Proceedings
of the
31st Conference
of the
International Group for the Psychology of Mathematics Education

PME 31

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Volume 1

Plenaries, Research Forums, Discussion Groups, Working Sessions, Short Oral Communications, Posters

Editors

Jeong-Ho Woo **Hee-Chan Lew**
Kyo-Sik Park **Dong-Yeop Seo**

The Korea Society of Educational Studies in Mathematics

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INTRODUCTION

PREFACE

Since the first meeting in Utrecht, the Netherlands in 1977, PME Conference has become one of the most important research conferences in mathematics didactics all over the world. It is a great honor and pleasure for us to organize the 31st Conference of PME in Seoul, Korea.

The theme of PME31 Conference and the plenary panel is “School Mathematics for Humanity Education.” One of the major problems of today’s school mathematics is the issue of founding ‘mathematics for all.’ The nature of mathematical knowledge demands strongly school mathematics to become a main subject for humanity education going beyond the practicality. To provide an opportunity to rediscover the idea of mathematics education as cultural education and to explore the way that today’s mathematical pedagogy should first turn to in order to realize such an idea, the Local Organizing Committee of PME31 has adopted this theme. It is towards this goal that we encourage all participants from the countries all over the world to bring up and share their valuable perspectives on mathematics education for humanity building.

And the participants will be ready to become immersed themselves in problems and discussions across the spectrum of all the aspects of mathematics education. We hope that PME31 Conference will prove a venue for a rich variety of mathematics educational researches with the single goal of developing the quality of mathematics education around the world and act as a moment for many exciting researches to come in the near future.

The papers in the 4 volumes of the proceedings are grouped according to the types of presentations; Plenary Lectures, Plenary Panel, Research Forums, Discussion Groups, Working Sessions, Short Oral Communications, Poster Presentations, and Research Reports. The research forums papers appear according to the order of presentation. The papers of the discussion groups and working sessions are sequenced according to their number codes. For other types of presentations, the papers are sequenced alphabetically by the name of the first author within each group.

I would like to extend our thanks to the members of the Program Committee of PME31 and to the reviewers for their respective roles in working with the papers in these proceedings. And particularly I wish to express my appreciation to Professor Park Kyo-Sik for his dedication devoted to the preparation of the proceedings.

Last but not least, I would like to pay my tributes to the cooperation of the members of the PME31 Local Organizing Committee and many Korean colleagues for their sharing with me so willingly the responsibilities, and the efforts of PME Project Manager Ann-Marie Breen and the members of Hanjin PCO who worked so hard to make this conference possible.

Woo Jeong-Ho, Conference Chair

WELCOME OF CHRIS BREEN, PME PRESIDENT

I would like to welcome all of you who are attending PME31 in Seoul, Korea. One striking and attractive feature of PME is the way that the organisation flows and grows through the changing landscape of its annual conference. This year we move on from an enormously large conference in Europe to a smaller conference in Seoul situated in Korea, a country with a rich heritage befitting an ancient civilization.

Our thanks must go to Professor Jeong-Ho Woo and his conference team who have been hard at work preparing to give us a unique conference, and I am delighted that so many PME members are availing themselves of this wonderful opportunity. Our thanks also go to those whose hard work has resulted in the publication of this set of proceedings.

In recent times PME has embarked on a period of revision, and PME members have an opportunity to participate in and form this changing agenda by attending the Policy Meeting on Monday 9th July as well as the Annual General Meeting on Thursday 12th July, where members will vote in the new PME President.

To me, it seems extremely appropriate that my reign of office as PME President should end in the East – the same part of the world where I attended my first PME conference in Tsukuba, Japan in 1993. I trust that many PME31 participants will share my experience of that first PME in Tsukuba, by leaving this PME31 conference with the warm pleasure of having sat around a well-stocked table and shared a great feast with friends-to-be to an extent that will inspire them to keep returning to PME in the years to come.

Chris Breen, PME President

INTERNATIONAL GROUP FOR THE PSYCHOLOGY OF MATHEMATICS EDUCATION (PME)

History and Aims of PME

PME came into existence at the Third International Congress on Mathematics Education (ICME3) held in Karlsruhe, Germany in 1976. Its former presidents have been Efraim Fischbein (Israel), Richard R. Skemp (UK), Gerard Vergnaud (France), Kevin F. Collis (Australia), Pearla Nesher (Israel), Nicolas Balacheff (France), Kathleen Hart (UK), Carolyn Kieran (Canada), Stephen Lerman (UK), Gilah Leder (Australia), and Rina Hershkowitz (Israel). The current president is Chris Breen (South Africa).

The major goals of PME are:

- To promote international contacts and the exchange of scientific information in the field of mathematics education.
- To promote and stimulate interdisciplinary research in the aforesaid area.
- To further a deeper and more correct understanding of the psychological and other aspects of teaching and learning mathematics and the implications thereof.

PME Website and Membership

Membership is open to people involved in active research consistent with the Group's goals, or professionally interested in the results of such research. Membership is on an annual basis and requires payment of the membership fees (USD50) for the year 2007 (January to December). As from 2007, PME members will be required to use the online facility on the PME webpage (www.igpme.org) to sign up as members and manage their own membership details. Those who are not able to attend a particular conference are encouraged to continue supporting PME by signing on as Members for the year.

For participants of the PME31 Conference, the membership fee is included in the Conference Deposit. Enquiries concerning PME and membership should be directed to Ann-Marie Breen, the PME Project Manager (ambreen@axxess.co.za).

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PME PROCEEDINGS OF PREVIOUS CONFERENCES

PME International

The tables indicate the ERIC numbers of PME conference proceedings.

No.	Year	Place	ERIC number
1	1977	Utrecht, The Netherlands	Not available in ERIC
2	1978	Osnabrück, Germany	ED226945
3	1979	Warwick, United Kingdom	ED226956
4	1980	Berkeley, USA	ED250186
5	1981	Grenoble, France	ED225809
6	1982	Antwerp, Belgium	ED226943
7	1983	Shoresh, Israel	ED241295
8	1984	Sydney, Australia	ED306127
9	1985	Noordwijkerhout, Netherlands	ED411130 (vol.l), ED411131 (vol.2)
10	1986	London, United Kingdom	ED287715
11	1987	Montréal, Canada	ED383532
12	1988	Veszprém, Hungary	ED411128 (vol.l), ED411129 (vol.2)
13	1989	Paris, France	ED411140 (vol.1), ED411141 (vol.2), ED411142 (vol.3)
14	1990	Oaxtepex, Mexico	ED411137 (vol.1), ED411138 (vol.2), ED411139 (vol.3)
15	1991	Assisi, Italy	ED413162 (vol.1), ED413l63 (vol.2), ED41364 (vol.3)
16	1992	Durham, USA	ED383538
17	1993	Tsukuba, Japan	ED383536
18	1994	Lisbon, Portugal	ED383537
19	1995	Recife, Brazil	ED411134 (vo1.l), ED411135 (vol.2), ED411136 (vo1.3)
20	1996	Valencia, Spain	ED453070 (vol.1), ED453071 (vol.2), ED453072 (vol.3), ED453073 (vol.4), ED453074 (addendum)
21	1997	Lahti, Finland	ED416082 (vol.1), ED416083 (vol.2), ED4l6084 (vol.3), ED416085 (vol.4)
22	1998	Stellenbosch, South Africa	ED427969 (vol.1), ED427970 (vol.2), ED427971 (vol.3), ED427972 (vol.4)
23	1999	Haifa, Israel	ED436403
24	2000	Hiroshima, Japan	ED452301 (vol.1), ED452302 (vol.2), ED452303 (vol.3), ED452304 (vol.4)
25	2001	Utrecht, The Netherlands	ED466950

26	2002	Norwich, United Kingdom	ED476065
27	2003	Hawai'i, USA	http://onlinedb.terc.edu
28	2004	Bergen, Norway	ED489178 (vol.1), ED489632 (vol.2), ED489538 (vol.3), ED489597 (vol.4)
29	2005	Melbourne, Australia	
30	2006	Prague, Czech Republic	

Copies of some previous PME Conference Proceedings are still available for sale. See the PME web site at http://www.igpme.org/view.asp?pg=conference_proceed_past or contact the Proceedings manager Dr. Peter Gates, PME Proceedings, University of Nottingham, School of Education, Jubilee Campus, Wollaton Road, Nottingham NG8 1 BB, UNITED KINGDOM, Telephone work: +44-115-951-4432; fax: +44-115-846-6600; e-mail: peter.gates@nottingham.ac.uk

PME North American Chapter

No.	Year	Place	ERIC number
2	1980	Berkeley, Califomia (with PME2)	ED250186
3	1981	Minnesota	ED223449
4	1982	Georgia	ED226957
5	1983	Montreal, Canada	ED289688
6	1984	Wisconsin	ED253432
7	1985	Ohio	ED411127
8	1986	Michigan	ED301443
9	1987	Montreal, Canada (with PME11)	ED383532
10	1988	Illinois	ED411126
11	1989	New Jersey	ED411132 (vol.1), ED411133 (vol.2)
12	1990	Oaxtepex, Mexico (with PME14)	ED411137 (vol.1), ED411138 (vol.2), ED411139 (vol.3)
13	1991	Virginia	ED352274
14	1992	Durham, New Hampshire (with PME16)	ED383538
15	1993	Califomia	ED372917
16	1994	Louisiana	ED383533 (vol.l), ED383534 (vol.2)
17	1995	Ohio	ED389534
18	1996	Panama City, Florida	ED400178
19	1997	Illinois	ED420494 (vol.1), ED420495 (vol.2)
20	1998	Raleigh, North Carolina	ED430775 (vol.l),

			ED430776 (vol.2)
21	1999	Mexico	ED433998
22	2000	Tucson, Arizona	ED446945
23	2001	Snowbird, Utah	ED476613
24	2002	Athens, Georgia	ED471747
25	2003	Hawai‘i (together with PME27)	
26	2004	Toronto, Ontario	
27	2005	Roanoke, Virginia	
29	2006	Yucatin, Mexico	
30	2007	Lake Tahoe, Nevada	

Abstracts from some articles can be inspected on the ERIC web site (http://www.eric.ed.gov/) and on the web site of ZDM/MATHDI(http://www.emis.de/MATH/DI.html). Many proceedings are included in ERIC: type the ERIC number in the search field without spaces or enter other information (author, title, keyword). Some of the contents of the proceedings can be downloaded from this site.
MATHDI is the web version of the Zentralblatt fur Didaktik der Mathematik (ZDM, English subtitle: International Reviews on Mathematical Education). For more information on ZDM/MATHDI and its prices or assistance regarding consortia contact Gerhard König, managing editor, fax: (+49) 7247 808 461, e-mail: Gerhard.Koenig@fiz-karlsruhe.de

THE REVIEW PROCESS OF PME31

Research Forums. The Programme Committee and the International Committee accepted the topics and coordinators of the Research Forum of PME31 on basis of the submitted proposals, of which all but one were accepted. For each Research Forum the proposed structure, the contents, the contributors and the role of the contributors were reviewed and agreed by the Programme Committee. Some of these proposals were particularly well-prepared and we thank their coordinators for their efforts. The papers from the Research Forums are presented on pages 1-121 to 1-180 of this volume.

Working Sessions and Discussion Groups. The aim of these group activities is to achieve greater exchange of information and ideas related to the Psychology of Mathematics Education. There are two types of activities: Discussion Groups (DG) and Working Sessions (WS). The abstracts were all read and commented on by the Programme Committee, and all were accepted. Our thanks go to the coordinators for preparing such a good selection of topics. The group activities are listed on pages 1-183 to 1-190 of this volume.

Research Reports (RR). The Programme Committee received 180 RR papers for consideration. Each full paper was blind-reviewed by three peer reviewers, and then these reviews were considered by the Programme Committee, a committee composed of members of the PME international mathematics education community. This group read carefully the reviews and also in some cases the paper itself. The advice from the reviewers was taken into serious consideration and the reviews served as a basis for the decisions made by the Programme Committee. In general if there were three or two recommendations for accept, the paper was accepted. Where a paper only had one recommendation for accept, two members of the IPC took a further look at the proposal and made a final decision as to whether the report would be accepted or rejected. Of the 180 proposals we received, 109 were accepted, 14 rejected, 50 were recommended as Short Oral Communications (SO), and 7 as Poster Presentations (PP). The Research Reports appear in Volumes 2, 3, and 4.

Short Oral Communications (SO) and Poster Presentations (PP). In the case of SO and PP, the Programme Committee reviewed each one-page proposal. A SO proposal, if not accepted, could be recommended for a PP and vice versa. We received 104 SO proposals initially, of which 74 were accepted and 30 were rejected; later an additional 50 SO proposal were resubmitted from RR. We received 32 initial PP proposals, of which all of 32 were accepted; later an additional 7 PP proposals were resubmitted from RR. The Short Oral Communications and Poster Presentations appear in this volume of the proceedings.

LIST OF PME31 REVIEWERS

The PME31 Programme Committee Thanks the following people for their help in the review Process:

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The papers listed below are research reports indexed by research domain. The domain used is the first one that authors listed on their research report information. The papers are indexed by the first author and page number.

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PLENARY LECTURES

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ON HUMANISTIC MATHEMATICS EDUCATION: A PERSONAL COMING OF AGE?

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INTRODUCING RESONANCE

It is an established PME custom that the retiring President of the organisation is invited to give one of the plenary papers at the conference that ends his/her term of office. The conference venue for PME31 had been decided before I began my term of office as PME President in 2004, and the theme of the PME31 conference was selected as usual by our conference hosts without influence from PME's International Committee.

This means that outside factors (quite correctly) totally beyond my control have led to my giving this plenary paper in Seoul at a conference with the theme 'School Mathematics for Humanity Education'. I mention this as I have become increasingly aware of some remarkable synchronicities that this situation has created that resonate strongly with several aspects of my involvement over the years in the field of mathematics education.

A welcome PME-linked resonance comes from the fact that I attended my first post-apartheid PME conference in the nearby city of Tsukuba in Japan in 1993 and have not missed a PME conference since then. However, the main resonance takes me back to July 1986 when, for the first time after my appointment to the University of Cape Town, I presented papers at the national mathematics education conference.

Varela, Thompson and Rosch (1991, xv) describe the term 'structure' as a fluid and temporal self, which is formed 'by the combined influence of one's biological constitution and one's history of interaction with the world' (Davis 1996, 9). The enormous influence that this has on the way in which each one of us sees the world is emphasised in the following quotation:

> In the enactive approach reality is not a given: it is perceiver-dependent, not because the perceiver 'constructs' it as he or she pleases, but because what counts as a relevant world is inseparable from the structure of the perceiver. (Varela 1999, 13)

So my relevant world of mathematics education is both formed and informed by my history of interaction with the world. In thinking about the topic for my plenary paper, I decided to honour the synchronicities of the occasion by re-entering that time of my life in 1986 and then reflecting on the journey I have taken since then. Having made this decision, I noticed that July 1986 is exactly 21 years ago. In my country, many youngsters hold 21st birthday (or 'coming of age') parties to mark the occasion of their 'becoming adult' and are often presented with the key of the front door of the house as a symbolic recognition of this rite of passage. I will leave it for you to

2007. In Woo, J. H., Lew, H. C., Park, K. S. & Seo, D. Y. (Eds.). *Proceedings of the 31st Conference of the International Group for the Psychology of Mathematics Education*, Vol. 1, pp. 3-16. Seoul: PME.

decide at the end of this paper whether you think that I have come of age to a sufficient extent to be given a key!

1986 – 21 YEARS AGO

I had been appointed as the first full-time mathematics educator at the University of Cape Town towards the end of 1982. In 1986, the national congress of the Mathematical Association of South Africa (MASA) took place at the nearby University of Stellenbosch and as a newly appointed colleague from a neighbouring university I was expected to make a contribution to the academic programme of the conference. I submitted two papers for consideration by the review committee and both papers were accepted.

Resonance One: Humanity Education

In many ways, my first paper was a manifesto of planned classroom practice that traced the influences that had informed and formed my teaching practice and beliefs. It dealt with the challenge of introducing pupil-centred activities into schools (Breen 1986a), and it was dominated by a section headed 'Humanising Mathematics Education' in which I outlined the powerful influences that had led me to be interested in this approach. In the first place I drew on the work of Caleb Gattegno (1970, 1974, and 1987) who had emphasized the importance of subordinating Teaching to Learning by stressing the following:

- a deep respect for and acceptance of the capabilities of learners;
- an acknowledgement that in the teacher/student dyad, the learner is central;
- the recognition that it is the learner who must do the learning, and that the teacher's function is to create situations and experiences that focus the learner's attention on the key concepts of the mathematics being presented;
- the discipline to provide the learner with the minimal essentials for understanding to occur, to not 'tell' the learner everything, or almost everything, in the belief that 'telling' fosters learning;
- the further recognition that conversations among and between learners is a valuable tool in a teacher's instructional repertoire;
- the understanding that teaching is subtle work in terms of it being delicate, restrained, and finely grained;
- an appreciation that "only awareness is educable" (Gattegno 1987, p.vii), by which is meant that learners can only acquire knowledge of that of which they are aware.

My plea for the use of learner-centred materials owed a great deal to the work of David Wheeler (1970, 1975) who had stressed the need for humanizing mathematics education. According to Wheeler (1970, 27) the task of the teacher is to "accept the responsibility of presenting them with meaningful challenges that are:

- not too far beyond their reach

- not so easy as to appear trivial
- not so mechanical as to be soul-killing
- but assuredly capable of exciting them"

In my paper I argued for the foregrounding of the mathematics through the use of activities which offered students an entry into the essence of the mathematical concept to be covered, so that whatever work they then did would inevitably be focused on that core concept (an early constructivist approach). I pointed to Trivett's warning at the problems that could arise if teaching methods were prioritized at the expense of the mathematics:

> I began to see what I had been doing over the previous years: glamorize the mathematics, obscure it, … to make it attractive and pleasing to the learners. I had dressed up the subject matter and the learning of it with the subtle implication that real mathematics is hard, is dull and is unattainable for the majority of boys and girls and that the best we teachers can therefore do is to sweeten the outward appearance, give extraneous rewards and indulge in entertainment to sweeten the bitter pill. (Trivett 1981, 40).

Resonance Two: School Mathematics for Humanity Education

My second paper was very different and reflected the context in which mathematics education was taking place in South Africa in 1986. The struggle against the apartheid government had moved into the schools in the Western Cape the previous year and many schools were unable to run normal classes for extended periods during the year. Matters came to a head at the end of 1985, when members of the Western Cape Teachers Union refused to administer examinations on the grounds that the current climate meant that learners would be severely disadvantaged by being asked to write examinations on work that had not been covered during the year. In the end examinations had taken place in some schools under armed police presence (Breen 1988).

During 1986, the security police increased the severity of the action taken against students and teachers. One of my mathematics student teachers of the class of 1986, a quiet, socially responsible, liberated and tolerant woman, who was taking a year's break from school teaching to get her teacher's diploma, was arrested that year:

> On 12 June at 1am she was detained from her home by the police. As I write this it is 30 July. Jane is still in Pollsmoor Prison. Her detention was acknowledged in the newspapers for the first time this week. She has had medical treatment during this time and was returned to prison against medical advice. No reasons have been given for her arrest… (Breen 1986b, 60).

This was the background against which the annual conference of the white-dominated Mathematical Association of South Africa had taken place in 1986. I remember many heated discussion with my Dean before I was convinced that I should participate in this conference which would almost certainly not address the pressing political and contextual issues of the day. In the end it seems as if I made my own sort of peace at attending this conference by contributing a very different second contribution.

This second paper was titled Alternative Mathematics Programmes (Breen 1986c) and signaled my attempt to ensure that mathematics educators attending the conference paid some attention to the broader contested political struggle that was taking place in schools. In particular I focused on the call that was being made for the development of 'alternative' subject programmes in schools as I believed that mathematics educators could not stand outside this debate. In the conference paper, I explored the issue of whether mathematics was neutral and culture-free, and how one might begin to construct a non-trivial programme that tackled issues such as a global perspective on the historical development of mathematics, content issues, the use of different contextual examples to challenge dominant realities portrayed in textbooks and classroom teaching methods.

I ended the last section on classroom teaching methods with the view that an emphasis on humanistic mathematics education would 'provide an exciting basis for teaching in a way that will combat elitism, racism, and sexism as a by-product while focusing attention on the deep structure of mathematics' (Breen 1986c, 187). This seems a plea for 'Humanistic Mathematics Education for a Future Society' – a strong resonance for me with the PME31 conference theme of 'School Mathematics for Humanity Education'.

Multiple lives?

In order to demonstrate the way in which dominant societal realities could be challenged through the use of different contextual examples I provided conference participants with a collation of some worksheets that students (including Jane) had completed for an assignment at the start of 1986 (Breen 1986d). Students had been asked to design mathematics worksheets from materials found in the newspapers. Several groups of students decided to base their worksheets on some of the radical anti-apartheid community literature that was being circulated at the time - some of which was banned by the government soon after publication. Reaction to my second conference paper unsurprisingly centred on these worksheets rather than on the main body of the paper, with one respondent describing the worksheets as 'inflammatory material which fans the flames of revolution' and expressed his 'grave reservations about introducing politics into the mathematics classroom' (De Villiers 1987, 21).

Looking back 21 years later, I can revisit the struggle between the various parallel lives that I was experiencing at the time. In one of these lives, everything was reasonably safe and orderly and I could go about my business of exploring the teaching of classroom mathematics, focusing on content and pedagogy without having to deal with the outside context. In another life, I tried to acknowledge the broader educational school realities and align these with mathematics education debate and practice. In yet another life, I tried to come to terms with the broader political struggle and rapidly deteriorating context. My various contributions on humanistic mathematics education, alternative mathematics programmes and the alternative mathematics worksheets seem to represent aspects of these various lives.

However, I am now aware that the inclusion of the worksheets in my conference presentation had the inevitable effect of deflecting attention away from the important debate that I wanted to begin about the role of mathematics educators in addressing urgent contextual projects in the greater educational arena. It seems as if I had decided in advance that my linking the current political context to mathematics education at the conference was certain to ruffle feathers, so I might as well go the whole way and make a radical intervention. This would allow others who took up the debate later to appear to be much more reasonable.

Developing a New Methodology.

The very act of writing and presenting these fractured conference contributions forced me to re-think my practice and 1986 became a watershed year in my teaching methodology for pre-service students. It was the last year in which I used a traditional lecture format with an appeal solely to logic and rationality. 1986 was also the last time that I ran my lecture sessions in a formal venue of tiered fixed seating. By the end of 1986, I felt the hopelessness of trying to lay some foundations for debate for a future democratic education system in our country through an appeal to the research literature and personal experience. Many students resisted any attempt to bring the social and political into the teaching of mathematics, saying that they had chosen to teach mathematics precisely because it was politically neutral and they would not have to get involved in the tensions of addressing the larger political context. The enormity of the obstacle of trying to challenge and influence the beliefs and assumptions of the student teachers through reason became starkly evident for the first time and it was clear that I needed to change my strategy entirely.

My fundamental project at that stage was to prepare student teachers for a very different non-racial and democratic education system. In trying to develop a teaching style and methodology which would address the need to prepare teachers for a new society I decided that the core of the problem lay in the fundamental beliefs that were entrenched in each one of us through our lived experiences. The aim of my sessions began to focus on causing what I called at the time cognitive conflict (Breen 1992). In a variety of ways including role play, students were asked to engage with activities and then to reflect on their own actions and responses in community with the rest of the class. Reflective journals had to be completed after each session to record the insights that each student had gained into herself as teacher, learner and mathematician. Central to each activity was the tackling of the subject of mathematics as a human interaction, and my aim was to ensure that the engagement with other's of different opinions would assist in opening up the student to a new range of possibilities.

TELLING TALES

One of the features of what I saw to be humanistic education was a focus on the stories of individual learners. In particular I was influenced by the work of David Kent who had written a series of articles in the 1970's which contained the name of the student in the title (for example, Kent 1978). In fact, I now see a reflection of the paradoxes of the time in 1986 that I used the same technique in entitling my article about political dimensions of mathematics teaching and Jane's detention without charge, *What has happened to Jane?*

In the rest of this section of the paper, I will introduce you to a small selection of the many memorable students who have crossed my path since 1986 and have shared their stories with me along the way.

The Context fights back.

The aim of my post-1986 methodology had been to sensitize students to the variety of challenges that would face them in schools as we moved towards a democratic future in our country. In particular, in accord with my second paper, I wanted them to foreground a humanistic mathematics education approach as articulated in my 1986 papers. It proved to be an unexpectedly difficult challenge for many.

For example, Catriona wrote to me of her dismay at the effect of the school's testing regime on her class. Writing to me as her class were busy writing a test in front of her, she described her battle to control and teach mathematics to this particular bottom set in her privileged school as she struggled with an overlong and overcomplicated syllabus. She wondered how much bleeding these shapes in blue uniforms in front of her were doing as a little more creativity and natural freedom was stamped into conformity each day. She then said words that remain with me vividly even though they were written 20 years ago:

> Chris, do you know what really hurts – I realise that over the past year and a half I have been one of the causes of their pain, The odd smile, laugh and sometimes even touch can never make up for the frustration, the worry, the anger (whatever) they must feel. (Breen 1987, 45)

Several years later I decided that it was time to research the effects of this curriculum on the lived realities of the students who had passed through my hands. The results were generally favourable in terms of the educational experience but there were indications in some of the stories that the reality of the teaching situation had claimed several victims amongst those who were attempting to bring changes into the school system (Breen and Millroy 1994).

For example, Thabo left the university all fired up to play a major role in the upliftment of his community. He entered the teaching profession with enthusiasm and showed remarkable staying power in overcoming the lack of teaching aids in the school as well as the challenge of teaching 5 classes of the same year with an average class size of 70. His class motto was 'take risk' and he pushed his students into using group work. He compensated for the extra time taken to introduce these new methods by offering Saturday classes. Thabo soon became extremely popular with the students and the Headmaster decided to sit in on a lesson to see at first hand what was happening. The Head became excited and earmarked Thabo for accelerated responsibility and promotion. By the end of the next year he was promoted to the position of Head of Department. During the end of year break things started falling apart for Thabo. The other more senior teachers objected to his promotion and agitated in the local community and succeeded in getting Thabo suspended from the school. The Head was subsequently transferred to a new post and only after support from the students was Thabo re-instated to his post but this time it was at a junior level.

Assessment and the Psychology of Fear

Doing mathematics has long been linked to achievement in tests and examinations and the pursuit of marks carries with it all sorts of normative consequences. One of the highlights of a school learner's achievements in mathematics is to win an award in the local Mathematics Olympiad competition. One year at the prize-giving I tried to open up the way in which such a competition focused one's gaze on the competition rather than on the inherent qualities of the subject and the joy of learning (Breen 1990). Trivett (1981) spoke of the way his teaching methods were distracting the learners from the core of the subject being learnt. In a similar way, achievement in tests and the subsequent positioning that it brings to one's position in society creates a barrier between oneself and learning and the subject.

This insidious nature of this positioning became more evident in a piece of research I did a few years ago where student teachers wrote a test to examine their knowledge of the school syllabus (Breen 2004a). Students were asked to predict which of them would achieve the top marks and which of them would do badly. They were quite happy to enter into this artificial game even though they had no evidence other than their class interactions of the past few months. In general the class was somehow able to position each other reasonably accurately. However they were all surprised at the success of Nkosinathi, a quiet Black African student whose silence did not fit into the apparently dominant expectation of extroversion for achievement. More worrying for me was the fact that Nkosinathi also quite clearly did not value his own ability and did not consider that he might feature in the top group of achievers.

The tragic consequences of early failure in mathematics can quite readily be seen in classes of pre-service primary school teachers who are forced to take mathematics as one of their subjects because primary school teachers are required to teach each subject in the curriculum. I have previously described the case of Marissa who was one of those most scarred by her previous experiences in mathematics (Breen 2004b). Initially she could not pass even a basic junior school content test on operations and got headaches or became physically ill whenever she felt under pressure in the mathematics classroom. During the year's mathematics course we battled together to turn her anxiety and sense of failure around and at least get her to pass the year, and were both excited when she eventually passed the course.

Patience suffered similarly as a very weak student in mathematics, but had the added disadvantage of being a music student from a disadvantaged background. She had to work in a local supermarket after hours to earn sufficient money for her studies but her plight was largely ignored by our department. Some lecturers chided her publicly for her poor work but did not provide any assistance or guidance. Her fellow students took up her plight and confronted lecturers with her lived reality and the need to give her proper support. As it happened the last timetabled lecture of the academic year was Mathematics Method. When I arrived for class that day, I could sense an enormous energy in the room as they came to the end of what had been for them a very difficult year – especially as they had stood up together against the staff of the department on behalf of Patience and several other students who were in a similar position. I offered them the opportunity to take the time at the end of my class to bring the year to completion by taking turns to say goodbye and anything they wanted to say. The students and I stood in a circle and held hands and went around taking turns. Each student took this ritual very seriously and showed a powerful grace, compassion and unity in expressing their thanks to each other for their interactions during the year. Patience's turn to speak came at the very end as she was standing next to me. She gasped a few times and then collapsed to the ground with a heart-rending large wail as the frustration of her silenced voice of the year could even now not find an outlet.

