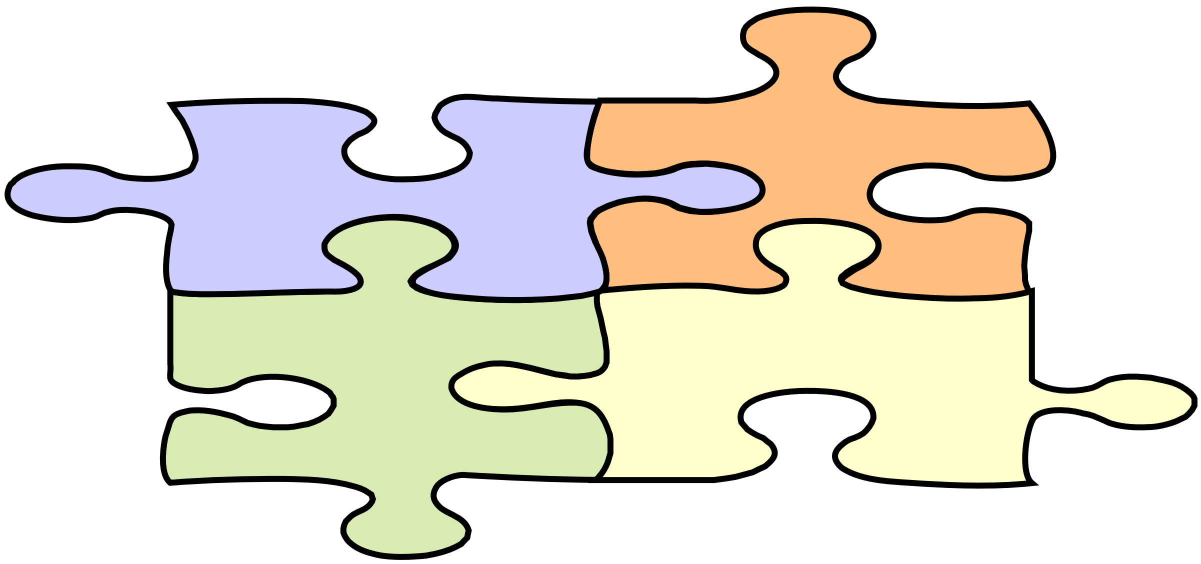
“Education is no substitute for intelligence. That elusive quality is defined only in part by puzzle solving ability. It is in the creation of new puzzles reflecting what your senses report that you round out the definition.”

Frank Herbert
Chapterhouse Dune



Algebra for Babies

Exploring natural numbers in simple arrays

Occasional Paper Five

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Acknowledgements

As Jerome Bruner once pointed out no project or book stands alone. When we read a book, for example, we are reading the minds of millions who have gone before the author.

Made of research and instructional projects, “Algebra for Babies: Exploring natural numbers in simple arrays” is no exception. Many minds joined. All deserve thanks.

First, Ervin and Jennifer Collier, parents of the three marvelous children involved in the thematic project, supported the effort to make sense of children’s understanding of algebra at an early age.

Then, Ingrid Fluellen, an educational specialist, co-taught the children giving them enrichment activities in the children’s section of a public library while an educational psychologist facilitated the algebra sessions with each child individually in a conference room. Begonia Collier, another educational specialist, provided additional instructional materials to enrich the children’s mathematical reasoning.

Mr. Rouse and the librarians at the University Branch of Jacksonville Public Library supported the project as well. They provided a conference room for the sessions and supportive materials including floor puzzles and computers.

Fanta and Jua Fluellen, outstanding young teachers in Philadelphia participated in the talk at the Research Forum. Jay and Loyda Fluellen provided resources and technical support.

Joy Cooke and Faraday del Rosa, the Instructor and Coach respectively for “Teaching algebra in elementary school” (Harvard University GSE) provided most of the activities explored in the thematic project side of this work. They gave feedback on reflections about the activities and responses to readings. They encouraged the online community to think more deeply about what counts as algebraic thinking in young children.

Finally, God provided all knowledge. Without God, “Algebra for Babies: Exploring natural numbers in simple arrays” would have been impossible. With God, kindergarten children engaged mathematics beyond the Sunshine State standards.

