=group, 3=vary, 4=compare situation), while the remaining eight represent combinations of the four problem structures. Namely, these columns include items involving two additive structures, one additive and one multiplicative, one multiplicative and one additive, two multiplicative structures, three additive structures, two additive and one multiplicative, two multiplicative and one additive and three multiplicative structures (columns 5-12, respectively). Both figure 1 and the item fit map for the 398 items fitting the model reveal that all the items of the tests have a good fit to the measurement model.

.

		1	2	3	4	5	6	7	8	9	10	11	12
HIGH	***ACHIEVEMENT***	***DIFFICULT ITEMS***											397
3.0					87				385			375	398

											376	
X				86								
										373		
X												
X				74								
										353		
XX												
	37				364							
XXX					360 340			230				263
XX	152				390							378
XXXX				99			367				395	262
2.0 XX	89											
XXXX								250				
XXX				75 161		361			257			
XXXXXX				63		255			371			
XXXXXX				36 151	344	370					220	
XXXXX				28	209	289	347				240	
XXXXXX			32						392			222
XXXXXX				30					391 217			223
XXXXXXX	153				380			210	372		200 396	282
XXXXXXXXX						355 350	386					182
XXXXXXXXXXX	65		39				212		351	394		336
XXXXXXXXXX	90	31 35		64		288	387		237	393 354	356	358
1.0 XXXXXXXXXX	128		71			363 235		274		179 219	327	243 377
XXXXXXXXXXXXX	33 77	38		150			171		216		261 330	242
XXXXXXXXXXXXX	13 120				368		211		197		280	338
XXXXXXXXXXXXXX				159	245	381	292	270 305			180	335 333
XXXXXXXXXXXXXXXXX		5		40 76	348 225	275	294	365 170	352		281 332	
XXXXXXXXXXXXXXXXXXXX	53	81	70 107	149	164 184	341		389 312	277 318	259		
XXXXXXXXXXXXXXXX	29		158		169	175	172 192	190	177	199	221	203
XXXXXXXXXXXXX	1		83	72 112	388 249	195		174	256 316	279		
XXXXXXXXXXXXXXXXXXX	14 66			51 163	193 229 265	383			176		241	357 183
XXXXXXXXXXXXXXX	78 114	69	19	139	165	268 266				278		
XXXXXXXXXXXXXXXXXXX	41	17 45		12	189	208	251	234 254		238	328	202
XXXXXXXXXXXXXXX				52 147	269		231 293		236		181 201	
0.0 XXXXXXXXXXXXXXXXXX	25			24 1 11	382 207		191					
XXXXXXXXXXXXXXX	101		157	84 49	362 167	215	252					
XXXXXXXXXXXXXXX	141 140		27	23 160	173 187 300		232 271	349	196			
XXXXXXXXXXXXXXX	15 54	57		60 138	213	246 284	272	345				
XXXXXXXXXXXXXX	2 42	113	18	115 110	359 204	226		214		276		337
XXXXXXXXXXX	118	155	7	50	224 227	228	346			178		
XXXXXXXXXX		144	59 146	11	339 185	188 285			315	322		
XXXXXXXXXXXXXX		16	82 95	2 1 96	233 253	287						
XXXXXXXXXX				9 10 48		168				218		
XXXXXXXXXXXXX			47	61	342 267	166		194				
XXXXXXXXX	55 129			87 388	247	343 186						
XXXXXXXXXXXX			106	20 148	299	264					331	
XXXXXX			658	137	273							
-1.0 XXXXXXXXXXXX	67 130		122 124		206	206		307	319			
XXXXX	102	116		108				310				
XXXXXXXXX		143		22 123	301		291		317			
XXXXXX	43		94	62 127	304							
XXXXXX		44	145	136 121				308				
XXXX	117			125		290			314		329	
XXXXXX	79 142				298		296					
XXXX	3 91	4	46	100				311				
XXXX										323		
XX										321		



Moreover, pupils' scores range from –3.62 to +3.58 logits and the item difficulties range from -3.20 to +2.99 logits. This implies that the 398 items of the test are relatively well targeted against the pupils' measures, though a set of both more and less difficult items could be given to 19 students placed at the two opposite ends of the ability scale (12 pupils' scores were over +2.99 logits, and 7 pupils had lower scores than –3.20 logits).

The following observations arise from both Figure 1 and Table 1. Firstly, as concerns posing one-step problems (columns 1-4), items 25-40 (PP by selecting the needed data and posing a proper question to reflect problems of a given structure) are among the most difficult items of the test. In contrast, PP based on complete diagrams provides adequate guideline, and thus PP items of this type (items 128-151) turn out to be easier than items of the previous type and of many PS items, as well. In the case of multiple step problems, only the second type of PP was included (items 339-378). Figure 1 reveals that PP of this type is more difficult than solving problems of the analogous structure. There is only one exception in the 12th category (problems involving three multiplicative structures), where students had more difficulties in solving rather than posing problems.

As regards solving one-step problems, columns 1-4 reveal that the three types of knowledge cover a wide spectrum of PS abilities. At the one end of the spectrum (difficult items end) one may observe items related to the planning knowledge. This is more obvious for non-consistent problems (i.e., problems with inconsistency between their wording and the operation needed to arrive at a solution), such as items 152, 153, 159 and 161. The "difficult items end" is also occupied by items related to the 5th level (the second type of elaboration knowledge). Specifically, these items concern filling in the proper diagram in order to represent the structure of a given problem sufficiently. Items related to choice of the proper representation (items 41-64) appear somehow lower rather than the previous items, even lower to items related to identifying the structure of a given problem (levels 1-2). Items linked to setting the proper equations appear at the lower end of the scale, except of those connected to inconsistent problems (such as items 107, 111, 112, 114, 120). Finally, the distribution of items in columns 1-4 suggests that the problem structure interacts with the three types of knowledge, since change and compare problems cover a wider spectrum of abilities, in comparison to vary and group problems.

Regarding multiple step problems, columns 5-12 suggest that planning knowledge items (379-398) can be considered as lying at the hardest end of the ability scale, as in the case with one-step problems. Likewise, filling in the proper representation items (224-263) appear above items related to the identification of the problem structure (items 164-183) or to the selection of the most suitable representation (items 184-243). Moreover, items related to setting the correct equation for a given diagram (items 264-338) appear somehow below items of the aforementioned levels. Finally, the distribution of items in the two final columns suggests that problems involving more than one multiplicative structure can be considered as more difficult than those involving mainly additive structures.

DISCUSSION

The findings of the present study provide support to results of relevant studies related to problem solving and posing (Mayer & Hegarty, 1996; Goldin, 1998; English, 1997; Silver & Cai, 1996; Kyriakides, Philippou & Charalambous, 2002). Analytically, achievement in problem posing seems to be influenced by the type of given information.

Complete diagrams aid the construction of problems in contrast to PP by selecting and combining given statements. However, in the case of multiple step problems, even though pupils were provided with complete diagrams, PP activities turned out to be harder than PS items. The distribution of items in Figure 1 also suggests that a number of variables are interwoven in the PS process. The structure of the problem, the cognitive processes involved in solving problems (i.e., types of knowledge), the consistency between the wording of the problem and the suitable operation, as well as the number of needed steps for solving a problem (one vs. multiple steps) are some of the variables affecting PS achievement. However, a relatively consistent pattern concerning the type of knowledge involved in the PS process emerges from Figure 1, both for one-step and for multiple step problems. Planning knowledge items are the most difficult, as it was expected, since achievement in these items demands the presence of the previous two types of knowledge. Using the correct representation properly also appears to be a critical element in the PS process. However, the selection of the proper representation is not sufficient in the PS process. Solvers need to place the given data and the unknown quantity in the correct position to form a complete representation that will guide the selection of the proper operation(s). Indeed, the present study suggests that setting the correct equation for solving a problem is of less importance than constructing a proper representation for a given problem.

It goes without saying that teachers should help students pay attention to the construction of proper representations. Teachers should also be aware that a number of variables are involved in the PSP process. Awareness of these variables can be helpful in both designing teaching interventions for eliminating related difficulties and measuring pupils' skills in PSP. Further research is also needed in order to specify the importance of each variable in the PS process. Item Response Theory models involving two or three parameters might be helpful in this direction since discontinuities in the levels of the specification table of the test can be assessed.

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MATHEMATICAL PROOF AS FORMAL PROCEPT IN ADVANCED MATHEMATICAL THINKING

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In this paper the notion of "procept" (in the sense of Gray & Tall, 1994) is extended to advanced mathematics by considering mathematical proof as "formal procept". The statement of a theorem as a symbol may theoretically evoke the proof deduction as a process that may contain sequential procedures and require the synthesis of distinct cognitive units or the general notion of the theorem as an object like a manipulable entity to be used as inputs to other theorems. Therefore, a theorem could act as a pivot between a process (method of proof) and the concept (general notion of the theorem). I hypothesise that mature theorem-based understanding (in the sense of Chin & Tall, 2000) should possess the ability to consider a theorem as a "formal procept", and it takes time to develop this ability. Some empirical evidence reveals that only a minority of the first year mathematics students at Warwick could recognise a relevant theorem as a "concept" (having a brief notion of a theorem) and did not have the theorem with the notion of its proof as a "formal procept". A year later some more successful students showed a concept of the theorem as a "formal procept" and their capability of manipulating the theorem flexibly.

INTRODUCTION

Mathematical proof is one of the most important aspects of formal mathematics. From most mathematics textbooks we can simply see the process of a mathematical proof as the development of a sequence of statements using only definitions and preceding results, such as deductions, axioms, or theorems. Theoretically the *process* of a mathematical proof occurs when the proof is built up and looked at subsequently as a process of deducing the statement of the theorem from definitions and the specified assumptions. A proof becomes a *concept* when it can be used as an established result in future theorems without the need to unpack it down to its individual steps. I choose to focus on this sequence of proof as a process of deduction becoming encapsulated as a concept of proof in a manner that would seem natural to most mathematicians. It is noted that there are alternative theories, for example, Dubinsky and his colleagues (Dubinsky, Elterman & Gong, 1988) focus on the use of quantified statements as processes becoming turned into mental objects by applying the quantifiers. Pinto and Tall (2002), in contrast, show how some students are capable of building formal proofs by reconstruction of prototypical imagery used in thought experiments.

ORIGINAL NOTION OF PROCEPT

Gray and Tall (1994) suggested the notion of "*procept*", which was taken to be characteristic of symbolism in arithmetic, algebra and calculus, defined in the following terms:

> An *elementary procept* is the amalgam of three components: a *process* which produces a mathematical *object*, and a *symbol* which is used to represent either process or object.
>
> A *procept* consists of a collection of elementary procepts which have the same object.
>
> (Gray & Tall, 1994, p.120)

The original definition was made in the context where the authors were aware of a wide range of examples and the definition was framed to situate the examples within the definition. In this primary consideration it is a “descriptive definition”, in the sense of a definition in a dictionary, rather than a “prescriptive definition”, in the sense of an axiomatic theory. However, if we consider the definition of “procept” in a prescriptive view, it seems applicable to extend the original notion of “procept” to the notion of formal proof, which can be called “*formal procept*”, by adding the following analysis.

EXTENDED NOTION OF FORMAL PROCEPT

It should be noticed that there are three components of an elementary procept: *process*, *object*, and *symbol*. Now we can put the frame of Gray & Tall’s “procept”, particularly in the form of an “elementary procept”, on the notion of formal mathematical proof. The *symbol* is the statement of what is going to be proved (which can be a theorem). The *process* is the deduction of the whole proof. And the *object* is the concept of the general notion of proof. Therefore, a theorem, for example, which is considered as a formal procept could act as a pivot between a process (method of proof) and the concept (general notion of the theorem). It should be stressed that the individual is not considered to conceive the real meaning of a theorem until the theorem has become a formal procept. With the above interpretation we could see the role of a symbol as being pivotal not only in elementary mathematical thinking but also in advanced mathematical thinking to allow us to change the channel between using a symbol as a concept to reflect on and link to other concepts and as a process to offer the detailed steps to deduce a proof. However, an immediate argument arises. It seems that the above corollary does not always follow because even mathematicians sometimes use certain theorems without fully understanding their proofs. However, I find this viewpoint an *advantage* to our analysis, for it simply shows that such individuals are *not* using theorems as formal procepts, they only have *part* of the structure, usually the statement of the theorem which they then use as an ingredient in another proof without fully understanding the *totality* of the structure. I consider the whole notion of a theorem to be grasped when the notion of proof of the theorem is also assimilated in the individual's understanding. Some evidence here shows that only a few students understand the notion of proof as a formal procept, but the empirical research also shows that, over time, more students grasp the subtlety of the idea.

HIERARCHY OF THE DEVELOPMENT OF SYSTEMATIC PROOF

Chin and Tall (Chin & Tall, 2000) postulated a hierarchy running through the development of systematic proof, in stages consisting of *concept image-based*, *definition-based*, *theorem-based*, and *compressed concept-based.* These stages show successive compressions of knowledge in the sense suggested by Thurston (1990). The first stage, which is concept image-based sees the student having a concept image of a particular concept built from experience, but very much at an intuitive stage of development. The transition to definition-based involves the first compression. From amongst the many properties of the concept-image, a number of generative ideas are selected and refined down to give the concept-definition. During the definition-based stage, the definitions are used to make deductions, all of which are intended to be based explicitly on the definitions. Many students, however, remain in the concept-image based stage, basing their arguments not on definitions and deductions, but on thought experiments using concept images (Tall & Vinner, 1981; Vinner, 1991). Bills and Tall (1998) introduced the term ‘formally operable’ definition (or theorem), proposing that:

> A (mathematical) definition or theorem is said to be *formally operable* for a given individual if that individual is able to use it in creating or (meaningfully) reproducing a formal argument. (Bills & Tall, 1998, p.104)

Tracing the development of five individuals over two terms in an analysis course, focusing on the definition of “least upper bound”, they found that many students never have operable definitions, relying only on earlier experiences and inoperable concept images. Furthermore, it was also possible for a student to use a concept without an operable definition in a proof using imagery that happens to give the necessary information required. Thus, we already know that the development from the concept-image based stage to the compressed notion of operable definition is a difficult one for many students. Even so, they are then expected to move on to the next, theorem-based stage, when theorems that have been proved by the process of proof are now regarded as being compressed into *concepts* of proof, to be used as entities in the process of proving new theorems. For this to be fully successful, I hypothesise that students who have developed mature theorem-based understanding should possess the ability to consider a theorem as a “formal procept”. I further hypothesise that individuals with this capacity to use theorems flexibly as processes or concepts are developing a compressed concept level of mathematical thinking that enables them to think with great flexibility and conceptual power.

EMPIRICAL STUDY

In the cross-sectional probe, 277 first year mathematics students, following a course in one of the top five ranked mathematics departments in the UK, responded to a questionnaire on equivalence relations & partitions” when just having learned the topic for several weeks. Thirty-six out of these 277 students were interviewed. In the longitudinal probe, fifteen selected students answered the same questionnaire and were interviewed during the first term in their second year. Their marks for the first year study are widely distributed — three are over 80, four between 70 to 79, four between 60 to 69, one between 50 to 59, and three between 40 to 49. This presentation is focused on two questions in the questionnaire which are generally designed to examine how the students manage to apply a relevant theorem to make their deductions. The plan of the study is to obtain a global perspective of the first year mathematics students’ general understanding of some relevant theorems, then to investigate whether and how the students’ understanding improves.

“EQUIVALENCE RELATION” AT THE THEOREM-BASED LEVEL

The following question is designed to examine if the students improve their understanding from the definition-based level to theorem-based level:

> A relation on a set of sets is obtained by saying that a set X is related to a set Y if there is a bijection $f: X \to Y$. Is this relation an ***equivalence relation***?

It is necessary to note specially that the following theorem, which can be directly applicable to this question, had been taught before the topic of “relations” was introduced in the lecture:

(1) The *identity* map is a bijection.

(2) The *composition* of bijections is a bijection.

(3) The *inverse* of a bijection is a bijection.

This involves compression of the proofs of (1), (2), (3) (as processes) into useable concepts (theorems).

In the cross-sectional probe, only a small percentage of the students (13%, 36 out of 277) tried to apply the above theorem to make their deductions. Nearly a half of the students (132 out of 277) still went back to examine the definitions step by step to answer this question (they were categorised as "definition-based"). More than a half of these thirty-six students (14 out of 36) only briefly referred to the theorem without giving more detailed interpretation. That is these fourteen students could only state the theorem but seemed not able to unpack its meaning. For these fourteen students, the notion of proof cannot be considered as a formal procept yet because they did not seem to know the process (method of proof) but only the brief concept (statement of theorem). In addition, it should be noticed that, within the thirty-six interviewees (six out of these thirty-six interviewees were categorised as “theorem-based”), thirty-three expressed that they had impression of the relevant theorem. It seems to suggest that most of the students should know or, at least, have some kind of impression of this relevant theorem, even though the majority did not manage to apply the theorem to the practical question.

In the longitudinal probe, twelve out of fifteen were able to apply the theorem in the second year, whilst only three were categorised as “theorem-based” in the first year. As was found in the cross-sectional probe, the students’ concept images of this topic were not solid at that time. Although most of the students seemed to know the relevant theorem, they did not really have a clear idea how to apply it to this practical problem. JULSON (68% for his first year study) was an example offering a definition-based response (as follows) but he vividly expressed in the first year interview — “I remember I learned it [the theorem] in the lecture a couple weeks ago, but I’m sorry I haven’t put it in my head yet.”

Always bij. $f: X \to X$, $\therefore$ set $X \sim$ set X
if bij $f: X \to Y$, then bij $f': Y \to X$
if bij $f: X \to Y$ and bij $f': Y \to Z$, then bij $f'': X \to Z$ (JULSON 68%, 1st year)

Compared with their former responses, the quality of these fifteen students’ deductions seems to indicate that the notion of the theorem had become more workable in their concept images. JULSON’s recent response (classified as “theorem-based”) could offer us some evidence.

$-$ id = bij : $X \to X$
$-$ if f = bij : $X \to Y$ then f^{-1} = bij : $Y \to X$
$-$ if f = bij : $X \to Y$ and g = bij : $Y \to Z$
then gf = bij : $X \to Z$

(JULSON 68%, 2nd year)

In the second year, JULSON not only stated the theorem but also explained how the theorem can be proved (in the interview). Thus he clearly showed that the notion of proof of this theorem had become a "formal procept" in his concept image as he knew both the statement of theorem (as general concept) and the method of proof (as process).

The following quoted conversations recorded in the interview with DIAHUM might offer us some more delicate insight into how the successive moves — from informal to definition-based, then on to theorem-based conceptions — happened with the individual.

DIAHUM (48% for his first year study) gave the following response (classified as "informal definition-based") in the first year:

$a \to a$
If $a \to b$ $b \to a$.
If $a \to b$ $b \to c$ have same no. of elements
$a \to c$.

(DIAHUM 48%, 1st year)

He cleared up what he meant in his response as follows:

> I was trying to apply the definition of equivalence relation to make the answer more formal. But I don't think my answer was formal enough because I didn't really know how to apply the definition even though I can remember it. And another problem is I can't recall the definition of bijection. What I can remember is a bijection is one-to-one and onto. That means the two sets have the same number of elements (he explained later that this idea was from what he learned at A-level).

He also expressed that he knew the theorem which is directly relevant to this question in the interviews. But the theorem seemed to be something only in his understanding in a theoretical manner rather than in his intuition which can be freely referred to at any time.

In the second year, he responded in terms of the relevant theorems as follows:

reflexive: Yes since $\exists$ a bijection $X \to X$.
symmetric: Yes since if $\theta : X \to Y$ then $\theta^{-1} : Y \to X$.
transitive: Yes since if $\theta : X \to Y$ $\phi : Y \to Z$
then $(\phi\theta)(X \to Z)$.

(DIAHUM 48%, 2nd year)

Although he did not use the term "identity" to mention the bijection mapping from the set *X* to itself, he could precisely write down the composition of two bijections whilst some others mentioned it in the wrong order. In addition, he could explain the idea to prove the theorem in the interview. When being asked why he answered in this way this time, he gave the following explanation:

> Well, I think it's fairly natural for me to make the deduction like this. When I faced the question, the theorems burst upon my head and I just wrote down the proof.

DIAHUM's case seems to suggest that he cannot freely apply a formal conception until it is assimilated in his concept image as an embodiment. When DIAHUM could only recite the formal definition of equivalence relation but was still struggling with the implication of it, it is natural for him to consult the relevant ideas he learned at school to make his first deduction because they were more embodied and secure in his concept image. Having a year of time to digest all these notions, the theorem, which he only knew about before, had been assimilated into his concept image as a formal procept that he could recall intuitively in the second test.

SUB-SUMMARY

In the students' (written or oral) responses, we can see that most students seemed to apply the relevant theorem directly to this practical question in the second year whilst most of them only gave a definition-based response in the previous year. This kind of result is consistent with the successive move from definition-based conceptions to theorem-based conceptions over time during which the ideas are being used formally (Chin & Tall, 2000). From the improved quality of the students' deductions, I consider, at least for some students, the notion of proof of the theorem seemed to have become a "formal procept" in their concept images. They only appeared to know the general concept (statement of the theorem) but not the process (method of proof) of the notion of proof of the theorem before. But, a year later, some students seemed to be able to unpack the notion of the theorem to the proof process and to apply the theorem to the question more flexibly.

LINKAGE BETWEEN "EQUIVALENCE RELATIONS" AND "PARTITIONS" (AT THE COMPRESSED CONCEPT-BASED LEVEL)

Theoretically the notion of "equivalence relations" is linked to the notion of "partitions" as there is a theorem stating that "an equivalence relation can produce a partition of a set and vice versa" which is always formulated as the conclusion of the topic. The following question is asked in order to examine whether the students appreciate the idea practically.

> Write down <u>two</u> different ***partitions*** of the set with four elements, $X=\{a,b,c,d\}$. For the first of these, please write down the ***equivalence relation*** that it determines.

In the cross-sectional probe, the students' reponses to this rather easy question with only four elements in the set reveal that only few students (sixty-one out of 277) show there is a workable linkage between the two notions in their concept images. The others gave two correct partitions with incorrect or without corresponding equivalence relations, or incorrect partitions with incorrect or without corresponding equivalence relations, or totally wrong answers. However, all the thirty-six interviewees said that they knew there is a theorem linking the two notions "equivalence relations" and "partitions" together, whether they appreciated it or not. It seems fairly clear that being aware of the statement of a theorem does not mean that the theorem is operable in one's concept image. I consider that the notion of proof has not become a "formal procept" yet, since the students could only remember the statement of the theorem as general concept but did not have the access to proof as process, the method of proof. Thus they could still not apply the theorem to this practical problem in the first year.

In the longitudinal probe, there were only five out of the fifteen subjects being able to apply the idea of the relevant theorem by successfully giving two correct partitions with a correct corresponding equivalence relation in the first year, and the number increased to eleven in the second test. As to the other four students, three gave two correct partitions without corresponding equivalence relation and one even failed to offer two correct partitions without giving any corresponding equivalence relation. Please note that all the fifteen expressed that they remembered they had seen, in the lecture, the theorem which links the two notions together.

HELTON, getting 61% for his first year study, can be a representative of those who failed to offer a correct response before but solved the question successfully in the second test. In the interviews, he expressed that he could just remember the theorem without really understanding the meaning of it. But when preparing the examination, he studied how the theorem is proved and then grasped the idea of the theorem. Thus he could simply solve the problem in the second year. However, MAUHAM (71% for her first year study), offering two correct partitions without giving the corresponding equivalence relation twice, is someone who confessed that he only recited the statement of the theorem and had no idea how the theorem can be proved.

SUB-SUMMARY

The result of this question appears to parallel the former question in many instances. All the students sensed the relevant theorem linking the two notions together but only a few could practically apply the theorem to the question in the first year. A year later, some students' understanding had progressed to reach a more mature theorem-based level. The theorem was no longer a "general concept" only but also a "process" which suggests the method of proof to make the whole notion of proof of the theorem as a "formal procept" in their concept images. However, only trying to recite the statement of a theorem without understanding the notion of proof of the theorem is not helpful for improving the student's understanding.

DISCUSSIONS AND CONCLUSIONS

The proceptual encapsulation in advanced mathematics seems to be slightly different from that in simple arithmetic (Gray & Tall, 1994), in which pupils appear to build up the notion of proceptual structure from encapsulating various processes, to obtaining the concept, then on to forming the procept of a symbol. The empirical data of this presentation reveal that most students, at the university level, seem to have the product (the statement of a theorem) first, then to develop the notion of proof if possible. There is evidence that being stuck in processes of calculation seemingly prevents pupils from obtaining the concept (e.g. Blackett, 1990, Gray & Pitta, 1997). However, the use of the computer to carry out the process, and so enable the learner to concentrate on the product, significantly improves the learning experience (Gray & Pitta, op. cit.; Gray & Tall, op. cit.). This kind of evidence suggests that concentrating on the product first, then to develop the notion of procept is possible and also helpful for improving student's learning.

The notion of formal procept is applicable to trigonometry and calculus. Many trigonometric formulae and theorems, for example, Mean Value Theorem and

Intermediate Value Theorem, in calculus can be seen as formal procepts. If the students only recite the product (the statement of the theorem) without understanding the idea of the proof, they could not be able to apply these formulae or theorems to solve practical problems flexibly. Besides, when more and more formulae and theorems are learned, the less able students will become trapped in reciting all these products which increase the burden upon an already stressed cognitive structure.

The empirical evidence presented in this paper gives us confidence to make a conclusion that the notion of procept of Gray & Tall can be extended to advanced mathematics. At the beginning, most students just have the product (the brief notion of the theorem) in their concept images only. But they cannot grasp the essence of the theorem and have more flexible thinking until they perceive the notion of proof of the theorem. Therefore, the ambiguity of process and product represented by the notion of formal procept also provides a more natural cognitive development at the university level which gives the students enormous power to develop more flexible mathematical thinking.

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A METHOD FOR DEVELOPING RUBRICS FOR RESEARCH PURPOSES[1]

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A methodological approach that emerged during the design of task-specific research rubrics to code large sets of open-ended survey data fills the void in scholarship about developing rubrics for research purposes. A brief rationale for using this method rather than other, often-used, data analysis methods is provided, with a description of the methodology, using an example to support the description. Finally, recommendations are included for those who plan to undertake the task of rubric development for research purposes.

RATIONALE AND CONTEXT

Beliefs are a central construct to those interested in research on teaching and learning mathematics. At least as difficult as defining *belief*, is devising instruments that operationalize a definition. We describe the process we used to develop research rubrics to code large sets of belief-survey data, but we first state the four components of beliefs that guided us in the development of our instrument: Beliefs influence perception; they are not all-or-nothing entities; they are context specific; and they are dispositions toward action (for more information about the beliefs we assessed, see Ambrose, Philipp, Clement, and Chauvot, 2003)

Mathematics education researchers have typically used case-study methodology to analyze teachers' beliefs related to mathematics teaching and learning (e.g., Clarke, 1997; Cooney, Shealy, & Arvold, 1998; Raymond, 1997). This approach provides rich descriptions of the beliefs of a small number of preservice elementary school teachers (PSTs), typically no more than four in one study. Its strength is that it relies on thick data sets that include multiple observations, interviews, and surveys that are collected over a long period of time. The findings from this research inform the community about details of the conceptions of small numbers of teachers, with conclusions that have multiple data points to support findings. These rich reports of a small number of cases are important for theory building, but theory testing often requires tools for studying larger groups of individuals.

One means for studying the beliefs of large groups of individuals is through Likert scales, which are often used in a pre- and post-testing to measure change before and after some treatment (e.g., Bright & Vacc, 1994). Likert scales are used in many fields, and are widely accepted research instruments. However, the Likert scales typically used in mathematics education use statements that are decontextualized, so that results are difficult to interpret, while the voices of the individuals go unheard, thereby making inferring what perceptions may have guided the responses difficult (for more information

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about our rationale for not using Likert scales, see Ambrose, Philipp, Clement, and Chauvot, 2003).

As part of our large-scale research and design project, Integrating Mathematics and Pedagogy (IMAP), we needed to measure beliefs, about mathematics and children's learning of mathematics, held by large groups of PSTs. Because we were interested in studying changes in beliefs of large numbers of PSTs, case-study methods were neither manageable nor appropriate. Our concern about limitations of Likert-scale surveys caused us to seek a different approach. We subsequently designed an open-ended, computerized survey that provided various contexts to assess PSTs' beliefs. Using piloted data, we then developed rubrics to assign numerical codes to the responses. One advantage of using this kind of instrument is that it can be used for dual purposes. Written responses of individuals can be used to provide insights into their beliefs and interpretations. The numerical scores can be used to statistically analyze differences among groups in different treatments. We offer this description of rubric development to establish the validity of both the process and the rubrics that we developed. We also offer it so that other researchers who decide to develop their own rubrics can perfect the process, but have a clear idea of the intensity of this process before embarking on it. A paucity of scholarship about developing rubrics to be used for research purposes is available; therefore, describing this methodology is warranted. Sharing emergent methodologies is not a new idea. Glaser and Strauss (1967) called for others "to codify and publish their own methods for generating theory" (p.8, cited in Cobb & Whitenack, 1996), and others have published their methods for analyzing data (e.g., Cobb &Whitenack).

RUBRIC DEVELOPMENT

Gronlund (1998) provided a basic definition of the term *scoring rubric*: "a set of scoring guidelines that describes the characteristics of the different levels of performance used in scoring or judging a performance." We developed 19 scoring rubrics for eight items that assessed seven beliefs. Because of inadequate reliability measures on 2 of the rubrics, we eliminated them, along with the item with which they were associated, from our final data analysis.

The methods employed by our teams of researchers were different from those used by many who develop rubrics for use in classrooms. In particular, classroom rubrics are often created and then shared with students so that students have guidelines from which to construct responses. Also, classroom rubrics are often global in nature; for example, on a website dedicated to rubric development, a rubric is provided for assessing 8th and 10th graders' writing mechanics. The criteria for the highest score are "There are few or no minor errors. There are no major errors." We needed to develop rubrics that captured detailed information about respondents' beliefs about mathematics and mathematics teaching and learning. Because we inferred their beliefs from responses to scenarios to which respondents reacted, either in the role of the teacher or in commenting on teaching scenarios, we needed to develop *task-specific* rubrics, specific to a particular segment about a particular belief (Moskal, 2000).

Two research teams met two to four times per week in 3–4-hour sessions for 6 months. During this rubric-development phase, the 2–4-member teams each had two members who formed a consistent core, to provide grounding to the team and expertise with development. To get a wide range of responses for each item, we began with data gathered from three groups: prospective-teacher participants in a pilot of an experimental treatment (pre and post data), expert mathematics educators, and mathematics education graduate students. We later gathered responses from prospective teachers in a second pilot of the treatment; thus, our rubric development was based on a set of about 80 responses.

We adopted a grounded-theory approach (Glaser & Strauss, 1967), using pilot data, to develop each rubric. To begin the process, each person on the team independently analyzed the entire set of responses on a particular item with a particular belief in mind and sorted them into categories. Those responses that provided the greatest evidence of the belief were placed into one category, whereas those that provided no evidence of the belief were placed into another category. The remaining responses were placed into one, two, or three groups, depending on how each team member categorized responses. To determine the appropriate category for each response, the team members looked for degrees of evidence of the belief in question.

After individual team members had sorted the responses, they met to compare their categories and to develop descriptions for the categories. During these first meetings, team members tended to agree on the responses that showed the greatest and least evidence of the belief but had greater difficulty coming to consensus on responses that provided only partial evidence of the belief or, as was sometimes the case, responses that provided evidence of the belief in response to one part of an item only to provide disconfirming evidence of the belief for another part of the same item. For example, in one segment designed to assess the belief that a person might be able to perform a procedure without understanding, respondents were asked to state whether a student (Carlos) who could perform the standard algorithm for addition could understand and explain another student's (Sarah's) compensating strategy. One respondent wrote, "Yes because Sarah and Carlos show they understand although Carlos might not understand and might just know how to carry a 1." This response provides conflicting evidence about what Carlos understands; team members had to make decisions about how to categorize such responses.

During these discussions, descriptions of grouped responses emerged. Quite often, the group agreed on which responses belonged together in a particular category but had difficulty developing a written description for the category.. The challenge became to make the implicit features of the category explicit. We needed descriptions that were both robust, describing aspects of the belief that the responses provided, and procedural and concrete, so that others using the rubrics could code the responses with a high degree of reliability. This rubric-negotiation process was lengthy, and the development process took approximately 72 person-hours per rubric (4 weeks X 6 hours per week X average of three people per team). Sometimes negotiations concerned the number of categories, whereas in other cases, negotiations concerned the descriptions of the categories. We often traversed the terrain from the theoretical to the practical. We described categories to one another, then re-analyzed the data to check that the descriptions provided the glue that held the category together with regard to the belief in question.

Once we had reached consensus on a rubric's categories and descriptions, we re-analyzed the data to check for interrater and intrarater reliability. We then shared the rubric with the other team to test for coherence, reliability, and validity—a critical component of our work. The other team's members used the rubrics to code the data; we then compared the development group's codes with the testing group's codes to determine interrater reliability. We sought at least 80% agreement; if we did not achieve that level, we further clarified the rubric descriptions. We also discussed issues of validity to ensure that the scores were representative of the claimed amount of evidence for the belief we were claiming to measure.

AN EXAMPLE OF RUBRIC DESIGN

Using two different rubrics, we measured one belief (Belief 6) about children's learning of mathematics:

> The ways children think about mathematics are generally different from the ways adults would expect them to think about mathematics. For example, real-world contexts, manipulatives, and drawings support children's initial thinking whereas symbols often do not.

We used responses to a survey segment about fractions to infer the respondents' support (or lack thereof) for Belief 6 (see Figure 1 for Segments 8.1 and 8.2). The greatest challenge in developing this rubric was to appropriately describe each of the three categories, particularly the middle category. We struggled to describe responses like the following, which we had placed in the middle category:

Explain Item c Comparing 1/5 and 1/8	Explain Item d Word problem	Choose *c* or *d*	Explain choice of which item is easier
I think this problem is pretty simple once the child has it explained to him/her. They could use visual aids or any other method of viewing which fractions are larger and smaller.	This story problem paints the picture and is more understandable because you know why the answer is what it is.	d is easier	It illustrates the answer so that you can visualize the candy bar and the amount of children at the party which helps you visualize how much candy each child would receive.

In an early version of the rubric, we described it in the following way: "Says Item d is easier than Item c but has a weak explanation." Group members realized that the term *weak* was insufficiently clear to describe some explanations for future coders; in cases similar to the example above, the term *weak* did not capture the reasons that we determined that the response was a middle score. In another draft focused on the respondent's claims about Item c, we stated, "Says Item d is easier but tends to think that Item c is either relatively straightforward OR would be difficult for reasons that are NOT related to the ways children typically approach the problem." This description was later revised because the focus had shifted from the aspects that the respondent provided about the belief to aspects that the respondent did not provide. The final version (see Table 1) focused on specific ways that the respondent provided some evidence of the belief. We felt that the final version was more concrete than the earlier version and was more focused on the evidence that the respondent provided with respect to the belief.

8.1 Place the following four problems in rank order of difficulty for children and explain your ordering (you may rank two or more items as being of equal difficulty). NOTE. Easiest = 1.

a) Understand 1/5 + 1/8 Rank: ____ Please explain your rank:

b) Understand 1/5 x 1/8 Rank: ____ Please explain your rank:

Which fraction is larger, 1/5 or 1/8,
or are they same size? Rank: ____ Please explain your rank:

Your friend Jake attends a birthday party at which there are five guests who equally share a very large chocolate bar for dessert. You attend a different birthday party at which there are eight guests who equally share a chocolate bar exactly the same size as the chocolate bar shared at the party Jake attended. Did Jake get more candy bar, did you get more candy bar, or did you and Jake each get the same amount of candy bar?

Rank: _____ Please explain your rank:

Consider the last two choices:

8.2 Which of these two did you rank as easier for children?

___c is easier than d __d is easier than c ___both items are equally difficult

Please explain your answer.

Figure 1. Segment 8.1 and 8.2.

Score	Description	
0	• Says that Item c is easier (or that c and d are equally difficult), AND the explanation indicates no or little appreciation for the use of real-world context to support children's understanding OR	• Says that Item d is easier BUT gives either inconsistent explanations (that is, explanations that indicate that they think c might be easier) or a clear focus in 8.2 on the teacher's role in showing students how to do the problem[a]
1	• Says that Item c is easier BUT expresses great appreciation for real-world context OR	• Says that Item d is easier (or that c and d are equally difficult) AND expresses some appreciation for real-world context or says that Item d is easier to visualize
2	• Says that Item d is easier (or that c and d are equally difficult) AND notes that Item d is easier because of the real-world context or says that d is easier to visualize AND • notes that Item c is more difficult because of the abstractness or confusion of the numerals (or says that Item c is more difficult to visualize)	

Table 1. Final rubric for Belief 6, Segments 8.1 and 8.2

Note. When the responses in 8.1c or 8.1d appear to conflict with the response in 8.2, give MORE weight to the response in 8.2.

We successively devised at least six versions of this rubric, each more detailed and more focused than the previous one. In rubric development our first concern was validity; we asked ourselves the question "What does this kind of response tell us about the extent to which this respondent holds this belief?" Our second concern was reliability; thus, we sought to develop rubric descriptions that would be clear for others using them. For this rubric, our coders (external to the project) achieved 87.5% interrater reliability (the target for interrater reliability is typically set at 80%). The mean interrater reliability for all 17 rubrics was 84% for the responses of our 159 participants who completed this survey before and after taking part in one of five treatments; 20% of the responses were coded by two coders, and all responses were blinded.

[a]Our perspective is that a respondent who writes, "I would teach the children how to think about this idea," does not hold the belief that children think about mathematics differently from the way adults might expect, because, from this respondent's perspective, children need to be shown how to think about the mathematics.

CONCLUSIONS

Our definition of *belief* guided the development of our instrument by causing us to operationalize four components of beliefs in our rubrics. Our items are situated within contexts. We assume that people hold these beliefs at varying levels, and we infer these levels either by observing to what respondents attend in contexts or by placing respondents in positions to act and inferring the extent to which they hold a belief by their purported actions within these contexts. We have provided the reader a specific example of this process.

We have found the use of rubrics for research purposes to be quite promising, because the responses offer a rich data set that is typically unheard of when studying conceptions of large numbers of participants. Yet, we provide a word of caution. The work of rubric development requires the resources of time, money, and large numbers of persons qualified to develop and code rubrics. The end result, however, is a survey that can be used for a variety of purposes, both quantitative and qualitative. We end with recommendations for those who may be considering developing their own rubrics.

RECOMMENDATIONS FOR DEVELOPING RUBRICS

- Be clear on the definitions of your constructs, because these constructs serve as the foundation to which you must return when you attempt to operationalize these constructs.
- Use a team of two to four people when developing research rubrics. Rubric development cannot be done alone. Interpretations of responses can vary widely, but one person cannot know how others will interpret responses unless others are simultaneously examining them—at least three per team is highly recommended. Also, having two coding teams allowed us to have others who were experienced coders but not familiar with the particulars of a rubric code the data.
- Decide on a particular number of categories beforehand, but do not feel constrained to use that number of categories; for example, begin with four categories: responses that provide no evidence, weak evidence, evidence, and strong evidence of the belief. Then decide on particulars for that belief. We did not use this strategy, but we think that the approach could have facilitated our work, because it might have guided our initial conversations about responses.
- Accept that when dealing with the written word, some responses will be challenging to code. From any 100 responses, we typically found that 5–10 could fall into one of

two categories, either because of differences in interpretation of the response or because an insufficient amount of information was provided by the respondent.

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CURVED SOLIDS NETS

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The transformation of a solid to its net is based on something quite different from simple perceptual impression. It is a mental operation performed by manipulating mental images. The aim of this study was to observe pre-service and in-service teachers' ability to visualize the transformation of a curved solid to its net and vice versa, and to try to classify and to analyze students' mistakes and difficulties related to this kind of visualization. The most significant conclusion is that when lacking any experience, many students are unable to imagine the process of unfolding a curved surface. The findings show poor performance in attempts to produce nets or judge drawings as being possible nets of a cylinder or a cone.

INTRODUCTION

The transformation of a solid to its net is not a copy of the corresponding perception, but rather a mental operation performed by manipulating mental images. It may, therefore, significantly contribute in developing students' visual ability, which is an important goal of mathematical education. The aim of this study was to observe pre-service and in-service teachers' (herein referred to as students) ability to visualize the transformation from a curved solid to its net and vice versa, and to analyze and classify students' mistakes and difficulties related to this kind of visualization.

The net (sometimes called development) of a solid is the 2-D shape obtained from the solid by "unfolding" its whole surface so it lies in one plane. Visualizing solids nets, in general, requires a level of mental imagery students often lack, but can benefit by experience, awareness and analytical analysis. However, there is a difference in mental processes needed for visualizing curved solids nets as opposed to polyhedra nets: in polyhedra nets, every surface of the solid (which is a flat polygon) appears in the exact same shape. In curved solids, such as cones and cylinders, there are curved surfaces that look totally different when rotated to lie on a plane.

Because we are dealing with adults, we have to take in account that although they often know how to draw the standard net of a cylinder or a cone, this does not mean that they are able to visualize the process. They "know" the expected shape only because they have seen it in books or in class. This phenomenon is especially common among teachers. In this study, there is an attempt to overcome this problem by asking students to draw additional nets, or to identify nonstandard nets. When doing so, they function very much like those who have never seen a net of a cone or a cylinder before, and we can obtain a better idea of their actual ability to mentally fold and unfold surfaces.

THEORETICAL BACKGROUND

Folding and unfolding a solid is considered part of visual ability by many researchers: McGee (1979) includes it among "spatial visualization" which in his classification is one of two kinds of spatial abilities. Bishop (1989) differentiates between two abilities in visualization: the *visual processing* (VP), and the *interpretation of figural information* (IFI). The process of folding and unfolding solids belongs to the VP, because it involves the manipulation and extrapolation of visual imagery, and the transformation of one visual image to another. In the organizing framework suggested by Gutierrez (1996), the VP and IFI mentioned by Bishop are seen as processes of visualization, that is, mental or physical actions where mental images are involved.

Piaget (1967), in a chapter dedicated to rotation and development of surfaces, sees visualizing solids developments (i.e. nets) as "something quite different from simple perceptual impression… Between perception and imagination there lies a whole series of increasingly systematized actions, internalized in the form of images" (p.272). He explains that to pass from perception of the intact solid to an image of a drawing of its development (i.e. net), it is necessary to perform a mental action, and at the same time, to mentally co-ordinate the different points of view. This ability is reached, in Piaget's terms, only in sub-stage IIIB, that is, the second sub-stage of the concrete operational stage (age 9-11). Before that stage, although they have all the necessary prerequisites for drawing developed and rotated surfaces, children are limited in mental operations such as internalizing concrete actions in the form of symbolic images. At Piaget's stage IIA (about age 5-6) children are limited to reproducing as the net their drawing of the intact solid, expecting that their drawing could be cut and folded to reproduce the solid with all its sides. At Piaget's stage IIB (age 6-7) there is a first attempt to distinguish between the intact and the developed solid (by descriptions, "hints" in the drawings or gestures), but children still cannot rid themselves of their current perception. At Piaget's stage IIIA (age 7-9), there is a beginning of coordination between actual viewpoints: children represent some phases of the unfolding process, but they are unable to predict the final outcome and to represent the mental unfolding movement to the stage that all surfaces are arranged in one plane. They are unable to distinguish between different viewpoints and co-ordinate them well, and thus we can still find confusion between perspective view and net. Only at stage IIIB are they able to project all the parts of the curved surfaces on one single plane..

Piaget asserts that imagining the rotation and development of solids depends largely on the actual experience of folding and unfolding solids. "The child who is familiar with the folding and unfolding paper shapes through his work at school is two or three years in advance of children who lack this experience" (p. 276). In our case, it seems that students, who lack this experience, act in many ways on a level corresponding to stages IIB or IIIA. This supports Van Hiele's claim that the level of geometrical performance depends on experience, teaching and learning, and does not necessarily develop spontaneously with age (see Fuys et. al., 1988).

There is some more recent research dealing with solid nets, but most investigate polyhedra nets, and focus mainly on children's performance (cf., Mariotti, 1996, Meissner, 2001). Meissner argues that a solid net can be seen as a *Procept*, as defined by Tall (Tall et. al., 2000), because it has both a static and a procedural aspect, and is both process and product. This study accepts this opinion. As Piaget (1967) demonstrates, the development of a solid is definitely not a perceived object, like most geometrical objects are, but rather a conceived object. As such, it requires a pseudo-empirical abstraction, deriving from the process that the individual performs on the object. Those processes are encapsulated, as Meissner shows, into a static object. The term “net” serves as the symbol, representing both the process and the object. Those objects can be manipulated, as in this study, when students worked on nets of regular cones and cylinders as elementary objects. (A discussion of this matter is outside the purvey of this paper.)

THE STUDY

The study was carried out on 43 pre-service teachers (all learning to be mathematics teachers in elementary or junior high school in a college of education), and 78 in-service mathematics teachers in elementary schools.

The study uses a combination of quantitative and qualitative methods:

1) Two questionnaires were given to the whole population: one open and one closed.
2) The questionnaires were followed by a teaching unit of 2.5-3 hours. The unit began with group discussions and manipulating pre-cut paper shapes. Students were encouraged to describe their discoveries and their mistakes verbally. Then, the teacher conducted class discussions and additional activities dealing with some general aspects of non-polyhedral solids nets (such as investigating how various shapes of flat surfaces look when “folded” to surround a flat disc, what the connection between the net of a solid and the way this solid rolls is, which curved surfaces can be unfolded to flat nets etc.) The discussions dealt with generalizing students’ discoveries, analyzing their own beliefs and mistakes and some didactic aspects of the activity. Some of those units were videotaped and analyzed.
3) Ten of the students were interviewed and videotaped, some singly and some in pairs.

This paper focuses only on analyzing the finding from the two questionnaires, and not on the teaching unit.

The open questionnaire:

As a first step, the students answered an open questionnaire in which they had to draw several nets of cylinders and of cones. For each of the solids, they had to draw one net, and then three more, as different as they could from the first one. It was expected that their first drawing would usually show if they were familiar with the standard net (the prototype), whereas the three others would reflect their ability to visualize the “unfolding process”. It was made clear to the students that we would accept any kind of net, even if the “cutting” was done in nonstandard ways (through the middle of the circle, along a broken or a curved line etc.), but it had to be “in one piece” (possibly connected by one point). They were told to think of wallpaper on which they could draw their net so that it could be cut and pasted onto the entire solid surface (they had the solids at hand). This requirement of non-conventional nets enabled us to make the students actually try to visualize the process, instead of just producing familiar drawings.

The closed questionnaire:

In the closed questionnaires, students had to identify drawings as being or not being possible nets for a cylinder or for a cone (separately). Almost all the drawings in this questionnaire were drawings produced by students or by school pupils in previous studies (some of them as answers to the open questionnaire in a pilot study). The aim of this stage was to examine students' ability to visualize the process of folding the shapes to create the solids, and to detect typical mistakes, beliefs and difficulties in doing so.

CLASSIFICATION OF TYPICAL MISTAKES

The theory and classification suggested here is based on both the quantitative findings and an analysis of the interviews and of the discussions. It is not an exclusive classification. Each type of mistake deals with one aspect, so that the same mistaken drawing can fit several types. Most types are relevant for both cylinders and cones. All the examples are taken from students' answers to the open questionnaire.

Type1: Confusion between the perspective view of the solid and its net

1-A) "***Ellipses***": drawing ellipses instead of discs as bases. (cyl.**1**, cyl.**2** , cyl. **3** in Fig.1)

1-B) "***Proportion as seen***": drawing the lateral surface[1] the same width as the disc. Here we can detect a mix-up between the perspective view of the solid intact exterior and the imaginary net of its surface. (cyl.**4**, cyl.**5**, and cone**6** in Fig1)

1-C) "***Front and back***": although taking into account that the solid has a back side, students sometimes draw the width of the back and the front "as seen". This means that they are able to control some points of view, but not all of them. (**7** and **8** in Fig1)

Fig. 1

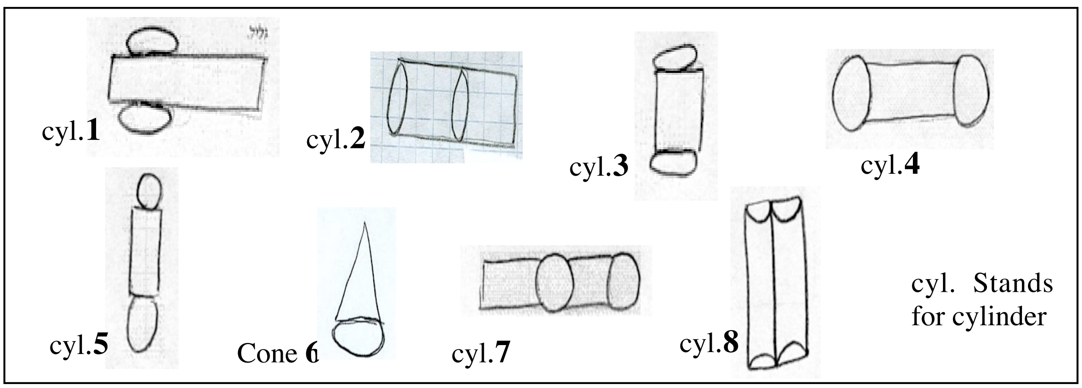

Figure 1.

Type 2. Joining the disc and the lateral surface along a line.

Students have a clear tendency to join the disc to the developed lateral surface along a line, and not only at one point. Here only clear cases were considered mistakes. In many cases, although considered here correct, we can see a slight distortion or overlapping of the disc that might result either from the above or from inaccuracy.

2-A) "***Overlapping disc***": drawing the disc partly overlapping the lateral surface (**9** and **10** in Fig.2). In many cases, students drew the lateral surface, then the disc (or discs), and finally erased the part of the drawing "under the disc". The *overlapping disc* creates a curved joining line. In the teaching unit, students discover why this is not possible.

2-B) "***Peeling without detaching***": drawing pieces of the lateral surface around the disc as if it was peeled, but without detaching the pieces from the disc. This mistake, like 2A, creates a

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[1] When speaking of the drawing, the meaning of 'lateral surface" will be the unfolded lateral surface.

curved joining line. (In fact, it can be seen as a variation of 2A, but students often judged it as different.) (**11**, **12** and **13** in Fig.2)

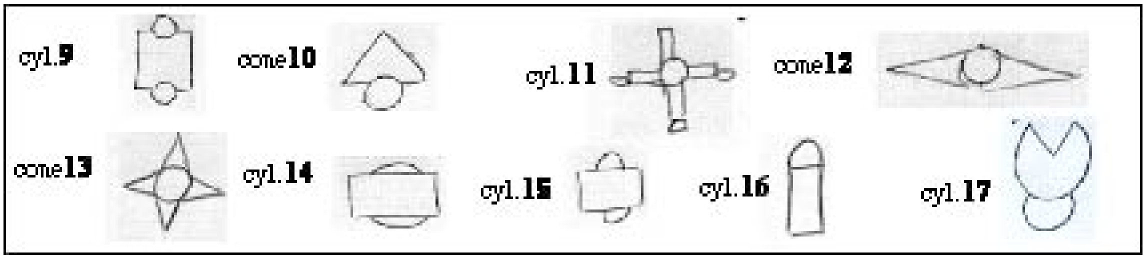


Figure 2.

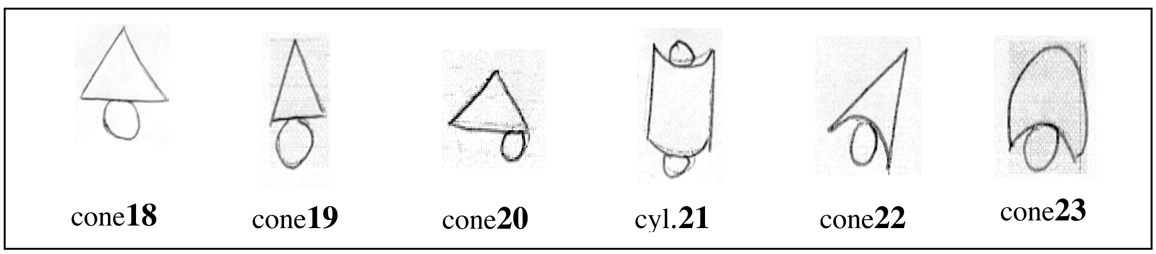
2-C) "***Partial disc***": drawing less than a whole disc (so that the lateral surface's drawing remains intact). This creates a straight joining line. (**14**,**15**,**16** and **17** in Fig.2)

Type 3. Wrong form of the edge to be joined

3-A) "***Triangles***" (relevant only for cones): drawing the lateral surface as one or more triangles (**18**, **19** and **20** in Fig. 3).

3-B) "***Embracing curve***": drawing the edge as a "concave curve", as if is embracing the disc (**21**, **22** and **23** in Fig. 3). Students explain that it feels right that the curve bend towards the disc, because when folded, it must surround it. This mistake can also be connected to type A – confusion with the perspective view.

Fig. 3

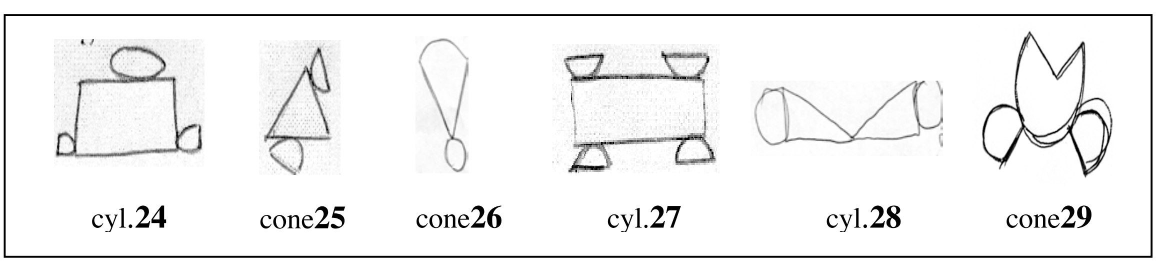


Type 4. Wrong placement of the parts

Students sometimes fail to imagine how the net's parts should be connected to one another. They are not able to control the whole process simultaneously.

4-A) "***Wrongly placed bases***": placing the bases or the half bases on wrong sides of the developed lateral surface (**24**, **25**, and **26** in Fig. 4). This phenomenon sometimes occurs when trying to draw an interesting "other net" by splitting the base.

Fig. 4



4-B) "***Uncontrolled parts***": drawing parts (half discs or half lateral surfaces) without taking in account how they should take their place when folded. (For instance: ignoring that those half discs may turn over and overlap instead of integrating to a whole disc.) (**27**, **28**, **29** in Fig. 4)

Type 5. Other mistakes

5-A) "***flexible surface***": imagining the unfolding of the lateral surface as if it was made of a sheet of rubber (**30**, **31**, **32** and **33** in Fig. 5).

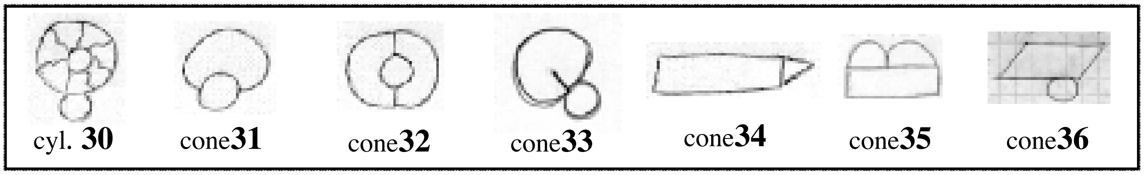


Figure 5.

5-B) "***No idea***": some mistakes attest to a total inability to visualize the outcome of the development process. In this study, it happened only in the cone case, and was accompanied with reflections like "I have no idea how to do it". (Of course, we can detect in those drawings elements of other types of mistakes)(**34**, **35** and **36** in Fig. 5)

ANALYZED RESULTS AND EXAMPLES

Examples from the opened questionnaire

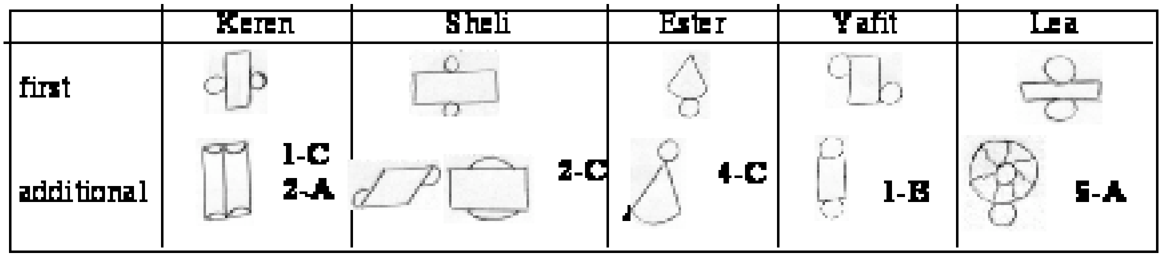
The examples given in the last section give an idea of the variety of drawings students made in the open questionnaire. Of course, there were many creative correct drawings, but in this paper, we focus only on the mistaken ones. Drawing, according to Piaget, can act as an intermediary between perception and imagination for it is "an imitative representation which remains external and material in character, whilst at the same time laying a basis for internalized images" (Piaget, 1967, p. 272). Through their drawings, we have a clue to students' mental processes of visualization.

As mentioned before, students had to draw one net, and then three additional nets for each solid. In the analysis, we have to distinguish between the first drawing and the additional ones. The first drawing is very significant only for those who never saw such nets. The additional drawings may bring to light difficulties of those who gave a correct first drawing "just because they knew" the answer, without actually trying to visualize the process. Hence, the interesting cases in those additional drawings are those in which there is at least one wrong drawing, after a correct first drawing ("wrong after correct").

Table 1: Percentage of mistaken answers in the opened task:

	Pre-service teachers (n=43)	In-service teachers (n=78)	Total (n=121)
First cylinder	21	21	21
Wrong after correct cylinder	58	30	37
First cone	54	34	40
Wrong after correct cone	57	20	28

Let us examine some examples. As can be seen from the table, cones are less well known than cylinders (less mistakes in the first drawing).

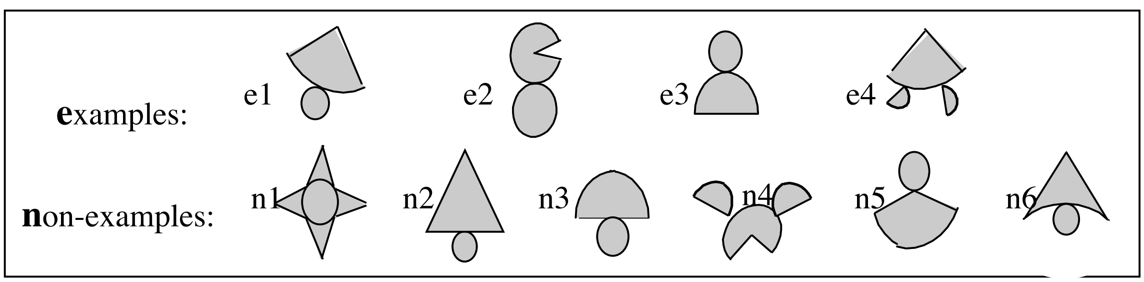


One might think that visualizing cylinders nets is an easy task for adults, but the picture changes dramatically when we come to the “Wrong after correct”: it seems that many of those who automatically drew a rectangle with two discs, do not really have the ability to visualize the process (among pre-service teachers - 57% of the “correct answerers”).

In all cases illustrated, the first drawing would lead us to believe that those students are able to unfold those solids correctly. However, their additional drawings reveal that this is not so! When they produce a drawing after really trying to visualize the process, we can see their difficulties and confusion. The most striking case was of Leah, who explained that she “peels” the lateral surface all around the bottom base of the cylinder (the central disk). It seems that, like young children, she maintains topological properties and neglects Euclidean properties (proportions and distance).

Examples from the closed questionnaire

The following examples are taken only from the cone questionnaire, in which 26



drawings had to be judged as examples or non-examples of possible nets of a cone.

Table 1: Percentage of mistaken answers in the closed questionnaire:

	classification	Pre-service teachers (n=43)	In-service teachers (n=78)	Total (n=121)
e1	Prototype A	14	4	7
e2	Prototype B	40	17	25
e3		74	62	67
e4	**4-B**	75	60	64
n1	**2-B**	72	58	63
n2	**3-A**	65	29	42
n3	**5-A**? **4-A**?	27	15	20
n4	**4-B**	45	60	56
n5	**4-A**	20	15	17
n6	**3-B**	49	31	37

As can be seen from the table, there is a substantial difference between pre-service and in-service teachers. This difference is probably caused by the fact that in-service teachers are familiar with cones' and cylinders' standard nets. Many of them had even taught this recently in their classes. But the difference is impressive only in the classical cases: the two kinds of prototypes, (especially prototype B), and the "popular mistake" – the triangle – again because many of them encountered this mistake when teaching the subject). In unconventional cases, their knowledge did not help them, and they had to try visualizing the whole process, which was, as we can see, far from being easy for them. The case of e3 and a3 is very interesting. While most students do not believe that a half disc could be folded to a cone's lateral surface (e3), most of the remaining students visualized it the wrong way (n3). Some of them reported thinking that the vertex should be at the center of the arc (attesting to mistake 5-A).