Teacher Research

In 2000 I introduced a new taught Masters module called 'Re-searching Teaching' which took its much of its theoretical framework from enactivism and included in its methodological tools Mason's Discipline of Noticing (see Breen 2002). The emphasis in the course has been for students to become more aware of their practice in the moment and to lay their own authentic path while walking. Several students took to this approach and wanted to continue these ideas in their research dissertations. In different ways, Neil, Agatha and Kendal insisted on following their own paths and inserted their own interests and histories into their research with unusual yet successful results. In a paper presented at the 2005 conference of Complexity Science and Educational Research (CSER) (Breen 2005a), I outlined these different paths and showed how these different paths had brought them into different degrees of opposition with my academic colleagues who had taken on the task of being the traditional gatekeepers of research. In the CSER paper I described a series of nine different dilemmas that arose along the way for me as supervisor as I was placed in the position of having to mediate their continued progress through the lens of traditional practice, which was often framed in the guise of official policy. At each step I found myself having to face the probability that what I considered to be the appropriate and authentic course of action would bring me into conflict with my colleagues in the department.

LIGHT AND SHADOW

> We notice that when sunlight hits the body, the body turns, bright, but it throws a shadow, which is dark. The brighter the light, the darker the shadow. Each of us has some part of our personality that is hidden from us. Parents, and teachers in general, urge us to

> develop the light side of the personality — move into well-lit subjects such as mathematics and geometry — and to become successful. (Bly 1988, 7)

I used the above quote in an article I wrote for a special edition of *For the Learning of Mathematics* (FLM) which focused on psychoanalytic and therapeutic approached to mathematics education. It is interesting to remember that David Wheeler played a significant role in the creation of the journal and that the journal's name draws inspiration from Gattegno's entreaty to subordinate teaching to learning.

In this FLM article (Breen 1993), I tried to respond to an accusation from a Jungian psychologist that Plato's emphasis on Logic … encourages separateness, class war and apartheid. ..Thinking without feeling is not the God Plato thought it would be; it is closer to the Anti-Christ (David 1992). My aim was to try to explore ways of creating a classroom environment for mathematics teaching in which the teacher could hold the tension of the opposites between the light and shadow of mathematics and classroom teaching.

Many of the people who have been introduced to you on the pages of this plenary paper have introduced aspects of the shadow into my attempts to keep mathematics education in the light. At a time when I thought that I had developed a teaching methodology which would sensitize and prepare teachers for a future non-racial and democratic South Africa, Catriona and Thabo told stories which emphasised the fact that individuals do not operate outside of a context – the social and political are ever-present in our teaching.

Nkosinathi shows the powerful positioning role that mathematics places on an individual, both internally and externally – unquestionably a shadow consequence of all assessment practices. At first attempt I tried to script Marissa as an example of a wonderful successful remedial teaching process. With some prompting from Dick Tahta, I tackled the task of looking into the shadows to explore different dimensions of what was going on in our interactions, some of which even now I am still choosing to leave in the dark (Breen 2007, in press).

As I tell my story, I am aware that these stories have a much greater impact and consequence than the personal. Patience brought a different challenge to my department that we were not able to grasp and we kept her in the dark as long as we could. It took the support of her fellow-students to bring her out of the shadows, but we had already done the damage. Certainly the responses to the different research approaches that Agatha, Kendal and Neil wanted to pursue brought forth a strong reaction and pressure from colleagues for them to stay in the light.

Thinking about PME, one might argue that PME conferences have always been held with the express purpose of annually celebrating the light. Our aim is to share new knowledge with each other and discuss the way forward with as much certainty as we can manage. We each have our own template of what that light looks like and how it should be explored, and we judge each other's contributions against this template in our search for certainty. We often just seek out 'sound bites that confirm our position'

(Wheatley 2005, 210). Only on momentous occasions has something lurking in the shadows been brought to our attention and caused great consternation. This desire for the light is, I think, exemplified by the example of one reviewer who wanted Marissa to stay in the dark and recommended that the proposal that I had submitted discussing her experience as a case study should be rejected because someone so poor at mathematics should not be allowed to become a primary school mathematics teacher.

Living between the opposites.

> To live between the opposites means that we not only recognise opposites, but rejoice that they exist…Living in the opposites does not mean identifying with one side and then belittling the other… (Bly 1990, 175).

In this paper I have taken a reflective look at some of the events of the past 21 years since I presented those two contrasting yet separate papers at my first mathematics education conference. Looking back I do not believe that my passion for the learning of mathematics has diminished. I also believe that my sense of promise for the approach discussed in the paper on pupil-activities has not diminished as shown by my recent development and re-conceptualising of these ideas against an enactivist theoretical framework (Breen 2001). I think that what has changed is that I have come to believe that it is important to see mathematics education as both light and shadow and the challenge is to live between the opposites and consciously foreground one or other aspect without belittling the other side.

In 2004, the organisers of PME28 chose a theme that asked us to look into the shadows and consider the theme of Inclusion and Diversity. At PME29 in 2005, members of PME at the AGM voted to change the aims of PME to allow a broader range of topics and research fields to be presented at our annual conference. At PME30, one of the Discussion Groups considered whether these changes meant that it was no longer necessary to hold a separate Mathematics Education and Society conference. For me these developments indicate a welcome willingness on PME members' part to look beyond the light of mathematics education and embrace the shadow as an integral part of our field.

Willing to be disturbed.

> Noticing what surprises and disturbs me has been a very useful way to see invisible beliefs. If what you say surprises me, I must have been assuming something else was true. If what you say disturbs me, I must believe something contrary to you. My shock at your position exposes my own position… If you're willing to be disturbed, I recommend that you begin a conversation with someone who thinks differently than you do. Listen as best you can for what's different, for what surprises you. Try to stop the voice of judgment or opinion. Just listen. (Wheatley 2005, 212)

The difficulty is that I have needed much more than a willingness to embrace both light and shadow in thinking about mathematics education. My habits and beliefs have been formed and entrenched in the light over a long period and it is extremely difficult to 'notice what one fails to notice' (Goleman 1997, 24).

I have been exploring ways of tackling this in the Master's module by bringing critical incidents that concern us to the rest of the group for comment using the strategy of accounts-of (Mason 2002). Nicky had collected accounts that highlighted her frustration at the lack of completed homework being done by her extra lesson pupils (Breen 2005b). She shared a typical account with the rest of the class and stepped back to listen to the variety of different responses and ways of handling the situation that came from the rest of her colleagues. Their responses highlighted for the first time the singular and fixed way that she had been looking at the problem of homework and she could begin to plan a new response. However, even this next step proved to be complex, and her description of her evolving process of realisation of her anticipated way forward and the degree of discipline and awareness that was necessary to put a new plan into operation were extremely sobering.

For me, a willingness to acknowledge both the light and shadow is only the first step. Our skills at listening will provide a necessary entry into further awareness. Davis (1996) introduces us to three levels of listening, two of which are very familiar to us all. It is the third type that he lists, that of hermeneutic listening which has provided me with the greatest challenge to employ. 'This manner of listening is far more negotiatory, engaging, and messy, involving hearer and the heard in a shared project' (Davis 1996, 53).

Maturana points out an attitude that is necessary to adopt if one wants to be open to new perspectives.

> When one puts objectivity in parenthesis, all views, all verses in the multiverse are equally valid. Understanding this you lose the passion for changing the other….If the others can also put objectivity in parenthesis, you discover that it is easier to explore things together, because one is not denying the other in the process of exploration.
>
> (Maturana 1985)

I have reported elsewhere (Breen, Agherdien and Lebethe 2003) on the approach that two in-service mathematics education field workers had made to me to assist them in their workshop teaching practice. We found ourselves attempting to follow Maturana's example of exploring things together, but we learned a crucial lesson in the process. The defining moment came when as the one with power, I chose to make myself vulnerable by inviting them to comment on my teaching practice rather than sit in on their lessons and talk down to them. The expected practice would have taken me as expert to pronounce on their competence. Instead we placed my teaching under the spotlight and they explored aspects of my teaching that they selected as being important to them.

One of the most important lessons that I learned from working with Agatha, Neil and Kendal as they attempted to forge their own paths in the research journey is that students and colleagues can be wonderful sources of perturbation if one is willing to be disturbed and open up the space to trust and listen to them. I tried to reflect on this insight by encouraging teachers to remain true to their research goals in the concluding section of a chapter on Teachers as Researchers.

> The teacher research movement can assist by causing dissonance and trouble. Trouble that comes from conviction based on evidence drawn from research by those in the field who know that we haven't got education right and who are prepared to put their energies into getting something changed. The minute teacher research becomes comfortable; someone else needs to take over… If your research endeavour is uncomfortable, you know you are close to the edge, and you can be sure that beneficial learning is taking place. (Breen 2003, 541-542).

I think my challenge is the same as that of our PME community but I will address the questions to myself in the first place. To what degree am I willing to be disturbed? Am I willing to stop the voice of judgment and listen? Am I prepared to enter into a joint project of communication with the person whose ideas have surprised or disturbed me? Am I truly willing to learn from my students and embrace both light and shadow? This seems to me to be the essence of the challenge that I face 21 years later.

A COMING OF AGE?

> I like the idea that the work a person does on his shadow results in a condensation, a thickening or a densening, of the psyche which is immediately apparent, and which results in a feeling of natural authority without the authority being demanded. (Bly 1988, 54)

In ending this plenary paper, I look back to those two 1986 papers and see the presence of both light and dark. The first paper was entirely in the light as it sought to further the learning of mathematics through a humanistic perspective that foregrounded both concept and learner. The second paper was more complicated. It was created with the express purpose of getting participants to talk about the issues that were very present in society but had been consigned to the shadow in conference presentations. In two distinctly different and paradoxical moves in that second paper, I assumed that many would be angered by my introduction of this shadow into the conference so went a whole step further by introducing radical material that ensured that those who did not want to listen would have good cause to rationalize their opposition. The second move seemed to try to trivialize the extent of the problem and tried to push it into the light by concluding that salvation would come from taking on a humanistic mathematics education approach and so provide an exciting basis for teaching in a way that will combat elitism, racism, and sexism as a by-product while focusing attention on the deep structure of mathematics.

My hope is that I have been more direct in describing my journey over the past 21 years and the challenges that currently face me, so that you will decide that I have indeed come of age. I hope that some of the things that I have said have surprised or even disturbed some of you and that you will be able to show your willingness to be disturbed by engaging in Maturana-like hermeneutic conversations with each other.

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(Since many of my own articles will be hard to access, I hope to post most of them with permission on www.chrisbreen.net).

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CERTAINTY, EXPLANATION AND CREATIVITY IN MATHEMATICS

Michael Otte

I. Introduction

The New Math reform intended to bring mathematics education nearer again to theoretical mathematics. It tried to narrow the gap between research and teaching, erasing the distinction between lower and higher mathematics and establishing mathematics in general as a reality *sui generis* and as independent from all "metaphysical" concerns. The most significant innovation of this reform in the curriculum "was the new emphasis on axiomatic structure and rigorous proof" (Hanna, 1983, 21). In a report of the Conference Board of the Mathematical Sciences of 1966 we read, for instance:

"The emphasis on structure and proof in algebra is the fundamental component of a change that has taken place in school mathematics in the U.S. at the secondary level during the last ten years. This change is so profound and far-reaching that it can only be described as a revolution" (quoted from Hanna 1983, 21).

This revolution contained, however, a paradox which nearly nobody had noticed, namely that "rigorous formal proof" did not mean proven knowledge in the classical Aristotelian sense as exemplified by Euclid's *Elements* of geometry. "Euclid most go!" This was a well-known slogan by the Bourbakists. Truth and proof became as unrelated as pure and applied mathematics or even problemas and theories.

At about the same time, in 1962, Thomas Kuhn published his essay *The Structure of Scientific Revolutions,* which had a great impact not only on the philosophy and historiography of science but also on the new educational policy, didactic and cognition theory and vice versa (Kuhn referred extensively to the work of Piaget, for example). And Kuhn's conception of theory and Hilbert's axiomatical mathematics are really two of a kind and resemble each other very much (Otte 1996, 214).

In a contribution to a conference on Kuhn's work Lakatos wrote:

"For centuries knowledge meant proven knowledge [...]. Wisdom and intellectual integrity demanded that one must desist from unproven utterances [...].Einstein's (or Hilbert's; my insertion M.O.) results turned the tables and now very few philosophers or scientists still think that scientific knowledge is, or can be, proven knowledge. But few realize that with this the whole classical structure of intellectual values falls in ruins and has to be replaced: one cannot simply water down the ideal of proven truth – as some logical empiricists [...] or [...] some sociologists of knowledge do" (Lakatos 1970, 92).

That Lakatos is right can be seen today in all quarters of our technology based knowledge-societies. Metaphysical realism appears as sterile and futile as straightforward positivism or pragmatic functionalism. With respect to mathematics education or the philosophy of mathematics the notion of "rigorous proof" has come under attack and a distinction between "proofs that prove and proofs that explain"

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(Steiner 1978; Hanna 1989; Mancosu 2001) was introduced, tentatively reassuming the Aristotelian model of science in this context. Such endeavors remain extremely ambiguous and unclear, however, not least because the unity of universal and individual, essential to the Aristotelian model, cannot be resuscitated in the same static manner within modern societies.

In trying to educate the younger generation within to-days technological "knowledge society", it seems nevertheless worthwhile to remember that knowledge fulfills two major roles in human culture: a practical one and a philosophical one. Education is to be based on proven scientific knowledge not the least because "it seems that science came into being with the requirement of [...] coherence and that one of the functions it performs permanently in human culture consists in unifying [...] practical skills and cosmological beliefs, the *episteme* and the *techne* [...] despite all changes that science might have undergone, this is its permanent and specific function which differentiates it from other products of human intellectual activity" (Amsterdamski 1975, 43/44).

The essential point concerns the unity of universal and particular conceived of simultaneously as evolutionary process of the human subject and of the cultural growth of knowledge. The outlook of research mathematicians is extremely individualistic, emphasizing a separation between the "context of discover" and the "context of justification". When Thom affirms that the real problem which confronts mathematics teaching is the problem of "the development of meaning, of the 'existence' of mathematical objects" (Thom 1973, 202) then it is essentially this question of the interaction between general and particular meant, simply because no general idea can be given directly. On the other hand, no deep problems produce the means of their solutions out of themselves. Meaning being, however, also a subject's category, it follows from Thom's statement that the knowing subject has to conceive of itself as a socio-cultural as well as a individual being.

Even reductionistic and foundational approaches to the question of meaning have difficulties with this problem, not least, because they seem totally obedient to Frege's distinction between three worlds. "The first world is the physical world; the second world is the world of consciousness, of mental states ..; the third world is the Platonic world of objective spirit, the world of ideas" (Lakatos 1978, 108).

It is beyond the possibilities of this paper to cure this disease, but we want at least analyze it in what follows from a historical and philosophical point of view. And with respect to the central problem of unity of general and individual we believe that a semiotic approach to mathematical cognition might be helpful. Man is sign, Peirce had famously stated. But human subjects are also concrete individuals. Peirce might not have put forward an elaborated theory of social practice, but his semiotics actually introduces, besides general meanings, two types of non-symbolic signs, icons and indices. Differently from symbols these other signs seem to be independent from social convention with respect to their relations to objects and they play a fundamental role in mathematical reasoning (Otte 2006a).

Mathematical activity is not direct concrete action, but is semiotic activity after all. It is not just mental experiencing either. In thought the difference between possibility and actuality seems to collapse, as soon as one strikes upon an idea the problem seems solved. Neither in real world situations nor in semiotic contexts exists such a coincidence. Semiotic activity seems actually to be an intermediary level between our inner world and objective reality. So it seems worthwhile to try and find out what kinds of objectivity are involved in semiotic activity; how, for example, circumstances and structure interact in the creation of meaning.

II. An analytical Scheme

A rather superficial glance is sufficient to perceive that the distinction of “foundationalism vs. anti-foundationalism” serves well as a first classificatory scheme for the various philosophies of mathematics along history. We do not claim, however, that this opposition is necessarily exclusive; people may pursue both goals, although perhaps with different emphasis.

Foundationalism ruled the “Classical Age” (Foucault) from Descartes to Kant. Beginning from the Renaissance people had been concerned with evidence and certainty of knowledge and had attributed to mathematics therefore a certain privileged epistemological status. At the turn of the 18th century mathematics came, however, under pressure from two sides essentially, with respect to its status as secure knowledge as well as concerning its adequacy for an understanding of the historical world.

Kant’s epistemology was, like all of 18th century Enlightenment, referring to the notion of a static and immutable human subject, which alone is active, whereas the objective world is passive. Knowledge thus becomes a construction of the human subject according to its own conditions and limitations and the meaning of mathematics comes to be defined in terms of the ways of human understanding. The Enlightenment, writes Lovejoy, was “an age devoted, …, to the simplification and standardization of thought and life” (Lovejoy 1936/1964, 292), this uniformity being conceived of as the true purpose of Nature. John Stuart Mill (1806-1873) still emphasized that the mathematics and natural science are to be based on the “Uniformity of Nature”.

This situation changed in the early 19th century and variation and diversity came to be seen as the essence of being and excellence. Hegel’s outlook, for example, is historical and process oriented. His philosophy begins with the idea of simultaneous evolution of subject and object, as hypostatized in the historical evolution of Spirit; such that the general conditions of knowledge at any moment in time are only relative and become also the object of knowledge again in the course of further development. Circular interactions between conditions and conditioned, between general and particular, between, for example, concept and object are essential. Kant`s famous “revolution” of epistemology is based on the claim that “the objects must conform to our cognition” (Kant, Critique, Preface to the Second Edition), rather than the other way around. Hegel would say both ways of adaptation must occur.

In this Hegelian spirit Gilles Deleuze could be deduced to claim that philosophy distinguishes itself from mathematics by the fact that philosophy constructs concepts, prototypes or metaphors, whereas mathematics begins with definitions. But mathematics employs metaphors too and what concerns mathematical definitions, they have sometimes rightly been called "definitions of definitions" (Karl Menger), rather than definitions proper. Grassmann described how a formal expression may attain concrete meaning by looking for all expressions that are equivalent to the one given, looking upon them "like the species of a genus not like the parts of a whole" (Grassmann 1844, 108). The idea that knowledge should be conceived of as concept development gained generally ground since the early 19th century.

Classical mathematical foundationalism came thus under attack in course of these developments from two sides, namely from anti-foundationalist perspectives on knowledge, on the one side and as well from the opposite desire to make the foundations of mathematics more secure by establishing pure mathematics as a branch of knowledge in its own right. One can see this from the different modifications of the principle of continuity, that is, the principle of the uniformity of Nature, in the views of Cauchy, on the one side, and Poncelet, on the other. It is, in fact, this problem of continuity what characterizes the mathematical "crisis" at the beginning of the 19th century, a crisis which in some sense did not end so far.

Rather than conceiving of continuity in terms of variation and invariance, Bolzano and Cauchy thought of it in arithmetical terms. The program of rigorization by arithmetization searched to solve the foundational problems in a reductionistic manner, by defining all mathematical concepts in terms of some basic entities, ultimately the natural numbers. Anti-foundationalist positions, like Poncelet or Peirce, in contrast, tried to employ, so to say, a top-down strategy, solving the problem of the objectivity of mathematics by extending and generalizing its relational structures and its rules of inference. Mathematics became in this manner to be understood as dealing with "ideal states of things" (Peirce, CP 3.558), rather than with an approximation to the actual world conceived of as a static set of objects.

In consequence all parts of mathematics dealing with continuity became geometry and geometry itself lost its status of an independent discipline. Even analysis, having been a stronghold of arithmetizing rigor since Bolzano and Cauchy, became "geometrized" with Borel and Lebesgue, after the Cauchy-Riemann approach had exhausted its possibilities (Otte 2007).

Generally speaking modernity since Schelling (1775-1854) and Hegel (1770-1831) or Bolzano (1781-1848) has become obsessed with the integrity of thought and thus with meaning and in consequence with language and philosophical conceptualism. This often implies the view that history is "logical", and reasonable and that every fact must have an explanation. Contingencies are to be negated or ignored. As Dummett sees it: "The theory of meaning … is the foundation of all philosophy, and not epistemology as Descartes (or Kant, my insertion M.O.) misled us into believing.

Frege's greatness consists, in the first place, in his having perceived this" (Dummett 1981, 669).

But this development began much earlier and Dummett could have equally well indicated Hegel or Bolzano. Meaning is, however, a subject's category which could be used in service of foundationalist or anti-foundationalist interests, either searching for ultimate meanings or trying to enlarge and transgress given meanings. The debate about the continuity principle showed exactly that. And much depends on how the human subject is conceived of, either in eternal terms of logical universalism or in terms of existential diversity and as immersed into the general course of history. As Norbert Wiener writes:

"To us, nowadays, the chief theme of the mathematicians of the Romantic period may sound most unromantic and repelling. The new mathematics devoted itself to rigor. …. What the new generation in mathematics had discovered was the *mathematician*; just as what the Romantics had discovered in poetry was the poet and what they discovered in music was the musician" (Wiener, 1951, 96). The Romantics had, however, an equally strong disposition for bold analogies and metaphors, destined to generalize and enlarge conceptual thinking (Caneva 1974, Otte 1989).

Hegel and Bolzano seem as strange bedfellows (see Bolzano WL, § 394), as later Carnap (1891-1970) and Heidegger (1889-1976), but they came close to each other in some aspects of their philosophies because of what Hintikka (1997) has called the "universalist conception of logic and language". Hintikka has on many occasions drawn our attention to the importance of this distinction between "the view of language as the universal medium … and the view of language as a calculus" (Hintikka 1997, 21ff) and he has pointed out that one of the consequences of the universality of language is the ineffability of semantics and truth. His does not make a difference in case of an anti-foundationalist position, but proves disastrous to foundationalist projects, like those of Bolzano, Frege or Russell.

The rigor movement of arithmetization seems to have nowhere been as strong and pronounced as in Germany, in fact. From a recent survey article about philosophies of knowledge in France (Sinaceur 2006) one gets the impression that positivistic and Hegelian type philosophies, struggled for dominance in French philosophy of mathematics and science among the generation after Brunschvicg (1869-1944), with Cavailles (1903 -1944) as the most important figure. Cavailles has to a certain extent been influenced by Hilbert, but was more thorough in his Hegelian rejection of the particularities of the subject, as endorsed by intuitionism or phenomenology. "Le sens veritable d'une theorie est non pas dans un aspect compris par le savant lui-meme comme essentiellement provisoire, mais dans un devenir conceptual qui ne peut s'arreter" (Cavailles 1976, 23).

Even modern axiomatic came in two different versions, a foundationalist, Euclidean one, as exemplified by M. Pasch`s *Vorlesungen ueber neuere Geometrie* of 1882 and a postulatory one, which is represented by Hilbert`s *Grundlagen der Geometrie* of 1899. Pasch's project amounts to nothing else but a rigorous and logically refined

version of Euclidean axiomatics, whereas Hilbert's enterprise was essentially anti-foundationalist (Lenhard/M. Otte 2002, 259-292).
According to the Hilbertian view axioms are organizing principles, framing the scope and structure of a theory and as such they are meta-mathematical statements, rather than mathematical ones and are bold generalizations intuiting possibilities. An axiomatic theory in this sense is more like an instrument of research than a foundation of knowledge. With respect to the relation between mathematical theory and the objective world the axioms are framing a particular perspective onto that world, which has to prove its validity in terms of its fertility and productiveness for the research process itself. Cavailles expressed, as we have just seen, such views. The foundations of such an axiomatized theory are to be seen in the intended applications of that theory and thus lie in the future, so to speak. But the future is actually always influenced by the past and the postulates of such a theory are therefore not completely arbitrary, but are framed or influenced to a certain degree by past theories. As the thinking subject is, on the other hand, not be fully transparent even to itself, there cannot be clear and absolutely certain a priori foundations of knowledge.
The old Euclidean axiomatic conceptions which had dominated the scene for more than 2000 years were radically overturned. Frege did not consent to these developments and he searched therefore for absolute logical and set theoretical foundations of axiomatized mathematical theories and Russell followed him in this. "A moral conviction supported by many successful applications is not enough"' Frege said (*Grundlagen der Arithmetik,* §1). Frege and Russell objected that the axiomatic method is incomplete as unspecified terms occur within the axioms such that it becomes impossible in Peano's arithmetic, for instance, to say what symbols like "1" or "number" etc. really mean.
Philosophers have painted a similar picture of the historical development of their discipline in its relation to the sciences. Richard Rorty, for instance, set up a great part of philosophy in terms of foundationalists and anti-foundationalists in his important and influential book, *Philosophy and the Mirror of Nature* (Princeton UP 1979), and he counted Descartes through Kant, Frege, Russell, as well as, phenomenology from Peirce to Husserl among the first group, whereas Heidegger and Wittgenstein are in the second group. Rorty's assessment remains somewhat ambiguous and incomplete in some cases, we believe. Hegel or Schelling and the Romantics impeded the establishing of epistemology as a foundational enterprise in Germany, as Rorty himself is quite ready to recognize (Rorty 1979, 133), such that there was an anti-foundationalistic wave in early 19th century against which Bolzano struggled vehemently, for example. However, Rorty's existential individualism is opposed to Hegel's philosophy by nature and as a matter of principle. Hegelianism produced, according to Rorty, "an image of philosophy as a discipline which somewhat both completed and swallowed up the other disciplines rather than grounding them" (Rorty 1979, 135).
Hegel was an anti-foundationalist and a universalist, and so were Wittgenstein or Heidegger, who tried "to construct a new set of philosophical categories which would

have nothing to do with science" (Rorty 1979, 5). Wittgenstein, on his part, reduced "the basis of our thought to linguistic etiquette" (Gellner 1979, 23), affirming that the only alternative would be to endorse logico-platonistic theories of meaning in the sense of Bolzano or Frege.

These kinds of universalism pass without being thoroughly analyzed and assessed or even without being noticed in Rorty's book, because he himself is eager to separate philosophy from science or mathematics and wants to replace epistemology by hermeneutics, *Bildung* (edification) and existentialism and "the hermeneutical approach stands and falls together with the thesis of the universality of language" (Hintikka 1997, XV). Rorty dismissed epistemology because, on his view, the aim of knowledge is not to represent the world, but is in the individual's capacity to cope with reality. Rorty is right in emphasizing that know-how is more than knowledge. But he misses the point that know-how does not develop sufficiently in modern society without guidance from mathematics and science, that is, 'knowing that' is as important as 'knowing how', and truth is as relevant as efficiency. Rorty does not perceive such things because of various reasons, which, however, cannot be spelled out in detail here.

Now, analytical philosophy might consider the views of Hegel, Heidegger or even Husserl as irrelevant to mathematics, whereas Bolzano and Frege belong among those who have created a new conceptual ideal of mathematics and have established new standards of mathematical rigor. Further on, Bolzano was opposed to Hegel and German idealist philosophy for largely the same reasons that motivated Carnap to become a political and philosophical adversary of Heidegger. And with respect to knowledge and experience Bolzano, Frege and Carnap were foundationalists, whereas Hegel and Heidegger endorsed anti-foundationalism and interested themselves primarily in the historical and political dynamics of knowledge, rather than in its justification.

But nevertheless and despite of all these differences they all found themselves within the very same boat of "universalism", that is, they all believed that the world in which we live can be conceptualized and can thereby be dominated by conscious thought and reflective analysis. Everything should have a reason and a logic and contingent facts have no place in this idealistic picture of reality. Hegel did strongly criticize Euclidean mathematics from such a point of view in the introduction to his "Phenomenology of Spirit", complaining about the arbitrariness of the auxilary constructions necessary for carrying through the argument of a proof in geometry.

All shared the belief that the world was a kind of mental creation and thus thoroughly transparent, intelligible and explainable. Interpretation and meaning therefore become the main concern, not representation, that is, universalism conceived of signs and representations exclusively in terms of their function, ignoring the importance of form. We shall have much more to say about this difference of form and function later. For the moment it might suffice to just give a single example of mathematical form based reasoning.