Jerry E. Fluellen Jr.
24 March 2008

Ingrid Fluellen, Educational Specialist, and three participants in the thematic project, ended with a post session pose. The children explored definitions of a square and growth patterns in simple squared natural numbers. They made arrays with concrete and symbolic materials. They examined abstract mathematical models of the arrays in T-Charts. All sessions for the thematic project were held at the University Branch of the Jacksonville, Florida Public Library.

Abstract

In 12 audio taped sessions, three kindergarten children engaged algebra in a thematic project. Toni, Asa, and Cornel had one-on-one lessons dealing with simple natural numbers, patterns, and relationships.

Along the way, each child studied one of Toni Morrison's *Who's Got Game* books for children to explore repetition patterns in well written literature. Then, each child "algebrafied" a Liberian folk tale and a Chinese folktale to explore number, pattern, and relationship in simple arrays. Finally, they engaged square numbers in mathematical models.

Children met the primary understanding goal set for the thematic project. They independently created arrays for simple square natural numbers one to four.

On the practitioner research side of the project, the inquiry was this: what happens when kids explore natural numbers in simple arrays? This descriptive inquiry guided an ethnographic like collection of data as the participant observer facilitated personalized workshops and audio taped each session for reflections (3 workshops per session times 12 sessions, 36 workshops in all).

In brief, the study found that three kindergarten children in a systematic algebra project made generalizations, added to the web of knowledge, used mathematical memory, and displayed flawed reasoning as a springboard for success.

In all, the paper presented what counted for learning and what counted for research—two themes of the 29th Ethnography and Education Research Forum at University of Pennsylvania.

What counts for learning

To develop a deeper understanding of how kindergarten children learn algebra, an educational psychologist designed a thematic project. Harvard Project Zero Research Center's teaching for understanding (TfU) framework organized the instructional plan around the National Council for Teaching Mathematics (NCTM) Standards and Principles.

Additionally, assigned readings and thought demanding tasks from "Teaching algebra in the elementary school" (a Harvard GSE World Wide professional development course) informed the thematic project.

On their first report card, for example, each of the three kindergarten participants earned at least "S" for meeting the Sunshine State Standard for mathematics. One earned a "S+" grade; thus, one is operating above proficient.

These are simple facts. No causal statement can be made about the thematic project and high grades.

In any case, as of session eight in the thematic project, each child could count natural numbers to ten in two ways--by ones and by twos. Each child could make arrays for 1x10, 10x1, 2x5, and 5x2. Each child could respond to a set of thinking routines used for literature including the "How many…?" question.

By session twelve, each child could make arrays for 1, 2, 3, and 4 squares, respectively and could define the concept of square. Each could recognize a growth pattern in square numbers from 1 to 4.

These were all above the Sunshine math standard set for kindergarten.

On the other hand, each child could not as yet generalize a definition of square shapes regardless of size or square numbers regardless of size. Nor could anyone of the children contrast a square and rectangle or square and a circle. None could generalize about growth patterns.

But as brain research suggests, because they made early neuronal connections about squares and square numbers now, they will have a foundation to draw on later in their mathematical development.

Finally, as a surprise, each child can identify repetition and alternation patterns in the real world and in Jacob Lawrence paintings. Toni can create alternating patterns as well.

In one of the final sessions, Toni lined up 16 chess pawns on a chess board. She made a row of 8 pawns alternating white and black pieces. She made a second row directly beneath it--again alternating colors. She created a 2x8 array, independently. That showed transfer of thinking, a "performance of understanding" from the perspective of Harvard Project Zero researchers.

At a meta level, Toni, Asa, and Cornel experienced Jerry Fluellen's power teaching prototype which connected standards, teaching for understanding, culture of thinking, and teacher inquiry into a 21st century whole. (Fluellen, 2007; Fluellen, 2006)

That meant the NCTM provided disciplinary standards. Harvard Project Zero Research Center provided the teaching for understanding framework used to plan the thematic project. Tishman, Perkins, and Jay provided the culture of thinking approach serving as the deep structure for the thematic project, and the teacher inquiry added value to reflections. All together the four interactive factors were power teaching.