CONCLUSION

The most significant conclusion from this study is that when lacking any experience, students sometimes behave as young children do, and are unable to imagine the whole process of unfolding a surface simultaneously. They show some static, incomplete aspect of the action, and often confuse the perspective drawing of the solid with its net. When comparing students' drawings with children's drawings in Piaget's work, we can find correspondence to stages IIB (age 6-7) or IIIA (age 7-9). Some of the drawings produced by the students seem exactly the same as some children's drawing illustrated in Piaget's book. However, it is not enough to know that there is a problem. We must make sure that teacher training includes activities that develop this aspect of visual imagery.

The study, including its teaching unit, also has pedagogical aspects: analyzing and reflecting on their own thought patterns, may help teachers to reach a better understanding of their pupil's possible learning processes.

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PRESERVICE ELEMENTARY TEACHERS' SOLUTION PROCESSES TO PROBLEMATIC ADDITION AND SUBTRACTION WORD PROBLEMS INVOLVING ORDINAL NUMBERS AND THEIR INTERPRETATIONS OF SOLUTIONS

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In this study, we examined 139 prospective elementary teachers' solution processes to additive word problems for which the solution is 1 more or 1 less than the answer produced by the straightforward application of the addition or subtraction of the two given numbers. For each problem, five aspects of their solution processes were examined: (a) the modeling strategy or procedure, (b) execution of procedures, (c) the solution, (d) the type of errors, and (e) the implicit or explicit interpretation of the solution produced by the mathematical model or procedure. The major findings of the study were that a majority of prospective elementary teachers' responses (about 91%) contained incorrect solutions to such problems and that about 93% of the errors were ±1 errors. That is, errors due to preservice elementary teachers' failure to interpret correctly the solution produced by the straightforward addition or subtraction of the two numbers given in each word problem.

PURPOSE OF THE STUDY

Story problems have played and will likely play a prominent role in elementary school mathematics. Verschaffel, Greer, and De Corte (2000) mention, among others, the following reasons for the inclusion of word problems in the mathematics curriculum: (a) word problems provide practice with real life problem situations where students will apply what they learn in school, (b) word problems motivate students to understand the importance of the underlying mathematical concepts because they will use such concepts and abilities to solve problems in the real world, and (c) word problems help students to develop their creative, critical, and problem-solving abilities. However, as argued by several critics (e.g., Gerofsy, 1996; Lave, 1992) word problems, as currently presented in instruction and textbooks, fail to accomplish these goals. The failure is due, in part, to their stereotyped nature (Nesher, 1980) and unrealistic approach needed to solve them: Most, if not all, word problems that students are asked to solve require the application of a straightforward arithmetic operation (Verschaffel, Greer, & De Corte, 2000). As a consequence, when faced with word problems in which the situational contextual plays an important role in the solution process, students fail to connect school mathematics with their real-world knowledge.

Children’s lack of sense making (Silver, Shapiro, & Deutsch, 1993) or suspension of sense-making (Schoenfeld, 1991; Verschaffel, Greer, & De Corte, 2000) has been examined in several studies. However, we do not have empirical data about prospective elementary teachers’ (PETs) solution processes when solving addition and subtraction word problems whose solution is 1 more or 1 less than the one produced by the straightforward addition or subtraction of the two given numbers. In this paper we attempt to bridge this gap. First, we examine prospective elementary teachers' modeling strategies to solve problematic or non-routine word problems involving addition or subtraction of ordinal numbers. Second, we analyze their interpretations of the solutions produced by the mathematical models or procedures. For purposes of this paper, a problematic problem is a problem in which the solution provided by the arithmetic procedure involving the given numbers does not represent the solution to the problem.

THEORETICAL AND EMPIRICAL BACKGROUND

We will refer to the process of representing aspects of reality by mathematical structures as mathematization or modeling. There are several schematic diagrams to represent this process (e.g., Verschaffel, Greer, & De Corte, 2000; Silver, Shapiro, & Deutsch, 1993) but Silver et al.'s (1993) referential-and-semantic-processing model will suffice for our purposes (Figure 1).

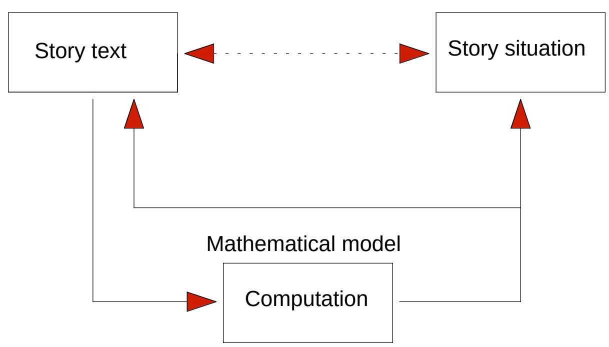

Figure 1: Silver et al.'s (1993) referential-and-semantic-processing model

The process of mathematization or modeling can be described briefly as follows: (a) to understand the structure of the mathematical situation embedded in the story text of the problem, (b) to construct a mathematical model or to choose a mathematical procedure to obtain the solution to the story problem, (c) to execute the computations or procedures called by the mathematical model, and (d) to interpret the outcome or solution produced by the mathematical model in terms of the situational context described in the story text or the constrains of the real-world story situation. Students' realistic solutions to problematic word problems could be enhanced by paying particular attention to the appropriateness of the mathematical model and the appropriate interpretation or meaning of the outcome produced by the mathematical model. Silver, Shapiro, and Deutsch hypothesize that many unsuccessful solutions use the model displayed in Figure 2.

Several researchers (e.g., Cai & Silver, 1995; Greer, 1993, 1997; Reusser & Stebler, 1997; Silver, Shapiro, & Deutsch, 1993; Verschaffel & De Corte, 1997; Verschaffel, De Corte, & Lasure, 1994; Verschaffel, De Corte, & Vierstraete, 1999) have examined children's lack of use of their real-world knowledge to solve problematic word problems. Silver, Shapiro, and Deutsch (1993), examined 195 middle school students' responses to the following augmented-quotient division-with-remainders problem: The Clearview Little League is going to a Pirates game. There are 540 people, including players, coaches, and parents. They will travel by bus, and each bus holds 40 people. How many buses will they need to get to the game? Silver, Shapiro, and Deutsch found that about 22% of the students correctly performed an appropriate procedure but provided an incorrect solution without explicit interpretation. Most of these students interpreted the result of the division (e.g., 13 or 13 with another number) as the number of buses needed to transport the people to the game. In another study, Verschaffel, De Corte, and Vierstraete (1999) investigated 199 fifth and sixth graders' difficulties in modeling and solving problematic additive word problems involving ordinal numbers. The pupils were administered a paper-and-pencil test that included six problems whose solution is 1 more or 1 less than the numerical answer provided by the addition or subtraction of the two given numbers. The researchers found that about 83% of the errors made on these 6 problems were ±1 errors.

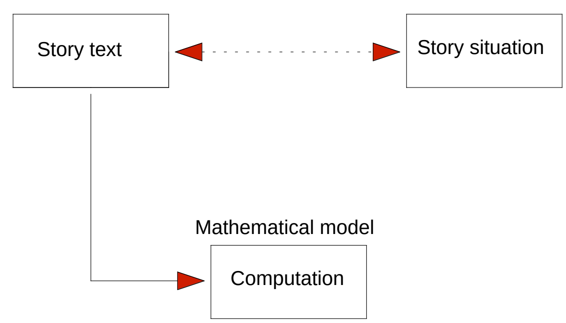

Figure 2: Silver et al.'s (1993) referential-and-semantic-processing model for unsuccessful solutions

In this paper we extend Verschaffel, De Corte, and Vierstraete's (1999) research to PETs. Since teachers are one of the key factors in the instructional environment, it is critical to examine their use (or lack of use) of realistic modeling assumptions in dealing with problematic word problems. Some may argue that there is no reason to believe that PETs will perform better than Verschaffel et al.'s subjects because, when they were children, PETs solved non-problematic word problems. However, PETs are, arguable, a more mature population, and, hence, the possible generalizability of the findings of Verschaffel et al. has to be established *empirically* for PETs.

METHOD AND SOURCES OF EVIDENCE

The sample consisted of 139 pre-service elementary teachers (PETs) from five sessions of their second mathematics content course for elementary teachers in a State University in the USA. The participants were administered a paper-and-pencil test during regular class and told that they would have enough time to complete the test. The directions asked PETs to show all their work to support each of their responses. As in the Verchaffel, De Corte, and Vierstraete's (1999) study, the test contained 9 experimental items and 7 buffer items. The experimental items were adapted from Verchaffel, De Corte, and Vierstraete's (1999) investigation and are displayed in Table 1. The first three of the experimental items can be solved by the straightforward addition or subtraction of the two given numbers while the other six are problematic in the sense described above.

Although written protocols contain some intrinsic limitations when compared to verbal protocols, several researchers have validated the use of written data to uncover cognitive processes (e.g., Hall et al., 1989). In fact, we did not face any difficulty in determining the strategies that PETs used to solve the problems. Written responses are also the most feasible method to collect data on large samples of subjects. However, interviews are being conducted to gain additional insights into the nature of PETs' thinking and reasoning when solving non-routine addition and subtraction problems involving ordinal numbers.

Table 1: The nine experimental items

Type	Item	Required operation(s)*
I-L	1. In January 1985 a youth orchestra was set up in our city. In what year will the orchestra have its twenty fifth anniversary?	S + D
I-D	2. Our youth club was set up in September 15, 1970. I became a member in September 15, 1999. How many years had the club already existed when I became a member?	L - S
I-S	3. In March 2000 it had been 34 years since our school had held its first annual school party. In what year was the school party held for the first time?	L - D
II-L	4. In September 1975 the city's youth orchestra had its first concert. In what year will the orchestra have its fiftieth concert if it holds one concert every year?	(S + D) - 1
II-D	5. Last October (2001) I participated for the first time in the great city running race that is held every year. This race was held for the first time in October 1959. How many times has the race been held?	(L - S) + 1
II-S	6. In November 1994 the twenty fifth annual school party took place. In what year was the school party held for the first time?	(L - D) + 1
III-L	7. There was a summer market in our city every summer from 1950 up through 1969. Since then the summer market was cancelled 30 consecutive times. In what year did the summer market restart?	(S + D) + 1
III-D	8. For a long time the city held a fireworks display every year on the last day of the October festival. In October 1982 we had our last fireworks, and thereafter there was no fireworks display. In October 1999 they restarted the tradition of the annual fireworks display. How many years did we miss the fireworks?	(L - S) - 1
III-S	9. In December 1999 our sports club held its annual election for its officers. Because of a lack of candidates, there had not been elections for the 23 years preceding 1999. Prior to this election, in what year did the last election occur?	(L - D) -1

ANALYSIS AND RESULTS

Preservice elementary teachers' (PETs) written responses were examined with respect to five aspects of their solution processes related to the process of mathematization: (a) procedure or mathematical model (strategies), (b) execution of procedures, (c) numerical solution to the problem, (d) errors, and (e) interpretations of the solutions produced by the mathematical model. The analysis of errors helped us to better understand PETs' interpretations of the solutions produced by the procedure or strategy.

PETs produced a total of 1247 responses out of 1251 possible. We found that 1243 (99%) of the responses to the items contained an appropriate modeling strategy or procedure. A strategy, model, or procedure was judged appropriate if it could potentially lead to the correct solution to the problem. That is, a procedure that, if executed correctly, could yield either the correct solution or 1 more or 1 less than the correct solution. The remaining 4 responses contained an inappropriate procedure. Regarding PETs' types of modeling strategies for the solution processes containing an appropriate procedure, 30 responses contained strategies using counting techniques, one contained the use of a picture, and 25 contained only answers, which suggests that PETs performed the operations mentally. All of the other responses (1187 or 95%) involved strategies that relied on formal methods (i.e., adding or subtracting the two given numbers). We did not find any evidence of other useful heuristic strategies such as solving an analogous simpler problem. With respect to the execution of the procedures, PETs performed 1179 (95%) appropriate procedures correctly.

Regarding the numerical solution to the problems, PETs performed well on the three non-problematic items (Table 2). Only 35 (8%) responses were incorrect. On the other hand, PETs' performance on the problematic items was poor. The percentage of correct responses for each problematic word problem varied from 4% to 14% (Table 2). Overall, only 9% of the solutions to the problematic items was correct.

Problem	Number of correct solutions	Percentage of correct solutions
1	127	91%
2	129	93%
3	126	91%
4	7	5%
5	12	9%
6	5	4%
7	19	14%
8	12	9%
9	18	13%

Table 2. Percentage of correct solutions for each experimental word problem

Given that 781 (95%) of the responses to the problematic word problems contained an appropriate procedure executed correctly, we hypothesized that PETs' lack of success with the problematic word problems was due to the interpretation that the addition or subtraction of the two given numbers produces the correct solution. To verify this hypothesis, we conducted an analysis of errors for the problematic word problems (Table 3). As Table 3 indicates, in all but one case, at least 90% of the errors were ±1 errors. On average, about 93% of the errors were ±1 errors. PETs also made ±1 errors with execution of procedures. These errors are errors due to procedures executed incorrectly for which, in addition, PETs did not adjust their responses. That is, even if PETs had executed the procedures correctly, they would have still made ±1 errors. In this sense, these types of errors are potential ±1 errors. Combining these two types of errors, we conclude that for each problematic word problem, at least 94% of the errors were

±1errors. Overall, about 97% of the errors made by PETs on the problematic word problems were ±1errors or potential ±1 errors. About 3% of the remaining errors were due to other factors such as adjusting both numbers, using the inverse operation (subtraction instead of addition), executing procedures incorrectly, etc. We conclude that ±1 errors and potential ±1 errors were influenced by PETs' interpretation that the result of the addition or subtraction of the two given numbers yields the correct solution to the problematic word problems. As one PET said: "this is just a subtraction problem. We get the correct answer by subtracting the numbers."

Table 3: Type errors for each of the six problematic word problems

Type of problem	Required operation(s)	Type of ±1 error	± 1 errors (%)	± 1 errors and execution of procedures (%)	Execution of procedures (%)	Other (%)
II-L	(S + D) - 1	+ 1 error	83%	14%	2%	1%
II-D	(L - S) + 1	- 1 error	94%	2%	2%	1%
II-S	(L - D) + 1	-1 error	90%	8%	0%	2%
III-L	(S + D) + 1	-1 error	98%	0%	1%	2%
III-D	(L - S) -1	+ 1 error	98%	0%	1%	1%
III-S	(L - D) - 1	+ 1 error	93%	1%	1%	5%

DISCUSSION AND CONCLUSION

While we expected, based on previous research with school children, that some prospective elementary teachers (PETs) would not provide the correct solution to the problematic word problems, it was alarming to find out that a high percentage of the total responses to the problematic items contained unrealistic solutions. In fact, only 9% of the solutions to the problematic items were correct. An analysis of errors revealed that about 97% of the errors were either ±1 errors or potential ±1 errors. We offer several explanations to understand this finding. One explanation might be that PETs are used to approach word problems in a superficial or mindless way because the word problems posed in the traditional instructional environment can be solved by the straightforward application of arithmetical operations. In this sense, PETs probably expected that all the test problems were of that kind. As stated by one PET “I didn’t even know that this type of problems existed.” Another explanation may be PETs’ insufficient repertoire of useful heuristic strategies (such as thinking of an easier analogous problem or making a diagram). This was evident during class instruction where PETs were asked to solve problematic subtraction and addition word problems using an analogous simpler problem or making a diagram. Some PETs stated that these heuristic strategies helped them to understand and solve the problems. On the other hand, we also may explain some of the PETs’ unsuccessful solutions as due to PETs’ lack of understanding of heuristic strategies. This was also evident during classroom instruction. A few PETs stated that they did not know how to pose an analogous simpler problem. Others said that using a picture was confusing. Another explanation for PETs’ unsuccessful solutions may be an insufficient understanding of the enumeration process to solve subtraction and addition word problems involving ordinal numbers. This lack of understanding might have prevented PETs from interpreting that we need to adjust by one the solution produced by the addition or subtraction of the two given numbers. This explanation is similar to the one offered by Silver, Shapiro, and Deutsch (1993) to account for some of their students’ responses involving 13 or a combination of 13 with another number. The researchers advanced the explanation that these responses reflected a lack of interpretation of the remainder of the division of 540 and 40. That is, these students' faulty solutions were influenced more by a lack of understanding of the meaning of the remainder than by an understanding that the solution to the word problem can be represented by the division of the two given numbers. In this case we know that if the remainder is greater than zero, then an extra bus is needed. In the present study, however, the solution to the problematic word problems is not represented by the addition and subtraction of the two given numbers. Successful solutions involve understanding something else: the connection between the nature of the enumeration process needed to find the solution and the answer produced by the addition or subtraction of the two given numbers. There is not an algorithmic hint we can give students to obtain the correct solution to this type of problems. Understanding the nature of the enumeration process of addition and subtraction word problems involving ordinal numbers is cognitively more complex than understanding the meaning of the remainder of division word problems. Evidence to support this claim comes from observing, during classroom instruction and on tests, that some PETs chose to use counting techniques because they did not know what numbers to use or how to adjust the solution produced by the addition or subtraction of the two given numbers. The use of a picture to understand the enumeration process was too confusing for some of these PETs.

The low percentage of correct solutions to the problematic word problems is alarming. Since a high percentage (about 97%) of incorrect responses contained ±1errors or potential ±1 errors, it seems that PETs need some type of instructional intervention to learn how to model whole number addition and subtraction word problems in which the solution is 1 more or 1 less than the sum or difference of the two given numbers. We will assess the effects of both minimal (such as giving students a hint that some of the given problems are problematic or "tricky" or creating a cognitive conflict by asking them to solve similar simpler problems) and more intensive interventions on PETs' modeling strategies to solve problematic addition and subtraction word problems involving ordinal numbers in a sequel to this study (Authors, in preparation). The cognitive conflict technique was used during classroom instruction and during interviews. Some PETs realized that the addition and subtraction of the two given numbers does not always produce the correct solution to word problems involving ordinal numbers.

This study contributed to our understanding of PETs' modeling strategies and errors when solving subtraction and addition word problems involving ordinal numbers. Other studies have examined children's solutions to other types of problematic word problems (e.g., what will be the temperature of water in a container if you pour 1 l of water at 80° and 1 l of water of 40° into it? (Nesher, 1980), John's best time to run 100 m is 17 sec. How long will it take to run 1 km? (Greer, 1993)). To better understand PETs' thinking when solving problematic word problems, we also need to examine the modeling strategies that they use to solve such problems. This is certainly a fertile area for further research. Other problematic word problems for which we lack information about PETs' thinking and modeling strategies are problems similar to the ones investigated in this study but involving multiplication and division (e.g., A farmer wants to fence the front of a square field whose side measures 1000 feet. How many posts does he need, if he wants to place a post every 20 feet?).

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OPEN-ENDED REALISTIC DIVISION PROBLEMS, GENERALISATION AND EARLY ALGEBRA

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This paper explores the role of open-ended realistic division problem in the development of algebraic reasoning. A written test was administered to 672 students. From the results of this test students were selected for semi-structured interviews. The students were interviewed in pairs and were asked to explain to each other how they would solve the problem. The results of the written tests and the semi-structured interviews indicated that both students understanding of division and the real world context of the problem played an important role in abstracting generalisations.

A principle focus of the mathematical reform movement has been to embed mathematics in reality (Heuvel-Panhuizen, 1996). This embedding in reality serves as a source of application and learning, with the links to the real world providing practical knowledge (Treffers, 1993). This enables the students to learn to mathematise their world from the early years using mathematical sign systems interwoven with natural language (Filloy & Sutherland, 1996). The students can draw on understandings that they have constructed in relation to everyday life.

When realistic problems are open ended in nature as well as contextual, they have an added advantage. They draw on the same content but allow the possibility of the students investigating the situation for themselves and so coming to a better appreciation of the concept as a result of their own thinking (Sullivan, Warren, White, & Suwarsona, 1998). They also produce quite different classroom interactions in that the students reporting on their own insights and the variety of solutions they find becomes a part of instruction. This can lead to the development of a classroom culture that supports the processes of justification and deliberate argumentation (Brown & Renshaw, 1999), another principle focus of mathematics reform classrooms.

This paper reports on a study of Grades 7 and 8 students' responses to an open-ended realistic division problem that required them to generalise in a real world context. In this generalisation, the problem reflects those used in the modern practice of introducing algebraic thinking through problems situated in 'real world' contexts (e.g., the handshake problem, making fences and the number of poles required). It underpins the development of algebraic reasoning in its several forms by facilitating processes of justification and argumentation that are believed to accompany acts of generalising and formalising (Kaput & Blanton, 2001). It can be seen to be an example of algebrafication of existing arithmetic problems by transforming them from one-numerical answer arithmetic problems to opportunities for pattern-building, conjecturing, generalising and justifying mathematical facts and relationships (Carpenter & Franke, 2001; Kaput & Blanton, 2001). Thus, the paper explores how students' solutions to an open-ended realistic division problem in order to: (a) identify students' ability to generalise from a 'realistic' open-ended division situation; and (b) delineate students' thinking that supports and interferes with generalising and formalising, that is, thinking algebraically.

The trend to algebrafy the primary curriculum as a result of the recognition of the difficulties many beginning algebra students have with the domain is worldwide (Carpenter, & Levi, 2000; Carraher, Schliemann, & Brizuela, 2001; Warren & Cooper, 2001; Kaput & Blanton, 2001). Such algebrafication is not about introducing formal algebra in the early years but rather about developing arithmetic reasoning in conjunction with algebraic reasoning. Its focus is deeper understanding of mathematics by fostering fundamental skills in expressing and systematically justifying mathematical generalisation not formal manipulations of symbols that is characteristic of secondary algebra (Kaput & Blanton, 2001). It emphasises sense making and understanding, the abstraction of structures from computation, relations and functions, and provides the generalisation and formalisation that, along with the syntactic manipulation of symbols, is seen as the essence of algebra (Lamon, 1998). It is an important part of the transformations within and across representations, the translations between mathematical signs systems and non-mathematical signs systems and the consolidation, simplification, generalisation and reification of actions that are the basis of sign systems such as mathematics (Kaput, 1999).

However, the use of open-ended realistic problems is not without contention. An overemphasis on mathematics in everyday life can (almost inevitably according to Filloy & Sutherland, 1996) result in an under emphasis on algebra. Also, the use by many elementary students of informal methods, while appropriate at a beginning level, can result in avoidance of the later development of formal algebraic methods. Thus, this paper will also explore whether such avoidance is evident in the students' responses to the division problem.

METHODOLOGY

The study used a mixed method approach. A large number of students completed the problem in a written form from which responses a small number of students was chosen for interview. As stated earlier, the problem was open-ended, realistic and directed the students to generalise. It contained two parts to complete:

> *Sarah shares $15.40 among some of her friends. She gives the same amount to each person. a. How many people might there be and how much would each receive? (Give at least 3 answers.) b. Explain in writing how to work out more answers.*

This problem was situated in the "real world" context of sharing money and fits the criteria of a 'good question' as it requires more than remembering basic facts and has several answers (Sullivan & Liburn, 1997). It represented a transformation of a one-numerical answer arithmetic problem. It was chosen for its realism, openness and its support of algebraic reasoning (Kaput & Blanton, 2001) that occurs in the generalising and formalising processes attached to part b, that is, in articulating how multiple answers can be generated by simply dividing the amount of money by the number of friends. Its solution involves translation between language and symbol system and generalisation of the process required to generate solutions.

The division problem was administered as part of a larger written test comprising of 6 tasks to 672 students aged from 11 years to 14 years, with 82% of the sample aged 12 or

13 years, attending six different co-educational schools in Brisbane (2 primary and 4 secondary). Each school was located in a low to medium socio-economic area. The sample was spread across two different grade levels, Grade 7 (n=169) and Grade 8 (n = 503), which straddle the primary and secondary school years (in Queensland, students complete 7 years of primary school before commencing secondary school). A follow up semi-structured interview was administered to 24 students from the 2 schools that made up the Grade 7 cohort. These students were chosen from the school end of year results. Twelve were in the high range, 8 in the medium range and 4 in the low range of achievement. The focus of the interview was to gain insights into the written responses and probe for reasons why students appeared to be experiencing difficulties. The students were interviewed in pairs and the interviews were videotaped. Each problem was presented on a card. They were asked to read the question out aloud and explain to each other how to solve the question. Students of similar ability were interviewed together (e.g., a high achieving student was interviewed with a high achieving student).

RESULTS

The test responses were marked and coded in terms of the number of correct responses (part a) and in terms of the form of the explanation (part b), whether it was attempted, trivial, incomplete and valid. The videotapes of the interviews were transcribed and the students' discussions categorised in terms of the central issue being considered. The results are given in terms of the written and interview responses.

Written responses

The problem involved recognizing that the operation required was "division" and that the application of this operation would generate more than one answer. Students were asked to provide more than 3 answers to the problem (part a) and to explain a process used to generate answers in general (part b). The students' responses to part a, in terms of the number of answers provided, are provided in Table 1.

Table 1. Percentage of responses to part a, *"how many people are there and how much might each get?"*

No of correct responses	No of students (percentage)
0	341 (50%)
1-3	339 (48%)
4-5	11 (2%)

Only 50% of the students responded to part a of the problem, the lowest response rate for all the tasks in the written test. This could reflect student's adversity to answering 'word problems' (this was the only word problem on the test).

The students' written responses to part b, "Explain in writing how to work out more answers", were classified into four categories: i) no response, ii) trivial, iii) incomplete, and iv) valid. Trivial responses simply reflected a single statement. Incomplete responses gave a direction but did not complete how the answers would be found. Examples of incomplete explanations were: *"Try dividing more numbers into $15.40"*, *"I just divided the number by 4. I guessed I got it right (write)"*, and *"Half it…Quarter it…Third it …"*.

Valid responses provided were complete, providing sufficient to explain how many different answers could be found. Examples of valid responses were as follows: *"Divide $15.40 by how many friends you want and then split the money"*, *"You divide the number of friends to $15 and then afterwards you can divide the 40 between them"*, and *"If you have 4 friends and you divide $15.40 between them so each of your four friends get money … If you have 5 …"*. The students' responses to part b, in terms of the categories, are provided in Table 2

Table 2

Percentage responses to part b, "explain in writing how to work out more answers."

Explanation	No of students (percentage)
No explanation	426 (63%)
Trivial explanation	28 (4%)
Incomplete explanation	137 (20%)
Valid explanation	90 (13%)

Two hundred and ninety two students (43%) did not answer both parts, indicating that some students only answered the first part and some only answered the second part. Only 90 students (13%) offered a valid explanation with 11 of these giving 4 or 5 examples of possible solutions.

Responses to the interview

The transcripts of the conversations in the interviews were classified into five broad categories. Each of these categories is described with an example of a conversation from an interview

1. *Not enough information to solve the problem*

(Both low students - **L= Laura, J = Jane, I = Interviewer**)

L Wouldn't be able to work it out. Not enough information to work it out.

I Jamie?

J You can't -You need the number of people.

In this situation, the openness of the problem interfered with these students' ability to reach answers. For them, the problem needed the amount of money and a number of friends, that is, two numbers. The meaning of 'sharing' also caused some concern.

2. *Sharing means continually dividing by 2*

(Both medium students - C = Cheryl , D = Danni)

C I reckon you take 15.40 you could divide it by 2 and keep dividing by 2 and how many times it divides is how many friends. And when you get down to as low as you can, that's how much money each person gets.

I Ok

D Divide 15.40 by two and keep dividing it till you get to zero.

15.40
− 2
15.38
− 2
15.36

C writes it down on paper to show her reasoning.

C continues to talk through the answers as she works out the solutions $15.38 but then she shares it to her other friends, but then the other person gets 15.36 so you keep doing that.

I So how much does each person get?

C 2 cents.

I Ok it just goes on forever, what do you think about that Danni?

D I think that you find half of $15.40 and share it with a friend. Like she gives one half to her friend and the other half to herself. $7.70 or something.

I So what do you do from then on?

C You get 7.70 and share it among two friends.

D Then you subtract 7.70 again and keep taking it off until zero.

C Half is divided by 2. I think I muddled it up.

E Can you have more than 2 friends.

C Don't know.

D I reckon she'd have just her and a friend. Maybe it could be like 3. 35 and share it with another 2 friends.

Cheryl and Danni seemed to view division, initially, as repeated subtraction of 2 and then as repeated division by 2. Sharing seemed to evoke a "twoness" in their solutions to the problem.

3. ***Must go evenly into $15.40***

(Both high students - M = Michelle, S = Sarah)

M Have to figure out how many friends she has and how much they will get. They have to get the same amount of money each and it has to equal 15.40.
I Ok so give me a solution.
M Ok you start with the number say 5 and you do 15.40 and you divide it by 5 and if 15.40 can be divided by 5 then she would have 5 friends and you would have to work out how much each gets.
M Yeah probably …..would it be $3.08 each.
I What about if she had 3 friends?
M No you can't do it, so she can't have 3 friends.
I She can't have 3 friends?
M&S No
M Will have to be divided into 40.
I How many answers do you think there are?
M So I think there is only 1- 5 cause that is the only number that can go into 15 <u>and</u> 40.
I Only one, is it what do you think Sarah?
S Ten can but not sure.
M Not ten and ten doesn't go into 15 evenly. Cause its 1 and a half. It says that she gives the same amount to each person.
I So what do you think Sia?
S I guess she is right.

Michelle separated the dollars from cents and shared each separately. Sarah did express some doubts about Michelle's reasoning, but it seemed that Michelle's confidence in her responses acted against any challenges by Sarah. This is not uncommon when students engage in open discussions (Goos, 2000).

4. ***Context of the problem - must share evenly and can't have 1c or 2c coins*** (Both high students - R = Robin, A = Allan)

R Basically I would keep dividing it, like 3 divided by that, you could try that.
A Wouldn't work - you'd have a couple of cents left over. You can’t divide by 3 without having some cents left over.
I Oh ok -What are some numbers?
R No, oh yeah 5 it would be $3.08
A Can't get 8c in money.
R Don't have 1 or 2 cent coins
I Ok so what else works?
R 7
A $2.20.
I What else?
A The highest you can get, because you can't divide by 8.
A $ 2.20 I’d say she’d have 6 friends and herself.
I Can’t think of any other solutions?
R No

5. ***General solution ignoring the context***
(Both high students - J= John, D = David)

J She had 154 and you could divide 15.4 like 10 cents each.
I Oh ok, among 15 people at 10 cents each.
J and if there were two people it would be $7.70.
I How many people could you have?
J 1,540 and you could get 1 c each.
D Or you can get a fraction of a cent.
I And can you get a fraction of a cent do you think?
J Wouldn’t pay them much.
I Wouldn’t pay them much. So what other numbers could you have?
J 1,540
I Yeah, go smaller.
J You could have 5.
D You could have lots of numbers. Just divide

John and David ignored the context of the problem and focussed on the general process of division in order to reach an “infinite” number of solutions. In this, they differed from the students described in categories 3 and 4 who situated their solution in a real world context versus students who.

DISCUSSION AND CONCLUSIONS

With respect to division, the interviews provided many insights into how students’ experience and understanding with division, particularly for partitive division (or “sharing”) problems (Greer, 1992), gives rise to so much difficulty in the written test. First, the difficulties with absence of information seemed to be a consequence of word problems commonly not being open-ended and usually having two numbers. Second, difficulties with “twoness” and remainders appeared to reflect limited intuitive models

(Mulligan & Mitchelmore's, 1997) and restricted cultural conceptions of sharing. Third, difficulties in finding many solutions appeared to be a consequence of embedding the division problem in a real worlds context of money. In realistic contexts, many students seemed to assume that money is discrete and those sharing cannot receive fractions of coins in their solutions (a belief that prevents students going beyond the context of the problem and identifying the structural aspects of “to share you simply divide the among you have amongst the number you are sharing with”). Overall, students’ simple early understandings of division and their assumptions about the realistic context of the division problem appeared to act as a cognitive obstacle to later mathematics learning, a conjecture that deserves further research.

With respect to generalising, both the intuitive understandings of division and the real context of the problem impacted on the generalisation process. While the students’ discourse supported the notion that the division problem does stimulate argumentation and justification, its impact on algebraic thinking (identifying the underlying structure of arithmetic) is of concern. The context of the problem appeared to evoke certain conceptions of division that restrict algebraic thinking (an interaction that deserves further investigation). Students’ attentions seemed to be on the solution itself rather than the method used to solve the problem. While this assisted them in reaching realistic solutions to realistic problems, its-long term effect on thinking algebraically needs further investigation. Is algebraic thinking easier in context-free situations? And does the continual use of open-ended realistic problems result in the avoidance of the difficult transition from arithmetic to algebra (Filloy and Sutherland, 1996). Both these questions need further research.

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A COGNITIVE MODEL OF EXPERTS' ALGEBRAIC SOLVING METHODS

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We studied experts' solving methods and analyzed the nature of mathematical knowledge as well as their efficiency in algebraic calculations. We constructed a model of the experts cognitive functioning (notably teachers) in which the observed automatisms were modeled in terms of schèmes and instruments. Mathematical justification of transformation rules appeared as the basis of teachers' algebraic competencies. Our research led us to consider the management of mathematical justification as a fundamental teachers' task.

The mastery of literal elementary calculations concerns the conceptualization of mathematical objects (equation, inequation, systems…) and efficiency in using transformations rules. We studied experts solving methods (notably teachers' solving methods); the nature of their competencies and efficiency. On the one hand, subjects constructed mathematical knowledge and on the other hand, mental structures (schèmes) which allowed them to rapidly perform algebraic calculations. Mathematical justification of transformation rules appeared as the basis of the efficiency.

The identified invariant task in algebraic calculation and the different types of justifications given by experts and students, led us to take the theoretical concept of the operational invariant further than previous authors. The different types of operational invariants involved in algebraic calculations concerned operational knowledge and the efficiency of solving methods. Experts' solving methods included analysis tasks which allowed choosing a pertinent transformation. Observed automatisms in the subjects' cognitive functioning were modeled by means of the theoretical concept of instrument.

Our theoretical framework is fundamentally based on the research of Vigotsky, Piaget and Vergnaud in the area of psychology; and the research of Rabaldel and Pastré in the field of cognitive ergonomics.

THEORETICAL FRAMEWORK

Transformation rules state a mathematical property on the one hand, and on the other, they propose an action. Most transformation rules, given by teachers or textbooks, state mathematical properties which are considered to be self-evident (self-justified). But later, students adopt transformation rules that are more "economical", closer to the written transformations. For example, the rule *"Add (or subtract) the same number to each member of the equation"* is replaced by another: *"Transfer a term to the other side of the equation by changing its sign"*. The latter states a mathematical property that is not at all self-evident, but it does describe the transformation that students perform on paper. Also, other rules (used in factorization, development, numerical calculation, etc.) are sometimes totally obscure. Using the rule, the student develops a mental structure that enables him to take action in a structured manner.

The utilization schème of a rule is constructed by a subject through using the rule and through writing transformations. This schème (the concept of schème was introduced by Piaget) allows performing a transformation quickly and without focussing attention on the action at hand. Thus, subjects can acquire some distance with the task. Consequently, *the utilization schème of the rule may be described as a controlled automatism.* Schèmes can be observed in the fact that the subject performs transformations quickly and his notations are stable. In general, human beings construct adapted schèmes for any type of task. The utilization schème of the rule organizes the way the transformation is written and thus leads to new rules ("economics rules") which correspond to the transformations as written.

Instrument : the utilization schème of a rule is constructed using the rule, but it is not the rule itself. The given rule (symbolic object) and the utilization schème (subjects' mental construction) form a whole that we call an "instrument". P. Rabardel (1995) introduced the theoretical concept of instrument in the field of cognitive ergonomics.

The solving methods. Experts identify mathematical objects and define relevant goals (goals are not the same in solving equations, inequations, systems…) and a solving strategy. Consequently, it can be said that subjects construct a particular solving method for each type of mathematical object. At the level of the solving method, subjects analyze the particularities of the mathematical object and choose relevant transformations, which are performed by means of instruments. For each transformation, experts use an instrument and the same instruments are used in solving different mathematical objects. Then, instruments are not associated to a particular mathematical object and *they establish continuity between methods.*

Solving methods are "instrumented" schèmes, another type of schème, which involves analyzing task. The efficiency of solving methods depends on the constructed operational invariants. If the method is well constructed, subjects can reconcile speed with efficiency in performing transformations.

THE THEORETICAL CONCEPT OF OPERATIONAL INVARIANT.

The concept of operational invariant was introduced by Piaget. For example, Piaget (1950) considered conservation principles in physics (e.g. the conservation of energy) as operational invariants of physical thought: *"conservation principles constitute both absolutes of the considered reality and operational invariants of the deductive processes used to analyze this reality...".* Later, Vergnaud (1990) identifies two types of operational invariants concepts-in-action (concepts pragmatically constructed by subjects) and theorems-in-action (for example, mathematical properties, true or false, pragmatically constructed by students. In the present research other types of operational invariants were analyzed.

In Cortés A & Pffaf N. (2000) mathematical justifications of transformation rules were identified as operational invariants. The most general justifications are the conservation of the truth-value and the conservation of an identity.

The concept of numerical function and the conservation of the identity. In all definitions of numerical function two types of variables can be distinguished. For example, in the equality f(x) = 50 (x-2), x is the independent variable and f(x) is the dependent variable. The definition of this function is the algebraic expression given, which allows calculating the corresponding value of f(x) for a given value of x. We analyze the meaning of the equal sign as an “identity”: the numerical value of the algebraic expression will be the numerical value of f(x), for all given values of x.

Allowed transformations (factorization, development, numerical term reduction, etc.) conserve all couples x => f(x). For example, f(x) = 50 (x-2)= 50x-100. Both algebraic expressions have the same numerical value for all given values of x: *the transformation conserves the identity*. Such transformations can be analyzed as “rewriting”, at least the most elementary among them.

This analysis can be generalized to all the “rewriting transformations” used in algebraic calculations. Transformation made on only one member allow the equation to be rewritten: (e.g. (-6) (4x + 20)= 3x+100□ -24x-120= 3x+100), an implicit function is conserved (the left-hand member). The fact of focussing attention on a particular term implies constructing an implicit function, which is transformed (e.g. $x^2 □ 4 + (x □ 2) \geq 0$ $f(x) = x^2 □ 4 = (x+2)(x □ 2)$; $(x+2)(x □ 2) + (x □ 2) \geq 0$) The implicitly constructed identity is conserved.

The conservation of the truth-value. The left and the right sides of an equation are different algebraic expressions (e.g. (-6) (4x + 20) = 3x+100). For given numerical values of x, the numerical value of each side varies according to its own law. In general, the equation can be seen as the equality of two functions. The equal sign has a different meaning in comparison to the equal sign of functions. This equality is only true for a particular value of x (the solution of the equation): the equal sign expresses equivalence for the solution of the equation. However, x can take other values and the equality then becomes false. Consequently, the concept of solution leads on the one hand to particular values of x and on the other, to the truth-value of the equation. The truth-value is a fundamental property of the equation.

Transformations which change the direction or the unequal sign of inequations (-6x < -3 ; 6x > 3) can be justified by conservation of the truth value in numerical expressions (-6 < -3 ; then 6 > 3). Allowed transformations always conserve the truth-value of the initial expression. The concept of truth-value is implicitly used by students in calculating solutions to trivial equations, by giving values to the unknown (without making transformations). Likewise, students implicitly use the truth-value of equations when they check the calculations they have made.

EXPERIMENTAL WORK

Our research is based on individual interviews concerning the solving of equations, inequations and systems of equations. We conducted: a) Ten recorded interviews with “experts” in algebraic calculations (engineers and scientists). b) Five recorded interviews with high school mathematics teachers. c) Ten recorded interviews with with10th grade technical high school students. In all cases the number of subjects was small, but was enough to identify the main characteristics of the subjects’ solving methods.

The different types of operational invariants constructed by experts for algebraic calculation.

In our research, we take the theoretical concept of operational invariant farther than previous authors. Indeed, we have identified invariant tasks, in other words, tasks that the expert always undertakes (implicitly or explicitly) in algebraic calculations. Each task is undertaken by means some specific piece of knowledge or by means a competence, that we call the operational invariant of the task.

Five invariant tasks were identified: 1) The identification and the analysis of the mathematical object. 2) The respect of the priority of operations. 3) The checking of the validity of the transformation. 4) The checking of transferred terms in a new written expression. 5) Numerical calculations.

1) The identification and the analysis of the mathematical object. For these tasks the operational invariant is a concepts (equation, inequation, function, system) which allow subjects to choose a particular solving method (to define goals and a particular transformation strategy). A relevant transformation is always chosen after the analysis of the particularities of the mathematical object.

2) Operational invariant concerning the respect of the operation to be given priority. The identification (usually implicit) of the operation to be given priority allows choosing a relevant transformation (for example, factorization and development change the priority of operations and allow other transformations). There is a multiplicity of operations (square root, power of a number, additions, etc), and the operation to be given priority depends on the situation. Subjects need knowledge concerning each pair of operations involved in a particular situation; in this sense, this knowledge (operation invariant) is composite. Similar operational invariants were find by Pastré (1997), in the area of cognitive ergonomic.

3) Operational invariants concerning the checking of the validity of transformations. The mathematical justification of a transformation (for example the conservation of an identity) makes a link between the mathematical properties of the transformed object and the rule used. We consider the mathematical justification of rules as the most important operational invariant because it allows subjects to check the validity of transformations. Different types of justifications were identified:

> *a) Operational invariant of the type "principle of conservation".* All allowed transformations conserve the truth-value of mathematical expressions (invariant property) which allows checking the validity of transformations (operational for the thought). The conservation of the truth-value (or the conservation of the identity) constitutes the most general filiation and justification of transformations; teachers do not explicitly use it. In France, only conservation of solutions is explicitly used.
>
> *b) Operational invariant of the type "self-justified or evident mathematical property"* justifies one or several transformation rules. For example, transformation rules given in 8th grade school textbooks (e.g. "One adds the same number to each side of the equation") later justify the "economic" rules used by students and teachers (e.g. "transfer a term to the other side of the equation by changing sign"). Another example: the reversibility of factorization justifies development and vice-versa.
>
> *c) Operational invariants of the type "theorems-in- action".* For many students, transformation rules do not have mathematical justification. We consider these rules

(true or false) as theorems in action (mathematical properties without justification). Some times teachers who are sure of using right properties were not able to justify the transformation rules they used; these rules are "theorems in action".

4) The operational invariant concerning the checking of transferred terms in a new written expression is a competence: subjects must exhaustively check the written terms in the new mathematical expression. Executing this task, subjects' attention frequently moves from the previous expression to the new one and vice-versa.

5) ***Numerical calculations*** were made by means of mental calculation instruments (we do not analyze these calculation in this article).

THE EXPERTS' ALGEBRAIC SOLVING METHODS

The teachers were asked to solve different mathematical objects (equations, inequations, systems d equations and inequations…). They always began by identifying the mathematical object and choosing goals (for example, to isolate the unknown in solving equations and inequations, to obtain a single unknown equation in solving systems of equations…). The analysis of the particularities of the expression allows subjects to choose relevant transformations. For each mathematical object, particular goals were defined and a particular solving method was employed (a particular organization of the solving task).

The situation is "evident" For most exercises, experts and teachers immediately chose a solving strategy. They have reached a very high degree of expertise and they do not need to justify the "economics" rules they use. Teachers use "economic" rules daily and these rules become "evident": they have the intimate convection (they believe) that they are true, its allows them to work quickly.

In solving a system of equations, for example $x + y = 24$; $50x + 100y = 1750$, teachers first verified the existence of the solution (analysis of the particularities task): the system was considered as two straight lines and teachers verified that they were not parallel. Then, a goal was stated : to obtain a single unknown equation. After transformation, the expression $y = 24\text{-}x$ (which prepares a substitution) was considered as an equivalence; x and y were considered as the coordinates of the solution point. This transition was totally implicit.

In France, teachers do not explicitly use the truth-value property of mathematical expressions. But, only the explicitation of the truth-value of the system allows exploring the double meaning of the system (obscure for most students): the points of a Cartesian representation are the truth-values of each equation and both equations are true only for a couple of values (x and y, called solution). Allowed transformations conserve the system's truth-value, then the solution.

By substitution, a single unknown equation was obtained: $50x + 100\ (24\text{-}x)= 1750$ and a goal was (implicitly or explicitly) stated: to isolate the unknown. The choice of a relevant transformation (development) implied the analysis, in general implicit, *of the priority of operations*. The development transformation was made by means of an instrument (evident rule, plus a utilization schème) and numerical calculations were made by means of mental calculation instruments. A first check was made taking into account the rule used: teachers verified multiplied terms as well as the signs of numerical results.

The mathematical justification of development was "it is the same thing"; which is to say, that the conservation of the identity (f(x)= 100 (24-x)= 2400-100x) was implicit. The checking of the transferred terms was made at the level of the method (independently of the used rule), once the equation was written. These checking tasks were made very rapidly and often subjects were not conscious of doing them. Other transformations can be analyzed in the same manner.

The situation is not evident and the property used must be justified.

Some situations are not evident for teachers and experts. For example, for certain teachers, numerical calculations of the type $((n^s)^p)$ were not "evident" problems, which correspond to an "evident" rule. Subjects hesitated between power addition ($s+p$) or power multiplication (s•p). Then, they constructed a simple numerical model in order to justify the relevant operation. In this type of situation, teachers (or experts, in general) always justified the transformations they used. We consider mathematical justifications as fundamental.

When teachers were confronted with a choice of transformations (among several), they explicitly checked the validity of the transformations*: they used an "evident" of self-justified mathematical property, which they believed was true (intimate conviction)*. This aspect of the expert's functioning constitutes a very interesting model for teaching processes, because students are often confronted with the choice of the relevant rule.

WEAK STUDENTS' ALGEBRAIC SOLVING METHODS.

Most 10th grade students used "economic" transformation rules without mathematical justification (Cotés and Pffaf (2000)). Students' solving methods resemble algorithms. For example in the solving of equations: first, they processed additive transformations according to the rule *"transfer to the other side by changing signs"*. When they had obtained an $ax=b$ type of equation, they then applied a multiplicative transformation: *"the coefficient transfers by dividing"*.

In general, students did not analyze the particularities of the expression to be transformed. Thus, an equation containing a product of factors (e.g. 50x + 100 (24-x)= 1750) became an insurmountable difficulty for some students. They always began with the same transformation, *"transfer to the other side by changing signs"* (e.g. 50x + 24-x= 1750-100). Students could verify that the number 100 had changed signs, but they were not able to check the validity of the transformation. Furthermore, this error suggested that the absence of a justification for the rule, which is to say, the absence of a link between the property stated by the rule and the properties of the equation prevented students from taking the priority of operations into account.

Nevertheless, most of the students were able to solve the previous equation. They used a prescription, *"first develop the parenthesis"* (teachers often make this prescription). Which is to say that the "prescription" replaced the tasks of analysis which makes it possible to choose a relevant transformation. In general, most of these students had improved their performance by adopting prescriptions or new rules without justification.

Thus, in most classes, success in solving inequations could be attributed to the adoption of a new transformation rule without justification: *"when one multiplies or divides by a* *negative number the unequal sign changes direction"*. We observed in Cortés and Pffaf (2000) that the rule was rapidly forgotten by most students and that they treated inequations and equations in the same manner.

Likewise, systems of equations were solved, in most classes, by adopting news rules without justification. In Cortés A. (1995) we observed that many students were only able to treat a particular form of system (e.g. $ax + by= c; áx + b'y= c'$) and that they failed in treating systems containing other types of expressions (e.g. $y=ax$).

In general, student progressed by matching new transformation rules to new mathematical objects; and often, the rules used were redundant. Which is to say, students often used a particular transformation exclusively for treating a particular mathematical object. Students were successful for a short time only.

CONCLUSION□

Experts' solving methods, main characteristics. Subjects identified the mathematical object and defined goals, which conditioned the solving strategy. The analysis of particularities and the analysis of the priority of operations were always rapidly and often implicitly made in order to choose a relevant transformation. Thus, the change of the operation to be given priority (introduced by several rewritten transformations) and the conservation of an identity remain implicit.

Experts' thought often consists in applying an "evident" transformation (held true by intimate conviction). Consequently, teachers did not need justify used transformations. These competencies can not be directly transferred.

For each transformation, experts used an instrument and the same instruments were used in solving different mathematical objects. Then, instruments are not associated to a particular mathematical object. Solving methods are "instrumented" schèmes.

Conclusion concerning teaching processes. Some situations are not evident for teachers: they must justify their transformations. The checking of the validity of transformations stops when subjects consider the mathematical property used to be "evident".

Our analysis of experts' solving methods led us to consider that the management of mathematical justifications is the basis of theirs methods. Indeed, justifications allow checking transformations and the checking processes involve identifying the operation to be given priority as well as the checking of transferred terms. But, which justifications should be proposed to students? We observed that certain justifications proposed by teachers (at the beginning of the learning processes) were later forgotten by most students.

In Cortés and Pffaf (2000), we analyzed a course on solving inequations; we used general justifications (conservation of the truth-value and the conservation of an identity). All students progressed in learning a new subject and weak students also progressed in a previous subject (equations): general justifications give a filiation to transformations. Otherwise, conservation of the truth-value allows justifying the transformation of inequations, systems of equations, and also allows establishing a link with numerical expressions (used as justifications).

In France, the best students acquire their skills through doing many hours of exercises at home. How can algebraic calculations be taught without asking students to make an extraordinary effort? Daily use of the most general justifications can improve the efficiency of teaching courses, because:

a) At the beginning of learning algebra, the justification must be different from the “evident” transformation rule statement. It is the only way to preserve the justification in the student’s memory, because rules change.
b) General justifications can be used in all chapters and allow establishing filiations between chapters: it is possible to teach the solving of a new mathematical object and remediate previous ones.
c) General justifications may constitute the basis and the starting point of the justification process of a particular mathematical property involved in transformations.

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LEARNING TO INVESTIGATE STUDENTS' MATHEMATICAL THINKING: THE ROLE OF STUDENT INTERVIEWS

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Interviewing students, though a common and popular activity in teacher education programs has been scarcely researched as a strategy to prepare prospective teachers for mathematics teaching. This study explores the role of interviews as occasions for prospective elementary school teachers to learn three essential practices in the teaching of mathematics—questioning, listening, and responding. Two orientations to interviewing students used by the participants in the study are described and illustrated. Their reported insights focused more prominently on the practice of listening and interpreting students' mathematical ideas which raises questions about structures and designs of the interview experience so that the practices of questioning and responding become more prominent

Teaching in the ways that are envisioned in the mathematics reform documents (NCTM, 2000) where teachers ask for, listen to, and make sense of students' ideas require a host of skills, knowledge and dispositions that are not well understood. They require teachers to "comprehend students' thinking, their interpretations of problems, their mistakes … and they must have the capacity to probe thoughtfully and tactfully" (Cohen, 1989, p. 75). Learning about students and their ways of thinking, therefore, is inarguably one of the most important domains of knowledge for teaching (Shulman, 1987). From planning lessons, to asking questions, to facilitating class discussion, teachers' knowledge of and ability to investigate students' thinking, can make a difference in the kinds of learning opportunities that are offered to students in the classroom (Henningsen & Stein, 1999; Fenemma et al, 1996). How and where teachers learn these competencies, however, has proven to be an elusive question.

Conducting interviews with students is one strategy that has been proposed to help prospective teachers practice and learn questioning techniques (Moyer & Milewicz, 2002) and learn about students' mathematical thinking (Schorr & Ginsburg, 2000). Similar to previous proponents we consider interviews as a valuable strategy in preservice teacher education. Although conducting interviews differ in significant ways from the challenges and demands of actual classroom practice, it is a context for learning and practicing skills such as questioning, listening, and responding that are essential in the classroom. Although research in recent years has seen a surge of interest in strategies to help prospective teachers learn these essential strategies (e.g., first author, 2000; second author, 1999), interviews with students though a very common and popular strategy in teacher education courses, have been scarcely researched both in terms of their design and their effectiveness or impact. In this study we explore preservice teachers' approaches to interviewing students as well as the kinds of insights they report as having gained from their interview experience.

Data Sources and Analysis

To explore the question of how interviewing students could become occasions for prospective teachers to learn to investigate students' mathematical ideas we draw upon our own experience teaching elementary and secondary mathematics education courses. The data we report here comes from one group of prospective teachers in the first author's elementary mathematics methods course. This course is field-based, that is, the TE students are placed in a local school 2 hours/week and attend class 3 hours/week. The participants of this study were attending a course that was offered during the fall of one academic year. Participants were 18 elementary preservice teachers attending the course in their senior year as undergraduates and prior to their year-long internship placement.

The Interview Task

The preservice teachers conducted mathematics interviews with children from their local field placement elementary schools. Number sense was the topic chosen for the interviews, in particular, the interview protocol provided to the prospective teachers focused on a doubling task (Kelleher, 1996) which investigates students' mental computational strategies when doubling numbers. The interview task was presented to the students as follows:

Look in your field classroom and in their mathematics textbook to learn about how these help students develop number sense and collect student work to find out about how children make sense of numbers.

Write:

(1) Describe an activity from your field placement classroom, or from their textbook and/or teachers' guide, and explain how it helps students develop number sense.

(2) Write about what you learned from interviewing a student about what they understand and can do with numbers. Use the sample interview in the back as a resource and use Chapter 6 and 9 to help you interpret your findings.

(3) Talk about what you learned, found challenging and insightful as an interviewer

The sample interview provided to the preservice teachers did not include the specific questions that they were to ask or in what order they were to ask them, instead it stated the goal of the interview and what they might investigate with the task. It also included some advice about interviewing, such as a reminder to consistently use probes, to give students adequate time to think, and avoid validating student's responses by saying "That's right", or "Good!" If you are compelled to say something, you may use less evaluative feedback such as "That's interesting."

Sample Interview Task

This task asks students to mentally double numbers. It is designed to help us learn about the students' comfort and facility with numbers as well as to explore the students' strategies for doing mental computations. You might begin by asking the students to double 2, then 4, then 8, and so on. Always remember to ask students to explain how they figured their answers. Beware that some students might need clarification of what is meant by doubling and might need an example (using fingers or counters). In terms of what to look for in your analysis, it is interesting to note:

(a) What is the largest number that the student can double mentally?

(b) How does the student handle the numbers that are "easier" to double (e.g.:, multiples of 5 and 10) as opposed to the "tougher" numbers (e.g., numbers that require "carrying" or regrouping such as 17)?

(c) Is the student's strategy a broad or a limited strategy?

(d) What sort of manipulatives (including how the student uses his/her fingers) does the student use to figure out the question?

(e) What do the student's facial expressions and non-verbal cues suggest about his/her level of confidence and engagement with the task?

In preparation to their interviewing experience, several in-class experiences were designed to help preservice teachers plan and prepare their interviews. Preservice teachers watched three sets of video clips showing one-on-one interviews with students. The first two clips were two 5-7 minutes of videos of the first author interviewing two first grade students with the doubling task. The third clip was a video from the MACT (1990) materials where an interviewer is asking fourth through sixth grade students to calculate subtractions mentally and with paper and pencil. Following the viewing of the videos, the preservice teachers and instructor discussed examples of questions that gave good insight into students' thinking, whether the students' responses were conceptual or procedural in nature, and to discuss which questions they would like to ask the students that were not asked by the interviewer.

Data sources and analysis

The class discussion around the aforementioned videoclips was audiotaped. Observation notes from two graduate students were also collected to gather impressions of the participants' orientations towards interviewing. These first impressions were used as an analytical lens and guide to the later analysis of the written reports. These reports were typically 3-5 pages in length. These written reflections were read and examined for constructs, themes, and patterns in preservice teachers' orientations towards interviewing students and their reported insights. The researchers coded the themes using a constant comparative method (Strauss, 1987). The data were clustered around the most salient and recurring theme across the 18 participants. A framework that has been previously used by the authors to look at preservice teachers' learning in other contexts and that focus on their "questioning, listening, responding" practices (see second author, 1999) also emerged as a useful framework to organize and cluster the patterns that arose in this context.

RESULTS

Our initial observations and impressions of what preservice teachers' attended to and ignored when they watched others conduct an interview were similar to those we have made in our previous courses and that have been made by others. The preservice teachers' reactions to the "mental vs. paper and pencil interview" were of surprise that the young students in the video could mentally figure out the subtraction problem much more easily than with paper and pencil. This tends to be a surprise because the written algorithm is an explicit part of the elementary mathematics curriculum, whereas mental computation and estimation is something that students tend to learn on their own and not as part of the mathematics curriculum that is taught in schools. Another reason, is that having been schooled by traditional mathematics, our preservice teachers tend to rely on computational procedures than their own sense making.

Seeking explanations as to why this phenomenon might happen, the preservice teachers tend to raise questions about the validity of the interview process, for instance whether the students were nervous or felt on the spot. They also say that the interviewer was not as supportive or encouraging when the students got an incorrect answer and this could have contributed to the students' difficulty with the written task. They are also outraged that the interviewer asks some of the students (who cannot tell whether their answer is right) whether they would like to check it with the calculator. Many of our preservice teachers tend to see this question as an attempt to embarrass the student for being wrong rather than as a way of challenging the students to rethink their solution.

Yet after those initial reactions subsided, and with further questioning by the instructor, the preservice teachers discussed the students' thinking beyond whether it was correct or not, and wondered about what other questions the interviewers could have asked. For instance, one preservice teacher analyzed the strategy used by one of the students in the doubling interview and brainstormed questions they could further ask the students.

> *I thought it was interesting how he did 19 plus 19 cause like at first I thought he was going to do how Julianne did it: 20 + 20 is 40 and subtract two to get 38. But he did it quite differently, he broke down the 19, and said 19+10 is 29 and then counted up nine more.*

Orientations to Interviewing

Preservice teachers' approaches to interviewing revealed two distinct patterns or orientations: evaluative and inquiry. These terms are meant to reflect the preservice teachers' focus either on the product or the process of the students' thinking. In this study, approximately two thirds of the participants used an evaluative orientation, and one third conducted inquiry-oriented interviews. The evaluative approach is one that is similar to Stigler and Hiebert's (1999) characterization of some teachers' teaching practice as "rapid-fire questions," and that Moyer and Milewicz characterize as "checklisting." In this evaluative mode, the interviewer moves quickly through the interview and asks few or no follow-up questions. This interviewer may also be observed instructing rather than assessing by either explicitly showing students or by asking leading questions.

The preservice teachers in the inquiry orientation on the other hand, were focused on gaining access to the students' thinking and used probing questions regardless of the correctness of the students' response. Consider the following excerpts from two reported interview transcripts (PT: preservice teacher and S: student). In the first example, the preservice teacher moves the student quickly to calculating on paper and pencil after asking only two questions that the student is unable to answer correctly, whereas in the second example, the preservice teacher continues to probe into the students' response to her first question.

Excerpt One

PT: Can you double 45 for me?

S: (long pause) I don't really know, but it's interesting.

PT: Okay, can you double the number 14 for me?

S: 14 is ……16.

PT: You think that the double of 14 is 16, can you try that problem on paper?

Excerpt Two

PT: What is Fifteen doubled?

S: …thirty.

PT: Thirty. What did you do with your hands? You were doing something.

S: I did this.

PT: So, hm…how…So, you counted…

PT: With my fingers…

PT: How did you count? How did you count with your fingers?

S: I go like…hm, fifteen, sixteen, seventeen, eighteen, nineteen, twenty, twenty-one, twenty-two, twenty-three, twenty-four, twenty-five, twenty-six, twenty-seven, twenty-eight, twenty-nine, thirty.

PT: Good. So, how do you know when to stop counting?

S: Because when I get to…when I got to ten I know five more.

Learning and Insights

Regardless of how well or not preservice teachers conducted the interviews or their orientation to interviewing, all the preservice teachers in this study had opportunities to reflect on their learning and insights. Many talked about their insights into students' thinking and could be seen analyzing in more or less detail the kinds of responses the students gave to their questions. Others talked about their insights into the interviewing process and discussed how they were challenged by it. Preservice teachers' insights were categorized using the questioning, interpreting, responding framework alluded to earlier. Most preservice teachers' reflections focused on the "listening" part of the framework that is preservice teachers mostly wrote about what the students said, seemed to understand or be confused about. The following example serves to illustrate.