Mathematical thought, as Aristotle already said, begins with the Pythagoreans, with *theoremata* like the following, "The product of two odd numbers is odd". Or, "If an odd number divides an even number without remainder, it also divides half that number without remainder". These are theorems that, as one says, go beyond what can be experienced concretely because they state something about infinitely many objects. Actually, they do not state anything at all about objects (e.g., about numbers); instead, they are analytic sentences that unfold the meaning of certain concepts. How do we prove, however, those analytic propositions, like "the product of two odd numbers is odd"? We represent certain activities.
We say, for instance, if an odd number is divided by 2, there will by definition remain a remainder of 1. From that, we infer that there is for each odd number X another number N such that $X = 2N + 1$. If we now have two odd numbers represented in this way, and if we multiply them, theorem will result quasi-automatically by applying the distributive and commutative laws and observing that the product has exactly the same <u>form</u> as multiplier and multiplicand. The mathematician typically proceeds by constructing (algebraic or geometric) diagrams and by observing them. The value of diagrammatical reasoning is due to the fact that human brains are good at recognizing broad categories and strategic positions in a general way and that this helps to reduce the problem at hand to a manageable size. The significance of diagrams becomes much more obvious as soon as the continuum is involved, however.
In summary, we claim that the distinction between explicitness of meaning vs. representation as form corresponds more or less exactly to Hintikka's complementarity of the universality of language vs. language as a calculus and that "explicitness vs. intuition" could thus serve as a second classifying distinction, thereby enriching and refining our grip on the situation. The choice of words does not import too much and "collective vs. individual" might have been an alternative. What matters are more the relationships indicated by these distinctions. We interpret in particular the "linguistic turn" and the trend in the development of mathematics to replace seeing and evidence with conceptual proof as meaning essentially three things:

First, an increase in foundational interests trying to transform mathematics into a more or less coherent and closed system of propositions and theories (this does not mean that an interest in theory must necessarily be foundational!). There is no science without a system and no system without a foundation, such was the belief. Besides a kind of positivism took hold making people believe that mathematics has finally reached its final form.

Second, explicativistic mathematics takes logic and language as the primary context of mathematical knowledge. Mathematical objects are to be exhaustively defined and are conceived of in terms of arguments of propositional functions. That some x exists means that a propositional function "x is P" is sometimes true. And what can serve as a premise in a mathematical proof must have a propositional content. Therefore the first question is: What does this or that mean? Frege or Bolzano make everything depend on the meanings of propositions and modern analytical philosophy of mathematics has followed their example.

Third, there exists the desire to make mathematical concepts as distinct as possible and reasoning as detailed and explicit as possible. This goal is a consequence of the wish to indicate the "objective connections" between truths. Foundationalists are more concerned with judgments, while anti-foundationalists concentrate on concepts and concept evolution.

Bolzano once indicated the greater care in raising everything to higher levels of distinctness as being his main strength. "It is only through this on the path to more precise definitions of concepts, that I have come to all of the distinctive doctrines and views you encounter in my writings (even the mathematical ones)", he writes to Romang (quoted from Sebestik 1997, 33). Mathematical progress, according to Bolzano, is due to the deepening of meaning by discovering the true composition of concepts (Proust 1989). And Frege's lifelong concern for the number concept is due not least to the conviction that "arithmetics in the widest meaning of the term produces concepts of such delicacy of composition as it rarely occurs in other sciences", as he says.

Frege made all points quite clear in the preface to his *Grundgesetze der Arithmetik*.

He first describes it as one of the principles of the ideal of a strictly scientific method in the sense of Euclid to indicate those propositions which are used without proof, because it is impossible to prove everything, such "that we can see upon what the whole construction is based". Then he comes to the requirement of being completely explicit, "going further than Euclid", as he says, and also proceeding more in-depth than Dedekind, who has employed quite a number of non-logical concepts in his *"Was sind und was sollen die Zahlen?"*. Frege writes:

"One is generally satisfied if every step in the proof is evidently correct, and this is admissible if one merely wants to be convinced of the truth of the propositions to be proven. If it is a question, however, to provide an insight into the nature of this evidence, this way of proceeding does not suffice, and we must write down all intermediate steps in order to let the full light of consciousness fall upon them. Mathematicians normally are concerned with the content of the theorem and with its being proven. Here the new thing is not the content of the theorem, but the way in which the proof is constructed, and on which foundations it rests" (Frege 1962, VI-VIII; our translation).

We may schematically present our conclusions so far by the following diagram (it needs hardly mentioning that every such classification is abstract and cannot be completely true to reality!):

	explicit	intuitive
foundationalist	**Bolzano**	**Kant**
anti-foundationalist	**Hegel**	**Peirce**

III. The Aristotelian Legacy

Mathematical explicativism, that is, the quest for proofs that really "explain" something or "provide an insight into the nature of mathematical evidence", as Frege had stated it, certainly grew out of the camp of philosophical or logical universalism. How could a model-theoretical conception of truth after all become involved with the question of final and definite explanations?

There exists nowadays an extended and unsurveyable discussion about the problem of explanation, suggesting in particular that the prevailing understanding of the notion comes down to us from Aristotle. Aristotle's *Posterior Analytics* is the first elaborated theory in the Western philosophical and scientific traditions of the nature and structure of science and its influence reaches well into our times. It had long been accepted with such a degree of unanimity that nobody even thought of imputing special merit to Aristotle for his establishment of it.

Nearly all the authors who have during the last thirty years or so concerned themselves with the question of explanation and mathematical proof (Steiner 1978, Hanna 1989, Mancosu 2000, 2001) have alluded in one way or other to Aristotle's notion of "demonstrative science" and in particular to Aristotle's distinction between knowing of the fact and knowing of the reasoned fact, that is, between 'knowing that' and 'knowing why' as a basis for a distinction between explanatory and non-explanatory mathematical proofs, and nearly nobody failed to indicate that Bolzano has been the first modern writer to come back to Aristotle's distinction.

Bolzano seems, in fact, to have been the first modern author pleading for demonstrations "that show the objective connection and serve not just subjective conviction". His "Wissenschaftslehre" (WL; Doctrine of Science; 1836/1929) contains a distinction between proofs that verify, being intended to create conviction or certainty, and others, which "derive the truth to be demonstrated from its objective grounds. Proofs of this kind could be called justifications (Begruendungen) in difference to the others which merely aim at conviction (Gewissheit)" (Bolzano, Wissenschaftslehre, vol. IV, §525). In an annotation to this paragraph Bolzano mentions that the origin of the distinction goes back to Aristotle and the Scholastics, who have, however, as he adds, attributed an exaggerated importance to it by affirming that only justifications produce genuine knowledge, but that the distinction had fallen into neglect in more recent times.

Consulting our scheme of the last chapter one might suppose that Peirce's views on the matter of proof and explanation are opposed to Bolzano's. And Peirce, in fact, claims that mathematicians, in contrast to the philosophers, who hold that no demonstration is “thoroughly satisfactory” except when it is a ‘demonstration why’, “entertain a contempt for that style of reasoning and glory in what philosophers stigmatize as mere … demonstrations that” (Peirce, CP 4.233). Mathematicians also do not trouble themselves “minutely to dissect those parts of method whose correctness is a matter of course” (Peirce, CP 4.240). There is no necessity, Peirce says “for supposing that the process of thought, as it takes place in the mind is always cut up into distinct arguments” (Peirce, CP 2.27). And trying to cut it up so would amount to a paradox “like the Achilles and Tortoise argument of Zeno”. Further one, the human mind can carry out visual processing even when it does not have time or the means for a conceptual analysis, such that perceptual judgments are, on the one hand, the starting point of all explicit argumentation and are, on the other hand, not to be explicitly derived from the immediate percept, because this would again amount to a situation like Zeno’s paradox.

The distinction Steiner and others have drawn between proofs that explain and proofs that merely prove or verify is stimulated, as was said, by the Aristotelian model of science, as it is exemplified, for instance, by Euclid’s *Elements* of geometry and the interest by philosophers and math educators in proofs “that explain” contributes certainly to the current interest in this model. E. Casari had even claimed “that whereas the new axiomatics … can really be seen as a tool … of set-theoretical thinking which grew out of the 19th century analysis, axiomatics in the sense of Euclid, i.e., logical analysis and organization of intelligible concepts and meaningful sentences, seem to remain an irreducible, fundamental tool of our thinking” (Casari 1974, 61).

The Aristotelian model has been described by Th. Heath (1949), E. Beth (1965) and more recently by R.D. McKirahan (1992) and W. de Jong (2001).

An Aristotelian science, according to these descriptions, comprises of a system of fundamental concepts such that any other concept is composed and is definable in terms of these fundamental concepts and it also contains a system of fundamental propositions such that all other propositions are grounded in and are provable from these first premises. And the fundamental concepts or premises should be indubitably clear and certain such that no further justification is called for. Explanation in such a context meant reduction to a set of ontologically as well as epistemologically privileged principles and propositions. Demonstrations in the sense of Aristotle are also explanations. “To prove *that* a conclusion holds is simultaneously to show *why* it holds. Thus proofs do more than indicate logical relations among propositions; they also reveal the real relations among facts, and scientific knowledge involves not only that but also why” (McKirahan 1992, 4). Aristotle did not believe that merely logical reasoning could explain anything. And he discusses the difference between knowledge of the fact and knowledge of the reason by the following example.

“Let C stand for *planets*, B for *not twinkling*, and A for *being near*. Then it is true to state B of C ... But it is also true to state A of B; ... Then A must apply to C; and so it has been proved that the planets are near. Thus this syllogism proves not the reason but the fact, for it is not because the planets do not twinkle that they are near, but because they are near they do not twinkle" (Aristotle, Post. Analytic, Book I, chapter 13, 78a-b). Bolzano uses a quite similar example resp. argument (WL, §198).

Neither the logician nor the man of mere experience possesses real knowledge of causes. McKirahan has drawn attention to another important aspect of Aristotelian science, namely that it is to be defined by its special subject matter. Each science treats, according to Aristotle "a limited range of things or phenomena, its subject genus, as arithmetic studies numbers and geometry spatial magnitudes. The subject genus can be regarded as a structured collection of subjects and attributes in certain relations. … Further, science deals primarily in the universal and necessary, not the particular and contingent. It treats individuals not in their own right but as falling under universals" (McKirahan 1992, 3/4).

We have knowledge of a particular thing, according to Aristotle, when we know what it is (Metaph. B 996b). This means that the universal determines the particular. Analytical philosophy in the sense of Frege would say that existence makes sense only as a second order predicate and contingent fact is not a matter of mathematics or logic. To examine the principles of mathematics belongs among the tasks of philosophy, according to Aristotle, "because mathematics investigates its subject matter not qua things as such, but only insofar it represents a continuum of various dimensions" (Metaphysics, 11. Book 1061a). The continuum represents the unity of being and quality as given in the immediate percept and mathematics is therefore a part of first philosophy which deals with the metaphysics of continuity.

From the primacy of subject matter results the exigency that demonstrations should be "pure", that is, should not be constituted by means that are foreign to the subject matter, "like to prove a geometrical proposition by arithmetic" (Aristotle, Posterior Analytics, I.VII). This homogeneity between method and object was important to Bolzano too, but it is totally alien to modern axiomatic mathematics! On occasion of a lecture by H. Wiener in 1891 Hilbert made a remark which has become famous and "which contains the axiomatic standpoint in a nutshell: It must be possible to replace in all geometric statements the words point, line, plane by table, chair, mug" (Reid 1970, 264). On such an account axiomatic mathematics can be called a doctrine of forms, rather than meanings or substances.

According to Peirce it has exactly been this change of (geometrical) axiomatics which finally dethroned the Aristotelian model. "Metaphysical philosophy may almost be called the child of geometry", he says. "Aristotle derived from the study of space some of his most potent conceptions. Metaphysics depends in great measure on the idea of rigid demonstrations from first principles; and this idea …. bears its paternity on its face. … The absolute exactitude of the geometrical axioms is exploded; and the corresponding belief in the metaphysical axioms, considering the dependence of metaphysics on geometry must surely follow it to the tomb of extinct creeds. The first to go must be the proposition that every event in the universe is precisely determined by causes according to inviolable law" (Peirce, CP 1.400-402).

Today complexity theory in the sense of Kolmogorov and Chaitin claims that even in pure mathematics there are theorems which are true for no reason at all. Furthermore, rigorous mathematics had been accustomed to consider the incompleteness phenomenon as something on the fringes of real mathematical practice, something irrelevant and happening in very unusual pathological circumstances only. With the advent of the computer things have changed, however. A real number, for example, as a rule is not computable, because the computable numbers form a countable subset only (presupposing Turing's Thesis about computability). Randomness becomes a central notion in foundational considerations (Mumford 2000) and frequently now mathematical facts are discovered, which are random and have no proof at all and thus "are true for no reason. They are true by accident!" (Chaitin 1998, 54). Such kinds of views were certainly stimulated by the experiences of computer scientists. Computer mathematics is often compared to a game of chess, by pure mathematicians, and as a rule they do not fail to observe that chess is "somewhat trivial mathematics" (Hardy 1967, 88; Davis/Hersh, Mathematical Experience). Humans think in ideas rather than in terms of pieces of information, it is said. But what does that mean? Are ideas mental experiences or mirrors of a given world? Peirce would say, they are signs. Whatever they are, one should certainly avoid the conclusion that ideas per se dominate reality, such that having an idea would be sufficient to solve the problem at hand.

Aristotle's model of science became questioned when "reality" meant no more something statically given either "out there", or in Platonic 'heaven', but reality began to be conceived of in terms of the system of human (cognitive) activity and practice itself. Descartes opposed Aristotle's conception of science vehemently. He believed in the primacy of problem solving and construction and considered the matter exclusively from a methodological point of view. Already before Descartes there existed controversies about the explanatory character of Euclidean proofs. The proof of Euclid's theorem I in Book I, or very similar proofs, have been indicated over and again as examples of mathematical proofs that do <u>not</u> explain, because the constructions at hand do not follow, it was criticized, from the essence of the figures themselves (Schueling 1969; Mancosu 1996).

Constructions using compass and straightedge have a long history in Euclidean geometry. Their use reflects the basic axioms of this system. However, the stipulation that these be the only tools used in a construction is artificial and only has meaning if one views the process of construction as an application of logic. Descartes used new special instruments which he called "new compasses" for tracing curves as means of construction, demarcating between geometrical and non-geometrical curves. The acceptable geometrical curves were those that had an algebraic equation (Bos 2001, 336). Descartes method was twofold, comprising an analytical and a foundational part. For the analytical part algebra and the use of algebraic curves was responsible, whereas it is the continuity of the curves constructed and thus geometry, which secured the existential claims at stake, not the fact that they can represented by algebraic equations.

Kant's notion of mathematics was also constructive and foundational in this geometrical sense. But by making space a form of apriori intuition he made the dependency of mathematics from our ways of world experiencing more explicit. Mathematics according to Kant is not analytical knowledge from mere concepts, but requires the construction of these concepts in the intuition of space and time. Kant compared Platonic idealism to the "light dove" which "cleaving in free flight the thin air, whose resistence it feels, might imagine that her movements would be far more free and rapid in airless space" (Kant, Critique, B 9).

Kant was no empiricist, however, but was a constructivist, considering the laws of mathematics legislative with respect to empirical experience. A "new light" (Kant) must have flashed on the mind of people like Thales, when they perceived that the relation between the length of a flagpole and the length of its shadow enables one to calculate the height of the pyramid, given the length of its shadow. "For he found that it was not sufficient to meditate on the figure as it lay before his eyes, …. and thus endeavor to get at knowledge of its properties, but that it was necessary to produce these properties, as it were, by a positive a priori construction" (Kant, *Critique of Pure Reason*, Preface to the Second Edition 1787; see also *Critique,* page B 744). And indeed, the flagpole in itself has no positive relationship whatsoever to the pyramid as such.

Bolzano said that Kant, in claiming that mathematics is essentially diagrammatic reasoning, had confounded mathematics in itself with the way we humans might come to know and to develop it. Bolzano's attitude was strictly non-psychologistic. What Bolzano had in mind when looking for proofs that explain has nothing to do with the psychologistic colorings the term "explanation" has received from recent philosophy and education. It just means what Frege had spelled out so clearly: purity of system and utmost explicitness of reasoning.

Diagrams may facilitate our reasoning, Bolzano admits. But all of Kant's claims with respect to the role of visual diagrams, Bolzano continues, he "need not necessarily see, …. , but could deduce them from concepts. And if we take into account that the seeing in question is not an immediate perception, but must be (unconsciously) deduced from such perception by adding various geometrical truths, we shall hardly elevate such judgments, gained without distinct consciousness to the status of a separate source of knowledge" (Bolzano WL 3, p. 187). Bolzano may be right from his point of view, although his insistence on conceptual explicitness is more a logician's interest, than a mathematical one and it represents conviction in the autonomy of abstract reason. It shows moreover that Bolzano did not consider the possibility of implicit knowledge and experience, that could not be transformed into explicit propositional form, but would nevertheless have effects on human action and reasoning.

Kant was more concerned with questions about the applicability and objectivity of mathematics and his diagrams might perhaps better be interpreted as models and models are something different from systems of propositions. Bolzano seems no more interested in epistemological questions. He seems to have been the first in history not considering mathematics as a means of mediation between the human subject and objective reality, as Kant had done (see, for example, his introduction to the Second Edition of the *Critique of Pure Reason* of 1787), but seems to have understood science and mathematics as realities *sui generis*, and as objects of study in their own right. We need, according to Bolzano, no external certainties, as Descartes or Kant do, in order to begin to philosophize. Considerations of logical consistency are sufficient to convince us, says Bolzano, that there are “truths in themselves” and this is all that is needed. Therefrom results the title of his monumental work, *Wissenschaftslehre* (doctrine of science). Questions of semantics not epistemology became the fundamental concern of philosophy and logic (Coffa 1991). Perhaps for the first time, writes Jean Cavailles, “la science n’est plus consideree comme simple intermediaire entre l’esprit humain et l’etre en soi, … mais comme un objet *sui generis*, original dans son essence, autonome dans son mouvement” (Cavailles 1976, 21).

Bolzano wanted a new definition of science, in terms of texts or treatises, each comprising of a system of truths about a specific area of knowledge, and he framed a new conception of logic, namely as “doctrine of science”. Bolzano seemed astonished that apparently nobody before him, had given these “simple definitions” (Bolzano, WL vol. I, 18). We conclude therefore that Bolzano’s alleged Aristotelism is a myth, or at least, is only half true. Aristotelian science, differently from Kant`s, was determined by its object, by what was thought and the coherence of an Aristotelian science is guaranteed fundamentally by the connection of its concepts and propositions to a specific domain of objects. Even de Jong acknowledges that “Bolzano is rather critical” on these point (de Jong 2001, 332). He also mentions Bolzano’s assertion that basic truths need by no means be self-evident (p.333).

One might doubt, moreover, that the above characterization of Aristotelian demonstrative science is complete or even that it captures the essential aspects. Although Aristotle is recommending the axiomatic of geometry as a paradigm of demonstrative science in his *Posterior Analytics,* he does not necessarily conceive of its proof procedures in exactly the way as they are commonly understood nowadays. Aristotle is most often regarded as the great representative of a logic and mathematics, which rests on the assumption of the possibility of clear divisions and rigorous classification. “But this is only half the story about Aristotle; and it is questionable whether it is the more important half. For it is equally true that he first suggested the limitations and dangers of classification, and the non-conformity of nature to those sharp divisions which are so indispensable for language [...]” (Lovejoy 1964, 58). Aristotle thereby became responsible for the introduction of the principle of continuity into natural history. “And the very terms and illustrations used by a hundred later writers down to Locke and Leibniz and beyond, show that they were but repeating Aritotle's expressions of this idea" (Lovejoy loc.cit.).

It also seems that in Greek mathematics occurred two different kinds of proof. "During the first phase of Greek mathematics there a proof consisted in showing or making visible the truth of a statement". This was the epagogic method. "This first phase was followed by an apagogic or deductive phase. During this phase visual evidence was rejected and Greek mathematics became a deductive system" (Koetsier 1991, 180f; and the bibliographic reference given there).

Epagoge is usually translated as "induction". But it is perhaps not quite what we think of as induction, but is rather taking one individual as prototypical for the whole kind. Such kind of reasoning is often called "abductive reasoning". Abduction is also characterized as inference to the best explanation and is based on some law of continuity or uniformity: that this triangle has property *A* may best be explained by assuming that *A* holds in general and for all triangles, that is, this triangle is a general or prototypical triangle. Aristotle writes with respect to epagoge: "The consideration of similarity is useful both for inductive arguments and for hypothetical reasoning [...] It is useful for hypothetical reasoning, because it is an accepted opinion that whatever holds good of one or several similars, holds good also for the rest" (Topics 108b 7). So we translate epagoge as abduction or hypothetical reasoning as based on continuity.

We have presented elsewhere an epagogic proof of the theorem about the Euler line of a triangle and have shown (Otte 2006) that a demonstration in the Aristotelian sense typically was conceived of as a whole continuous process, in which the individual steps of inference are based on associations of ideas and remain largely implicit, rather than as a series of distinct logical inferences. Analogy, metaphor, model-based and abductive reasoning, all played an essential role in constituting *epagoge.*

Although continuity seems widely banned from the Euclid's *Elements,* even the diagrams of Euclid could nevertheless be interpreted in two complementary ways, as I Mueller has argued.

"Under one interpretation the statement (to be proved, my insertion) refers to a definite totality.... and it says something about each one of them. Under the other interpretation no such totality is supposed and the sentence has much more conditional character" (Mueller 1969, 290). Euclidean diagrammatic reasoning could from such a point of view better be described in terms of the notion of "thought experiment", that is, "involving an idealized physical object, which can be represented in a diagram" (Mueller 1969, 291). A general diagram in this sense is like a natural kind.

Now this kind of seeing a geometrical diagram was prevalent in epagogic proof and it gained much in weight during the 19th century and with the development of projective geometry on grounds of the continuity principle (see Poncelet (1788-1867); Hankel

(1839-1873)). Geometry itself was transformed in fundamental ways, becoming essentially theory of space, rather than an inquiry into the properties of geometrical figures situated in a “metaphysical nowhere land” (Bochner).

Analytical philosophy and modern mathematics eliminated continuity and with it metaphors, prototypes and natural kinds because the underlying similarity relations are completely alien to formal logic and set theory. Quine recognizes, for example, that “a sense of similarity or of kind is fundamental to learning in the widest sense” (Quine 1969, 129), but he suggests that “it is a mark of maturity of a branch of science that the notion of similarity or kind finally dissolves” (p. 121). Modern scientific or mathematical disciplines construct each their own specific notions of similarity or equivalence, because each of them represents only its particular and partial perspective on our world. The branches of science could perhaps be classified, writes Quine, “by looking to the relative similarity notion that is appropriate to each. Such a plan is reminiscent of Felix Klein’s *Erlanger Program* in geometry” (p. 137). Our proof of Euler’s theorem could have perhaps been cut short in this manner by classifying the theorem as belonging to projective geometry, but this would have been difficult or even impossible for the average learner. The complementarity of space or continuity, on the one side, and of structure and set, on the other, could perhaps be of interest to a genetical view on mathematics. We shall come back to this issue in the next chapter.

Since the 19th century ever greater parts of mathematical activity were devoted to proof analysis. Modern mathematics became more and more interested in finding simpler, shorter and clearer ways of proving results which are already known. The essential interest of such work is in logic and structure and is due to the question of whether certain results are in the reach of certain methods. And with respect to these methods clear understanding and obvious meaning is the concern, like in the case of Descartes. Writing down a formula leads now to the question, what does it mean, what are its necessary prerequisites. Writing down a Taylor series or a Fourier series may stimulate investigations into the meaning of notions like continuity, differentiability or integrability.

Fourier constructed new representations of a very large class of functions using the integral and during the rest of the 19th century mathematicians spent a good deal of effort determining the range of functions for which Fourier’s assumptions are valid. And the essential chapter of Riemann’s famous *Habilitationsschrift* begins with the classic statement: “First then, what do we understand by $\int f(x)dx$?” Riemann by asking what the symbol $\int f(x)dx$ means, searched for necessary conditions of integrability, rather than some sufficient properties of the functions to be integrated, like continuity, as Cauchy did and thereby was able to generalize the notion of the integral. Riemann always searched analytically for necessary conditions, primarily to exclude arbitrariness of hypotheses. But Riemann also assumed a definitely anti-foundationalist attitude, emphasizing repeatedly that “der Begriff der Endursachen ganz aus dem Spiel bleiben kann" (that we need not bring into play the concept of final cause).

Or look at Frege, who was a foundationalist: "The problem (concerning infinitesimals) is not … to produce a segment bounded by two distinct points whose length is dx, but to define the meaning of an equation like df(x)=g(x)dx" (Frege, Grundlagen Par. 60). In Frege or Bolzano this quest for meaning had a foundationalist flavor and impetus, certainly, but in general this need not been so. It depends whether one concentrates on propositions or on concepts. Riemann's intention was to construct concepts in the first place.

Lebesgue somewhat later modified and generalized Riemann's conception, basing the theory of integration on a theory of measure and admitting infinitely additive measures. Lebesgues' notion of the integral actually was not only a generalization of Riemann's conception, but "opened up a completely new field and defined a new framework for the concept of integral" (Th. Mormann, Towards an evolutionary account of conceptual change in mathematics, in: G. Kampis et.al. (eds.), *Appraising Lakato*s, Kluwer Dordrecht 2002, 139-156, 151). It also forced the presentation of such a theory in axiomatic form, which was somewhat against Lebesgues' own wishes (Otte 2007).

Mathematical concept evolution can, in fact, perhaps best be understood in terms of axiomatic concept variation. Grassmann's definition of non-commutative vector product is an early important example, suggested by the mathematization of electricity. Axiomatic or form based diagrammatic reasoning can be seen from a foundationalist as well as from an evolutionary perspective.

IV. Diagrammatic Reasoning and Creativity

Peirce has claimed that all necessary reasoning is essentially diagrammatic and is as such a mixture of analysis and generalization, or analysis and synthesis. Mathematics is not simply straightforward conceptual thinking because mathematical reasoning deals essentially with the relation between general and particular!

Peirce saw well that what imports in mathematics is the thorough understanding of particular situations and individual examples. "The source of all great mathematics is the special case, the concrete example. It is frequent in mathematics that every instance of a concept of seemingly great generality is in essence the same as a small and concrete special case" (Halmos, 1985). Furtheron thorough study of examples or problem situations may suggest the essential concepts. Mathematical generalization is primarily to be conceived as the introduction of new abstract objects by means of abductive or hypothetical reasoning. Peirce writes: "Hypothesis substitutes, for a complicated tangle of predicates attached to one subject, a single conception" (Peirce, W3 337). Such hypotheses are meant to help carry through deductive argumentation. Peirce would, however, negate that theory is purely instrumental and would rather claim that useful hypotheses express objective possibilities, that is, are objective universals.

One uses the general and abstract to explain the particular and concrete, or seemingly concrete, in exactly the same manner in which Newton's laws are used to explain simple mechanical phenomena, or Ohm's law is used to explain the facts of electricity or Peano's axioms explain the facts of arithmetic. The general, as used in scientific explanations of such kind, in our case, for instance, the associative law of algebra, is less sure from a concrete empirical point of view and less positive than the individual facts to be founded on it. The less certain is used to explain the more certain; so, neither science nor mathematics have definite foundations, although every argument must start from some knowledge accepted as true.

Mathematical deduction thus becomes "really a matter of perception and of experimentation, just as induction and hypothetical inference are; only, the perception and experimentation are concerned with imaginary objects instead of with real ones" (Peirce, CP 6.595). Peirce calls mathematical reasoning that depends on generalization *theorematic.* In theorematic reasoning the mathematician must handle an abductive strategy capable of integrating the missing information. The mathematician constructs and manipulates or modifies a diagrammatic representation of the premises in order to find out that *foreign idea* — to use Peirce's expression — which must be added to the set of explicit premises already available in order to carry an argument forward.

The opposition of the theorematic and deductive or corollarial reasoning comes down ontologically to the famous antinomy of the indivisibility and the infinite divisibility of Space and Matter, which had been known from the days of Xenon. This antinomy consists in the fact that discreteness must be asserted just as much as continuity. The antinomy is expressed in the opposition between Leibniz two fundamental principles, the "Principle of the Identity of Indiscernibles", on the one hand, and the "Principle of Continuity", on the other. Leibniz tried to resolve the conflict by distinctions between the real and the ideal, or the factual in contrast to the merely possible. The possible, not being fully determined is a continuum and a general and as such is opposed to unexplainable contingent fact and fully determined existence.