Of particular note for exploring what counts for learning is the Harvard Project Zero (PZ) teaching for understanding framework used to plan the thematic project.

Teaching for Understanding Framework

To state what children need to know, what they do to show that they know, and how they will know they know, Harvard University Project Zero Research Center's teaching for understanding (TfU) framework provided a powerful curricular tool. It organized thinking around just five ideas: generative topic, throughline, understanding goals, understanding performances, and ongoing assessments.

A "generative topic" tells what is specific to a disciplinary standard and is of interest to both the students and the teacher.

The "throughline" states a theme repeated explicitly or implicitly all along the instructional sequence.

"Understanding Goals" and "Understanding Performances" frame the connections between desired disciplinary knowledge and what students do to show what they know as well as build new understanding.

"Ongoing assessment" means that almost every moment of a session yields insight about student understanding because each activity, makes thinking visible, representing understanding or misunderstanding. Ongoing assessment does not wait for summative assessment at the end. Instead, it collects and gives feedback all along whenever children make thinking visible. (Blythe, 1999)

Often, the feedback will be "value neutral." As Grant Wiggins explains, telling learners what they did with an eye on opportunities to <u>self adjust</u>—improves their performance of understanding. Value neutral feedback is more useful than evaluative statements with no opportunity to get better. (Wiggins, 1997)

The thematic project reflects these five TfU features.

Generative Topic

Algebra for Babies:
Exploring natural numbers in simple arrays

Throughline

"All learning integrates thinking and doing."
Peter Senge

Understanding goals

1. **Learners will understand how to make simple arrays that represent number, pattern, and relationship.**
2. **Learners will understand how to use their simple arrays to construct definitions of square and recognize growth patterns in square numbers?**

Note that the understanding goals cohere with Sunshine State Standards for mathematics in kindergarten. But more importantly, they speak to NCTM standards and principles for teaching algebra, PreK to 12, as well as a growing research base asserting that children must be taught to think algebraically early in order to grasp the full value of mathematics later in their lives.

Understanding Performances

To connect the dots among the two understanding goals, a series of lessons in 12 sessions provide a sequential exploration. The simple arrays become one kind of pattern. They model number, pattern, and relationship. Thus, the sessions engage the children in mathematical reasoning, particularly as they co-construct mathematical models of data collected on T-charts.

They are encouraged to make generalizations to deepen disciplinary understanding.

To paraphrase Susan Jo Russell, they develop mathematical knowledge; they create a web of mathematical ideas; they increase mathematical memory; and they use flawed mathematical reasoning as a springboard for new mathematical knowledge. (Russell, 1999)

This process spirals and is often recursive.

Along the way in this thematic project, students explore algebraic properties of multiplication and addition. They play with natural numbers (whole numbers that increase by one). Thus, they are introduced to algebraic thinking at an early age.

What follows is the sequence of instruction stating the key understanding performances.

- **Kids will construct summaries of Who's Got Game Toni Morrison books for children with thinking routines (What's going on here? What makes you say so? How many? What's at the core? What is the pattern?). The summaries will represent several narration patterns including problem-solution, main characters and minor characters, and event sequence. (Each child will study a different Toni Morrison book as a narrative entry point.)**
- **Kids will explore two ways of counting in a Liberian folktale *Two ways of counting to ten* and make a simple 2x5 array from food stuff to illustrate both counting natural numbers by 1 and skip counting by 2.**
- **Kids will construct simple multiplication arrays from food stuff, chess pieces, chess boards, and blocks. They will use an activity from Two of Everything Chinese folktale to add value to their opening experience of arrays with the Liberian folktale.**
- **Kids will use chocolate squares and other concrete objects to make simple arrays for squares of 1 2, 3, and 4. They will explore the properties of arrays and collect data about their observations and inferences, paying particular attention to simple arrays of square numbers.**
- **Kids define square and recognize growth patterns in simple square natural numbers.**
- **Kids will make equations to represent simple square natural numbers.**
- **Kids will summarize what they have learned about simple arrays in this project Vis a Vis the two understanding goals in a final performance of understanding. Note that the audio taped sessions become data collected about children's mathematical reasoning in algebra for the ethnographic study.**
- **In addition to observations and inferences about the mathematical reasoning of young children, down the road, Piaget's reflecting abstraction model will become a theoretical lens to analyze the data represented in the audio tapes covering the final assessment of understanding.**