> *One interesting answer I got was when I asked one of the girls what 12 + 12 is, and she answered 24, and I asked her how she got that. She told me that she knows what 10+10 is, and what 2+2 is, so she just put them together. This tells me that she has those 5 and 10 "anchors" in her head, and she knows how to build from them. On the other hand, when I asked a bit more difficult question—what is 19+19—all three of the children had trouble with doing it mentally.*

Reflections on their questioning and responses to students' answers, on the other hand, were less prominent. Only 8 preservice teachers explicitly wrote reflections that could be placed in those two other categories. One preservice teacher, for instance wrote: “If I were to do another interview, I would plan to ask more follow up questions in hopes of learning more about the students’ thinking.” And another one wrote: “I learned that I need to work on how to better explain and simplify questions for younger students and meet them at their level of understanding.” Still a third wrote:

> *I also learned some important lessons about myself through the math interview. I was often tempted to guide Virginia’s thinking the way I wanted it to go. I had to hold myself back not to interfere. Fortunately, that fleeting moment of frustration was quickly replaced with admiration for Virginia’s careful response. She had made perfect sense of the problem despite my inner concern that she had mixed things up by not seeing it my way.*

CONCLUSION

This study supports the claim others have also made (Moyer & Milewicz, 2002; Schorr & Ginsburg, 2000) that opportunities to conduct interviews with students around a mathematical task provide prospective teachers with multiple opportunities to learn about students’ mathematical ways of thinking and about their unspoken teacher tendencies. We offer that the two approaches to interviewing uncovered in this study might be used as indicators of preservice teachers’ teaching practice before they teach in a real classroom. We can imagine further refining these categories to provide preservice teachers with feedback and further experiences that would help move their orientations towards inquiry rather than evaluation of students’ thinking. Results of this study show that interviewing students provide preservice teachers with opportunities to learn and practice questioning techniques, analysis of students’ mathematical work, and to reflect on these practices. Careful design, structure, and support of the interview experience, however is very important. The design and structure put in place for this study proved to be insufficient to move all the participants towards an inquiry orientation, and to focus everyone’s attention onto their questioning and responding practices. Structuring opportunities that focus more prominently on these elusive aspects of the experience would greatly increase the potential of interviews as contexts for preservice teachers’ learning.

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AFFECTIVE ASPECTS ON MATHEMATICS CONCEPTUALIZATION: FROM DICHOTOMIES TO AN INTEGRATED APPROACH

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The present paper aims to propose a theoretical reflection in order to overcome a strong tradition in psychology concerning the analysis of cognition and affectivity as dichotomic processes explaining human behaviours. A general theory of the human subject is presented to discussion, followed by the proposition of a new unit of analysis for the study of mathematical activity, integrating affectivity and cognition. The choice of such a unit of analysis takes into account the specificity of conceptualization and learning in mathematics, the interest of pre-conceptual competencies-in-action, and the need of studying culturally meaningful situations. This theoretical effort is considered especially relevant for increasing the contribution of psychology of mathematics education in the research context of mathematics education.

INTRODUCTION

How the soul and the body act one against another.

(Descartes, in *The Passions of the Soul,* head-title of the 34th Article)

Theoretical and methodological efforts have been made towards the inclusion of affectivity as a valid explicative variable concerning cognitive abilities and competencies in general (e.g., Ginsburg, 1989), competencies at school (e.g., Frias and cols., 1990) and particularly competence in school mathematics (e.g., McLeod, 1992). This is an important issue for most of those interested in complex psychological processes such as *learning*, *development* and *conceptualization,* since it concerns crucial points in the general (and hopefully less fragmentary) domain of psychology, as well as *urgent* questions in psychology of education in specific domains, such as mathematics (Hazin and Da Rocha Falcão, 2001; Da Rocha Falcão, 2001).

In fact, we seem to have overcome a three-century tradition of opposing affectivity and cognition (see, for example, Descartes, 2003), the first seen as a source of perturbation, a kind of disturbing screen between a rational mind and the real world. In a second moment, we have seen important theoretical efforts in order to emphasize a conjoint effect of affectivity AND cognition (as explicative variables) on specific aspects of complex behaviours (school abilities, achievement and adaptability in work contexts,

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etc.). This approach implies a valorization of affectivity, not only seen as a disturbing aspect but also as a cooperating one (at least until an 'optimum level of excitation', as exemplified by the 1908' formulation of the Yerkes-Dodson Law of the relationship between anxiety and cognitive performance (Yerkes & Dodson, 1908, also referred by Evans & Tsatsaroni, 1996)). Nevertheless, it is important to admit that even in the context of this contemporary approach, the same Cartesian dichotomy is still present. A strong tradition of splitting human phenomena in rational / spiritual aspects (*res cogitans*) and somatic / emotional / animal aspects (*res extensa*) (Descartes, 2003) nourished theoretical systems in psychology stressing one of these poles (to the detriment of the other one), without an integrative approach to show the functional interconnection between affectivity and cognition (Damasio, 1994).

There are central questions to be addressed in order to progress in the theoretical-psychological debate about the topic under discussion. Firstly, it is important to conceptually clarify what is meant in psychological literature by the words *affects*, *affectivity*, *cognition*; discussing this issue implies in clarifying theoretical choices, connected to specific focuses of interest in the domain of psychology of mathematics education. Secondly, it is *fundamental* to take into account that the theoretical choices referred above must be coherently based upon a *theory of the human subject*, which requires the combination of previously mentioned psychological-theoretical approach with a philosophical-epistemic analysis. Once these two aspects have been minimally worked out, it is possible to think about *research methodology* in a more sophisticated context. In the present paper, the intention is to offer some hints for the reflection about the three points mentioned above, towards an *integrative* approach of affectivity and cognition in terms of a third, post-Cartesian approach.

AFFECTS, AFFECTIVITY, COGNITION: WHAT IS THIS ALL ABOUT?

There are two important theoretical systems in psychology which are good examples of the contemporary approach of affectivity and cognition: Piagetian genetic epistemology and Freudian psychoanalysis. Cognition, from a Piagetian point of view, is related to a biological need of equilibration, where affective aspects are seen as "combustible" for logical structures (the "engine"): "(...) *affectivity is considered as the energetic pole of behaviour*" (Piaget, 1972). Cartesian heritage of dualistic approach of affective and cognitive aspects is clearly present here. Freud, for his turn, will stress unconscious pulsional (libidinal) aspects as central in the theoretical explanation of human behaviour, viewing cognition (or *epistemophilic motivation*) as a derivative of libidinal impulse by sublimation or neurosis (Freud, 1910).

There are certainly other important theoretical contributions addressing this specific aspect of an integrative affect-cognition view of human behaviour (see, for example, Henry Wallon, Donald W. Winnicot and John Bowlby's works on child development); none of them, nevertheless, seem to propose theoretical elaboration concerning specific aspects of cognition (i.e., specific knowledge domains or *conceptual fields* (Vergnaud, 1990)) taking into account affective aspects as constitutive, not merely adjuvant.

A pervasive approach concerning ***affectivity*** proposes that this complex psychological process encompasses *changing states of feeling* (local affect) as well as *more stable,* *longer-term constructs* (global affect) (DeBellis and Goldin, 1999, pp.250; italics added). In a similar approach, D.B. McLeod suggests three dimensions of variation in affect: *intensity*, *directions* (positive vs. negative) and *stability* (McLeod, 1992). According to these dimensions, beliefs and attitudes would be seen as "cool" and "stable", while emotional reactions would be classified as "hot" and "unstable" (Schlöglmann, 2003). In a recent effort to systematize theoretical contributions to the present issue, J. Evans and A. Tsatsaroni mention four theoretical models in order to link cognitive and affective domains in educational research: two cognitivist models (Individual-differential and 'Constructivist' models), a Psychoanalitic (traditional-Freudian) model, and a Post-structuralist (Lacanian) model (Evans & Tsatsaroni, 1996).

This diversity of approaches explain the variety of research efforts, in terms of the choice of units of analysis and methodological tools for the study of the dipole affectivity-cognition/mathematical abilities: *emotions* concerning mathematical experience (Breen, 2000; Weyl-Kailey, 1985), psychoanalitic *transfer* and *counter transfer* phenomena in student-teacher relationship (Cabral & Baldino, 2002), *self-esteem* and *self-concept* and performance in school mathematics (Hazin & Da Rocha Falcão, 2001; Ginsburg, 1989), *attitudes* and *beliefs* towards mathematics (De Brito, 1996; Pehkonen, 2001), *personality traits* and *cognitive styles* in mathematical problem-solving (Régnier, 1995; Ginsburg, 1989). In fact, an important aspect to take into account when choosing one or more approaches among the various possibilities briefly mentioned above is: what *theory of the human subject* underlies these various theoretical and methodological propositions? This is the central issue of the next session.

SOME RELEVANT ASPECTS CONCERNING A THEORY OF THE HUMAN SUBJECT

The contemporary contributions of cultural-historical psychology strongly emphasize the need of taking into consideration a developmental perspective, crossed with a psychology of human acts interested in here-and-now phenomena, including classroom scenarios (Valsiner, 2001). According to L.S. Vygotsky, there has been two main philosophical approaches concerning this issue: a dualist/pluralist approach, in which the human subject is segmented in various spheres or aspects, like: biological (endogenous, nature) vs. cultural (exogenous, nurture); brain vs. mind; cognitive, rational vs. affective, passional; individual vs. social, and the like; on the other hand, there is a minor effort towards a *monist* approach in which a unified consideration of human consciousness and/or activity in real contexts is proposed (Vygotsky, 1996). Most theoretical/methodological approaches, as mentioned in the previous section, include the idea of a subject splitted in two or more aspects. In this context, affectivity (in its various instances) is clearly seen as a variable that could be isolated.

In the context of the proposition of an integrative approach, Valsiner & Van Der Veer give priority to the critical revision of the antinomy individual-society. For these authors, many others dichotomies could be also overcome by the critical deconstruction of this central antinomy. On the way to this critical revision, Valsiner & Van Der Veer propose to overcome the approaches of the individual's *fusion* in or *captured* by the society, towards the consideration of an ***inclusive separation***, a co-construction of both the subjectivity and intersubjectivity (Valsiner & Van Der Veer, 2000). J. Valsiner introduces the crucial theoretical idea of *semiotic mediation of affective processes.* According to this idea, the affective experience starts from the most simple level, called level 0 (zero), concerning the 'inner state of excitation', followed by bodily emotions ("general immediate feeling tone", level 1); these basic levels are followed by a crucial level 2, where specific categories of emotions are labeled by words like "sad-sadness" and "fear", and where it is important to mention a co-construction of a subjective experience semiotically mediated (then culturally embedded) by language; at level 3, generalized categories of feeling are construed, once more through discursive actions like the speech construal "I feel bad"; finally, at the most elaborated and complex level 4, over-generalized feelings are semiotically construed, as denoted by speech construals like "I just feel... can't describe it"; at this level, "(...) the person 'just feels' something – but cannot put that feeling into words" (examples of this experience are aesthetic feelings – 'catharsis experienced during a theatre performance', or in 'interpersonal situations of extreme beauty'). According to J. Valsiner, the experiences above "(...) can be seen as examples showing that human affective field can become undifferentiated as a result of extensive abstractions of the emotions involved, becoming overgeneralized to the person's general feelings about oneself or about the world" (Valsiner, 2001, pp. 164). At this level, the meaning of the affective experience cannot be analyzed in terms of the individual *or* the societal-cultural world; this is the very theoretical contribution of the notion of *inclusive separation*, mentioned above: most psychological phenomena must be analyzed in terms of a dialectical *co-construction*. Under the enlightenment of these considerations, some methodological consequences must be emphasized.

FROM RESEARCH ON AFFECTIVITY AND MATHEMATICAL ACTIVITY TO RESEARCH ON MATHEMATICAL SENSE-MAKING

Two central points must be emphasized here: first, it is always valid reaffirming that methodology is not accepted to pre-establish approaches, choices, limits, targets; theory is the riverbed in which methodology flows. Second, mathematical activity is not more or less complex than any other activity in diverse cultural contexts (Rogoff & Lave, 1984). Nevertheless, mathematical activity has a *specificity* that must be taken into account. The teacher is expected to *consider* the mathematical activity in its complexity, *including* for example the need – of the student – to have approval and love from the teacher; at the same time, a challenge to both teacher and psychologists of mathematics education is: how to be open to these aspects and at the same time not to change the classroom into another cultural place (e.g., familiar or therapeutical place).

The psychological contribution to mathematical activity of sense-making must take into account the systemic complexity of this and any other human activity, keeping in focus the epistemic specificity of mathematics. A psychological approach that loses sight of this last aspect offers a poor contribution to the domain of psychology of mathematics education (Da Rocha Falcão, 2001); on the other hand, "operational splitting" of human activity would ascribe psychological contributions to a tradition of theoretical oversimplification (mostly dictated by epistemological choices concerning psychology as a "valid" science). These considerations lead to a central question concerning the link between theoretical and methodological aspects, in the context of the integrated psychological approach of mathematical activity: which should be the *minimal unit of* *analysis* to the study of this specific activity? This is a complex question, for which we propose four central points to be considered: 1. A theory of reference concerning learning and conceptualization: according to G. Vergnaud, learning and conceptualization always refer to specific domains (Vergnaud, 1997), and learning of mathematics has certainly its specificity. As a consequence, any research proposal on learning cannot avoid a previous epistemic analysis of the conceptual field explored. 2. The consideration of pre-conceptual competencies-in-action: we refer here to human competencies that have two major characteristics: firstly, they are *effective,* in the sense that they help people dealing with daily, culturally situated situations; secondly, these competencies are very hard to *express* by any symbolic means (natural language, graphic representations, mathematical models, and so on). Examples of these competencies are those shown by handicraft workers, but also some competencies of very highly school-educated intellectual workers (researchers, engineers, specialized technicians, and so on – see, on this subject, Samurçay & Vergnaud, 2000). Taking these competencies into account in mathematical conceptualization implies in connecting school activity to other socio-cultural contexts. 3. The integrative approach of cognitive-affective aspects: delimitation of affective and cognitive poles reflects a philosophical perspective on human nature that cannot be considered as theoretical *a priori*. Most researchers on the domain of affectivity and mathematical activity, as discussed on section 2 above, have stressed that affective states can vary from very "hot", emotional, "irrational" states to very "cool", attitudinal, cognitive-like states (Schlöglmann, 2003). On the other hand, there are references to "aesthetic feelings and motivation" in mathematical activity (Gadanidis, Hoogland & Hill, 2002), "mathematical intimacy and integrity" (De Bellis & Goldin, 1999), and considerations issued from neuroscience on "memory about emotions as a cognitive memory" (Schlöglmann, 2002). Even though we can always refer distinctly to "affective" and "cognitive" systems (in the context of neuropsychology, for example), it seems that it would be highly productive to overcome this dichotomy in the context of the building of a new unit of analysis in psychology of mathematics education. 4. The proposition of situations to be analyzed in a diachronical process: finally, this new unit of analysis should result of an effort to "bring complexity of psychological phenomena into the analytic focus of psychology" (Valsiner, 2001). This effort implies in considering that culture is a part of the systemic organization of human psychological functions, but at the same time, human beings can distance themselves from any current setting through such cultural (semiotic) means, and yet they remain parts of the setting, as suggested by J. Valsiner (Valsiner, 2000, pg. 59). Psychological unit of analysis, hence, must consider meaningful cultural situations, without evacuating individual *subjectivity*. The consideration of these cultural situations, on the other hand, cannot be reached by the only analysis of discursive production; we assume that the development of conceptualization is based simultaneously on the extraction of regularities in empirical world, on the construction of predicates and inferred, non-directly observable objects, and finally on the establishment of relations between linguistic invariants and operational invariants (Vergnaud, 1997). In other words, if it is not possible to reduce thinking to empirical world, it is not possible to reduce it to specific structures of language either. Finally, an important tool to be extensively used to explore such a complex unit of analysis is *interpretation*, through comparison and generalization. Interpretation, however, must be circumscribed to the specific research context (i.e., psychology of mathematics education) in which it is exercised, without problematic amalgam with other contexts of use (e.g., the clinical-psychotherapeutic context).

FINAL REMARKS AND CONCLUSION

The main target of the present paper was to offer hints in order to contribute to the proposition of a new unit of analysis concerning mathematical activity, overcoming the traditional-Cartesian dichotomy between affective and cognitive aspects, among other dichotomies. This presumably more productive unit of analysis targets cultural situations, in the context of which a mathematical activity takes place, involving a set of identifiable epistemic contents (a conceptual field). Individuals can act in the context of these situations in various ways, with a common goal; psychological analysis should be able to show both diversity and generality of these phenomena, through the interpretation of situated actions. This interpretation covers discursive acts as semiotic productions necessarily framed by socio-cultural contexts (e.g., mathematical didactic contract in the classroom of mathematics, workplace culture). Furthermore, individual emotional states and conceptual metaphors can only be adequately reached by including the analysis of individual bodily gestures (as developped by Lakoff and Núñez, 2000).

Let us emphasize that all these aspects should be considered in a dialectical, conjoint approach, according to the theoretical concept of *inclusive separation*, discussed above.

Affect and cognition are in fact ways of looking at the same phenomena: human activity. The specific contribution of psychology, in the context of the community of mathematics education, is the proposition of an integrative approach of the human subject as a mathematics learner *possessor of a* subjectivity that is always embedded in culture, but never subsumed by this same culture. Considering this same discussion under a methodological point of view implies in focusing on smaller and *yet* complex enough situations, in order to be able to tell a better *narrative* about people doing mathematics. This is a valuable research target for the coming years.

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WHAT CAN WE LEARN ABOUT COGNITIVE LEARNING PROCESSES BY ASKING THE PUPILS?

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The focus is on the metacognitive awareness of ten high-achieving high school pupils from Denmark and England and their cognitive learning process. Un-structured interviews in smaller groups investigates how they say they learn a mathematical concept that is new to them. I develop the "CULTIS model for analysis" (Consciousness, Unconsciousness, Language, Tacit, Individual, and Social). These are six themes in which various psychological learning theories express themselves. I conclude that the pupils can talk about their learning using own words and I can identify fitting theoretical notions. The pupils each have their own way of learning, however, there are similarities. Seemingly contradictory theories "works" within single pupils.

INTRODUCTION

This reports a recently finished Ph.D. study (Dahl, 2002). It investigates how ten high school pupils (age 18-20) say they learn a mathematical concept that is new to them. There are four Danish pupils (*gymnasium, A-niveau*) and six English (*A-level, AS Level Mathematics*). The pupils have been taught mathematics at the highest level possible and are high-achieving. Various psychological learning theories are used to get a greater understanding of what the pupils say.

METHODOLOGY

Schoenfeld discusses the concept of metacognition, which can be understood either as knowledge about or regulation of cognition (Schoenfeld, 1992, p. 334). Knowledge about cognition means to have relatively stable information about one's own cognitive processes. This knowledge develops with age and "performance on many tasks is positively correlated with the degree of one's metaknowledge" (Schoenfeld, 1985, p. 138). Metacognition, understood as regulation of cognition, includes the planning before beginning to solve a problem and the monitoring and assessing "on-line" during problem-solving and learning (Schoenfeld, 1992, p. 355). Furthermore, good problem solvers "maintain an internal dialogue regarding the way that their solutions evolve" (Schoenfeld, 1985, p. 141). I therefore assume that high-achieving pupils have knowledge of how they learn mathematics and it will be relevant to know the "internal dialogue" as this might give a picture of the learning process.

The teachers have been asked to select the best pupils. The four Danish pupils are interviewed as one group and the six English pupils are interviewed in pairs. Each session with the English pupils consists of three phases, but the Danish pupils do not meet the second phase:

> *1. Semi-structured group interview*. The pupils are asked to describe a usual mathematics lesson, what they *do* when they meet some new mathematics, how they know if they have learnt it, and they have been asked to describe their learning process. This lasts about 10 minutes.
>
> *2. Participant observation*. They get a sheet (Nelson & Wilson, 1990) with some basic knot theory they are supposed to get *some* understanding of. They have the opportunity to involve the observer in the learning process, but they do not. They also get a sheet with questions to make them think about what they do while working with the mathematics. This lasts about 15 minutes.
>
> *3. Unstructured qualitative group interview with open-ended questions*. They are asked what they *did* in order to learn, why they did the things observed, how and why it helped, what they would *do next* to fully understand this mathematical concept, if (how) this learning situation is different from the usual, and how they would present this mathematics to the class. The method is to listen and probe to "open-up" in-talk and reformulate or challenge their answers to get a more elaborate explanation. The attempt is to try not to ask leading questions and to keep an open mind. Periods of silence are allowed to not interrupt anyone's chain of thoughts. The pupils give examples from either the knot theory or some other mathematics they have met. This lasts about 30 minutes. The style of the interview is chosen to avoid a self-fulfilling process if both interview and model for analysis are strongly influenced by theory.

THE PSYCHOLOGY OF LEARNING MATHEMATICS

To have a range of possible explanations of what the pupils tell, mainly the following theorists are used: Glasersfeld (1995), Hadamard (1945), Krutetskii (1976), Mason (1985), Piaget (1962, 1970), Polya (1971), Sfard (1991), Skemp (1993), and Vygotsky (1962, 1978). To some extent they have a similar focus as this study and they are "classics". Below I explore in which themes these theories express themselves and I develop the so-called CULTIS model for analysis. The pupils' explanations are thus classified into these six different themes. The six themes are also divided into three binary opposite-pair. The fact that there is, for instance, a theme named 'social' means that when I read through the transcribed interviews, I label some of their remarks 'social' without, at first, going any deeper into what the pupil means. The themes are thus overall boxes to sort various topics the pupils mention. The themes "cut" the theories into "modules". The themes are: <u>C</u>onsciousness-<u>U</u>nconsciousness; <u>L</u>anguage-<u>T</u>acit; <u>I</u>ndividual-<u>S</u>ocial (CULTIS). To some extent the themes interact and overlap each other but each have their own identity.

First pair of themes; Theme 1: Consciousness

According to Polya (1971) and Mason (1985), working with mathematics has three phases. First: 'enter' the problem, understand the problem and device a plan. Second: carry out the plan, and the third is a revision of the whole process.

Polya states that a good idea of a plan is "based on past experience and formerly acquired knowledge. Mere remembering is not enough for a good idea, but we cannot have any good idea without recollecting some pertinent facts" (Polya, 1971, p. 9). The pupil must furthermore understand the problem before starting to work on it, and the pupil should "desire its solution" (Polya, 1971, p. 6), or in other words, be motivated. In Activity Theory motivation plays the major role (Mellin-Olsen, 1989, p. 16-17), but is in this paper mentioned as one item of many. According to Polya, it is a practical skill to be able to solve problems and since we require all practical skills by imitation and practice, this also applies for solving mathematical problems (Polya, 1971, p. 4-5). Also Sfard (1991, p. 18) states that operational understanding is the only way to 'get in touch' with abstract constructs. Mason writes that practice is important but without reflection it may leave no permanent mark, and that it also needs time. Mason also states that to support mathematically thinking one needs a questioning, challenging, and reflective atmosphere (Mason, 1985, p. 153).

First pair of themes; Theme 2: Unconsciousness

Hadamard (1945, p. 56) states that there are four stages in learning: preparation, incubation, illumination, and verification. Conscious work is preparatory to the illuminations. Polya states that "only such problems come back improved whose solution we passionately desire ... conscious effort and tension seem to be necessary to set the subconscious work going" (Polya, 1971, p. 198). The illumination is generally preceded by an incubation stage where the solving of the problem is completely interrupted (Hadamard, 1945, p. 16). The first stage in solving a problem is therefore to work in a very concentrated manner on it. What is experienced as sudden inspiration "is the result of previous protracted thinking, of previous acquired experience, skills, and knowledge" (Krutetskii, 1976, p. 305).

Second pair of themes; Theme 3: Language

Different theorists discuss the indispensable role of language, words, and concept formation in learning. Vygotsky describes language as the logical and analytical thinking-tool (Vygotsky, 1962, p. viii) and that thoughts are not just merely expressed in words but come into existence through the words (Vygotsky, 1962, p. 125). Furthermore, as mathematics in itself is a language (Pimm, 1990, p. 2; Dahl, 1996), it becomes important also to discuss concept formations. In relation to the learning of mathematical concepts, a basic principle is that all concepts, except the primary ones, are derived from other concepts and they take part in the formation of other concepts (Skemp, 1993, p. 35). This conceptual structure is called a schema, and a schema is therefore a tool for learning as it integrates existing knowledge (Skemp, 1993, p. 37). Similarly, Tall (1991, p. 9) refers to the notions of *assimilation,* a process by which an individual adopts new information and *accommodation,* which signifies that the individual's cognitive structure must be changed. Thus, it seems that language is not essential for the creation of the basic concepts. But higher order concepts build on the basic concepts and to learn the higher order concepts, other concepts are necessary. Furthermore (Skemp, 1993, p. 29-30), an integrated conceptual structure is easier to remember than unconnected rules.

Second pair of themes; Theme 4: Tacit

There are also more negative views of language as a tool for learning. To Hadamard, "thoughts die the moment they are embodied by word" (1945, p. 75), but, however, “signs are necessary support of thought” (Hadamard, 1945, p. 96). Piaget (1970, p. 18-19) states that “This, in fact, is our hypothesis: that the roots of logical thought are not to be found in language alone, even though language coordinations are important, but are to be found more generally in the coordination of actions, which are the basis of reflective abstraction”. Thus, actions are the root of logical and mathematical thought. In relation to tacit knowledge, one can observe that a person has a certain kind of knowledge, but “on questioning, it appeared that he did not know he was doing this. Here the subject got to know a practical operation, but could not tell how he worked it” (Polanyi, 1967, p. 8). The “negative” arguments are thus centred on the general uselessness of words in thinking and learning, that language merely “supports thinking”, as well as the lack of ability to describe what one is doing.

Third pair of themes; Theme 5: Individual

The individual perspective of learning is represented by for instance Glasersfeld whose epistemology is that “knowledge, no matter how it is defined, is in the heads of persons, and that the thinking subject has no alternative but to construct what he or she knows on the basis of his or her own experience. … all kinds of experience are essentially subjective (Glasersfeld, 1995, p. 1). Piaget asks what mathematical-logical knowledge is abstracted from and he finds that the basis of abstraction comes from the action itself, not the object (Piaget, 1970, p. 16-18). The individual who is learning is therefore active and the acknowledgement comes as the individual manipulates with the objects and reflects on this manipulation. Piaget talks here about reflective abstraction, which among other things means the transposition from one level of a hierarchy to another, and it means the mental process where a reorganisation of thoughts takes place. Reflective abstraction is based on coordinated actions, not individual. This therefore means that (1) language is not the main thinking-tool, (2) both individual actions and coordinated ones are performed by the individual and they both lead to abstraction, but it is the latter that leads to reflective abstractions and then to logical-mathematical knowledge. Piaget therefore finds that logical-mathematical abilities do not arise from language or linguistic competency, but from the ability to coordinate actions and operate with objects.

Third pair of themes; Theme 6: Social

In this theme, social interaction plays a fundamental role in shaping pupils’ internal cognitive structure (Schoenfeld, 1985, p. 141). This process has two levels: “first between people … and then inside the child ... All higher functions originate as actual relations between human individual” (Vygotsky, 1978, p. 56-57). This process of internalisation is gradual. In the beginning a teacher controls and guides the pupil’s activity, but later they begin to share the problem-solving functions, and here it is the pupil who takes the initiative while the teacher corrects and guides. At last, the pupil is in control and the teacher’s role is mainly supportive. According to Vygotsky, the potential for learning is furthermore limited to the “zone of proximal development (ZPD)” (Vygotsky, 1978, p. 86). ZPD is the area between the tasks a pupil can do without assistance, and those, which require help. It is therefore essential that pupils are active and have the opportunity to be guided by a knowledgeable person. Verbal thinking is an example of a social activity. When the pupil speaks aloud, the “audible speech brings ideas into consciousness more clearly and fully than does sub-vocal speech” (Skemp, 1993, p. 91-92). Vision is therefore individual, while hearing is collective (Skemp, 1993, p. 104).

A possible synthesis of different psychological theories?

Above is seen that the various theories are sometimes contradictory. The dualism is mainly in Theme 4 to 6. Two representatives of this are Piaget and Vygotsky, particularly about the role of the individual and the social and of language. According to Lerman (1996, p. 133), “Vygotsky’s and Piaget’s programs have fundamentally different orientations, and that the assumption of complementarity leads to incoherence”. However, Piaget and Vygotsky have a mutual sympathetic, yet critical, view of each other (Piaget, 1962; Vygotsky, 1962). I therefore discuss the possibility of a synthesis between them as well as the possibility of a grand-theory (Dahl, 2002). The conclusion was, for now, to settle with Bohr’s Principle of Complementarity, but here talk about what I express as *odd complementarity* denoting that neither psychological theory is complete, but they might not be equally *dis*-complete.

SAMPLE DATA, DISCUSSION, AND CONCLUSION

The pupils’ statement can mainly fit the CULTIS model. To a certain extent they mention the same things, but there seems to be a difference between them on areas such as the role of practice, language, and individual-social issues. Below is an example from the English interviews with Pupil D and E. The essential is underlined, and in the column to the left is noted which theme the remarks have been identified to belong to. ‘I’ is the interviewer and ‘Q’ means question.

Q I: So, what could they have done, let’s say, if they wanted to describe knot theory.

6 D: I think this is, this is the kind of thing where it is very difficult to talk about in a book and to represent 3-dimensional object within a 2-dimensional way, and it’s where it would help to have a teacher explaining something and say pointing all this is the vertex and this is an edge (E: (inaudible)) (inaudible)

I: Pointing at, I mean

6 D: Yea, drawing it or tiny little knots and say this is

5 E: Depends who they are targeting it at (1 sec silence), don’t be so (1 sec silence) so
4 stuck up (inaudible) (laughs) and so you get to a, I don’t know, don’t use such big
words, they are aiming to people who don’t understand it (I: mmm, well it’s part of
er) and use basic, yea (I: so I mean) no, I know, but I it would seem a bit sort of (1
4 sec silence) if it if it was in a GCSE and A-level course (D: mmm) all this would
have, language, it wouldn’t be right, it’s the the way they approach it, the language,
it’s just too, people would struggling with the language when they are suppose to be
learning the maths.

Q I: So is there a diff, I mean, er, so maths has nothing to do with the language? Or, can you learn maths without language.

4 D: Yea.

3 E: No, but you can use different language, simple language to convey a point.

4 D: <u>Cause the maths in it is quite easy</u>, I think, well, it's not (E: I'm sure it is (giggles)) (laughs) What do you mean it is nothing really difficult what it is saying is this is what a knot is, this is (E: Yea) what a link is, and, OK, that really really simplistic, it takes a long time (laughs (inaudible)) <u>it took me a long time to work out what they were trying</u> (E: <u>Yea what they were explaining</u>) whereas the fact <u>as soon as I, kind of translated it, I thought oh well, that's what a knot is, find that's easy</u>.

Q I: What did you translated it (inaudible)

4 D: <u>Into simple language</u> (laughs) er, it er (inaudible)

Q I: You translate it before you understand it, er, so (D: (inaudible)) if you have understand, then, it, you don't need to translate it.

6 E: I think <u>it here would be easier if the author translated</u> (D: Yea (laughs)) <u>rather than er leaving the reader</u> to er (1 sec silence) <u>to do it</u>, I mean.

4 D: <u>You have you have do the two together, you have to translate while you're trying to understand</u>

The theoretical understanding of Pupil D and E, as well as the eight others, in terms of the Theme 3 and 4 is thus that some pupils say that language is the main thinking-tool, others that it hampers thinking, others that language seems to have a dual nature as it both facilitates learning and hampers learning and this does also depend on the type of language. For Theme 1, almost all the pupils talk about motivation, but there are various views of its nature. The cognitive drive is mentioned as important. A more "outer" motivation is to be forced. An "inner" motivation comes from being confused by something. Some pupils also explain that lack of motivation can be caused by lack of self-confidence. There is also a "show-off" effect. All the pupils talk about doing exercises as important for the learning process, but there are some variations. In Theme 2, most pupils talked about Hadamard's (1945) three phases. About Theme 5 and 6, it seems that most of the pupils argue that learning has both a social and an individual side. The value of the social side is mainly when one experiences problems trying to learn by self-study. After input from others, one can move on alone. This is the case for eight of the pupils. The two others emphasise the social side. Theme 5 and 6 thus complement each other "odd".

There are some utterances that did not fit into the themes. Half the pupils mention that how one is used to learn/being taught, influence on how one learns later on. For instance: Pupil Z explains that how they learn is influenced by the fact that they have been trained to have a visual cognition, and therefore they learn most things through their eyes. Pupil A says that the learning strategies one knows and uses, are connected with the ways one has been taught to do things. Pupil D explains that it is hard to adapt to a different teaching style. Thus, the teaching methods must be part of, what I would express as a *zone of proximal teaching (ZPT)*, inspired by Vygotsky's ZPD. Similarly, one could here state that if a (new) teacher (perhaps a new school) uses teaching methods that are too "far away" from the teaching styles the pupils are used to, the pupils might not learn. Furthermore this leads to a conclusion that a change of teaching styles ought to be gradual. A pupil's previous experience of learning, his learning history, does therefore to some extent influence how he later on is able to learn. This means that the single teacher's method of teaching is a factor that has consequences for the pupil's later learning successes. As this study confirms that some pupils are able to talk about their learning processes, one might argue that a discussion and greater awareness of this between pupil and teacher might, improve the learning.

Another further result is that different learning theories seem to fit different branches of mathematics and the types established go across country and gender.

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KNOWLEDGE SHARING SYSTEMS: ADVANTAGES OF PUBLIC ANONYMITY AND PRIVATE ACCOUNTABILITY

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This paper explores the benefits of the design elements of public anonymity and private accountability in classroom Knowledge Sharing Systems (KSS). The major findings of this study indicate that classroom KSS have the potential to allow for greater equity of input, reduce academic anxiety, increase teachers knowledge of student understanding and improve student participation. .

INTRODUCTION

This paper explores the benefits of the design elements of public anonymity and private accountability in classroom Knowledge Sharing Systems (KSS). These systems allow participants in a class to input information into a computer database and makes that information available to others. For this paper, classroom KSS are defined as computer mediated systems that allow for free response student input, perform some sort of input aggregation, and make the input data available back to the classroom in a meaningful format. Allowing for free response student input specifically refers to allowing the students the ability to enter many different types of data (text, functions, graphics). Public anonymity refers to the ability of the student to submit data to be viewed by the entire class in a way that their identity is not revealed to fellow students. The teachers ability to access who made each submission is private accountability. This paper will focus on a classroom KSS which allows students to do synchronous data input. The author draws on four years of classroom observations using a prototype classroom KSS.

Until recently, in Education research, the predominance of research with classroom KSS which allow for anonymous input has been done using asynchronous data input (Cohen & Scardamalia, 1998; Hoadley & Linn, 2000; Hsi & Hoadley, 1997; Scardamalia & Bereiter, 1992; Scardamalia, Bereiter, McLean, Swallow, & Woodruff, 1989). The exception to this would be work done using the ClassTalk system designed by Louis Abrahamson (Abrahamson, 1998; Mestre, Gerace, Dufresne, & Leonard, 1996). ClassTalk allowed students to do parallel data entry and aggregated the responses in a group display. In the field of Communication research, systems allowing for synchronous data input have been the predominant focus (Connolly, Jessup, & Valacich, 1990; Gallupe & Cooper, 1993; Jessup, Connolly, & Galegher, 1990; Scott, 1999; Valacich & Dennis, 1992). The systems used in Communication research grew out of research on brainstorming in group settings and will be described in greater detail later in the paper.

The major findings of this study indicate that classroom Knowledge Sharing systems have the potential to allow for greater equity of input, reduce academic anxiety, increase teachers knowledge of student understanding and improve student participation.

THE TECHNOLOGY

The technologies being in the classrooms described in this paper are two different versions of a prototype network of handheld devices in development by Texas Instruments. Both prototypes were completed to a “proof of concept” level for experimentation. The goal of placing the early prototypes in a limited number of classrooms was to gain insight into what uses teachers would find for a network of handheld devices. The first prototype version of the classroom network was a wired network comprised of a series of hubs linked together from group-of-desks to group-of-desks. Each of the hubs allowed up to four handheld devices to be connected. The devices used were the TI-83, a graphing handheld. The TI-83 is a stateless device that can be programmed but has no flash memory. The daisy-chained series of hubs were connected to a computer at the front of the room. The teacher accessed a computer side interface to send and get data from the devices.

The second prototype version of the system was architecturally different from the first. This system was wireless and designed with a server-side database. The server was located in Dallas, Texas and the schools using the system gained access via the Internet. This system was composed of nine wireless hubs, an access point, and a concentrator gateway. Each hub allowed up to four handheld devices to connect to it. The devices used were the TI-83 Plus, a graphing handheld, which unlike the TI-83, has Flash ROM. The wireless hubs communicated with the concentrator gateway via an access point. Activities were created through a combination of server side XML scripts and device programs. Unlike the previous system where the send, get, and aggregate commands originated on the computer interface, data movement protocols for the second prototype are built into a device side application. In the second prototype, the classroom computer is used to cue activities and view individual student data. The devices allow for text and numeric input and are fully programmable through a Basic-like language. Both systems use an overhead ViewScreen™ panel that displays the teacher devices’ screen.

OVERVIEW OF COMMUNICATION RESEARCH

The research on knowledge sharing systems in the field of Communications grew out of research done to find the most effective group brainstorming environment. Brainstorming groups have been used for many years in businesses. Ideally, a small group will meet, call out ideas, get inspired by each others ideas, and produce a wealth of beneficial information for the company. Key things which hinder this are, Production Blocking - waiting your turn, someone saying what you were going to say, deciding your input was "stupid" and not saying anything and Evaluation Apprehension - peoples anxiety over what others may think of what they are saying (Connolly et al., 1990; Jessup et al., 1990; Valacich & Dennis, 1992). Starting with early group brainstorming research, it was thought that groups interacting verbally would produce a greater quantity of higher quality ideas than individuals working in isolation (Osborn, 1957). It has been found that the dynamics of waiting your turn to speak and politics inherent in group situations actually make verbal groups less productive for idea generation (Connolly et al., 1990). Research has shown that face-to-face verbal brainstorming groups are not as effective as nominal groups (groups where individual participants write ideas on paper and then submit their papers at the end of the designated time). Face to face groups fall behind in both number of ideas submitted and the quality of those ideas(Valacich & Dennis, 1992). Growing from Osborn's early speculation that seeing other peoples ideas should help groups create more and better ideas. Electronic Brainstorming Systems (EBS) and a Computer Decision Support Systems (CDSS) were created(Gallupe & Cooper, 1993; Jessup et al., 1990). Both allow for all individuals in the meeting to input ideas simultaneously, anonymously and see what others have submitted. Research using Electronic Brainstorming Systems has found that these systems become more and more effective the larger the group size. Studies done with groups up to 18 showed that larger groups were the most productive creating a greater quantity of higher quality ideas (Gallupe & Cooper, 1993). Previously, verbal groups were shown to peak for effectiveness around 4-7 members with additional members actually decreasing productivity. With knowledge sharing systems, larger and larger groups became more effective (Valacich, Dennis, & Connolly, 1994). The parallel to classroom situations is interesting. With classes typically having twenty-five students or more, a classroom KSS could add tremendously to the effectiveness of the communication.

PUBLIC ANONYMITY

Greater Equity of Input

In the networked classroom, students can submit answers to be considered by the class without their identity being associated with that information. Teachers can identify whom the individual information comes from on the computer monitor, but in the group display space, the responses are anonymous.

Anonymity facilitates the ability to explore answers in a non-threatening way. It gives the ability to ask questions like, "What do you think the person who sent in this point was thinking?" or "Who can defend this answer." without tying the identity of the student who sent in the response to the question. This gave the possibility for non-threatening discussions of the ideas. Students can discuss the thought that went into an answer independent of assigning that answer to a specific person. Freed from *who* sent in the answer, they are able to explore *what* the answer might mean.

Teacher: Where with Navigator, I can see the various equations and the differences in the equations and then that promotes discussion. Well what's different? - well obviously the numbers are different but what do these numbers represent, why is it different, and why would somebody have that. It just promotes a lot of discussion and everybody's free to discuss it because kids can be criticizing an equation that they themselves wrote that nobody would know. And they do sometimes, sometimes they'll say, and that's the surprising thing, is after a relatively short time the kids are very open about saying, "Oh that was me, and the reason I did that is...". Um and, it's interesting because it gives, even if, if I'm the one who got the right equation, it's interesting listening to somebody who got a wrong equation because that sorta solidifies, perhaps solidifies my correct concept of why my, why I choose what I choose. But, sometimes kids get somewhat lucky when they're making choices and as they're listening to somebody else, and this has happened, they're listening to somebody else's explanation and they'll go, "Oh no, I didn't do it that way." And so they got the right answer for the wrong reason. Um, and that's, that's interesting because they're, they're really truly understanding what's going on by listening to other people. And, it also helps them see that somebody else may have gotten a totally wrong answer, but in trying to analyze what was wrong about it they were just off on a totally, on a different track. Obviously a wrong track, but its, its not that the person was, was stupid in what they did, they just misinterpreted something. So I think it helps a lot, by having discussions.

Research done using other computer mediated knowledge sharing systems, on the effects of anonymous input, has found that allowing students to submit new ideas or respond to previously submitted ideas, anonymously creates a more equitable environment where boys and girls participate equally (Hoadley & Linn, 2000; Hsi & Hoadley, 1997; Scardamalia & Bereiter, 1992). These three research studies were done using knowledge sharing systems which allowed students to submit data asynchronously. Findings from these studies show that allowing the option of anonymous input creates a more equitable environment for participation (Hoadley & Linn, 2000; Hsi & Hoadley, 1997; Scardamalia & Bereiter, 1992).

Reduces Academic Anxiety

Students identify with their response, icon, data, etc., that shows up in the group display. They want to see their data up front. The anonymity allows them to choose if they identify their representation to others, but all are very conscious of seeing themselves in the group display. Additionally, seeing their response in the group display has made students more accountable to the class.

Where's my point? Who am I? That's my answer! are all common exclamations to hear when running a networked activity with students.

With time, this representation of self in the group space can give the students a sense of how they are doing relative to the class as a whole. As one example, in the class in Islandtown, the students come into class every day and enter their responses to a subset of the homework problems. As a class they then look at the responses and discuss the problems that were the most difficult. This daily activity of seeing how many people got which problems correct, helped the students to feel more comfortable with the idea that some days you understand and some days you don't. They articulate how this let them feel more comfortable in class and more confident to ask the teacher for help.

Interviewer: Why do you think the system is important in the classroom?

Student 12: It just helps everybody open up, and everybody interact, and it really just opens up the classroom because then you know what you need to study, you know where you stand and you know how everybody else is standing and it makes you feel comfortable because you're kind of involved in everybody else and how they're doing in the class. So it makes everybody kind of closer in this class. Cause I know in other classes, I have no idea how anybody's doing. Sometimes I feel like I'm the only kid who's getting bad grades. And I'm the only kid slipping behind, but here I know what's going on and it just makes it more comfortable definitely to come here everyday.

Interviewer: Does that, in the other classes where you don't know how other people are doing (Student 12: Right), you don't know if you're the only one (Student 12: Right), does that raise your anxiety level any...?

Student 12: Oh Definitely! Yeah it's scary, because I think I'm the only one...I'm looking at my test, I think I'm the only one who got a 60 or whatever. And the couple of kids around me I'll know what they got but then I have no idea how anyone else is doing, because it's all privately done. Not that I need to know their test grades, but I'd like to know, how I stand. Am I the only one who needs help? And then you feel embarrassed to be the one raising your hand all the time, be the one staying after class because you think you're the only one. So, here, it's a lot more comfortable. You're not embarrassed in front of the other kids.

Interviewer: So it's really helpful to know when you're the only one, but it's also really important to know when you're not the only one (Student 12: Right) because it kind of gives you the courage to (Student 12: Exactly) ask questions more often?

Student 12: Yeah and then you feel like you're not a failure in the class it's not a big deal if you can't understand it, you just work harder because other kids are having the same problem...

In a visit subsequent to the one where the quotes in this paper where gathered, the teacher related to me some of her observations. She was surprised to find that the community effects of the system were not persistent. When the system was not working, she noticed that the students went back to not asking questions. In one of her classes that day we staged a mini experiment. The students reviewed the first half of their homework without using the network and the second half using it. Without the histogram display of responses, the class discussion was poorer. At the end of class we held a question and answer session with the class focusing on why the students thought they participated differently when the group display was available than when it wasn't. Here is a synopsis of their comments.

1. Without seeing the histogram of everyone's responses, if you get a question wrong, you are afraid to ask why.
2. Without the system, whoever speaks up first, wins the argument. If the person who speaks first seems to be agreed with by the majority of the class, others get insecure and won't talk about other possible solutions.
3. With the histogram, if you see that at least one other person in the class selected what you did, it gives the confidence to defend the answer. Without the histogram, you are afraid that you are the only one.
4. With the histogram, the answer is out there to defend, it doesn't even have to be yours.

Without the aggregated view, the students felt that they did not have the information they needed to fully discuss the homework.

There was a second school in the pilot site community where data was also gathered. A feature of the KSS prototype was that it allowed teachers to create their own activities. The teachers at the second school saw no reason to create their programs with an aggregate data display. It just did not occur to them that the students would gain anything from seeing the aggregate display. The activities that they created had the student results sent to the teacher computer and rarely sent an aggregate of the class data to the display. In interviews, the benefits of anonymity as seen in the "Assessment" and "You can't fake it" transcripts were universal across both sites. But one of the key features of the system, the ability of students to identify with data in the display space was missing. Absent from the student comments from this site was the powerful sense of community and meta analysis of understanding seen at the Islandtown site.

PRIVATE ACCOUNTABLITY

Increase Teacher Knowledge of Student Understanding

Assessments are only meaningful if the results can be interpreted in a manner and in a timeframe useful to the teacher. A KSS that allows for anonymous, parallel response to questions by all students and gives the teacher tools to analyze those responses, allows for many more meaningful assessment opportunities.

The teacher in Islandtown, taught Advanced Placement Calculus. Her reality was that the entire course was to get the students prepared to take the AP Calculus test which is mostly multiple choice. For this reason, all of the homework that she assigned came from AP practice tests and was in Multiple Choice format. She used the classroom KSS daily to facilitate discussions during review of the students homework.

> Teacher: It's great to know, where the kids are, actually it's not always great because sometimes it's pretty depressing to see where the kids are. There was something I did this year in one of my classes and I asked if there were any - I thought I had done a fine job - I asked if there were any questions, nobody had any questions and I just had an inkling, And I said okay well log on and lets check. And I believe two kids got it right so obviously they didn't have a clue what they were doing and I went back and retaught.
>
> (Later in interview)
>
> Teacher: I feel really strongly that this product is an invaluable tool for educators. As teachers, I keep going back to assessing, which is not the only thing that TI-Navigator does and I'll address that in a minute, but as teachers we need to assess our students and ourselves and it's instantaneous and its real and its so important to know where the kids are at. And what you think you taught, and what you feel you explained really well, is not always what they received and to wait for a test two weeks from now, meanwhile you've built on that concept, and if a child has had difficulty with the concept in the beginning and you're building on it, everything is going to fall apart. And with TI-Navigator you cannot only check that concept, you can go back and check very basic concepts. There are all sorts of almost game-like activities that you can do with the kids where you can get a real good sense for what's going on with out the intimidation of a test and the pressure of a test, and that's wonderful.

Being able to gather all student responses gives the teacher options for how to proceed in class. The teacher could ask questions after a lesson is completed to find out if the topic is understood or must be re-taught. Student responses to a pre-test could be gathered before a new unit is taught to gauge students' prior knowledge. Foundational concepts could be reviewed or introductory lessons could be skipped depending on what the result of the pre-test indicate. A teacher could ask content knowledge questions and then use that information to form cooperative groups with greater confidence that student ability needs and strengths were matched (Bellman, 2002). The ability to gather responses on all questions from all students is important because of the knowledge it gives the teacher, and what the teacher is then able to do with that knowledge.

Improve Student Participation

The ability for all students to answer all questions is powerful for what it allows the teacher to understand. It is equally powerful for what it allows for the student. Not raising your hand or avoiding eye contact no longer lets a student off the hook for participation. The network enables all students to be more engaged in the classroom.

Student 11: It kind of forces you to do your homework because if the number of responses you know, don't match the number of people in the class, you know (Student 10: And she can check), and she checks, (Student 9: Yeah). So it kind of makes you keep on top of yourself also,

Student 9: Yeah

Interviewer: Is forcing you to do your homework a good thing?

Student 10: Probably

Student 9: Yeah, definitely.

Interviewer: Is it?

Student 11: That's one of the classes I'm the most prepared in. I think that TI-Navigator does help, because, it kind of forces you to do it. Things are...

Student 9: She can tell if...inaudible...)

Student 11: She can tell if you using this if you're not doing it and if your not, you know...

Student 10: You can't fake it.

Student 11: You can't fake it

With a classroom KSS the teacher can see who has and has not submitted a response. Because the responses are anonymous to the rest of the class, which mitigates student embarrassment, it is okay to “force” all students to answer.

CONCLUSION

For a long time, the benefits of classroom knowledge sharing systems have been reserved for classes working in conjunction with university based research projects. As these systems are now becoming commercially available, we will soon be able to see their benefits across a much greater population. There will be a need for professional development to help teachers integrate the functionalities of these systems into their classes. Simple things like showing the aggregate results of student responses back to the class are easily overlooked as not being important if their impact in not explained. With the insight given to all students about their understanding and how others are doing, classroom KSS have a powerful impact on how students experience even simple activities like reviewing homework.

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SECONDARY SCHOOL STUDENTS' IMPROPER PROPORTIONAL REASONING: THE ROLE OF DIRECT VERSUS INDIRECT MEASURES

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A systematic series of studies by De Bock et al. revealed a strong, deep-rooted tendency among secondary school students to apply the proportional model in non-proportional problems involving lengths, areas and volumes of similar geometrical figures. In these studies, however, items were used involving direct measures for area and volume as well as indirect measures (e.g. the time to manure a piece of land as an indirect measure for its area), assuming that this would make no difference. The current study confirmed that there were no significant differences in performance related to the presence of direct or indirect measures in the items, but there were some differences in the applied solution strategies. These findings confirm the internal and external validity of the earlier studies on students' illusion of linearity.

THEORETICAL AND EMPIRICAL BACKGROUND

It is known that many students have a tendency to apply proportional or linear solutions "everywhere", also in situations where they are not applicable. This so-called "illusion of linearity" has been exemplarily described in several mathematical domains, such as elementary arithmetic (Verschaffel, Greer, & De Corte, 2000), algebra (Matz, 1982), and (pre)calculus (Leinhardt, Zaslavsky, & Stein, 1990), and recently also in probability (Van Dooren, De Bock, Depaepe, Janssens, & Verschaffel, 2002).

The best-known case of the overreliance on the linear model, however, is situated in the domain of geometry: many students of different educational levels believe, for example, that when the sides of a figure are doubled, the area and volume will be doubled too (National Council of Teachers of Mathematics, 1989). In the past years, we performed a series of empirical studies to evidence this particular tendency in secondary school students. In these studies (see De Bock, 2002; De Bock, Verschaffel, & Janssens, 1998), large groups of 12–16-year old students were administered (under different experimental conditions) written tests consisting of proportional and non-proportional word problems about the relationship between lengths and perimeters/areas/volumes of different types of similarly enlarged and reduced figures. The following item is an example of a non-proportional problem about the area of a square: "*Farmer Carl needs approximately 8 hours to manure a square piece of land with a side of 200 m. How many hours would he need to manure a square piece of land with a side of 600 m?*" The majority of the students in these studies gave a proportional answer on this type of non-proportional problems, thinking that the time to manure the large piece of land would be tripled too. Even with considerable support (such as the provision of drawings, enhancing metacognitive awareness, and/or embedding the problems in an authentic problem context), only very few students appeared to make the shift to the correct non-proportional solution. A further in-depth investigation using individual interviews (De

Bock, Van Dooren, Janssens, & Verschaffel, 2002) showed that students' unwarranted proportional reasoning was due to a set of closely related factors: an intuitive approach towards mathematical problems, particular shortcomings in geometrical knowledge, inadaptive beliefs and attitudes, and a poor use of heuristics.

In many of the studies by De Bock et al., it was not explicitly stated that the problems were dealing with the perimeter, area or volume. Instead, an indirect measure for these quantities was used. For example in the above "farmer Carl"-item, the problem statement mentions the *time needed to manure* a certain piece of land, *not the area* of the piece of land itself. Of course, it can reasonably be supposed that this time is directly proportional to the area, so that "time to manure" can be considered as an appropriate indirect measure for the area. Analogously, the time to dig a ditch around the piece of land was used as an indirect measure for its perimeter. However, during the in-depth interviews study (De Bock et al., 2002), the suspicion arose that the use of indirect measures in previous studies might have strengthened students' tendency towards unwarranted proportional reasoning. For example, some students had difficulty in immediately recognising the problem offered during their interview as dealing with area. And since the problem in the interview study asked about the millilitres of paint needed to cover a particular figure, some students were even more confused because millilitres reminded them of volume rather than of area. In the present paper, we report an empirical study aimed at investigating whether the use of direct or indirect measures for area in non-proportional problems has an influence on students' solution processes and performances.

THEORETICAL INDICATIONS FOR THE INFLUENCE OF THE NATURE OF THE MEASURE

Besides our own anecdotal experiences during the above-mentioned interview study (De Bock et al., 2002), the research literature on (mathematical) problem solving yields several indications that the type of measure in the problem referring to the perimeter, area or volume indeed may have an influence on students' solutions, and therefore possibly on the occurrence of unwarranted linear reasoning.

Rogalski (1982) investigated elementary school children's reasoning about lengths and areas. She reports that some students overgeneralised the properties of "unidimensional" lengths in a figure (e.g., the side of a square in metres) to "unidimensional" area measures (e.g. the amount of paint needed to cover that square in litres), while this was not the case for problems with "bidimensional" area measures (e.g., cm^2). In other cases, students overgeneralised the properties of direct measures of length (e.g., cm) to direct measures of area (e.g., cm^2), by proportionally relating the area of a figure to its length.

Another indication is the finding that students sometimes tend to use "key word strategies" (e.g., Verschaffel et al., 2000). Superficial characteristics of the problem text (the presence of certain words) immediately guide the choice for a particular operation. In our case, the presence of the words "perimeter", "area" or "volume" or expressions with direct area or volume units (such as cm^2 or cm^3) might remind the student to apply another strategy than the most straightforward proportional solution scheme, while the student might not be reminded to do so if an indirect (though proportionally related to the area or the volume) measure is mentioned.

Finally, a rational task analysis indicates that problems formulated with indirect instead of direct measures for perimeter, area or volume essentially involve extra thinking steps: students need to notice that the indirect quantity is related to the perimeter, the area or the volume of the figure under consideration. Moreover, they need to know that the relationship between the indirect and the direct measure is a direct proportional one. Because of these extra steps, more errors can appear when solving problems with indirect measures. Therefore, the use of problems involving indirect measures in our previous research might have strengthened students' tendency towards unwarranted proportional reasoning and have had a negative impact on their performances. An empirical study was conducted to find out whether the measure in the problem has indeed this influence.

RESEARCH QUESTIONS AND HYPOTHESES

The current study aimed at investigating the influence of the nature of the measures in a word problem on students' tendency to (improperly) apply proportional solution methods. Can we identify differences in students' performances on and solution processes for non-linear word problems, when they are formulated in terms of direct or indirect measures?

Based on the theoretical indications described above, we hypothesized that students would perform better on problems in which the measure of perimeter/area/volume is explicitly mentioned, and especially that they would less often apply a proportional solution method when it is not applicable. Moreover, we expected that the strategies used by students to solve problems involving direct measures would differ from the strategies used to solve problems involving indirect measures. For example, for problems explicitly expressed in area measures, we expected students to apply more often previously learnt knowledge and strategies for calculating areas because they were provoked to do so by key words in the problem statement, while other strategies (such as applying internal ratios or the "rule of three") would appear more often with problems involving indirect area measures (wherein such key words are lacking).

METHOD

A paper-and-pencil test was administered to 145 secondary school students aged 15–16, attending two different typical schools for general secondary education in Flanders (Belgium). All participants received two problems: a (proportional) item about the perimeter of an enlarged irregular figure (where the perimeter of the smaller version was given), and a (non-proportional) item about the area of that enlarged irregular figure (where the area of the smaller version was given). These problems were presented in random order.

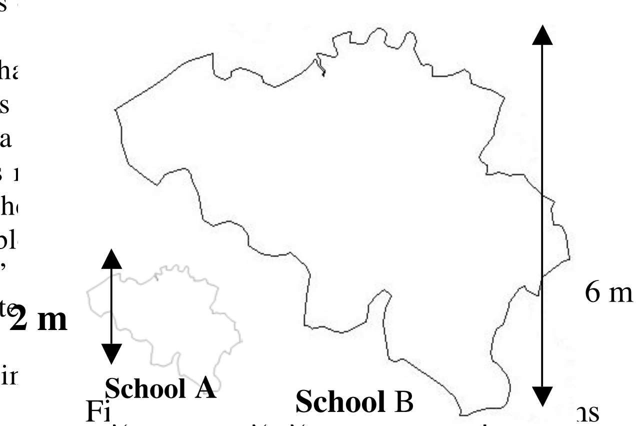
Students' solutions were analysed in two ways. First, they were scored with 1 or 0, depending on whether the response was correct. The interrater agreement of this categorisation was □=□.933. Next, the underlying solution strategy was identified, using

	Direct version	***Indirect version***
Intro	In school A, pupils made a chalk drawing of the map of Belgium. The drawing had a height of 2 m.	In school A, pupils made a chalk drawing of the map of Belgium. The drawing had a height of 2 m.
Perimeter problem ***(propor-tional)***	In the mathematics lessons, the pupils figured out that the drawing had a perimeter of 11 m. In school B, pupils drew a map of Belgium on a bigger scale: it was 6□m high. What would be the perimeter of this map?	Afterwards, the pupils put coins of 1 euro on the chalk lines of the map. They needed 3 kg of coins to do that. In school B, pupils drew a map of Belgium on a bigger scale: it was 6□m high. How many kg of coins would they need to put on the chalk lines?
Area problem ***(non-pro-portional)***	In both schools, the students also made an estimation of the area of their map. In school A, pupils estimated that their map had an area of 3 m^2. What would be the area of the map in school B?	In both schools, the students also paved the whole map of Belgium with euro coins, turning it into a coin carpet. In school A, pupils needed 40 kg of coins to do that. How many coins would the students in school B need?

Table 1: Examples of "direct" and "indirect" versions of the word problems administered to the students

a qualitative categorisation scheme based on previous research findings. (For the categories in this scheme, see the results section.) The interrater agreement for this part of the analysis was □ = 0.802.

Two mathematically equivalent versions were developed, and students were assigned to one of two conditions: ha students received a test with problems mentioning the terms perimeter and area the direct measures. The other students r test with problems containing only the measures for perimeter and area. Tabl examples of the "direct" and "indirect" were given to the students. To guarante participants interpreted the word correctly, the test also contained an ir Figure 1).



Fi ns

RESULTS

Table 2 presents an overview of the performances of the students on the proportional and non-proportional items in the two conditions. These performances were analysed by means of a 2 × 2 repeated measures ANOVA, with "type of problem" (proportional or non-proportional) and "condition" (direct or indirect measures) as independent variables and the number of correct answers as the dependent variable.

	Direct		***Indirect***		***Total***	
	Mean	**SD**	**Mean**	**SD**	**Mean**	**SD**
Proportional	0.792	0.410	0.876	0.331	0.848	0.360
Non-proportional	0.208	0.410	0.237	0.427	0.228	0.421
Total	0.500	0.503	0.557	0.498	0.535	0.499

Table 2: Mean performances (and standard deviations) of the students on the proportional and non-proportional problem in the "direct" and "indirect" condition

First of all, the ANOVA revealed a significant main effect of "type of problem", $F(1, 43) = 55.83$, $p = 0.0001$. It appeared that students performed much better on the proportional item than on the non-proportional item: for the two conditions together, about 85% of the students answered the proportional item correctly, whereas only 23% gave a correct answer to the non-proportional item. This result is in line with our previous research findings (De Bock, 2002; De Bock et al., 1998, 2002), confirming students' overwhelming tendency to improperly apply proportional solutions on non-proportional word problems.

Second, the ANOVA revealed no additional significant effects, neither the main effect of "condition", $F(1, 43) = 1.35$, $p = 0.2473$, nor the "type of problem" × "condition" interaction effect, $F(1, 43) = 0.33$, $p = 0.5694$. This indicates that there was no difference in performance depending on whether problems involved direct or indirect measures, neither for the proportional item (involving the perimeter), nor for the non-proportional item (involving the area). Therefore, our hypothesis that students would perform better on non-proportional problems if they were expressed with direct measures for the area was rejected.

Despite the absence of an effect of "condition" on students' performance, the question remains whether students in the "direct" condition applied other solution strategies than the students in the "indirect" condition. As mentioned above, a qualitative analysis was performed on the notes of the students on the test. Each correct and incorrect solution[i] was categorised using the scheme in the left column of Table 3, to determine the strategy that was applied to obtain the answer. We will illustrate the categories using the direct non-proportional problem given in Table 1.