The essential problem of theorematic reasoning, and of creative behavior generally, is to see an A as a B! How to decide whether $A = B$? Or even to see the meaning of $X = X$? There are essentially two types of operations involved in such a kind of problem, that is, to perceive similarities and to draw distinctions. In a continuous world of mere possibilities everything seems related to everything. In a universe of distinct individuals, in contrast, relational thinking seems unlikely. We interpret an equality like $A=B$ as the goal to find that seemingly foreign idea or theory, which helps to establish an objectively viable relation between A and B. The situation is analogous to the establishing of proof by means of theorematic reasoning.

Theories have, as we have seen, quoting Quine, their own specific equivalence relations, but theories are not theories of their own application. Arithmetic applies when it applies, Lebesgue had famously said. And when the child in a New Math classroom was asked how many things were in the cage after a wolf, a rabbit and an apple had together been put into it, arithmetic did not apply actually and did not help to answer the question.

Logic does not much help either, contrary to what Frege and Russell had believed. A girl was terrified in the biology exam because she did not see the equivalence between the statement *A*: *"Little round penguins do not freeze as much as tall and slim ones"* and the other *B*: *"In hot summer little fat man are sweating more than tall slender ones"*, both, *A* and *B* being meant to be consequences of mathematical theory about the relationship between volume and surface of geometrical bodies, rather than as depending on biological knowledge. Biological experience might be necessary, however, to see whether the mathematical argument applies or not.

There are no universal and context independent criteria to make decisions like the one with respect to the meaning of $A = B$! Is the catenary a parabola, for example? Galileo believed that it is one; others like Huygens, Leibniz or Bernoulli proved him wrong. Topologically, however, the catenary is indeed a parabola, as everybody can see easily. But topology does not matter much, neither in algebraic calculus nor in civil engineering! There we have to deal with distinct measures and forms or with forces and accelerations. If we define parabola and catenary in analytical terms the difference is easily seen, considering the growth rate and other characteristics. Theory and problems are circularly connected as we also perceive from this example.

Algebra constructs properties of mathematical objects rather than these objects. Cardano might have known from the continuity principle, for example, that a cubic equation must have a real root, but in certain cases his well-known formula did not yield this root. Algebra shows that mathematics is essentially intensional, that is, its formulas represent activities and thus represent hypostatic abstractions, abstractions from action. S. Bochner has considered this, that is, "abstraction from abstraction, abstraction from abstraction from abstraction, and so forth" as the decisive feature of "modern" mathematics since the 16th/17th centuries. And he continues: "On the face of it modern mathematics … began to undertake abstractions from possibility only in the 19th century; but effectively it did so from the outset" (Bochner 1966, 18, 57).

Algebra shows what Frege has called, in his famous essay on *"Über Sinn und Bedeutung"* [On Meaning and Reference] the mode of presentation, it maps, in fact, the mathematical activity itself and thereby becomes an analytical tool so far as analysis can be made explicit. Algebra is algebra on algebra, as the English mathematician J.J. Sylvester (1814-1897) once remarked. In dynamic geometry software systems, like Cabri, making continuity directly accessible this analytical role of algebra comes out even more clearly (see Otte 2003a, 206ff). Algebra is certainly not absolutely indispensable to solve geometrical problems or to frame geometrical theories, but algebraization expresses a radically instrumental view of mathematical knowledge, which has its advantages and disadvantages, especially from a genetical point of view..

To have seen the dependence of mathematics on form makes up an essential part of Descartes' and Leibniz' achievements which resulted in Leibniz' creation of formal mathematical proof in the modern sense (Hacking 1984). In the 19th century this achievement led to an ideal of mathematics which asked before trying to solve a problem whether such a solution were in fact possible relatively to certain means. Rather than trying to construct a mathematical relationship in a certain manner, one first asks, "whether such a relation is indeed possible", as Abel stated in his memoir *On the Algebraic Resolution of Equations* of 1824, in which he presented one of the famous impossibility proofs of modern mathematics.

Assuming the quintic to be solvable in radicals, Abel deduces the form of a solution, showing that it must involve rational functions of the roots; he then uses a theorem of Cauchy, on the values a rational function of five quantities can assume when those quantities are permuted, to reach a contradiction. Abel's procedure of indentifying the essential form of a solution is the very same used in other impossibility proofs. In order to prove, for example, that the Delian problem, the doubling of the cube is impossible with Euclidean constructive means, on represents the "constructible numbers" in algebraic form showing that the third root of 2 cannot have this form and thus cannot be a constructible number. What seems missing here are the ideas of continuity and transformation of forms and structures into another. But Lagrange already brought on this problem of the solution of algebraic equations by radicals "new methods, involving attention to transformations or mappings and their invariants" (Stein 1988, 240).

On the basis of relational thinking the continuity principle served, for instance, to assure the existence of a solution of the Delian problem, which cannot be solved constructively with ruler and compass alone, but can be solved if one uses the continuity principle, admitting, for example, conics as legitimate instruments of construction (as Descartes did). But because of lack of interest in logic and theory it took until 1834, when Wantzel showed by diagrammatic means the constructive insolvability of the Delian problem. Mathematical solutions are just images or forms. All impossibility proofs up to Goedel's famous incompleteness results proceed more or less in this manner.

Diagrammatic reasoning, as based on the idea of continuity (or of transformation) and invariance, corresponds more or less to what Hintikka has called "language as a calculus" and it is opposed to foundationalism. The following example tries to illustrate this claim. The well-known Gestalt psychologist Max Wertheimer (1880-1943) thought completely different and made some commentaries on the presentation and solution of Zeno's paradoxes by means of a geometric series that is current in present day mathematics. Rather, he comments on the current proof of the convergence of that series, which is accomplished by multiplying the series by a and subtracting afterwards. Set $S = 1 + a = a^2 + ...$ Then $S - aS = 1$ or $S = 1/(1 - a)$.

Wertheimer writes: “It is correctly derived, proved, and elegant in its brevity. A way to get real insight into the matter, sensibly to derive the formula is not nearly so easy; it involves difficult steps and many more. While compelled to agree to the correctness of the above proceeding, there are many who feel dissatisfied, tricked. The multiplication of $(1 + a + a^2 + a^3 + ...)$ by a together with the subtraction of one series from the other, gives the result; it does not give understanding of how the continuing series approaches this value in its growth.”

Wertheimer wants an intuitive demonstration. Intuition is, however, essentially the seeing of the essence of a thought or object as a form or object itself and therefore the diagrammatic proof which Wertheimer does not accept as satisfactory, could be called an intuitive proof. Only intuition is now directed towards the diagrammatic representation itself and to its form and it becomes therefore a means of cognition, rather than being a foundation of knowledge.

Let us dwell on these different conceptions of intuition trying to briefly sketch an anti-foundationalist view of knowledge from such a perspective.

Peirce, being convinced that meaning and thought cannot be reduced to either quality or feeling, on the one hand, and mere reaction, on the other, but rather requires mediation between the arbitrariness of intuitive associanism and the absolute determinism of external compulsion, speaks of the necessity of a third mediating element which he called “synthetic consciousness” (Peirce, CP 1.377). This kind of synthesis, which is neither the fruit of mere associations by resemblance nor of mere necessity, is stimulated by the creative constructions accomplished by the artist, the mathematician or the man of science in representing and solving a problem and is thus mediated by representations, like diagrams, models or works of art. There is, in fact, we believe, no creative process or activity without a product, a work of art or a theory or whatever. Thinking occurs by means of signs or in signs.

And everything we have constructed is just done and is there in the plain light of the day. It means per se nothing, it is just there; Hegel would have said, it is abstract! But is poetry so abstract, Peirce asks. The present, whatever one might think of it, “is just what it is regardless of the absent, regardless of past or future” (Peirce, CP 5.44). A work of art or a mathematical theory is anti-narrative and it neither needs nor deserves interpretation or commentary. Such a commentary or interpretation would just be another creation and would add nothing to the thing created and given. To exist in this way it must only have a certain consistency. “Consistency belongs to every sign, so far as it is a sign; therefore every sign, since it signifies primarily that it is a sign, signifies its own consistency” (Peirce, CP 5.313-15).

An action is an action, a work of art is just a work of art, a theory is just a theory. It must be grasped as a form *sui generis,* before we can inquire into its possible meanings or applications. In artistic drawing what we achieve is a line, and the line does all the work, and if it fails to do so no philosophical commentary will rescue or repair a bad work of art. In literature or philosophy, it is the word or the sentence, in mathematics the new concept or the diagram which carry the entire weight, etc. etc. Mastery, Paul Valery, says, presupposes that “one has the habit of thinking and combining directly from the means, of imagining a work only within the limits of the means at hand, and never approaching a work from a topic or an imagined effect that is not linked to the means” (Valery, 40).

Everything just is and thus means itself: P=P! This principle of identity lies at the heart of art as at that of mathematics or exact science and it is obviously directed against any historical or evolutionary concerns. P just means P! No comment or historical investigation, no psychological or philosophical consideration shall be able to add anything to the matter. “Painters despise art-critics and mathematicians have usually similar feelings”, wrote G.H. Hardy and he added somewhat exaggeratedly, “Exposition, criticism, appreciation is work for second-rate minds” (Hardy 1967, 61).

Looking on mathematics in this way, however, leaves it as set of completed works and finished theories that might reveal their secret beauty to the talented discoverer sometimes, but which could not be taught nor learned. Being a mere form of reality, or a reality *sui generis,* it has nothing to do with human activities. One might feel its consistent presence and internal harmony but does not know to master it or develop it further. Such a view does not allow, for example, considering unresolved problems.

This is not good, because great problems and practices of their analysis and investigation amount to the greater part the “real” history of mathematics, such that it might become especially important to identify the most appropriate ways to set problems up, as well as the proper contexts in which to address them. This might then seduce people to aspire for utmost explicitness. Mathematics should be considered from the perspective of application, that is true. But the application of general knowledge to particular situations is not to be completely anticipated and is not governed by explicit rules or laws. Bourdieu has called this the “fallacy of the rule”. Application and the know-how it requires are a different kind of thing that cannot be spelled out explicitly. Something must be done in good faith and with critical awareness.

And in the process of application a theory shows in fact its meaning. At this very moment it assumes a very different nature. It becomes a sign or an idea, which helps to orient problem solving activity. That the meaning is not contained in the idea or theory itself can be seen from the fact that the latter’s intended applications are not predetermined and given once and for all. Therefore the desire of searching for ever more “fundamental” meanings is disastrous or detrimental. The common insistence on absolute “What is”-questions is of no use at all. “What is the Number One?”, is one such question, having become popular after Frege.

There exists, in fact, among philosophers and humanists a widespread conviction that to understand means to interpret and that the rules of such interpretation are given by actual linguistic custom. It is believed that everything must have a definite meaning. Susan Sontag has called this belief a “revenge of the intellect upon the world”

(Sontag 2001, 7). The modern style of interpretation, she writes, "excavates, and as it excavates, destroys; it digs behind the text, to find a sub-text which is the true one" (Sontag 2001, 6). Susan Sontag is right: an obsession with meaning or meaningfulness characterizes those who believe that they deserve more than the world seems prepared to offer them.

The meaning of a thing is, however, nothing but the meaning of a representation of that thing and the latter is thus just another representation. To interpret is to construct another representation; a representation that might finally dissolve the clouds around a given problem. The semiotic description of creativity also intends to convey the idea that creativity is not a consciously preplanned and mentally controlled process, where the real production amounts to nothing more than an ex post execution of some ideas in the head, as if the medium or representation were nothing more than a rather arbitrary clothing. And conversely, reality is not something "out there" and invariably given, but consists in the subject-object interaction itself. Reality might be the sign process itself, depending on the type of activity. Paraphrasing Collingwood, who had said that all history is the history of thought, one might claim that all mathematical history is the history of form or formal representation. And the formal impossibility proofs show exactly that.

The common tendency to regard Gödel's incompleteness results, for example, as vindicating those who have emphasized the primacy of intuition, as opposed to those who emphasize with Hilbert, Gödel or Kolmogorov the importance of formalism, proves rather superficial, because it ignores "that the very meaning of the incompleteness of formalism is that it can be effectively used to discover new truths inaccessible to its proof-mechanism, but these new truths were presumably undiscoverable by any other method. How else would one discover the 'truth' of a Gödel sentence other than by using a formalism meta-mathematically? We have here not only the discovery of a new way of using a formalism, but a proof of the eternal indispensability of the formalism for the discovery of new mathematical truths" (Webb 1980, 126/127).

Axiomatics and formal proof have little to do with founding a discipline, even though that could have been the motivation for establishing them. They are simply ways of organizing some field and thus to make its frontiers and alternatives or possible generalizations clearer or even imaginable in the first place. Axiomatics in the traditional sense seemed to furnish foundations. But when Euclid axiomatized geometry what he really accomplished could be interpreted as the exhibition of the possibility of alternative, non-Euclidean geometries and thus of mathematical generalization.

The essence of knowledge is its growth and insight begins at the frontiers of knowledge. Mathematics and science surprise established expectations more often, than they confirm them. Such insights are not always welcome, however, not even among mathematicians.

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I NEED THE TEACHER TO TELL ME IF I AM RIGHT OR WRONG

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This talk presents some thoughts about the possible reasons of students' tendency to rely on teachers for the validity of their solutions and of their lack of sensitivity to contradictions in mathematics. Epistemological, cognitive, affective, didactic and institutional reasons are considered in turn.

SOME FACTS TO EXPLAIN THE CONTEXT OF THE QUOTATION IN THE TITLE

The facts come from a research on sources of frustration in adult students of pre-university level mathematics courses required by a university for admission into academic programs such as psychology, engineering or commerce (Sierpinska, 2006; Sierpinska et al., 2007).

Fact 1. In a questionnaire[1] used in this research there was the following item:

> *I need the teacher to tell me if I am right or wrong.*
> Agree Disagree Neutral

Of the 96 students who responded to the questionnaire, 67% checked "Agree".

Fact 2. Six respondents were interviewed. One of them, female, about 21 years old, a candidate for admission into commerce, was required to take a calculus course. She failed the first time round. She re-took the course in the summer term and passed, but found the whole experience extremely frustrating. Here is what she told us, among others:

> *My teacher in the summer, he was a great teacher, he explained well and everything, but it's just that I could never grasp, like I couldn't be comfortable enough to sit down in front of an example and do it on my own, instead of looking back at my notes. Okay, what rule was it and why did I do this? I just never understood the logic behind it, even though he was a great teacher, he gave us all possible examples and he used very simple words…, start from the very easy and try to add things on to make it more difficult. But… how I studied for the final? I was looking at the past finals… All by memorizing, that's how I passed [this course] the second time.*

Fact 3. In items 74 and 75 of the questionnaire, students were asked for their preference regarding two kinds of solutions, labelled "a" and "b", to two inequalities with absolute value ($|2x-1|<5$ and $|2x-1|>5$, respectively). Solutions "a" could be called "procedural"; they are commonly taught in high schools and consist in reducing the solution of an inequality to solving two equations and then following certain rules to write the final solution to the inequality. Solutions "b" resembled those taught in

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introductory undergraduate courses focused on logic; they referred explicitly to properties of absolute value and use logical deduction. In item 75, solution "a" ended with an incorrect answer: a condition on x which contradicted the initial inequality. Nevertheless, in both items, there was a clear preference for solutions "a" (69% chose this solution in item 74 and 62% in item 75). Only about 1/5 of the respondents chose solutions "b" in each item. The choice of these solutions was almost always justified by reasons other than correctness: "clearer", "easier", "simpler".

Fact 4. Four instructors of the prerequisite course were interviewed. All reported students' dislike of theory and proofs and preference for worked out examples of typical examination questions. They said that students prefer to memorize more rules and formulas than to understand how some of them can be logically deduced from others and memorize fewer of them. They reported eventually giving in to students' preference and avoiding theory and proofs in their classes.

One of the instructors (female, PhD student) told us:

> *[Students don't want to reason from definitions about] those rules, [although] all the rules come from the definition (...) It's especially true when we learn (...), the seven rules of exponentia[tion]. Sometimes, I just try to let them know that [it is enough to just] know four [rules], or even three, if one knows the definition well. You don't need to put so much time on recalling all those rules in your mind. But when I try to explain those things, they don't like it. They ask me 'Why, why you do this?'*

Another instructor (male, PhD student) was telling us that he would do very little theory in class, replacing proofs by graphical representations, and giving significance to theorems, formulas and methods by using historical anecdotes:

> *Actually, to be honest, I don't do much theories, or proof or anything like this, you know, in such classes. I try to avoid it as much as I can, but let's say about the integral thing that I just did, I filled out all the rectangles with colours, and then I told them this is fun, then I put the definition of definite integral, then I put a remark: 'In fact this is a theorem, you know... and it was proven by Riemann...'. Then I told them about Riemann a little bit, they were happy, and that he still has problems and it's worth many dollars to solve this. So I made the mood and then I moved to fundamental theorem of calculus, just the statement without proof, without anything you know, then start giving examples. (...) I don't think they like proofs because in a proof you cannot put numbers or anything, you have to do it abstractly and this they hate. Yeah, they don't like this.*

EDUCATIONAL VS ECONOMICAL GOALS OF MATHEMATICS TEACHING

Why teach mathematics? According to Ernest (2000), the answer depends on who is speaking. From the perspective of economic theories of education, mathematics may be seen as contributing to general purposes of education such as,

- Building human capital by teaching skills that directly enhance productivity;
- Providing a screening mechanism that identifies ability;

- Building social capital by instilling common norms of behaviour, and
- Providing consumption good that is valued for its own sake.

(Gradstein et al., 2005: 3).

Most educationists eschew using such pecuniary terms when they speak of the purposes of education although they do realize that words such as "capital" and "consumption good" might better reflect the current reality than their ideals. After all, they do invoke financial issues when they discuss the reasons of the difficulty of achieving their preferred goals of education, or deplore the tendency of some universities to become "'corporate entities', where students are 'clients' and traditional values – 'raising the better-informed citizen' – are losing ground to job-training" (Curran, 2007).

Educationists prefer to continue viewing the purpose of education as "to provide rich and significant experiences in the major aspects of living, so directed as to promote the fullest possible realization of personal potentialities, and the most effective participation in a democratic society", based on "reflective thinking", with mathematics contributing to its development by providing the person and citizen with analytic tools especially appropriate for dealing with "quantitative data and relationships of space and form" (Committee on the Function of Mathematics in General Education, 1938: 43-45)[2].

Is actual education in general, and mathematics education in particular, anywhere near achieving this lofty goal? Actual mathematics teaching is often blamed to foster rote learning of computational and algebraic techniques, geometric formulas and textbook proofs, thus failing to contribute to the education of critical citizens and reflective thinkers. This leads to reform movements and curriculum changes. But consecutive reforms don't seem to change much. For example, the lament over rote learning was used both in promoting the famous "New Math" reforms of the 1960s and in their criticism later on (Kline, 1973; Freudenthal, 1963; 1973; Thom, 1970; 1972; Chevallard, 1985).

EFFORTS AT ACHIEVING A CONCEPTUAL MODEL OF MATHEMATICS TEACHING

Since the (in)famous "New Math" reforms, there has been relentless theoretical and experimental work on the design and study of classroom situations engaging students in mathematical thinking and reasoning (e.g. Brousseau, 1997). The emphasis on independent, creative and critical mathematical thinking and mathematical reasoning in educational research and ideology appears to have made it to curriculum development, if only in the form of rhetoric. But in some countries, classroom activities aimed at these goals have been institutionalized or are prepared to be institutionalized (e.g. in Québec[3]). The "situational problems" start with a general description of a situation (intra- or extra-mathematical) supposed to provoke students to formulate their own questions, propose solutions and defend them in small groups and whole classroom discussions. Observers of mathematics classrooms where such activities take place are usually duly impressed by students' engagement. Still, in-depth analyses of students' productions and the content of teacher-student interactions bring disappointment as to the level of critical and autonomous mathematical reasoning actually done by the students (e.g., Brousseau & Gibel, 2005).

The constructivist movement in North America tried to eliminate the rote model by a fundamental change in teachers' and educators' epistemology of mathematics. This resulted in bitter "math wars" (Schoenfeld, 2004) but rather not in the desired epistemological changes. Representatives of the opposed camps speak at cross purposes: one side attempts to prove that schools with more traditional curricula and methods of teaching produce better scores on standardized tests (Hook, 2007), while the other argues that knowledge developed in reformed schools cannot be measured by such tests.

Reformers (of constructivist or other profession) in North America are, however, far from saying that replacing the rote model by the conceptual model is easy. It is even hard to convince some people that it is necessary, especially if the rote model "works", in the sense described by Goldin in the quote below.

> At all socioeconomic levels, [the US] society persists in setting low educational goals. In wealthy, suburban communities, where the intellectual and physical resources for quality education are generally available, there is a disturbing tendency for schools to coast, particularly in mathematics and science. Here it is easy for school administrators to cite high achievement levels, evidenced by standardized test scores and students' admissions to prestigious universities, as hallmarks of their schools' successes – although these may be due more to the high socioeconomic status of parents than to high quality education. Why push our children if they are already doing fine?... Why take risks, when bureaucracy and politics reward stability and predictability? (Goldin, 1993: 3).

What doesn't work, according to Goldin, is the teaching of mathematics. Maybe something else is being taught but not mathematics, which is conceptual knowledge:

> For students to go beyond one- or two-step problems in mathematics requires conceptual understanding, not the ability to perform memorized operations in sequence; in removing the development of this understanding from the curriculum, *we have removed the foundation on which mathematics is built*. (Goldin, 1993: 3; my emphasis)

However, the conditions formulated by the author for the replacement of the rote model by the conceptual model appear very costly in terms of funding, organized human effort, and cultural changes that would also take a long time to stabilize.

> The [reform] initiatives that have been undertaken must be increased drastically if we are to arrive at a new cultural context – one in which elementary and secondary school teachers have seen in some depth pure and applied mathematical research, and move easily in the university and in industry; one in which research mathematicians and scientists know some of the problems of education, and move easily in schools; a context based on one large community of mathematical and scientific researchers and educators, rather than the disjointed groups we have now. (Goldin, 1993: 5)

THE CONTEXT OF MY CONCERN WITH THE CONCEPTUAL MODEL OF MATHEMATICS TEACHING

I became interested in this problem because of the facts mentioned at the beginning of this paper: the apparent tendency, among students of the prerequisite mathematics courses to rely on teachers for the validity of their solutions and to remain insensitive to even obvious mathematical inconsistencies; and approaches to teaching that seem to enhance these attitudes. Similar phenomena were observed by other researchers in other groups of students (Lester et al., 1989; Schoenfeld, 1989; Stodolsky et al., 1991; Evans, 2000; FitzSimons & Godden, 2000).

These are disturbing results from the point of view of the educational goals of teaching mathematics that we cherish. In the case of the prerequisite mathematics courses, it seems even hypocritical to force candidates to take these courses by telling them that they will need the mathematical theory and techniques in their target academic programs, and then fail to even develop their independence as critical users of mathematical models. Do we have any use for financial advisers who are not critical with respect to the predictive mathematical models they are using and blind to the mistakes they are making? How credible are reports of psychologists who use statistical methods in their studies but do not understand the theoretical assumptions and limited applicability of the methods they are using? Is it necessary to mention engineers who design wobbly pedestrian bridges because they fail to notice that the computer program they were using for their design assumed only vertical and not lateral vibrations? (Noss, 2001).

The prerequisite courses certainly serve the purposes of academic selection in the administrative and economical sense of reducing the number of candidates to such levels as the human and material resources of the respective university departments are capable of handling. These courses are also a source of financial support for the mathematics departments who staff them with instructors, markers and tutors, recruited from among faculty, visiting professors and graduate students. Their existence is, therefore, institutionally guaranteed.

The question is if it is possible to make these courses serve educational as well as administrative and economical purposes, by modifying the teaching approaches and convincing students of their value for their future study and professions. *Realistically* possible, that is, which means respecting the constraints under which these course function. They must be short and intensive because students are adults who may have jobs and families: they cannot spend a lot of time in class and they can't wait to have the prerequisites behind them and start studying the core courses of their target programs. Classes are large and there is pressure to make them even larger, for economical reasons; universities are always short of money. There is also the lack of professional pedagogical knowledge or experience among the instructors (graduate students, professors), who, when they were university students themselves, have rarely if ever experienced any other form of teaching than a lecture, occasionally interrupted by questions from the students or short problem solving periods. At the university, in mathematics departments, there is no pressure and certainly no requirement to teach otherwise. This may not be the most effective method of teaching but it is the least costly one in terms of intellectual and emotional effort. Graduate students of mathematics are not experienced and confident enough, neither in mathematics, nor in classroom management skills (not to mention language skills), to conduct an investigation or a mathematical discussion. Professors are usually more interested in a neat organization and smooth presentation of the mathematical content than in knowing what and how students in their class think about it. Indeed, they may not want to know, for fear of losing morale. It is more pleasant to live in the illusion that students think exactly the way we think ourselves. Grading tests and examinations is usually a rude awakening, which depresses teachers for a little while. But it is better to be depressed just for a short while than all the time, realizing in every class that whatever one says is understood by the students in a myriad of strange ways, most of which have nothing to do with the intended mathematical meaning.

I had the following hypothesis, which I probably shared with many of fellow mathematics educators. If students in the prerequisite courses were lectured not only on rules, formulas and techniques of solving standard questions but also on some of the theoretical underpinnings of these, then they would have more control over the validity of their solutions and would be more interested in the correctness of their solutions. Knowing the reasons behind the rules and techniques would allow them to develop a sense of ownership of mathematical knowledge. Teachers and students would be able to act more like partners in front of a common task. There would be a possibility of a discussion between the teacher and the student about the mathematical truth. If the student only follows the teacher's instructions, discussion of mathematical truth is replaced by the verdict of an authority: the teacher decides if the student is right or wrong. The theoretical discourse would distance the student from his or her self-perception as someone who either satisfies the expectations of an authority, or stands corrected. There would be no reason for the student to delegate all responsibility for the validity of his or her solutions to the teacher. Moreover, since justified knowledge is more open to change and adaptation in dealing with novel situations, it is more easily transferable to other domains of study and practice and not good only for solving the typical examination questions (Morf, 1994) and therefore more relevant; it is worth teaching and learning.

A teaching experiment

I planned to use a teaching experiment to explore this hypothesis: a mathematical subject (I chose inequalities with absolute value, to keep the same topic as in the research on frustration) would be presented in a short lecture using different approaches, some stressing effective procedures for solving a type of problems, others - the underlying theory. Subjects (recruited from among students of the prerequisite mathematics courses) would then be asked to solve a series of problems (the same for all groups). The experiment would end with a “task-based interview” (Goldin, 1998), where students would be asked questions about their solutions to the given problems, and about their views and habits relative to checking the correctness of their solutions. In particular, they would be asked questions such as: how do you know this is a correct answer? When you solve assignments or do test questions, how do you know you are right? Are you even interested in knowing this? I was hoping to see if there is any relationship between the teaching approach used and students' control over the correctness of their results.

At the time of writing this paper, only 13 students have been interviewed. It is too early to draw conclusions. But the results, so far, suggest a far more complex reality than my naïve hypothesis had it. Students have many reasons for checking or not checking their answers and they do it in a variety of ways. Students following the theoretical approach lectures were not clearly more likely to care about the validity of their answers; in fact, after the procedural lectures, more students seemed concerned with the validity of their answers than after the theoretical approach lectures.

Why could it be so that the teaching approach doesn't matter so much? In the rest of this paper I am going to offer some hypotheses about the possible reasons for this state of affairs, which state is still quite hypothetical, of course, but also made more plausible by the hypotheses.

More specifically, I will be talking about the possible reasons of students' dependence on teachers for the validity of their solutions (abbreviated "DT") and their lack of sensitivity to contradictions ("LSC").

It is difficult to find a single theory that would explain the DT and LSC phenomena although there have been attempts in educational research to capture as much as possible of the complexity of teaching and learning (e.g., Illeris, 2004; Chevallard, 1999). My reflection will therefore be eclectic, borrowing ideas from a variety of theoretical perspectives. I will organize it along the following categories of possible reasons: epistemological, cognitive, affective, didactic, and institutional.

EPISTEMOLOGICAL REASONS

[DT] *Much of mathematics is tacit knowledge*. Dependence on the teacher might be something that is specific to mathematics not *in fact*, but *in principle*. Essential aspects of mathematical ideas and methods cannot be made explicit (Polanyi, 1963). It is difficult to learn mathematics from a book. There are non-verbalized techniques that are learned by interacting with a master; doing a little and getting quick feedback. There is a lot of implicit schema building for reasoning and not just information-absorbing and deriving new information directly by association or simple deduction. (Castela, 2004).

[DT] *A mathematical concept is like a banyan tree*. The meaning of even the most basic mathematical concepts is based on their links with sometimes very advanced ideas and applications that are not accessible to the learner all at once and especially not right after having seen a definition, a few examples and properties. Initial understanding is necessarily fraught with partial conceptions, over- or under-generalizations or attribution of irrelevant properties (some of which might qualify as epistemological obstacles; Sierpinska, 1994). The student is quite justified in feeling uncertain about his or her notions and in looking up to the teacher for guidance.