Ongoing assessments

Given the view expressed in Grant Wiggins' article about the role of feedback and learning plus Robert Marzano's summary of research about the effect of feedback on student achievement and the view of Harvard Project Zero Research Center, ongoing assessment becomes particularly important. Telling students what they do and do not do--eye to eye with a description or reference to the desired quality of understanding (criterion) is at the heart of ongoing assessments. (Wiggins, 1997; Marzano, 2004)

Each item in the set of ongoing assessments listed below incorporates Grant Wiggins' idea that learning takes place best when the feedback is value free, examining responses with an explicit or implicit rubric describing the desired performance.

For examples, in the "Algebra for Babies…" project, ongoing assessment shows up in the following:

- **Recitations with the thinking routines during dramatic readings and creative dramatics games with the Toni Morrison books, Ruby Dee's Liberian folktale, and the Two of Everything Chinese folktale**

> **Note that kids work with a set of age appropriate thinking routines to seek patterns in the literature as well as with the activities algebrafying the literature:**
>
> 1. **What's going on here?**
> 2. **What makes you say so?**
> 3. **How many?**
> 4. **What's at the core?**
> 5. **What's the pattern?**
> 6. **What's missing?**
> 7. **Is that so?**

- **Recitations from the creation of arrays (concrete, symbolic, abstract)**
- **Reflections at the end of sessions eight through 12 (What did you learn? What surprised you? What new questions do you have?)**
- **Audio taped responses to most of the 12 sessions make thinking visible for further reflection**
- **Reflections on student performances of understanding in terms of mathematical reasoning about number, patterns, and relationships**

> Note that each personalized session for the children used Howard Gardner's MI approach as a bread and butter method for delivering teaching for understanding. Thus, each session had an <u>entry point</u>, <u>powerful analogy</u>, and <u>multiple representations</u> for deeper understanding.
>
> Also, the entire 3x12 project roughly followed entry point, power analogy, and multiple representations leading to achievement of the two related understanding goals. The project opened with each child studying a book from Toni Morrison's Who's Got Game series. These sessions became an entry point as children practiced responding to thinking routines including "How many?" "What's the pattern?" and "What's at the core?"

Algebrafying Ruby Dee's retelling of the Liberian folk tale Two ways of counting to ten introduced uses of arrays as well as skip counting by twos. It became a powerful analogy and created an analogy children could experience: an array is analogous to multiplying two factors to find a product or summing addends to find a sum.

Algebrafying the Chinese folktale Two of everything introduced representing number, pattern, and relationships in both arrays and a T-chart—a mathematical model. This set up the work with simple square natural numbers represented in arrays made from concrete materials, symbolic materials, and abstract materials, namely the T-chart as a mathematical model for the square numbers 1 to 5. All the final sessions, then, were multiple representations yielding deeper disciplinary understanding.

Consistent with NCTM standards and principles, kindergarten children were reaching for an understanding of number, pattern, and relationship in simple arrays and becoming more aware of simple algebraic properties of multiplication along the way.

In summary, two projects intertwine in "Algebra for Babies: Exploring natural numbers in simple arrays." On the one hand, a 12 session thematic project instructed kindergarten children in algebraic thinking. On the other hand, a teacher inquiry reflected on the thematic project and paid even more attention to the conference theme what counts for research.

What counts for research

What happens when children explore natural numbers in simple arrays?

At the core of this teacher inquiry is the idea of learning. It is not enough to say natural numbers are those whole numbers that increase by one or that a simple array visualizes a product and two factors. With a systematic thematic project in place, the question becomes what do children do?

Toni's work on a set of arrays at the end of the thematic project illustrates mathematical reasoning defined in terms of generalizations, web of knowledge, mathematical memory, and flawed reasoning.

She examined the model of one square, then drew arrays for 2 square, three, square, and four square independently. She was able to explain, for example, that 4x4=16. She wrote an equation on the T chart for 4 square as well as five square—one she did not represent in an array.