	Direct			***Indirect***		
Type of solution strategy	**Proportional**	**Non-proportional**	**Total**	**Proportional**	**Non-proportional**	**Total**
Proportionality	**54.2**	**66.7**	**60.4**	**80.4**	**64.9**	**72.7**
Interrnal ratio	45.8	60.4	53.1	66.0	51.5	58.8
External ratio	6.3	6.3	6.3	5.2	5.2	5.2
Rule of three	2.1	0.0	1.0	9.3	8.2	8.8
Reducing figure	**22.9**	**16.7**	**19.8**	**4.1**	**12.4**	**8.2**
General principle	**0.0**	**4.2**	**2.1**	**0.0**	**12.4**	**6.2**
Other	**22.9**	**12.5**	**17.7**	**16.5**	**10.3**	**12.9**
Total	**100.0**	**100.0**	**100.0**	**100.0**	**100.0**	**100.0**

Table 3: Overview of the solution strategies (in %) used by the students to solve the proportional and non-proportional problem in the 'direct' and 'indirect' condition

A first category captures three solution strategies relying on proportionality. These strategies are correct for the proportional perimeter item but incorrect for the non-proportional area item. In the example, students using the "internal ratio" reason that the ratios of the heights of the maps (2 m / 6 m) should apply to the areas of these maps too. Relying on the "external ratio" strategy means reasoning that the ratio between the height and area in the first map (2 m / 3 m^2) should be the same in the second map. The "rule of three" strategy first reduces one of the quantities to its unit, e.g. 3 m^2 for 2 m, thus 1.5 m^2 for 1 m, thus $6 \times 1.5\ m^2 = 9\ m^2$ for 6 m. A second category ("reducing the figure") refers to strategies in which the irregular figure under consideration (the outline of a map of Belgium) is reduced to a more regular figure, such as a rectangle or a right-angled triangle. A third category ("general principle") comprises those solutions that explicitly refer to the general principle governing the similar enlargement of geometrical figures: if a figure is enlarged with factor k, its perimeter enlarges with factor k, and the area with factor k^2. A fourth category contains all remaining solutions, such as unanswered problems and (sometimes correct, but mostly incorrect) solution processes that were difficult to understand or to categorise in one of the other three categories.

Table 3 shows that most of the problems in both conditions were solved with a strategy relying on proportionality, mostly the "internal ratio" strategy. This explains why most students performed well on the proportional item (they correctly reasoned that the perimeter was tripled because the height was tripled), but failed on the non-proportional item (since the area was not tripled). Only a minority of the students thought of applying an approach whereby the irregular figure was reduced to a regular one, and even less students applied the general principle. A comparison of the "direct" and "indirect" condition shows that there were some small but interesting differences between the strategies used by the students. As expected, proportional strategies such as the "internal ratio" or the "rule of three" were more often applied to solve the word problems in the "indirect" condition than in the "direct" condition. When students in the "indirect" condition recognized the non-proportional character of the area problem, this happened sometimes because they knew and activated the general principle, sometimes because they reduced the irregular figure to a regular one. In the "direct" condition, however, considerably more students applied a "reducing the figure" strategy. These findings are in line with our expectations. The presence of direct measures for perimeter and area in the problem statement seems to trigger other strategies in some students: it reminds them to apply previously acquired knowledge about areas of rectangles or triangles, to work on the drawing, etc., whereas problems with indirect measures elicit more often more "general" approaches for solving word problems, such as the application of "internal ratios" or the "rule of three".

CONCLUSION AND DISCUSSION

Earlier studies have shown that secondary school students have a strong tendency to apply proportional solution strategies even in situations where they are not applicable. More specifically, a systematic line of research by De Bock et al. (De Bock, 2002; De Bock et al., 1998, 2002) has shown that many students believe that if a figure enlarges with factor *k*, not only the perimeter but also the area and volume of that figure increase with that factor *k*. In many of these studies, however, problems were used wherein the quantity under consideration was an indirect – though proportionally related –measure for the perimeter, area or volume, e.g. the weight of an object as an indirect unit for measuring its volume. Implicitly, it was assumed that this would have no significant influence on students' solutions. Recently, however, suspicions arose on this assumption. The use of indirect measures in many earlier studies might have strengthened students' illusion of linearity, and influenced the research findings on the factors influencing this misconception.

The current study explicitly addressed this issue by experimentally manipulating the measures in the problem statement: half of the students solved two items involving direct measures for perimeter and area, while the others solved isomorphic items with indirect measures. A comparison was made of students' performances as well as their solution strategies. We found no significant differences in the performances on the two types of problems. Apparently, the type of measure used in the problem statement has no significant influence on students' performance in general, and on the occurrence of improper proportional reasoning in particular. This seems to confirm the internal and external validity of the findings of the earlier studies on students' illusion of linearity. A qualitative analysis of the underlying solution strategies, however, provided some interesting differences. For items with indirect measures, more students applied a strategy based on the general application of linearity, whereas items with direct units for perimeter and area elicited more content-specific strategies such as working on the graphical representation and the application of formulas for perimeter and area.

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THINKING IN ORDINARY LESSONS: WHAT HAPPENED WHEN NINE TEACHERS BELIEVED THEIR FAILING STUDENTS COULD THINK MATHEMATICALLY

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This paper is about improving mathematics achievement among the lowest attaining students in some mainstream secondary school through focusing on thinking in ordinary lessons. The research is funded by the Esmee Fairbairn Foundation (grant numbers 01-1415 & 02-1424) and involves three academics and nine teacher-researchers. A research perspective has been taken throughout the project. In this paper we report on the commonalities which emerged from the teachers' varied practices, and the creation of ways to interrogate mathematical tasks.

BACKGROUND OF THE PROJECT

The foundational belief of the Improving Attainment in Mathematics project (IAMP) is that attainment can best be improved by concentrating on the development of ways to think with, and about, key ideas in mathematics, rather than focusing on repeated curriculum coverage. We are working within a climate of frequent testing and politically-imposed targets of supposedly appropriate achievement. Schools have been supplied with materials intended for use with students who have below-target achievement levels on entry to secondary school; these consist mainly of worksheets which provide reminders and practice of previously studied topics. The package (called 'Springboard') conveys the impression that the mathematics which students have previously failed to learn over several years can suddenly be learnt successfully through the supplied approaches, exercises and activities.

In contrast, our approach offers a way to work with such students which values and uses their proficiencies, rather than using a discourse of deficiency (Watson, 2001). It is important to remember that these students have already been labelled as 'failures' and are together in teaching groups consisting of others in similar circumstances.

Research shows that low attaining students can and do think in ways which are similar to those described as mathematical (e.g. Ahmed, 1987; Harries, 2001). For example, some have shown the ability to use examples and counter-examples, to generalise, to develop efficient methods of working, to abstract. Research associated with the development of thinking skills in mathematics suggests that achievement *can* be improved through explicit use of thinking skills and cognitively well-structured lessons (Adhami, 2001). There is also a growing body of research evidence that students from educationally, socially and economically disadvantaged backgrounds can benefit from mathematics teaching which allows them to exercise and develop their thinking, and that they also do better in standard tests as a result (Silver, 1993; Tanner and Jones, 1995; Boaler, 1998). Published studies tend to have three identifiable dimensions: intervention with methods and materials, intervention with professional development activities, and high levels of teacher commitment. In studies which involve innovation, results which purport to be about particular methods may indeed say more about the commitment of teachers and the professional development benefits of research. Boaler's study is a rare example in which there is no intervention from outside in the more successful of two schools, and the commitment of some of the teachers is not particularly high, but the unusual way they teach seems to benefit students more than methods used in the more traditional school.

In this study there is high commitment from teachers and an inevitable professional development effect, but no imposed innovative methods or materials. We are instead learning about the teachers' and pupils' abilities to incorporate thoughtful activity into *every* lesson, of whatever type, while avoiding a skill-focused remedial approach. We are also learning about how high commitment translates into action.

DEFINING MATHEMATICAL THINKING

We did not define 'mathematical thinking' (MT) or 'key ideas', nor provide teachers with descriptions of what they ought to be doing. Rather, we asked project teachers what they felt they could do, within their current practice, to develop MT. Like Ruthven (1999) we value teachers' practices and aim to contribute to a warranting process of teachers' knowledge through 'triangulation of implementation against intention; experience against evidence; internal participants and external standards; continuing analysis and evaluation of model in light of evidence and development' (p.210)

Whereas all the teachers could agree on *some* aspects of MT, such as the importance of generalisation, there was much early discussion about its full meaning throughout school mathematics. For example: is 'ordinary thinking in the domain of mathematics' a more useful activity on which to focus? Can choice of operation in a word problem be described as MT? Discussions around these issues were rich but inconclusive (Pitt, 2002). We chose to take an empirical approach and compare the practices of teachers who had deliberately taken the decision to work with the target students in ways which (a) did not follow the provided materials and methods and (b) included specific attention to development of MT, *whatever meaning a teacher attached to that phrase*.

Thus the meaning of MT is grounded in researched practice and emerges through a process of co-configuration (Engestrom, 2002) in which differently positioned participants in the process create situated knowledge together in response to a crisis in the existent system. In this case, the crisis is the sustained underachievement of a significant number of students within a prescriptive curriculum and assessment system. The teachers are largely self-selected, committed to evaluating their work and they also recognise the value of working within a supportive group; details of their selection and operation of the group are outside the scope of this paper and can be found elsewhere (Watson, De Geest & Prestage, 2002)

METHODS

Data are collected about the practices of the teachers and their students' responses in the classroom. Some teachers are deliberately introducing new (to them) strategies into their work with the target groups, others are working to become more articulate about their existing practice. Data are collected about pupils' written work, test scores, oral responses and interactions in lessons. Further data include teacher diaries, lesson plans and evaluations, interviews, recorded lessons and observation notes of lessons, and recordings of discussions between teachers and other researchers.

Analysis is ongoing using, at first, techniques of grounded theory to identify commonalities and differences between teachers. These are fed back into group discussions to see if more differences emerge, or more commonalities can be articulated. Thus our role as researchers contributes to the co-configuration of what counts as 'knowledge' in the group. Our aim in this process is to see what can be said, if anything, *in general* about teachers who are working towards similar aims. In addition, we are developing descriptions of individual teachers' practices and the mathematical activity of their students in classroom settings. Thus at some time in the future we shall present portraits of secondary classroom micro-cultures in which teachers offer, and pupils take up, opportunities for mathematical thinking.

Data from classroom observation, teacher notes and group discussions led to the creation of descriptions of the types of pupil activity that the teachers in the group identified as evidence of mathematical thought (Watson *et al*, 2002). These are summarised in Table 1 into two types, those which were specifically prompted and those which occurred unprompted in classroom settings. Many of those which occur in both columns as actions teachers can prompt but for which some students eventually take responsibility.

Prompted	**Unprompted**
Choosing appropriate techniques Contribute examples Describing connections with prior knowledge Finding similarities or differences beyond superficial appearance Generalising structure from diagrams or examples Identifying what can be changed Making something more difficult Making comparisons Posing own questions Predicting problems Working on extended tasks over time Dealing with unfamiliar problems	Choosing appropriate techniques Contribute examples Describing connections with prior knowledge Finding similarities or differences beyond superficial appearance Generalising structure from diagrams or examples Identifying what can be changed Making something more difficult Making extra kinds of comparison Generating own enquiry Predicting problems Changing their mind with new experiences Creating own methods and shortcuts Initiating a mathematical idea Using prior knowledge

Table 1: Observable actions indicating mathematical thought

The list is similar to descriptions of the mathematical behaviour of high achieving mathematical learners yet arises solely from the data of this project with weaker students. Rather than expecting this kind of behaviour spontaneously, as one might from stronger students, the teachers deliberately framed and organised the classroom environment to make it more likely that students would behave in these ways.

CHARACTERISTICS OF TEACHERS' PRACTICES

Analysis of the project classrooms has enabled us to identify several characteristics which are seen by the teachers to be central to higher achievement of the target students.

Guiding learners into mathematical cultural practices

Teachers see their task in terms of structuring teaching to enable students to make contact with mathematics using their powers of thought. They recognise the learners' entitlement to access mathematics as an established cultural artefact and not to be limited to watered-down, concrete, procedural versions, or a version constrained by the learners' current context. They do not simplify mathematics to make it more accessible. Teachers focus on adapting the habits of the learner and classroom so that the learner may be enculturated into the world of mathematics. For example, they offer situations in which there are dimensions of choice so that students learn how to choose appropriately; they ask for examples in order for exemplification to become a habit; they find playful ways to elicit more mathematically sophisticated responses.

Making connections

All the teachers want students to view mathematics as connected rather than as separate topics, using structural links between topics to design tasks. Some teachers were explicit with students about making links within and across topics. Sara reorganised the imposed scheme of work so that links were obvious; Anthony explicitly encouraged students to express any connections they saw with previous work, with other mathematics or with other contexts.

Preparing to go with the flow

Teachers deliberately plan to 'go with the flow' of student response. This may entail planning a range of approaches to allow flexibility so that practical, spatial and numerical approaches are all possibilities and the teacher can decide which approach is appropriate, or when to move between them. They also respond to students' moods, using them constructively to mould the progress of lessons rather than battling against them.

Allowing thinking time

All teachers find they were giving students a long time to think, including long wait-times with whole-class questions, and in general throughout their work and interactions. Giving more time, creating space rather than imposing pace, is seen in the project as having emotional, behavioural and cognitive effects. Class discussion, with space given for individual thought, was used more and more but not without difficulty (see below).

Varying task-type

All found themselves, either deliberately or incidentally, using fewer worksheets and textbooks and more activities, developments from starter tasks and students' own questions.

Extending duration of tasks

There has been in general a shift towards longer tasks in the project, if for no other reason than the fact that teachers are building more thinking time into their expectations. However, for a few teachers this is a deliberate major move in order to create an atmosphere in which students are embedded in a mathematical situation for several lessons. This goes completely against the normal belief that such students 'cannot concentrate' and need to be offered task variety. Some project teachers have therefore focused on developing ongoing questions and enquiries. This is particularly well-developed in Becky's case. For example, she gave pupils straws to make and discuss angles and their relationships by intersecting the straws. Students managed to find all the usual angle rules for intersecting lines, triangles and parallel lines over a two week period. This method allowed them to generalise because it offered them unlimited possible angles and the space to make conjectures, experiment and think things through. In contrast some others introduced shorter, high-concentration tasks to enable students to learn that they *could* concentrate, so that they could build on this new behaviour later with more extended tasks.

Creating own examples

All our teachers used 'create your own example' tasks as part of their everyday lesson structure. For example, Andrea deliberately included some blank places in an otherwise teacher-driven activity so that students could create their own examples with which to work. Several teachers use 'if this is the answer, what is the question?' tasks. One student said:

> Making my own examples makes me think. I think about half the time in class now.

Respecting learners

Overwhelmingly, the teachers respect students as learners. They do not guess where the students are in their mathematical development, they ask and listen. The teachers provide the scaffolding, the students construct. The curriculum does not dictate progress, the learners do.

Differences.

We have learnt that it is possible for two teachers to make apparently contradictory decisions about classroom norms, but to have and achieve similar aims. What seems to be important here is not the decision that is made, but the *purpose* of the decision. For example, all teachers thought it was important for students to discuss mathematics with their peers, be it in pairs, in groups, or whole class discussions, where the whole-class discussion provides models for how to discuss mathematics. They also believe that everything said in class is valuable and everyone should hear it. But for one, this leads to the practice of repeating everything which is said by students (ensuring everyone hears); for another this leads to the practice of repeating nothing and orchestrating discussion around what each student says (ensuring everyone listens).

All the teachers recognise a link between thinking and writing about mathematics. For one teacher, the act of writing is seen as forcing thinking because it has to be expressed in a linear form, using logical connectives like 'and, but, if, then, because, so, …' The effort required to communicate forces clarity – speech demands transformation of thinking into what makes sense to others. Others believe that writing gives you something of your own to look back at; a way to remind yourself what it is that you know. One, however, sees writing as a serious *distraction* from thought. She sees mental visualisation, and struggling to visualise, as acts which make future access to mathematical facts and methods easier because the memory has been activated by the effortful creation of an image. For all of them the purpose is to promote thought, but different decisions about writing arise as a result.

Difficulties

Of course persuading students to adapt to these ways of working is not easy. Many had developed powerful habits of rejecting the curriculum. Project teachers have not given up attempting to turn rejection into engagement but response can be slow and temporary. Sara reported that in one term the proportion of responsive students had changed from one third to two thirds, but her feelings about the class were dominated by the intransigence of the remaining ones. Andrea gave students the task of learning to take a piece of paper to and fro from home, which can be seen as a behaviourist approach to developing work habits and does not relate easily to her beliefs about how students learn mathematics, and how rewards might be intrinsic. Sian created a board-room arrangement of desks so that students could discuss more easily, but they had been used to being seated separately, all facing front, and it took time for them to cope with new expectations. Another teacher excluded students who quietly refused to work and kept those who were noisy and disruptive but took part in the mathematics. An important part of the study is the recognition of the realities of working with segregated groups of 'failing' students in secondary school, and that the processes involved are of re-enculturation.

STRUCTURE OF MATHEMATICAL TASKS

Apart from some of the techniques used to re-enculturate learners, much of the above is typical of the intentions behind reformed curricula. What is exceptional about these teachers is that they have reconstructed these characteristics for themselves by considering what is best for their most vulnerable students (for an example of an individual teacher's account of similar work see Tierney, 1997). Yet while we can describe the characteristics of teaching which contributes to classroom cultures within which thinking can flourish, we recognised that teachers were doing more than this.

We wanted to identify the nature and effects of the tasks and scaffolds which were initiating and framing thinking. Boaler and Greeno (2000) hypothesise 'a form of connected knowledge that emphasises the knower's being connected with the contents of a subject-matter domain**.'** (p.191). Our view of mathematics is that 'the contents of the subject-matter domain' are deeply connected within themselves through mathematical structure, and that enculturation into mathematical thinking involves becoming fluent with constructing, creating and navigating similar or isomorphic structures, that is, being intimately attuned to the ways in which mathematics is internally connected. We are influenced in this enterprise by Gibson's (1977) concepts of affordance (what sort of responses are possible to the sensory impacts of the mathematics lesson?) and constraint (how can a teacher usefully reduce that freedom so that the learner focuses on mathematical change and invariance?). We are also influenced by Marton's identification of dimensions of variation in learning (Runesson, 1997). To interrogate our data further we are using a set of analytical questions which arise from co-configured concepts of tasks and scaffolds which promote mathematical thinking.

The initial task: What is the task? How does it open/close possible responses? What is the purpose? Do students know the purpose? How do students work on it? Why this starting point? What representations are offered? What are the dimensions of variation? What is it possible to learn from this task?

Sustaining, motivating, extending: What is valued/praised, and how? How are the students encouraged while working to think more, or to make the work harder? How are incorrect answers dealt with? What are the constraints/freedoms? How are connections made? What is the generality with which they are working? What is it possible to learn with these interventions and emergent features?

Learning: Do the students self-correct? What choices do they make? What do they contribute to the lesson? What is evident in their written work? How much information, variation, elaboration do they give? What range of representations, mathematics, complexity? What evidence is there of awareness of generality? What dimensions have they chosen to vary and how are they varied?

These questions address what is provided by the teacher, what responses are likely, and what learners make of these mathematical affordances and constraints.

Affordances of mathematical tasks

Sara used a task from a published resource to ask students to create patterns using two colours and then express them using letters, for example: a,b,a,b…; a,a,b,a,a,b…. By using our questions above we were able to identify why the task failed to stimulate MT: students' attention had *first* been focused on choosing colours and making patterns, the easiest dimension of variation offered to them, rather than on expressing these as generalities about the resultant sequences. The way the task was presented failed to constrain learners to focus on mathematics, and structures in particular. Restructuring so that students had to make initial choices about sequence, rather than colour and pattern, was more successful, the dimension of variation being the letter sequences themselves.

Linda is enthusiastic about what students can do when given an 'answer' and asked to provide a question. She started using this strategy by imposing constraints designed to move them away from obvious decisions. For example, she excluded the use of '+' for numerical questions. She also introduced suggestions such as 'are there any shape questions which give an answer of 4?' Later she would ask 'can anyone make up something which is really tricky?' She structured the idea-sharing formally into her lessons. This helped them learn from each other what might be possible. She then 'faded' her support by asking 'what sort of questions could we ask?', thus beginning to hand over the responsibility for asking questions to the students. Within 45 minutes she had offered freedom to create questions, constraints to guide them away from easier questions and further constraints which could help them to explore further possibilities.

We are in the early stages of applying these analytical questions to *all* our types of data and are finding we can use them construct a coherent story about tasks and students' mathematics. Even where we have only teacher-reports and written work we can identify affordances and constraints of tasks.

It is too early to report the full implications of this study, but early analysis shows that there are common features in how committed teachers work when they reject a skills-based approach for 'weak' students and adopt a 'thinking' approach instead, and that task analysis as we have described it adds an important dimension to analysis of practice.

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TOWARDS A REDEFINITION OF THE MATHEMATICS CULTURE IN THE CLASSROOM

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The questions addressed in this paper are: how do students interpret the non-solution problems as a mathematical practice in the classroom? What kinds of arguments do the students offer? To deal with these questions the utterances of students working in the classroom with problems in which they had to look for general relations in order to argue such things as "there is no least positive fraction" or "it's impossible to tessellate a rectangle with some kind of pieces" are examined and discussed.

INTRODUCTION

The mathematics program for the secondary school in Mexico suggests the teaching of the discipline should be based on problem solving approaches. Among its objectives are that students develop their discovering abilities, recognize and analyze the components of a problem and that they formulate conjectures, communicate and validate them (Alarcón, et al., 1994).

To achieve these objectives, and also many of the objectives the NCTM (2000) has put forward, it's necessary to change radically the roles teachers and students usually enact in those classrooms where teaching and learning goes under the School Mathematics Tradition (Cobb et al., 1992)[1]. For instance, the students now explore open problems or situations in which they must pose questions, make conjectures and argue or criticize mathematical ideas.

In such situations they must recognize if it is possible or not to find or do what they are supposed to do. In this paper we discuss student's utterances that emerge when faced with what we call *non-solution problems*, that is, problems in which they must answer that it's not possible to find or do what they are asked to do, so they must find general relations within the problematic situation in order to argue such things as "the number 123 doesn't belong to that sequence", "there is no least positive fraction" or "it's impossible to tessellate a rectangle with that kind of pieces." The questions addressed here are: how do the students interpret the non-solution problems as a mathematical practice in the classroom? What kinds of arguments do the students offer?

COMPONENTS OF A CONCEPTUAL FRAMEWORK

The learning of mathematics is considered an individual process of construction and, also, an acculturation process into the mathematical practices of a wider society[2]. The use

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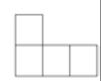
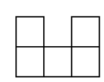
[1] Roughly speaking, this type of teaching is based on a teacher transmitting information and by the scheme eliciting-answer-evaluation (Mehan, 1979).

[2] We mean those practices accepted by math teachers, researchers in mathematics education, professional mathematicians, etc. Even though this *wider society* is not homogeneous, each

and meaning of the symbols and those mathematical practices are negotiated in the process of constituting a mathematical community in the classroom. (Cobb, Jaworski, & Presmeg, 1996; Yackel & Cobb, 1996).

The focus in this paper is on the argumentation process in the classroom. In this context, the notion of argumentation is related to offering an intentional explanation of the reasoning behind the solution. Several authors have reported the enormous difficulties students experiment when faced to the task of arguing (Fischbein, 1982; Balacheff, 2000). Beyond the social factors that may inhibit the student's processes of argumentation, Fischbein (1982) y Balacheff (2000) claim that the cognitive belief constitutes a major obstacle because students find no need to argue when they have intuitively seen the solution. In this perspective, the type of tasks implemented during the development of the study might function as a vehicle to promote students' presentations of arguments or convincing explanations.

Balacheff (2000) has differentiated two kind of proofs the students offer: *pragmatic proofs*, those in which students show the actions they performed finding the solution and *intellectual proofs*, based on the formulation of relations and characteristics of the problematic situation, for instance when they use a *generic example*, that is, when they refer to a particular case in order to show general relations. Pragmatic proofs imply the realization of a material task. This suggests an exploration theme for instruction; facing students with problematic situations where that material realization is not possible so that the student is being forced to search for general relations.

Topic	Problems
I Numerical patterns	• What's the position of number 311 within the sequence 3, 6, 9, ···? • What's the position of number 123 within the sequence 2, 5, 8, ···?
II Tessellations	• There are plenty pieces alike, formed by four 1-side squares as shown in the figure on the right. Make a 3 × 8 rectangle and a 5 × 10 rectangle using these pieces, without overlapping them or leaving holes.  • Make a rectangle using pieces like the one on the figure on the right. 
III Arithmetic	• Find the least positive fraction • Which number has the greatest number of multiples? • Which number has the greatest number of divisors?

teacher acts in his classroom according to his own interpretation on which of those mathematical practices are the accepted ones.

Table 1. Non-solution problems

METHODS AND GENERAL PROCEDURES

We worked with a seventh grade group of 30 middle class students in a school located in Mexico City. The students come from different primary schools (grade 1-6), which can be considered within the School Mathematics Tradition.

The sessions developed in a cyclic mode: 1) It starts with a set of different kinds of problems discussed within small groups of students (2 to 4 students). Each group writes down a report and one of them is audio recorded; 2) the next two or three sessions are dedicated to a plenary discussion of these problems, promoting the students to present different interpretations and solutions to the task; 3) Based on the analysis of the reports and the audio recording, a set of new problems is designed and a new cycle is initiated.

Within the school year, students worked with *non-solution problems* (Table 1). These problems were mixed with other kind of problems so they couldn't know, in advance, which type they were dealing with.

Two weeks after the students had worked with the last of these problems they were asked to write down, individually, the following items: *Describe non-solution problems and give examples. How come do they appear? Are they important?*

RESULTS

We discuss in separate sections each of the questions posed in the introduction of this paper.

Interpretation of non-solution problems

The teacher systematically promoted the acceptation and valorization of the mathematical practice of dealing with non-solution problems making explicit comments of how natural is to pose questions and problems in which we cannot know, in advance, whether they can or cannot be solved, and that recognizing that it's not possible to do or find something is an important form of knowledge. However, each student interpreted this practice in different ways.

The first time students worked with a non-solution problem, we observed, as we expected, difficulties in assuming that it was not possible to do what the teacher asked them to do; only one fourth of the groups answered that 311 is not in the succession 3, 6, 9, ···. The audio-recorded group made the division 311÷3, obtaining 103 with remainder 2 and reported 103 as an answer. Few minutes later, a student from another group *revealed* (whispering) "The first one is wrong":

> Pam: He is right, 'cause it's not a sequence
> Jim: Why is it not a sequence?
> Pam: Because 311 is a multiple of 3, so it can't be in a sequence that goes 3 plus 3, plus 3…

It's remarkable that Pam accepts without hesitation that their answer was wrong, as if she was not comfortable with 103 as an answer, but she didn't even consider the possibility of saying no to what the teacher had asked them to do. Only after the *revelation* she did establish, using a correct argument, that 311 is not in the succession.

Once the possibility of a negative answer became part of the students' answers repertory, it was observed that some students gave up the explorations after a few tries, concluding, "It's not possible." For instance, working with tessellation problems the answer "we tried and it can't be done" was relatively frequent[3]. The audio-recorded group, trying to make a 5 □ 10 rectangle with the L-shape pieces, failed for the second time:

> John: There are two squares left.
> Daniel: Two left! It can't be done.
> Mary: Let's write, "There are always two squares left" ... but why are there always two left? We have to write that too.

They quickly conjecture that it can't be done but they also recognize that they have to explain why it is so (they did explain it later). Something similar occurred when students searched for the number with the greatest number of divisors; after a few tries some groups concluded that there is no such number although they didn't offer any argument supporting their claim. Others didn't easily accept the legitimacy of that kind of answer; for instance, when trying to tessellate a rectangle with U-shape pieces (this was the fifth non-solution problem he had met) a student asked the teacher: "is it possible that it is not possible?"

This behavior is similar to the one observed by other authors: Schöenfeld (1992) concluded that students don't make long searches because they believe that "*you know the answer or you don't"* so there is no use in doing long searches. Also, as it has been reported (Balacheff, 2000), students usually make generalizations after analyzing a few particular cases. This author refers to this process as *naive empiricism.*

The analysis of the individual reports about what they think of non- solution problems made clear that, for most of the students, non-solution problems are viewed as anomalous or as a tool used by the teacher to make them aware of what they were doing, stopping them from acting automatically. Only a few considered that "it's normal that they appear when you ask questions and they help you to find new ideas and theories."

Most of the students showed correct examples of non-solution problems similar to those they had worked with. Even an example as "find the transformation represented by 3 □ 5, 5 □ 9, 9 □ 5", which is wrong, mathematically speaking, because there is an infinite number of transformations that satisfy the requested conditions, it is a good example in the context they've been working, that is, transformations of the type $ax + b$, $x^2 + a$ and $x^3 + a$. Others used inappropriate examples, again thinking in operating terms: "3 □ 2 is not equal to 15."

Arguments offered by the students

Number Pattern. Some groups operated with the numbers they had at hand, applying an algorithm that showed to be successful in another situation and paid no attention to the fractional quotient. This compulsory need to operate with the numbers in the problem, even in the most absurd situations, has been widely documented (IREM, 1980; Schubauer-Leoni, & Ntamakiliro, 1998).

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[3] In his interactions with the students the teacher systematically used the phrase "is it that *it can't be done* or that ***you** can't do it?*" trying to problematize their experience.

In both problems the students recognized the pattern + 3 governing the succession, but not all took into account the starting number and argued that 123 is the 41st number of the succession 2, 5, 8, ... because 41 $\times$ 3 = 123. Other students used a proportional approach stating that 123 is the 43rd number of the succession because 20 is the seventh, so 120 is the 42nd (120 = 20 $\times$ 6 and 42 = 7 $\times$ 6); some students simply generated the succession using a calculator until number 122 concluding that 123 is not in the succession, finally, some used a generic example as an argument: "Number 123 is not in the succession, we know it because 7 $\times$ 3 is 21 and because we started with 2 it's 20 (this is a way of saying that numbers in the succession are of the form $3n - 1$) and 124 is not a multiple of 3."

Tessellation problems. The problem of tessellating a rectangle using U-shape pieces turned out to be quite a complex one. All of the groups concluded that it's not possible to do so but none could find a valid argument. On the other hand, the problem of tessellating rectangles with L-shape pieces proved to be adequate. All the students could construct the 3 $\times$ 8 rectangle. The following discussion took place within the audio-recorded group just after they found there were two squares left, in their second try to construct the 5 $\times$ 10 rectangle:

> Mary: What were the dimensions of this one?
> John: 3$\times$8
> Daniel: And this one? Five times ten... I think that it's only possible when… three times eight, forty-eight, even number, five times ten, fifteen[4], odd number... Hey! I know why it can't be done (very excited). Because it's four (he refers to the four squares in the L-shape pieces), four and forty-eight are even numbers… that's why it can be done, or something like that, right?
> John. Five times ten... 50, 50 isn't it an even number?
> Daniel: Yeah, why?
> John: **Five times ten**
> D: Ay! Ay!

Daniels' strategy is to find a property that holds in the 3$\times$8 rectangle but fails in the other one. He first conjectured that the rectangle's area should be of the same parity as the piece's area, which is true but this property holds in both rectangles and he missed this point because of the mistake he made multiplying. He began conjecturing about multiples of three but this exploring line is interrupted by the (correct) argument offered by John that can be resumed as: the rectangle's area must be a multiple of the piece's area.

> Daniel: Maybe it's because of the multiples of three... Yeah, three times eight is 24 a multiple of 3...and 50 isn't. Let's try with…
> John: 24 divided by 4? Would it be **exact**?

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[4] In Spanish the words three (*tres*), six (*seis*) and ten (*diez*) rhyme and maybe this auditive association can explain why Daniel answered 48 (6$\times$8) as the result of 3$\times$8 and 15 for 10$\times$5. Three days after this episode took place, the teacher unexpectedly asked the student "3$\times$8?" and he quickly answered 48. A few days later, the teacher asked him again and he could answer correctly only after a brief pause.

Mary: Yeah.
John: 50 divided by 4? That is not exact. Four because of those 4-pieces
Mary: It can't be done, of course, because fifty, that is, five times ten…
John: And it isn't a multiple of four.

Arithmetic Problems. The three arithmetic problems can be argued by *absurdum reductio*. Searching for the least positive fraction the students realized that they can always find a smaller one: "if we think we have found it we add a zero to the denominator and we get a smaller one." Also a fraction with a denominator like 999··· or 1000··· was reported.

Trying to find the number with the greatest number of multiples, students noticed that the process of finding the multiples is potentially infinite. They also realized that not every number has the same number of multiples: 0 has only one and some of them claimed that 1 has more multiples than any number, arguing that, for instance, "multiples of one go one by one and the multiples of two go two by two." This claim led to an interesting discussion; many students argued that all, except zero, have the same number of multiples because they are generated multiplying them by 1, 2, 3, ···.[5]

The problem of finding the number with the greatest number of divisors turned out to be a more complex one. This time the search for the divisors doesn't lead naturally to the idea that you can always construct a number with more divisors. Only four out of 12 groups could argue using a generic example: "There is no such number with the greatest number of divisors. For instance, 36, if we multiply it by 2, 72 has the same divisors as 36 and also 72 as a divisor and we can continue multiplying."

CONCLUSIONS

Students are used to show the solution of a problem applying a known algorithm. The algorithm is constituted as the main generator of answers. But when they have to make an argument about the impossibility of something they can no longer hold on any known algorithm. Most of the non-solution problems proved to be adequate for promoting the search for general relations. In particular the problem of tessellating a rectangle with L-shape pieces turned out to be a fruitful exploring situation that can be easily generalized changing both the dimensions of the rectangle and the shape of the pieces.

Negotiation of new mathematical practices in the classroom is a process that takes time, especially when they contradict strongly attached beliefs of the students. Although most of the students got relatively quickly used to work with non-solution problems, they don't think of them as common and natural tasks when learning mathematics but only as a didactical trick.

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[5] Even older students *clearly see* that the succession 1, 2, 3, ··· has more elements than, for instance, the succession 2, 4, 6, ··· and it's hard to convince them that it's not the case, but the way these successions emerged made some of the students *clearly see* they have the same number of elements.

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SCALING UP STRATEGIES FOR CHANGE

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This research report builds on the results of the CATCH project[1](Webb and Dekker, 2002). The CATCH project is an effort to apply and scale up previous results and design a professional development program to bring about fundamental changes in teachers' instruction to support change in teachers' formative assessment practices. In this report we focus on two of the research questions: "What professional development materials support improved formative assessment across a wide range of schools?" and "How do teachers' conceptions and assessment practices change as a result of their participation in this development program?" Initial results show that a change in teachers' attitude towards formative assessment occurred. Moreover, changes in instructional assessment were easier to implement for teachers than ideas related to the pyramid model.

INTRODUCTION

When in The Netherlands a new mathematics exam program[2] was introduced in 1992, the possibility to change the mandatory central examinations at the same time proved to be of great importance. For teachers as well as curriculum designers the format of the questions (no more multiple choice questions), posed within a context, as well as the scoring guide that showed different strategies and sometimes even different possible (correct) answers, set an example for the daily classroom practice. As far as we know, no formal research has been conducted in The Netherlands to the extent of influence of central examinations on the teaching and learning process preceding it. Results from the Research in Assessment Practices[3] (RAP) project, preceding the CATCH project, showed that many teachers have limited understanding of formative assessment practices and, thus, provide students with incomplete information about their progress (Romberg, 1999). When teachers learn to utilize formative assessment practices in their classrooms as a consequence of appropriate professional development, there are positive effects on student learning and achievement (Black &William, 1998). When teachers retained conventional assessment techniques, especially focused on assessing basic skills, they also paid little attention to different strategies used by their students to solve a problem, to classroom discussions and, in general, to "teaching and learning with understanding", even when they used reform curricula.

It is on the basis of these past experiences that we assumed that changing the assessment practices of teachers already using reform curricula could play an important part to

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[1] CATCH Classroom Assessment as a basis for Teacher Change. Funded by the Office of Educational Research and Improvement (OERI PR/award R305A60007) 2000-2003

[2] for students of vbo (vocational) and mavo (theoretical level, leading to vocational education at middle level), central examinations are held at the age of 16.

[3] RAP was one of the projects of the National Center for Improving Student Learning and Achievement in Mathematics and Science (NCISLA), funded by OERI, 305A60007-98,1996-2000

enhance “teaching and assessing for understanding”. By posing questions and using test items that evoke mathematical reasoning and generalizing and by asking students to choose their own mathematical tools to solve a problem, it becomes clear what students are able to do instead of which facts, standard algorithms and definitions they know. If the process is as important as the product, if strategies used by students are important, it becomes important for the teacher to listen carefully to what students say in class and assess student work more closely. Teachers use the information gathered this way to guide instruction. Textbooks do not always provide good problems. So the next step for teachers to be taken is to adapt questions posed in the curriculum or to enhance their own assessment problems. Then the design of balanced assessments is being discussed, using problems at different competency levels. Assessing becomes a continuous process, an integrated part of the teaching and learning process instead of something from outside, interrupting this process but nevertheless inevitable. We feel that this results in a more student centered instructional environment, more likely to improve student achievement (Bransford, Brown & Cocking, 1999).

PROFESSIONAL DEVELOPMENT: MATERIALS AND STRATEGY

The ideas on which the CATCH project, meant for mathematics teachers at middle grades, was based and which served as a starting point for the design of professional development materials find their origin in the work of the Freudenthal Institute. The ideas are based on studies on assessment by researchers at the Freudenthal Institute, (de Lange 1987, Van den Heuvel P, 2000). They place assessment as an essential part of the teaching learning process and emphasize assessing for understanding. An essential
part of the principles underpinning the assessment principles is the distinction of mathematical competencies at three levels (De Lange 1996). These ideas were further expanded into the so-called pyramid model (Boertien en Verhage 1993,Verhage en de Lange 1995, de Lange 1999) when designing a National Option for the TIMSS[4] study (Kuiper, 2000).
These ideas, incorporated in the Framework for Classroom Assessment in Mathematics (de Lange 1999), that forms the theoretical framework for the CATCH project, are aligned by those of the OECD[5] Programme for International Student Assessment (PISA), (OECD, 1999).
In the aforementioned model three levels of mathematical competency are discerned:

Level 1: Reproduction, procedures, concepts and definitions;

Level 2: Connections and integration for problem solving;

Level 3: Mathematization, mathematical thinking and reasoning, generalization and insight.

A course for teachers as an elaboration of the framework was developed: *Great Assessment Problems (GAP)* (Dekker & Querelle, 2002). An assessment tool, *AssessMath!* (Cappo, De Lange & Romberg, 1999), developed earlier was also used.

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[4] Third International Mathematics and Science Study

[5] Organisation for Economic Co-operation and Development

Teachers taking part in the project were continuously asked to give feedback on the materials being developed.

The strategy used in the project is based on the "train the trainer" model. The initial seminar, a three-day conference in Utrecht for about 20 teachers as well as administrators from two different school districts, was meant to be used by the CATCH lead teachers as a model for their own CATCH summer institutes. The Dutch CATCH team consequently developed a model seminar, consisting of different elements taken from these summer institutes. We want to take teachers seriously and value their experience, thus neither work "top down" nor "bottom up" but both ways at the same time. During the summer institutes CATCH lead teachers in the two districts used examples from their own students and from their own classrooms.

We expected the teachers taking part in the project to go through the following steps in their professional development:

1. During the initial CATCH seminar, lead teachers and administrators critique and develop greater understanding of existing assessment instruments using resources such as *Great Assessment Problems* and *AssessMath!* together with colleagues from their own district and/or school;
2. Teachers select and adapt assessment instruments for their own use with students and report about results during monthly and ad hoc meetings with colleagues;
3. In considering and using these instruments, teachers examine the role and function of assessment instruments versus the desired learning outcomes and the potential for positive feedback;
4. Scoring and grading of student work is used to provide insight in student (mis)conceptions that guide instruction;
5. Teachers implement design principles for classroom assessment and learn how a series of items can be constructed to design balanced assessments that reflect a hypothetical assessment trajectory. This provides students with the opportunity to demonstrate the full range of mathematical competencies including making mathematical arguments, non-routine problem solving, developing their own models and inventing new strategies;
6. Teachers explore assessment opportunities embedded within instructional contexts, learn how to balance the use of formal and instructional assessment, and examine the relationship between classroom assessment and student achievement on external assessments;
7. Teachers inform their colleagues during successive summer institutes, thus helping ideas and outcomes of the CATCH project to "travel" to new classrooms, schools and districts.

While passing through this trajectory we expected teachers to (a) recognize problems at different competency levels, (b) use or even design more higher level problems in their own assessments, (c) understand and use instructionally embedded assessment and (d) use a more varied set of assessment instruments in general.

It has been amazing, even to the members of the research team of the project, both in The Netherlands and the US, how fast ideas of the CATCH project "traveled". The two year project has not yet come to an end but in district A, a small urban/suburban district, serving over 3,000 students predominantly European American (85%) with approximately 30% free or reduced-cost meals, members of the CATCH team have gained visibility and assumed greater responsibility in school and district leadership in enacting changes in use of curricula and fostering greater consistency in classroom assessment practices at each grade level. CATCH lead teachers in this district noted that the summer institute was a decisive turning point in re-directing colleagues to enjoin in collaborative decision making toward modifying instructional resources and improving classroom assessment instruments during monthly grade-level meetings.

In district B, a large urban district in Eastern United States, where middle grade teachers work with predominantly African American and Hispanic students, with more than 75% receiving free or reduced-cost meals, "traveling" has taken place to an even greater extent. The district is now making plans to expand the CATCH program to K-12 teachers of mathematics as well as science. Lead teachers in this district felt the pyramid model could also be adapted for other grades and other subjects.

ANALYSIS OF QUALITATIVE DATA

Qualitative data[6] for this project from 14 lead teachers [7] was gathered through an initial survey, three rounds of classroom observations and teacher interviews, and through collecting assessment portfolios. Observation protocols were adapted from instruments used by Horizon Research[8]. An exit survey for participating lead teachers will follow by the end of the project. Analysis of the interviews is being conducted, using the Multiple Episode Protocol Analysis (MEPA)[9] program. Interjudge agreement is secured by having analyzed and categorized the data independently by at least two members of the CATCH research team. Since this is only a relatively small study, the results of the second round of interviews were used to assess and adapt the codes used. Observations and assessment portfolios will be used to analyze individual teacher development and to validate the research implications derived from the interviews. The research questions, posed in the project were:

1. What professional development materials will be required to disseminate principles for improving formative assessment across a wide range of schools?
2. What support do school personnel and teachers in various school contexts, who are adapting these principles to local conditions, need to ensure that changes in formative assessment are sustained?
3. How do teachers make decisions about what assessment instruments to use, when to use them, and which reasons motivate their choices?
4. How do teachers' assessment practices change as a result of their participation in this professional development program?

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[6] quantitative data, based on standardized test scores of students taught by CATCH teachers, are also being collected and will be analyzed later.

[7] 6 teachers from district A, 8 from district B.

[8] see http://www.horizon-research.com/LSC/manual/0203/existing.php#6

[9] MEPA was designed by Gijsbert Erkens, University of Utrecht, The Netherlands.

5. How are changes in teachers' assessment practices reflected in their students' achievement?

By taking into account research question 4, we formulated a set of variables to be used for the first analysis of the second round of interviews where some change may already be expected.

V5 variable 1 to 4, **main codes** (alphabetic order), version 2

ccp	changes in classroom practice
cta	changes in teachers' attitude towards assessment
ss	important support from different sources

V6, **sub codes** (alphabetic order), version 2

cta – asw	analysis student work and strategies, scoring and grading, record keeping
cta – ie	instructional embedded assessment, more observations, more discussion, different role of homework
cta – lt	less tests, less quizzes, less homework checks
cta – oai	critique own assessment instruments, use of other formats
cta – pyr	more levels, assessment pyramid
cta – sr	emphasis on student responsibility
cta – und	better understanding of and more confidence in assessment issues in general

The same sub codes (except '- und') were used with main code ccp.

ss – am	AssessMath!
ss – catch	CATCH Team
ss – col	colleagues, amongst them lead teachers, leadership team
ss – cur	curriculum materials used (for example balanced assessments)
ss – gap	GAP-book
ss – mat	background materials used, e.g. balanced assessments MiC
ss – prin	principals, administrators
ss – time	release time
ss – utr	Utrecht seminar and info about assessment pyramid
ss – web	CATCH website
ss – work	workshops, summer institutes, conferences

V7, **sub code**

ns	non-success

sa	success student achievement
Furthermore sub codes to compare the answers of the second interview round with those of the first and third were used here.	

Figure 1, codes

After the initial analysis using the codes we found the interrater reliability was not sufficient, Cohen's Kappa of 0.4 was too low. In a small group codes were discussed and adapted. The second version of the codes used is shown in Figure 1. Furthermore a new column, V7 was added for coding (non) success and records that are important for comparison with earlier surveys/interviews. Codes were discussed until agreement was reached between two members of the research team, before importing them in the files.These codes will be used for the analysis of all three rounds of interviews. In the table of figure 2, results from the second round of interviews are shown:

variables (V6)	*abs.freq.*	*valid perc.*		*abs.freq.*	*valid perc.*
cta-asw	11	4.44%	**ss-am**	1	0.40%
cta-ie	18	7.26%	**ss-catch**	19	7.66%
cta-lt	1	0.40%	**ss-col**	27	10.89%
cta-oai	13	5.24%	**ss-cur**	5	2.02%
cta-pyr	20	8.06%	**ss-gap**	4	1.61%
cta-sr	1	0.40%	**ss-mat**	6	2.42%
cta-und	5	2.02%	**ss-prin**	3	1.21%
ccp-asw	18	7.26%	**ss-time**	3	1.21%
ccp-ie	37	14.92%	**ss-utr**	5	2.02%
ccp-lt	5	2.02%	**ss-web**	5	2.02%
ccp-oai	18	7.26%	**ss-work**	10	4.03%
ccp-pyr	5	2.02%	other/no code	2236	
ccp-sr	8	3.23%	Total	2484	100%

Figure 2, first results

Some remarks can be made by looking at these results:

1. The pyramid model (as one teacher noted in the initial survey: "Sweet, short and simple") proved to be an important model to initiate changes in teachers' ideas about assessment practices. However, whereas the pyramid was mentioned 20 times in relation to change in teachers' attitude towards assessment, only 5 times teachers reported a change in classroom practice in relation to the pyramid. Apparently this model is very appealing to teachers but not easy to put into practice. This is in contrast with the frequencies found in the coding of change towards instructional embedded assessment: Change in attitude towards instructional embedded assessment was mentioned 18 times, whereas 37 times a change in practice towards instructional embedded assessment was reported. Our conclusion is that this type of change is much easier for teachers to implement than the ideas related to the pyramid.

2. Often actual changes in classroom practice were mentioned. 37 times (14.9%) instructional assessment was mentioned, 18 times (7.3%) the use of other formats for assessments, 18 times (7.3%) the analysis of student work and keeping record of informal assessment and 5 times (2.0%) teachers stated the number of tests changed (quality for quantity).
3. Responsibility for their own work is a key word for teachers as well as for students. Teachers state they have a better understanding now of assessment issues (5 times, 2.0%) and some either say they want their students to have a greater responsibility (8 times, 3.2%) or that they have already achieved that (1, 0.4%).
4. The support given by the CATCH team (19 times, 7.7%) and colleagues (including the CATCH leadership team) proved to be important (27 times, 10.9%). Building a community of collaborating teachers was one of the goals of the CATCH program; these results show that goal was reached to a great extend.
5. Some teachers report they have not been successful (yet) in implementing CATCH ideas. They mention different reasons why, in their situation, or with their students, changes are impossible.

During our presentation at PME we expect to present more results since the results of the first and third round of interviews will be available by then. We will discuss our findings with the attendants.

CONCLUSIONS

Experiences in the CATCH project show that teachers as well as their students can profit from an enhanced insight in formative assessment practices. The pyramid model, as presented in the CATCH professional development program designed to support changes in teacher's formative assessment practices, proved to be an important model to change teachers' attitude towards "teaching, learning and assessing for understanding". To put these ideas into practice, however, was easier for instructional embedded assessment compared to their use in the design of teacher made tests. The ideas on which the project was based "traveled" faster and are now used by a much larger group of teachers than was expected. Issues that need further exploration are (a) the impact on student achievement on standardized tests, (b) the support offered by school personnel and (c) the way individual teachers make decisions about the assessment instruments to use, when they use them and which reasons motivate their choice.

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REPRESENTATIONAL ABILITY AND UNDERSTANDING OF DERIVATIVE

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The ability of students to interpret from, represent mathematical problems in, and interact with, a number of differing representations appears to be a crucial part of understanding mathematical concepts. Here we consider the role of representation in student understanding of derivative and present an outline framework for determining its influence. Analysis of questionnaires and interview data from case studies of two students on the basis of the framework revealed qualitative differences in their understanding of derivative and its associated concepts, and suggesting that representational fluency may be a key marker for this difference in thinking.

Since mathematical objects are conceived with the support of tools such as language, symbols, graphs, and artefacts, it has been considered (Brown, 2001) that representation cannot be divorced from the process of mathematical understanding. For example, an important indication of emerging understanding is the capability of the learner to represent a problem in a number of different ways, allowing them to approach solutions from different perspectives (Sigel, 1999). Indeed it has been suggested (Cifarelli, 1998) that problem solving success may be due to the ability to build appropriate representations of the problem, using these to construct meaning for the given problem situation. In addition, the ability to perform mathematical operations is contingent on the ability to recognise the syntax and rules of production in a chosen representational system (Hiebert & Lefevre, 1986) and to interact with representations according to these rules (Thomas & Hong, 2001). However, each representational form will support and enhance differing aspects of mathematical thinking and learning, and hence the formation of integrated multiple representations for the same phenomenon, or representational fluency (Lesh, 2000), is most likely to encourage meaningful understanding.

One crucial variation in thinking which interacts with the formation of representational ability is the process/object duality of mathematical conceptions described in the literature. While the ideas vary somewhat depending on the researcher's perspective, it is generally agreed that a dynamic process can become viewed as a static object (Sfard, 1991). This occurs, in the words of Dubinsky (1991) via reflective abstraction, and *encapsulation* of the process as an object. In this present research we have considered a theoretical perspective on possible differences between action (or procedural), process and object views of derivative, and how these might be characterised in terms of their relationship to graphical, algebraic and tabular representations.

A REPRESENTATIONAL FRAMEWORK OF KNOWING DERIVATIVE

Other recent research has tried to categorise student activity associated with derivative. For example, Kendal and Stacey (2000) have developed a *Derivative Competency Framework* that enables the monitoring and comparing of student achievement in differentiation across three representations (symbolic, graphical and numerical).

However, that framework concentrates on the solution processes employed by students and the relationship of these to the representations, analysing these in terms of the activities of formulation, interpretation and translation (and combinations). Our perspective here is that it would be useful to construct a framework allowing student understanding to be related to qualitatively different dimensions of knowing and to representational ability. We describe below five dimensions of knowing arising from a process-object theoretical approach to advanced mathematical thinking, and then present the framework in matrix form, describing possible abilities of students within the different representational modes (symbolic, graphical and numerical).

Five Dimensions of Knowing

We have attempted to describe a classification of knowing (and understanding) in terms of the representational abilities which they mediate. The five dimensions of knowing we have identified are: *procedure-oriented, process-oriented, object-oriented, concept-oriented* and *versatile*. It is important to state that these are not claimed to be mutually exclusive categories; we are trying to describe the dominant perspective of an individual.

- *Procedure-oriented knowing* —can successfully obtain an answer by following a sequential set of rules (a procedure), which may or may not be meaningful for them, to solve a given problem. Procedural knowing is not always meaningless and limited to an application of known procedures (Hiebert & Lefevre, 1986). It includes the ability to interpret and represent a problem in a particular representational system, and to know when and how to use the procedure (Skemp, 1976).

j. *Process-oriented knowing*—the learner has condensed and interiorised a procedure in its totality. Rather than being step-oriented and sequential it is global and holistic. The learner has an idea of what process may be used to solve a given problem and when it is appropriate. It includes the ability to describe and reflect on procedures without necessarily performing them (Cottrill *et al.,* 1996).

k. *Object-oriented knowing* —a process can be operated upon as an object. The learner can reflect on the process, but can also generate a mathematical entity out of it. They can perceive a representation as portraying an operation and the result of the operation.

l. *Concept-oriented knowing*—the level where the learner has created a 'bigger picture,' comprising schemas containing procedures, processes, and objects arranged in a relational manner. The learner with concept-oriented knowing can provide answers to why certain procedures and processes work, is able to create conceptual links across representations and relate process and object tools used in solving problems.

m. *Versatile knowing*—the learner has a sufficiently wide range of the four types of knowing described above to enable choice in problem solving, along with sufficiently developed metacognitive ability to choose an appropriate perspective at any given point in time, and ability to move fluently between the chosen perspectives as required.

From a consideration of these theoretical levels the *Representational Framework of Knowing* matrix in Table 1 was constructed. It describes general criteria, along with brief examples of the kind of representational abilities that might be associated with thinking about derivative and differentiation in that domain.

Dimensions	Representations		
	Symbolic	Graphical	Numeric/Tabular
Procedure-oriented	Manipulate symbols according to rules e.g. $\frac{d(x^n)}{dx} = nx^{n-1}$	Calculate from graphical forms. e.g. the gradient of a tangent at a point or a chord from two given points	Use procedures to obtain numerical results from tables. e.g., interpolation for approximating an average rate of change
Process-oriented	Interpret the meaning of symbols $\frac{dy}{dx}$, $f'(x)$, y' as a differentiation process	Have a pointwise approach derivative of graphs e.g. can draw the graph of a derived function given the graph of a function. Can understand second derivative as rate of change of gradient.	Understand and apply rate of change, differentiation and gradient processes in a tabular setting. e.g., use $\frac{f(x+h)-f(x)}{h}$ for small h from a table.
Object-oriented	Operate on the symbols $\frac{dy}{dx}$, $f'(x)$, y' as objects, e.g., assign meaning to $\frac{d\left(\frac{dy}{dx}\right)}{dx}$ Interpret nth derivatives as functions	Interpret derivative graphs as representing functions Operate on graphs as an entity e.g., relate the derivatives of $f(x)$ at $x = m$ and $f(x - a)$ at $x = m + a$ $\forall x$	Interpret a table of values as representing a discrete approximation to a continuous function e.g., transform a table of values for $y=f(x)$ to $y=f(x-a)$, for all x.
Concept-oriented	Relate the differentiation procedures, and processes applicable in one representation to each other and to relevant concepts. Make procedural and conceptual connections between corresponding differentiation procedures, processes and objects in different representations. Identify and operate on conceptual objects such as derivative and function presented in different representational forms.		
Versatile	Have sufficiently well-formed differentiation procedure-, process-, object- and concept-oriented knowledge to be able to identify and use appropriate objects, processes and procedures in their various representational manifestations. Choose appropriate representational system perspectives to solve a differentiation problem. Move seamlessly and fluently between the chosen perspectives as required.		

Table 1. The Matrix for a *Representational Framework of Knowing Derivative.*

METHOD

The case studies reported on here describe two male Form 7 students (aged 18 years), James and Steven, from a private and a public school, respectively. Both schools are situated in high-level socio-economic catchment areas in Auckland, New Zealand. The two students are part of a larger study of understanding of derivative currently taking place with students from four schools where the teachers agreed to ‘integrate’ graphic calculators in their calculus teaching. The students were selected on the basis of a test given to their respective classes, with James selected from among the higher achievers, and Steven from the lower achievers. Each of the two subjects completed a pre-test, a pre-intervention interview, a module of work on derivative using the TI-89, a post-test, and a post-intervention interview. The pre-intervention interviews were video-taped, while the post-intervention interviews were audio-taped. Both interviews were transcribed and the data analysed together with the results of the tests.

RESULTS

On the pre-test James scored 12.5 (2nd highest) and Steven 3 out of 31. However, it was clear by the second interview that Steven and James not only differed in test scores, but more importantly had qualitatively different types of thinking about derivative. For example, Steven, though appearing to be confident with his rules of differentiation has difficulty understanding what derivative and differentiation are. When asked if he understood what a derivative is he stated “Hmm..I know how to do that. I don't know how to describe it. Let's say it's related to graphs." Asked what the derivative gives you he replied "I don't know really." He recognises that he has had problems with his understanding and recall of the subject, "I did differentiation. I understood it. I haven’t done it since a little of last year…I’m not sure how much I remember. I hardly remember doing it.” It seems that his understanding of derivative is iprocedural and he does not see derivative as a process, or as an object resulting from a differentiation process.

One major disadvantage of a procedural approach occurs when procedures are incorrectly remembered, or used in an inappropriate context. This was apparent when Steven was asked to differentiate the function presented algebraically as $y = \sqrt[3]{(x^2+4x)^2}$ in the second interview. His responses given aloud included “First rule: rearrange the cube root, so you have [writes an answer]. Yeah, that’s right. And then I’ll expand that out first. So…[continues writing]…then I expanded the bracket out…so then I’ll use the chain rule.” While it may appear that Steven was confident that he had the correct procedures to employ to solve the problem (namely expansion and the chain rule), we can see from his written work (Figure 1) that he has errors in both the bracket expansion and in the differentiation procedure.

$$\left[\left(x^2+4x\right)^2\right]^{1/3} = \left(x^4+4x^2\right)^{1/3}$$

$$y' = \frac{1}{3}\left(x^4+4x^2\right)\left(4x^3+8x\right)$$

Figure 1. An example of Steven's procedural errors.

A similar problem emerged when he was asked to differentiate the function $q = 3m^2 - \frac{7}{m^3}$, where his answer was $\frac{dm}{dq} = 6m - \frac{7}{3m^2}$. He seemed to be confused by the unfamiliar symbols and so used the symbol $\frac{dm}{dq}$ instead of $\frac{dq}{dm}$. In addition he exhibited a poor knowledge of the procedure for differentiating powers. When his mistakes were pointed out, he responded: “It’s not actually mistakes. So, just like that…” His confidence seems to be based on his level of “knowledge of the procedure”, since he knows this has worked well in previous problems.

In contrast when James was asked how he would explain derivative to himself he gave a conceptual answer related to a graph, saying that:

> Well, derivative is sort of like…it is the graph of the gradient of this other graph …I think it's more a fact that it's the gradient at a given point on the graph. That makes more sense when I think about it. So, yeah, but the graph gets you just all of these with the corresponding *x-y*, why makes it a lot easier to read.

However, he was also able to see it as a function in its own right generated by a process, and as a rate of change, commenting that "They're all functions in their own respect…It's … rate of change." This provides evidence of both process- and object-oriented knowing. One of the areas where characteristics distinguishing procedural (action) or process thinking about derivative from object thinking can occur is regarding appreciation of $f'(x)$ and $\frac{dy}{dx}$ as objects. Seeing $\frac{dy}{dx}$, with its two apparent parts, as a single entity can prove difficult. Indeed it has previously been noted that some students find difficulty in the transition from the representation, $f'(x)$ to $\frac{dy}{dx}$. Chinnappan and Thomas (2001, p. 158), for example, record the words of a student who said "I couldn't do $\frac{dy}{dx}$, and as soon as I hit university, it changed… $f'(x)$ was, I mean, that's what I worked with. I liked that… I got completely stumped… and it took me a long time to figure it out, but once I did it was great, like a revelation."

Previously (Delos Santos & Thomas, 2001) we found that only 45% of 22 students could make any interpretation of $\frac{dy}{dx}$ in $z = \frac{d(\frac{dy}{dx})}{dx}$, and only 1 thought that it had anything to do with rate of change or gradient of a tangent. In agreement with Thomas (2002) we believe that this is because students perceive $\frac{d^2y}{dx^2}$ as a repeated application of the differentiation process, but are unable to see $\frac{d(\frac{dy}{dx})}{dx}$ this way, thus they meet a cognitive obstacle. This latter form appears to require one to see the application of the differentiation process to the derivative object $\frac{dy}{dx}$, and this gave us a starting point for an investigation of a difference in thinking. In the context of the framework there is a distinction between the former process-oriented view and the latter object-oriented perception in the algebraic domain.

In the second interview, at the end of the study, when asked to describe his understanding of a number of symbolic forms, James spoke of $\frac{d(\frac{dy}{dx})}{dx}$ as "The differential of…I don't know. Hold on, hold on…or derivative of first derivative, derivative with respect to *x*, second derivative. So it's a rate of change, derivative of *y* with respect to *x*…aha, aha.. rate of change…'cause the 'd' is 'delta'…interesting stuff", while $\frac{d^2y}{dx^2}$ was "the second derivative of the function *y* … the derived function of the first derived function which gives the nature of the turning point". They were clearly not the same to him. In contrast Steven found the former too difficult, saying "I'm not sure… I got confused by that, over this side. Sort of keeps me…" This was not probed any further since he appeared irritated by his lack of insight, lacking a relevant contribution from his concept image of derivative.