[LSC] *Contradiction depends on meaning*. Consider the expression: "$-x < 2$ and $x < -2$". This expression represents a contradiction if it is understood as representing a conjunction of two conditions on the real variable x. There is no contradiction if the second term of the expression is understood as the result of an application of the rule "if $a < b$ and $c \neq 0$ then $a/c < b/c$" to the first term of the expression, understood as an abstract alphanumeric string, and not as an order condition on a real variable. The rule is a "theorem-in-action" (Vergnaud, 1998: 232) that seems to be part of many students' mathematical practice.

[LSC] *Contradiction presumes there is meaning*. The above assumption that contradiction depends on meaning implies that there *is* meaning, that is, if a statement is meaningless for you, the question of its consistency does not exist for you.

Let us take the example of absolute value of a real number. For the mathematician, the absolute value may be associated with situations where only the magnitude – and not the direction – of a change in a one-dimensional variable is being evaluated. It may thus be seen as a particular norm, namely the two-norm in the one-dimensional real vector space, R^1: $|x| = \sqrt{x^2}$. This notion makes sense only if numbers are understood as representing the direction and not only the magnitude of a change, i.e. if "number" refers to both positive and negative numbers. Absolute value is an abstraction from the sign of the number. If "number" refers to magnitude, it has no sign and there is nothing to abstract from. Moreover, the notion is useless if only statements about concrete numbers are considered; it offers a handy and concise notation only when generality is to be expressed and algebraically processed (for a historical study of the concept of absolute value, see Gagatsis & Thomaidis, 1994). The commonly used definition

$$|x| = \begin{cases} x & if \quad x \geq 0 \\ -x & if \quad x < 0 \end{cases}$$

encapsulates all these meanings, of course, but doesn't make them explicit. For a student with a restricted notion of number and un-developed sense of generality in mathematics, this definition is meaningless. It is therefore not surprising that he or she does not see contradictions in statements such as $|x| = \pm x$ or $|x+1| < -3|x-1|$ (for more information on and analyses of students' mistakes in the domain of absolute values, see Chiarugi et al., 1990; Gagatsis & Thomaidis, 1994).

[LSC] *Contradiction requires rigour in definitions and reasoning*. "Contradiction" applies to statements where the meaning of terms is stable in time and space. Thus it applies to rigorous texts whose discursive function is closer to *objectivation* of knowledge rather than its *communication* (Duval, 1995; see also Sierpinska, 2005).

But the function of mathematics textbooks, at least at the pre-university or undergraduate level is communication, not objectivation. In such textbooks, the boundaries between definitions and metaphors, illustrations and proofs, are often blurred.

The aim of teaching at those levels is to help students develop "some sense" of the concepts and a few basic technical skills, with the hope that, if needed, the concepts will be reviewed in a more rigorous manner at the graduate level, for those who will choose to study mathematics for its own sake. Even the fathers of mathematical rigour in Analysis as we know it today, Bolzano (1817/1980) and Dedekind (1872/1963), conceded that too much concern for rigour and proofs in the early stages of its teaching would be misplaced.

To illustrate the confusion between definitions and metaphors in didactic texts, let us look at the introduction of the notion of absolute value in a college level algebra textbook (Stewart et al., 1996: 17). The section "Absolute value and distance" starts with a figure (below) and the following text:

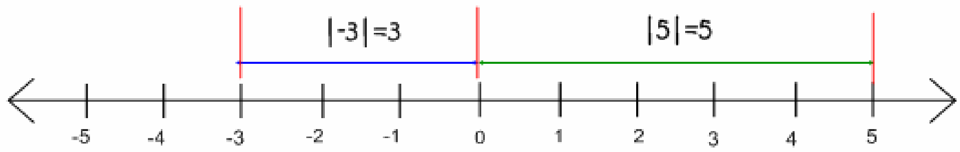

Figure 1. Reproduction of "Figure 9" in (Stewart et al., 1996: 17)

> The absolute value of a number *a*, denoted by |*a*|, is the distance from *a* to 0 on the real number line (see Figure 9). Distance is always positive or zero, so we have |a| ≥ 0 for every number a. Remembering that – a is positive when a is negative, we have the following definition [the two-case formula is given next in a separate paragraph which is also centered and bordered]. (Stewart et al., 1996: 17)

The first sentence of the text reads as a definition; it has the syntax of one. Yet it is not one because it uses the term "distance" as a term borrowed from everyday language, and thus as a metaphor. Distance in everyday language means something different than in mathematics, where it is assumed that the meaning of this word is fully determined by the following three properties and only these properties: 1. distance from point A to point B is the same as distance from point B to point A (so that the orientation of the movement between A and B is ignored); 2. that distance is not the path but a measure of the path and that this measure is an abstract number and not the number of centimeters or inches or other units (i.e. that it is a ratio); and 3. that going from A to B and then from B to C we cover a distance that is not less than the distance from A to C (so that $|a+b| \leq |a| + |b|$ can appear obvious later on). If, for the reader, distance is the number of units, and not a pure number, then, at this point, he or she may well think that $|-3| = 1.5$ cm and $|5| = 2.5$ cm according to the accompanying figure. Of course, the annotations on Figure 9 aim at eliminating this ambiguity: they suggest that the distance of a point representing a number on the number line is to be measured in the unit chosen for this representation and not in some other units, and that the distance is the number and not a number of units. Thus a lot of information is contained in the figure if only one knows what to look at in the figure.

Grasping the intended meaning of this text requires also certain conceptualizations that it may be unrealistic to assume in the readers.

One is the correspondence between numbers and points on the number line, which is not an easy concept (see, e.g., Zaslavsky et al., 2002). In everyday life, distance refers to places in space, not to numbers, so talking about distance between numbers doesn't make sense. Yet, this correspondence is taken for granted in the text: in the first occurrence of the symbol "a", it refers to a number; in the third, without warning – to a point.

Another assumption is the algebraic understanding of the symbol $-a$. The conception that this symbol represents a negative number is well entrenched in students, even at the university level (Chiarugi et al., 1990).

COGNITIVE REASONS

[LSC] *Sensitivity to contradictions in mathematics requires theoretical thinking.* (Sierpinska & Nnadozie, 2001; Sierpinska et al., 2002 – Chapter I, section, "Theoretical thinking is concerned with internal coherence of conceptual systems"). Theoretical thinking is not a common mental activity; it is not the first one we engage in when confronted with a problematic situation. When the situation is mathematical, theoretical thinking may be common among mathematicians but not among students.

The object of theoretical thinking is an abstraction from the immediate spatial, temporal and social contexts; these contexts, on the other hand, are in the centre of attention in practical thinking. Questions such as, *Is this statement true? Is it consistent relative to the given conceptual system?* make sense from the perspective of the theoretical mind, but not necessarily from that of the practical mind. Here, it is more natural to ask, *Does this technique work?, Is this answer good enough? Is this argument sufficiently clear, convincing, acceptable, under the circumstances?* The practical thinker is oriented towards acting in the situation, solving the problems at hand with the available means; he or she does not reflect on the various interpretations of the situation, the hypothetical solutions and their logically possible consequences.

For the action-oriented student, obtaining correct answers is guaranteed by "doing what one is supposed to do" according to examples provided by the instructor or a book.

Let me illustrate this point with a story from my teaching experiment.

Student AD (female, 21-25 years of age, candidate for a major in mathematics and statistics) was following a procedural approach lecture on solving inequalities with absolute value. She then solved six exercises, the second and third of which were, respectively, $|x-1| < |x+1|$ and $|x+3| < -3|x-1|$.

In both she followed the procedure shown on the example of $|x-1| < |x+2|$ in the lecture. Her solutions are reproduced in the *Appendix*. She obtained a correct solution in exercise 2 but her solution to exercise 3 was "∞" by which she meant that the inequality holds for all real numbers. She did not check her solutions by plugging in concrete numbers into the initial inequalities and so was unaware of the mistake in exercise 3. The interview included the following exchange:

AS: Tell me how you did the second one ($|x$-$1|<|x+1|$)

AD: I just did it the same way as they did here. I solved for zero on both sides.

AS: You solved for x?

AD: I put x-1=0, x+1=0, so x=1, x=-1, and then I put it in a chart and I solved if x is smaller than -1, if it's in between and greater than 1. And I solved it in these three cases. In the first case, both are negative, which means I put negative in front, before the bracket and I came out with that.

AS: How do you know you are right?

AD: Because that's how you are supposed to do it? I don't know!

In the interview about exercise 3, the student was encouraged to verify if the inequality is true for some concrete numbers, like 1, 2, -1. When she found that the result is false in each case, her reaction was, "I don't know what I did wrong". She was not satisfied until she found where she failed in applying the procedure. The object of her thinking was not the internal consistency in a set of mathematical statements, but her actions in relation to a task.

[LSC] *Noticing a contradiction in conditions on variables is harder than in a statement about concrete numbers*. Some students never miss a contradiction in statements such as -1 > 2, but have no qualms about "simplifying" the condition "$-x < 2$" to "$x < -2$". In the teaching experiment (procedural approach), one of the students (LA, male, over 30 y.o., applying for admission into computer science), would start by numerical testing of the given inequalities, and only when he arrived at exercise 4, he looked back at the notes from the lecture and attempted algebraic processing. He never made any mistakes in judging the validity of his numerical statements. With regard to the inequality $|x+3| < -3|x-1|$, he started by testing it for -3, 1, 2, and 3, always getting a contradiction. Then he looked up the lecture notes, and, as he said in the interview, he "finally understood what he was supposed to do". So he engaged in "analysing cases". His algebraic work was full of mistakes. There were careless mistakes (e.g., re-writing the right-hand side of the inequality as "x+1" instead of "x-1"; dividing 6 by 4 and getting 2/3, etc.). There were systematic mistakes such as not changing the direction of the inequality when dividing it by a negative number, and logical mistakes (not taking into account all possible cases and ignoring the conditions on x defining the intervals in which each case would be valid). His conclusion was $x < -\frac{2}{3}$. He crossed out his first numerical calculations and left the algebraic nonsense as his final answer.

[LSC] *Noticing a contradiction in a longer mathematical message is demanding on cognitive functions* such as attention, information processing, and memory, especially "mathematical memory" which doesn't seem to be particularly common among students and is considered to be a gift (Krutetskii, 1976). These cognitive faculties appeared rare among the subjects of our teaching experiment. Some admitted that their minds "wandered away" during the lecture and they missed some essential points. In fact, only one student (YG, female, 26-30 y.o. candidate to commerce) became completely absorbed in listening to the lecture (procedural approach), so that nothing mathematically essential for the presented method escaped her attention. She was the only one, among the 13 students, whose solutions to all exercises were complete and correct. She told us she tested her algebraic solutions with numerical calculations but did not bother writing these calculations down. She also told us that, in the mathematics courses, "most of the time I can understand the stuff during class. Actually, I seldom do questions after class. I use time during the class efficiently. So after class, at home, I seldom do mathematics. But just before examinations, I will study sometimes".

AFFECTIVE REASONS

[DT] [LSC] *The school mathematics discourse* (Moschkovich, 2007) *uses expressions such as "right" and "wrong" rather than "true" or "false",* normally reserved for courses in logic. But "right" and "wrong" are emotionally laden, especially when uttered in relation with a student's work and not – mathematical statements independently from who had said or written them. They are an element of an assessment and it is the teacher's job to assess, not the student's.

[LSC] *Relying on gut feeling about having got the "right answer*". Several students in our experiment mentioned "feeling" when asked, "How do you know you're right?" Even YG mentioned feeling, although she also checked her answers by numerical substitutions. But this appeared to be "double-checking", not the primary or only checking. Here is an excerpt from our interview with her:

> AS: When you solve a problem, how do you know you're right?
>
> YG: (Silence)
>
> AS: Are you making sure it's good?
>
> YG: I think when you do the mathematic problem and you are on the right track, you have the feeling that you're on the right track.
>
> AS: (…) Many people have the feeling they're right yet they get their answers wrong.

YG: Sometimes, when it's wrong, you will get a conflict, a contradiction. Actually, when I do some problems, I want to compare the answers, to make sure I'm right.

AS: Here, when you were doing these problems did you check your final answers?

YG: Yes, I would choose a number in this area and test, check the answer.

AS: You did that?

YG: Yeah. In heart, not write down.

[LSC] [DT] *Not verifying one's answers for fear of losing morale*. Here is what one student (BK, male, below 21, applying for admission into mechanical engineering), told us after having listened to a theoretical approach lecture and solved the 6 exercises. He solved half of them correctly. Each algebraic solution was followed by numerical substitutions for two numbers, but it turned out that the student did not attribute the status of verification to these calculations. He believed that these calculations were part of the expected solution. Contradictions between his numerical calculations and his algebraic work went unnoticed. He explained why he doesn't check his answers in the following terms:

AS: When you do your mathematics assignments, how do you know you are right?

BK: If doing it was smooth// if it was a smooth process, like I didn't find it was difficult, or// it was just flowing. Anyway, I never go back to check.

AS: Here, you may have felt that everything was going smoothly even if it was a little bit tedious, but still, not all your answers were right.

BK: Whenever I check and I realize I got it wrong, I start losing the morale. So I'd rather finish and then, if I feel I need to check, then I check. But if I checked in the middle and found a mistake it would have affected the way I was doing the other problems. (…) I just tend to believe I got everything right. I'd rather just receive the paper and be told what I got right and what I got wrong. I'm like, okay, I did the best I could. But if I'm at number 6 and I know I did the first three wrong then I start doubting the others, and I would not be happy after the test, I'd just walk sad, and I won't even want to receive my paper back. So I'd rather just not know.

DIDACTIC REASONS

[DT] *In didactic situations, the task is given by the teacher* and the decision if it has been satisfactorily completed is the teacher's responsibility; such are the rules of the *didactic contract* (Brousseau, 1997). The student's job is to produce answers, to the best of his or her knowledge. Under this contract, the "verification" or "check" part of working on an equation or inequality that the teacher demonstrates before the students does not have the function of reducing uncertainty, because the teacher is assumed to know the correct answer. It may appear to the students as part of a "model solution text" (as in the case of BK above). Some students, however, like YG, are able to see the epistemological difference between a solution and its verification. This student, while not indifferent to the rules of the didactic contract – she told us she wasn't sure if writing only an answer she had figured out mentally without "showing all her work" was acceptable – did not see it necessary to write down the numerical checking she did in her head.

[DT] *There are many tasks in school mathematics where it may be impossible or difficult for the student to verify the answer*. Ironically, proofs belong to this category. Students may be able to notice a blunt inconsistency, but they may not suspect the existence of a counterexample to one of their "theorems-in-action" if they don't know enough theory yet. Moreover, how much detail a proof should contain is a rather arbitrary decision and we have students asking us such questions as, "may I assume known that $\sqrt{p}$ is irrational for p prime, or do I have to prove from scratch that $\sqrt{19}$ is irrational in this particular exercise?" We normally proceed by local and not global deduction in presenting the material to students, and it is not always clear what can be assumed as proved, known, and what must be proved in a given problem. But even in research mathematics, the decision whether a proof is correct or not belongs to a group of experts; there is always a possibility that the author has overlooked an inconsistency. The completeness of a proof submitted for publication is decided by reviewers and editors and depends on the standards of rigour and detail of the particular journal.

[LSC] *Depriving students of opportunities for noticing a contradiction for the sake of "fairness" of assessment*. In the interviews with students following the teaching experiments, a few told us that they realize they made a mistake when problems are linked together so that the answer to the next depends on the answer to the previous one and they get something unexpected in the next one. They may not be able to elaborate on their reasons for knowing what to expect, (as in the transcript below) because this requires a meta-reflection on one's thinking processes and they may have no linguistic means to express themselves). However, to have expectations about the result of one's mathematical work must be based on some theoretical knowledge (even if it is based on theorems-in-action that are not all consistent with the conventional mathematical theory). The excerpt below comes from the interview with SC (male, less than 21 y.o., candidate to computer science), after a theoretical approach lecture.

AS: What methods do you use to check? How do you know you are right?

SC: Sometimes the equations are linked together, and when you get the wrong answer, you will not get the result expected in the second answer.

AS: But how do you know what to expect?

SC: (Silence)

AS: Can you give us an example of such a situation?

SC: Most of the time it appears when you are doing derivatives. We use the derivative to, uh, there is something we use, we use the derivative (pause). But the questions are linked, and if you don't get the good derivative, you have some problems to find the, the maximum or some other derivatives (pause).

AP: You get some contradiction in the table where you put intervals of the function increasing, decreasing, no?

SC: Yes, yes, exactly!

This points to the benefits of the didactic organization of exercises into interrelated sequences so that a mistake made in one exercise produces nonsense in the others and may give motivation for checking an answer. However, in marked assignments or on exams, for institutional reasons, questions are disconnected, so that the answer to the next exercise does not depend on the answer to the previous question. It is considered "not fair" to the students to link questions like that, so that mistakes carry over.

INSTITUTIONAL REASONS

[LSC] [DT] *In school, validity = compliance with institutional rules and norms*. In school practice, mathematics becomes, in fact, a collection (often a loose collection) of types of tasks (exercises, test questions, etc.) with their respective techniques of solution, where the form of presentation (e.g., "in two columns") often has the same status as the mathematical validity. Techniques are justified on the basis of their acceptability by the school authorities, not on their grounding in an explicit mathematical theory. It is not truth that matters but respect of the rules and norms of the didactic contract related to solving types of problems.

In the context of absolute values, school mathematics (in some countries) had developed a whole praxeology (in the sense of Chevallard, 1999), with specialized monographs on the subject for the use of teachers, where tasks were codified into types, methods of their solution exposed and justified internally relative to the definition of absolute value, without regard to the uses of the notion in domains of mathematics other than school algebra (Gagatsis and Thomaidis, 1994). In the process of didactic organization of the material for classroom teaching, some elements of the theoretical justification would inevitably disappear as too advanced for the students, or appear in the curriculum in ways that would eliminate their use as means of validity control (for examples of this phenomenon in the context of teaching elements of mathematical analysis, see Barbé et al., 2005).

FINAL REMARKS

In mathematics education we commonly blame students' poor knowledge of mathematics and negative attitudes to its study on procedural approaches to mathematics teaching and we claim that mathematics taught that way is not worth teaching or learning. We constantly call for reforms that would support conceptual approaches to mathematics teaching. I, at least, blamed students' dependence on teachers for the validity of their solutions and their lack of sensitivity to contradiction on the "rote model". But what guarantee is there that those ills would be removed by adopting the conceptual model? A lot of money and human effort could be spent on implementing the desired model and the results might be quite disappointing. The expected students' interest, autonomy and mathematical competence might not materialize not because of lack of teachers' competence or good will but because of epistemological, cognitive, affective, didactic and institutional reasons that are independent of their knowledge and good will. These reasons have their roots in the nature of mathematics, in human nature, in the very definition of a didactic situation and in what makes a school a school rather than a Montessori kindergarten.

Perhaps conceptual learning can occur and actually occurs also in procedural approaches? In my modest experiment, 3 out of the 5 students who followed a procedural approach checked their answers; only one out of the 8 students who followed the theoretical approach did so. The only student, who checked her answers effectively and had all her solutions correct, followed the lecture with the procedural approach. But this student (YG) did not apply the method showed on an example in the lecture uncritically in her solutions. She was gaining new experience as she was solving the problems and was finding useful shortcuts. After having applied the taught procedure to the inequality $|x+3| < -3|x-1|$, she realized that it wasn't necessary because the inequality contained an obvious contradiction. After she solved $|2x-1| < 5$, she knew what to expect in $|2x-1| > 5$ and its solution only served as a verification of the first one. In the interview after the experiment she described what kind of teaching approaches she considers "effective". Below, I give an excerpt from the interview where she describes her experience. I will close this paper with this student's words. They are worth thinking about.

AS: The lecture you listened to, was it very different from what you are used to?

YG: I think it is almost the same. In class the professor also give you some example. They just, they didn't explain you the theory, they just give you example: "after the example you will understand what I am telling you, the definition".

AS: So there is no theory, just "here is an example; here is how you solve it".

YG: I also think this is an effective way. Actually, earlier this semester I met – I think it was a MATH 209 professor – his way of teaching was totally different from other mathematics teachers. He put more emphasis on explaining the theory. And the most strange thing was that he'd write down everything in words. Normally, in mathematics, the teacher never writes the words, just the symbols, but he wrote everything in words. So in class we just took down the words. It was like a book, a lot of words to explain. And I don't think this way of teaching is good for me, so I changed to another professor. (…) I think mathematics is not literature.

AS: Was it the writing that was bothering you, or the theory that he was using, justifying everything. What was it that you didn't like, words or the theory?

YG: No, we spent most of the time, just writing, not writing, copying, so you don't have the time to think and to understand. So that's what was not good.

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3. I thank the undergraduate students who volunteered to participate in the experiment as students.

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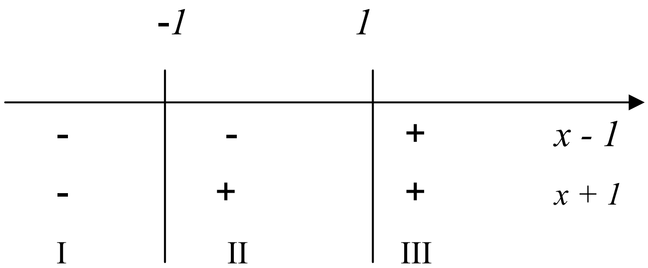
Appendix

Exercise 2. Reproduction of student AD's solution

$|x-1| < |x+1|$

$x - 1 = 0 \qquad\qquad x + 1 = 0$

$x = 1 \qquad\qquad x = -1$



Case I	$-x + 1 < -x - 1$	$1 < -1$	$\varnothing$
Case II	$-x + 1 < x + 1$	$0 < 2x$	$0 < x \leq 1$
Case III	$x - 1 < x + 1$	$-1 < 1$	$x > 1$

$| x - 1 | < |x + 1 | \Rightarrow x > 0$

Exercise 3. Reproduction of student AD's solution

$|x + 3| < -3\,|\,x - 1|$

$x + 3 = 0$ $\qquad$ $x - 1 = 0$

$x = -3$ $\qquad$ $x = 1$

	-3		1	
-		+	+	$x + 3$
-		-	+	$x - 1$
I		II	III	

Case I	$-(x + 3) < +3\,(x - 1)$	$-x - 3 < 3x - 3$	$x > 0$
Case II	$x + 3 < 3x - 3$	$0 < 2x$	$-3 > x > 3$
Case III	$x + 3 < -3x + 3$	$0 < -4x$	$x < 0$

$|\,x + 3\,| < -3\,|x - 1\,| \;\Rightarrow\; \infty$

Endnotes

[1] The questionnaire, together with raw frequencies of responses, can be viewed at http://www.asjdomain.ca/frequencies_table.html

[2] The notion of "democracy" invoked by the quoted Committee was based on Dewey's (1937: 238) description: "Democracy… means voluntary choice, based on an intelligence that is the outcome of free association and communication with others. It means a way of living together in which mutual and free consultation rule instead of force, and in which cooperation instead of brutal competition is the law of life; a social order in which all the forces that make for friendship, beauty, and knowledge are cherished in order that each individual may become what he, and he alone, is capable of becoming."

[3] See, e.g., the document, "The Québec Education Program – Secondary Education", where the basic goals of teaching mathematics are stated as the development of the following three transversal competencies: 1. To solve a situational problem. 2. To use mathematical reasoning. 3. To communicate by using mathematical language. The document is available at http://www.learnquebec.ca/en/content/reform/qep/ .

SCHOOL MATHEMATICS AND CULTIVATION OF MIND

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INTRODUCTION

Why do we teach and learn mathematics? A high school teacher in Seoul, who has taught mathematics for 15 years since she graduated from the college of education, recently, has had a serious concern with this problem. She tried to tell her students, most of whom did dislike studying mathematics, that mathematics is actually a very useful subject, but failed since it seemed that her claim was not convincing enough. It was because she thought it would be difficult to argue that mathematics is actually practical subject to the ordinary people. Even worse, the teacher herself could not answer to the question, "How important high school mathematics can be for the ordinary people?" This doubt seems to be common to all mathematics teachers.

Since the modern period, with the spread of useful computation techniques and the development of science and technology based on mathematics, the practical aspects of mathematics have been highly valued. It has been thought that the usefulness of mathematics in solving many problems and developing science and technology is unquestionable, and that would be the main reason which made mathematics as a core subject for public. However, a serious problem lies in the fact that not everyone is going to be a scientist or a technologist, and it is difficult to insist that most of what we teach particularly in middle and high school mathematics classrooms is necessary for leading the life of today

Then why do we have to teach such knowledge whose practicality is doubtful for the ordinary people? The traditional eastern education, which was based on the Four Books and the Three Classics of ancient China, had focused on the educational value of 'theoretical knowledge' that is far from practicality. This was also true in the tradition of the Greek education for the free citizens and in the medieval seven liberal arts education led by the Scholastics. The traditional ways of education, which put focus on theoretical knowledge, was to develop the conceptual means or the eye of our mind to see the phenomena and to find true ways for our spiritual life.

Standing against this tradition, and emphasizing the social aspects of education was Dewey (1916), who argued the value of 'practical knowledge' to nurture the competent citizen of a democratic society.

> "The value of any fact or theory as bearing on human activity is, in the long run, determined by practical application (McLellan & Dewey, p.1)."

His experientialism tried to find the origin of the mathematical knowledge in the process of solving everyday-life problems, claiming that mathematical knowledge should be taught as a means for the solutions in relation to the actual problems

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(McLellan & Dewey, 1895). After Dewey, the focus of the education has shifted from 'remote' theoretical knowledge to 'immediate' practical knowledge. Particularly in mathematics education, efforts to teach the practical knowledge and develop problem-solving ability have continued.

However, from the Greek period till now, if there is anything that has not changed in school mathematics, it will be the fact that theoretical knowledge, which goes far beyond practical knowledge and which is not very much motivated with practical problem solving, has been continued to be taught. Despite numerous reform efforts, it has been scarcely changed that the theoretical knowledge has been firmly centered on the practice of mathematics education. Therefore a question on the educational value of the theoretical knowledge-based school mathematics for ordinary people can be raised.

Meanwhile, we should pay attention to the significant change of the perspective on mathematical knowledge: the advent of non-Euclidean geometry, the development of modern abstract mathematics after Hilbert's *Grundlagen der Geometrie* and the advent of quasi-empiricism after Lakatos' *Proofs and Refutations*. Today, so-called social constructivism (Ernest, 1991) which claims that certainty does not exist, and mathematics is just the result of construction of a subject and social negotiation processes, seems to be regarded as the trend of the times. What does it mean by letting 'young children' construct by themselves and negotiate their own mathematics? Can we undervalue the education led by the teacher as the embodiment of awe-inspiring mathematical knowledge? We should carry out a thorough analysis of the true face of mathematics educational thoughts of constructivism, and reexamine the traditional basis of education. Confucius (2002) said,

> "溫故而知新 可以爲師矣(A man is worthy of being a teacher who gets to know what is new by keeping fresh in his mind what he is already familiar with (pp. 12-13)."

Despite the pragmatist and recent constructivist view on the education, it is hard to deny that the essence of education lies in developing a 'noble-minded person' through the change of his/her internal eye and teacher takes the lead in education, regardless of the Orient and the Occident. The issue here is that, paying attention to the traces of mind cultivating in mathematics education; it is required to find a way to rediscover such a tradition and advance it further. For this purpose, here we are going to look at the strands of thoughts, which were advocated by Pythagoras, Plato, Pestalozzi, Froebel, and Bruner, on the mathematics education for cultivating mind.

According to them, mathematics should be the subject for the education of human beings who possess the sense of truth and pursue truth, who explore the mysteries of the nature through mathematics, and who pursue the Reality that dominates the phenomenal world. Here, we will try to reexamine that school mathematics can be, and should be the best way to realize the ideal of education, that is, cultivation of mind. Also, we will claim that we need to pursue the idea in new perspective.

As for this, it can be claimed that the history of mathematics is a process of gradual revealing the Reality by human beings, and that learning mathematics should be a process of encountering with the Reality. We endeavor to find the clue for realizing that idea especially from the thoughts of F. Klein and H. Freudenthal, two most distinguished figures who had much influence on mathematics education of the 20th century. It is considered that their thoughts on mathematics education which claimed historical-genetic approach (Klein, 2004, p.268), reflection in connection with the historical and philosophical point of view to the mathematics learned (Klein, 1907, S.216) and teaching/learning mathematization (Freudenthal, 1973, pp.131-146), and tried to reveal the knowledge structure of school mathematics through didactical analysis (Klein, 2004; Freudenthal, 1973, 1983), can open a new perspective on the pursuit of the idea of mathematics education, that is, cultivation of mind.