Her prior knowledge from the Chinese folktale set up these performances of understanding.

Mr. Haktak found a pot as he dug up the garden. He placed his purse with five gold coins in the pot and took it home to his wife. She dropped her only hairpin in the pot. She pulled out two. She also found two purses, each with five gold coins. A quick study, Mrs. Haktak figured they could make a lot of money by doubling and doubling and doubling the gold coins.

Toni had recreated the doubling rule with a game based on Mr. Haktak's pot. She put in one piece of candy in a round box (the pot) and got out two. She put in two pieces of candy and got out four. She predicted that three pieces of candy would yield six.

When she reached 5 x 2 = 10, she was able to independently create a 2 x5 array with candy pieces on a chess board.

She seemed to understand that doubling meant more of something and this set up her work with squaring.

The next round of Mr. Haktak's pot involved squaring instead of doubling. Toni put in one piece of candy in the box and predicted that two would come. In her web of knowledge, more candy comes out of the pot than what went in. But one square yielded just one. Her flawed reasoning became a springboard. She put in two pieces, four came out. She put in three pieces, nine came out. She put in four pieces, 16 came out. She went one step further and put in five pieces. 25 pieces came out.

To test her mathematical memory, she engaged the task of making drawings of squares one to four with arrays. Toni drew all four arrays accurately without help from the interviewer.

Lastly, she discussed a pattern of growth in square numbers 1-4 on a T-chart (mathematical model) and wrote an equation for five square (5x5=25).

From a performance of understanding perspective, Toni demonstrated what she knew—her mathematical memory based on generalizations and a web of mathematical knowledge.

So what counts for research is the systematic collection of data to address at least one question. In the case of Toni, Asa, and Cornel, the research examined what they did along the lines of mathematical reasoning, particularly about arrays of square natural numbers.

While the audio tapes revealed numerous examples of mathematical reasoning in literature, arrays, and mathematical models represented in t-charts, the less obvious observations rest in the realm of Piaget's reflecting abstraction process applied to analogical thinking as children engage algebra.

To what degree, for example, was Toni engaging empirical abstraction when she said a square piece of chocolate candy had equal sides? If she had said, the chocolate square was the same as a square on a chess board, would that have been reflecting abstraction? If she had said, squares and rectangles make equal angles, would that have been reflected abstraction? If she had said, the total number of degrees in a square equals the total number of degrees in a circle, would that have been metareflection?

Piaget's reflecting abstraction model will be the stuff of a future exploration.

Annotated Bibliography

Bay-Williams, J. (2001) What is algebra in elementary school. Teaching Children Mathematics (December).
Identifies themes from NCTM Principles and Standards.

Blythe, T. (1998). *The teaching for understanding guide.* San Francisco: Jossey-Bass.

A practical entry into the teaching for understanding framework emerging from research and practice at Harvard University Project Zero Research Center and classrooms around the world, this guidebook instructs readers in each of the components of the framework.

Fluellen, J. (2007).The Titmouse effect (power teaching in 2054—a meditation on the 2007 Urban Sites Conference of the National Writing Project). ED497503

A comprehensive presentation of the power teaching prototype completing its first year of development at an urban magnet school in the South, this paper suggests four interactive factors of educational reform with an eye on student achievement defined in terms of high stakes tests and Gardner's five future minds.

Gardner, H. (1999b). MI approach. In C. Reigeluth (Ed.) *Instructional-design theories and models: A new paradigm of instructional* theory. Mahwah, New Jersey: Lawrence Erlbaum Associates, Publishers.

An application of multiple intelligences to create a simple, yet powerful method for helping students to understand disciplines more deeply, the MI approach stands tall among new paradigm instructional design theories because it rests on a landmark theory not just a collection of studies.

_____. (1998). Melding traditional and progressive perspectives. In M. S. Wiske (Ed.), Teaching for understanding: Linking research with practice (pp 345-350).

Taking a performance view of understanding instead of schema perspective, the author argues that understanding might best be understood as a transfer of knowledge from one situation to a situation for which that knowledge is appropriate.

_____. (2006a). Multiple intelligences: New horizons. New York: Basic Books.

Gardner reviews and extends his 1983 theory of multiple intelligences.