For $\frac{dy}{dx}$ James referred to it as the "gradient function…the first derivative of the function y with respect to x, and that it is used to find points, stationary points on the graph and turning points." Asked about $f'(x)$ he replied that it is "the first derivative, which is what we use in gradient, same as derivative of y with respect to x…kinda similar [to $\frac{dy}{dx}$], sort of interchangeable." When asked why he thought that there are two symbols used to represent derivative he said $\frac{dy}{dx}$ "could be graphical because y is involved…the function y on the vertical axis and that is defined by what x is doing", while $f'(x)$ "could not be as much like graphical". However, he still is a little confused, as seen in his remark that "$f'(x)$ is not implying the gradient of the graph…[it is] implying the derivative of the function $f(x)$…even though they both imply the gradient." Clearly there is a representational aspect here to James' thinking. He associates $\frac{dy}{dx}$ more with the graphical representation and $f'(x)$ more with the algebraic, even though it can be used to find the gradient of a function. This reinforces our view that the context in which symbols are first met has a strong influence on the way we think about them. In spite of this James has a concept-oriented perspective across the representations. He talks about concepts such as derivative, stationary points, turning points, gradient, axes, graphs, etc in a manner which portrays a richer conceptual structure than Steven.

In contrast with James' view Steven has a noticeably different perspective on the symbols. When asked about $\frac{dy}{dx}$ he immediately responds in a process-oriented way, saying "I must differentiate." Pressed for what it stands for he responds "Differential of y over differential of x" and appears unable to proceed further. His comments also stress the importance of context to meaning, since when asked why he thinks both $\frac{dy}{dx}$ and $f'(x)$ are used he replies "if there's no y or x in the equation then you can't put one there, so you'll have f'. It is a purely instrumental, context-dependent decision for him.

We decided to examine the students thinking further by taking them into uncharted waters, presenting them with unfamiliar algebraic constructions. While both of the students had used composite functions previously and were familiar with the form $f(g(x))$, we employed the unusual notations $f(f'(x))$ and $f'(f'(x))$ to access their thinking. For the first of these Steven responded "the original function times the differential of the original function", and proceeded to illustrate this by multiplying the function $f(x) = 2x^2 + 1$ by its derivative $4x$. He followed through consistently for the second describing it as "the differential times the differential" Thus when faced with an unfamiliar representational form Steven's recourse is to interpret the contiguous arrangement of f and f' (and later of f' and f') as a known operation or process, namely multiplication. James on the other hand displays the ability to think of the symbol $f'(x)$ as an object, what he calls the derivative function. He states that "this might be the first function here of the derivative function" and in order to elaborate further, he tried to interpret the symbol using specific functions, of the form $f(x) = x^n$, showing that he is able to take $f'(x)$ and treat it as an object, applying the function *f* to it. He wrote

$$f(x) = x^2 \qquad f'(x) = 2x$$

$$f(f'(x)) = (2x)^2 = 4x^2$$

and

$$f(x) = x^3 \qquad f'(x) = 3x^2$$

$$f(f'(x)) = (3x^2)^3 = 27x^3$$

and using the graphic calculator sought to generalise this, getting $n^n x^{(n-1)n}$, and saying "this is always gonna happen…variable *x* to a power, an even power". Moreover, he provided a graphical interpretation saying that "It's going away…it's always gonna be steeper than this original function…it's gonna be steeper…it's also gonna be concave up". Interestingly, when asked to describe $f'(f'(x))$ he said "that does imply second derivative". Hence instead of applying the same composite function thinking he had only seconds previously he now saw this as the second derivative $f''(x)$. It seems possible that this could be the result of a strictly linguistic interpretation of the symbolism. Reading $f'(x)$ as *f*-dashed of *x*, may cause one to read $f'(f'(x))$ as *f*-dashed of *f*-dashed of *x*. This in turn leads to James statement that "It's the derived function of the first derived function.", and hence the second derivative. Whatever the case it again stresses the importance of context in student thinking about symbolic forms. In this case even in the same algebraic representation the student treats expressions with *f( )* and *f'( )* in quite different ways.

CONCLUSION

We recognise that we have only just begun the process of providing evidence for the cells in our matrix of understanding of derivative and more work will come later. However we believe that, based on the data obtained from the two students we can summarise some of the characteristic differences between them that relate to the framework:

- Steven displays primarily procedure-oriented thinking in contrast to James' often concept-oriented approach. e.g., James sees $\frac{dy}{dx}$ and $f'(x)$ in terms of concepts; Steven as an instruction to carry out a differentiation procedure.
- Notation is often linked to representations, e.g., $\frac{dy}{dx}$ is seen as graphical and $f'(x)$ as algebraic.
- Interpretation of a representation appears to be anchored in the context of first exposure and this influences current interpretation of new representational forms. The introduction of new representations should be integrated with previous forms. This is shown in the need to make explicit links between $\frac{dy}{dx}$ and $f'(x)$.
- $\frac{d(\frac{dy}{dx})}{dx}$ and $\frac{d^2y}{dx^2}$ are not perceived as the same by a procedure-oriented thinker. The former requires an object-oriented perspective on derivative.
- Procedure-oriented thinking is more easily forgotten, and it is more difficult to monitor solutions, as Steven's example shows. Concept-oriented thinking is more flexible, allowing different perspectives from other representations to be related to solutions and to inform conjectures, as we see from James' work.

- Concept-oriented thinking is more extensible into new areas. e.g., James coped well with $f(f'(x))$ by applying the composite function concept, whereas Steven invented a procedure.

Since the ability of a student to understand conceptually may be anchored in the ability to represent an object in different forms, and to relate representations to the object and to other representations of the object, the student who has more representations may have more strategies that can be applied to the problem, and thus, more solution paths. This research suggests that developing richer representational ability could provide a useful means to move from procedure-oriented towards concept-oriented understanding.

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USING STUDENTS' WAYS OF THINKING TO RE-CAST THE TASKS OF TEACHING ABOUT FUNCTIONS[1]

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Recent research suggests that the examination of students' work may lead to changes in teaching practice that are more effective in terms of students' mathematical learning. However, the link between the examination of students' work and the teachers' actions in the classroom is largely unexamined, particularly at the secondary level. In this paper, I present the results of a study in which teachers had extensive opportunities to examine students' ways of thinking as the students developed models for exponential growth and decay. I describe two related aspects of the practice of one teacher: (a) how she listened to students' alternative solution strategies and (b) how she responded to these strategies in her practice. The actions of the teacher supported extensive student engagement with the task and the students' revising and refining their own mathematical thinking.

INTRODUCTION

The knowledge of subject matter alone is insufficient for effective teaching; subject matter knowledge is just one descriptor among many that attempt to capture the complexity of the nature of the knowledge base that is needed for teaching (Hiebert, Gallimore & Stigler, 2002; Shulman, 1986). Recent research on teachers' professional development would suggest that, among other things, effective teachers need to attend to students' ways of thinking about mathematical tasks (Schifter & Fosnot, 1993; Simon & Schifter, 1991). When teachers understand how students might approach a mathematical task and how their ideas might develop, this would seem to provide the basis for the teacher to support the student in ways that will promote student learning. However, most of the research on both students' ways of thinking and on teachers' understandings of student methods have focused on tasks in elementary mathematics, including important ideas in numeracy, rational numbers, and geometry (Ball, 1993; Jacobsen & Lehrer, 2000; Fennema, Carpenter, Franke, Levi, Jacobs, & Empson, 1996). Little research has been done on teachers' understanding of students' ways of thinking about tasks in secondary mathematics, such as functions, algebraic equations, Euclidean geometry, and data analysis.

THEORETICAL FRAMEWORK

In earlier work, I have argued that the focus of research on teacher knowledge needs to shift from examining what it is that good teachers *do* in particular situations to investigating how it is that good teachers *think* about particular situations (Doerr & Lesh, 2002). In other words, teaching is much more about seeing and interpreting the tasks of

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teaching than it is about doing them. A distinguishing characteristic of excellent teaching is reflected in the richness of the ways in which the teacher sees and interprets her practice not just in the actions that she takes. It is precisely a teacher's interpretations of a situation that influence *when* and *why* as well as *what* it is that the teacher does. Understanding teaching means knowing how teachers interpret the complexity and the situated variability of the practical problems of the classroom, how those interpretations evolve over time and across settings, and how and when those interpretations influence decisions and actions in the classroom. It is not enough to see what a teacher does, we need to understand how and why the teacher was thinking in a given situation, that is, interpreting the salient features of the event, integrating them with past experiences, and anticipating actions, consequences, and subsequent interpretations.

Increasingly, researchers on the development of teachers' knowledge have come to recognize two important perspectives on that knowledge: (1) that the knowledge for teaching is a complex and ill-structured knowledge domain (Feltovich, Spiro, & Coulson, 1997; Lampert, 2001) and (2) that to a large extent such knowledge is situated and grounded in the particularities of the contexts and constraints of practice (Borko, Mayfield, Marion, Flexer, & Cumbo, 1997; Lave & Wenger, 1991). These two perspectives on teachers' knowledge suggest that expertise in teaching is not a single, uniform image of a "good" teacher. Rather, expertise in teaching is highly variable, both across and within individuals and across multiple settings. This means that expertise can vary from teacher to teacher and that a given teacher's expertise can vary with different groups of students and with the same group of students in different situations. This is not to suggest a rejection of notions of accountability or assessment of good teaching. On the contrary, it is suggesting that teaching needs to be viewed as evolving expertise that will grow and develop along multiple dimensions in varying contexts for particular purposes.

One aspect of teachers' knowledge that has been the subject of much research (particularly, as I noted earlier, at the elementary level) is teachers' understanding of the landscape of children's conceptual development. In this paper, I emphasize that knowing this landscape is not the same as understanding one way of thinking or one way of developing or a particular learning trajectory. Rather, teachers need to recognize that within a given classroom children can be engaged in multiple ways of interpreting a problem situation and have multiple paths for refining and revising their ideas. The task for the teacher, then, includes seeing the multiple ways that children might interpret a situation, understanding that their ideas might be revised along various dimensions (while not being tested or refined along other dimensions), and acting in ways that will support the children's development towards more refined, more generalized, more flexible, and more integrated ways of thinking. The knowledge that teachers need consists of at least the mathematical understanding of the idea, an understanding of multiple ways that children's thinking might develop, the knowledge of typical mis-conceptions that students might have, an understanding of appropriate representations and the connections among those representations, and a knowledge of pedagogical strategies that will support that development.

In this research project, I examined teachers' knowledge of children's conceptual development by focusing on how the teachers saw and interpreted the events in their mathematics classrooms. I wish to recognize the complexity of those interpretations and the extent to which the teachers' interpretations are grounded in the context and constraints of practice. At the same time, I also wish to conceptualize the teachers' knowledge as that of evolving or emerging expertise along the multiple dimensions of practice. The central questions for this study are:

- How do teachers think about students' ways of thinking about exponential functions?
- How do teachers' interpretations of students' thinking influence their actions in the classroom?

By examining the particular characteristics of the teaching of an experienced teacher, I intend to generate an account of practice that moves beyond a description of what a "good" teacher does to an understanding of how teachers are seeing, interpreting and thinking about the tasks of teaching and evolving in their expertise.

DESCRIPTION OF THE STUDY

This particular study is part of a larger research project on the development of effective pedagogical strategies for teaching modeling tasks in technology-enhanced environments. The overall project includes modeling tasks that are intended to elicit the development of students' models (or conceptual systems) of linear change, exponential growth and decay, and periodic functions. The overall research design is that of the multi-tiered teaching experiment (Lesh & Kelly, 1999). This design enables researchers to examine the interpretations of teachers as they engage in understanding their students' interpretations of particular mathematical tasks.

The mathematical task that the students worked on is a well-known problem in exponential growth and served as the introductory lesson for the larger unit on exponential growth and decay. The task posed to the students was to investigate the pattern of pennies on a checkerboard when one penny is placed on the first square, two pennies on the second square, four on the third square, and so on. This simple recursive doubling pattern is very easy for the students to see. What is considerably more difficult for the students, despite their familiarity with the algebra of exponents and exponential functions from previous coursework, is to move from a recursive view of the function to an explicit form of a function that expresses the number of pennies on each square as a function of the number of the square.

Participants. The participant in this study was an experienced secondary teacher, who was teaching this particular lesson for the second time as part of a two-year project. This teacher had 30 plus years of experience and had strong content knowledge about exponential functions. The teacher, along with 11 others, had participated in two summer workshops where they explored the mathematics of exponential growth and decay and discussed various strategies that students might take in approaching this and other tasks in the sequence. The teacher also participated in monthly meetings during the school year in which student thinking and teaching strategies were discussed with colleagues and the researcher. Particular attention was paid to (1) listening to and identifying the different ways students might think about a problem and (2) supporting students so that they develop and revise their own solution strategies.

The teacher worked in a sub-urban public school with middle class students, with block scheduling, where the classes met for 75 minute periods for five times over a ten day period. There were 19 students, aged 16-18, with most taking this course as their final high school mathematics course. All of the students had graphing calculators and were generally familiar with their use from previous courses in mathematics.

Data Sources and Analysis. The teacher was video-taped during the lesson in which the Pennies Task was taught. The video-taping focused on the teacher and her interactions and exchanges with the students in her class. Field notes recording the researchers' observations were taken during the lesson. The video tape of the lesson, the transcript of the video tape, and the fieldnotes comprised the primary data sources for the analysis.

The data analysis was completed in two stages. The first stage of analysis involved open-ended coding (Strauss & Corbin, 1998) of the transcripts and field notes for the lesson. This coding was revised and refined through comparing the meaning of codes across similar lessons taught by other teachers in the study. This was followed by viewing the video tapes for the lesson, and adding annotations and clarifications to the transcript that were visible from the video tape. The coding of the transcript was then revised and refined in light of these annotations and clarifications.

The second stage of the analysis consisted of finding clusters of codes that defined the critical features the lesson. These features describe the dominant events that governed the lessons. In clustering these codes, I re-examined the data to find and interpret all those instances when the teachers were listening to and interacting with students for the purpose of understanding the students' thinking. This led to detailed descriptions of the teacher that were grounded in the data sources described above.

RESULTS

In this section, I briefly describe the implementation of the lesson by the teacher followed by the critical features of the lesson. The implementation began with Mrs. C asking the students to read the task and then to think about it independently. After a few minutes, she encouraged them to organize themselves into groups to work on the problem. From the outset, Mrs. C appeared to have a clear understanding that the central difficulty for the students will be in finding the equation that describes the number of pennies on each square as a function of the number of that square. As the lesson progressed, she repeatedly urged them to "think hard" and that what she wanted them to do was to "find the equation." Mrs. C allowed the students to engage in finding the equation for an extended amount of time. When the students had developed two different solutions to the problem, she asked both of those students to put their solutions up on the board and to explain how they arrived at their solution. She then posed that the differences in the form of the equation is a difficulty that the students must resolve.

The analysis revealed six critical features of Mrs. C's lesson:

(1) Setting an expectation for student thinking. Throughout the lesson, Mrs. C explicitly indicated that the problem did not have an immediate and obvious solution and that the real task for the students was to work hard and think about the task. Since this particular

lesson was the first task in an extended sequence of modeling tasks, Mrs. C. wanted to make clear that the rules of the game were changing; she would not provide them with easy answers, but would give them challenging tasks that they could productively think about. Throughout the lesson, she explicitly encouraged them to "work on the hard part" and "you've got to think now. That's the hard part."

(2) Focusing the task. From the teacher's perspective, the central mathematical difficulty for the students would be to find the closed form equation to describe the pattern of the pennies. As the students began work on the task, they generated tables and graphs on both paper and in their graphing calculator. The teacher encouraged this work, but repeatedly focused their attention on "finding the equation."

(3) Understanding students' ways of thinking about the task. Throughout the lesson, various groups of students attempted to model this exponential growth situation by using linear functions, investigating the rate of change using slope, finding quadratic functions, examining the behavior of the data at the origin, and exploring the patterns of perfect squares. In all of these cases, the teacher listened to how the students' were thinking about the task. The data revealed that the teacher was confident in her own ability to understand the diversity of students' thinking. She had anticipated the linear and quadratic approaches that students might take, but she also listened to and responded to new approaches:

Mrs. C	Okay. And that's what I want you to do is come up with an equation. [to Sara] Are you getting anywhere with it?
Sara	Does it have anything to do with perfect squares?
Mrs. C	[repeats to self] Anything to do with perfect squares...<pause> Like ... what are you looking at?
Sara	Like all the odd ones
Mrs. C	Okay <pause>
Sara	[cannot hear]
Mrs. C	Well, that's something that showed you something. Well, the even ones aren't perfect squares, though.
Mark	Yeah
Mrs. C	Well, so figure out why that might be.

The teacher was surprised at Sara's observation of the pattern of perfect squares, as can be seen in how she repeated the observation to herself. She probed to try to understand the student's thinking by asking "what are you looking at?" Mrs. C quickly realized that every other entry in the table was a perfect square, something she had not thought about before. She then reflected back to the students that not all the entries were perfect squares and encouraged them to continue thinking about why that might be so. Significantly, the students' investigation of the pattern of perfect squares and another group's investigation of the behavior of the data at the origin were novel ways of student thinking from the teacher's perspective. The teacher assimilated these approaches into her overall schema for students' ways of thinking about the task.

(4) Asking for student descriptions, explanations, and justifications. The teacher's response to the various ways of students' thinking was to ask the students to describe, explain and, on occasion, to justify their interpretations and reasoning. As we saw in the case with Sara (above), Mrs. C asked the student to explain her thinking so that the teacher could understand it. Mrs. C further encouraged the students to pursue a reason why their observation might be true. In the next example, when a student investigated slope, Mrs. C asked him to describe how he found the slope and to explain how he was using that to find an equation to fit the data. In the course of the student's explanation, he recognized that "the slope is always changing." This led him to consider a quadratic function rather than a linear one:

Mrs. C	Yes. It is the slope. Didn't you do a change in y over a change in x? [Bill looks on at Jerry’s work]
Mrs. C (to Bill)	No, he [Jerry] did the slope correctly.
Jerry	Well is there something wrong here?
Mrs. C	No, it is the slope, right?
Bill	Okay, okay, okay, I understand what you [Jerry] did. All right.
Mrs. C	It is the slope of the line connecting this [point] and this [point]. But [to Jerry] what did you just say?
Jerry	the slope changes, well, yeah
Mrs. C	So in other words, if you used different points than he did….
Jerry	you're going to get a different slope each time
Mrs. C	Which means?
Jerry	then, 'cause it's not a straight line. [Mrs. C shrugs affirmatively] So this will not work.
Mrs. C	So if you used this point and this point, you won’t get the same slope.
Jerry	So we have to figure out a parabola like
Mrs. C	[nodding] Right. You could figure out an equation, but what did you just say it wasn’t?
Jerry	a line
Mrs. C	Okay, so you've ruled that out anyway. Right?
Bill	It's challenging stuff
Mrs. C	Yes, I know. [Mrs. C moves to another group of students]

In asking for descriptions and explanations, the teacher created a situation wherein the student himself could refine his thinking and shift to a new way of thinking about the problem. In this instance, Jerry's initial investigation of slope led him to conclude that since the slope changes, the equation he is seeking cannot be that of a straight line. Jerry shifts his thinking to examining the possibility of a parabola. Mrs. C encourages him to pursue this new line of thinking! This gives the responsibility to the student to continue to seek explanations and justifications.

(5) Sharing and comparing solutions. After extended time on this task, the students had arrived at two different, but equivalent, solutions to the problem of finding an equation. One student wrote y=2^x/2 and the other student wrote y=2^(x-1). The teacher made these two solutions public on the board in the front of the classroom and asked two students to describe in considerable detail how they thought about the data in order to

arrive at their solutions. She intended to bring the students' description of their own reasoning as a part of the shared thinking of the class. She engaged the whole class in an extended discussion of why these solutions were the same and how to justify that claim.

(6) Anticipating development. Throughout the lesson, the teacher saw many ways that the students' ideas about exponential growth could develop. In her discussion with one student about the changing slope, the teacher encouraged him to explore this line of reasoning. Although it was not an immediate solution to the problem, the student was able to revise his own way of thinking himself, rather than simply being told that the function wasn't linear. The teacher recognized that the changing slope of an exponential function was an important characteristic of that function that would continue to be developed throughout the unit. This suggests that the teacher was anticipating the overall development of student ideas across the unit and within the lesson.

DISCUSSION AND CONCLUSION

This analysis reveals two ways that the teacher thought about students' ways of thinking about exponential function. The first way is about expectations: this teacher expected the students to think about the mathematics of the task. She clearly saw and explicitly communicated an expectation that the students should think hard about the task and that she expected them to spend time and to keep thinking even when they encountered difficulties. The second way of thinking about students' thinking is through having a flexible schema for how student's might approach the task. In the case of Mrs. C, she had a clearly developed set of schema that students might use that included using linear functions, investigating the rate of change using slope and finding quadratic functions. As the lesson progressed, she added two new ways of thinking about the problem through listening to unexpected student approaches: examining the behavior of the data at the origin and exploring the patterns of perfect squares.

The teacher's interpretations of students' thinking influenced her actions in the classrooms in two ways. First, having a well-developed schema for how students might approach the task enabled the teacher to press the students to explore and express their own ideas about the task. Even when students offered unfamiliar solutions (such as the pattern of perfect squares), she continued to ask the students for their descriptions and explanations. In asking for descriptions and explanations, the teacher created a situation where the student himself could refine his thinking and shift to a new way of thinking about the problem. Second, Mrs. C gave the students an extended amount of time to investigate their ideas about the function that could be used to describe the pennies data. She asked them to explain their ideas to her and, later in the lesson, to the whole class. She encouraged them to test their ideas by comparing the graph of their equations to the graph of the data. The teachers’ understanding led her to ask for the comparison of solutions as a way to make public and shareable the students’ ways of thinking about the task.

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FROM ORAL TO WRITTEN TEXTS IN GRADE I AND THE APPROACH TO MATHEMATICAL ARGUMENTATION

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The aim of this paper is to elaborate on (and provide some experimental evidence for) the following hypothesis: during the first grade students' approach to writing texts, an appropriate educational setting (based on both social interaction managed by the teacher and students' involvement in well chosen concrete experiences) can give them the opportunity of developing important skills related to mathematical argumentation.

INTRODUCTION

In the last decade the early development of students' argumentative skills progressively became a subject of major concern for mathematics educators for different reasons: the need for an early approach to skills that are relevant in the proving process (under the pressure of curricular changes that brought to a re-evaluation of mathematical proof in school: see NCTM Standards, 2000); the exploration of the potential of social interaction in developing mathematical knowledge and skills (see Krummheuer, 1995; Yackel, 1998); the importance of argumentative skills in curricula aimed at enhancing students' intellectual autonomy (see Maher, 2002). On the other side, what we know about argumentative skills (starting from Piaget, 1923; 1947) implies that they cannot be developed within the narrow borders of one discipline (in particular, mathematics): students' argumentative potential needs to be nurtured across different activities, demanding a large amount of time. The contributions by E. Yackel, C. Maher and others mostly concern classroom argumentative activities about *mathematical* subjects in grades 3 or 4 and onwards. How to prepare the ground for these argumentative activities in earlier stages? This paper aims at elaborating (and supporting through experimental evidence) a working hypothesis concerning the early development of argumentative skills that are relevant for mathematical argumentation through a suitable management of the appropriation of written language by first grade students. Some educational implications for our hypothesis will be outlined in the Conclusion.

THEORETICAL FRAMEWORK

Argumentation and mathematical argumentation

I shall use the word "argumentation" both for the process that produces a logically connected (but not necessarily deductive) discourse about a subject, and its product (cf Boero, Douek & Ferrari, 2002, p. 250). *Mathematical argumentation* can be characterised as that peculiar kind of argumentation, which deals with mathematical objects and skills (including general logical skills that are relevant in the mathematical discourse, like the management of the hypothetical reasoning). Hereafter, I consider some general attributes of argumentation (cf. Plantin, 1990) that are specially relevant in the case of *mathematical* argumentation.

i)- Production of a proposition that will be under discussion, in particular an interpretation, a guess, a plan, etc.; it may be produced to initiate an argumentation, or appear later as a partial result of the argumentation.

ii)- Production of reasons ("arguments") to validate the proposition - or question it. The reasons are taken from a reference corpus (in our experimental situations, the shared knowledge of the class - see Boero, Douek & Ferrari, 2002, p. 250, 256) they can be expressed using a number of representations (verbal statements, experimental evidences, drawings…). They can concern the use of peculiar tools (in the case of the discussion of a plan), the production of counter-examples (in the case of the discussion of a conjecture), etc.

iii)- Arguments and the proposition under scrutiny are held together by reasoning aimed at justifying, raising doubts, contradicting, refuting, interpreting, drawing new conclusions.

iv)- There is a global structure that needs to be maintained for the argumentation to be followed and understood. Verbal organisation is the visible aspect of such structure.

v)- The cognitive activity of a subject elaborating an argumentation is both conscious and voluntary; it presupposes the internalisation of an "other" who is in a position to control or regulate the logic of the reasoning, the truth of the statements, and the treatment of the signs involved.

Oral and written texts

There are two main (and rather coherent) references, one coming from Vygotsky's seminal work about the child's transition from oral communication to written text, the other related to Duval's investigation on the specificity of the written text in comparison with the oral text. Shortly, in Vygotsky's work (see Vygotsky, 1985) the transition from oral to written text is considered as a prototype for the transition from common knowledge to scientific knowledge in terms of consciousness, intentionality and systematic organisation. This is related to the fact that a written text must address a distant "other" in quite different conditions from oral communication, where the partners can understand each others with hints, gestures and various means of non verbal communication. Duval in his contribution (Duval, 1999) points out some characteristics of cognitive processes underlying writing (compared to oral communication): writing needs a greater conscious control than speaking; writing needs a reorganisation of the oral text; writing allows to escape the constraints of the oral text (as concerns short term memory, evidences, etc.)

Both Vygotsky's and Duval's contributions support and legitimate the hypothesis that the transition from oral to written texts could provide an opportunity to develop some argumentative skills relevant to mathematical argumentation (cf iv, v).

<u>Social interaction in the classroom</u>

In this paper I will consider social interaction as it works in communication situations designed with the purpose of developing linguistic representation of knowledge. This aspect has been widely considered in current literature over the last two decades (cf. Steinbring et al., 1998 for a representative set of orientations in the field of mathematics education). I will consider communication as a condition for cultural development for the individual (see Episode 2) and for the group of which one is a part (see Episode 3). In a Vygotskian perspective, communication reflects and influences the development of thought (see Vygotsky's comments about Piaget's internal language: Vygotsky, 1985, Ch. 2). In particular, argumentation in communication situations can enhance the development of students' argumentative skills considered before (specially, see **v**).

Context

Wedege (1999) discusses the use of the word "context" in educational literature and proposes a distinction between "situation context" (e.g. workplace, classroom social context, computer learning environments, etc.) and "task context" (e.g. everyday life situations evoked in a problem-solving task). We can remark that some studies concerning the opportunities offered by "task contexts" are coherently conceived from a Vygotskian perspective of "social construction of knowledge", i.e. take also the "situation context" into account as a relevant issue (for an example, see Bartolini Bussi et al, 1999). This will be also the case of the study reported in this paper. Concerning the potential of the task context in terms of the development of argumentative skills related to logical reasoning (cf **ii** and **iii**), I will refer to Guala & Boero (1999) and Arzarello (2000). They have pointed out the potential of time and space constraints inherent in the task context for the development of logical skills

THE MAIN HYPOTHESIS

The preceding theoretical considerations legitimate the following hypothesis: an interactive management of students' approach to writing can offer students the opportunity of approaching argumentative skills, relevant for mathematical argumentation, provided that the following conditions are fulfilled:

- 1-1 teacher-student interactions and classroom discussions orchestrated by the teacher are aimed at transforming students' utterances into pieces of written texts through explicit prompts by the teacher (and/or more competent peers) motivating the changes to be made;

- suitable tasks are chosen, based on concrete operations related to familiar task contexts that ensure the possibility of an immediate feedback for students' mistakes and incomplete texts, and are rich in logical connections related to space and time constraints.

EXPERIMENTAL EVIDENCE

Source of data

The Genoa Group for research in mathematics education has developed an innovative methodology for the approach to writing in first grade classes, within their project for an integrated teaching of mathematics and other disciplines in primary school (see Boero, 1994; Boero et al., 1995). This Project is conceived in the perspective of "research for innovation" (see Arzarello & Bartolini Bussi, 1998). Teachers interact individually with students about their experiences (mainly everyday life experiences - like the use of machines, or easy productions of objects and food - which ensure a concrete feedback for what children say). The student's utterances are interpreted by the teacher, who gradually helps the student to improve them in order to get a text suitable for writing. At the end of this process, the student dictates the oral text to the teacher; finally, the student copies the text, written by the teacher, on his copybook. Frequently, some of the texts produced during the 1-1 teacher-student interactions become an object of discussion for the whole class, in order to further improve them. Sometimes the production of a text related to a common experience (mainly in the "technological" domain) is proposed as a collective task for the whole class, through a discussion orchestrated by the teacher (who tries to get interactively a written text starting from the students' utterances: see Episode 3). This gradual introduction to written texts with an essential role given to the teacher's mediation and its rooting in student's shared concrete experience is an educational setting inspired by Vygotsky's elaboration on the dialectics between "ordinary" knowledge and "scientific" knowledge (cf the theoretical framework; see Boero et al., 1995).

The following three episodes come from first grade classes that adopt the Genoa Group Project. The first episode fulfils only a general introductory function.

Some episodes

Episode 1: End of January, grade I. Interaction between the teacher (T) and Maria (a low achieving student who is approaching the production of written texts). Students have already learned to report orally on some easy, concrete procedures.

(T): Tell me, Maria, how you have produced the soap bubbles

(Maria): I put the soap solution in the glass, then I blew into the glass, and the bubbles came out

(T): But if you blow into the glass, no bubble comes out: do it!

(Maria): I have forgotten to say that I blow into the soap solution through a straw

(T): OK; now you can tell me with precision how you have produced the soap bubbles

(Maria): I put some soap solution in the glass, then I took the straw and I blew through it, and the bubbles came out

(T): (repeats Maria's phrases) OK; now you can dictate your text to me. Remember that you must speak slowly, in order to give me enough time to write.

(Maria dictates a text that is very near to her oral text, and the teacher writes it down)

(T): (slowly reads the whole text, then concentrates on a particular sentence): Maria, pay attention: "I took the straw and I blew through it": If I take the straw and I blow through it, (the teacher performs the action) no bubble comes out!

(Maria): I have not said that the straw was put into the glass...

(T): Into the glass?

(Maria): No, into the soap solution!

(T): Did you put the whole straw into the soap solution?

(Maria): No, only the end!

(T): OK, now you can dictate the right sentence (etc.)

Comments about this episode: Let us consider the whole dialogue as a discourse. The teacher kept the line of the discourse (cf **iv**): all the data (elements of the experience) that played a role had to be made explicit, and in the right order; logical reasons (cf **ii**, **iii**) were given by the teacher to justify his requests (some of them may be seen as verbal logical reasons, some as related to the "logic" of the events). No argumentation is produced by the student, although the activity is not only an introduction to the production of written texts, but also to some rules for producing texts and related skills that are relevant for mathematical argumentation.

Episode 2: End of March, grade I. Students have to plan how to assemble a toy windmill; they can see a toy windmill already assembled, and the pieces to be assembled to get another toy windmill. Stefania is an average-achieving student; she is already able to write texts; the 1-1 interaction with the teacher concerns a text written by her.

(Stefania wrote: *"I will put the nail into the wood stick, then I will put the propeller and the washers"*).

> T: Let us try to do it. I take the nail and I put its tip into the wood stick...
>
> (she mimics the action)
>
> (Stefania): It does not work... How can I put the propeller?
>
> (T): Explain why your text does not work.
>
> (Stefania): Because if I put the nail into the wood stick, then I cannot put the propeller, because the head of the nail does not allow the propeller to go over the nail. And also the washers cannot go over. I must put one washer, then the propeller, then the other washer... Then I can put the nail into the wood stick.
>
> (T): OK, write a new text, including the explanation of the reason why you need to postpone putting the nail into the wood stick.
>
> (Stefania's new text: "I will put one washer over the nail, then the propeller, then the other washer. Then I will put the nail into the wood stick. If I had put the nail into the wood stick at the beginning, it would have been impossible to put the washers and the propeller over the nail").

Comments about this episode: the episode shows how in a first grade class the teacher can provoke awareness about logical requirements concerning a space-time situation. The teacher suggests an argumentative style to the students, and the student internalises it when she is writing the second version of her text. Here the student produces an argumentation that involves relevant skills in a mathematics education perspective (cf **ii**, **iii**, **iv**). Reasoning (even at the metacognitive level) is consciously carried out by her, under the pressure of the teacher who leaves to her part of the responsibility for leading the argumentation: Stefania has to explain why her text does not work. The teacher helps her to find the reason why it does not work, as if she suggested a counter example through material facts. The student puts it in words and draws conclusions about the structure the text should take; she becomes more conscious of the logical role of the elements of the experience within the text, as in the case of the construction of an argumentation aimed at validating the solution of an applied mathematical problem.

Episode 3: End of March, grade I. A classroom discussion regarding how to prepare a report for an absent schoolmate about some activities that concerned temperatures.

> (T): during the last week Fatima was ill; now she has recovered, and in two or three days she will be back with us. Last week we have made a lot of work with the thermometer. It would be good to prepare a text to explain to her what we have done, and why. Remember: we must explain why we have decided to use the thermometer, and how. Her mother will help her

understand the text, but her mother was not here, so it is necessary to be very clear in preparing our text **(1)**. What should we write in this text?

(as usual in the construction of a synthesis about classroom work, the teacher goes to the big blackboard and writes down what students propose)

(Debora): Dear Fatima, we have learned to use the thermometer

(Ugo): But Fatima does not know why we have decided to use the thermometer

(Bianca): We could write: we have decided to use the thermometer to get a true... to be able to say if it is true...**(2)**

(Ugo) That today it is warmer than yesterday

(Daniele) Because there was somebody who told that it was warmer, and somebody who told that it was colder **(3)**

(Luca) And somebody told that it was warm, while I told that it was rather cold

(T): Now we can try to write the first sentence, about why we have decided to use the thermometer

(Debora): We have decided to use the thermometer in order to establish if it is really cold or warm, out of our impressions

(Ugo) And if today it is colder or warmer than yesterday

(the teacher writes the whole sentence, coming from Debora's and Ugo's contributions, on the blackboard; then she reads it very slowly)

(T): but... we must compare... we must only establish that today it is warmer or colder than yesterday... And next monday?

(Daniele) Today ... not only today, anyone day **(4)**

(Patrizia) Otherwise Fatima could imagine that we have made a lot of work only for one single day. **(5)**

(Matteo) It is.. it is for all days: Monday, Tuesday, ...

(T): How to modify our sentence? (she reads the sentence slowly)

(Ivan: he reads the text on the blackboard): We have decided to use the thermometer in order to establish if it is really cold or warm, out of our impressions, and if ...if one day is colder or warmer than yesterday...

(Daniele): Not, not yesterday... the day... the day that comes before...

(Ugo): like yesterday for before today, and the day before yesterday for yesterday

(Daniele): The preceding day **(6)**

(Ugo): Yes, the preceding day

(T): Daniele, please, say the whole sentence [...]

Comments about this episode: This fragment shows how the task of producing a classroom report (in a co-operative style, with an emotionally shared communication purpose) about the "why" and "how" of a shared activity can provide students with the opportunity of an intensive argumentative activity, during the elaboration of the text and, in particular, the production of a sentence that carries generality. The final text is rich in argumentative skills (specially related to **ii**) that are relevant for mathematics education, in particular in the case of mathematical argumentation dealing with mathematical modelling and applied mathematical problem solving. In particular we may observe (cf. **(2)** and **(3)**) how the meaning of a tool and its contextualisation come into play. This aspect is very important in a mathematical argumentation when arguments for the adequacy of a tool to an activity (or a plan, or a solution) must be produced. Also skills related to the management of the generality of mathematical statements are involved. At the points **(4)**, **(5)**, **(6)** we can observe a progressive focusing on a problem of generality, up to its condensed, appropriate expression. Concerning **v)**, we may observe at the point **(1)** how the didactical contract takes on the role of the idealised internalised "other" that controls the production of an argumentation.

CONCLUSION

Both theoretical reasons (in particular, Vygotsky's and Duval's contributions about writing in comparison with speaking) and experimental evidence seem to support the working hypothesis stated in this paper. Indeed an early, intensive approach to argumentative skills, relevant for mathematical argumentation, seems to be possible through an interactive management of students' approach to writing and classroom discussions about produced texts (provided that suitable tasks are chosen, based on concrete operations that ensure the possibility of an immediate feedback for students' flaws). The episodes show in particular how space and time constraints inherent in the task context intervene in the development of students' argumentative skills and their reasoning, either as sources of mistakes or incomplete statements to be detected and overcome (see the second and the fourth episode), or as opportunities to deal with generality and express it (see the third episode).

The hypothesis dealt with in this paper opens an interesting perspective for the intervention on students' argumentative skills: indeed students' access to writing texts is the crucial goal for teachers in first grade. A synergy between the achievement of this goal and the development of students' argumentative skills, relevant for mathematics education, is one possible outcome of the study reported in this paper.

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YOUNG CHILDREN'S UNDERSTANDING OF GEOMETRIC SHAPES: THE ROLE OF GEOMETRIC MODELS

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In this paper, we explore the role of polygonal shapes as geometrical models in teaching mathematics, so as to elicit and interpret children's geometric conceptions and understanding about shapes. Primary pupils were asked to draw a stairway of figures (triangles, squares and rectangles) each one bigger than the preceding one. Pupils use two different strategies to tackle this task: (a) conservation of shape by increasing both dimensions of the figure and (b) increasing mainly one dimension of the figure. Each strategy seems to reflect a different way of reasoning, possibly corresponding to a different level of development in geometric thinking. Implications of findings for teaching geometry are discussed.

INTRODUCTION

Previous research concerning children's geometric conceptions has provided useful foundations affecting the development of mathematics education. In the present study three dominant lines of inquiry, which have been based on the theories of van Hieles and Clements, as well as, the use of geometric models, are taken into account. According to the van Hiele theory, children initially conceive a shape as a whole and not as a sum of its parts (visual level) (Hannibal, 1999). At the descriptive level children are competent in recognizing and expressing in words the components and properties of familiar shapes. At the next level, the informal deduction level, properties are logically ordered; this means that they are deduced from one another (van Hiele, 1999). Van Hiele supports that students' progress through levels of thought in geometry is more dependent on instruction rather than on age or biological maturation, and that types of instructional experiences can foster or impede development (Clements et al., 1999; van Hiele, 1999).

Clements et al. (1999) found that a prerecognitive level exists before van Hiele level 1 ("visual level"). At this level, children perceive shapes, but are not able to identify or distinguish among several shapes (Clements & Sarama, 2000). They propose that children develop stronger imagistic prototypes and gradually gain verbal declarative knowledge. Consequently, the visual level is reconceptualised as syncretic (a synthesis of verbal declarative and imagistic knowledge) (Clements et al., 1999).

Gagatsis and Patronis (1990) approached children's geometric conceptions from a different perspective, associated with geometrical models and their use. They argue that a geometrical model of _ is a collection S of points, lines or other figures in n-dimensional Euclidean space, representing a system _ of objects or a situation or process, if the intrinsic geometric properties of the elements of S are all relevant in this representation, i.e. they correspond to properties of the system _. Therefore a "polygon", according to Gagatsis and Patronis (1990), is a convex polygon, i.e. the convex hull of a finite set of points in the plane. In a more general sense, a "polytope" is the convex hull of any finite set of points in n-dimensional Euclidean space E^n. As for the dimensions of polytopes: 0-

polytopes are points, 1-polytopes are line segments, 2-polytopes are (convex) polygons. Polytopes are, therefore, considered as geometrical models, since we are not interested in a particular polytope as a specific set of points in space, but rather in all polytopes similar to it. Similarity is an equivalence relation in the set of polytopes. A specific example that constitutes a geometrical model of polygonal shapes is the class of equilateral triangles, which is actually a similarity class whose members have the properties of an equivalence relation: reflexive, symmetric and transitive.

However, according to Robertson (1984), in the space of all similarity classes of n-polytopes, any shape of polytopes can be continuously deformed into any other with the exception of 0-polytope. This kind of intuitive thinking, which involves a continuous variation process, is called “dynamic intuition” (Castelnuovo, 1972). Furthermore, Castelnuovo (1972) pointed out that children do not easily observe figures when they are steady, but rather when they move or vary in a continuous manner.

In this context, an investigation concerning a specific family of geometrical models has been conducted in order to identify the specific ideas that young children develop about geometric figures and explore their dynamic intuition. More specifically, the aims of the study were: (a) to investigate the extent to which children would conserve the shape (constant path) of a polygon or to what extent they would implicitly use some model of variation of shape (continuous path), (b) to examine how the above processes would vary with children's age, and (c) to identify implications of findings for theoretical descriptions of children's geometric thinking, for advancing children’s understanding of shapes and thus for teaching geometry in the early childhood years.

METHODS

The sample of the study consisted of 99 children ranging in age from 4.1 to 7.2 decimal years. All subjects were asked to "draw a stairway of triangles, each one bigger than the preceding one" and to repeat the same procedure with squares and with rectangles. The results concerning children’s reactions to the above tasks were codified in three ways:

(a) "T" was used to represent “conservation of shape”, i.e. pupils attempt to increase both dimensions of the figure (complete series of figures).

(b) "O" stands for pupils attempt to increase mainly one special dimension (complete series of figures: possibly producing rectangles in the series of "squares" or a square in the series of rectangles or an isosceles triangle in the series of "equilateral triangles").

(c) The symbol "N" was used to show that pupils produced a defective series (i.e. very irregular figures or non-increasing at all in a regular way).

This paper is focused on the first two types of responses. Moreover, implicative analysis (Gras, Peter, Briand & Philippé, 1997) was used in order to identify the relations among the possible responses of students in the test tasks. Therefore, six different variables representing the extent to which students used a specific strategy when they attempted to solve the three different types of test tasks emerged. More specifically, the following symbols were used to represent the six variables involved in this study:

a) Symbols “Trt”, “Sqt” and “Ret” represent the production of a series of similar triangles, squares and rectangles, respectively, of continuously increasing dimensions

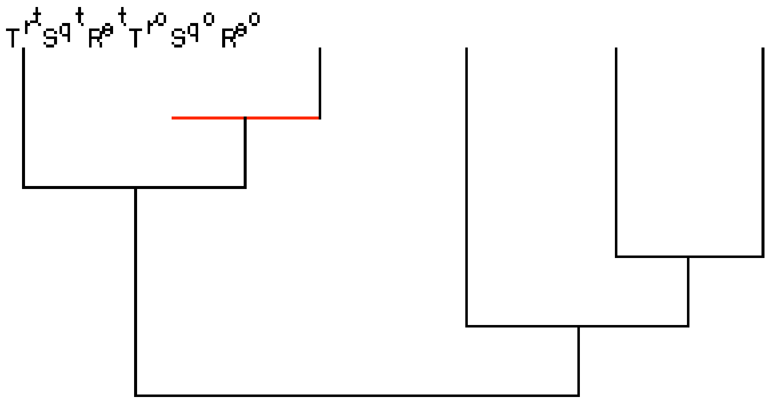
b) Symbols “Tro”, “Sqo” and “Reo” represent the production of a series of triangles, squares and rectangles, respectively, by increasing mainly one dimension of the figures.

For the analysis and processing of the data collected, implicative statistical analysis was conducted by using the computer software CHIC (Bodin, Coutourier, & Gras, 2000). A similarity diagram and a hierarchical tree were therefore produced. The notion of ‘supplementary variables’ was also employed in the particular analysis. Supplementary variables enable us to explain the reason for which particular clusters of variables have been created and indicate which objects are “responsible” for their formation. In our study, pupil’s age was set as a ‘supplementary variable’. Consequently, we were able to know which age group of children contributed the most to the formation of each cluster.

RESULTS

The similarity diagram is shown in Figure 1. We can observe that pupils’ responses to the tasks can be classified according to the strategy they applied. More specifically, two clusters (i.e., groups of variables) can be identified. The first group consists of the variables “Trt”, “Sqt” and “Ret” which represent the application of T-strategy (i.e., increasing both dimensions of the figures). The second group consists of the variables “Tro”, “Sqo” and “Reo” and refers to the O-strategy (i.e., increasing one dimension of the figures). The emergence of these two clusters are in line with the assumption of our study and reveal that children tend to approach the variations of all three kinds of figures in a similar way; that is, they apply the same strategy for all the paths of figures. The most contributing variable to the establishment of the group of variables representing the use of T-strategy is the group of pre-primary pupils, while the most contributing variable for the establishment of the class of responses concerning O-strategy is the group of primary pupils. In order to clarify further this finding, graphs 1 and 2 illustrate the percentages of pupils who use the two strategies according to their age. Graph 1 shows that the percentages of primary pupils who used T-strategy for the tasks involving triangles and rectangles are lower than the corresponding percentages of pre-primary pupils. On the other hand, the percentages of primary pupils who use the strategy of increasing one dimension of the figures are higher than the percentages of the pre-primary pupils. Chi-square test revealed that the six differences in the percentage of the two age groups of pupils who use each strategy are statistically significant ($p<.01$). This finding seems to be in line with the fact that the variable age was the most contributing variable for the creation of the two clusters emerged through the similarity diagram.

From the similarity diagram, we can also observe the existence of the same pattern of grouping in the test tasks students were asked to deal with. More specifically, students´ responses to tasks dealing with squares and rectangles are closely related. This close connection may indicate the children's intuitive (or taught) conception that rectangles and squares are “similar” geometric shapes and are therefore “completely different” from the shape of triangle. It is, however, important to acknowledge that there is only one statistically significant similarity at level 99% and this refers to the responses of pupils who use the T-strategy and “solve” the tasks concerning rectangles and squares.



Note: Similarities presented with bold lines are important at significant level 99%.

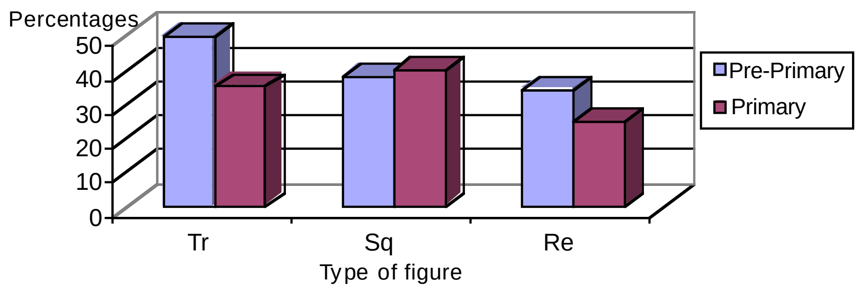


Figure 2. Percentages of pupils who used the T-strategy for solving the three types of tasks according to their age

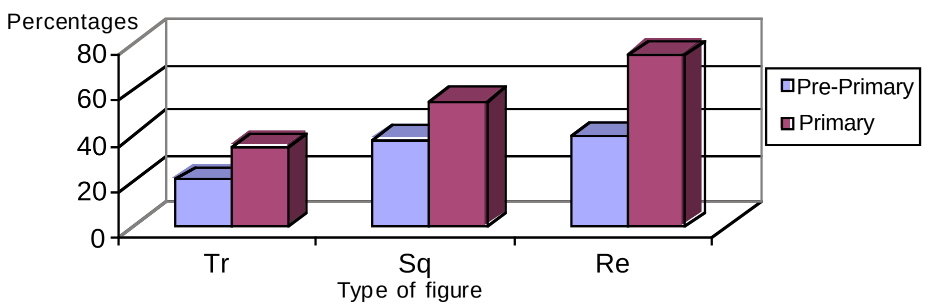
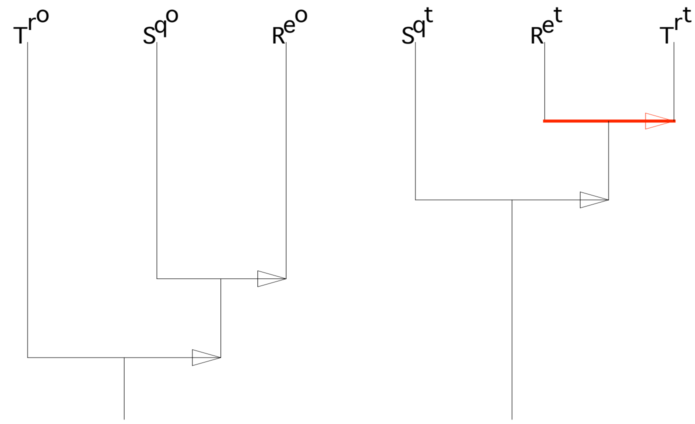


Figure 3. Percentages of pupils who use the O-strategy for solving the three types of tasks according to their age

The hierarchical tree, which shows significant implicative relations between variables of our study, is illustrated in Figure 4. The following observations arise from Figure 4. First, two groups of implicative relationships can be identified. The first group of implicative relations refers to variables concerning the O-strategy, while the second one provides support to the existence of a link among variables concerning the use of T-strategy. This finding is in line with the findings emerged from the similarity diagram. The formation of these groups of links indicates once again the consistency that characterizes children's reactions and strategies towards the tasks for different figures. Second, the implicative relationship (Sqo, Reo) indicates that the construction of a series of squares by increasing mainly one special dimension implies the application of the same strategy for rectangles. An explanation for the certain result is that, in the case of rectangles the dimension that children selected to increase was the longer one, whereas in the case of squares they were not sure which side to increase (since all of them had the same size) and this might caused them some further difficulty in dealing with this task. Eventually, however, they produced rectangles instead of squares.



Note: The implicative relationships in bold colour are significant at a level of 99%.

Figure 4: Hierarchical tree illustrating implicative relations among the six variables

DISCUSSION

The results of our study reveal that children tended to use a specific strategy in a consistent way in their attempt to solve the problems. This kind of behaviour was expected for two reasons: (a) each task was given to them as an open-ended one for all kinds of figures and (b) children were at different levels of thought in geometry. This observation leads us to the rejection of the traditional view of learning, which assumes that identification of a member of a class of shapes from other figures (Clements et al., 1999) or its construction can be apprehended by all pupils in the same instructional approach.

The application of O-strategy may indicate the function of mental operations (corresponding perhaps to the prerecognitive level), which allowed children to construct a path of "polytopes". Such kind of affined transformation may imply that there is a close relationship between the geometrical model of all similarity classes of polytopes and pupils drawings, which represent two alternative strategies. More specifically, a path of continuous deformation of any shape of polytopes into any other corresponds to O-strategy, whereas a constant path in the space of all similarity classes of polytopes, i.e. a path confined into one similarity class is closely associated with the use of T-strategy.

Children, who applied T-strategy, seem to have the characteristics of the "syncretic" level (Clements et al., 1999), since they conserved the components and the properties of the figure they transformed. Consequently, the application of T-strategy indicates that they didn't judge a figure merely by its appearance, but they were able to recognize and make use of all the components and properties of the figures giving equal importance to each one of them.

Children's behaviour toward the tasks was expected to correspond to their developmental level. For example, younger children were expected to increase mainly one dimension of a figure more frequently and to increase both dimensions of it less frequently than older children. However, the results of our study do not provide support to this assumption. Although further research is needed in order to identify the extent to which these findings can be generalized, it can be argued that our results reveal that good opportunities to learn are probably more important than the developmental level when it comes to children's learning about shapes (Van Hiele, 1999).

Some further implications for teaching geometry can be also drawn. Various studies seem to agree at one important point in relation to geometry instruction: teaching geometry needs to begin early, since young children's concepts remain constant after six years of age, without necessarily being accurate (Clements & Saramas, 2000). The consistent behaviour of older children towards the tasks of our study seems to provide further supports to the above argument.

Moreover, our findings concur with van Hiele's (1999) notion that in order to develop accurate conceptions teachers must provide teaching that is appropriate to the level of children's geometric thinking. To achieve that, teachers need to elicit, uncover and use the initial knowledge of shapes that children entering primary school already have and should build on children's ideas. Var de Sandt (2001), also, found that a learner-centred approach leads pupils to better outcomes and higher order thinking in geometry than a teacher-centred one.

The fact that pupils were found to deal in a similar way with rectangles and squares may be an indication of children considering rectangles and squares as closely related classes of shapes (Kyriakides, 1999). Thus, the particular finding support that, geometry instruction for young children, should emphasize shape properties and characteristics, as well as, the interconnective and hierarchical commonalities and differences among shapes {i.e., introducing the general case (e.g., rectangles) and then moving to the specific case (e.g., squares)}. Moreover, teachers should use accurate terminology, label shapes correctly and explain relative properties (Oberdorf & Taylor-Cox, 1999).

An example of a practical teaching intervention, which may contribute significantly to enhance children's conception, that a square is a special form of rectangle, makes use of children's "dynamic intuition". In fact it is based on the non-classical continuous path of varied polygonal shapes (O-strategy), rather than the classical (from mathematical and didactical point of view) constant path in the space of shapes (T-strategy) (Gagatsis & Patronis, 1990). This kind of learning can be achieved, if children following the former strategy are encouraged to realize that a square may occur among the rectangles in a very natural way, as an instance in a continuous variation of a rectangle.

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PERSPECTIVE-TAKING IN MIDDLE-SCHOOL MATHEMATICAL MODELLING: A TEACHER CASE STUDY

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Traditional word problems have not fulfilled the goal of mathematical sense-making for many students. Some studies have shown that authentic contexts, such as model-eliciting tasks, have the potential to engage students in making sense of realistic situations. However, there has been little research on the kinds of knowledge needed by teachers to support this type of student learning activity. In this paper, we report on the results of a case study that investigated the ways in which teachers respond to students' thinking while engaged in a model-eliciting task in data analysis. We describe how one teacher used perspective-taking to initially engage students with the task, to explain and justify their models, to assess the quality of their models, and to make connections to other mathematical ideas.

INTRODUCTION

Helping students understand the meaningfulness or significance of their mathematics learning is a major goal of education (Bransford et al., 2000). When students see the potential relevance of their mathematical experiences, they are more likely to engage in sense-making in their learning activities. The challenge for teachers lies in implementing authentic and mathematically rich learning experiences that are relevant and meaningful for students. Traditionally, one of the main ways in which teachers have attempted to bring meaning to students' mathematical learning is through word problems that comprise verbal descriptions of problem situations presented in a "real-world" context. The implicit idea behind the instructional goal of these problems is "to bring reality into the mathematics classroom, to create occasions for learning and practising the different aspects of applied problem solving, without the practical …. inconveniences of direct contact with the real world situation" (Verschaffel, 2002). At the elementary and middle school levels, solving such problems usually involves the application of one or more mathematical operations on the numbers contained in the problem (Verschaffel, Greer, & deCorte, 2000).

Numerous studies over the years, however, have indicated that traditional word problems have not fulfilled the goal of sense-making; that is, reality and school mathematics continue to remain as separate entities for many students. Students simply apply one or more operations without giving thought to possible constraints of the realities of the problem situation that may make such application inappropriate (e.g., Boaler, 2000; Lave, 1992; Schoenfeld, 1991; Verschaffel et al., 2000). Verschaffel (2002) reviewed several studies in which students were required to use judgement based on real-world knowledge and assumptions rather than the routine application of arithmetical operations (e.g., *John's best time to run 100 meters is 17 seconds. How long will it take him to run 1 kilometer*? p. 67). Not surprisingly, findings across several nations have revealed students' tendency to respond to such word problems in a stereotyped and non-realistic manner.

Related studies (e.g., De Franco & Curcio, 1997) have indicated that authentic problem contexts, where students participate directly in the problem situation, significantly improve students' inclination to apply their real-world knowledge to the solution process. One of the principles behind the model-eliciting problems that we used in the present study is the "Reality Principle" (Lesh, Hoover, Hole, Kelly, & Post, 2000). This principle, which governs the meaningfulness of the problems, emphasizes the importance of students making sense of the problem situation based on extensions of their personal knowledge and experiences. At the same time, the contexts of model-eliciting problems are designed to expand students' interests, rather than just catering to them.

A powerful, yet little researched, way in which teachers can support students' sense-making in a problem-solving situation is through perspective-taking. We define perspective-taking as understanding how the reality of a problem situation might be perceived from multiple points of view. This can be achieved by considering the problem situation from one's own perspective or from the perspective of others, such as the central characters within a problem context. We propose that when students are encouraged to adopt a particular perspective, they are being asked to create an imagistic system (Goldin, 1998). We take such systems to include not only the configurations of mathematical objects, but also how differing configurations might appear from different points of view. As such, imagistic systems play a significant role in interpreting, solving, and evaluating the solutions of modeling tasks.

We posit that students' development of significant mathematical models is dependent on the nature of the problem activities given and on the ways in which these problems are conceived and dealt with by the teacher. However, most research has focused on the nature of the problems, with considerably less research on the conceptions and strategies of the teacher. In this research, we have focused on the teachers' learning and reasoning as they implemented a sequence of modelling tasks. In particular, we were interested in understanding the ways in which teachers interpreted the tasks of teaching modelling problems in data analyses so that students engage in meaningful sense-making of the context and processes. In this paper we report on how a middle-school teacher, who participated in a teacher development program, initiated the use of perspective-taking as a way of promoting her students' mathematical modelling and sense-making activity.

DESCRIPTION OF THE STUDY

Participants

Seven middle-grade teachers and their classes participated in our study. They were from a co-educational private school situated within a middle class neighborhood in Brisbane, Australia. Neither the students nor their teachers had experienced modelling problems of the type implemented in this study. The teachers welcomed the opportunity to explore new ways to engage students in meaningful problem-solving activities. All the middle-grade mathematics teachers in the school, along with the head of the mathematics department and the school principal, were enthusiastic about participating in the project.

Program

The teacher development program comprised four teacher meetings, which were attended by all the teachers (except one who missed two meetings) and by the head of department. These meetings were intended to familiarize the teachers with the problem sequence by engaging them in a discussion of their own solutions to the problem as well as anticipated student solutions. The teachers' current practice included only limited use of group work, so pedagogical strategies for interacting with students in groups were discussed. The primary emphasis in terms of teaching strategies was to encourage and allow the students to develop, evaluate, revise and generalize their own solutions to the problems. The complete problem sequence was initiated over a period of 10 -12 lessons (depending on the teacher).

The student activities were a sequence of model development activities comprising five problem situations that require students to create usable rating systems in a range of contexts (cf. Doerr & English, in press). The core mathematical ideas were centered on notions of ranking, selecting and aggregating ranked quantities, and weighting ranks. For each problem, the students worked in small groups to analyze and transform entire data sets or meaningful portions thereof, for the purpose of decision making. The sequence of activities was designed so that the students could readily engage in meaningful ways with the problem situation and could create, use and modify quantities (e.g., ranks) in ways that would be meaningful to them and in ways that could be shared, generalized, and re-used in new situations. Our focus in this paper is on the first problem of the sequence, namely, the *Sneakers Problem.*

In the *Sneakers Problem*, the students encounter the notion of multiple factors that could be used in developing a rating system for purchasing sneakers and the notion that not all factors are equally important to all people. Students were asked "What factors are important to you in buying a pair of sneakers?" This generated a list of factors where not all factors were equally important to the students; the students then worked in small groups to determine how to order these factors in deciding which pair of sneakers to purchase. The students naturally produced different lists. The teacher then posed the problem that the sneaker manufacturer needed a single set of factors that represents the view of the whole class; in other words, the group rankings needed to be aggregated into a single class ranking. As we report below, the context of this problem, beginning with the point of view of the manufacturer, provided the teacher with the opportunity to use perspective-taking to support the sense-making efforts of her students.

Data Collection and Analysis

Of the seven teachers, we chose two for in-depth observation based on grade level, on prior observations of their lessons, on their mathematics background, and on their willingness to participate in this research. In this paper, we focus on the data collected from the first two lessons of one of these teachers, namely, a seventh-grade teacher whose primary teaching subject was biology not mathematics. This teacher frequently expressed to us her lack of confidence in teaching mathematics and how she felt that her mathematics teaching was more rote and less investigatory than her science teaching.

Each lesson was videotaped and audio-taped by the authors and detailed field notes were taken. The video-taping focused on the teacher and her interactions and exchanges with the students in her class The teacher meetings were audio-taped. The data analysis was completed in two stages. The first stage of analysis involved open-ended coding (Strauss & Corbin, 1998) of the transcripts of each lesson. Each author did this coding independently. This was followed by viewing the videotapes for each lesson, and adding annotations and clarifications to the transcript that were visible from the videotape. We then met to compare our coding and made revisions and refinements where needed.

The second stage of the analysis consisted of finding clusters of codes for each teacher that defined the critical features or characteristics for each lesson. These features describe the dominant events that governed the lessons, such as the ways in which the teacher encouraged student thinking, the ways in which she employed representations, the incidents in which she asked for meaning, explanations, and justifications. One feature that was prominent in the present teacher's interactions with the students' was the varied ways in which she encouraged perspective-taking. This feature was not emphasized in our teacher development program and was not evident in the other teacher's interactions with her students.