Here we will try to advocate that learning school mathematics should involve true sense of wonder and impression of encountering with the Reality, and awake the students to the sense of truth, beauty, and innocence. These experiences could become opportunities to let them realize the mathematical-ness of the universe, and lead the lives to devote themselves to reveal the mystery of it with the virtue of modesty. As a necessary consequence, school mathematics would be linked to a virtuous life. Here, we try to look at the practices to realize this educational thought through the discussion on the teaching of the Pythagorean theorem, negative numbers and functional thinking, and the methods for its realization.

THE TRADITION OF MATHEMATICS EDUCATION FOR CULTIVATING MIND

In the light of the Buddhism theory of mind, the Neo-Confucianism theory, and the theory of knowledge of Kant, Lee Hong-Woo (2001) argues that our mind forms a 'double-layer structure' which consists of the empirical layer and the transcendental layer, and claims that the goal of education should be the recovery of our transcendental mind, as followings. The Suchness of the Buddhism theory of mind, the Nature or the Equilibrium from *the Doctrine of the Mean*, and the Idea of Kant, all refer to the Reality which is the logical cause of things, the standard of everything, the Truth, the Goodness, and the Beauty itself. The goal of the discussion on the Reality is in finding out what kind of mind we should live with, and it is a common thought that the Reality refers to transcendental, 'primordial mind' at which our phenomenal, empirical mind should aim. It is the most fundamental hypothesis of education that school subjects intend to lead students to form their primordial mind by contributing to the recovery of transcendental mind. The knowledge we believe necessary enough to make it as a school subject is the one that has possibility of recovering the transcendental mind contained inside, if students would learn it well. According to the tradition of education, school subjects should aim to make students aware of our primordial mind, and encourage them to long for, and recover it.

To recover the original meaning of mathematics education, it is necessary to, above all, have an appreciative eye of it, and accordingly make a renewed appreciation of the importance of teaching and learning methods of mathematics. Here, first of all, we need to look into the history of mathematics education to find the traces of pursuing mind-cultivating education.

Pythagoras

In the western civilization, which has its roots in that of ancient Greek, mathematics was traditionally located at the core of knowledge education. If we trace back the western ideological history up to Greece, we could infer the real reason for learning mathematics. The English word 'mathematics' came from Latin 'mathematica' which came from Greek 'mathematikos', which in turn was based on 'mathesis' meaning 'mental discipline', 'learning' or 'to learn' (Schwartzman, p.132).

We can find the records as follows in the history of mathematics (Loomis, 1972; Burton, 1991). Pythagoras, who has been called the father of mathematics, went to Egypt following the advice of Thales. He learned a lot from there and later in Babylon, the center of world trade at that time. After coming back to Greece he influenced many people with his speech about morality, frugality, soul, immortality, and transmigration. "He changed the study of geometry into a form of liberal education (Fauvel and Gray, p.48)" pursuing the abstract properties of number and geometrical figure in itself beyond the Egyptian practical mathematics, and focused on the search for the harmonious mathematical forms lying inside the phenomenal world. And he declared that number rules universe. He believed that everything in the world itself reflects certain mathematical order, and one could purify his/her soul by contemplating mathematical order that lies inside the world, and organized a religious, academic community which pursued purification and salvation of the soul through mathematics. His students formed a school that learned 4 mathemata - geometry, arithmetic, astronomy, and music - and philosophy from him. This we call the start of mathematics, and the fact Pythagoreans called their subjects mathematics clearly tells us, what was the original idea that mathematics education should look for.

Plato

Plato, a student of Socrates, delivered the thoughts of his teacher through the *Dialogues*, and we can find the first record about teaching mathematics in *Menon*.

Plato assumed the Idea, a metaphysical world beyond the physical world. To him, a valuable person is the one who perceives the world of Idea by stepping over the shadow of the phenomenal world, and who has recovered their pure spirit by freeing themselves from the limitation and confinements of their body. To Plato, education is a way of making people turn their eyes from the phenomenal world, which is like the inside of a cave, to the realization of the truer world, the world of Idea. This is to recover the memory of the soul about the 'lost' Idea, that is, the world of Reality, and lead people to the life that adheres to 'the form of life' that contemplates the precious, harmonious order of nature. Plato cited arithmetic as the first useful subject to lead

the spirit to the Idea, and geometry as the second. By learning mathematics, he believed, one could get a clue for seeing the world of the Idea through one's mind's eye (Yim Jae Hoon, 1998).
Mathematics is a discipline that started as a study for the real human education, and a subject initiated for teaching the attitude for life, mind cultivating, and human education. The one who didn't know geometry was regarded as the one not living a philosophical life, thus not allowed to enter the Academy.

> "the 'Elements' in classical Greek education … used as a reference text for an introduction to hypothetic-deductive analysis as the specific method of *dialectic philosophy* … best way of explaining that *mathematical truth* is eternal in nature … best prepare the mind for *understanding the world of ideas* … which is the utmost goal of higher education (Steiner, 1988, p.8)."

"It is probable that Euclid received his own mathematical training in Athens from the pupils of Plato (Burton, 1991, p.143)." Thus we can say that *Euclid's Elements* was the book for Plato's idea, and mathematics education through *Euclid's Elements* was not different from mind cultivating. Furthermore, 7 medieval liberal arts education can be considered as moral building education that inherited this tradition of pursuing the Truth, the Goodness, and the Beauty.

Pestalozzi

The great educator who set up mathematics education as human discipline (Menschenbildung) is Pestalozzi. His educational thought is called 'die Idee der Elementarbildung'. He believed in the educational value of mathematics as the stepping-stone to cultivating mind and tried to naturally cultivate the three basic powers of human being – 'Sittliche, Intellektuelle, Kunstliche' and to realize the development of 'the power of high humanity' through mathematics education. To him, mathematics was the way of building morality, nurturing the sense for truth, developing the form of perception, cultivating thinking power and developing productive power, and to divide 'the spirit of computation and the sense of truth (Rechnungsgeist und Wahrheitsinn)' was to divide what God has combined. And he claimed that mathematics is the operation of elevating the potential of human reason with the power of reason, and the basic 'educational gymnastics (Erzehungsgymnastik)' through 'thinking and learning (denkend lernen)' and 'learning and thinking (lernend denken)' (Kim Jeong Whan, 1970).

> "Although not a dedicated mathematician, Pestalozzi, with his gifted intuition, comprehended the characteristics of mathematics, and, by making the best use of the characteristics, established the meaning of mathematics-as-discipline. The fact that he regarded the highest stage of mathematics education as …'die Operationen der Seele' in his own term, could be the result of the influence of the tradition of having listed mathematics as one of the 7 liberal arts since Greece" (Kim Jeong Whan, 1976, p.367).

We particularly need to pay attention to his insight that we should help students feel the sense of truth in computation. Computation, without a doubt, is a necessary tool for everyday life and the most fundamental among practical mathematics. However, can it just stay as the tool for solving everyday problems, at the level of empirical mind? The method of computation is an algorithm deduced from the basic rules of notation and operation, and an amazing mathematical form seeking truth, where the discovery of the answer and its proof is conducted together. The computational mathematics, which attracted Descartes, who sought for the universal method to solve problems, and Leibniz, who even said that as God calculates so the world is made (Vizgin, 2004), is the most certain method of inquiring truth together with axiomatic mathematics.

If we cannot feel 'the computational spirit and the sense of truth' as Pestalozzi said, when we look at the formula of the roots of quadratic equation or the fundamental theorem of calculus, we do not 'see' the true form of mathematics. Modern algebra, dealing with 'partial arithmetic' systems and 'full arithmetic' systems, was abstracted from fundamental computational rules, and modern mathematics is algebraic mathematics, which focused on those computational rules (Dieudonné, 1972).

The spirit of computational mathematics ought to be the basis of national education making students pursue the sense of truth and become a virtuous man. We should teach the students to realize that the method to compute is by no means a mechanical operation but rather the most wonderful method that human beings created to inquire the truth. I believe it would not be possible to discuss the truth without computational mathematics.

Froebel

With the manifestation of modern science by Galileo, Kepler, etc., in the 17th century, mathematics has become the essential tool for the inquiry of nature especially since Newton and Leibniz invented differential and integral calculus, and the basic property of nature that fundamental rule of physics would be always described in beautiful and powerful mathematics has been revealed.

> "The leading figures of the Modern Era - Kepler, Galileo, Descartes, Newton, and others - emphasized in the most elevated terms the Pythagorean idea of the divine mathematical-ness of the world." ... Einstein returned to Hilbert's claim of the pre-established harmony between physical reality and mathematical structures ... He used the example of conic sections, realized in the orbits of celestial bodies, to explain the sense of this harmony, and summarize as the following: "It seems that before we can find a form among things, our mind should organize it independently." (Vizgin, p.265)

The awe-inspiring nature of the mathematical knowledge that constitutes the order of universe obviously demands that mathematics become a subject for the moral education of the students, going beyond a practical tool for solving real life problems. Froebel was an educational thinker in the middle of the 19th century who highly

appreciated the value of mathematics for humanity education with the ideological background that the harmony of mathematics and physical world is the proof of the existence of God. He investigated the problem in the nature of human education and made the following assertions (Han Dae Hee, 2000).
Through mathematics a man can conceive the divinity that exists inside the universe, both human and nature. Through studying mathematics, he knows that there is a mathematical order inside the natural world, and that this order is expressed through the law of speculation of the pure human spirit. Here, he becomes conscious of the divinity inside man and nature, and becomes a valuable person who by realizing the spirit of God, believes in the existence of God and lives by the will of God. Therefore, to Froebel mathematics must be an educational means to perceive the spirit of God in nature and man, and it should be a subject that has essential meaning in 'human education'.

> "Human education without mathematics is nothing but a rag with patches (Froebel, p.208)".

This nature of mathematical knowledge demands strongly the school mathematics to become a subject for humanity education going beyond the practical usefulness.

Bruner

According to the structure-centered curriculum, suggested by Bruner (1963) in the middle of the 20th century, the content of curriculum should be the structure of knowledge implicit in the outer layer of knowledge. Bruner used the structure of knowledge as a synonym with 'general idea that makes up the foundation of the subject', 'basic notion' and 'general principle', which means 'the eye to look at matters' or 'the way of thinking' that defines each subject. According to the structure centered curriculum, to learn a subject means to be able to internalize the structure of the knowledge and see reality with the eye of the structure (Park Jae Mun, 1998).
Here we need to notice the point of view of Lee Hong-Woo (2002) as followings. 'The structure of knowledge', 'the form of knowledge' both can be used as a translation of Plato's Idea, the Reality, and 'logical forms' which exist on the other side of facts, providing the basis of the perception of phenomena. They are not the ideas we can actually figure out, but standards with which we see things, that we are trying to figure out or reach. Like the Idea, knowledge is such thing that is impossible to reach completely, so that the harder we try to pursue it, the more ignorant we realize we are.
We can understand this by considering, for example, how difficult it is to grasp the authentic meaning of the notion of function, functional thinking, or group despite much discussion. Since the structure of knowledge is the complete form and the logical hypothesis we cannot reach, it is only possible to approach to it through the spiral curriculum that continues ever-deeper discussion as the level rises.
Although structure-based curriculum theory revealed that to learn mathematics

should be to acquire the structure of knowledge within it, it did not argue that to emphasize the structure of knowledge would be a kind of efforts to approach the Reality, and a mind cultivating way through which one could recover transcendental mind, the Reality, the Truth, the Goodness, and the Beauty. It seems that from the period of Froebel till now, due to the influence of Dewey's experientialism curriculum, above all, we forgot about the virtue of mathematics for cultivating mind. 'The Standards' of NCTM, which has had quite large influence on the worldwide mathematics curriculum for the last 15 years, shares practical features with Dewey's thoughts.

> "today's society expects schools to insure that all students have an opportunity to become mathematically literate. …, and become informed citizens capable of understanding issues in a technological society. … In summary, the intent of these goals is that students will become mathematically literate (NCTM, 1989, pp.5-6)."

However, we should not disregard the idea of mathematics education for encountering with the Reality through mathematics, developing the 'high power of humanity' which pursues truth, seeing the order of the universe and exploring the form of life.

SCHOOL MATHEMATICS AS A WAY TO CULTIVATE MIND

The form of mathematics and encountering with the Reality

It is said that mathematics is the science among sciences and awe-inspiring school subject. Why is it so? Mathematics has pursued abstract form that is the essence of phenomena. Then can we clearly define the meaning of the Pythagorean theorem? What does $(-1)(-1)=1$ mean, and why is the concept of function so inclusive? What is the structure or form behind the surface of such mathematical knowledge? We ought to say that teaching mathematics is the process intending to transfer 'the fundamentally inexpressible one' which is hypothesized 'logically' in the mathematical knowledge.

Kant listed mathematical knowledge as a typical 'a priori synthetic', which we can tell the truth without considering experiences, thus let us know something about the phenomenal world and is universally true at the same time. Kant said that

> "In the earliest times to which the history of human reason extends mathematics ... already entered upon the sure path of science. ... The true method, ... was not to inspect what he discerned either in the figure, or in the bare concept of it, and from this, as it were, to read off its properties, but to bring out what was necessarily implied in the concepts that he himself formed a priori, and had put into the figure in the construction by which he presented it to himself. If he is to know anything with a priori certainty he must not ascribe to the figure anything save what necessarily follows from what he has himself set into it in accordance with his concept (Kant, 1956, p.19)."

To Kant, mathematics is the knowledge based not on objects, but on reason itself, and comes from the subjective condition of cognition, that is, the Idea itself. This kind of viewpoint can be found in Piaget's genetic mathematical epistemology. According to Piaget, mathematical concept is a secondary, operational 'schémes' constructed through, based on the schémes with which 'sujet épistemique' is born, the invariable functions of assimilation and accommodation and reflective abstraction (Beth, E & Piaget, J., 1966). It is noticeable that Vygotsky (1962), who, emphasizing the linguistic, social, and educational aspects in intellectual development, argued against Piaget, sees scientific concept as product of development that evolves through self-consciousness under the mutual influences of the 'non-spontaneous concepts' in the zone of proximal development with internal origin which starts from the primitive idea, 'spontaneous concept'.

According to Kant (1951), the Idea performs constitutive role and regulative role. The Idea constitutes the concepts, while it regulates to make the concepts acquire the utmost unity and aim at encountering with universally valid cognition, that is, the Idea itself. And according to him, the forms of mathematical knowledge contain such an indescribable Idea, the Suchness in the term of *the Awakening of the Faith*. Therefore, mathematics subject should have the position of the way for orientation towards the Idea. Based on this, we can say that the act of learning mathematics would be a way of heading for the Idea, that is, the Suchness. School mathematics should try to let students face the form of mathematics. The more students get involved in mathematics learning, the clearer the form of mathematics will become. We hope that through the continuous learning process, all at once students meet the moment when their minds become the form of mathematics itself and encounter with the Reality (Lee, Hong-Woo, 2001).

> "It must be stressed, however, that the Reality, by virtue of being un-manifested, is not an object for anyone to 'encounter', but a standard that makes us aware of our own deficiencies as human beings. ... by learning school subjects we are imbued with such humility as can be acquired only through good education (p.25)."

Pursuit of the purposive-ness of nature and moral life

Finding out that empirically different rules of nature can be united into one mathematical form could cause joy and admiration words cannot express. The laws of the movement of heavenly bodies observed by Galileo and Kepler are consistently described as conic sections. In addition, among natural phenomena, we can often find the phenomena that something momentarily decrease in proportion to itself, like the disintegration of a radioactive isotope, the cooling of heat, the decrease in sound wave intensity, the discharge of electricity, and the absorption of lights, and, surprisingly, these natural laws are consistently described as an exponential function $y = ce^{ax}$, which has the feature that the derived function is in proportion to the original one, that is, $\frac{dy}{dx} = ay$.

To explore physical rules goes beyond the practical aspect, and is an activity that pursues ‘the purposive-ness of nature’. However, even with the best of our efforts, we are only to be left with the fact that we are too far from reaching the goal. To pursue the laws of nature arouse our feelings of awe and respect to nature. Newton and Einstein could not help but admire at the purposive ness of nature expressed by the mathematical laws. The important thing is, such a mind of Newton or Einstein supposed in their physics should be the goal that all the students of physics share. Kant (1951)’s concept of the connection between nature and freedom can only be possible through the reflective judgment, that is, the process of our mind’s trying to reach the unreachable goal of the Reality that is the ultimate universal concept. The ultimate meaning of learning school mathematics is to endlessly pursue the ‘godlike state’ that actually cannot be reached. This means, by learning mathematical knowledge and through its deeper extension, the standard of our mind keeps uplifted, and at the same time the experience of our deficiency will come more acutely. We can say that a life in the middle of this learning process is a moral life (Chang Bal-Bo, 2003).

Gradual Encounter with the Reality through school mathematics

We can find, in Piaget's genetic mathematical epistemology, the point of view that mathematical cognition takes place by the alternation of content and form. Piaget (Beth & Piaget, 1966) hypothesized the potentiality of objective mathematical knowledge inherent in sujet épistemique but not realized, and the invariable developmental route. He claimed as followings;

> “Il y a donc ainsi une alternance ininterrompue de réfléchissements → réflexions → réfléchissements; et (ou) de contenus → formes → contenus réélaborés → nouvelles formes, etc., de domains toujours plus larges, sans fin ni surtout de commencement absolu (Piaget et collaborateurs, 1977-1, p.306).”

It reminds us the Polanyi (1962)’s view on knowing. He argues that cognition should be regarded as the unification of implicit ‘subsidiary awareness’ and explicit ‘focal awareness’, and maintains hierarchical realism, which claims that focal awareness, if internalized by subject, also becomes subsidiary awareness thus can be used to search higher focal awareness, that subject, with his internalized knowledge, passionately and devotedly pursues to encounter with ‘hidden reality’, and that, from the perspective of evolution theory, there exists various levels of reality, that previous stages become the basis of next stages and they move towards the perception on more vivid reality (Eom, Tae-Tong, 1998). Polanyi’s view on knowing, though seems to be relativistic thus stand against Kant’s transcendental realism, can be seen as the process of gradual conscientiousness of intrinsic order, which lies hidden as subsidiary awareness, through the substitution of content and form. Here, mathematical cognition on each level will be able to be understood as the discovery of more vivid Reality attained by the substitution of subsidiary and focal awareness, and partial perception on the structure of latent knowledge, and regarded as a partial meeting with the Reality.

We can find similar views in Freudenthal's theory of teaching and learning mathematization (1973, 1978, 1983, 1991) and the theory of geometry learning levels of van Hiele (1983). Freudenthal argued that mathematics is an activity rooted in common sense and a science that begins from common sense and pursues certainty. To him, mathematics is the mathematising process that pursues the noumenon organizing phenomena. He saw it as the starting point of the study of mathematics education to reveal such knowledge structure of school mathematics through didactical analysis. To regard mathematics as the process, not as resulting knowledge, can be considered as a view that sees mathematics as the process of revealing 'something that cannot be expressed'. According to van Hiele, mathematical thinking is the mathematising process which undergoes through the visual level, descriptive level, the theoretical level that perceives local logical relations, the level that perceive the formal deductive logic, and the level that insights the nature of the logical laws. What was implicit at the previous level becomes conscious and gets clearer, and mathematical thinking at each level studies the internal order of that of the previous level. In this light, mathematical thinking develops with the substitution of content and form, of subsidiary awareness and focal awareness, of tacit knowledge and explicit knowledge, in Polanyi's terms.

According to these, encountering with transcendental mind, the Reality, occurs in stages, and the more one studies mathematics, the more deep emotion one can get from encountering with the Reality. Since intellectual perception of mathematics contains the Truth, the Goodness, and the Beauty, though partial, authentic knowledge of school mathematics accompanies the emotional aspect. The problem is that most mathematics educators are focusing only on the level of empirical mind, without the consideration of encountering with transcendental mind, that is, the level of mind cultivating. However, these approaches of the mathematics educators can be re-interpreted as practical methods of mathematics education pursuing the noumenon of phenomena and recovery of the transcendental mind, the Idea.

Representation of wonder and impression of mathematical knowledge

Mathematics education is an activity of leading students to sense the deep form of school mathematics from the process of learning the surface content. As for this, recovering the vivid feelings that went out at the moment when the form of mathematics was written and described into the content back into the school mathematics would be a fundamental way of internalizing school mathematics. The only way to put the feelings back into school mathematics is 'the genetic approach' to recapitulate dramatically or indirectly the process where the insight into the form of mathematics took place, as Toeplitz (1963) tried. Teacher is the only one who can recapitulate that process, and that would be the reason of existence of him/her indeed.

Teacher must have “the attitude of passionate concern about truth.”

> “such activities will only be seriously engaged in by those who have a serious concern or passion for the point of the activity … a teacher, must have an abiding concern that this sort of attitude should be passed on to others. … It is mainly caught from those who are already possessed of it, who exhibit it in their manner of discussion and in their teaching (Peters, pp.165-166).”

A teacher is not just a messenger of information, but an initiator of judgment within him/herself. The main interest of a teacher is to inspire the judgments in students. Handing down judgments can only be done indirectly during the process of transferring information. The process of revealing the form of school mathematics within the mind of a teacher contains his/her belief and attitude to school mathematics, along with excitement, anxiousness, and passion the teacher felt when he/she learned it. Through the expression of such emotions, the teacher shows students that the form behind the content corresponds to its archetype. A teacher can be viewed as a person who leads students to sense the form of mathematics, and to pursue it, by revealing the intrinsic form, with his/her expression of emotions, among contents. The only one who already possessed it through his/her own way of thinking could transfer the passion of the quest for truth (Park Chae-Hyeong, 2002).

Polanyi (1964) criticizes the process of reception of pre-existing knowledge through textbook that it eliminates intellectual passion and excitement, which is originally supposed to be the most important aspect of science. Also, he maintains that knowledge education should be the process of enlightenment that helps students to encounter with reality at a new level through the internalization of ‘tacit knowledge’, that is, ‘subsidiary awareness’. According to him, in that process, affective factors like teachers’ persuasive passion and students’ heuristic passion are critical, and a teacher and a student should share the value of knowledge and the sense of truth, and together commit themselves to education and learning. Whitehead (1957) said,

> “the great romance is the flood which bears on the child towards the life of the spirit (p.22). ... the rhythm of these natural cravings of human intelligence ... It is dominated by wonder, and ... in a deeper sense it answers to the call of life within the child”(pp.32-33).

To encounter with the Reality through school mathematics is accompanied by wonder, and can guide students to impressive and respectful mathematics, pure mind, innocence, sincerity, enlightening experiences, intellectual modesty, and the pursuit of moral life.

Encountering with the Reality takes place successively through the stages of mathematization. Mathematical awareness of each level can be viewed as the exposed form of the part of the Idea by the medium of phenomena, and the focal awareness of some part of the subsidiary awareness like the tip of the iceberg, and the records of the result of partial facing the Reality. School mathematics, starting from the mathematics that gave the first impression, should be genetically developed so that it can provide experiences of wonder and awe at each level. However, mathematics that gave mathematicians impression may not do the same to students. Euclid's Elements was impressive to mathematicians, but not to students. Neither was Birkhoff's axiomatic system of Euclidean Geometry. Developing the Pythagorean theorem in the ways of Euclid or the geometry textbook based on the Birkhoff's similarity postulate (Birkhoff & Beatley, 1959) will be hard to impress them. The historic-genetic development of Clairaut (1920) and Toeplitz (1963), which tried to return to the first encounter, the root of the idea, thus go back to the solution for the burning question and the earnest goal of inquiry of that stage, so that the fresh, vibrant life could be reborn, would be appropriate. In addition, we need to emphasize the intuitive thinking, and lead students to move from the first intuition to the second intuition as Fischbein (1987) claimed.

> "The teacher has a double function. It is for him to elicit the enthusiasm by resonance from his own personality, and to create the environment of a larger knowledge and a firmer purpose. ... The ultimate motive power ... is the sense of value, the sense of importance. It takes the various forms of wonder, of curiosity, of reverence, or worship, of tumultuous desire for merging personality in something beyond itself. This sense of value imposes on life incredible labors ... (Whitehead, pp.39-40)."

Didactical analysis of school mathematics

The point of view that emphasizes 'the structure of knowledge' or 'the forms of knowledge' is to prescribe the meaning of knowledge education in the light of the nature of the knowledge. One of the main difficulties in mathematics education is the difficulty in understanding school mathematics in that way.

As Polya (1965) emphasized in his Ten Commandments for mathematics teachers, we cannot teach what we don't know and what we have not experienced. A deep understanding of school mathematics on the part of the mathematics teachers is the alpha and omega of mathematical education.

> "A teacher with profound understanding of fundamental mathematics is not only aware of the conceptual structure and basic attitudes of mathematics inherent in elementary mathematics, but is able to teach them to students (Liping Ma, 1999, p.xxiv)."

For mathematics to be taught as the subject for cultivating mind, didactical analysis of mathematical knowledge is required. However, school mathematics is a formal 'closed' knowledge that does not reveal the nature. To 'open' this and realize it into educationally meaningful knowledge through the didactical analysis, that is mathematical, historical-genetic, psychological, linguistical, practical and educational analysis of the structure of school mathematics may be the most important task for the teaching of mathematics for humanity education. The noteworthy ones here are the analysis of elementary mathematics from an advanced standpoint of Klein (2004), the historical-genetic developments of the school mathematics of Clairaut (1920), Branford (1908), Toeplitz (1963) and others, and the didactical analysis of the school mathematics of Freudenthal (1973,1983,1991). These are thought to have its goal in trying to open the closed school mathematics to make it become a didactically meaningful knowledge. This task, as Freudenthal insists, may be the starting point to mathematics educational research. The didactical phenomenological analysis of number concept, ratio and proportion, group, function concept and others that Freudenthal attempted, the analysis of complementarity of mathematical thinking that Otte (1990, 2003) attempted, the analysis of the epistemological obstacles of function concept that Sierpinska(1992) attempted, and the didactical analysis of the probability concept that Kapadia and Borovenik (1991) attempted, show how hard it is to 'see' the structure, form and nature of mathematical knowledge. How can teacher who does not see it properly or even does not try to see it, think about teaching it? These researches show how the mathematical knowledge that we are teaching and learning is only at the surface of the knowledge and how we are missing the much more important 'educationally' essential viewpoints. The mathematic educational research, first of all, ought to start from the inquiry of the 'structure' of school mathematics.

And we also should not undervalue the role of the oral and written language in education. Recently the discovery and constructive approach through the learner-centered activities using the concrete materials are emphasized, but it should not be overlooked that the using of language by the teacher in education has a more important role than anything else. The Socratic obstetrics in the Plato's *Dialogues*, *the Analects* of Confucius and Polya's modern heuristic are written in the dialogue style centered on the teacher. And the sentence that is the record of knowledge may be a kind of residue of the realization as satirized in a Chinese classic, but it is the only clue that makes us guess what the realization is. Thus we can do nothing but try to get the realization through the sentences.

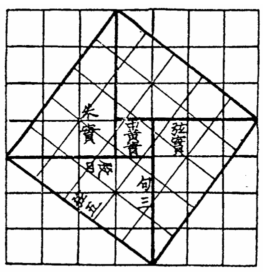
SCHOOL MATHEMATICS AND ENCOUNTERING WITH THE REALITY; IMPRESSIVE SCHOOL MATHEMATICS

Next, we are going to think about the Pythagorean theorem, the negative number and the functional thinking. With these three, we are going to look at the hidden forms of school mathematics, the process of gradual development that reveals its mysterious characteristics and the forms, and the organization of the teaching for cultivating mind.

The Pythagorean Theorem

In mathematics, one of the most famous and widely used theorems, and one of the bases of middle and high school mathematics is the Pythagorean theorem. What meanings does it have? It is a form that reveals the structure of plane, thus one of the essences that reveal the purposive ness of nature. Without it, there is no trigonometry and the related things. Since analytical geometry is an algebraic Euclidean geometry, the Pythagorean theorem has also been the basis of analytical geometry and thus the basis of modern mathematics. Since *Euclid's Elements*, as the matrix of mathematics, is a mathematical model of the things around us, the Pythagorean theorem, which is the basis of it, is almost everywhere in mathematics and its application. Accordingly, the Pythagorean theorem has been traditionally taught as the main content of school mathematics, thus one can hardly find anyone who had graduated high school without the knowledge of the Pythagorean theorem. However, the harder to find would be the one who was impressed by it, and who had surprising experience of its mysterious form.