_____. (2006b). Five minds of the future. Cambridge, MA: Harvard Business School Press.

Argues that five minds of the future need to be developed in the schools of today: disciplined mind, synthesizing mind, creating mind, respectful mind, and ethical mind. Asserts that most schools in our nation have not been developing these capacities in spite of a global need for such citizens.

Marzano, R., Pickering, D. and Pollock, J. (2004). Classroom instruction that works: *Research based strategies for increasing student achievement.* Association for Supervision and Curriculum Development.

Offers ten research based strategies for improving student achievement in a companion to the workbook applying the strategies to real classrooms.

Perkins, D. (1986). *Knowledge as design.* Hillsdale, New Jersey: Lawrence Erlbaum Associates.

A simple, yet powerful method for metacognition, knowledge as design serves as a tool of reflection for students of all ability levels and ages. This book details ways in which a human made object or idea can be discussed in terms of purpose, structure, model case, and argument (explanatory, evaluative, deep explanatory). In addition, this method for critical and creative thinking invites learners to go beyond the four features and invent one's own design when the occasion demands.

_____. (1995). *Outsmarting IQ: The emerging science of learnable intelligence.* New York: The Free Press.

Often overshadowed by the more popular multiple intelligences theory, this book presents a new theory of intelligence, namely, learnable intelligence. The author connects three kinds of intelligences: the traditional IQ, experiential intelligence, and reflective intelligence. The author argues that while native intelligence represented in IQ scores once seemed to be immutable, it can change significantly as the learner gains experience in a domain and practices strategies for reflection.

_____. (1998). What is understanding? In *Teaching for understanding: Linking research with practice*. Martha Stone Wiske, editor. Jossey-Bass Inc. San Francisco.

Presenting a new perspective in the cognitive development view of understanding, the author argues that schemas do not go far enough to capture understanding. From a performance view of understanding, a learner must create an intellectual product to show that understanding and build new understanding.

Perkins, D. and Unger, C. (1999). Teaching and learning for understanding. In C. Reigeluth (Ed.) *Instructional-design theories and models: A new paradigm of instructional theory.* Mahwah, New Jersey: Lawrence Erlbaum Associates, Publishers.

Presenting Harvard University Project Zero research center's teaching for understanding framework as a new paradigm instructional design theory, the authors argue that effective teaching includes a sound method of planning—one that connects generative topics, throughlines, understanding goals, understanding performances, and ongoing assessments.

Perrone, V. (1998). Why do we need pedagogy of understanding? In *Teaching* for *understanding: Linking research with practice.* Martha Stone Wiske, editor. Jossey-Bass Inc. San Francisco.

Providing historical context for improving education in the United States, the author argues that few schools teach for power and consequence. Most students do not get the kind of education that leads to literate citizens capable of solving or posing complex problems with created works.

Piaget, J. (1977). *Studies in reflecting abstraction.* Philadelphia: Taylor and Francis Group.

Providing a theoretical explanation of what it means to understand, the authors present a series of studies leading to the invention of a model—the reflecting abstraction model. Beginning with empirical abstraction the model suggests a spiraling succession of understanding at increasingly complex levels. Thus, empirical abstraction, reflecting abstraction, reflected abstraction, and meta-abstraction all explain a performance view of thinking and understanding in learners of all ages.

Reigeluth, C. Ed. (1999). *Instructional-design theories and models: A New paradigm of instructional theory.* Mahwah, New Jersey: Lawrence Erlbaum Associates, Publishers.

A compendium of approaches to teaching and learning, this book offers a range of methods (instructional design theories) to suit many classrooms in the nation.

Russell, S. (1999). Mathematical reasoning in the elementary grades in Developing mathematical reasoning 1999 yearbook. ED430 788.

Suggests that mathematical reasoning consists of generalizations, web of knowledge, mathematical memory, use of flawed reasoning as a springboard for new insights.

Wiggins, G. (1997). Feedback: How learning occurs. AAHE Bulletin, November V50 3, American Association of Higher Education.

Argues that value neutral feedback must do more than lead to student self assessment. Such feedback best leads to student self adjustment—making the project better.