RESULTS: THE TEACHER'S USE OF PERSPECTIVE-TAKING

Our analysis of the data revealed a number of ways in which the teacher used perspective-taking to promote her students' mathematical modelling during the Sneakers Problem. These included using it as a means of introducing and focusing the problem for the students, as well as encouraging the students to adopt multiple perspectives for various purposes throughout the problem activity. These purposes included: to construct generalized models, to explain and justify models, to anticipate consequences, to assess, revise, and refine models, and to make connections.

Focusing the Problem

After the students had suggested a number of factors that they considered important in buying a pair of sneakers, the teacher used perspective-taking to draw the students' attention to the next component of the problem:

> A particular target group in the community are young teenagers like yourselves…Shoe companies actually want *your* input into the sorts of things you think are most important when you decide to buy a pair of shoes….. Now that doesn't necessarily mean whether Mum or Dad would agree but if you had the money and it was *your* decision, what things are most important for you to consider when you want to buy a pair of shoes?

As the teacher observed each group making their lists, she again reminded the students of the perspective they were to adopt ("Now is this a list based on what you believe or what your parents might believe?"). In this way, she was encouraging them to order the factors from their point of view, which was important to the shoe company.

Once the students had generated their various lists of ordered factors, the teacher again highlighted the perspective of the shoe manufacturer to direct the students' attention to the differences in the lists and thereby introduce the next component of the problem:

Teacher: Ok girls. You've just made a really important decision but I don't know how I'm going to go back to this manufacturing company with these lists because what do you notice about them?"

Students: They're different.

Teacher: They're all different. But you're all 12 year-old girls and surely you all think the same when it comes to fashion and shoes?

Students: No.

Teacher: Well, what's the dilemma here? Do we have a problem?

In this instance, the teacher takes on the perspective of the shoe manufacture to pose the essential difficulty in the task, namely, that the different lists of each group will need to be combined into one list for the manufacturer. This, then, is used to engage the students in the mathematics of combining ranked lists of data.

Encouraging Explanation, Justification, and Generalized Models

The teacher displayed a strong focus on the students' construction of generalized models and on their ability to explain and justify their models. In doing so, she frequently posed different perspectives for the students to consider, including that of a marketing researcher, a shoe manufacturer, and a mathematician. For example, one group gave a subjective explanation of how they arrived at their model:

> Student: Most people thought that size was more important than comfort… you should get a size that suits you and then see if it's comfortable.

The teacher prompted the students to consider the perspective of the manufacturer:

> Teacher: OK. But if a shoe company had that information to work with, would they use that same logic to decide on a final list?

In this way, the teacher encouraged the students to re-consider the logic of their justification that was based on their opinions rather than the information in the lists. In response to a student's suggestion that more lists were needed a "bigger data" set, the teacher continued to challenge the students' justification by posing the perspective of a mathematician engaged in sense-making. One student, however, could not see the purpose of adopting such a perspective:

> Teacher: Ok, how do you think the mathematicians who are trying to make some sort of sense out of those lists, how do you think they would go about trying to figure out which came fist, which came second? Do you think they would use this method?
>
> Student 1: Probably not.
>
> Student 2: Anyway, we're not looking at what mathematicians would do.
>
> Teacher: Well, ultimately, we're after a list that is best for the shoe company to market the shoe, so they need to know whether they've got to focus on fashion, or whether they've got to consider price as the most important.

In other interactions with the students, the teacher emphasized the importance of a mathematical justification. For example, after observing one group of students use a subjective approach to working the problem, the teacher asked, "Are you sure that this isn't just what you think the order should be? Are you sure that this has been justified mathematically?" In questioning another group, the teacher asked, "Are you confident you're going to be able to get up there [before the class] and explain this [their model] as marketing researchers with degrees in mathematics?"

Assessing Models

After the groups had generated their models, the students presented their work to the class. Once the students had described and displayed their solutions, the teacher asked them to compare the final lists of factors they had produced. In doing so, the teacher adopted the perspective of a market researcher as she posed the question: “Which lists do you think the market research people would go with?” The students chose a list that had been created by three of the groups, namely, 1. price (most important), 2. size, 3. fashion, 4. style, 5. comfort, 6. quality, 7. color, 8. purpose, 9. brand (least important). One student explained why this list is preferred:

> Because they actually make sense, like, compared to the others. Some people might not be able to understand the bottom ones [of the displayed lists] whereas they can understand the ones at the top.

In response, the teacher asked the students what they would have anticipated as a solution, given their original lists of factors:

> So you think the market research people would go with that list because three groups came up with that list? Now what would we have expected, considering that our original six lists were very different from each other? What would we have expected to get from working on the ultimate list—this new list—what should we have seen?

By asking the students the above, the teacher was helping them appreciate the importance of developing a generalized model. In the class discussion the students were able to explain that each of the three lists was generated by the same model, namely, ranking, summing the ranks, and then re-ranking (some students also averaged the ranks).

Making Connections

On several occasions, the teacher focused on connections between the students’ existing mathematical knowledge and their new modelling experiences, as well as on connections between the modelling experiences and the real world. One example of this occurred on conclusion of the modelling program. The teacher asked the students to reflect on their modelling activities and to indicate whether they had applied some of their prior mathematics learning to these new experiences. This led to a discussion about their recent work on data and chance and the use of surveys. The teacher then posed the perspective of a market researcher: “How do you think market researchers use mathematics to work out what people like in a community?” Since one of the students was to take part in a weekend survey conducted by market researchers, a unique opportunity arose for further use of perspective-taking. A class discussion followed in which the students considered the mathematical nature of survey work from the perspective of marketing companies.

DISCUSSION

In these lessons, we observed how the teacher initiated the use of perspective-taking as a means of promoting her students’ mathematical modelling. The context of the modeling problem facilitated multiple perspective-taking, which encouraged students in their realistic sense-making efforts. The teacher’s use of perspective-taking served a number of purposes.

The teacher used the perspective of the shoe manufacturer and the students' own perspective on desirable features in a pair of sneakers as a way of focusing the students' attention on the problem. In the initial part of the problem, the teacher directed the students to consider desirable features from their own perspective, rather than that of others. The teacher then switched the perspective-taking focus from the students to the shoe manufacturer. The purpose here was to highlight the difficulty faced by the manufacturer when there are multiple lists of different features to consider. This naturally led to the next component of the problem, namely, to develop a model that would enable the lists to be aggregated.

In her efforts to encourage the students to construct generalized models, the teacher again used the manufacturer's perspective, along with that of market researchers and mathematicians. When students used a subjective approach to aggregating the lists, the teacher encouraged them to consider the perspective of the manufacturer. That is, she was prompting the students to reconsider the logic of their justification, given the needs of the manufacturing company. When students continued to justify their subjective methods, the teacher challenged their thinking by posing the perspective of a mathematician engaged in sense-making.

The perspective of a market researcher was used on several occasions, including to prompt the students to assess the models generated by the different groups, as well as to draw connections between the students' modeling experiences and real life. The ways in which market researchers use mathematics formed the basis of a concluding class discussion.

The teacher's decision to employ perspective-taking for multiple purposes reflected her awareness of the importance of students' sense-making as they worked the modeling problems. This decision also highlighted the knowledge and understanding the teacher had gained about implementing model-eliciting problems. For example, she stressed the need for students to construct generalized models, to explain and justify their models, and to assess, revise, and refine their models. Finally, the teacher's efforts in helping the students connect their new modeling experiences both to their prior learning and to the outside world were enhanced through perspective-taking.

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BRIDGING MATHEMATICAL KNOWLEDGE FROM DIFFERENT CULTURES: PROPOSALS FOR AN INTERCULTURAL AND INTERDISCIPLINARY CURRICULUM

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Teaching mathematics in a multicultural scholastic context is a current research theme in countries where the phenomenon of immigration is becoming important. There is a project being carried out in some southern European countries which aims to identify the needs of maths teachers in the lower secondary schools. The theoretical framework of the project is ethnomathematics and mathematics education in multicultural contexts. Bearing in mind the needs expressed by the maths teachers and the problems pointed out by them, three extensive didactic proposals have been prepared based on the activities of craftsmen in those countries with the highest percentage of emigration towards the South of Europe. In the final part of this article some of the initial indications following the introduction of one of these proposals into some Italian schools are briefly described.

THE SOCIAL AND SCHOLASTIC CONTEXT

The phenomenon of emigration towards countries in the south of Europe, such as Italy, Portugal and Spain, is a recent if somewhat significant one. The socio-cultural context is changing slowly but clearly. This change also affects the school context, especially compulsory schooling. In terms of percentage the figures are not yet at the levels of many northern European countries but, for instance, in Italy immigrant pupils represent 2.3% of the school population at present and are expected to reach 8% by 2011 with a higher percentage in primary and lower secondary schools (CARITAS, 2002).

While studying the phenomenon of immigration it is important to note the great diversity and variability which accompanies it. A diversity in the areas and countries of origin (in Italy more than 180 countries are represented), a diversity of distribution over the country, a variable in time and an inequality in the numbers present in a specific area of certain communities. All this makes it difficult to put into action a program both in the social and educational sphere which allows the transition from a state of emergency to the following of a project.

Nevertheless, when we talk about education and school and what has been done to give the teachers a basic knowledge which can be of use to them in facing the new professional reality of classes with pupils of different cultures, we can say that up to now the main aim has been to create a culture of cordiality which helped the immigrant pupil to settle down better in the class.

Little or nothing has been done to give the teachers (with the exception of L1 teachers) the tools to help them to use didactic methodology and to prepare curriculum proposals which take into account the new cultural context in the classroom. This lack of attention to the didactics in multicultural contexts is strongly felt today even in those subjects which were once considered to be culture-free and universal, like mathematics, especially by those teachers looking for educational strategies which respect the identity of each single pupil while being didactically effective for the whole class.

THE IDMAMIM PROJECT – ITS THEORETICAL FRAMEWORK

The European project IDMAMIM[1] - *Innovazione Didattica Matematica e sussidi tecnologici in contesti Multiculturali, con alunni Immigrati e Minoranze* is placed in the above socio-educational context. It is a three-year project in the final stages which is being carried out in Italy, Portugal and Spain.

The project, targeted for maths teachers in the lower secondary schools, is organised as follows:

- appropriation of information on the state-of the-art as far as maths teaching is concerned in multicultural contexts in the project partner countries;
- analysis of the opinions and experiences described by the teachers in their answers to a questionnaire (Favilli, César, M. and Oliveras, in press) and a semi-structured interview (César, Oliveras and Favilli, in press) as well as identification of their didactic needs;
- a course of seminars, aimed at teachers, on the principal indications which emerge from the international research on ethnomathematics (Powell and Frankenstein, 1997) and mathematics education in a cultural context (Bishop, 1988), which form the theoretical framework of the IDMAMIM project (Oliveras, Favilli and César, 2002);
- projecting, elaborating, experimenting, consolidating, evaluating and finalising (on paper and CD) formative material for teachers and didactic proposals for mathematics, especially destined for use in the multicultural context.

As just said, the theoretical framework of the IDMAMIM project is ethnomathematics and didactics situated in a cultural context. In their article Vithal and Skovmose (1997) underline how between the four strands which can be seen in research into ethnomathematics - the challenge to the traditional history of mathematics, the analysis of the mathematics of traditional culture groups and indigenous people, the exploration of the mathematics of different groups in everyday settings, the relationship between ethnomathematics and mathematics education - the last one is *the unifying strand as it pulls together the other strands:* even when it is *an under-researched area compared to the other strands* (p. 135). We agree with them and add that it is important that, in the programming of didactic activities in maths classes, the findings of such research would be taken into consideration not only in traditionally multicultural countries or in those countries where the culture of the colonising country has taken over and replaced local culture, but also in countries where the socio-educational context is undergoing a rapid and continuous transformation as described above for our countries. It is not at all easy to modify the maths teacher's approach to the curriculum in the different western school systems, but we believe that the presence of pupils from different cultures in the same class makes it indispensable for maths teachers to be made aware of the fact that

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[1] Partially supported by the European Commission under the Socrates Programme.

mathematics, as a product of human intellect, is by its own nature under the influence of the cultural context in which it is produced both in its creation and development (Bishop, 1998). Instead of talking about mathematics we should be talking about mathematical activities which can grouped under the same categories but are not the same on a practical level; the existence of these mathematical activities, which have been developed in radically different ways, must be known to the teacher who must also be able of bearing in mind the different contributions that other cultures have given and continue to give to mathematical knowledge when carrying out class activities (Zaslavsky, 1991)

Members of these *minority* cultures (macro- or micro-cultures as they may be) are not always aware of this contribution and of their mathematical knowledge, which are often used implicitly; intercultural mathematical education (Oliveras, Favilli and César, 2002) can give teachers the opportunity to help their pupils, even those belonging to this same culture, discovering these contributions and knowledge. In this way pupils coming from minority cultures see their cultural heritage enhanced, while the others can be made to realise that knowledge, even mathematical knowledge, does not stem from one culture alone (Joseph, 1992).

THE MICRO-PROJECTS

It is in this light that the projects for the didactic proposals, which are being tested with the help of some teachers in lower secondary schools, in the IDMAMIM project were chosen. These teachers meet periodically with the project national steering groups.

The proposals were made taking as an example the activities in class developed around micro-projects, as shown in (Oliveras, 1996). Ideas were taken from some craft activities which were significant and representative of cultures from geographical areas with the highest number of emigrants towards southern Europe. The entire process of making a zampoña (pan pipes) in the Andes, African craftsmen from the sub-Sahara area making Batik cloth and north African rug makers were video-taped. After studying these videos, the various stages of the craftsmanship were identified for each activity and for each stage the mathematical knowledge and notions used by the craftsmen were identified both implicitly and explicitly. The mathematical knowledge and notions were handed down by older more expert craftsmen or were gained from their own direct experience: in fact, these notions were rarely gained in a scholastic context as these craftsmen did not have the possibility of attending school, at least no further than the first classes. In addition to these mathematical notions that the craftsmen use, we tried to identify others that we could generically term the mathematics that a researcher can *see* and/or associate with the various phases of the activity.

As it is only natural, the sum of knowledge and mathematical notions identified in the development of the working process of the craftsmen are difficult to place in sequence in a standard school curriculum for one class; in some cases some notions are better introduced at different times or even in different classes with respect to other notions used immediately before or after by the craftsmen. For this reason the didactic proposals programmed and elaborated were presented to the teachers in an open-ended fashion, not as a set pack with indications on how and when they should be followed.

The teacher-researchers, who are taking part in the empirical work, were advised to follow the viewing of the video showing the craftsman at work with a practical session in class where the students actually make the object either individually or in small groups. This activity which appears to be of an entirely technical nature and only mimics the activity of others, actually obliges the pupils to think, often in an unexpected manner, along mathematical lines.

These reflections, both collective and individual – but always shared with the other pupils (César, 1998) – are brought about by the necessity to face and solve small practical problems or by abstract considerations that the pupils make during the elaboration.

This first stage, dedicated to the construction of the object and present in each of the three didactic proposals is designed to be carried out by various teachers, not only the maths teacher, in a co-ordinated and collaborative fashion:

- the technical education teacher has to provide help in the construction activity, not only from a practical point of view, but also by providing information on the choice of materials and tools needed;
- the geography teacher can give the pupils a lot of information about the geo-physical aspects of the area in which these objects are made;
- the history teacher can provide news and comments on the principal events which have occurred in that area from the past to the present;
- the human science teacher can help to define the cultural context of the area by providing information on the roots and principle characteristics of the local language, its use and development, on the method used to transmit the culture, on the principal representatives of the culture (writers, poets, artists, scientists, etc.).

Depending on which project is being carried out it would also be possible to involve the music teacher and the science teacher (for the zampoñas) or the art teacher (for the batik material and the rug). We believe that this interdisciplinary didactical methodology should also be followed systematically in the teaching of mathematics, especially during compulsory schooling. The transmission of mathematical concepts should not be seen to be an end in itself but rather to stem from the necessity to solve problems in real life, following the essence itself of mathematics, because as Galileo Galilei said in his "Saggiatore": *chi non la conosce non può leggere il grandissimo libro dell'universo (*those who don't know it [mathematics] can not read the great book of the universe).

This type of didactic proposal means to instil in the pupil the knowledge that mathematics is a cultural product give them an example of a mathematical activity which is not scholastic and not linked to a western culture help them work in an intercultural manner interest and motivate the entire class enhance other cultures through an exchange of knowledge promote the academic performance in mathematics of the foreign students and those from a minority culture.

THE MICRO-PROJECT OF THE ZAMPOÑA (PAN PIPES)

The micro-project elaborated and tested in Italy is the one which makes use of the construction of the zampoña (pan pipes from the Andes), a wind instrument usually made of two series of seven and six pipes placed side by side. This instrument is part of the traditions found in the population of Ecuador, Peru, Bolivia and Chile. The micro-project is based on the construction of a zampoña done by a craftsman from Cuzco in Peru. The micro-project follows three steps:

1. introduction and construction (discovering the zampoña!)
2. qualitative analysis (getting to know it better!)
3. quantitative analysis (let's make one bigger or smaller!)

By watching the video and the direct construction of the zampoña we move straight to step 2. Numerous references to mathematical concepts are used even unknowingly by the craftsman or may be linked to various phases of his activity by the researcher.
Here are some of these concepts and mathematical activities:

Relation - function - sequence - ordering - classifying - measuring - average, mode and median - cylinder - circle - ratio - proportionality.

In the following part of this article we will present some comments which came out of the development in the classroom of the micro-project with reference to step 3 - *a quantitive analysis,* and in particular to ratio and proportionality.

The problems linked to the introduction by the teacher and the appropriation by the pupils of notions of ratio and proportionality have been the object of great study and research, due to the intrinsic difficulty of such concepts usually introduced to pupils aged 12 -13, during their seventh year of schooling. Following the studies carried out by Piaget and his collaborators on proportionality (Piaget, Grize, Szeminska and Bang, 1968), which led to believe that once pupils had understood linear functions they would be able to solve problems of proportionality whatever the problem situation, Vergnaud's studies (Vergnaud, (1983), however, suggest that in order to understand the concept of proportionality the nature of the problem situation plays an important role. Nunes, Carraher and Schliemann (1993) in the chapter on ‘Understanding proportions’ complain that:

> Little attention is given in math textbooks to connecting the mathematics with the problem situation, and the initial phases of teaching involve mostly formal demonstrations. The formal demonstrations are followed by exercises in application of the procedure. In the application it is assumed that the procedure just learned is appropriate; therefore students do not concentrate on a discussion of what connections there may be between mathematical models and empirical situations. (p. 86)

We can confirm that, at least as far as Italian textbooks are concerned, the situation is identical!

The micro-project on the zampoña goes exactly in the direction wished for by Nunes, Carraher and Schliemann (1993): to create the desire in class to solve a specific problem, to stimulate debate and an exchange of ideas, reflections, observations and proposals by the pupils and finally to create the need in the class for the introduction of new mathematical concepts which may be essential to solve the problem assigned to the pupils, as well as the opportunity to discuss at home about out-of-school mathematics (Hughes, 2001).

In order to build the zampoña the pupils have two tables of figures which refer to the measures, taken by the craftsman at the end of his construction, of the length and diameter of each of the two series of tubes cut by him and used to make the musical instrument. In actual fact the craftsman *measures* the length of the tubes using a plank of wood which is in proportion to the length and breadth of the zampoña he wishes to make and with markings which correspond to musical notes. This way of measuring, which relies on a sort of graduated scale, bases on the craftsman's experience and is in itself a point for reflection and comment in the class because of the implicit knowledge and mathematical activities put into play by the craftsman. It was only after a specific request from the researcher that the measurements were taken using a ruler on completion of the construction. This confrontation between mathematics used in and out of a school context and the various mathematical abilities used by both illiterate workers and students has already been noted by Nunes, Carraher and Schliemann (1993) and by Masingila, Davidenko and Prus-Wisniowska (1996).

	Ray	Fah#	Lah	Doh	Mi	Soh	Ti
Length	18.1	14.5	12	10.2	8.0	6.6	5.5
Diameter	1.2	1.1	1.1	1.0	0.9	0.8	0.7

Figures for the series with 7 tubes

	Mi	Soh	Ti	Ray	Fah#	Lah
Length	16.2	13.9	10.8	8.7	7.3	6.2
Diameter	1.2	1.15	1.0	0.9	0.8	0.75

Figures for the series with 6 tubes

These measurements are obviously only approximate, especially as far as the diameter is concerned: for this reason the teachers are asked to make their pupils reflect on the significance of measurement, error, average, etc. and to take little notice of the diameter of the tubes (considering the modest values) but rather to choose them in order of decreasing diameter as the tubes get shorter or even of similar diameters. The pupils are then asked to build a zampoña of a different size for example bigger. They are given a table - with incomplete numbers - showing the lengths of the six tubes which make up part of the instrument.

Notes	Mi	Soh	Ti	Ray	Fah#	Lah
Length	28		18.2	14.8		

As the students use this table they realise, with a little help from their teacher, that there is a constant relationship between the length of two tubes representing the same notes in both the instruments of different lengths; in fact

Mi: 28 / 16.2 = 1.73 - Ti: 18.2 / 10.8 = 1.69 - Ray: 14.8 / 8.7 = 1.70

The values obtained are slightly different and this is due both to inaccurate craftsmanship and to the difficulty in obtaining precise measurements; the calculations of the relationship between the values is an excellent way of introducing some simple statistical notions. In particular, as far as the figures recorded are concerned, it is possible to calculate the average (1.71) and the median (1.70).

In this way the pupils are given a simple tool which allows them to find a more accurate value in the relationship between the two musical instruments.

The presence of an almost constant relationship is also useful in calculating the length of the other tubes; following the example above, if 1.71 is taken as being the value of the ratio between the two zampoñas (the one built by the pupils and the one we have some figures for) it is possible to deduce the length of the tubes of the other notes; for example, Soh: 13.9 x 1.71 = 23.77.

Therefore the construction of the zampoña appears to be a concrete way of treating the concepts of ratio and proportion!

In particular, the teacher can focus attention on the fundamental properties of the ratio (multiplying both members by the same number does not alter the ratio). Further, finding the ratio between the two classes of quantities (the tubes of the two zampoñas) is the key to using the measurements of the first elements to find the length of the second and viceversa.

SOME FIRST INDICATIONS

The micro-project of the zampoña has already been used in some classes of lower secondary schools in Italy during the past school year (involving four teachers and around a hundred pupils, some of whom were immigrants) and is being more widely used during this school year.

The first indications referred back by the teachers in their reports and by the pupils themselves in their comments in class seem to be very positive.

The pupils seem to be highly motivated by the will to construct something either individually or in small groups: *it's the first time I've built anything!*

Interdisciplinary teaching is a surprising novelty for many pupils (and teachers…): *What has mathematics to do with the zampoña?… I think it's useful to bring together two such nice subjects, maths and music, in one task!*

Mathematics is seen under a new light and appears less boring: It was interesting because it is nice to do maths like this! *The lessons were useful and helped us find out more about the zampoña and they were also fun…*

Mathematics turns out to be less of an ordeal than it usually is: *The lessons were much nicer also because it was a more fun way to reason, to think up original solutions… The mathematical notions were very easy to understand…*

At the end of the second cycle of using of this didactic proposal a more detailed analysis of the impact it has had on the pupils will be done, in particular using a comparison with students of other classes where the concept of the ratio and proportionality will have been introduced in a traditional manner. Furthermore, we will try to assess not only the greater interest shown in learning mathematics but also the real didactic effectiveness of the proposal itself.

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METAPHORS AS VEHICLES OF KNOWLEDGE: AN EXPLORATORY ANALYSIS

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The paper considers a teaching experiment carried out with secondary school students (9th grade), who face modelling tasks to approach some basic concepts of algebra and early calculus. The focus is on an embodied analysis of students' cognitive processes. The analysis highlights the use of metaphors as a means of sharing knowledge.

L'article présente une ingénierie menée au niveau de l'enseignement secondaire où les élèves (14-15 ans) ont été confrontés à des activités de modélisation pour l'introduction de concepts basics de l'algèbre et de premiers éléments de l'analyse. L'objectif est l'analyse des processus cognitifs des élèves en termes de 'embodied cognition'. Celle-ci met l'accent sur l'utilisation des métaphores comme moyen pour partager de connaissances.

INTRODUCTION

This paper focuses on the genesis of metaphorical thinking to talk about mathematical objects within a long-term *teaching experiment*. The experiment is concerned with the problem of introducing algebra at the beginning of secondary school level. From the didactical point of view this is a crucial issue in teaching and learning algebra in order to overcome some cognitive difficulties related to the differences between algebra and analysis. Particularly, the experiment is based on the concept of function and its multiple representations, like graphs, and number tables. The students participated in activities based around modelling mathematical or physical situations (for example motion), using both paper and pencil and technological environments. The experiment shares the Vygotskian perspective of the social construction of knowledge through *discussion* as a means of communication (Bartolini Bussi, 1998). The *social interaction* arises from two subsequent periods of each activity. Firstly, students work in groups, sharing their ideas with their peers; secondly, a mathematical discussion led by the teacher involves the entire class. An embodied approach is used to analyse students' cognitive behaviours. In particular, attention is drawn to the use of *metaphors* in the evolution of mathematical understanding.

THEORETICAL FRAMEWORK

"Any function in the child's cultural development appears twice, or on two planes. First it appears on the social plane, and then on the psychological plane. First it appears between people as an interpsychological category, and then within the child as an intrapsychological category" (Vygotsky, 1981, p.163). Vygotsky (ibid.) states the importance of a social nature of human learning. An interaction between individuals takes part in every kind of discussion; hence, an interpersonal activity comes before an intrapersonal interiorisation. According to this perspective, more recent research studies the role of the *mathematical discussion* in the ways individuals construct meanings (see e.g. Bartolini Bussi, 1998). A mathematical discussion is defined as "a polyphony of articulated voices on a mathematical object (either concept or problem or procedure or belief) that is one of the motives of the teaching-learning activity" (Bartolini Bussi, ibid.). Within this frame, "communication plays a central role. But communication is not seen as a disinterested communication. The individuals communicate between themselves to carry out goal-oriented activities having culturally motivated goals" (Radford, 2001). The teaching experiment taken into account in this paper is an example of such a situation: the problems are planned to favour communication processes through social interaction. Another component of the framework is represented by the *embodied* nature of mathematical ideas, that comes from cognitive linguistics and the theory of *embodiment* (Lakoff & Johnson, 1980; Lakoff & Nùñez, 2000). Such researches argue that human thought processes are deeply metaphorical and metaphors "are not random but instead form coherent systems in terms of which we conceptualize our experience" (Lakoff & Johnson, ibid.). Lakoff & Nùñez (ibid.) work within the mathematics field, asserting that abstract notions are conceptualised in terms of concrete notions through precise inferential structures: the *conceptual metaphors*. Metaphors project the inferential structure of a source domain onto a target domain, which can be both mathematical or not. There are different kinds of metaphors: in particular, when the source domain is outside mathematics, we speak of *grounding metaphors*. Such an approach pursues further connections to other fields which study the behaviour and the structure of the brain; these studies emphasise the fact that neuro-biological constraints seem to affect mathematical knowledge (see e.g. Berthoz, 1997; Dehaene, 1997).

THE RESEARCH PROJECT

The teaching experiment is part of an ongoing research project I am involved in. The project is focused on a *meaningful* approach to some basic topics in calculus (functions, limits, derivatives, integrals) and involves secondary school students, from the 9th grade onwards. From the didactical point of view, meaningful refers to the hypothesis that the use of devices and manipulative materials within different *fields of experience* (Boero et al., 1995) can support pupils in the construction of knowledge (see e.g. Arzarello, 2000). On one hand, devices and materials allow students to touch, perceive and observe changes as consequence of their actions, to feel things about the activities they face. The body as a whole, senses, intuitions and gestures are involved in the activities. On the other hand, devices and materials are *artefacts* (Verillon & Rabardel, 1995). According to an instrumental approach, artefacts are particular objects with their intrinsic characteristics and can become *instruments* if considered with a well-adapted use (Mariotti, 2002). A cultural artefact is important because it "becomes efficacious and transparent, by its use in the context of specific types of social interactions and in relation to the transformations it undergoes in the hands of user" (Meira, 1995). In this perspective, the research considers the transition from perceptual-experimental level to conceptualisation, from a cognitive point of view, and studies the ways in which students construct meanings in a social context.

THE TEACHING EXPERIMENT

The teaching experiment discussed in this paper involved 25 first year students (14-15 years old) of an Italian scientifically oriented high school (Ferrara & Robutti, 2002) [1], during the school year 2000-01. It lasted 30 hours; in the classroom sessions, first the students worked in *small groups* (of two-three-four pupils) and then participated in a *classroom discussion* (featured by the institutionalisation phase).

Activities

Some paper and pencil activities preceded other activities in which technological tools (sensors and symbolic-graphic calculators) were used. The first kind of problems has the aim of gathering and interpreting measurement data. The second one, related to the modelling of motion, features the core of the experiment. The students (of each group) moved in front of a sensor (CBR) to reproduce various kinds of motion (uniform, accelerated, periodic…), using their body or toy objects, e.g. a bouncing ball. The CBR functions as a motion detector: each tenth of a second, its inner chronometer sends an impulse towards the moving object and registers its position. Pupils observe the building of the graph (representing the time law) in real time on the screen of a calculator connected to the sonar. Time and position data of motion are stored in the memory of the calculator. The students may organise this data in a number table and in a graph. Pupils are asked to describe the movement both in a qualitative and quantitative way. Firstly they use informal language to explain the movement and the graph; secondly they are asked to interpret the shape of the graph from a global point of view. The third step is represented by a local interpretation through the calculation of the graph's slope at different points. These are not traditional problems; their didactical effectiveness lies in the possibility of relating the data of a phenomenon with the function that describes it. Generally, all the experiments in which students can interact with a tool to create phenomena, help them to understand the mathematics connected with those phenomena (see e.g. Nemirovsky et al., 1998). After these activities (which focus on the construction of a model starting from motion), the third phase of the experiment is related to the inverse passage, from models to motions. A paper and pencil activity represents the final phase; it has the purpose of finding the model of two different mathematical tasks.

Methodology

During the experiment, four people were present in the classroom: the two teachers (one of mathematics and one of physics), a university student and a researcher, who had planned the activities together. A video-camera recorded group and classroom discussion. The analysis was conducted using the video-tapes and the written notes of students. Therefore, pupils' actions, gestures and language are the ingredients by which their cognitive learning processes are studied.

EXAMPLES OF GROUNDING METAPHORS IN THE PROTOCOLS

The analysis discusses excerpts of the following situations (two letters are used as reference in the excerpts of protocols. As appeared below, they respect time order) [2]:

- *paper and pencil activities*: interpreting a graph of temperature measurement during the day (TR);
- *activities with technology*: (a) modelling uniform motion of a student (UM); (b) modelling accelerated motion of a student (AM); (c) modelling periodic motion of a student (PM); (d) reproducing a given time law with a movement (RA).

Gestures and words show a richness of different grounding metaphors in students communication. I group examples from the problems considered above, through the qualities of each metaphor.

A first case: the *Temperature is a Moving Object* metaphor

From TR (working group: C, I, S)

Tr17 S: In which time interval of one hour has the maximum decrease in temperature measured? [she is reading the statement of the problem] (…)

Tr21 I: Temperature went down more.

Tr22 C: Hmm, did temperature go down more? (…)

Tr28 I: For example, if it has gone down more between ten and eleven o'clock. (…)

Tr31 I: This! (…)

Tr39 I: …Because here it goes up [she is pointing to the part of the graph in which temperature increases]…and here it goes down. (…)

Tr47 I: Then that is the maximum decrease we were looking for.

Tr48 C: Listen, let's look at the temperature!

Tr49 I: But, it is clear…because on this side [she is pointing to the two ends of the vertical change with her fingers] did it go down from...? [she is positioning the ruler on the graph, parallel to the horizontal axis, in order to check the temperature in the two points considered]

The three girls are observing a graph of measurements of the temperature of a room, at different times during the day. They are trying to find the time interval corresponding to the maximum decrease in temperature (*Tr*17). Most of their sentences highlight the richness of verbs, like "to go up" and "to go down", to indicate a change in the values of temperature data (*Tr*21-*Tr*28; *Tr*39; *Tr*49). "Temperature" is the subject of these sentences. The subject is implicit, when "it" is used, but deictic gestures indicate the girls refer to temperature (*Tr*39; *Tr*49). Actually, the students are thinking of temperature in terms of the phenomenon they are used to experiment. This phenomenon is represented by the mercury column of the thermometer, which moves up or down with respect to a previous level, when the temperature of the room changes. This change is an increase or a decrease; therefore, it acquires spatial features (referred to the mercury level), to be described. On one side, these features are static if referred to the spatial structure of the mercury column (for example, level, height). On the other side, they are dynamic, when temperature is conceived of as a moving object (with respect to the movement of the mercury, up or down). As a consequence, we may state the existence of a metaphor. A (non mathematical) source domain (temperature) is thought of in terms of a (mathematical) target domain (space). For this reason, we can define the metaphor Temperature is a Moving Object. This is the "cognitive dimension" of the metaphor. Moreover, the metaphor has also a "social dimension" in the dialogue. From *Tr*21 on, it becomes a means of communication between I and C. Soon, the language introduced by I is acquired and used by the group mate C, first in order to understand which the question is, then to interpret the graph (the sign).

The *Time is a Moving Object* metaphor

From UM (working group: F1, F2, G1, G2)

*Um*155 G1: Time is increasing, space is also increasing. (…)

*Um*177 G1: If you look at seven [he is considering the first seven time values]…time is always increasing, all right…space: 42, 43,50. [he is reading some digits of space values on the table] (…)

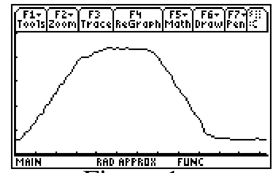

Figure 1.

*Um*296 F1: Time is always increasing, but not until a certain point.

*Um*297 G2: …Anyway, it is an observation…Space…

*Um*298 G1: But they increased together until a certain point!

*Um*299 G2: Then space started to decrease, while time continued to increase. (…)

*Um*439 F1: …I stopped and when I stopped…

*Um*440 G2: It goes on [she is pointing at the curve], doesn't it?

*Um*441 G1: Because if you consider a straight line [he is raising his pencil in a vertical position]...if time is increasing [he is raising his pen further]...if time is increasing and space is increasing too [again his pencil up]...we cannot get a curve [he is drawing a curve in the air]...for me it is a straight line. (...)

*Um*443 G1: Because they both are increasing. (...)

*Um*445 G1: If you always go on at the same velocity, they [time and space] both increase.

<u>From AM (working group: F1, F2, G1, G2)</u>

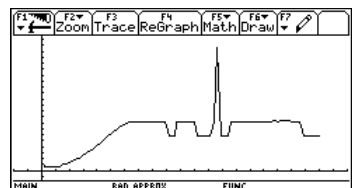

Figure 2.

*Am*102 G1: Time is increasing, space is increasing.

*Am*103 F1: Yeah!

*Am*104 G1: Both…

*Am*105 G2: They are increasing in a progressive way.

*Am*106 G1: Both space and time.

In UM and AM, the students are engaged with the CBR. They are asked to reproduce a back and forth uniform motion and an accelerated motion respectively. The excerpts show examples of a behaviour which characterises the entire experiment. In both situations, the students have to find the qualitative relationship between space and time. In the first case, pupils use both the number table and the graph (fig.1). The interesting point is their pervasive use of the verb "to increase" both for space and time data. Such a verb is properly used for a changing quantity. This is the nature of an object with spatial features. For example, its length or height may increase or decrease. Actually time passes or goes on, it does not increase. Instead, G1 early conceives of time as an increasing quantity (*Um*155, *Um*177). The same language is acquired by the other group mates (*Um*296-*Um*299). It features the discussion up to the end, even when the students have to interpret the shape of the graph (*Um*441-*Um*445). Time and space seem to behave similarly, as if they were the two domains of a metaphor. Time (as the target domain of the metaphor) is grounded on the structure of space (which becomes the source domain). The metaphor has both a social and a cognitive dimension. The students focus on the changes of a quantity; from their point of view, the "spatial features of time" (like length) are static, because they refer to the structure of the space. However, time can also have dynamic features, like in the sentences "We are approaching to Christmas" or "Christmas is approaching". In both cases, we think that a temporal period (Christmas) moves, as pointed out by the *Time is a Moving Object* metaphor (Lakoff & Johnson, 1980). The excerpt from AM shows the same cognitive behaviour in a different situation. In this situation the students only take into account the graph (fig.2). There is a second remark to do on UM; at *Um*440 the curve that “goes on” indicates the presence of a *fictive motion*: “a line is the motion of a traveller tracing that line” (Núñez et. al., 1999). G2 conceives the graph (a static sign) in a dynamic way. Hence, fictive motion allows a line to be thought in terms of motion. We can find examples in everyday or mathematical language: “the path goes across the wood”, “the graph riches a minimum”, and so on.

Towards a *blended space*?

<u>From PM (working group: F1, F2, G1, G2)</u>.

*Pm*77 F1: How many times did I cover this? [he is referring to back and forth run]

*Pm*78 F2: How does it [the graph] travel?...Look.

*Pm*79 F1: It is uniform more than one time.

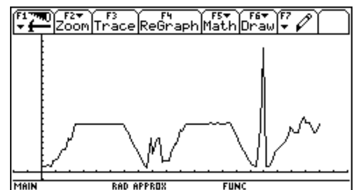

Figure 3.

<u>From RA (classroom discussion: T=teacher, A)</u>

*Ra*81 A: It [m] is not a constant.

*Ra*82 T: It is not a constant. According to you does it change positively or negatively?

*Ra*83 A: In theory...

*Ra*84 T: I mean, does it increase or decrease?

*Ra*85 A: In theory, it increases.

*Ra*86 T: Why in theory? (...)

*Ra*89 A: It has to slow down, I mean it has to, I mean it does not continue to accelerate [he is drawing a parabolic curve in the air], then it suddenly stops [he is drawing the horizontal stretch], it has to slow down a little [he is drawing like a hunch]... If we consider that rate, I mean that rate does not increase where it slows down.

PM refers to the activity in which the students are asked to reproduce a periodic motion (fig.3 shows the graph). Instead, in RA pupils have to realise a motion represented by a given model. In the first situation the students are explaining the shape of the graph; in the second one they are discussing with the teacher about the changes of the slope (m) in a particular time interval (from six to eight seconds). In both excerpts, it is suggestive to note the trend of pupils towards linking the graph with motion (*Pm*78, *Pm*79, *Ra*89). They think of the graph in terms of the experiment they lived, when they moved in front of the CBR. A metaphor can describe the situation: we may call it the *Graph is a Moving People* metaphor. The source domain is given by the motion experiment, while the graph becomes the target domain. However, there is something more in the information embedded in this case (also with respect to the previous situations). The students seem to identify the graph and the motion. This cognitive behaviour is marked by the use of verbs of motion (“to travel”, “to be uniform” “to slow down”, “to accelerate”) when talking about the graph (“it”; also A’s gestures refer to a curve). We can interpret this process as the constuction of a *blended space* (Lakoff & Núñez, 2000). In this space the features of the two domains (motion and graph) are merged in a unique new sign: the *motion-graph*.

CONCLUSIONS

The exploratory analysis points out that metaphors seem to support the students and show an evolution in their conceptualisation processes. Moreover, metaphors appear fruitful to construct a shared language within a group. Further research in the field is needed: recent trends are going to study how a technological artefact can affect the use of metaphors (see, e.g. Robutti, accepted) to build meanings for advanced mathematical concepts. An interesting open problem is the dialectic between metaphors and symbolic manipulation, in activities in which signs are deeply involved, as, for example in the most recent works by Tall (see e.g. Tall, 2002).

Notes

1. The school is a Liceo Scientifico, in which students attend five mathematics classes and three physics classes per week.
2. I do not present the entire problems because they are not so important for the analysis.

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EQUITY AND BELIEFS ABOUT THE EFFICACY OF COMPUTERS FOR MATHEMATICS LEARNING

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Members of society appear to have great faith in the educational value of computers. It is widely believed that computer use will promote learning. Unsupported by research evidence, many contemporary mathematics curriculum documents include statements advocating computer use and the benefits to be derived. As part of a larger study in which equity issues and perceptions of computer use for secondary mathematics learning are being explored, teachers' and students' beliefs about the benefits of computer use for mathematics learning were examined. The student data were also analysed across several equity dimensions. The findings are presented and discussed in this paper.

INTRODUCTION

In the developed world in particular, people seem to have great faith in the power of the computer to enhance students' learning. Selwyn, Gorard and Williams (2001) argue that "societal trust in the technological fix has been well established" (p.256). Contemporary mathematics curriculum documents include statements about incorporating technology (calculators and computers) into mathematics classrooms and of the related benefits to students. Reasons for the claims that computer use will enhance learning are put forward, but there does not appear to be strong research evidence in support of them.

One example of such claims is found in the technology principle, one of six underpinning principles of the USA's *Principles and Standards for School Mathematics* (National Council of Teachers of Mathematics [NCTM], 2000) which reads as follows:

> *Technology*. Technology is essential in teaching and learning mathematics; it influences the mathematics that is taught and enhances students' learning. (p.16)

A slightly less blatant perspective is put forward in the Victorian (Australia) *Curriculum and Standards framework [CSF] II* (Victorian Curriculum and Assessment Authority [VCAA], 2001a, website).

> The CSF encourages full use of the flexibility and value for teaching and learning programs provided by the increased application of information and communications technology (ICT).
>
> The CSF acknowledges that through the effective use and integration of ICT students are quickly developing new capabilities and that teachers have greater choice in creative teaching, assessment techniques and connections to students learning at home.

An *Information and Communications Technology [ICT] chart* (VCAA, 2001b, website) accompanies the *Mathematics CSFII – online*. The ICT chart reveals that students in grades 7–10 are expected to use a range of applications and computing skills for mathematics. The implied message is clear: computer use, will benefit students' learning.

To what extent, do secondary students' and teachers' beliefs match the rhetoric of the proclamations and expectations found in the curriculum documents? Do secondary teachers use computers for mathematics teaching? Do they believe that computer use improves their students' mathematics learning outcomes? To what extent do students believe that computers enhance their mathematical understandings? Are there differences in these beliefs among students categorized by a range of grouping factors?

As part of a larger study, data were gathered on teachers' and students' beliefs about the benefits of computer use for mathematics learning. The data sources included: large and small scale surveys of teachers and students, classroom observations, interviews with teachers and students, and students' post-lesson reflections. Due to space constraints, only findings from the survey data are presented in this paper.

PREVIOUS RESEARCH

There has been some research on beliefs and attitudes in relation to computer use in education with respect to equity factors: gender, socio-economic background, and race/ethnicity. Less relevant research has been undertaken in mathematics education.

Forgasz (2002a) summarised research findings on gender differences with respect to computers in general as follows:

> Compared to males, females are generally reported to be less positive about computers, like them less, perceive them as less useful, fear them more, feel more helpless around them, view themselves as having less aptitude with them, and show less interest in learning about and using computers; females are also less likely than males to stereotype computing as a male domain, to have received parental encouragement, to use computers out of school or to own one. (p.2-369)

Clarke (1990) reported gender differences favouring males in overall computer use, in course enrolments, and for programming and game playing. The gender differences were partially attributed to: expectations based on cultural beliefs about competence; associations of computing with mathematics, technology and maleness; and to the attitudes of parents and teachers.

Rather than serving as an educational panacea, Hanson (1997) maintained that computer use frequently exacerbated inequities for non-white students and for students from low socio-economic backgrounds. In a study focusing on computer use in grade 10 mathematics and science, Owens and Waxman (1998) found that females were less likely than males to report using computers for mathematics, and that African American students reported using computers more often than white and Hispanic students. Positive attitudes were postulated as the explanation for both findings. The latter finding, the researchers maintained, appeared to challenge previous claims that minority students had fewer opportunities to use computers than white students. In the UK context, Selwyn, Gorard and Williams (2001) question the assumption underpinning the UK government's claim that "providing access to technology for previous non-participants in learning will automatically lead to increased learning and decreased social exclusion" (p.262).

In a study of Australian mathematics teachers, Norton (1999) found that computers were considered equally or more effective than traditional instruction for doing calculations or providing basic skills practice. Few teachers considered computers useful in developing conceptual understandings; most argued the opposite with explanations for how computers might hinder understanding. One secondary mathematics teacher did not use computers for teaching mathematics because of beliefs about secondary level mathematics, teaching, personal teaching practices, and assessment issues (Norton & McRobbie, 2000).

THE STUDY

In this paper, findings are reported from data gathered in the first year of a three-year study [1]. The overall study aims include: (i) determining the effects of using computers on students' mathematics learning outcomes, (ii) identifying factors that may contribute to inequities in learning outcomes, and (iii) monitoring how computers are being used for mathematics learning in grades 7–10. Data on attitudes and beliefs about using computers for the learning of mathematics were gathered in the first year of the study.

Sample, instrument, and data gathering methods

Participants included grade 7–11 students and grade 7–10 mathematics teachers from 29 co-educational schools in Victoria (Australia). There were 17 metropolitan and 12 rural schools from across the three Australian educational sectors: government (19), Catholic (4), and Independent (6). Of the 29 schools, 8 were located in high, 16 in medium, and 5 in low socio-economic areas [2]. The sample sizes of the grade 7–10 and grade 11 students respectively were 2140 (F=1015, M=1112, ?=13) and 519 (F=237, M=281, ?=1). There were 96 (F=52, M=44) grade 7–10 mathematics teachers. Other characteristics of the student samples pertinent to the analyses undertaken are summarised in Table 1. None of these characteristics were relevant for the grade 7–10 mathematics teachers.

<table>
<tr><th></th><th colspan="12">Grade 7–10 students (N=2140)</th><th colspan="6">Grade 11 students (N=519)</th></tr>
<tr><td>English/Non-English speaking background [ESB/NESB]</td><td colspan="6">ESB
1643 (77%)</td><td colspan="6">NESB
491 (23%)</td><td colspan="3">ESB
365 (70.5%)</td><td colspan="3">NESB
153 (29.5%)</td></tr>
<tr><td>Aboriginal /non-aboriginal [ATSI/non-ATSI]</td><td colspan="6">ATSI
42 (2%)</td><td colspan="6">non-ATSI
2079 (98%)</td><td colspan="3">ATSI
7 (1.4%)</td><td colspan="3">non-ATSI
507 (98.6)</td></tr>
<tr><td>Student socio-economic status [high/medium/ low]</td><td colspan="4">High
500
(24.2%)</td><td colspan="4">Medium
1185
(57.4%)</td><td colspan="4">Low
381
(18.4%)</td><td colspan="2">High
88
(17.3%)</td><td colspan="2">Medium
328
(64.4%)</td><td colspan="2">Low
93
(18.3%)</td></tr>
<tr><td>Laptop/Desktop computers used in mathematics</td><td colspan="6">Laptop
197 (9.2%)</td><td colspan="6">Desktop
1943 (90.8%)</td><td colspan="6">NA</td></tr>
<tr><td>Grade level</td><td colspan="3">Gr 7
558
(26.1%)</td><td colspan="3">Gr 8
538
(25.1%)</td><td colspan="3">Gr 9
522
(24.4%)</td><td colspan="3">Gr 10
522
(24.4%)</td><td colspan="6">NA</td></tr>
</table>

Table 1. Student characteristics: Frequency and valid percentages

Three different survey questionnaires were administered in semester two of the 2001 school year: (i) to grade 7–10 students, (ii) to grade 11 students, and (iii) to grade 7–10 mathematics teachers. The following items were included in each version of the survey:

Q1 Do you believe that using computers for learning mathematics helps **people** to understand mathematics better? **Yes / No / Unsure** (Please circle)

Why do you think this?

Q2 Do you believe that using computers for learning mathematics helps **you [your students]** understand mathematics better? **Yes / No / Unsure** (Please circle)

Why do you say this?

Analyses, results and discussion

The three groups of respondents

For each question, the distributions of valid responses (frequency and percentage) for each category of response (Yes/No/Unsure) for the grade 7–10 students, the grade 11 students, and for the grade 7–10 mathematics teachers are shown in Table 2.

	Do you believe that using computers for learning mathematics helps people to understand mathematics better?			*Do you believe that using computers for learning mathematics helps you [your students] understand mathematics better?*		
	Yes	**No**	**Unsure**	**Yes**	**No**	**Unsure**
Gr 7-10 students (Q1: N=2072[a]) (Q2: N=2009)	598 28.9%	602 29.1%	872 42.1%	525 26.1%	816 40.6%	668 33.3%
Gr 11 students (Q1: N=513) (Q2: N=381)	159 31.0%	134 26.1%	220 42.9%	122 32.0%	161 42.3%	98 25.7%
Gr 7-10 teachers (Q1: N=93) (Q2: N=80)	59 63.4%	9 9.7%	25 26.9%	49 61.3%	7 8.8%	24 30.0%

[a]Frequency of total valid responses to each item

Table 2. Frequencies and valid percentages of responses to the two items for grade 7–10 students, grade 11 students, and grade 7–10 teachers

The data in Table 2 reveal that:

- for grade 7–10 and grade 11 students, the percentages of "Yes" responses are very similar across the two items; the percentages of "No" and "Unsure" responses are reversed across the two items.
- for grade 7–10 teachers, the percentages of "Yes", "No" and Unsure" responses is very similar across the two items.
- grade 7–10 mathematics teachers are more convinced than both groups of students that computer use enhances people's and students' understanding of mathematics.

An interesting difference in the patterns of responses is that both groups of students appear more convinced that their own understanding of mathematics was not promoted by using computers (approximately 40%) than their views about the effects of computer use on other people (under 30%).

Reasons for beliefs that computers help in understanding mathematics

Enjoyment and the speed at which computers display ‘solutions’ were common reasons provided by teachers who believed that computers helped their students’ understanding:

> Enjoyment, different to just doing paper calculations, produce accurate good looking graphs etc. – make predictions (gr.9 teacher, M)
>
> Particular software saves time and verifies their understanding. Computers allow them to carry out problems/exercises/questions quicker (gr.7 teacher, M)
>
> Students have different learning styles and many are very familiar with using a computer. This can then be used as a tool for learning mathematics (gr.9 teacher, F)

Most students did not provide explanations for their beliefs about computers helping their understanding. Among the grade 7–10 and grade 11 students who did (many more females than males) and who believed that computers had helped, common responses included: computers were fun; made the work more interesting; provided alternative perspectives; helped more than teachers; and/or had assisted in the understanding of particular mathematical concepts. Representative examples included:

> Because they are motivating + fun but you learn at the same time (gr.9, F)
>
> They put things in different ways (gr.7, M)
>
> Because it’s visual (gr.8, F)
>
> Because computers explain more than the teacher does. When the teacher says something the students might forget it (gr.8, M)
>
> Because in some programs (eg., Excel) it is a good way to see/understand patterns & algebra (gr.7, F)
>
> It helped me with graphs, circular functions and with triangles, tangent, cos, sin (gr.11, M).

Reasons for beliefs that computers do not help in understanding mathematics

Teachers who used computers in mathematics teaching but did not believe that they helped their students’ understanding of mathematics were somewhat cynical and negative about computers, and about students’ behaviour with them. Typical examples included:

> It is just an instrument to arouse enthusiasm (gr.9 teacher, M)
>
> The students still see a computer lesson as a ‘slack’ lesson – or a ‘fun’ lesson. Because they mostly need to read instructions, they rarely understand exactly what we are trying to get them to master (gr.8 teacher, F)
>
> These students learn just as well without computers (gr.10 teacher, F)

The reasons provided by students were more perceptive and realistic than those provided by the teachers; the students had used computers in mathematics classes. Their views reflected: limited use of computers or inadequate computer skills; lack of appreciation of how the computer (software) works; the computer made no difference; a preference for teachers to assist; and a preference to work problems out for themselves. Representative examples included:

> We didn’t really use them enough & I prefer to see how the equation is done (gr.11, M)
> ‘Cos I didn’t know how to make the computer do it properly (gr.11, F)

Because the computer does everything I don't need to think & therefore I don't learn (gr.10, F)
Because I still don't understand it, and it's just the same as doing it on the board or calculator (gr.10, F)
It is no difference to work with computers or maths books (gr.7, F)
Because I would prefer if someone told and explained it to me in person (gr.8, M)
I find it easier by hand (gr.9, M)

Differences in beliefs by various grouping factors

Chi-square tests were used to explore for differences in distributions to the categorical response data for each item by a range factors including several equity groupings.

For the grade 7–10 and grade 11 student data, the responses were analysed by:

- school factors: type (government/Catholic/Independent), school location (metropolitan/rural) and socio-economic location (School SES) [2]
- student factors: gender, socio-economic status (Student SES) [2], and two ethnicity factors: language background (non-English speaking/English speaking) and Aboriginality.

It should be noted that:

1. for grade 7–10 students only, the data were also analysed by: use of lap-top or desk-top computers in mathematics classrooms; and grade level, and
2. for grade 7–10 mathematics teachers, the data could only be analysed by: school factors (as above); and teacher gender.

The results of the chi-square analyses for the two items are summarised in Table 3.

		Gr 7-10 students		**Gr 11 students**		**Gr 7-10 teachers**	
Factor	**Grouping**	**Q1**	**Q2**	**Q1**	**Q2**	**Q1**	**Q2**
School factors	School type (Gov/Cath/Indep)	**	**	ns	ns	ns	ns
	School location (metro/rural)	ns	ns	ns	ns	ns	ns
	School SES (high/medium/low)	ns	*	ns	ns	ns	ns
Personal factors	Gender (male/female)	***	***	***	*	ns	ns
	Student SES (high/medium/low)	ns	**	ns	ns	-	-
	English/non-English speaking	ns	ns	ns	ns	-	-
	Aboriginality	ns	ns	ns	ns	-	-
Other factors	Grade level (7/8/9/10)	*	***	-	-	-	-
	Lap-top/desk-top computer	*	ns	-	-	-	-

* = p<.05 ** = p<.01 *** = p<.001 ns = not statistically significant

Table 3. Results of Chi-square tests for each of the two items by grouping factors: Levels of statistical significance

The data in Table 3 indicate that:

- there were no statistically significant differences in teachers' views about the efficacy of computers for helping people or their students understand mathematics by teacher gender or by any of the three school factors
- among the comparisons by common grouping factors, there were more statistically significant differences in the responses to both questions among grade 7–10 students than among grade 11 students

- gender appears to be the most significant equity factor in respect of students' differing beliefs about computers promoting mathematical understanding.

Among grade 7–10 students, differences in views on the efficacy of computers helping their own understanding of mathematics (Q2) were as follows:

- government school students were more convinced than Catholic and Independent school students that computer use helped their mathematical understandings (Yes: 28.3% cf. 23.6% & 21.2%); Independent school students were more convinced than the others that computers did not help (No: 45.2% cf. 41.0% & 38.0%)
- the higher the SES location of the school attended, the more convinced students were that computers did not assist their mathematical understandings (No: high – 45.6%; medium – 39.4%; low – 35.1%); a similar pattern was evident with respect to students' background SES (No: high – 47.1%; medium – 40.1%; low – 35.7%).
- males were more convinced than females that computers helped their understanding of mathematics (Yes: M – 31.3%; F – 20.3%); females were more convinced that they did not (No: F – 44.8%; M – 36.9%)
- as grade level increased, the students seemed less convinced that computer use helped their understanding of mathematics (Yes: grade 7 – 29.4%; grade 8 – 29.1%; grade 9 – 24.2%; and grade 10 – 21.5%)

Among grade 11 students, there was only one grouping factor on which statistically different views on the efficacy of computers helping their own understanding of mathematics. Males were more convinced than females that computer use had helped their understanding of mathematics (Yes: M – 34%; F – 29.8%); females were more convinced that they had not (No: F – 29.6%; M – 21.3%). It is interesting to note that across Australia at this grade level and beyond, more males than females are found to be enrolled in the most difficult mathematics courses.

CONCLUDING COMMENTS

Computer use for mathematics learning in Victorian (Australian) secondary schools is fairly widespread – about 70% of the participating teachers reported using them at some time (Forgasz, 2002b). Whether or not computers had been used in classrooms, the teachers appeared to believe more strongly than their students (approximately 60% cf. 30%) that mathematical understandings are enhanced by doing so. Many of the teachers' reasons seemed to be contradicted by the students' reasons for disagreeing. When students' beliefs were examined by various grouping factors, statistically significant differences were found in response distributions to the question on the effect of computer use on their mathematical understanding by school type, school location, grade level, students' SES, and gender. Statistically significant gender differences were also found for the beliefs of grade 11 students. These findings raise issues that invite further exploration: why do teachers' and students' beliefs differ so widely; are there specific mathematics content areas for which computer use does/does not enhance conceptual understandings; how can the effects of the grouping variables by which students' views differed be ameliorated? Answers to these questions will have important implications for the future implementation of computer applications for mathematics learning.

Endnotes

1. This study is funded through the Australian Research Council's Large Grant scheme. My thanks are also extended to Nike Prince for assisting in the data analyses.

2. The Australian Bureau of Statistics [ABS] provides an index of socio-economic categories – high, medium, and low – based on area postcodes (zip codes). Data on school location postcodes and students' home postcodes were gathered.

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IDENTIFYING A RESEARCH AGENDA: THE INTERACTION OF TECHNOLOGY WITH THE TEACHING AND LEARNING OF DATA ANALYSIS AND STATISTICS

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This paper provides an overview regarding the need to identify a research agenda by addressing the following questions: (1) What do we know about the content of data analysis and statistics to be developed at different levels, K–12? (2) In what ways can technology tools enhance current and new directions in teaching and learning data analysis and statistics? (3) What is the role of empirical research in clarifying the interactions between software development and use and the teaching and learning trajectories K–12 in data analysis and statistics? (4) What are the needs and directions that can help frame a research agenda?

Data analysis and statistics have emerged as major topics in primary and secondary (K–12) school mathematics curricula during the 1990's (NCTM, 1989; NCTM, 2000). Statistics has lacked definition at the K–12 levels. The lack of clarity about what content to address has resulted in initial work focusing on how we might take more traditional statistics and translate the content for use with younger students. Increased attention has been given by researchers and curriculum developers to setting better directions for what we want K-12 students to know and be able to do with respect to data analysis and statistics and to defining the nature of instruction needed to support these directions.

More recently, the interaction of technology with efforts to redefine both the content and instructional practices regarding data analysis and statistics in K–12 has provided new directions. Educational technology affords us a greater variety of strategies for *teaching* statistics and, at the same time, offers us new ways of *doing* statistics (Garfield & Burrill, 1997). The role of research must be addressed now, and the opportunity for defining and teaching a new content area with this kind of technological support must be grounded in research as this content is incorporated into school curricula.

What do we know about the content of data analysis and statistics to be developed at different levels, K-12?

Statistics is a vital, albeit relatively new, part of the K–12 curricula. Since the *Standards* (NCTM, 1989) have been in place, statistics and probability have become recognized topics in the K–12 curricula. Before 1989, most statistics and probability coursework and research occurred at the post-secondary levels, with an emphasis was on research about the understanding of probability concepts (Shaughnessy, 1992; Shaughnessy, Garfield, & Greer, 1996). The *PSSM* (NCTM, 2000) contains recommendations about specific expectations for each of four grade ranges (pre-K–2, 3–5, 6–8, 9–12). Schaeffer (2000) summarizes recommendations for content across K–12 curricula from NAEP and adds, as well, his own suggestions. In the elementary grades, there is an emphasis on a process of data analysis, making graphs, and using measures of center. At the middle grades level, the emphasis continues on these topics, with more sophisticated uses of graphs and introduction of association and sampling. At the high school level, content includes distinctions between univariate and bivariate data; regression coefficients, regression equations, and correlation coefficients; sample statistics versus population statistics; simulation; and integration of data analysis with content such as algebra.

Recent attention and emphasis has been given to characterizing and defining the big ideas that need to be considered (Figure 1). Data analysis may be characterized as an iterative, four-stage process that includes asking a question, collecting the data, analyzing the data, and forming and communicating conclusions. Within the context of data analysis, there are several big ideas related to statistics that must be considered, several of which are detailed in Figure 1.

In addition, technology is hinted at in recommendations, but assumptions about its appropriate use and availability to support teaching and learning have been limited by vision, versatility of software, and accessibility to hardware. Most school-appropriate technology tools used for data analysis and statistics fall into the category of spreadsheet software (e.g., Appleworks, Excel) or graphing tools (e.g., graphing calculator, Cricket Graph) that offer similar functions, including limited plotting, graphing, and analysis capabilities. They restrict the user to conventional displays, and emphasize numerical over categorical data.

In what ways can technology tools enhance current and new teaching and learning data analysis and statistics?

While technology has long been available to analyze statistics, the role of technology in teaching and learning statistics at the K–12 levels is still in its infancy. Serious integration of technology data tools in teaching and learning statistics provides a catalyst for an array of other changes, including changes in curriculum, classroom discourse, and students' ways of learning.