The following fact proves that, before human beings met the Pythagorean theorem and its converse, there had been implicit knowledge about it. It is told that the fact that a triangle which has the proportion of its three sides 3:4:5 is a right triangle, had been known to ancient Chinese, Babylonian, and Egyptian, and used practically by carpenters and stonemasons (Eves, 1976; Loomis, 1968). The figure at the right, from *the Chou-pei Suan-ching*, the oldest classic mathematics book of ancient China of the 6th century B.C., shows the Pythagorean theorem when the proportion of three sides is 3:4:5(Ronan, 2000, p.17).



General feature of the relationship among the three sides of the right triangle was first discovered and 'proved' by Pythagoras, thus named after him. This is an epoch-making historical event that encountered with the form of the plane that is a continuum. The fact that the proof of the Pythagoras' theorem is the culmination of Euclid's first book and that rather than a theorem to be proved from Euclid's axioms, it could be taken as itself being the characteristic axiom of Euclidean geometry equivalent to the fifth postulate suggests that the Pythagorean proposition is the most characteristic and fundamental feature, that is, 'structure theorem (Tall, 2002)' of Euclidean geometry. Furthermore, geometry is concerned with assigning measures and "$P_{=}(h^2 = a^2 + b^2)$ is the simplest sensible rule for assigning an overall measure to separations spanning more than one dimension (Lucas, J.R., 2000, p.60)."

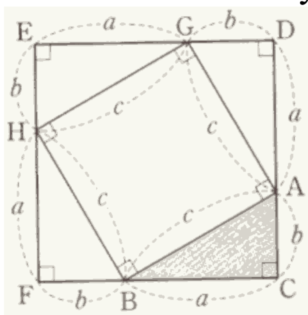
One of the most influential discoveries based on the Pythagorean theorem is the incommensurable magnitude, that is, the irrational number, and this has had great influence on the development of mathematics since Greece. The episode that Pythagoras offered an ox to a deity in honor of its discovery (Loomis, p.6) shows the depth of his implicit knowledge of the Pythagorean theorem.

What on earth is *Euclid's Elements*? Euclidean geometry is called as peripheral geometry. It can be viewed as an attempt that had been made by Greek mathematicians since Pythagoras to manifest the purposive ness of nature, through the geometrical language. In other words, it can be considered as an attempt to define the structure of the space in which we live, something 'indescribable', through the form of 'Euclidean axiomatic system'. *Euclid's Elements* can be seen as a manifestation of the pursuit of the internalization of the purposive ness of nature, the noumenon, the thing itself, the Reality, even if the part they discovered turned out to be the structure of peripheral spaces. If we think that the establishment of the fifth postulate was to support the Pythagorean theorem (Gould, pp.284-286), we cannot help but be impressed by Euclid's insight on the structure of a plane. There, Greek might have seen the core position of the Pythagorean theorem. They might have experienced the harmony between physical reality and geometrical system, the beauty of the axiomatic system and the Pythagorean theorem, and 'pure emotion' of beauty and sublimity, by the reflective judgment due to the 'Idea-oriented' mind. And they might have committed themselves in continuous studies of geometry and reflections to pursue the thing itself.

What do we intend for students to 'see' when we teach the Pythagorean theorem? The most important thing would be that it is the basic form that manifests the structure of the space, the thing that we can never know, where we live in. We should provide an opportunity that arouses awe and respect to the 'logical reason' the Pythagorean theorem is based on, and inspires ceaseless yearning for mathematics that has tried to inquire into it.

However, obviously students will not be impressed by the Euclidean approach until they realize the profound meaning of 'Elements' and experience the mystery of axiomatic organization. History of mathematics education is eloquent of it. Unlike the Greek ideal that mathematics education would lead students' eye of reason to the Reality, mathematics education after Euclid could not pass over the formal education that worshipped Euclid, thus dry, formal education was continued.

At present, the proof of the Pythagorean theorem using the ratio of similarity of the similar right triangles and Euclid's proof based on his axiom of parallel lines are not accepted due to the concern that young students might have too much difficulty in understanding them. Instead, based on the axiomatic system of S.M.S.G. Geometry (1961) in which the rectangle area postulate is included, we introduce the way to prove the Pythagorean theorem with the area formula of rectangle. For example, in the right figure (Cho Tae-Geun et al, p.28), since AGHB=FCDE - 4ABC,



$$c^2 = (a+b)^2 - 4\times\frac{1}{2}ab = a^2 + 2ab + b^2 - 2ab = a^2 + b^2$$

After that, there are problems such as finding the distance between the two points on the coordinate plane, the length of a diagonal line of a rectangular parallelogram. The problem is, these kinds of approach just provide the students with dry textbook knowledge and its application, in which the impressive experience of realizing that the Pythagorean theorem manifests the purposive-ness of nature will not be accompanied. The reason lies on the Birkhoff's metric postulates. Surely, this method vastly simplifies the logic of the Euclidean geometry, but it seems that this approach is unsatisfactory as an elementary approach to demonstrative geometry and geometrical thinking for pupils.

To ease the difficulty of Euclid's axiomatic system, school geometry has changed to Legendre's(1866) and Birkhoff's axiomatic system(1959) since the 19th century. Today's school geometry simplifies the logic of Euclid's geometry for students' better understanding. Particularly, to simplify the problem of the length and the proportion of the length related with the irrational number, one of the most tantalizing problems of Euclidean geometry, Birkhoff's postulates are accepted. As a result, by accepting the existence of the real number, which is the length or the proportion of the length of line segments, the metrical feature of a figure is added in school geometry, so that students would understand it more easily and simply. This approach might be surely more 'understandable' to students, however, would be considered as get rid of the mysterious and impressive features of geometric figures related with the continuity of space.

Years ago, for the first time, I have felt mysterious of the Pythagorean theorem after I read *Éléments de Géométrie*, written by Clairaut (1920). The fact that gifted mathematician like him perceived the problem of *Euclid's Elements* as a textbook, and wrote a geometry textbook which developed historic-genetically requires much attention and interest. His sparkling ideas in *Éléments de Géométrie* will provide valuable data to the teacher who wants their students to acquire the main ideas of geometrical knowledge through simple and natural process. Clairaut dealt with the Pythagorean theorem as the answer for the problem finding the square as big as the sum of two squares.

Pythagorean theorem tells us that the length of certain line segment can be represented as the sides of every right triangle which has its vertex on the circle whose diameter is the line segment, thus mutually 'independent' two smaller line segments. How mysterious is that every square can be represented as the sum of two squares whose sides are the same as those of such a right triangle, and, furthermore, every plane geometric figure can be represented as the sum of two similar geometric figures.

Negative numbers

I was really surprised when I read the sentence, 'How wonderful $(-1)(-1)=1$ is!' Until then, I had never felt mysterious or impressive for any mathematics that I had learned in school. How about you?

However, if we think even for a moment, we can see that the part of numbers and computations in school mathematics is full of surprises. How many mysteries natural number has? Gauss said that the number theory is the queen of mathematics. The number theory records the mysterious secret of integer that has impressed many gifted mathematicians. What part of it appears in school mathematics? How can we deal with it without any impression?

For all that, why is the explanation of the computation method of negative numbers in school mathematics so awkward? Positive and negative numbers, and their rules of addition and subtraction from the point of view of relative magnitude already appeared in the 1st century, on the equation chapter of *the Nine Chapters*, a mathematics book of Han dynasty of China (Katz, p.17). Among the books of Indian mathematician Brahmagupta in the 7th century, the computation rules for positive and negative numbers had been dealt with (Boyer & Merzbach, p.220). However, though they had already recognized that negative numbers could be useful in computation and solving equations, mathematicians had not accepted negative number until the 19th century. The main reason was that they regarded numbers as the magnitude abstracted from physical world, and that they could not find the consistent proper models of negative number and its computation in the real world. The history of negative number tells us how firm the prejudice that the origin of mathematical cognition lies on objects was. It was not until the mid-19th century, when the fundamental shift of view from the concrete view to the formal view took place, that the efforts to establish the existence of negative numbers succeeded. Hankel regarded negative number not as the concept that represents 'real thing', but as the component of formal computation system, and proved that the adjunction of negative numbers into the computation system shows algebraically consistent (Fischbein, pp.97-102). Hamilton made an attempt at a formal justification of negative and imaginary numbers in algebra. Mathematicians have become more clearly aware of the fact that, between the two fundamental elements of arithmetic, the objects of operation and the operational rules, the latter are really essential and the formal essence of mathematics. This is the start of the axiomatic algebra (Dieudonné, 1972). In today's mathematics, integer is defined as computational number that makes up the consistent formal system satisfying the arithmetic rules accepted as axioms; integral domain.

These facts show that negative number was originated from the 'common-sensual' computational behavior that is the expression of the natural mind of human beings. Negative number is believed to be the typical mathematical concept that underwent troubles for 1500 years (Fischbein, pp.97-102) due to the wrong prejudice, 'cognition follows the objects'. Since the ancient times, positive and negative numbers has been used as the relative numbers that represent the reversed situations as gains and loss. Algebraically, this is not different from defining it as a formal object, for example, defining negative number -3 as the solution of the equation $x+3=0$. The algebraic extension of number occurs as the formal adjunction of the solution of an equation. Numbers are extended by adding, for example, -3, $\sqrt{2}$, i as the solutions of the equations $x+3=0$, $x^2=2$ and $x^2=-1$, while they still satisfy the computation rules of the existing number system. Although there are no solutions in the existing number system, we hypothesize that such a number exists, and add it as x, which is a new formal object. If we add x such that $x^2=-1$, that is, i to the real number system, it extends to $a+bi$ type of numbers, that is, complex numbers. Really wonderful point of this is that this extended number system satisfies the fundamental theorem of algebra that every equation of nth degree has n-solutions. Freudenthal (1973, 1983) went so far as to call the 'permanence of arithmetic laws' that extends algebraic structure without changing the existing properties like this 'the algebraic principle'. This is the logic of analysis. We assume that, in the problem-solving situation, the value of the unknown is x. Then we set up equations under given conditions, and solve it assuming that x satisfies all the rules of existing number system.

The operation of numbers as the object of computation starts from elementary school, and undergoes the stage of intuitional operation, algorithmic operation, and operation as the solution of equations, and finally reaches the stage of global organization of it's algebraic structure. There, the sign rules like $(-a)b=-ab$, $(-a)(-b)=ab$ and $-(-a)=a$ are deduced from the fundamental rules of operations. In high school mathematics, though the term 'axiom' is not used for the real and complex number systems, number is dealt with by the axiomatic method.

At the stage of abstract algebra, negative number is prescribed formally as the object of computation that satisfies fundamental axioms, in the additive group of integer, ring and integral domain which are the systems of 'partial arithmetic (Dieudonné, p.112)', in the rational number field which is the system of 'full arithmetic', moreover, in the real number system which is the completed ordered field, and in the complex number system. According to this pure, formal, and algebraic theory, negative number is the formal object that makes up the arithmetic system. However, is negative number actually free in this formal number system? Gödel's theorem shows that every arithmetic system is incomplete.

All of the intuitive models of positive and negative number that we use in middle school mathematics are incomplete since the consistency that is the essence of mathematical thinking is damaged. As a result, the focus of the teacher's efforts shifts from mathematical knowledge itself to the pedagogical methodology, a typical situation called 'meta-cognitive shift (Brousseau, 1977)'. The instruction of negative number in middle school ends with incomplete, complex models. Students' understanding of computational principles of negative number is very low, and they end up memorizing the rules, thus accept $(-1)(-1)=1$ without any impressive moments. Despite the limits of models used in school, students, teachers, and even the writers of the textbooks cannot go far enough to reconsider the origin of the problem, and lack clear understanding of formal essence of the negative number.

If we pay little attention to the understanding of the formal essence of negative number that many mathematicians had hard time to accept, and satisfy with shallow understanding and exercising computation rules, we deviate from the essence of mathematics education. Freudenthal (1973, 1983), pointing out that integer is the first formal mathematics students meet, requires teaching them through inductive extrapolation and the formal approach faithful to the essence of negative numbers, instead of the concrete models whose consistency is damaged. He believed that, only with the formal approaches, students could understand the formality of negative numbers and have a new standpoint towards the mathematical thinking and the mysterious forms, as the mathematicians did. Is it really possible? If we assume that the solution of the equation $x+a=0$ is $-a$ and extend the number system to integer, and the laws of the operations of natural numbers are maintained according to the permanence of arithmetic laws, we can elicit the computation rules of integers.

This process is not different from introducing i as the x satisfying the equation and complex number as the number of $a+bi$ type, and deducing the computational rules of the complex number, based on the permanence of arithmetic laws. However, since negative number is introduced in the first grade of middle school, it is not easy to deal with it as formal characteristic. Because this process is important enough for

Freudenthal to call it as 'the algebraic principle', it is believed to be meaningful to provide students the opportunity of experiencing this process from an early age. Then integer and its computation will be the first opportunity to encounter with this process in middle school as Freudenthal said. Meanwhile, the unavoidability of the formal approach to negative number cannot but accept when we think about the connection to the next instruction, that is, teaching the linear equation.

Since the formality of negative number is manifested by the solution of equation, we need to look at how textbooks deal with the equation. Unfortunately, despite the fact that analysis and the permanence principle are basically included in the solution of equations, explicit statement for these factors can be hardly found, thus few students are aware of the formality of arithmetic system inherent in the equation solving process.

However, in the high school mathematics for the students who are 3 years older than middle school students, there is a sudden shift to the formal point of view, thus negative numbers is introduced from the point of view of real number field. Here, real number is dealt with formally from the aspect of axiomatic method and *-a* is defined as the additive inverse of a, although the term 'axiom' is not actually used. The sign rules of the multiplication of real number, like *(-a)(-b)=ab*, are proved from the axioms. Furthermore, students learn that, as the number system extends to the complex number, -1 is regarded as complex number $-1+0i=\cos\pi+i\sin\pi$, this can be represented as a point or a vector on the complex plane, and complex number system also satisfies the basic properties of addition and multiplication. The fact that $(-1)(-1)=\cos 2\pi+i\sin 2\pi=1$ and this can be represented, on the complex plane, meaningfully as the multiplication of two complex numbers is also dealt with. If we use Euler's formula, $e^{i\theta}=\cos\theta+i\sin\theta$ taught in the college mathematics, since $-1=e^{i\pi}$, we can beautifully represent that $(-1)(-1)=e^{i2\pi}=1$. When students learn such a characteristics of negative number, they will be surprised once again at the form of mathematical knowledge.

After all, we can say that negative number is the formal object that makes up the arithmetic system. Then, what is the arithmetic system? We can 'construct' it starting from the Peano's axioms, which are the result of formalization of the primitive intuition of the counting numbers. Thus, Freudenthal(1973) said;

> "The number sequence is the foundation-stone of mathematics, historically, genetically, and systematically. Without the number sequence there is no mathematics (pp.171-172)."

Freudenthal's "educational interpretation of mathematics betrays the influence of L.E.J.Brouwer's view on mathematics (p.ix)", but, from this standpoint, it seems that counting number system is the universal principle, the form of mathematics, common sense, and the thing that locates near the 'logical cause' of it, that is, the Idea that is our original mind.

If we accept the natural number arithmetic, then negative number might become a good opportunity to provide first experience of the formal essence of mathematics. If it is realized, students will give their serious thoughts to negative number, and be surprised at the wonder of mathematical form.

Functional Thinking

School mathematics, traditionally, had been classified as arithmetic, algebra, and geometry until the modern age. After the early 20th century, when Klein (1907) emphasized the importance of 'functional thinking' and advocated differential and Integral calculus as the peak of function concept, various functions and their differential and integral calculus have become one of the main parts in school mathematics.

"Klein claimed that the function concept was not simply a mathematical method, but the heart and soul of mathematical thinking"(Hamley, 1934, p.53) and attempted to organize school mathematics centering on functional thinking. Klein said that

> "der Funkionsberiff in geometrischer Fassung den übrigen Lehrstoff wie ein Ferment durchdringen soll(1904, S.4) ", "ich bin der Überzeugung, der Funktionsbegriff in geometrischer Form solte überhaupt die Seele des mathematischen Schulunterrichts sein!(1907, S.34). "

This attempt comes from his belief that functional thinking combined with geometric intuition connects algebra with geometry and is the fundamental, core idea that lies on the bottom of the whole mathematics including applied mathematics. Function is, surely, the mathematical thinking that acts as the archetype, that is, the foundation of mathematics, and a unifying concept that lies at the root of all mathematics. The Meraner Lehrplan, in which Klein took the lead, set forth as one of the aims of mathematical teaching to make the pupils more and more conscious of the continuity of the subject as they pass from stage to stage, and the unifying principle which made this continuity possible was defined to be 'education in the habit of functional thinking' (Hamley, 1934, p.53).

Today's school mathematics is literally filled with functions. In school mathematics, function-related contents are dominant enough to see the possibility of realization of Klein's argument that requires the organization of school mathematics centering on the functional thinking. The problem is what kind of education takes place.

Looking at the history of function, its first conceptualization began with the curve that represented movement in dynamics. It was adapted as the form of mathematical thinking to interpret, explain, and expect various changes in nature, and has been used, along with the development of science, as the central form of mathematics with which we explore and describe nature. The functional thinking is the mathematical form that reveals the purposive ness of nature, thus we can even say that the rule of nature is functional, and nature is described in the functional language.

Along with set and structure, function is a representative 'unifying concept' and 'organizing concept' of today's mathematics, and the fundamental concept that lies in the root of every mathematical concept. With the operation of functions like composite function and inverse function, mathematical thinking becomes more affluent and powerful. The operation of composite function and inverse function gave a birth to the group of transformation and permutation, and, moreover, the general theory of groups, and its core form, the automorphism group (Freudenthal, 1973). MacLane and Birkhoff (1968, p.1) mention,

> "Algebraic system, as we will study it, is thus a set of elements of any sort on which functions such as addition and multiplication operate, provided only that these operations satisfy certain basic rules."

According to Klein's Erlanger Programm, the group of transformation became a form that organized various geometries, thus gave birth to the transformation geometry. In the field of topology, continuous mapping and topological transformation has been the tool and the object of study. With the advent of category theory, also called 'Erlanger Programm of algebra (Eilenberg & MacLane, 1945)', mathematics started to be regarded as the study of mapping. The richness of today's mathematics can be said as being created by the functional view.

Function is, without a doubt, not just a mathematical concept, but the 'soul of mathematical thinking' that lies in the root of every mathematical thinking and reveals the purposive-ness of nature. In this light, Klein's argument that the heart of the doctrine of mental training is to be found in methods of conceptual thinking (Hamley, p.55) is thought to emphasize the point that function, by letting students pursue the overall understanding of mathematical thinking, can be, and should be the valuable channel to the root that enables all mathematical thinking, that is, the Idea. 'Rückblicke unter Heranziehung geschichtlicher und philosophischer Gesichtspunkte (Klein, 1907, S.216)' to the mathematical knowledge learned, which is given in the final grade of Meraner Lehrplan, is thought to be significant, and is regarded as an attempt to realize the mind cultivating value of school mathematics by providing the opportunity of 'reflective judgment' to pursue the ultimate meaning, that grasps the basis of mathematics and natural phenomena through the concept of function.

The concept of function, which has been refined for a long development process, is a very strong and fundamental concept that lies in the bottom of mathematics, and an inclusive concept whose essence is not easy to define. It has various aspects such as the geometric, computational, set-theoretic, and logical aspects (H-G.Steiner, 1982). M. Otte(1982, 1990, 2003), arguing that mathematical knowledge is a process of limitlessly revealing its essence and can be understood only by complementary co-ordination of various aspects of it, emphasizes that the concept of function takes an important part in revealing the complementarities of the structure of mathematical knowledge (Jahnke,H.N. & Otte,M, 1982).

It is a noteworthy fact that function is immanent in our everyday life experiences. According to Piaget (1977), the psychogenetic origin of the function concept is the schémes inherent in sujet épistemique. Historically, despite its relatively late introduction, the concept of function has existed from an early age, although it was not dealt with as a conceptual object. The tacit advent of function as the numeration table goes back to the ancient Babylonia, around 2000 B.C. (Eves, 1976). The fact that function is implicitly used in ancient mathematics, and that it has existed as a subsidiary awareness long before it got a name, requires our attention, particularly in relation to the transcendental mind.

Function is one of the most powerful unifying and organizing concepts of mathematics, and also the form that makes up the rules of nature, and as Klein (1904, p.15) pointed out, mathematical elements of today's culture is totally based on the concept of function and its development in geometrical and analytical aspects. What is the functional thinking that Klein emphasized? Is it the inherent structure of mathematics, or a mathematical cognition closer to the Reality? If the Reality is immanent in the mathematical cognition as its logical hypothesis, opens the eyes of students of mathematics to its existence, and inspires endless desire for it in the students, functional thinking can be and should be a shortcut to realize the Reality, the Idea, the primordial mind (Kang, Hyun-Young , 2007).

CONCLUDING REMARKS

The pendulum of mathematics education today has gone too far to the side of practical knowledge education for training the experiential mind. The important goal of learning mathematics goes beyond its practical application. It is to realize the humane education that enables students to be aware of the Reality that dominates the world of phenomenon, located inside human being and all things, by the knowledge of mathematics. For this reason, Plato and Froebel put much importance on the mathematics subject in the education process.

Clairaut, Branford (1908), and Toeplitz (1963) used historical-genetic development to grow students' ability and attitude to 'see' phenomena with mathematical eye and understand it. Especially Klein tried to do it through 'the education in the habit of functional thinking'. Bruner attempted to do the same through letting students discover the structure of mathematical knowledge, and Freudenthal tried to do it through providing mathematising experiences. Students can see the phenomena in wonder with mathematical eye only after they understand the mathematical knowledge related with it. But we should go one step further. Mathematics helps students to have a desire to see the harmonious appearance of the world of the Reality that can only be seen with our mind's eye, and to sense the spirit of God which lies within human being and nature, thus can be a noble mean for education that lead students to realize 'the standard of life'. Isn't it the idea that school mathematics as humane education and mind cultivating is in need of?

It is said that the reality is a parody of the idea. We argue that school mathematics should be a short cut to the recovery of primordial mind, and provide to students the wonderful and impressive experiences about mathematics. However, look at the editor's preface of McLellan and Dewey (1895)'s book. It says:

> "There is no subject taught that is more dangerous to the pupil (than mathematics) in the way of deadening his mind and arresting its development, if bad methods are used (p. v)."

What is the reason that *Euclid's Elements*, which pursues the idea of mathematics education of Plato, suffocates the mind of students? How can we teach mathematics so that the students understand the true meaning of mathematics and their human nature can be cultivated? This is the most important problem we, as mathematics educators, must solve.

Teacher determines education. One of the most immediate problems current mathematics education is facing is to set up an educational foundation of the mathematical subject. To change students' eye and their ways of living their lives, to realize the humane education through mathematics education, the mind of mathematics teachers should be changed first, and their educational philosophy should be established. The educational philosophy of a teacher is a main standard that determines the contents and methods of education. Mathematics teachers must endeavor to have profound understanding of the philosophy and epistemology of the East and the West, and the thoughts of philosophers and educators who provided the ideological root of mathematics education. This is not a mere problem of understanding a theory, but requires an understanding of what is right and valuable in education and life.

School mathematics, as records of knowledge, is a closed knowledge. Mathematics teachers need to open school mathematics through didactical analysis of it to enjoy the form and structure within its contents, revive the wonder and impression mathematicians felt when they first discovered it, and inspires it in students' mind.

To realize this sort of education, as the Scripture, which is the writing of the enlightened one, shows the Suchness much better than phenomena do, it is necessary to have appropriate teaching materials that can reveal the root of mathematical knowledge. Situations in mathematics classrooms are situations where one tries to transfer 'something that cannot be expressed' logically assumed in the mathematical knowledge contained in textbook. The methods that had failed to provide wonder to mathematicians can never attract students. In school mathematics, Euclidean geometry that includes the Pythagorean theorem reveals the essence of space around us can provide wonder and impression as our ancestors did experience. Regarding negative numbers, however, there is difficulty in leading students to perceive its formal beauty even mathematicians could not successfully do it until the mid-19th century. Among many parts of school mathematics, function is the fundamental form that lies in the root of all mathematical thinking including applied mathematics, thus 'the heart and soul of mathematics' as Klein claimed. It can be a shortcut to return to the Idea by unifying all mathematical thinking. We should help students 'see' the functionality as the immanent order of physical phenomena, and provide materials that guide them to the realization that unifying all mathematical knowledge is possible with function. We should let them experience the wonder in functionality. These problems of school mathematics are waiting for professional studies.

What does 'doing mathematics' mean? Let's look at Freudenthal's claim. "Dealing with one's own activity as a subject matter of reflection in order to reach a higher level"(1991, p.123) is an important mathematical attitude. To do mathematics is to observe one's or other's mathematical activity and reflect on it. Mathematical thinking is a mental activity that pursues the noumenon, which is the means of organization of phenomena, through continuous alternation of the content and form. The main body of such mathematical activity is reflective thinking.
The following claim by Gattegno(1974, p.vii) is also significant. "Only awareness is educable." Here, awareness can be interpreted as realization. Man becomes educated through realization. Mathematics, through the realization of the essence that lies within the contents, leads human being to the internal awakening, thus, for mind cultivating, mathematical knowledge is more important than anything else. Then, how is awareness formed? Our conclusion is that it is obtained by activities, discussions, and reflections. The following is from *Analects of Confucius* (1992).

> "學而不思則罔 思而不學則殆 (Mere reading without thinking causes credulity, mere thinking without reading results in perplexities(pp.18-19)."

If mathematics should be able to help students to realize 'the order which dominates phenomena', 'the form', 'the logical cause' and 'the existence of the Reality', and become a 'wise person' who wants to be enlightened, we should let them experience the authentic mathematical thinking and internal awakening. To do that, actual 'mathematical activity' and 'reflective thinking' are desperately in need of.
We should lead students to experience the self-awareness of pure and truthful mind, and ignorance through the impressive and awe-inspiring school mathematics, and lead the experiences to the intellectual modesty and pursuit of moral life. To do this, we need, as Klein tried, historical, philosophical, and reflective unifying stage in school mathematics, thus, ultimately, to be able to become an educational path for students to encounter with the Reality. As Polya (1965, p.104) points out, "What the teacher says in the classroom is not unimportant, but what the students thinks is a thousand times more important." Mathematics education for every student should be conducted as cultural education in the real sense.

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PLENARY PANEL

School Mathematics for Humanity Education

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Frade, Cristina

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INTRODUCTION TO THE PME PLENARY PANEL, ‘SCHOOL MATHEMATICS FOR HUMANITY EDUCATION’

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The theme of the PME 31, ‘School Mathematics for Humanity Education’, will addressed by a panel of four panellists. In this introduction, the theme and the position papers that the four panellists wrote in preparation for the panel are introduced briefly.

INTRODUCTION

The plenary panel of this conference will address the theme of the PME 31, ‘School Mathematics for Humanity Education’. This theme may be interpreted in various ways. We may, for instance, think of what mathematics is, and discuss whether or not mathematics is a human construction. Or we might consider what we want mathematics to be for our students. Here we may take Freudenthal (1973) as an example, who stresses that for him mathematics is a human activity, and that that is what it should be for the students too in his opinion. Another perspective would be to look at how mathematics education is experienced by students emotionally. Then issues such as appreciation for, or dislike of, mathematics come up. This theme could be elaborated further in terms of task and ego motivation, cultivating mathematical interest, and the role of identity. Finally, we might look at the goals of mathematics education from the perspective of humanity education. Should the emphasis be on practical problem solving, or on more formal mathematics, cherishing mathematics as cultural heritage, trying to cultivate appreciation for the beauty of mathematics, or focus on mathematical thinking and reasoning?

In preparation of the panel, each of the panellists wrote a short position paper, in which they briefly elaborate their own take on this issue. We may, however, discern one common tread in a shared concern for the negative way mathematics is valued and experienced by many students. This they point out is in conflict with mathematics for humanity education. Their diagnoses and remedies, however, vary. Cristina Frade emphases the cultural aspect, Willi Doerfler promotes mathematics as an activity of acting with signs, Martin Simon elaborates the importance of realizing the human potential, and Matasaka Koyama advocates mathematics education as a means for developing the students’ personality and humanity.

By way of introduction, I will briefly sketch the four positions.

Cristina Frade starts by challenging the notion of a dichotomy between ‘theoretical’ and ‘practical’ in regard to mathematics. She argues that traditional theoretical-practical dichotomy may lead to a ‘perverse hierarchy’ between school mathematics and mathematics that is developed out of school. This may be mirrored by

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a similar hierarchy in the relation between teachers and students, within which teachers and students can be thought of as belonging to two different cultures—with one dominating culture, that of the teachers. She shows how Wenger's notion of participation in a community of practice may offer an alternative.