With the increased capabilities and availability of technology tools, it is important to consider their most appropriate use in facilitating students' learning of statistics in different situations. Ben-Zvi (2000) describes how technological tools are now being designed to support statistics learning:

1. Students actively construct knowledge, by 'doing' and 'seeing' statistics.
2. Students have opportunities to reflect on observed phenomena.
3. Students develop their knowledge about their own thought processes, self-regulation, and control.

Bakker (2002) has articulated the need to distinguish between software used to do data analysis and software used to learn data analysis. He points out that professional statistical software packages are not suitable for use by students when they are learning data analysis. How can a user choose among histograms, box plots, or circle graphs if they do not yet understand what these representations are and when each would be useful? For this reason special software needs to be and is being designed (e.g., Tinkerplots, Fathom, Tabletop, Minitools) that enhances learning.

Ben-Zvi (2000) sees computers as cognitive tools, that is, as tools that help transcend the limitations of the human mind. Cognition tends to be situated in context; cognitive development involves both the individual mind and the development of knowledge through socially structured activities. He makes the point that this concept leads to specific ways of using computers in education (pp. 139-143):

Technology is an amplifier of statistical power: In learning environments that are not based on the use of technological tools, graphs or tables are either presented to students or constructed by students according to prescriptive instructions. With the use of multi-representational technological tools, many of the standard data manipulations are automatic operations. Students produce a variety of different representations, ones that often reflect their emerging understandings of the data and the context in which the data are situated.

Technology as a reorganizer of physical and mental work: The appropriate use of technology has potential to bring about structural changes in the system of students' cognitive and sociocultural activities, rather than just to amplify human capabilities. Such powerful tools bring about reorganization of physical or mental work in a variety of ways.

What is the role of empirical research in clarifying the interactions between software development and use and the teaching and learning trajectories K-12 in data analysis and statistics?

Integrating Research: Software Development and Students' Statistical Thinking

Research on statistical thinking with students in grades K–12 has been sparse if nonexistent for quite sometime. However, in the more than 10 years since the release of the Standards (NCTM, 1989), research in statistical thinking has emerged as an exciting option and has begun to yield models of students' conceptions that are detailed enough to have practical, pedagogical implications. Further, powerful new software tools designed explicitly for statistics education could make statistical thinking accessible to students in K–12 in ways not before considered.

The design of technological tools and contexts for their use to support statistical reasoning and learning is easier said than done. To design computational environments well requires the "intertwining of many different threads of thought" (Resnick, 1995, p. 31). Resnick identifies three major threads (p. 31).

- *Understanding the domain knowledge*: What are the knowledge and skills that make up the domain of knowledge? How might we approach this domain knowledge differently? In what ways do technological tools recast areas of knowledge, thus providing new ways of thinking about the domain's concepts and allowing learners to explore previously inaccessible concepts?
- *Understanding the learner*: What is the learner's existing framework? How will the learner integrate new experiences into this framework? In what ways might learners construct new concepts and new meanings and how might the technological tool provide direction and scaffolding to support this process?
- *Understanding computational ideas and paradigms*: Technology is not only the medium the computational designer uses to craft artifacts. The computational environment, itself, involves a set of powerful new ideas for students. So, the command structures and the relationships among actions in such software tools as Tabletop, Fathom, or Tinkerplots highlight some of the more generalized understandings of how one might function as a data analyst.

Rubin (2001) suggests two research methodological perspectives and their integration and interaction can be used to frame this work.

> Software development as a research-based endeavor: As argued by Clements and Battista (2000), the state of the art in both models of thinking and software design make it possible for "research and software design to be a more intimately connected, mutually supporting process" (p. 762). The research/design cycle can create a synergy that enriches and accelerates progress in both fields.
>
> The role of conjecture-driven research: As described by Confrey and Lachance (2000), conjecture-driven research begins with a "means to reconceptualize the ways in which to approach both the content and pedagogy of a set of mathematical topics." Such research is most often carried out in the context of a teaching experiment, during which the conjecture is continually revisited and modified in response to students' questions, discoveries, and insights. Conjectures are generative, not restrictive; they lead to more sophisticated conjectures, not necessarily to the proof of a hypothesis.

Selected Findings Related to Software Development and Use

The possibility of students' forming intelligent partnerships with technology in studying statistics gives them the potential to work at a level that may be impossible without technology (Jones, 1997). Clearly, the argument for integrating software development and research of students' knowledge and understanding is persuasive. To a great extent, recent work on the use of technology has addressed this need with varying levels of specificity. Fifty articles (ICOTS-5, ICOTS-6 proceedings; Conference on Research on the Role of Technology in Teaching and Learning Statistics; full reference list to be provided at PME) focused on the use of technology in teaching and learning statistics and data analysis have been reviewed. In addition, for selected articles that appeared to have a focus on student learning, the main findings have been summarized. The major content areas addressed in the research contributed to framing the "big ideas" summarized in Figure 1 (those that occurred most often). A variety of software tools were reported to be the focus of research.

When these articles were reviewed, reported evidence indicated that computer and calculator tools allow students to:

- Rapidly graph and display data for easier analysis.
- Easily access displays, multiple linked representations, simulations, and animated and/or interactive demonstrations of statistical concepts.
- Easily access large amounts of organized data from official sources.
- Problem-solve and receive immediate feedback.
- Use larger and more complex data sets than feasible when work done by hand.
- Creatively develop ideas and learn about structuring data by experimenting with tools before being presented with conventional methods.
- Focus on concepts instead of doing complex calculations by hand.
- Solve problems without having to know complex calculations.
- Rapidly simulate data for modeling.
- Rapidly transmit and share data.
- Explore real-life applications of statistics.

This list provides a good summary of the ways technology contributes—on a surface level—to students' access to and learning of data analysis and statistics. It is also more of behaviors and actions than the development of statistical understandings. For example, statements focused more substantively around big ideas were not the norm, such as a *possible* statement that might emerge on students' actions and thinking about lines of best fit supported by explorations using technology:

> The use of a movable line on a scatter plot helped students explore what finding a line of best-fit might mean. Student discussions following this experience were focused on the clustering or lack of clustering of points around an apparent line and what this might mean for the relationship between the two variables being investigated.[1]

However, there are bits of information that would link research findings related to software use back to the development of big ideas, for example:

- *Software tools with ready-made methods influence the way a subject matter problem is conceived of and is transformed into a "statistical problem" and into a "problem for the software" (Biehler, 1997).*
- Students working in pairs on problems dealing with graphing are more able than their classmates working alone to make the critical inferences crucial to learning from problem-solving experiences (Jackson, Edwards, & Berger, 1993b).
- On a project, students used more mathematics, and took less time, largely because technology took care of graphing and laborious calculations (Erickson, 2002).
- The graphing facilities of *Microsoft Works* are often overwhelming and confusing to students, as when the basic "new chart" menu command is selected, the program immediately confronts the user with a large number of simultaneous choices, thus requiring much advanced planning (Jackson et al., 1993).
- With *Cricket Graph*, the first choice made by the user, the type of graph, is irrevocable. Many students find this frustrating, as they are still learning about how to choose an appropriate graph, and they don't want to start from the very beginning again (Jackson et al., 1993).
- Many students will choose the default representation in the software without thinking about their choices (Jackson et al., 1993).
- Certain sequences of actions (e.g., sort the data, re-graph, move or resize, choose a different graph type) taken by students had correlations to whether the resulting graph turned out basically good or poor (Jackson, Berger, & Edwards, 1992).

Issues about student thinking and use of technology in teaching data analysis/statistics also surface:

- There is a need to make the interface visually simple and clear so that the students pay less attention to the tools and more to the task and target concepts.
- When presented with many choices, such as in an open construction tool, students may experience cognitive conflicts
- Ideas learned may be too deeply connected to specific tools (i.e., learning the tool rather than learning the concepts to be learned through the tool)
- Students often use the display methods offered by the software rather than thinking through the best representations for the purpose of the investigation
- The graphical representation used strongly affects a student's reasoning about a set of data
- The improved performance of a student who is in a computer partnership is necessary but not sufficient to demonstrate learning.

What are the needs and directions that can help frame a research agenda?

[1] This statement is hypothetical; there is no evidence for its validity in any research reviewed although there are software options that make this kind of investigation of student understanding a realistic option.

Identifying the questions

We have just begun to understand the questions we want to ask about what it means to know and be able to do data analysis and statistics at the K-12 grade levels. With a shift away from the automatic transfer of traditional content from post-secondary to K-12 and a move toward reconceptualizing this content and how it is learned, we find ourselves in a state of flux. When we add to this the need to consider the interactions between technology, content and learning, given the new kinds of software being developed, there are rich opportunities for framing questions. Suggestions will be provided during the session with opportunities for the audience to add to the list.

Knowing what we know

Part of the dilemma with respect to identifying questions includes the need to know what we know. In a recent funded proposal to NSF, Rubin (2000) provided a short but rich summary of the literature to date focused on the concept of variation in the context of technology use. We need to have these kinds of summaries available to the broader research and software development communities. This need can be met, in part, through access to reviews of the literature related to the big ideas (e.g., Friel, Curcio, & Bright, 2001; Meletiou, 2002), but such reviews are not quickly completed and updating to reflect new research must be a continual process.

Locating the reported research (that includes or excludes the integration of technology) is problematic as well. Much of the current research is reported in conference proceedings (e.g., *ICOTS-5, ICOTS-6, Garfield & Burrill, 1997*) that are sometimes not easily accessible. While more of these publications are being made available online, anyone looking for this information must have a knowledge of the organizations and ways of sharing information that currently exist within the community.

The ideal would be to have available an annotated bibliographic electronic database organized around the big ideas of statistics—a repository of abstracts of research (e.g., Huntley, Zucker, & Estey, 2000) or a more comprehensive database upon which to build a resource bank of related research references and resources. Such a resource would serve as an evidence-based repository that could be used to inform research directions and could be updated to reflect results of new work as it is added to the field.

New issues and ideas emerge when technology is used

Beyond addressing the basic needs of sorting out research questions, providing summaries of the research related to the big ideas in data analysis and statistics, and making the current literature readily accessible that would help in framing a research agenda, the interaction among technology, content and student thinking emerges as an arena that is rich in possibilities for research. The use of the technology itself surfaces perplexities about content and student thinking only because students now can work in such rich investigative environments (e.g. Fathom, Tabletop, Tinkerplots). The very capabilities that these software tools provide raise a multitude of questions about content and how to think about the big ideas in the domain of data analysis and statistics. Examples will be provided with demonstrations of software as part of this session.

We are in a state of flux about exactly what is the content to be addressed at what levels, K-12. We are only now gaining access to new software tools that will push for understanding of substantially richer conceptions of the big ideas of data analysis and statistics. There is a need to include research that clarifies *the impact on students' knowledge*. The process of theory building contributing to practice and vice versa is very much a real phenomenon in the arena of data analysis and statistics education.

Data Distribution Variability Trend Covariation Sampling Model Representations
Measures of central tendency or location Measures of spread and dispersion
Actions on data

Figure 1: Selected Big Ideas: Data Analysis and Statistics
(Cobb, 1999; Cobb, McClain & Gravemeijer, In press; Cziko, 2002; Garfield, 2001; Hancock, Kaput, & Goldsmith, 1992; Konold & Higgins, 2002, in press; and Konold & Pollatsek, 2002)

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TO PRODUCE CONJECTURES AND TO PROVE THEM WITHIN A DYNAMIC GEOMETRY ENVIRONMENT: A CASE STUDY

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This paper analyses a case study of a pair of students working together, who were asked to produce conjectures and to validate them within the dynamic geometry environment Cabri. Our aim is to scrutinize the students' reasoning, how the gap from perception to theory is filled, how Cabri influences the reasoning. We have singled out a sequence of phases in which the students pass from exploration to increasing degrees of formal reasoning. Our study reveals, among other things, that Cabri fosters the flexible use of methods close to analysis and synthesis.

INTRODUCTION

In this paper we report a case study concerning a pair of students who explored an open problem within the dynamic geometric environment Cabri-Géomètre I (henceforth Cabri). This activity has brought to the fore a variety of strategies and ways of thinking that makes it possible to detect and to analyze how the students came to prove. By scrutinizing the students' reasoning we single out continuity and leaps in the transition from perception (observation of figures in the screen) to theory.

To explain the context in which the studied activity has developed we state our ideas on proof in classroom. In this concern a recent paper by Herbst (2002) stresses two main issues:

> As proof is intimately connected to the construction of mathematical ideas, proving should be as natural an activity for students as defining, modeling, representing, or problem solving. Yet, important questions that must be raised concern what it takes to organize classrooms where students can be expected to produce arguments and proofs and what proof may look like in school classrooms. (p.284)

In analyzing the *two-column proving custom* Herbst (2002) complains that, under this method, "proving activities for students have often been closer to exercising logic to validate obvious and inconsequential statements […], than to building compelling arguments for the reasonableness of important mathematical ideas […]." (p.284). This criticism may be applied also to other methods of teaching proof practiced in classroom, which are dominated by a rote learning model. These methods push students towards schema of proving such as those classified by Harel and Sowder (1998) as "ritual", "symbolic", "authoritarian". These schemas are far from being suitable to make the activity of proving a meaningful activity.

Our paper refers to an approach aimed at promoting a smooth transition from argumentation to proof, see (Furinghetti et al., 2001). The elements characterizing this approach are the following:

- focus on the debate about the construction of theorems and proofs, with the distinction between the problems of conforming to the standards of exposition and rigor and those of construction, validation, and acceptation of a statement
- specification of the rules for stating a theorem and proving it

- reflection on the environments which seem to foster the production of hypotheses and their formulation according to logical connection
- possibility of singling out cognitive continuity between the processes of producing and exploring the statement of a theorem and the construction of its proof, with particular attention to the reference theories and the leaps inside a theory and among different theories
- the role of the social dimension of the learning as for knowledge on theorems and proof, with particular reference to mathematical discussion in classroom and the modes of using various mediators (history, technology,…).

We think that only through experimenting personally the construction of parts of a theory (under the guidance of the teacher and in situations carefully projected) students may give up, when necessary, the perceptive level and appreciate the meaning of theories. To make students to construct parts of a theory means to allow them to experience the construction of mathematical knowledge at different levels: the level of exploring within particular cases, those of observing regularities, of producing conjectures, of validating them inside theories (which may be already constructed or in progress). In developing this approach we are concerned with the transition from elementary to advanced mathematical thinking. Gray et al. (1999) have pointed out that the "didactical reversal – constructing a mental object from 'known' properties, instead of constructing properties from 'known' objects causes new kinds of cognitive difficulty." (p.117)

Nunokawa (1996) has discussed the application of Lakatos' ideas to mathematical problem solving. In our approach to proof we are thinking something similar. We see students as immersed in a situation similar to that termed by Lakatos (1976) *pre-Euclidean*, that is to say a situation in which the theoretical frame is not well defined so that one has to look for the 'convenient' axioms that allow constructing the theory. The didactical suggestion implicit in Lakatos' words is that it is advisable to recover the spirit of Greek geometers. When they made proofs they were not inside a theory in which axioms were explicitly declared. Initially antique geometry developed in an empirical way, through a naïve phase of trials and errors: it started from a body of conjectures, after there were mental experiments of control and proving experiments (mainly analysis) without any sure axiomatic system. According to Szabo, this is the original concept of proof held by Greeks, called *deiknimi*. The *deiknimi* may be developed in two ways, which correspond to analysis and synthesis. These ideas suggest a way of realizing cognitive continuity in our approach to proof in classroom. Also they suggest the means to reach this objective: socialization, discussion, sharing of ideas.

REALIZATION OF OUR APPROACH TO PROOF

The general ideas we have previously discussed need to be adapted to the classroom needs as well as to the present conditions of students' learning. This requires creating environments suitable to exploration, production of conjectures, validation of these conjectures. To this purpose we propose to students open problems. We take as characterization of open problems the following, see (Arsac et al., 1988):

- The statement of the problem is short, so that it can be easily understood, it fosters discovery and all students are able to start the solution process.
- The statement of the problem does not suggest the method of solution, or the solution itself, but it creates a situation stimulating the production of conjectures.

- The problem is set in a conceptual domain which students are familiar with. Thus students are able to master the situation rather quickly and to get involved in attempts of conjecturing, planning solution paths and finding counter-examples in a reasonable time.

We think that open problems promote the devolution of responsibility from the teacher to students. This is even truer when students work in group and participate to classroom discussion. This situation fosters creativity, e.g. the ability to overcome fixations in mathematical problem solving and to produce divergent thinking within the mathematical situation (fluency and flexibility), see Haylock (1987).
Another element characterizing our approach to proof is the use of Cabri. It is widely recognized that the exploration with this kind of software amplifies the potential of producing conjectures, see (Santos-Trigo & Espinosa-Perez, 2002). At the same time it stimulates to prove the validity of the produced conjectures, see (Arcavi & Hadas, 2000).

In the situation we have outlined the statements to be proved are not provided by an authority (teachers, books), but are the result of an autonomous research and it is Cabri which confirms that a conjecture produced by students is 'good'. Thus the motivations to prove are different from those found in the usual didactical situations, where the task given to students is on the form "prove that…» The motivations we provide are similar to those of mathematicians at work, see (Burton, 1999).

THE CASE STUDY OF ALEX AND LUCA

The experiment reported is an example of what may happen in classroom when our approach is proposed. The class begins with a work of exploration and observation, which leads to produce conjectures. The validation of these conjectures is performed through the dragging text with Cabri. The way of reasoning is similar to that employed in empirical sciences, e.g. induction, abduction, analogy. In this context the role of proof is to *explain* why the produced conjectures hold within a theory (in our case Euclidean geometry).

The experiment was carried out in a class of a Scientific Lyceum (an Italian high school with a scientific orientation), at the beginning of the school year. The students (17 years old) worked in small groups (2 or 3 persons per group, 8 groups) with one computer per group. They were acquainted with exploration of open problems and had worked in group quite regularly before the experiment. They mastered Cabri. The time allowed for the experiment has been one hour and a half. In the following we describe the main phases of the work of the pair composed by Alex and Luca. The report is based on fieldnotes taken by the teacher (who acted as an observer) and on the students' protocols.

The statement of the problem was given without the figure. Alex and Luca draw quickly and accurately the quadrilateral *ABCD* and afterwards the quadrilateral *HKLM* using Cabri, see Fig. 1 (all figures made by the students with the computer were in color).

The problem:

You are given a quadrilateral *ABCD*. Consider the bisectors of the four interior angles: be *H* the intersection point of the bisectors in $\angle A$ and in $\angle B$, *K* the intersection point of the bisectors in $\angle B$ and in $\angle C$, *L* the intersection point of the bisectors in $\angle C$ and in $\angle D$, *M* the intersection point of the bisectors in $\angle D$ and in $\angle A$.
Investigate how *KHLM* changes in relation to *ABCD*? Prove your conjectures.

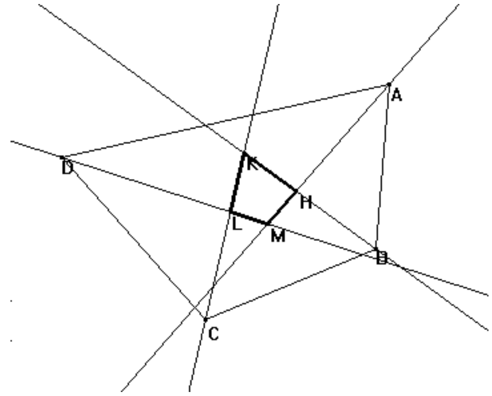

Fig. 1

PHASE 1

The students drag the vertexes *A*, *B*, *C*, *D* at random. This is the mode of dragging called "wandering dragging" by Arzarello et al. (2002): it is used when one is looking for ideas. This mode may be seen as almost static, since students drag the figures for a while and afterwards focus on the obtained figures kept still. During the wandering dragging they find the configuration reported in Fig. 2 in which the points *H*, *K*, *L*, *M* are almost coincident.

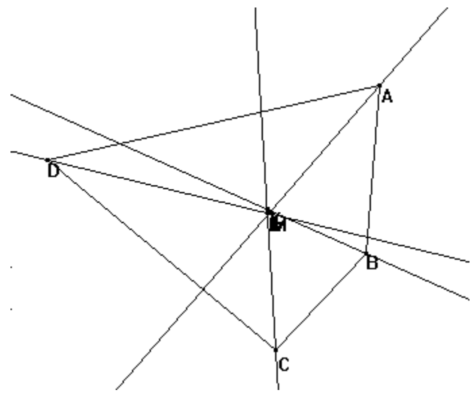

Fig. 2

PHASE 2

The mode of dragging changes significantly. Alex and Luca decide to focus on the internal quadrilateral *HKLM*. Of course, they can only act on the vertexes *A*, *B*, *C*, *D*, but they choose a particular configuration of *KHLM* (a point, a square, a rectangle, a rhombus, a parallelogram, a trapezium) and afterwards drag the vertexes *A*, *B*, *C*, *D* so that the quadrilateral *KHLM* keeps the particular configuration they have chosen. The students report only this part of the exploration in their protocol:

> To observe the changes of the figure we have considered the internal quadrilateral; that is to say first we have observed the particular cases of the internal quadrilateral and for each case we have looked at the changes of the external quadrilateral. With this method we have realized that different external figures correspond to each particular case of the internal figure: for example, when *H*, *K*, *L*, *M* are coincident, the external figure may be a square, a right-angled trapezium, or other figures.

These students are the only ones in the classroom using this mode of dragging based on the internal quadrilateral. This allows them to see very soon that not only squares and rhombuses generate internal quadrilaterals, which are points (In the case of squares and rhombuses the bisectors of opposite angles are coincident). The function of Cabri in this phase is to support transformational reasoning. We recall that Simon (1996) describes transformational reasoning as

> the mental or physical enactment of an operation or set of operations on an object or set of objects that allows one to envision the transformations that these objects undergo and the set of results of these operations. Central to transformational reasoning is the ability to consider, not a static state, but a dynamic process by which a new state or a continuum of states are generated. (p.201)

We stress that this phase marks a *leap* in the exploration. The mode of working goes back from the final result (a particular configuration of the internal quadrilateral) to the premises (the given quadrilateral and the bisectors of angles). This recalls the method of analysis, which is considered by many authors an efficient method of discovery. This method dates back to Plato and Greek mathematics. Hintikka and Remes (1974) describes it as follows:

> For in analysis we suppose that which is sought to be already done, and we inquire from what it results, and again what is the antecedent of the latter, until we on our backward way light upon something already known and being first in order. And we call such a method analysis, as being a solution backwards. In synthesis, on the other hand, we suppose that which was reached last in analysis to be already done, and arranging in their natural order as consequence the former antecedents and linking them one with another, we in the end arrive at the construction of the things sought. And this we call synthesis". (p.8).

Smith (1911) explains analysis as a method to solve problems and to prove theorems. He says that this method has several forms, but the essential feature

> consists in reasoning as follows: "I can prove this proposition if I can prove this thing; I can prove this thing if I can prove that […] until comes to the point where is able to add, "but I *can* prove that." This does not prove the proposition, but it enables [the student] to reverse the process, beginning with the thing he can prove and going back, step by step to the thing that he is to prove. Analysis is, therefore, the method of discovery of that way in which he may arrange his synthetic proof. (p.161-162)

Analysis leads to a construction and synthesis shows the validity of this construction. We note that schemas based on the triad analysis-construction-synthesis are present in the works of ancient mathematicians; in particular, on the method of analysis is based the development of modern algebra carried out in Viète's *In Artem Analyticem Isagoge* (1591). Gusev and Safuanov (2001) have argued that to solve problems requires various aspects of analytic-synthetic activities, and, in particular, analysis through synthesis.

PHASE 3

This is a phase of reflection made by students on what has been observed with Cabri. The students stop to explore. Through the reconstruction of some interesting configurations that they have obtained with Cabri they look for invariants. They write:

> We have looked for a relation among the figures obtained by means of the same internal figure. In this search we have discovered a theorem that we have formalized in this way: When all angle bisectors intersect in a point *P*, this point is the center of the circle inscribed in the quadrilateral.

In experiments with other students we have found that after having produced a conjecture the students continue to use Cabri to make figures which may prove the correctness of their conjecture. Our students show a different behavior: they are already convinced about the correctness of their conjecture and use Cabri only to refresh what they have done. The computer screen is no more an environment in which to conduct exploration and to take inspiration for conjecturing; it becomes a kind of fieldnotes keeper.

Our students show to be aware of the reverse path followed in their reasoning ("We have looked for a relation among the figures *obtained by means of the same internal figure* [italic added]"). Since they are convinced of the validity of their conjecture they are motivated to answer the question "Why the conjecture is valid?" Alex and Luca use the words "theorem" and "formalized", which evidence that they have definitely put themselves inside the theoretical framework of Euclidean geometry. They seem to perceive the function of proof as a process suitable to explain *why* a given conjecture is true. There is one sentence in their writing that shows the interlacement between the exploration ("In this search [made with Cabri]") and the theory ("we have discovered a theorem").

PHASE 4

Alex and Luca are ready to prove the conjecture produced. They abandon Cabri and use paper and pencil, also for drawing the figure on which their proof is based. [Fig. 3 reproduces accurately the drawing made by students with paper and pencil].

The proof is not complete and precise, but it may become acceptable with few amendments. In this phase it is clear that the mode of communication is changed. The focus has shifted from the facts observed in the screen to their justification in Euclidean geometry. The transition from the computer to paper and pencil marks the transition to the synthetic mode of proving (via the construction of Fig. 3).

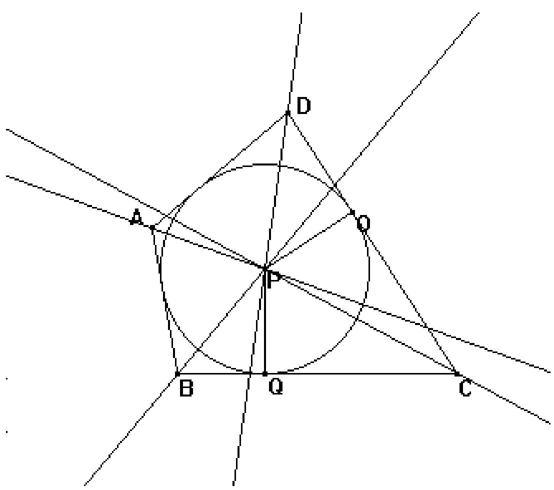

Fig. 3. (Made by students with paper and pencil)

We try to prove: to this aim we use two straight lines through the center which are perpendicular to two adjacent sides of the quadrilateral.

Hypothesis: $PQ \perp BC$ $PO \perp DC$ $\angle OCP = \angle QCP$

Thesis: $PQ = PO$ (radius of the circle)

Proof

We consider the triangles POC and PQC. We must prove that they are congruent. We know that the angles $\angle PQC$ and $\angle POC$ are right and congruent; also we know that $\angle QCP$ and $\angle OCP$ are congruent. Since PC is common to the two triangles, POC and PQC are congruent for the fourth criterion of triangles congruence. In particular, PQ and PO are congruent, so that they are two radiuses of the circle inscribed in the quadrilateral $ABCD$.

PHASE 5

Alex and Luca have sketched their proof. Since the statement of the problem given to them required studying the variation of *HKLM* in relation to the variation of *ABCD*, they change the statement just proved by them to emphasize the relation of dependence

> We take a step backward: we have observed that the only common element among the figures obtained through a particular configuration of *HKLM* (*H*, *K*, *L*, *M* coincident) is the theorem that we have just proved. We know the theorem stating that a quadrilateral may circumscribe a circle when the sums of its opposite sides are equal ($AB+CD=AD+BC$). Hence we may say that *H*, *K*, *L*, *M* are coincident when $AB+CD=AD+BC$.

Even if our students are working with paper and pencil, they refer explicitly also to the exploration with Cabri ("we have observed"). This confirms the interlacement of exploration and proof. We note that the original property based on the inscribed circle is visual and was obtained with a construction, while the property $AB+CD=AD+BC$ is the consequence of a theorem. Thus the final statement is expressed in a form ("…when $AB+CD=AD+BC$") that hides the steps through which students arrived to the statement. Definitely the students are in the synthetic mode of reasoning inside the Euclidean theory.

PHASE 6

Alex and Luca have produced, proved, and stated in a formal way a conjecture. Now they go back to the original problem given by the teacher to look for other results. Again they use Cabri to explore, in a way more systematic than that used initially. As done in the phase 1, they start from the external configuration *ABCD*. This coming back to exploration evidences the cognitive continuity between the phases of exploration, production of conjectures and proof. But they have to stop: the time is over.

FINAL COMMENTS

We summarize the steps of students' reasoning from conjecture to proof:

- *Reading the terms of the problem and translating it in the graphical language*: the role of Cabri is central in interpreting correctly the statement
- *Wandering dragging in search of inspiration for producing a conjecture*: this mode is close to empirical methods used in experimental sciences. This is a moment in which creativity has to be present: Cabri amplifies the students' creativity.
- *"Aha!" moment*: a property is discovered. This provokes a *leap* in the way of dealing with the problem. The mode of using Cabri is reversed: instead of going from the given quadrilateral *ABCD* to the resulting quadrilateral *HKLM*, our students start from *HKLM* and investigate on the facts that may have this quadrilateral as consequence. They apply a method recalling analysis.
- *Dragging with Cabri to search a way for proving the conjecture*: this is a phase in which the students need to reflect. Cabri provides many situations and the students have to find those suitable to their purpose, since, as Poincaré (1899) has observed, it is not enough to produce right situations, you have to choose among all possible situations. The way of thinking here is close to the analytic method.
- *"Aha!" moment*: the students make a construction that inspires the statement of a theorem. Again we feel that Cabri has amplified the students' creativity. Here we have another *leap* in the students' reasoning. The method they follow is mainly synthetic. This leap is marked by the use of paper and pencil instead of Cabri.
- *Inside the Euclidean theory*: the students are able to produce a new theorem through deduction from an Euclidean theorem.

In the strategies applied by our students it is remarkable the presence of methods close to analysis and synthesis, as well as the role of the construction as a pivot between the two methods.

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"IT IS POSSIBLE TO DIE BEFORE BEING BORN". NEGATIVE INTEGERS SUBTRACTION: A CASE STUDY

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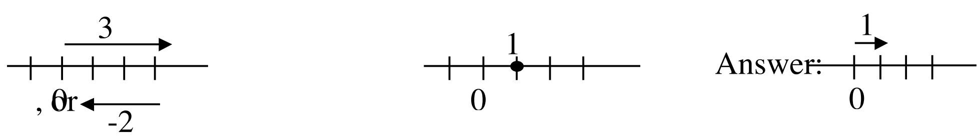
A Case Study is presented in this article, where there is a contradiction between pre-algebraic language semantic and syntax used to solve word problems through a negative integers subtraction.

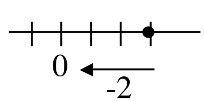
GENERAL BACKGROUND

The Case Study reported on this article is based on Vergnaud's research (1982) appearing on the arithmetic to algebra transition which has turned out to be far prolific for the comprehension of negative numbers difficulties students face. Vergnaud sorts additive relations in order to interpret the procedures students use in solving addition and subtraction problems. This classification developed from the fact that knowledge is organized in "conceptual fields" which cover set of problems, situations, concepts, relationships, structures, contents and operations of thought, connected to each other and likely to be interwoven during the process of acquisition. For example, the concepts of state, measure, addition, subtraction, time transformation, comparison relationships, displacement and abscissa on an axis and natural and relative numbers belong to the field of "additive structures". In order to interpret students' behavior when dealing with elementary arithmetic problems, Vergnaud finds it essential to distinguish two sorts of calculus: "numerical calculus" and "relational calculus". Numerical calculus refers to ordinary operations of addition, subtraction, multiplication and division. Relational calculus involves the operations of thought that are necessary to handle the relationships involved in the situation. However, it is highly important to point out that such operations of thought are not expressed or explained by students. As a matter of fact, they can only be hypothesized by observing students' actions. On the other hand, Janvier (1983) showed that signed numbers can be interpreted as two semantically differents sets of objects in the number line model. In this well-known model, numbers are either positions or displacements on this line. Ambiguities are introduced at the beginning. Addition then becomes a combination of different elements. Example:

3 + (-2) is



3 + (-2) is



Answer as above.

The reader has certainly noted the ambiguity between states and transformations. The difficulty then arises from the fact that adding may be the composition of two transformations or the transformation from a state to a state. Although students can learn a rule and follow it, this basic confusion creeps in when subtraction operation is introduced. On the theoretical basis of Vergnaud relational calculus and Janvier analysis, the following question is risen in the reported study reviewed in this article: How do students do, out of a word problem statement, a negative integers subtraction? This question is approached through the so-called Socrates problem, which statement is: "Socrates, the Greek philosopher was born in 469 B.C. and died in 399 B.C. How old was he when he died?" This problem can be solved by the following negative integers subtraction:

$-399 - (-469) = -399 + 469 = 469 - 399 = 70$ … (1)

Answer: 70 years old.

The two first members of the operation (1) do not belong to arithmetic language, since this language is not proper to represent signed numbers (Vergnaud, 1982). Besides, the horizontal representation of (1) belongs to pre-algebraic language. Thus, -399 – (-469) is a subtraction in the integers domain, while 469 – 399 is a natural numbers subtraction. Likewise, -399 – (-469) equals 469 – 399; that is, there is a syntactic equivalence between both expressions. However, these expressions are semantically different. -399 – (-469) means subtracting Socrates birth date from his death date. Many students solve the problem through the operation 469 – 399 = 70, which is meaningless.

A CASE STUDY

The results of a Study carried out with 41 eight grade students of 12-13 years old, who answered an exploratory questionnaire on fractions and integers, are reported in Gallardo & Novoa (2000). These students were classified in profiles or classes according to their performance in the questionnaire. Nine students from different profiles were chosen to carry out individual clinic interviews. The interviews protocol consisted of the same issues in the questionnaire, that is:

1. Operations with integers and fractions at the syntactic level and their representation on the number line.
2. Solving of word problems.

One of the most important results from this Study, regarding integers, consisted of identifying a "persistent attitude" by most of the students, who always insisted in using the signs multiplication rule, which allowed them to solve subtractions without considering them so. Thus, when using the rule (-) (-) = (+), the subtraction a – (-b) became the addition a + b. The extreme difficulty in the negative integers subtraction in the case of C, the student who best performed in the questionnaire is herein presented, and her actions during the process of solving "Socrates Problem" are analyzed. This problem turned out to be the most difficult item for students. C solved correctly operations in the form of $a \pm b = c$, where a, b, and c are integers both in the syntactic level and on the number line. She also solved the word problems described in the Appendix herein. However, C could not manage to transfer the operating capacity to Socrates Problem. Next, C and the interviewer's (I) dialogues are presented with the student's actions analysis. The analysis to each of C's answers is described in brackets [].

Formulation: Socrates, the Greek philosopher was born in 469 B.C. and died in 399 B.C. How old was he when he died? 1C. [She writes as in the figure below then she subtracts the birth date from the death date, even though it is wrongly done. She does not get surprised of the size of the number obtained (930) too big for an age, besides it is negative].

$$\begin{array}{r} \square 399 \\ \square 469 \\ \hline \square 930 \end{array}$$

Figure 1. C’s wrtten work on the problem of Socrates age at death

1I. WHAT HAPPENED?

2C. I subtracted, added, that is to say I added the death date minus the birth date and then I added. [*She first says subtract and then she says add because she knows a school rule that she puts into words in 3C. The rule expressed by the student is the following: “as both are minus signs, then it is like an addition but with negative sign”. Nevertheless, if she had followed the rule she would have gotten: (-399) + (-469) = -868 and not –930 as in 1C*].

2I. Did you add?

3C. Well, I am subtracting but as both are minus signs, then it is like an addition but with a negative sign. [*Contradiction. She makes a wrong subtraction and at the same time she uses an addition rule. Notice that in the vertical representation of 1C, there is no operation sign, because it is an arithmetic representation that is not adequate for operating signed numbers. First she chooses to carry out a subtraction according to the semantics of the formulation (death date minus birth date) and then she uses a sign rule for the numerical expression that appears in1C*].

3I.What is the difference between this formulation and the previous? [See Appendix, problem 1].

4C. That the one before dealt with positive quantities, while this one deals with negative quantities. [*Notice that she refers to quantities and not numbers*.]

4I Why are these negative?

5C. Because it says before the Death of Christ, then the dates before the Death of Christ are represented with a negative sign. [*Again notice that she does not talk about negative numbers but signs*].

5I. Then, how old was this philosopher when he died?

6C. Mmm, 930 years old. [*The answer she gives is a positive number even though she got a negative one*].

6I. Where did it come from?

7C. Because I added the quantities, I correct, I subtracted them but then it would be minus 930 years old. [*She repeats the complete school rule (3C), but this time she adds to 930 the words “years old”*].

7I. Let's see, can you live a negative quantity of years old?

8C. No.

8I. There is another difficulty here, is it possible for anybody to live 930 years?

9C. No.

9I. Is there anything that could help you to clear out the formulation?

10C. But the dates before the death of Christ are upside down that is to say first the big numbers and at the end the smaller ones, because they are negative quantities.

10I. How would you express that?

11C. Well, all the quantities before the Death of Christ, since they are with negative numbers and they have to be placed opposite, I mean, for example, not as year one, year two, but first bigger years and at the end the small ones. [*As saying year one, year two, there is no doubt that she refers to the number line*].

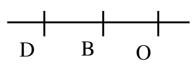
11I. What does your description look like?

12C. When the numbers on a line are represented or located.

12I. Very good. Could you draw the number line for this?

13C. Yes.

13I. Indicate more or less, he was born here, he died here.

14C. [She writes] 

[*Besides the explanation given in 10C and 11C, first she represents he dies (D) and then he was born (B)"*].

14I. Where was he born?

15C. Here [B]

15I. Where did he die?

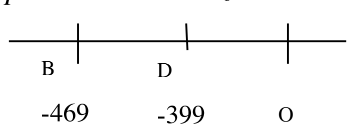
16C. Here [D]

16I. According to these facts, What is first, his birth or his death?

17C. His birth [*She takes the zero in the number line as origin*].

17I. Give it numbers, then.

18C. So it would be all the way around [*permutes B with D and ascribes –469 and –399 respectively*] first it would be his death and then his birth…so I would make a subtraction. [*Contradiction shows up with 17C where she states that his birth was first. However in 18C she states that "his death would be first". She exchanges B and D in 14C so, new D is now placed closer to zero on the number line:*



18I. Forward.

19C. [*Write*] $\begin{array}{r} 469 \\ -399 \\ \hline 70 \end{array}$. Then, he would be 70 years old. [*At 18C, the student says at the end of the discussion "… so I would make a subtraction". When moving from 1C (a vertical subtraction of signed numbers) to the representation on the number line, she does not give signs to quantities, obtaining a positive result: 70 years old, which describes a real fact].*

19I. Finally, he was born in 469 B.C. and die in 399, he was?

20C. 70 years old.

20I. How would you prove that the answer is 70 years old?

21C. Well, I would add 70 to his birth date and if the answer is 399, then the answer is correct. [Write] $\begin{array}{r} -469 \\ 70 \\ \hline 399 \end{array}$. I've got it! [*She said "add" , however she subtracted because she pictured it* $\begin{array}{r} -469 \\ 70 \\ \hline 399 \end{array}$ *as* $\begin{array}{r} 469 \\ -70 \\ \hline 399 \end{array}$. *This shows that she considered the addition of integers (-469 + 70 = 399) as a subtraction of natural numbers (469 – 70 = 399)*].

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21I. Did verification work?

22C Yes!.

22I. What did you say you were going to do to prove your answer?

23C. I was going to add 469 plus 70. [*She is still engrossed in natural numbers*].

23I. 469 plus 70, would be equal to more than 500.

24C. Oh! Yes! [*Here, the student realized she had used up her knowledge in this regard. She is in a vicious circle, in a dead end. At the beginning she had got 930 years old as an answer and now she would get 539 years if following the addition procedure as set out in 23C].*

CONCLUSIONS

C's actions are observed in the interview protocol: In 1C, she writes signed numbers in an arithmetic vertical representation, and she gets a wrong result: -930. In 2C, she does an operation with quantities in context (B.C. chronology). In 2C and 3C she is confused when choosing between subtraction and addition. In 4C, she refers to positive and negative quantities. In 5C, she relates the B.C. chronology to minus sign, instead of a negative number. In 6C, owing to the fact that the interviewer said the word "age", she makes her answer "positive", that is, she leaves out the minus sign and adds instead the word "years". She answers: "930 years". In 7C, she is again confused between addition and subtraction, and she makes her answer "negative". In 10C, 11C and 12C, the student uses "numbers on the number line". She had so far referred only to relative quantities and signs instead of numbers. In 14C, when transferring B.C. chronology to the number line, she represents on it dying (D) before being born (B). In fact, she does not interpret it so, since she reads it from right to left, starting at zero, reaching first B and then D on the number line. Thanks to the interviewer's participation C manages to represent in the right way the problem data (18C). However, 19C shows she does not do operations on the number line, but reads "from left to right", and then she writes a natural numbers subtraction in a vertical line. Both in 19C and 20C, she states: "70 years". In 21C, she says she will add 70 to the birth date in order to check her answer. However, she interprets the integers addition vertical representation as a natural numbers subtraction. In 22C, she is satisfied by her result (70 years old), since she thinks she managed to prove it in 21C. She says: "I've got it". In 23C, she sets out a natural numbers addition which in 24C makes her excitingly say: "Oh, yes!" She is stuck in arithmetic. Regarding Socrates Problem, as the student did not recognize the subtraction of signed numbers she subtracted the death date from the birth date.

To sum up, we can conclude from the dialogue in the interview that the student mostly used natural language throughout the protocol process. In 1C, the student writes a wrong subtraction of signed numbers and it is not until 19C that she writes once more a natural numbers subtraction. Even though she does not know it, it is the syntactic equivalence -399 – (-469) = 469 – 399 what helps her get "a possible and actually right answer" of the problem: 70 years old. In 4C, relative quantities (positive and negative) are mentioned. In 10C, when spontaneously using the number line model, she mentions the word "numbers" for the first time. In Gallardo (2002a) four levels of acceptance of negative numbers: subtrahend, signed numbers (plus or minus sign is associated with the number) relative number (idea of opposite quantities) and isolated number (result of an operation or solution of an equation) were abstracted from an empirical study with 35 pupils of 12-13 years-old. In the Case Study reported on this article, it was observed that C showed these levels as well. C's actions analysis suggests the need of re-defining integers and their operations beyond basic arithmetic. In Gallardo (2002a) we read "it is in the transitional process from arithmetic to algebra that the analysis of students' construction of negative numbers becomes meaningful. During this stage the students are faced with equations and problems having negative numbers as coefficients, constants or solutions".

The research approach in this article is in process. The results of a recent empirical analysis with 12-13 year-old students about difficulties due to ambiguity between states and transformations by using the number line model and the word problems solving context are reported in Gallardo (2002b). This empirical analysis shows the difficulties of telling the difference between transformations and states and therefore it does not allow to recognize the dynamic nature of the integers. Most of the students interpreted transformations as static states in themselves. This fact hid the dialectic relation between states (static relation) versus transformations (dynamic relation) of integers.

APPENDIX

Problems C solved correctly:

1. Benito Juarez was born in 1806. How old is he in 1857?

2. A watcher is 135m under the sea level. Where will he get after going up 15m?

3. A watcher is standing in a specific place. He goes up 100m to reach 50m over the sea level, what was exactly his position at the beginning?

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EVOLUTION OF FORMS OF REPRESENTATION IN A MODELLING ACTIVITY: A CASE STUDY

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The report describes a mathematical modelling activity of a natural phenomenon (transmission of hereditary characters in a codominance case) using the concept of model (as represented by the diagram in Fig. 1) as a theoretical instrument. The chosen tool enables us to show how the construction of a link between reality and a model is related to the evolution of the graphical representations adopted by the students. The lack of such evolution may be either an obstacle to the modelling activity or favour an inappropriate adoption of models generally used for other phenomena. Our analysis also compares the resolution processes enacted by students belonging to different educational environments.

THEORETICAL FRAMEWORK

Most research on probabilistic thinking concerns students' evaluation of a stochastic situation (prediction or interpretation of outcomes); few research studies concern probabilistic modelling, i.e. the construction of a model to interpret stochastic real world phenomena (for a general survey, see Borovcinik & Peard, 1996). As concerns semiotic aspects, Pesci (1994) and Dupuis & Rosset Bert (1996; 1997) referred (resp.) to Fischbein (1987) and Duval (1995) for general theoretical frameworks. They pointed out the crucial role of semiotic tools in probabilistic thinking, but did not deal with modelling of complex phenomena. The process of progressive schematisation of a phenomenon in order to get an interpretation in terms of a probabilistic model still needs to be carefully investigated. In particular it would be necessary to consider the nature of some specific semiotic tools (tree graphs, double entry tables, etc.) and the relationships that each of them allows to establish between the phenomenon on one side, and probabilistic thinking on the other.

In Dapueto & Parenti, 1999, the epistemological concept of model, represented in the diagram in Fig. 1, is discussed (in comparison with preceding studies: see Blum& Niss, 1991; Norman, 1993) and used in order to tackle some problems of situated teaching-learning. They describe the diagram with an example: *"Let R be a quadrangular field that we wish to evaluate economically an AR the extent of that field. If we note that the field is more or less rectangular, we can consider the lengths of two consecutive sides as the factors ER that determine the extent*. […] *Let us associate the lengths ER to two numbers,* a *and* b, *which express the lengths in a particular unity* U*;*[…] *Let us associate to AR* a*b*; this is AM.* […] *Hence, the model M is the multiplication (or rather the formula* S=a*b, *if* S *expresses the extent in square whose side is* U*)*". Subsequently the authors write *"in mathematical modelling (i.e. when artefacts* [used to build the model] *are mathematical objects) the passage from ER to EM is called mathematization. But also the use of mathematical artefacts in building a physical, biological,…model (for instance a physical law or a bionic model)l is a mathematization"*

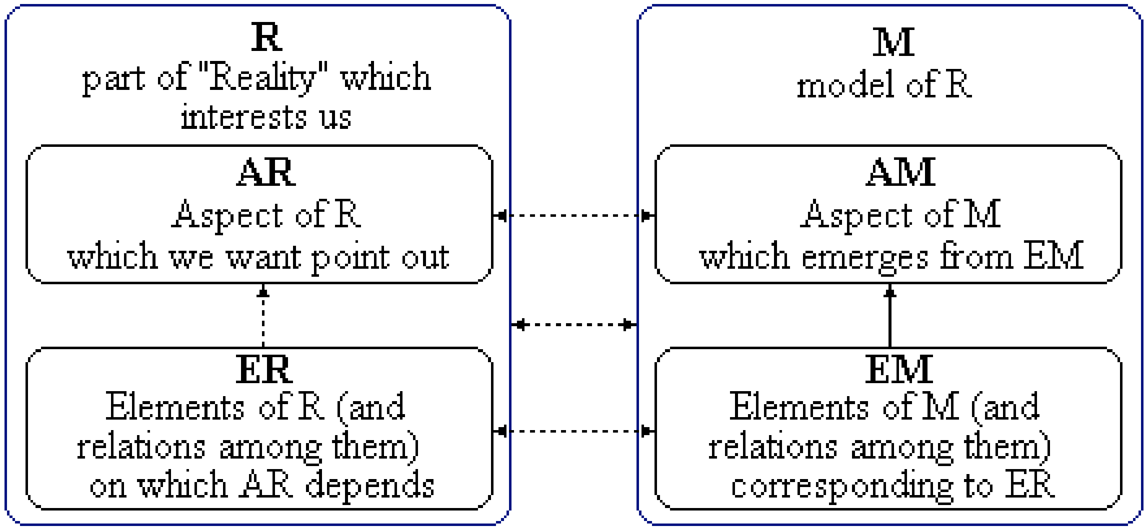

Fig.1

We used this representation of the concept of model in order to carry out a detailed analysis of VIII grade students' behaviour while performing a complex activity of mathematical modelling of a natural phenomenon, i.e. the transmission of hereditary characters in a codominance case. It enabled us to study how the construction of a link between reality and a probabilistic model is influenced by the evolution of the representations adopted by the students, particularly in passing from ER to EM (the representation of the phenomenon through schemata, graphs, symbols, concepts drawing on different branches of learning). The lack of such evolution can either hinder the modelling activity or induce an improper borrowing of representations generally used to analyse other phenomena.

THE TEACHING EXPERIMENT

The teaching experiment involved two VIII grade classes: the first located in the North of Italy with 20 students, the second located in Cataluña (Spain) with 26 students. It is important to remember that, in Italy, in the lower secondary school, Mathematics and Sciences are taught by the same teacher; in this case the teacher has been the same since grade VI. In the Spanish classroom the situation was the same. Both class teachers belong to the Genoa Research Group in Mathematics Education. In both classes the didactical contract implied the argued production of hypotheses and the comparison of such hypotheses through discussion. Particularly the students were familiar with various experiences of mathematical modelling of physical and natural phenomena. (see Boero & Garuti, 1994 and Boero et al.,1995) In the curriculum of the Genoa Group Project the study of Genetics represents the core of Sciences teaching in seventh grade and it is connected to Mathematics through the introduction of the probability model. At the beginning of the activity great attention is given to the students' conceptions on transmission of hereditary characters, and Mendel's Laws are reconstructed through the guided reading of the essay presented by Mendel to the Naturalistic Society in Brno in 1865 (original title: *Versuche über Pflanzen-hybriden*). Teachers greatly emphasise the fact that Mendel (differently from previous scientists who had already recognised phenomena of dominance and segregation) did not limit himself to a qualitative description of the problem but, by means of mathematical concepts, developed a theory that allowed him both to quantitatively describe the problem and to deepen the biological interpretations. In particular the relationship between the analysis of the experimental frequencies and the probabilistic model is carried out, referring both to the data collected by Mendel and to the flipping of two identical coins, as representation and simulation of the phenomenon (the genes become the sides of the coin) and as prototype for favouring the students' conceptualisation. The only difference between the two classes refers to the forms of representation used: the Italian students developed their own form of representation (essentially tree graphs, keeping always a strong link with a more iconic representation of the phenotype) and ended up in the use of pairs of letters to denote the genotype; while the Spanish students learned to use double entry tables, the so called "Punnett square", to represent the possible combinations of genes.

THE TASK: A PRIORI ANALYSIS

The Mendel's surprise

The Mirabilis Jalapa is an ornamental plant of which two varieties are known, one producing only white flowers and the other only red flowers. So this plant gave Mendel another excellent chance to study the effects of crossing over an isolated and flashy characteristic such as colour. Mendel prepares two groups of plants: one with red flowers only and the other with white flowers only. Then he crosses the two types of plants, waiting for seeds production. The next season all the seeds are planted and he waits for the plants to grow and for the flowers to bloom. At the blooming Mendel finds out that all the plants are covered with pink flowers. The original characters, white and red colour, had disappeared and seemed to have mixed up like water and wine. Like in the previous crossing studies Mendel decided to cross the pink plants. The following year some of the new produced plants carry pink flowers, some red and some white flowers.

- *When Mendel realises that the second generation produces pink, white and red flowers, he is able to calculate the percentage of each colour. According to your opinion, which are the percentages forecast by Mendel and why?*
- *Supposing you were Mendel, how would you interpret the observed results of the crossing?*

In the classical case studied by Mendel, the *Pisum sativum* case, one character is dominant on the other and after the crossing of two pure lines, for instance yellow seeds peas (YY) and green seeds peas (GG), in the next generation the so called hybrids (YG) show only yellow seeds plants, and in the subsequent generation the probability to obtain yellow seeds plants is 75% (three possible combination out of four: YY, YG, GY, while the probability to obtain green seeds plants is 25 %, one combination out of four, GG). This makes sense provided that the distribution is uniform, i.e. no gene is 'favoured' with respect to the other (in the case of the coins this corresponds to the hypothesis of their equality).

The case of *Mirabilis jalapa* represents a case which Mendel's first law cannot be applied (Law of dominance: *In a cross of parents that are pure for contrasting traits, only one form of the trait will appear in the next generation. Offspring that are hybrids for a trait will have only the dominant trait in the phenotype*). To correctly understand the results, it is necessary to switch from this model at the phenotype level, i.e. the shown characteristics, to that at the genotype level, i.e. the genes combinations. In this way the dominance can be singled out as the hypothesis subjected to the applicability of the law.

An interpretation of the new phenomenon can be built through a new process of modelling: starting from the genotype level; finding an explanation of the fact that the hybrid is not similar to either parents, but shows a new character, the pink colour of the flower; and producing a probabilistic evaluation for the distribution of characters in the next generation.

The use of the previously described diagram (Fig.1) can help to better recognise different steps in the representation and interpretation of the phenomenon. In our example R (the part of reality which interests us) is the transmission of the flower colour, AR (the aspect of R, which we point out) represents the way in which the flower colour depends on the colour of the flowers of the previous generation, and finally ER are the colours of the plants of the different generations. All these elements can be obtained from the verbal description of the problem. At this point the crucial step in the modelling activity consists of the representation of ER through pairs of letters (W and R), which symbolise the genes that control the colour of these flowers. The representation of the situation AM is then realised highlighting the possible coupling either through a tree graph (with the edges labelled by the relative probabilities) or with a double entry table (under the hypothesis that all cells have the same probability). Referring to this representation the model M can be described through a formulation in probabilistic terms (in the second generation a flower has a 50% probability to be pink, 25% to be red and 25% to be white).

To be able to interpret the phenomenon we need, therefore, to go through the building up of a representation of the different elements (and of their respective relationships) which characterise the phenomenon. This representation needs to be abstract enough to facilitate the analysis of the phenomenon, but at the same time, it needs to remain context-related in order to allow a natural managing of the interactions with the phenomenon that are necessary to the elaboration, discussion and realisation of the model and its possible revision. This is a fundamental aspect from a didactical point of view. The problem of the *Mirabilis jalapa*, proposed to the students one year after studying Genetics in VII grade, represents a challenge for them. The results of the crossings seem to be in contrast with Mendel's hypothesis: it is a case of codominance, that the pupils have never met before, in which the presence of two different genes determine an aspect of the character different from that of the parents. Furthermore, the fact that the pink colour can be interpreted as a mixture between white and red can make some students reconsider those pre-mendelian hypotheses strongly present in the class before introducing Genetics.

SOME OUTCOMES

An overall view of students' behaviour

- In the Italian class 13 out of 20 students produced a correct modelling of the situation and answered the first question (regarding the probability of the different colours). Only 4 students were able to formulate an explicit codominance hypothesis, the others limited themselves to describe the situation, without working out any conclusion.

- In the Spanish class 8 out of 26 students produced a correct modelling of the situation and answered the first question; in spite of this none of them suggested the idea of codominance.

Italian students' behaviour and evolution of their representations

All the students describe the situation through graphical representations or drawing flowers, or simply writing the names of the colours, reaching a schematic representation of the crossing, always at the phenotype level. They translate into images what is described in words in the text. They still are at ER level and they should pass to EM level. At this point the students' behaviour differentiate. Some students (7 out of 20) stop here (Fig. 2) and try to answer the first question, in some cases applying a probabilistic reasoning. As an example Giulia writes: *"Mendel will find equal percentages for pink, red and white flowers, because the three colours have the same probability"*. She extends the uniform distribution hypothesis to the colours, while this hypothesis is only valid for the pairs of genes. In other cases the answers are not connected to any probabilistic reasoning, but to personal conceptions. Sara writes: "*I think that Mendel foresees this percentages: 25% pink flowers, and the remaining white and red, because pink is not pure, so it is probable that in the first experiment the pink flowers are less than those of pure colours*". In this case the student is not guided by any model, but by an idea of purity that prevails. These students are not able to interpret the results of these crossings as Mendel did and only one student from this group shows surprise. Luca: "*If I were Mendel I wouldn't be surprised of the absence of white and red, but certainly more surprised of the coming out of pink colour. According to Mendel the characters are inherited from father or mother. By the way, the mixture theory came out during discussions in class. Often it happened to hear that in a family the father was dark-haired, the mother blond and the son brown-haired. But this never came out in Mendel. The explanation that I could give to myself if I were Mendel is that since the genes are both present, they appear mixed up*".

The rest of the students (13 out of 20), after describing the situation exactly in the same way as their classmates, associate a pair of letters, representing the pair of genes, to the flower colour and this allows them to connect themselves to the learnt theory, to find out the possible combinations and to calculate their probability (2/4, 1/4, 1/4 or 50%, 25%, 25%). The transition from an iconic representation (still at ER level) to the one using letters, consistent with the studied theory, (we are now at EM level), allows them to recognise the pink flower as a hybrid and to correctly calculate the percentage. (Fig.3). Andrea writes: "*The percentages are 50% pink, 25% red, 25% white. It's what Mendel thought because the pink flowers are normal hybrids, even if the colours are mixed up to form the pink colour (RW), the genes are equal to any hybrid. When crossing to different pure lines (WW and RR), I obtain a hybrid (RW). Then, crossing the hybrids, I will get two hybrids, one pure white and one pure red, out of four"*.

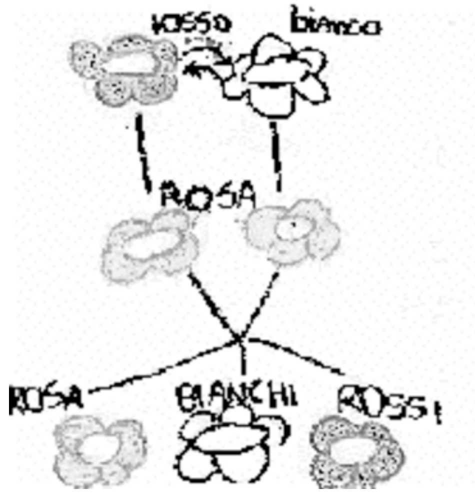

Fig.2

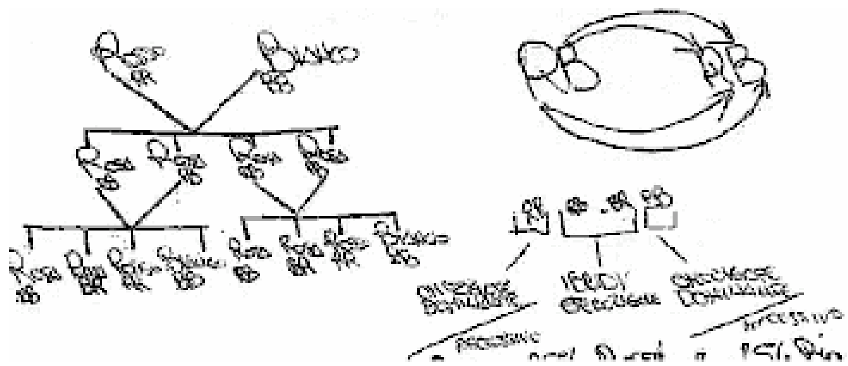

Fig. 3

The second question concerns the interpretation of the results of the crossing of the *Mirabilis jalapa* according to Mendel's theory. In order to be able to do so it is necessary not only to produce a model for the *Mirabilis jalapa* (passing from ER to EM), but also to model the case studied by Mendel (*Pisum sativum*) and compare the two. They are indeed different realities that can be interpreted by the same model, but the different results need to be interpreted (probability distribution 1/4, 2/4, 1/4 vs 1/4, 3/4). In the frame of our theoretical tool we need to pass from AM to AR. Only four students out of the thirteen of this group are able to formulate a codominance hypothesis and, again, the representation helps them. These students produce a graphic representation also for the known case of *Pisum sativum* and compare the two representations, realising in this way that those elements, showing in the case of *Pisum sativum* the dominant character with a 75% frequency, correspond to the 50% of pink flowers added up to the 25% of one of the other two colours in the case of *Mirabilis jalapa.* This observation allows them to make the codominance hypothesis. As an example Laura writes: "*In his previous experiments Mendel obtained 25% of recessive and 75% of dominants, but included in that 75% there was also a 50% of hybrids not visible. In this case the hybrids show not the dominant colour but the pink colour It seems that white and red can show up together, so there is neither dominant nor recessive, and this means that you can obtain not two but three types* (phenotypes). *In this way it is easier to understand what is hybrid and what is* *pure*”. Davide explains: “*The pink colour is like the red with a "mutation". This "mutation" has been created in the past or it is an effect that happens when the dominant characters are close to the recessive ones*”. The other students do not produce any hypothesis consistent with the results of the crossing and none of them produces a representation of Mendel’s experiment to be compared with the case under study.

COMPARISON WITH THE SPANISH STUDENTS

We recall that Spanish students had learnt to use the double entry table in order to represent the results of the possible crossing. Twelve out of 26 students do not utilise the learnt representation in order to model the situation, but apply the probabilistic model to the three different colours of the flowers, giving the same arguments as the Italians students. Six out of 26 students make the hypothesis of a gene controlling the pink colour, without realising that this hypothesis must be logically refused (it is not possible that the first generation plants, the ones with red and white flowers, carry a gene responsible for the pink colour, since they are pure). They pass from ER to EM since they use letters to represent the pairs of genes, they lean on the learnt representation, the Punnett square, but they stop at this level: they are not able to interpret AM, aspect of model, in relation to the real situation. As an example Josè writes: *Surely the second generation parents will be carriers of the pink gene, and that is the reason why the pink flowers come out. Let’s draw a table* (See Fig.4). *I would say that Mendel foresees that 0% will be red, 0% white, 25% pink and 75% carriers of red and white*”. We can see that what the student writes is contradictory with what is written in the text of the task; he is driven by an initial hypothesis, applies a learnt representation, but he does not control at all the meaning of what his modelling says (he cannot say which colours are the flowers of the remaining 75%). 8 out of 26 students give a correct prediction of the percentages of any type of plant, they use the double entry table fluently and establish causal relationships between pairs of genes (genotype) and aspect of characteristics (phenotype), passing from AR to AM and vice-versa. Nevertheless none of the students introduces an interpretation in terms of codominance.

	Y	R
B	BV	BR
R	RV	RA

Fig. 4

CONCLUSION

The epistemological concept of model, as represented in Fig. 1, was used to describe the process of progressive schematisation of a complex situation that needed to re-construct the probabilistic interpretation of the phenomenon under scrutiny. The concept was useful both in order to detect the points of that process, where the contact between reality and schematisation was lost by some students, and in order to understand some differences between two classes, where semiotic tools for schematisation had been introduced and used in different ways. Implications for teaching of modelling in the case of probabilistic models concern the need that students learn: to build and use different kinds of graphical representations (from those very near to the phenomenon, to those more suitable for calculation); and to keep under constant control the relationships between such representations and reality.

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KEY TRANSITIONS IN COUNTING DEVELOPMENT FOR YOUNG CHILDREN WHO EXPERIENCE DIFFICULTY

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This paper explores the Counting development of Australian children participating in the Early Numeracy Research Project who were identified as low-attaining using an individually administered assessment interview and a research informed framework of growth-points. The progress of Grade 1 and Grade 2 children who participated in an intervention program was compared to children who did not. Results suggest that the intervention was more effective for Grade 1 children, but that the effectiveness of the intervention seemed to depend on the growth point transitions children needed to make.

BACKGROUND

Counting is not only an everyday 'survival skill', but provides a basis for the development of number and arithmetic concepts and skills (Baroody & Wilkins, 1999). Although children need to develop more powerful strategies, being able to count a collection of about 20 items enables young children to solve many of the numerical problems they encounter. Learning to count collections is therefore an important development in mathematical learning. However, there is a group of young children who have difficulty developing this knowledge. These children are in danger of being "left behind" and of not benefiting from the curriculum provided in the regular classroom.

Teachers argue that it is often difficult to help children who have been *left behind* in the classroom. Most teachers do not have adequate time to single out children for significant periods of individual instruction. However, the children in danger of being *left behind* need opportunities to accelerate their learning; regular instruction that targets their individual needs. This is the purpose of intervention programs.

As part of the Early Numeracy Research Project (ENRP, Clarke, McDonough & Sullivan, 2002), a large scale project conducted in Australia from 1999-2001, an intervention program entitled *Extending Mathematical Understanding* was developed for Grade 1 (six year old) and Grade 2 (seven year old) children who were being *left behind* in their number learning. This paper explores the effects of the intervention program on Counting development, and insights gained about difficult progressions in Counting knowledge.

KEY GROWTH-POINTS IN LEARNING TO COUNT

As part of the ENRP, a research-based framework of six growth-points (see Figure 1) was created to describe the key developments, during the first three years of schooling, of children's counting knowledge. Similar to the work of Wright (1998), the ENRP Growth Points are concerned with children's production of number name sequences. However, the ENRP Growth Points focus also on children making the count-to-cardinal transition in word meaning described by Fuson (1992a) so that they are able to think about the number sequence to solve problems. The growth points do not describe children's use of counting strategies in addition, subtraction situations. These strategies are described in ENRP growth points pertaining to the addition and subtraction domain.

1. Rote counting: *Rote counts the number sequence to at least 20.*

2. Counting collections: *Confidently counts a collection of around 20 objects.*
3. *Counts forwards and backwards from various starting points between 1 and 100; knows numbers before and after a given number.*
4. Counting from 0 by 2s, 5s, and 10s: *Can count from 0 by 2s, 5s, and 10s to a given target.*
5. Counting from x (where x>0) by 2s, 5s, and 10s: Can count from x by 2s, 5s, and 10s to a given target.
6. Extending and Applying: *Can count from a non-zero starting point by any single digit number, and can apply counting skills in practical tasks*

Figure 1. ENRP Counting Growth-points

For some young children, the progression to *counting collections* (2) and *counting forwards and backwards from various starting points* (3) is prolonged or difficult. These growth-points relate to two of the counting levels described by Fuson (1992b), the *Unbreakable List Level*, and the *Breakable Chain Level*. These levels describe the development that occurs in order for children to *count collections*, or *count forwards and backwards* by ones. The *Unbreakable List Level* involves the number name sequence being broken into individual words, which are used in counting by relating each number word to a perceptual item to be counted (Steffe, von Glasersfeld, Richards, & Cobb, 1983). Children begin to relate the last word counted to cardinal meanings for the group of counted objects (the cardinality principle). They can then use *count-all* strategies to add two numbers.

The *Breakable Chain Level* involves children being able to start saying the number word sequence from any number word. They eventually use this ability in combination with an embedded cardinal-to-count transition in word meaning to add by a more efficient *counting-on* method, in which counting to determine the final sum begins with the first addend number word, instead of beginning the count from one.

These two levels, as they relate to *counting collections* and *counting forwards and backwards*, are not only important for children's counting development, but are also important for the development of numerical problem-solving strategies. It is the progression to these growth-points that is difficult for young children *left behind* in Counting.

IDENTIFYING AND ASSISTING CHILDREN *LEFT BEHIND* IN COUNTING

As part of the ENRP, all children took part in assessment interviews conducted by their teacher at the beginning and end of each year (March/November). The interviews were coded to determine the growth points each child reached in nine areas of mathematics, including Counting. The processes for ensuring the reliability of scoring and coding are outlined in Rowley and Horne (2000).

Table 1 shows the percentage of Grade 1 and Grade 2 children in ENRP trial schools who reached each of the Counting Growth Points in March 2000. These data enable the children *left behind* in Counting to be identified.

The distribution of children's counting ability across the growth points demonstrates a wide range in understanding, and highlights the challenge for teachers to cater for the range of abilities in classrooms. Further, the results suggest that a number of children being *left behind.* Eleven percent of Grade 1 children were not yet able to count a collection of 20 items, even after one year at school, and three percent of Grade 2 children were yet to develop this knowledge. A further 22 percent of Grade 2 children who could not yet count forwards and backwards by ones beyond 100 were also in danger of being *left behind* their peers and faced with a curriculum with which they could not adequately engage in order to learn successfully.

Counting Growth Points (March 2000)	Grade 1 *(n=1505)*	Grade 2 *(n=1544)*
0. Number names	5	1
1. Rote counting	6	2
2. Counting collections	56	22
3. Counting forward/backward by ones	15	14
4. Skip counting by 2, 5, 10 from 0	16	47
5. Skip counting by 2, 5, 10 from *x*	2	13
6. Extending and applying	0	1

Table 1: Percentage of Trial School Grade 1 and Grade 2 Children in 2000 Who Reached Each of the Counting Growth Points.

In order to assist the Grade 1 and Grade 2 children, who were being *left behind,* ENRP trial schools could elect to implement an intervention program. Twenty-one of the thirty-five schools elected to do so in 2000. The intervention program, *Extending Mathematical Understanding* (EMU) comprised daily 30-minute sessions for between 10 and 20 weeks, depending on the progress of students. Specialist teachers worked with groups of three or four students or with individual students. The program was not remedial in nature, but was built upon constructivist learning principles (see, e.g., von Glasersfeld, 1989). Children were engaged in experiences that required 'hard' thinking, and were required to reflect upon their activity and articulate what they had learnt and how they had learnt. The specialist teachers were trained to provide intensive instruction and feedback that was directed to the particular learning needs of each child.

Typically, each EMU session was structured to include 10 minutes of counting and place value activities, 15 minutes of rich problem solving activities (often with an addition, subtraction, multiplication or division focus), and 5 minutes reflection on the key ideas explored. Counting activities included: estimating the numerical value of large collections and then counting these collections; grouping items to emphasise the tens structure and meaning of number names using materials such as *ten frames;* using number charts and vertical number lines to emphasise patterns in the number sequence; and prediction games using the constant function on calculators, with justified argument required for the predictions.

COUNTING PROGRESS OF THE CHILDREN *LEFT BEHIND*

To determine the effect of the intervention program on the development of children's counting knowledge, the Counting growth of children in ENRP trial schools who participated in an EMU Program (the EMU Group) was compared to children in ENRP trial schools who had reached the same Counting Growth Point in March, but who did not participate in an EMU Program (the Comparison Group). Of particular interest is whether the EMU Program was more effective than the regular classroom program in assisting children to count collections, and whether children were able to advance further to counting forwards and backwards by ones from any number. These are important developments in Counting knowledge for those *left behind*.

Table 2 describes the growth for Grade 1 children who were not yet able to count a collection of 20 items at the beginning of Grade 1 (March).

Low-attaining Students	Counting Growth Points (November)				
	1	2	3	4	5
	Rote count	Count Collections	Forwards & backwards	Skip-counting	Skip-counting from X
EMU students (*n*=18)	0	50	6	33	11
Comparison Group (*n*=120)	10	58	13	18	1

Table 2. November 2000 Counting Results for Grade 1 Low-attaining Students who in March were not yet able to count collections (expressed in percentages)

The results indicate that Grade 1 children in the EMU group made better progress in Counting than the comparison group. There are 4 points to note. First, all children in the EMU group were able to count collections of 20 items by the end of the year. Second, half of the EMU group were at least able to count forwards and backwards by the end of the year, compared with about one-third of the comparison group. Third, children in the EMU group were more likely to progress further than *counting forwards and backwards* and be able to *skip-count*, or skip count from various starting points. The final point to note is that at least half of the children in both groups did not progress beyond *counting collections*. It appears that progressing beyond *counting collections* to Growth Point 3 is a prolonged transition for many children, even when children participate in a daily intervention program.

Low-attaining Students	Counting Growth Points (November)					
	0	1	2	3	4	5
	Not apparent	Rote count	Count Collections	Forwards / backwards	Skip-count from 0	Skip-count from X
EMU Group (*n*=9)	0	11	33	22	22	11
Comparison Group (*n*=19)	10	5	42	16	21	5

Table 3. November 2000 Counting Results for Grade 2 Low-attaining Students who in March were not yet able to count collections of 20 items (expressed in percentages)

Table 3 below shows the results for the 28 Grade 2 children who were not yet able to count collections of 20 items in March. Considering that there were more than 1500 Grade 2 children in the cohort, it is clear that these 28 children were being *left behind*.

A surprising result was that not all Grade 2 children in the EMU group learnt to count collections, whereas all Grade 1 children in the EMU group did. All Grade 2 children in the EMU group learnt to rote count, and a higher proportion of the EMU group reached each of the subsequent growth-points. As with the Grade 1 children, a large proportion of each group did not progress beyond *counting collections*. Overall, the children in the EMU group made better progress than the comparison group, but this was not pronounced. It is possible that the experiences provided by the EMU program were more effective for Grade 1 children who were not able to *count collections* than for Grade 2 children.

The results in Table 2 and Table 3 suggest that progression from *counting collections* to *counting forwards and backwards,* is prolonged for a large proportion of children, even when they participate in an intervention program that includes a focus on counting development. To explore this issue further, the growth of Grade 1 children who began the year being able to count collections was determined (see Table 4). This is the median growth point for Grade 1 children in March (*n*=1505).

Low-attaining Students	Counting Growth Points (November)				
	0/1	2	3	4	5
	Rote count or below	Count Collections	Forwards/ backwards	Skip-count from 0	Skip-count from X
EMU Group (*n*=21)	0	33	38	19	10
Trial School Group (*n*=756)	1	38	17	38	6

Table 4. November 2000 Counting Results for Grade 1 Low-attaining Students who in March were able to count collections of 20 items (expressed in percentages)

The results suggest that about one-third of these Grade 1 children did not progress to the next growth point by the end of the year. The progress of the two groups was similar, suggesting that there was little advantage for the children who participated in the intervention program. This highlights the difficulty of this progression for some children.

Low-attaining Students	Counting Growth Points (November)					
	0/1	2	3	4	5	6
	Rote count or below	Count Collections	Forwards / backwards	Skip Count from 0	Skip Count from X	Extending / Applying
EMU Group (*n*=37)	0	46	11	29	14	0
Trial School Group (*n*=276)	2	25	12	45	15	1

Table 5: November 2000 Counting Results for Grade 2 Low-attaining Students who in March were able to count collections of 20 items (expressed in percentages)

Table 5 shows the progress of Grade 2 children who were able to count collections at the beginning of the year. The median growth point for Grade two children in March was skip-counting (Growth Point 4).

The results show that at least one-quarter of Grade 2 children did not progress from *counting collections* to *counting forwards and backwards* by the end of the year. This indicates again the difficulty of this progression for some children. Further, the results suggest that Grade 2 children who began on Growth Point 2 were disadvantaged by participation in the EMU program with respect to Counting. Indeed, children in the comparison group were more likely to progress to at least Growth Point 3. It may be that

Grade 2 children who can count collections, but who are not yet able to count forwards and backwards from varying starting points, need the broader type of learning experiences provided within the regular classroom, rather than experiences geared precisely to their next growth point transition. The regular classroom program for Grade 2 children is more likely to emphasise *skip counting*, including skip counting from different starting points. It may be that *skip counting* learning experiences help children to construct knowledge about patterns in the number sequence that also assists the progression to counting forwards and backwards by ones. Children are less likely to have these counting experiences within the EMU program if they have not yet reached Growth Point 3. This suggests that children at a particular point in their counting development may be disadvantaged if the learning experiences in which they engage are too narrow.

Grade 2 children's progress in counting may also be influenced by their learning in other mathematical domains. For example, it could be that children who participated in the EMU program were being *left behind* in other mathematical domains. To explore this issue further, the percentage of Grade 2 children who had reached Growth Point 2 in March and who were below the median growth points for Grade 2 in Place Value, Addition and Subtraction and Multiplication and Division were calculated (see Table 6).

Low-attaining students	Place Value	Addition & Subtraction	Multiplication & Division
EMU Group (n=37)	90	72	70
Comparison Group (n=276)	65	42	40

Table 6. Percentage of Gr 2 children who had reached Counting Growth Point 2 in March and were behind in Place Value, Addition & Subtraction and Multiplication & Division.

Clearly, a greater proportion of children in the EMU group were behind in the other domains. This may explain why the children made less progress in Counting than the comparison group: their difficulties were broader in scope, and their learning may have been concentrated in other areas. Further research is necessary to investigate the interaction between these domains in the mathematical learning of children who are being *left behind* their peers.

CONCLUSION

The results reported in this paper suggest that the EMU Program was effective for increasing the counting knowledge of children in Grade 1 and Grade 2 who could not yet count collections of 20 items, although it appears that intervention in Grade 1was more effective than intervention in Grade 2. The extent of children's learning seemed dependent on the growth-point transitions children needed to make. This suggests that children may need different types of experiences, depending upon their age and level of understanding. It seems that Grade 1 and Grade 2 children who have reached the same growth point in Counting do not gain equivalent benefit from equivalent experiences.

The results also suggested that the progression to *counting forwards and backwards by ones* was prolonged for a sizeable proportion of the Grade 1 and Grade 2 children. More research is needed to explore the nature of this progression and how teachers can assist children to make this transition. The type of experiences offered by the EMU program did not seem to advantage children in this case. It appears that children learning to count forwards and backwards by ones beyond 100 benefited more from the broader type of experiences and interactions offered by the regular classroom program.

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ON LINE PROFESSIONAL COMMUNITY DEVELOPMENT AND COLLABORATIVE DISCOURSE IN GEOMETRY [1]

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In professional development, attention to both collaboration and critical thinking in the various interactive socialization processes of teaching practices should be essential strategic elements in a formative environment. This study presents contributions from the teleinteractive dynamic established in a virtual environment for the critical development of the professional content knowledge of the mathematics teacher.

INTRODUCTION AND BACKGROUND

We believe that technological mediation - through ICT[2] – has a crucial influence on the comprehensive development of domains of Professional Content Knowledge (PCK) in mathematics (Escudero and Sánchez, 2002). But how can criticality be developed in teachers' actions? Is it possible to get to reach a level of development with metacognitive characteristics in continuing education courses?

Within this frame, our aim is to ascertain how teachers use their own linguistic and cultural capital critically (Blanton, 1998) in a short course through the Internet: ***(1)*** by describing and analyzing how teleinteractive collaborative (Wood, 2001) communication develops (NCTM, 2001), in different communicative spaces, and ***(2)*** by reflecting about critical aspects of PCK.

In the realm of teacher thinking, we consider that critical reasoning (Kuhn, 1999) is a personal process built over a long period of time. It has aspects of a nature that is *declarative* (metacognitive realm), *procedural* (strategic) and *of principles* (epistemologic). In such a process we have considered the four teaching actions concerning criticality (Smyth, 1991) in relation to teaching – *descriptive, inspirational, confronting, reconstructive* - all of which do take place and develop in the course of the dynamic. We started from the knowledge originated in the teacher (Llinares, 1998, 2000). We considered affective aspects of the use of teachers' knowledge in the teaching situation, their professional perspectives, self knowledge and the development of criticality. We consider that there are three aspects- *geometric, strategic-interpretive and affective-attitudinal* - of PCK at work on an on-line course. In the *geometric* aspect there are the teachers' meanings and reflections about the process of thinking mathematically. As aspects of *strategic-interpretive* knowledge, we have considered the reflections on learning, instruction and interactive processes. As *affective-attitudinal* aspect, we have considered attitudes regarding both the teachers' and the students' own learning, awareness and socioculturization, flexibility, judgment, fairness and values in teaching.

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[2] Information and Communication Technologies (ICT)

METHODOLOGY

Our virtual geometric environment was structured around 6 hipertextual axes: **(a)** the activity that forces the teachers to review their own knowledge on geometry and professional activity, **(b)** observation of the role that everyday life plays in the different geometric activities, **(c)** reconstruction of cognitive processes of students in class, **(d)** determining the material for each subject, **(e)** organized summary of contents, and **(f)** continuous evaluation. The geometric content was developed in eight units (flat areas and forms, spatial relations, geometric constructions, angles, symmetry, similarities and reasoning).

A group of mathematics teachers worked on a 50 hour Internet Course over six months using a range of online interactive tools: e-mail, geometric "authentic" tasks, self-regulation inquiries at the end of each lesson, discussion forum and distributed chats. The focus of this research was to analyze written mathematical communication in deferred or real time (Giménez and Bairral, 2001). Thus, semi-structured interviews, text writings and videotaped experiences of teachers' classrooms were used to recognize changes-in-action in geometry by means of their a-synchronous productions.

Two case studies were conducted. In this study we are applying a semantic analysis on them using Webs of virtual interaction (Giménez, Rosich and Bairral, 2001). The first case stems from the information obtained in the course of the pilot study (2000) and the other one uses the information inherent to the development of the second course (2001). For the ***presentation of data*** we have: **(1)** selected among the teachers' contributions in different communicative spaces; **(2)** coded characteristic elements that are representative within the contributions; **(3)** exemplified and analyzed the contributions ***(a)*** by identifying and describing actions of criticality, ***(b)*** by analyzing other elements in course of critical reasoning (Kuhn, 1999), and ***(c)*** by identifying aspects (Llinares, 2000) of PCK; and **(4)** confronted the process in function of what has been observed (discussion forum, chats, ICQ messages, various tasks, e-mail, questionnaires and the researcher's diary).

RESULTS

In order to explain the characteristics of the observed reasoning, we are basically using the discourse of one of the teachers (figure 1) to explain our observations and results. See, as an example, how Joana uses a metaphor to emphasize that changing the shape means changing paradigms, thus showing some traits that are characteristic of a *reconstructive action of criticality* (Smyth, 1991). We refer to PCK elements comprehensively (Joana step 3b) by pointing out the following aspects: *geometric* (analyzing, structures, valuing communicative processes), *strategic-interpretive* (by attributing values, questioning and exemplifying about daily life, mathematics aspect and mathematics task) and *affective-attitudinal* (enhancing the negotiation of meanings). As for the *geometric* aspect, we stress a greater awareness on the use of models and content integration. With regard to the *strategic-interpretive* aspect, we observed an involvement and discussion on the part of teachers regarding their own approaches and their contribution to their mates' approaches.

> Joana (forum 3): ¨What **type of activities** can we propose to divert the focus of definitions to the understanding of the concept? **Integration** between the different branches of mathematics have to be given a priority. For **example**, how about working on the concept of area by associating it to the factorizing of a number?...¨

Regarding the *affective-attitudinal* aspect, we stress an open vision regarding the teaching-learning process of the teachers themselves and that of their students; no unnatural recalling of memories and the reflection about episodes in relation to their own personal-professional history as well as the importance of all this on the teachers' professional practice. See Carla's text.

> *¨I have been a **teacher for 25 years**. For me, these 6 months of course count as 6 years. I feel like an ¨old friend". **I have** often found myself discussing certain contents with my colleagues and remembering you, **I feel like writing to you to talk about it**. I don't feel embarrassed talking about my doubts and I feel safe with your directions¨ (15th August 2001, by e-mail)*

The development of a virtual teachers' professional development community in geometry was revealed by enabling for a collaborative discourse with the following characteristics: (a) constant expressions of respect and trust among all participants, (b) open expliciting of ideas and principles and personal knowledge as well as exchange of experiences about the practice, (c) dialogic immersion of the researcher in the group enhancing and incorporating teachers' contributions on the site (http//www.ufrrj.br/institutos/ie/geometria/), as well as other strategies, and (d) acknowledgment of a critical teleinteractive dynamic that favored -through a good thought provoking participation from the tutor- cognitive unbalance among the teachers and enabled a process of personal commitment and critical thinking with signs of PCK improvement.

> Joana (forum 7): ¨In unit 5 **I answered** the 1st[a] question, but **I am still in trouble** about filling in the chart. I have been doing a good deal of thinking about it today. At first I thought about giving a quick answer, but I just won't. **I want to make a more careful** analysis of the activities and concepts involved. I keep thinking, and **what do you people think**?

The flexibility of virtual work dynamic favors the constant negotiation of meanings (Horvath and Lehrer, 2000) with teleinteractions continually sustained or reconstructed in each specific educational context. So it is that, at a discussion forum (Joana step 4) there is some reflection on the cognitive process situated in the task proposed and the teacher even gets to set forth to the group her need for personal geometry study. Besides, an analysis on the communicative nodes in the debate interaction showed that even contents of a metacognitive nature are shared:

> Ana (final questionnaire by e-mail): ¨**I really** liked to participate in that way. I remember having considered Antonio's statement as being rather traditional or conservative, I don't know for sure. At school **we are often alienated** from what is going on around us, and we often don't challenge what happens. And **in our mathematics classes we** sometimes miss the chance to see the world in a more open way

Steps 1 - 2: *Interview3 (I3)* - teacher comparing geometric tasks

"... Shape influences the way we are hanging a picture, etc. That's a very interesting thing to explore and discuss, for instance the shape of boxes. What shapes do boxes usually come in? They are parallelepiped. That's very usual. Does this happen by chance? No, that's for things to fit in better, more adequately. So, when we try to change, it's not just a matter of changing the shape, we're changing the paradigm.

Step 3 a: Identifying and describing critical actions (Smyth, 1991)

Reconstructive action: reflecting on what I consider important, pedagogically, what could make a difference if changes were to be introduced in teaching.

Step 3 b: Analyzing elements in the process of critical reasoning (Kuhn, 1999) *Declarative:* it attributes values, discusses, plans

Identifies Principles: it compares what is real and day to day life, it negotiates meanings.

Step 3 c: Identifying aspects of professional content

Geometric: it analyzes structure, it values communicative processes.

Strategic-Interpretive: it attributes values, it challenges and exemplifies about what is daily life, what is mathematical, and about mathematical task.

Step 4: Confronting and analyzing the process

Forum8: "I think that my difficulty in responding to task 2 is in the way it was set out. As a matter of fact, that's a comment that I think is worth doing, The tasks that are being proposed here in this course have an interesting way, different from the usual, which is often: question and answer. That is also new to me…"

Figure 1. Example of part of the analysis of teacher Joana's discourse

A constantly activated feedback process was promoted. It emphasized different professional aspects in different discursive spaces: **e-mail** (it fosters more discussion with regard to individuals or small groups, the researcher acts according to the teacher's request and has a local control over the process, the teacher transforms the tasks hypertextually, teleinteraction is temporarily unlimited); **chat** (joint reflection with an action triggering a more immediate response, better controlled by the researcher); **questionnaires** (personal information, integration and attention to evaluation processes); **discussion forum** (space for greater integration, identification of the interaction aspects used, it acknowledges the value of social and motivational components, value of unit contents and communicative spaces, space allowing for a more flexible response over time, where the researcher has a global and local control of the process, need to trust the group in order to contribute collectively, socialization and sustained discussion of the practice); **informative links** (identification of curricular elements that favor integration, distributed knowledge over the development of tasks); **interview** (reasoning located in the task, immersion provoked and conducted by the researcher); **ICQ messages** (expounding and clarifying personal doubts and other professional exchanges); **experience narratives** (development of situated knowledge and, regarding context, attention to cognitive processes and to the complexity of the educational process).

Besides, teleinteractions could be grouped in four key formative moments (Goffree and Oonk, 2001): *(1)* sensitivity and prior acknowledgment of the team members, *(2)* acceptance and trust for the teaching negotiation, *(3)* critical adaptation and accommodation of practical knowledge, and *(4)* collaboration and awareness regarding theoretical orientation.

Initially, plenty of descriptive directions were given and a cooperative contact was made.

> (Joana, mail C5.2): ¨ **textbooks** gives a static view of angles and what **I liked about the activities is that they talked about** a dynamic view of angles, when they speak about opening. **But books** talk like that because later, they are going to talk about figures, polygons and although **those angles can be larger or smaller depending on the figures**...in the figure it is static.¨

The attempts at reflection are always related to very concrete situations (Llinares, 2000), even when the difficulties presented are centered in strategic elements about the learning process.

> (Antonio, Ev1a-c) [about difficulty in geometry] break the learning pace through formulae and memorization by heart on the part of students and **introduce the content through** studies where they would have to build their own knowledge, even if it had to be guided. [complement] **The reason for** breaking the teaching method the students were used to, is that the group isn't mine, since I am working as a counsellor. I asked the teacher of the group for some extra time and permission to be able to elaborate with them a little studying along with teacher Cámara's activities [facility]. After breaking the ice with the students in connection with the new subject, **I felt that** learning happened in a natural way.

In phase, or moment, 3, there are plenty of elements of confrontation and critical adaptation of professional knowledge. At this moment the teachers reason and discuss about their issues, or their colleagues'. There are also traits of reconstructive inclusions in their PCK.

> (Joana, interview 40) ¨... Discussing with students why these shapes appear so little. I think that this is **a discussion that has to be made, because** there, you are talking about a geometrical aspect, I mean, because why is it that something, a room shaped like a circle, why is it less functional? **You get to have a different architecture**¨.

And they get to express their process of situated professional development critically, with their discourse supported by examples of their geometry-in-practice performance.

> (Antonio, questionnaire S4 nº 6.1c): [following a researcher's suggestion] ¨As for the towns being aligned, the first answer **may be**, because the airport at the middle point would be in the smaller town, **no problems**. **It meets the demands** of the question, doesn't it? For the second, **I think that** if it is equidistant from the 3 towns, it doesn't make sense to have them aligned, does it?¨

Teacher's cooperation when establishing relations between events and facts in the process of working in the environment and the awareness towards a theoretical orientation of a local character, in which the different practices and personal experiences gain importance, are examples of professional discourse at formative moment 4:

> (Antonio, interview162): ¨...**When I asked** these questions, **I was really** in doubt: **do I have to** set forth a question for the student to notice some properties before using the software or for him to start dragging [stress] directly in the program and find out? **Because one thing is** letting the student work with the CABRI in these matters and him/her starting, deducting [stress], to draw a vertex, stretch a segment **to see** what is going on there¨

In this teleinteractive process, PCK develops with the use of professional knowledge situated in concrete situations in teaching. It is constructed by integrating characteristics of discourse and interactive processes in each discursive space; and it is a distributed knowledge, that is to say, it is managed hypertextually and personally by the teachers themselves. It can be continually socialized in each communicative space of the environment or in another formative context along the process of professional development.

CONCLUSIONS

It is not only the use of the ICT that will bring about improvements in the educational process, but rather the way in which they are integrated in the formational scenario and how an adequate use for them is developed that will serve the needs of certain approaches in teaching-learning. The communication among teachers that was established as a linguistic capital from the different communicative spaces and the different interactive processes in the constitution and consolidation of a ***collaborative professional discourse community*** have been noteworthy characteristics of the set up environment.

Virtual environment, in spite of presenting a course with restricted features and limited in time, revealed important for an attitude that would boost the teachers' work and awaken the need in teachers for a constant investment in their careers. The availability and multiplicity of tasks and suggestions (reading of articles, possible projects etc.) constituted a unique trait of the environment, allowing to involve each teacher in studies of personal interest in which the professional time devoted to action and reflection was significantly considered. Inter-subjective teleinteractions and professional knowledge shared in the course of the tasks of the environment allowed us to perceive a potential of the virtual environment to integrate, from the personal interests of the people involved, elements that were initially external to the environment (other books, other teachers, links to Webs, participation in events, etc.), which substantially enriched the PCK development process in all teachers.

Formative moments enabled us to ascertain that in the dynamic of virtual work, PCK: ***(a)*** develops with the use of professional knowledge situated in concrete situations in the teaching process, ***(b)*** is constructed integrating the characteristics of discourse and the interactive processes in each discursive space; and ***(c)*** is a distributed knowledge, that is to say, managed hypertextually and personally by the teachers themselves, as they can be continuously socialized in each communicative space in the environment or another formative context along the process of professional development. In spite of all this, we have to admit that the possibilities for in depth critical thinking with theoretical contrast are not easy in a short virtual course like the one that was carried about.

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ONE LINE PROOF: WHAT CAN GO WRONG?

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Having an ability to appreciate, understand, and generate proofs is crucial in being able to evaluate students' mathematical arguments and reasoning. As such, the development of this ability in perspective teachers is imperative. This study examines the work of a group of preservice elementary school teachers in their efforts to generate one-line proofs on closure statements. We provide a framework that allows us to carry out a fine grain analysis of students' proofs and also provides a tool for diagnosis and remediation.

Mathematics occupies a privileged position among the sciences as the discipline that is most pure, most exact. The facet of mathematics that is most directly responsible for facilitating this honor is that of proof. The mathematical proof provides the certainty that is demanded in a field where precision and exactness is the currency of practice. A well-constructed deductive proof offers humans the purest form of reasoning to establish certainty. As such, proof is an important part of not only mathematical practice, but also of mathematical learning and teaching (Hanna, 1989).

However, research has repeatedly shown that proofs and the ability to understand and generate proofs is difficult for students in general (Hoyles, 1997) and for preservice elementary school teachers in particular (Barkai, Tsamir, Tirosh, & Dreyfus, 2002; Martin & Harel, 1989; Simon & Blume, 1996). This may be due, in part, to the fact that proficiency with proof requires the coordination of a number of competencies, identification of assumptions, and organization (or tracing) of logical arguments. The teacher, as the person who establishes the expectations and norms of a mathematics classroom, plays a crucial role in development of such competencies (Yackel & Cobb, 1996). Furthermore, the teacher's own aptitudes are requisite in the evaluations of students' arguments and mathematical reasoning (Barkai et. al., 2002). Therefore, it is imperative that, as difficult as it has proven to be, the ability to understand and generate proofs be instilled in perspective teachers.

Having said that, however, it may be beneficial to the achievement of such goals to find instances where the rigours and demands of exactness required in a proof are mediated by a reduction in the length of the proof. As such, this study examines preservice elementary school teachers' efforts to generate one-line proofs. We examine the abilities of the participants in generating such proofs as well as provide a framework for the analysis of their efforts. The framework, which allows for a fine grain analysis of the participants' work, also gives insights into the complex coordination of competencies that is required even for the writing a very short proof.

THE STUDY

Participants in this study were preservice elementary school teachers (n=116) enrolled in a course "Principles of Mathematics for Teachers", which is a core course in a teacher education program. One of the issues that wove itself through all the topics of the course was the need for support of mathematical claims. Extensive discussions and exercises were aimed at helping students understand when and where a general argument, or a proof, was needed and when an example was sufficient.

During the course the participants were exposed to the concept of closure as part of the discussion of number systems. The formal definition was provided – a set is said to be "closed" under operation if and only if for any two elements in the set, the result of the operation is in the set. Further, a variety of examples of sets closed or not closed with respect to certain operations were provided and a variety of problems in which students had to prove or disprove closure were posed. This included the invitation to prove or disprove claims such as *even numbers are closed under addition*, *multiples of 5 are closed under multiplication*, *rational numbers are closed under division*, *prime numbers are closed under addition*, among others.

In this study we analyze two questions that sought a written response from the participants:

> *(Q1) The set of perfect squares is closed under multiplication. Prove the statement or provide a counterexample.*
> *(Q2) The set of odd numbers is closed under multiplication. Prove the statement or provide a counterexample.*

MATHEMATICAL ANALYSIS AND FRAMEWORK

For our purposes, we considered the "ideal" solutions (that is, proofs) of these statements to be:

(Q1): Let a^2 and b^2 be any two square numbers.
Then, $a^2 \square b^2 = (ab)^2$ *which is itself a square number.*
(Q2): Let $(2m+1)$ and $(2n+1)$ be two odd numbers.
Then $(2m+1)(2n+1) = 4mn+2m+2n+1 = 2(2mn+m+n)+1$, *which is itself odd.*

However, the generation of such seemingly simple and short proofs is deceivingly intricate, requiring an appreciation of the need for, and the coordination of many skills (see Figure 1).

First and foremost is the recognition that a proof is indeed required for the purposes of establishing the truth of a statement. From a mathematical perspective, such a requirement is obvious. The establishment of the validity of a statement requires the treatment of the statement in general, as opposed to the examination of a few particular cases. Once a need for a proof has been established, the students then need to be sensitive to the fact that treatment of the general case requires the selection of some form of representation. Representations play a crucial role in mathematics; they are considered as tools for communication, as tools for symbolic manipulation, and as tools that promote and support thinking (e.g. Skemp, 1986, Kaput, 1991). Furthermore, the choice of representation is often linked to students' understanding of the content (Lamon, 2001).

However, the recognition that a representation is needed is not enough. The students must select one that is both correct and useful for the purposes of generating a proof. For Q1(above), for example, choosing to represent the two square numbers as X and Y is in itself not incorrect, but for the purposes of generating a proof, it is completely useless. A much more effective (and natural) representation of two square numbers is a^2 and b^2.

Once such a representation is established, the students must then be able to work with it. That is, they must be able to perform correctly any manipulations necessary to transform the expression into the form that clearly represents the nature of the number. In the example of Q1 such a manipulation is not onerous. Q2, however, requires much greater adeptness with algebraic manipulation in order to mould the expression into one that clearly expresses its inherent ‘oddness’. There is an assumption in this last sentence, though.

STUDENTS’ RESPONSES

A complete and correct proof was provided by 19% of the participants for Q1 and by 37% of the participants for Q2. However, it is not our purpose to quantify student’s responses. Instead, we have organized students’ incorrect responses according to the framework provided above. What follows are exemplars of this organization.

<u>(Not) Recognizing the need for a proof</u>

Among participants who did not attempt a proof we recognize two kinds of arguments. One is a narrative style that reiterates the statement, at times explaining what is to be proven. For example:

> (Q1): *The set of perfect squares is closed under multiplication because no matter what 2 perfect squares you multiply together, your answer will always be another perfect square.*

Another is justification with a single numerical example. For instance:

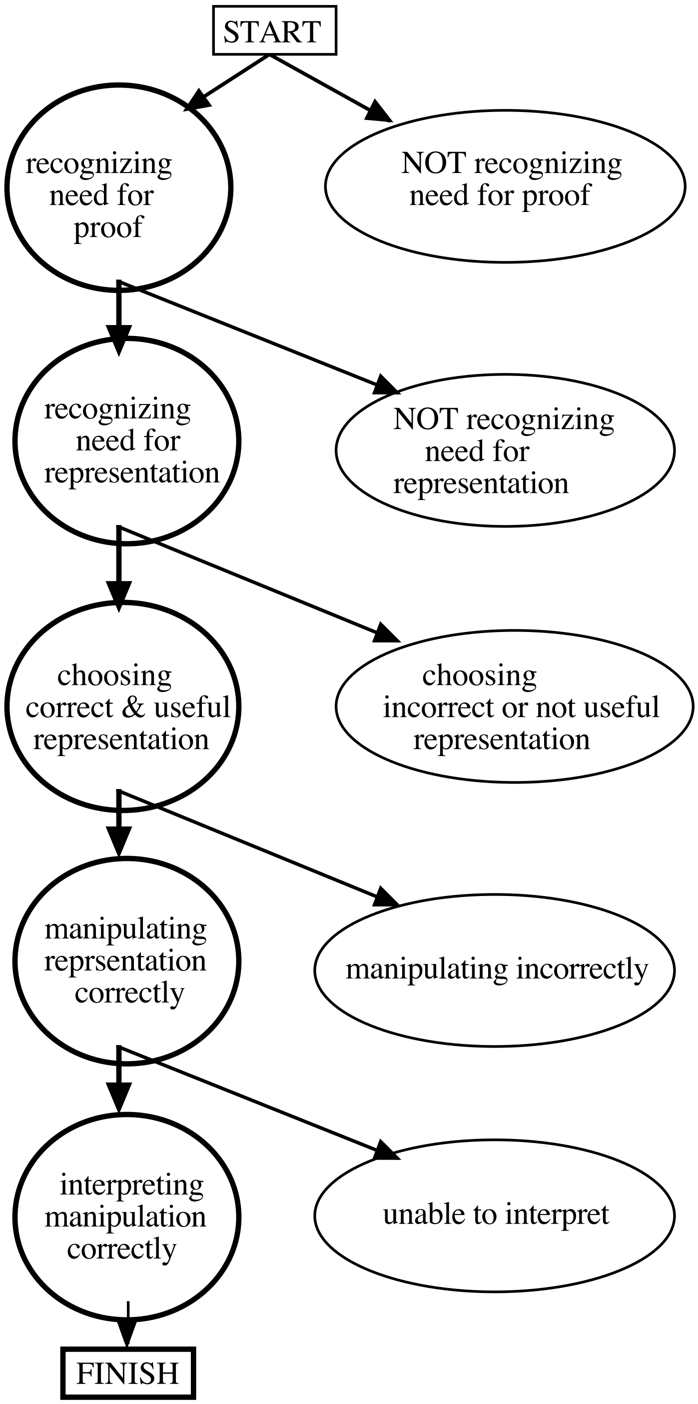
> (Q2): *{1,3,5,7,9, …}*
> *3×5 =15*
> *The product of 2 odd numbers is an odd number.*

<u>(Not) Recognizing the need for representation</u>

We believe that a general argument in this case requires a representation of the objects in question. However, some of the participants justified their decision of closure with an inductive argument, as exemplified below.

> (Q1): *The set of perfect squares is closed under multiplication because if you multiply 2 of the numbers inside the set you get another perfect square as a result.*
> *4×25=100*   *4×36=144*   *25×36=900*
> $\sqrt{100} = 10$ $\sqrt{900} = 30$ $\sqrt{144} = 12$
> *etc.*

> (Q2): *O={1,3,5,7,9,11,13…}*
> *5×7=35*   *11×9=99*   *15×13=195*
> *True, the set of odd numbers is closed under multiplication. Any two odd numbers multiplied together will result in an odd number.*



The phrase *clearly expresses* assumes that the students are able to interpret the result of their manipulation as representative of what they are aiming to show. This is the last step in the proof process. The students must be able to constantly interpret their manipulations in order to know what they have found, and when they have found it.

The left side of the diagram in Figure 1 represents the steps towards a complete and correct proof. The right side represents the potential obstacles at every step.

Figure 1. Pathway towards (and digression from) a one line proof

We distinguish justification by a single example from justification by a series of examples. In the former case a student may believe that one example is sufficient. In the latter case we recognize an attempt to build an inductive argument, which identifies students' empirical proof schemes (Harel & Sowder, 1998). While empirical verification is very useful in clarifying the problem, it is only a preliminary stage towards a proof. However, it is very common for students to prefer an empirical argument over any sort of deductive reasoning (Hoyles, 1997).

<u>(Not) Providing a useful representation</u>

Once the need for representation has been recognized, it is essential to choose a correct and useful representation. Examples below demonstrate students' choice of representation that is inappropriate for the task at hand.

> (Q1): *Let X equal any whole number, then* X^2 *equal a perfect square.*
>
> X^2 □ X^2 *also a perfect square?*
>
> X^2 □ $X^2 = X^4$ *, yes* X^4 *is a perfect square so the set of perfect squares is closed under multiplication.*
>
> (Q2): *k*□*W,* □ *k+1*□*W, k+3*□*W*
>
> $(k+1)(k+3) = k^2+3k+k+3 = k^2+4k+3 = (k^2+4k)+3$
>
> *adding 3 makes the # odd, so the set of odd #s is closed under multiplication.*

The above response to Q1 does not satisfy the generality; while X^2 is an appropriate representation for a square number, using it for two different numbers compromises the argument. In the response to Q2, *k+1* and *k+3* represent odd numbers only if *k* itself is even, but this constraint was omitted. Furthermore, consideration of consecutive odd numbers compromises the intended generality of a proof.

<u>(Not) Manipulating the representation correctly</u>

The ability to chose a correct and useful representation is a necessary condition, but not sufficient. The next step is the ability to manipulate the chosen representation successfully. Unfortunately, as shown below, manipulation of algebraic symbols presented an obstacle for some participants.

> (Q1): *True. X,Y are whole numbers*
>
> $a = X^2$ *= perfect square,* $b = Y^2$ *= perfect square*
>
> *Prove:* $ab = \text{number}^2$
>
> $ab = X^2Y^2 = XY^2$ *= perfect square*
>
> (Q2): *a and b and c are whole #s,*
>
> *(2a+1)* □ *(2b+1) = 2ab+1, this is odd, since ab must be whole (set of whole #'s closed under multiplication), and any whole # times 2 is even, so plus one must be odd. So any odd # multiplied by any odd # equals an odd #.*

Note the perfect structure of the argument in the second example. Unfortunately, it is based on an incorrect symbolic manipulation.

(Not) Interpreting the manipulation

The ability to manipulate algebraic expressions pays off only if a learner is able to interpret the result of such manipulation. However, the data show, that several students were on the brink of completing the proof, but did not recognize it. That is to say, they were not able to interpret the result of their manipulation. Consider for example the following response to Q2:

Let (2n+1) and (2m+1) represent odd numbers.

$(2n+1)(2m+1) = 4mn+2n+2m+1$

$2 \mid 4mn+2n+2m+1; \quad 2 \mid 2mn+n+m+1$

This student has chosen a useful representation and also manipulated it correctly. However, this correct manipulation is followed by rather random symbol pushing, and no conclusion with respect to the closure of the set in question is presented. It appears that this student wasn't sure how to proceed in interpreting her manipulation.

CONCLUSION

The purpose of this study is twofold. One, it provides a framework for analyzing short proofs related to the notion of closure. Two, it shows viability of this framework by providing an analysis of students responses. For examining the work of students that did not complete a proof, the framework assists in identifying the obstacle that threw the student "off track". In such it provides an avenue for remedial instruction.

The study demonstrates that the concept of closure was generally well grasped. That is to say, the majority of students understood that they were expected to show that the product of two perfect squares is a perfect square, and the product or two odd numbers results in an odd number. What is considered as proper "showing" is a more general issue, extensively discussed in prior research (e.g. Harel & Sowder, 1998).

However, it is troublesome that what prevented some students from completing the proof was not their understanding of closure, or appreciation of the need for a proof, but a poor ability to choose an appropriate representation or inability to manipulate the chosen representation. The latter draws the focus from undergraduate teacher education and invites regression to skills of simple algebraic manipulation. Lack of competence in these skills presents an obstacle not only for correct manipulation, but also for interpreting the meaning of manipulation, that is, the ability to represent the manipulated expression in a desired form.

We believe that experience with one-line proofs is a valuable tool for sharpening the proof skills. The content of closure provides appropriate grounds not only for generating these proofs, but also for appreciating the role of useful representation. Furthermore, we suggest that the framework that we developed is appropriate for analyzing a variety of short proofs related to number properties. Future research will determine the scope of applicability of this framework.

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DESCRIPTIONS AND DEFINITIONS IN THE TEACHING OF ELEMENTARY CALCULUS

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In this paper, we discuss the (potentially positive) pedagogical role of intrinsic limitations of computational descriptions for mathematical concepts, with special focus on the concept of derivative. Our claim is that, in a suitable approach, those limitations can act for the enrichment of learners' concept images. We report a case study with a first year undergraduate student and place this in a broader empirical and theoretical context.

INTRODUCTION

Giraldo (2001) defined a *theoretical-computational conflict* to be any pedagogical situation with apparent contradiction between the mathematical theory and a computational representation of a given concept. We have argued that the approach to the concepts of derivative and limit can be properly designed to prompt a positive conversion of theoretical-computational conflicts to the enrichment of concept images (Giraldo & Carvalho, 2002a, 2002b, Giraldo, Carvalho & Tall, 2002). In addition, we distinguish between a *description* of a concept, which specifies some properties of that concept and the formal concept *definition*. Descriptions commonly employed in mathematical teaching include numeric, graphic and algebraic representations that individually involve limitations that do not fully reflect the mathematical definition. We will argue that suitable use of these limitations can stimulate students to engage in potentially enriching reasoning.

RESEARCH FRAMEWORK

Our theoretical position is grounded in the theory of concept image and concept definition (Tall and Vinner, 1981). The *concept image* is the total cognitive structure associated with a mathematical concept in an individual's mind. It is continually being (re-)constructed as the individual matures and may (or may not) be associated with the *concept definition* (the statement used to specify the concept). Barnard and Tall (1997) introduced the term *cognitive unit* for a chunk of the concept image on which an individual focuses attention at a given time. Cognitive units may be symbols, representations or any other aspects related to the concept. A rich concept image should include, not only the formal definition, but many linkages within and between cognitive units.

In a strictly formal standpoint within a formal system of rules of inference, a mathematical object is perfectly characterized by its definition, so that the definition completely exhausts the object and, in this sense, a mathematical object *is* its definition.

However, the theory of concept image suggests that the teaching of a mathematical concept must include different approaches and representations to enable learners to build up multiple and flexible connections between cognitive units. The three main forms of representation for functions, numeric (tables), algebraic (formulae) and geometric (graphs), each have their own limitations. A table can have only a finite number of entries that does not necessarily determine the whole function, a formula may be presented in a way that does not mention the range or domain and a physical graph can only approximately present the information required for the formal function. Each of these is a *description* that lays stress on certain aspects of the concept, but also casts shadows over others.

The literature reveals examples of the *narrowing effect* (described in Giraldo, Carvalho & Tall, 2002) of the students concept image as a result of focusing only on certain aspects, particularly computational ones. For instance, Monaghan et al (1993) reported that students using Derive to study calculus explained the meaning of the expression $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ by replacing $f(x)$ with a polynomial and referring to the sequence of key strokes to calculate the limit. Research in Brazil (Abrahão, 1998; Belfort & Guimarães, 1998) reveals many instances of students accepting numeric and visual output of technology without query, even when software limitations produce results that clearly conflict with their prior knowledge.

However, we believe that the limitations of the various descriptions need not necessarily lead to a narrowing of the concept image. On the contrary, such limitations have a potentially positive role. Sierpinska (1992), for example, affirms that the awareness of the limitations of each of the forms of representation, when they are all meant to represent the same concept, is essential for the understanding of the concept of function. We believe that the emphasis on theoretical-computational conflicts can lead not to a narrowing, but to the enrichment of learner's concept images.

THE CASE OF THE DERIVATIVE

One of the most widely used descriptions for the derivative concept in elementary calculus courses is the following: The gradient of the function $f(x)$ at x_0 is the slope of the tangent line to the graph of *f* at the point $(x_0, f(x_0))$. However, as Vinner (1983) and Tall (1989) observe, the notion of tangency in students' concept images is often strongly linked to geometry problems about the construction of tangent lines to circles. The approach to those problems focuses on global geometric relationship of the curve and the line, particularly, on the number of points of intersections. Thus, the idea of being tangent—to "touch" in one single point—is featured in opposition to the idea of being secant—to "cut" in two points. This leads to a narrowing of the concept image of a tangent that is not consistent with the notion of tangent in infinitesimal calculus.

An alternative to the traditional approach, based on the notion of local straightness, has been proposed by Tall (e.g. Tall, 2000). This is grounded on the fact that the graph of a differentiable function 'looks straight' when highly magnified on a computer screen. Tall claims that local straightness is a primitive human perception of the visual aspects of a graph, deeply related to the way an individual looks along the graph and apprehends the changes in gradient, that is suitable as a cognitive root for the concept of derivative.

However, the notion of local straightness is also a *description* for the concept of derivative, since it comprises limitations that can trigger theoretical-computational conflicts. For example, floating point approximations made by computer software may cause unexpected results, as the one shown on figure 1. It displays the process of local magnification of the curve $y = x^2$ (in the neighborhood of (1,1)) run by software *Maple*. Until a certain stage of the process, the curve does look like a straight line, but afterwards (for graphic window ranges lower than 10^{-6}) it becomes polygonal.

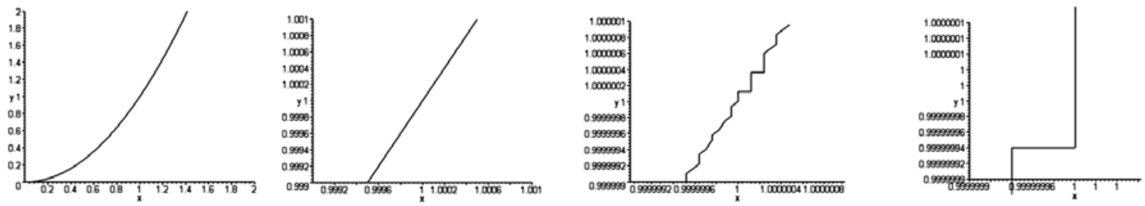

Figure 1. A theoretical-computational conflict observed on the local magnification process.

Theoretical-computational conflicts like this are deeply related to the fact that a finite algorithm is being used to describe an infinite limit process. These intrinsic limitations may lead to narrowed concept images, if computational descriptions are over-used. Nevertheless, our hypothesis is that a suitable approach, where theoretical-computational conflicts are not avoided, but highlighted, can prompt the positive conversion of these same limitations: they can make for the enrichment of concept images, by underlining that *the notion of limit, in the sense of infinitesimal calculus, is beyond computers accuracy, no matter how good it is, or, more generally, any finite accuracy.*

A CASE STUDY

The experiment reported in this section is part of a wider study, in which six first year undergraduate students from a Brazilian university were observed in personal interviews dealing with theoretical-computational conflict situations from different natures. We summarize the responses of one of the participants, Antônio (pseudonym) to four interviews, concerning the concept of derivative (translated from Portuguese).

Interview 1: Participants were given a few general questions concerning their conceptions about functions, continuity and differentiability.

Antônio was asked how could he decide whether a function is differentiable or not, given the algebraic expression. He stated that a function would be differentiable if he could apply known formulae to evaluate derivatives. Afterwards, he was asked how he could decide about the differentiability if the graph of the function on a computer screen was given, instead of the expression. He stated that he would zoom the graph in to have a more careful view, but it would be impossible to be sure, as computers are not flawless.

Interview 2: Participants were asked to gradually zoom in the graph of the function $y = x^2$ around the point (1,1) using the software *Maple*, and simultaneously explain what they were observing. They would obtain screens similar to the ones shown on figure 1.

At the beginning, Antônio declared he would see something similar to the tangent straight line, as he zoomed in on the graph. When the software started to display a polygonal for the curve, he claimed that the computer was wrong, as this was not the expected result. After thinking for a while, he explained the computer's error:

Antônio: It's because the computer hasn't got idea what it's doing. It's kind of messing up the points. […] As the computer sketches the graph by linking the points and these points are results of approximations, so it links without thinking. It links the points, and whatever it gets will be the graph for it, as it doesn't know what goes on.

Interview 3: Participants were asked to zoom in the graph of the blancmange function around a fixed point using the software *Maple*, and explain what they were observing. The blancmange function is defined in the interval [0,1] as the sum of an infinite series of modulus functions and is continuous but nowhere differentiable (figure 2). However, a finite truncation of the series was being used to draw the graph so that the function displayed was non differentiable at a finite set of points, rather than everywhere. The students were familiar with the functions and its properties, as they had studied it previously on calculus lessons.

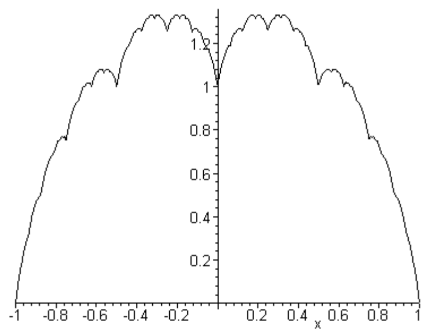

Figure 2. The blancmange.

Antônio started by explaining the construction of the blancmange function. He showed good comprehension of the process:

Antônio: […] You are taking a number and multiplying it by $1/2$, taking that one and multiplying by $1/2$, by $1/2$. So, it's a geometric progression with rate $1/2$. […] Then, it's the sum of a geometric progression. The sum of a geometric progression is a limit, then it converges to a point. […] Then each point there is a geometric progression, it's the limit of a convergent geometric progression. It's there. So you might say the curve is a sum of sums of geometric progressions. [he means the union of sums]. It's well defined.

He then started the process of local magnification and explained that, as the curve was not differentiable, the graph would become more wrinkled as he zoomed in. As the algorithm used a finite truncation of the series, it did not looked more wrinkled, as he expected, but quickly acquired a straight aspect. Antônio showed great surprise, and asked the reason for the unexpected result. After listening to our explanation, he commented:

Antônio: Oh, I see. You could sum a few more steps, but not until infinity.

After thinking for a few minutes, he proceeded, with increasing excitement:

Antônio: But it [the computer] can't make infinity. […] Hey! I think that nothing could make! […] It can't add until the infinite! There will be always an infinity missing. And nothing can represent the infinity, as a whole, but we can show that it goes to that place, that it tends to that. That's the infinite. […] It's impossible to represent it, not on the computer, not on a sheet of paper, and not in anything else! The computer only represents things that a human being knows.

Interview 4: Participants were asked to investigate the differentiability of the functions:

$$v_1(x) = \begin{cases} x\sin(1/x), & if\ \ x \neq 0 \\ 0, & if\ \ x = 0 \end{cases}$$ and

$$v_2(x) = \begin{cases} x^2\sin(1/x), & if\ \ x \neq 0 \\ 0, & if\ \ x = 0 \end{cases}$$

For that purpose, they were given the graphs of the curves $y = x\sin(1/x)$ and $y = x^2\sin(1/x)$ sketched by Maple in a neighborhood of the point (0,0) (figure 3).

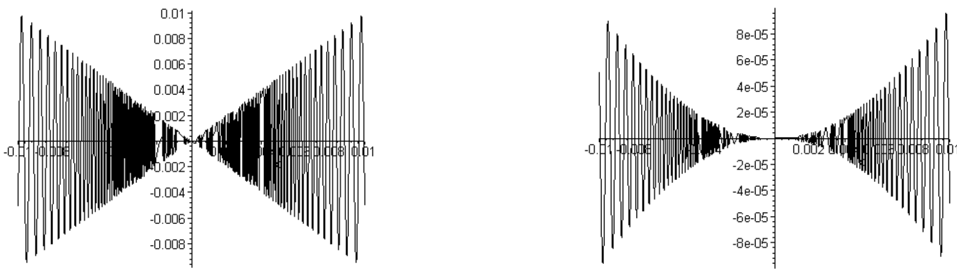

Figure 3. The curves $y = x\sin(1/x)$ and $y = x^2\sin(1/x)$.

Antônio said at first that both the functions should be differentiable, as the formulae he knew applied to the algebraic expression. He then started to zoom in the first graph around the origin, and the curve progressively looked more smudged. Antônio argued that again it should be due to an interpolation error, but the function v_1 should have a derivative. Afterwards, he repeated the process for the second graph. He commented:

Antônio: Look, when it gets closer to 0 it kind of tends to an area. But it's not. We can't see it, but it's the joining of two curves with […] the oscillation tends to zero, that's why we cannot distinguish.

We asked Antônio to conclude about the differentiability. He said:

Antônio: If it were $\sin(1/x)$, without anything else, they wouldn't be. They wouldn't be differentiable at 0, because $\sin(1/x)$ wouldn't be defined. But, for these functions the point (0,0) exists, so it's the joining of two curves there. […] Hey, wait a minute! I think v_1 is not [differentiable], do you know why? Because at 0, it's shaped by the joining of the two straight lines, $y = x$ and $y = -x$. […] When it gets closer to that point the parts approach each other within those lines! They will meet each other at that point, right? But it's a rough joining, it's kind of a corner. […] The other one [v_2] is different, it's a smooth joining. Here, the parabolas shape the curve, not the lines, that's the difference. For that reason, I think that one has a derivative and the other hasn't, v_1 has and v_2 has not. […] But I can't be doubtless sure just looking at the graph. Let me think.

Antônio concludes that the only way to be sure would be using the definition of derivative. He has a little difficulty in evaluating the limits, but reassures himself that it would be the only safe way, even if he could not do it.

DISCUSSION

Since the first interview, Antônio clearly expressed his preference for algebraic description. He states that the criteria for deciding about the differentiability of a function must be based on formulae. Moreover, he appears to be aware of the limitations of computational algorithms. Such mental attitude gave him means to quickly grasp the cause of the unexpected result on interview 2. In this sense, the theoretical-computational conflict involved (represented in figure 4) did not operate as an actual conflict, since it was almost immediately solved by the student.

THEORY The curve is differentiable therefore it can be approximated by straight lines.	COMPUTATIONAL DESCRIPTION: The curve does not look like a straight line when magnified

Figure 4. The theoretical-computational conflict in interview 2.

On the other hand, in interview 3 a theoretical-computational conflict (represented in figure 4), played a central role on Antônio's reasoning. In fact, Antônio's enthusiasm suggests the conflict actually triggered a new idea for him: *it is not possible to represent the concept of infinite by any physical means*. Moreover, he points out the reason for the impossibility: *infinity can never be attained*. The theoretical-computational conflict leads Antônio to grasp not only the limitations of the computational description, but of other forms as well; and to figure out a conceptual distinction between finite and infinite.

THEORY The curve is not differentiable, therefore it cannot be approximated by straight lines.	COMPUTATIONAL DESCRIPTION: The curve looks like a straight line when magnified

Figure 5. The theoretical-computational conflict on interview 3.

The theoretical-computational conflict involved in interview 4 was slightly more intricate than the ones observed previously, as figure 6 illustrates. In addition to that, the differentiability of the function could not be established by a careless use of the differentiation algebraic formulae, against Antônio's former dominant criteria. However, the confrontation of computational and algebraic descriptions—suggesting different conclusions—impelled him to follow another strategy: he states that *the differentiability of the function could only be doubtless concluded by means of the formal definition*.

THEORY One of the curves is differentiable and the other is not.	COMPUTATIONAL DESCRIPTION: Both of the curves seem to be differentiable

Figure 6. The theoretical-computational conflict on interview 4.

Antônio's mental attitude towards conflict situations contributed to the results reported in this paper. The outcomes of the interviews summarized above suggest that the conflict have acted as positive factor for the enrichment of Antônio's concept image of derivative and related notions. Nevertheless, other participants show quite different behaviors. In some cases, the conflicts do prompt students to engage into a rich reasoning. In others, the conflicts are barely noticed by students, as they are quickly solved (like Antônio did on interview 2). But some students very often cannot cope with theoretical-computational conflict situations at all. This obstacle can be due to a more general attitude towards technological devices, transcendent to their use as learning environments. The global results of the investigation in which this experiment is comprised are currently being analyzed. One of our aims is to understand more clearly in which situations conflicts do have a positive role for the enrichment of learners' concept images, in particular, in which sense and in which extent learners' previous attitudes and background determine that role.

The main goal of this work is to put forward an alternative model of approach, not purely grounded on formalism nor purely on imprecise representation forms. This propose does not mean to undervalue of the formalism, in relation to the imprecise. On the contrary, through the emphasis of limitations and differences, we intend to prompt the development of rich concept images, as well to stress the central role of the formal conceptualization on the construction of a mathematical theory.

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