Wilibald Doerfler takes the position that mathematics is a human activity through the core. He argues that mathematics is done and produced by human beings. His main point is that we should make mathematics more humane, and more mundane, by making students aware of the human origin and nature of mathematics. He elaborates this with the thesis that mathematics is an activity of designing and using signs, which, he argues, reveals its human origin, and highlights the aspect of mathematics as a social practice. He elaborates mathematics as a shared and social practice of sign use, as an alternative for mathematics as a purely individual mental activity with abstract objects.

Martin Simon approaches mathematics education from a different angle. He connects the notion of humanity education with the notion that humans have a potential for mathematical reasoning, knowledge, and communication. This should be realized through education. He contrasts this with the observation that students are often treated as if they have no ability and motivation to learn. Which brings him to the question of how to foster students' flexible use of their full complement of intellectual resources. He connects this with his research that aims at developing understanding of mathematics learning in a way that enables one to scientifically support students' abilities to learn—which he illustrates with recent work on understanding how students construct mathematical concepts through their own mathematical activity.

Masataka Koyama argues for the need for humanising mathematics education by taking his starting point in the Japanese cultural tradition of "GEI (Art)-esprit". Within this perspective, mathematics education is characterized as part of the way to develop students' personality. In line with those ideas, he depicts mathematics as a creative activity of the human mind, and promotes mathematics-as-an-activity as the way for students to develop their personality and humanity. He emphasizes the role of mathematics as a means for 'educating students' awareness' as a typical human quality, but warns that this does not mean that we may reduce or lower standards for the mathematical content. Instead, we are to help children to collaboratively meet the challenges they may encounter in their process of learning mathematics.

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HUMANIZING THE THEORETICAL AND THE PRACTICAL FOR MATHEMATICS EDUCATION[i]

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If we move towards a humanistic view of mathematics education we should then move to a reconceptualization of the traditional use of terms 'theoretical' and 'practical' to address learning contexts as socio-cultural practice, or even abandon these terms. I will reflect on this from three academic points of view.

The words 'theoretical' and 'practical' are traditionally used in philosophy and education to suggest two distinct forms of knowledge or two different ways in which people acquire knowledge/competences. The former is traditionally associated with conceptual and abstract knowledge/competences suggesting an exclusive mental movement or action of the mind. The word 'practical' is traditionally associated with *knowledge-in-use* which includes application of knowledge/competences in daily-life contexts and in professional performances; it traditionally indicates an engagement of a person in a perceptible activity, which often involves some movement or action of the body. Furthermore, this engagement is supposed not to require any reflection on what is being done in the sense that individuals often are not aware of the 'tacit' knowledge that underlies their action (see Frade & Borges 2006).

I found rethinking of this dichotomy between 'theoretical' and 'practical' challenging. I have taken it to suggest a separation between mind and body, and consequently detachment of what is seen as one form of learning (theoretical) from other forms of activity (e.g. living). This points to important social and educational implications that may not fit with any humanistic view of learning. Some of these implications are discussed by Jean Lave in the first chapter, *The practice of learning,* of the book *Understanding practice – perspectives on activity and context* (Chaiklin & Lave, 1996). Implications include: neglect of the lived-in world, uniformity of knowledge, learning detached from culture and affect, social positioning, power/control relationships, exclusion, individualistic and passive learning, cognitive hierarchy and failure to learn. If we take up a socio-cultural perspective of learning like that of situated cognition (Chaiklin & Lave, 1996), one that presupposes learning to be an aspect of everyday practices, not only does the dichotomy between the theoretical/mind and the practical/body collapse, but also the division between learning and other forms of activity. In fact, from such a perspective the word 'knowledge' moves to the word 'knowing' since the latter and learning are seen as engagement in changing processes of human activity. Jorge T. da Rocha Falcão and I have emphasized elsewhere (e.g Da Rocha Falcão, 2006) that this traditional theoretical-practical dichotomy may also lead to a perverse hierarchy between mathematical activity developed out-of-school (e.g. street mathematics) and mathematical activity developed at school, where the former is viewed as 'hierarchically inferior' from some academic perspectives. Perhaps we could better say that what is considered as being theoretical and practical depends on the practice?

The island metaphor. In his contribution to the PME 30 discussion group *Participation, thought and language in the context of mathematics education* (DG: P,T&L), Luciano Meira (2006) suggested an insightful reconceptualization for 'the theoretical' and 'the practical' using the island metaphor to describe power/discursive relationships between 'explorers' and 'natives'. This metaphor is an adaptation of a narrative of Bruno Latour in the book *Science in Action: How to Follow Scientists and Engineers through Society* (Latour, 1987). Through this metaphor Meira associates the practical to a person's life (natives living on an island), whereas the theoretical is associated to a representation of this person's life (explorers coming to the island and drawing a map). However, he points out that there is no good reason to think that the natives do not 'theorize' about the explorers' *forms of life* at the moment they are mapping the island. According to Meira's approach cultural conflicts arise inevitably both when the natives begin to live the lives 'imposed' by the map, and when they visit the homeland of the explorers and question the rationale for the map. It is not difficult to elaborate a correspondence suggested by Meira between the island metaphor and mathematics education. Let us suppose that the island corresponds to a mathematics classroom within a *strongly classified* curriculum (using Bernstein's terms) in which 'children-natives' live a great part of their lives. The mathematics 'teacher-explorers' 'impose' on them a map which includes the *vertical discourse* of mathematics – *via recontextualization* – and some established social and mathematical norms, which the children-natives are supposed to share and to follow. The teacher-explores' homeland would correspond to what Alan Bishop – who also participated in this discussion group – suggested might be called 'mathland'. Cultural conflicts arise, for example, when students question the rationale for this map or when they feel themselves to be 'outsiders' in mathland. Whatever the correspondence between Meira's metaphor and mathematics education, it should suggest a kind of 'dominator-dominated' relationship between teachers and students. This leads us to a reflection about the character of mathematics education in terms of humanity. In the discussion group Alan Bishop, in an attempt to humanize this imbalanced relationship, asked Luciano Meira: 'Why not set out to invite children into mathland and give them the tools to navigate it?'

A cultural-affective perspective of learning. Alan Bishop (2002a) proposes a distinction between 'Western Mathematics' and 'Numeracies', where the former is associated with 'the theoretical', and the latter is associated with 'the practical'. In the context of mathematics classrooms, 'Western Mathematics' and 'Numeracies' come from different individuals with histories of experience in different discourses, which converge to form the classroom discourse or *the borderland discourse* (see Gee 1992 in Bishop 2002b). If we change the word 'numeracies' to 'everyday "mathematical" knowing', the relationship between Bishop's reconceptualization of 'the theoretical' and 'the practical' leads to the recognition of the fundamental role of teachers' values, and suggests that the way in which teachers 'reveal in action' (using Bishop's words) their general education and mathematical values in classrooms can strongly contribute to students' mathematical learning and approximates either to a process of *enculturation* or to a process of *acculturation* (2002b). Bishop borrowed these terms from anthropology. *Enculturation* is the induction, by the cultural group, of young people into their culture, whereas *acculturation* refers to the induction into an outside culture by an outside agent. Often one of the contact cultures is dominant, regardless of whether such dominance is intended. According to Bishop, mathematics teachers are the main agents of mathematics *acculturation*. He considers two types of 'acculturator'-teacher: a) the teacher who does not make any reference to any mathematical knowledge out-of-school; b) the teacher who imposes what s/he wants through her/his privileged position and power. In both cases, says Bishop, the resulting cultural conflicts, although containing a cognitive component are infused with emotional and affective traces/nuances indicating deeper and more fundamental aspects than can be accounted for from a cognitive perspective.

Learning in communities of practice. Etienne Wenger (1998) elaborates a theory of communities of practice (CoP) in which the term 'practice' does not reflect any dichotomy between 'the practical' and 'the theoretical'. For him, communities of practice include all of these, even if there might be discrepancies between 'what we say' and 'what we do'. He observes that 'when a theory is a goal in itself, it is not detached but instead is produced in the context of specific practices. Some communities specialize in the production of theories, but that too is a practice' (p.48). Thus, in Wenger's approach the distinction between theoretical and practical moves to distinctions between kinds of enterprises rather than distinctions in qualities of human experience and knowledge. To address learning in communities of practice or between enterprises the author proposes talking in terms of 'participation' and 'reification', where the former could be thought of as replacing 'the practical', and the latter 'the theoretical'. However, participation and reification 'say' much more than this and represent a humanistic view of learning since both processes explicitly take into account people, interaction, community, identity, and so on. In fact, according to Wenger, participation is a process related to the social experience of living in the world 'in terms of membership in social communities and active involvement in social enterprises' (p.55). Participation includes, then, talking, doing, feeling and belonging. On the other hand, a process of reification is constituted when talking, doing, feeling, belonging, etc take form by producing objects that congeal such experience into what the author calls "thingness". Reification includes: designing, representing, naming, encoding, describing, perceiving, interpreting, using, reusing, decoding and recasting. Wenger emphasises that participation and reification should not be viewed as a dichotomy, rather these processes correspond to a duality; they are seen to be complementary processes *through a process of* negotiation of meanings. Furthermore 'participation is not merely what is not reified' (p. 66) and 'reification is not just objectification; it does not end in an object(...)these objects(...)are only the tip of an iceberg' (p.60). The distinction between 'the theoretical' and 'the practical' is not an important focus in Wenger's theory about CoPs, since what has been traditionally seen as 'theory' can be seen as the main practice of a certain CoP. Besides, the two forms of knowing that indeed characterise all CoPs are described in terms of the wide concepts of participation and reification, where both concepts have their own explicit and tacit dimensions. In the context of mathematics education these suggest a refocussing of the teachers' attention away from students' cognitive differences towards students' 'collective' cognition which is *now* strictly linked to their participation and identity formation in learning practices.

Final comments. My understanding of 'humanity mathematics education', theme of this PME 31, is based on three main humanistic aspects: 1) a conception of mathematics with a 'human face'; 2) which mathematics is good for people and why; 3) the way in which people are introduced to and learn mathematics. In this paper I have tried to focus on the third aspect. I chose to do so because I believe that the first two aspects may become reduced to empty discourses if education does not take into account that what is good for people is strongly dependent either on their culture or on the affective relationship these people develop with mathematics. It does not make much sense when we educators claim to believe in the powerful nature of mathematics whilst learners – here I am including all those who learn some mathematics for some use – neither get to recognize this nor see any sign of humanity in it.

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MAKING MATHEMATICS MORE MUNDANE – A SEMIOTIC APPROACH

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The trivial fact that mathematics is a human activity is interpreted by viewing it as a semiotic activity with (systems of) signs and diagrams. Organizing learning as the progressive participation in this social practice with and on signs is deemed to make mathematics more accessible and intelligible and less often a cause of anxiety and frustration.

Under all circumstances mathematics is done and produced by human beings. Any calculation, algorithm, proof, formulation of a theorem, drawing of a diagram, inferring a consequence from assumptions, etc., all this plainly and observably is carried out by somebody. And this somebody always is a member of a social context. Thus, mathematics is deeply and genuinely human. A question, often discussed and never resolved, then is if this human mathematics refers to, or is about, something essentially different from what we human mathematicians produce. As longs as mathematics, like in ancient Egypt or Mesopotamia, consists of a collection of recipes to solve (everyday life) problems this question is not asked. This might be so since the mathematical signs used have natural and immediate referents. But in more detached and generalized settings a kind of desire appears for genuine mathematical objects as referents for the used signs. This desire leads to well-known solutions ranging from Platonism to Empiricism. But, one should be aware that also all these kinds of ontology are devised by human thinkers, even if they postulate extremely non-human origins of mathematics. My first and basic thesis thus is:

- Make the learners aware of the human origin and nature of all of mathematics and of all that is said about mathematics.

To realize this goal, it will be necessary to have the learners do mathematics of whatever kind themselves and also reflect about what they are doing. Historical glimpses will be supportive to this end as well. Besides the notorious failure of so many students in mathematics at all grades and levels, another very detrimental phenomenon is the widespread anxiety and frustration on the part of many learners of mathematics. I consider this to be an extremely unacceptable situation which mathematics education practice and research must strive to change to the better. Anxiety and frustration are closely related to the feeling that one does not understand the mathematics, that it is beyond one's cognitive or intellectual grasp. Clearly, it is unavoidable that a piece of mathematics will be too complicated and inaccessible for somebody. But this is also the case for puzzles like Sudoku of which nobody is afraid. Of course, in the case of a school subject where one possibly might fail to pass an exam there are many reasons for anxiety. But in mathematics, I think, a specific

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feature is that learners consider themselves in principle unable to understand. A notorious non-understanding of this sort concerns (-1)(-1)=1 or the imaginary number $\sqrt{-1}$. In my view, one reason is the following. The common discourse, also in classrooms, suggests that the usage of those signs is determined by the objects (numbers) for which they stand. But those "abstract" objects (even if they existed) cannot be grasped by the learner who then thinks, "It is too abstract for me." Mathematics often is experienced as not making sense, as arbitrary and useless, as something for which you need a special aptitude and gift possessed only by a few.

Those phenomena of non-understanding, anxiety and fear, feeling of lack of aptitude appear as surprising and unjustified if one considers the widespread and also well accepted usefulness of mathematics (which is not doubted even by those who hate maths). All that is even more surprising if you confront it with the talk about the aesthetics and beauty of mathematics. If one takes this serious then something must be very wrong in how mathematics is presented to, and perceived by, the students. I cannot present a solution to this paradoxical situation. But I will offer some thoughts about principles for a way to alleviate the problems. I start from the widely accepted view that an important human ability is the production and usage of signs of all sorts, linguistic and non-linguistic ones. Much of our individual and social life is regulated and mediated by sign systems. As authors like Vygotskij and Peirce have emphasized: Our thinking and communicating are sign activities. Thus, the design and usage of sign systems is a deeply human quality and activity of which mathematics is a highly specialized and extremely powerful variant. Of special relevance in this context is the Peircean notion of diagram which is defined as an icon of relations which in the well-known semiotic triad may coincide with its object (Stjernfelt, 2000).

One important aspect of mathematics as a sign activity is that it produces symbolic structures that can be used to model situations and processes of many sorts. On the one hand, already available symbolic/diagrammatic structures (like the decimal number system, fractions, differential equations, combinatorial graphs, etc.) of mathematics can be used as functional models to describe non-mathematical situations, to make predictions or to prescribe structures and relations (normative models). All these contexts of applicability on the other hand serve as sources for the design and development of symbolic/diagrammatic structures. In programs like Realistic Mathematics Education those contexts are used with great efficiency for the development of appropriate sign systems within the class-room community. Manipulation of the mathematical structures, mostly in the form of symbolic systems, permits a kind of understanding since one can take the former as explanations for the observed phenomena in the modelled situations. For that it suffices to consider mathematical models as theoretical constructs which operationally simulate (within chosen degrees of accuracy) experiential observations without stipulating any kind of ontological correspondence between the two. This is made very clear by the fact that in many cases very different mathematical structures "explain" the same phenomenon by for instance making similar or compatible predictions. The main consequence for mathematics education from all that is that mathematics very efficiently empowers human thinking in many diverse areas. And mathematics education has to devise more and better methods to enable the learners to experience in an authentic way this empowerment by mathematical knowledge. This concerns using ready-made sign systems and the design of those in the classroom as well. Students must become aware of the fact that by the use of mathematics you can think and imagine what otherwise is completely unthinkable. It has to be emphasized that this goal can be attained also with simple mathematics. Just think of sociograms for describing or designing social group structures. Of course, not everybody will find that enticing, there is no guarantee for interest and motivation. A special form of this cognitive empowerment is presented by what can be called hypothetical thinking: designing possibilities, analyzing alternatives, answering "What-if-questions", and the like. This feature is very helpful for planning activities and for evaluating different "futures". Mathematics permits us to make concrete the assumption of something which is not yet the case and to draw conclusions and consequences from that assumption. And all that can be done together in a group. It is always a shared and social activity which can be scrutinized, doubted and discussed since it is based on or even involves the production, manipulation and interpretation of sign or diagram systems materialized by writing on a sheet of paper or on a computer screen.

The humanistic intentions of such a semiotic approach to learning and doing mathematics are now manifold. Firstly, it embeds mathematics into the general and basic human (individual and social) faculty of sign production and use. Then it emphasizes and makes clear the genuinely human origin and quality of mathematics which one can view as demystifying mathematics. But, most important perhaps, it turns mathematics into a social practice of a great variety of activities, actions and operations with signs presented by inscriptions, mostly on paper. Learning mathematics in such a view is not the acquisition of static knowledge (about objects like numbers or functions) but the progressive participation in the practice of sign activities. Thereby, the meaning of the signs is constituted by their usage within the practice and not by reference to a priori given objects outside and independent of the practice. Even the abstract objects of mathematics like numbers or functions are the emergent product of this sign activity. This does not make learning mathematics easy. Participating in a social practice is demanding in many respects. There are rules and conventions to be followed, many routines have to be acquired and demanding problems have to be solved. There clearly are ways of organizing the classroom which are more compatible with this view than others. And, what I consider being very important, is that the learners become aware of this trait of doing mathematics as a semiotic activity and that there is great value in becoming proficient in operating with the signs of mathematics.

I have so far emphasized the more utilitarian aspect of mathematics as a sign activity by using and interpreting the symbolic structures as models of and models for. But I want to conclude with the possibility of cultivating interest in the sign systems and diagrams as independent of potential interpretations. For school learning this could mean the exploration of properties and relationships of the symbolic/diagrammatic structures in their own right. In many cases this amounts to recognizing recurring patterns and regularities (like with figural numbers) or to investigate questions of the type "What happens if …?". The learners carry out experiments on objects (inscriptions on paper) according to rules which they know they themselves have designed or could have designed (Peirce speaks of diagrammatic thinking). This kind of activity I consider being an essential part of the social practice of mathematics where consequences of agreed upon conventions are explored, but now within the symbolic or diagrammatic structures of mathematics themselves which there are considered as the objects of mathematics. This can be started with basic number relations and be repeated again and again over the learning process. It is conceivable that through such activity a positive attitude towards mathematics is fostered even when one does not master some parts of the diagrammatic practices. This is based on the assumption that within the semiotic paradigm the activities and processes which might lead to some proficiency are easier to convey and to justify. The mathematical activities based on manipulating and designing inscriptions and diagrams can be demonstrated, observed and imitated, giving mathematical activities an aspect of a handicraft. Mathematics then appears not so much as a mental and individual activity with abstract objects but as a shared and social practice of sign usages. The connection between engagement and successful participation in the social practice should become more transparent, the more it is reflected upon and discussed explicitly in the classroom. Of course, it is necessary that the teacher shares this view. Some suggestions for further and related reading are given in the references.

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MATHEMATICS: A HUMAN POTENTIAL[i]

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The learning of mathematics is a human potential. To scientifically support students' abilities to learn requires understanding mathematics learning in the absence of teaching as well as learning that occurs in response to teaching. I briefly describe work that aims at explicating the mechanism(s) by which learners learn from their mathematical activity and reflective processes in the context of task sequences designed to support their learning.

I start with the premise that human's have the potential for mathematical reasoning, knowledge, and communication[ii]. As such, mathematical potential should be realized through education. However, what is the appropriate relationship between the science of education and the development of this human potential? I begin with an analogy.

Joseph Chilton Pearce (1992) described how western medicine developed a protocol for delivering babies that included putting the mother in a supine position, administering drugs to the mother, and putting the baby in the nursery following delivery. These techniques dramatically reduced the mother's role in the birthing process and resulted in the baby's need for resuscitation at birth -- frequently by being held upside down and spanked. Western scientists studied primitive cultures and found that babies delivered by their mothers, usually at home, did not need resuscitation, had lower incidence of infant mortality, and smiled and showed signs of intelligence up to two months earlier than babies born in Western hospitals. Pearce's point was not that we should return to home births without trained professionals, but rather that the proper role of medical science is to understand, support, and enhance the natural processes that are part of the species inborn abilities, and be prepared for medical emergencies that might occur. Mothers have inborn abilities to birth a baby and nurture it after birth. The role of medical science is to foster and support optimal expression of these abilities.

A similar story can be told of mathematics education. Students were brought into schools and told and shown new material that they were supposed to learn. Students' ability to solve problems and to exhibit conceptual understanding was unimpressive. Issues of how to motivate students became a frequent topic of concern. The system was treating students as if they had no ability and motivation to learn. Attention to children before they go to school reveals that these children are engaged in a continual process of rapid learning that includes two incredible achievements: learning to speak at least one language and developing a concept of number. Have we as educational professionals understood, supported, and elicited this incredible ability to learn? Are we teaching mathematics in a way that fosters students' flexible use of their full

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complement of intellectual resources to solve problems and build more complex understandings?

To scientifically support students' abilities to learn requires understanding mathematics learning in a way that is a useful basis for mathematics teaching. For explanations of mathematics learning to be useful, they must account for learning in the absence of teaching as well as learning that occurs in response to teaching. Teaching can be thought of as the intentional creation of carefully conceived opportunities for students to use their learning abilities to develop powerful mathematical concepts and reasoning in relatively short order.

> The difficulty of studying learning—and teaching—lies, in my view, in the fact that it demands the study of the processes by which children come to know in a short time basic principles … that took humanity thousands of years to construct. (Sinclair, 1990, p. 19)

Sinclair's comment can be understood as pointing to the need to harness students' potential through attention to the social and the cognitive aspects of learning. Two main theoretical frameworks have provided a foundation for thinking about mathematics teaching and learning: socio-cultural theory, based on the work of Vygotsky, and constructivism, based on the work of Piaget. Each frames an inquiry into the question of supporting students' ability to learn mathematics. Socio-cultural perspectives have focused on the human processes of learning that derive from viewing learners as social beings interacting in cultural settings. This perspective has highlighted learning through participation in groups, the appropriation of cultural tools, the negotiation of meanings, and the characteristics of learning communities (e.g., classrooms) that foster mathematics learning.

Bereiter (1985) comment can be seen as an important bridge between studies done from these two perspectives:

> How does internalization take place? It is evident from Luria's first-hand account (1979) of Vygotsky and his group that they recognized this as a problem yet to be solved. (p. 206)

Constructivist perspectives background some of the social issues and focus on the internal processes of the learner as culturally established knowledge and functioning are constructed as individual abilities. Piaget's construct of *assimilation* provides a way to think about which students can appropriate what knowledge under what circumstances. Research based on a constructivist perspective has provided information on how learners reason at different stages of learning particular concepts, providing a rudimentary map of the conceptual terrain.

My recent work[iii] focuses on understanding how students construct mathematical concepts through their own mathematical activity. This is in line with the goal of understanding, supporting, and enlisting students' abilities to learn. The rationale is that if we understand how students construct new abstract concepts through their activity, we can generate a set of principles for the design and sequencing of mathematical tasks. A constructivist perspective provides the principal framework for this investigation. Our work is oriented by Piaget's (2001) claim that the development of more complex understandings evolves through the learners' activities and their inherent ability and tendency for reflection. *Reflection*, which is often not conscious, is the natural tendency and ability for learners to identify commonalities in their experience (von Glasersfeld, 1995).

Bereiter (1985) observed.

> The areas in which instruction has proved most uncertain of success have been those areas in which the objective was to replace a simpler system by a more complex one. (p. 217)

Current thinking in mathematics education embraces the posing of mathematical tasks as an integral part of the teaching/learning process. However, what informs the design and sequence of mathematical tasks? At least in the areas where mathematics teaching has been "most uncertain of success," a scientific approach to the design of task sequences is needed. Towards this end, our work is oriented by the goal of explicating the mechanism(s) by which learners learn from their mathematical activity and reflective processes.

In order to study these learning processes, we have adapted a teaching experiment methodology with individual subjects. In these teaching experiments, the role of the researcher/teacher is restricted to posing problems that are part of a designed task sequence, negotiating the meaning of the problems, probing the subject's thinking, and asking for justification of the subject's actions and statements. The researcher does no direct instruction, gives no hints or suggestions, and asks no leading questions. The methodology is aimed at allowing the researchers to have consistent access to the learner's activity and to minimize the influences of others on the learner's thinking.

Of course, it is never possible to study human activity and learning independent of socio-cultural factors. Thus, it is always important to have socio-cultural lenses, as well as other lenses (e.g., affective), ready at hand during the interpretation of data. Our strategy is to minimize social interaction while studying a variety of learners as they learn a variety of mathematical concepts. This strategy does not eliminate the social aspect of thought, language, interaction, and tool use, but rather makes the students' activity more prominent as compared to the impact of the interaction between student and researcher.

In our first, empirical study of this type, we worked individually with 3 prospective elementary teachers to develop understandings of division of fractions. These subjects developed an understanding of the meaning of division of fractions and reinvented a common denominator algorithm for division of fractions based on understanding the invariance among quotients across changes in the (common) units of the divisor and dividend.

For development of the algorithm, the task sequence began with the student drawing diagram representations of division-of-fractions word problems using rectangular wholes. This starting point was selected based on our anticipation that the students would be able to solve the problems in this form (without any instruction on this during the study) *and* that this student resource could be a useful basis for reinvention of the algorithm with understanding. This choice of a starting point is consistent with the Realistic Mathematics Education principle of guided reinvention (Gravemeijer, 1994).

The successful learning by the students, in response to the sequence of tasks, provides evidence of the powerful effect of a carefully engineered sequence that fosters students' abstractions. Further it provides one case that we were able to analyse in terms of the process by which the students came to that abstraction and the related key aspects of the task sequence. Many more such examples with different age students learning different mathematical concepts are needed in order to have an adequate data set based on which we could elaborate a mechanism(s).

I offer a glimpse of the data from Erin's reinvention of the common denominator algorithm. Erin was asked to draw diagrams to solve first division-of-fraction word problems and then context-free problems. When she was quite competent in doing so and explaining her work, she was given additional context-free problems with large denominators (23/25 ÷ 7/25, 7/167 ÷ 2/167) and asked to not draw a diagram, but "anticipate what you would get if you would draw it." For two consecutive problems of this type, Erin was not able to determine an answer directly, but was able to solve the problem by narrating step by step the diagram drawing process she would have used. These problems were followed by a third problem that had the same numerators as the second problem (7/103 ÷ 2/103). Erin immediately gave the answer of "3 ½." Erin had anticipated the commonality in her activity. From this point, Erin had a curtailed strategy for solving division of fractions problems that she could explain and justify upon request. Her explanations demonstrated her understanding of the invariance among quotients across changes in the (common) units of the divisor and dividend.

Humans have an amazing ability to develop new mathematical abstractions. Greater understanding of their abilities to learn mathematics can result in design principles for curriculum development and related principles for instruction that use and supports learners' learning abilities.

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[ii] *I make no claim that this potential is innate. Rather, this is an observation about human's of normal intelligence living in mathematically sophisticated cultures. It seems useful to think about these abilities as being a combination of biology, development, and socio-cultural influences.*

[iii] *This work has been done in conjunction with Drs. Ron Tzur, Margaret Smith, Karen Heinz, Margaret Kinzel, Luis Saldhana, Tad Wattanabe, Ismail Zembat, Gulseren Karagoz Akar, and graduate student, Evan McClintock.*

NEED FOR HUMANISING MATHEMATICS EDUCATION

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The title of PME31 Panel is "School Mathematics for Humanity Education." The purpose of this paper is to make some contribution to the discussion on it by insisting the need for humanising mathematics education mainly from a Japanese perspective.

FUNDAMENTAL CONCEPTION

The role of education is extremely important because education seems to be the basis of all social system from a political perspective in each country. For example, in Japan at the end of last century the Ministry of Education had undertaken educational reform along with four major points of the "Program for Educational Reform (1998)" in order to develop Japan as a country with vitality and to build a nation based on creativity of science and technology and a culturally oriented nation (Ministry of Education, Science, Sports and Culture, 2000). All people shall have the right to receive an equal education corresponding to their ability and the people shall be obligated to ensure that all boys and girls under their protection receive ordinary education, as provided by the constitution and the fundamental law of education. School mathematics has been a major subject for the education for all. Education in general shall aim at the full development of children's personality. Therefore mathematics education in school is expected to contribute to a part of children's development of personality. In that sense, we may characterise mathematics as a school subject for humanity education (Woo, 2005).

SCHOOL MATHEMATICS

What is school mathematics for humanity education? Woo (2005) insists on the importance to open a formal 'closed' mathematical knowledge and develop it into educationally meaningful knowledge through the didactical analysis, and that the mathematical, historic-genetical, psychological, linguistical, practical and educational analysis of the structure of school mathematics may be the most important task for the teaching of mathematics for humanity education (p.7). In a unique perspective of "GEI (Art)-esprit" as a Japanese cultural tradition, Hirabayashi (2006) characterizes mathematics as a subject of common education as follows. "Mathematics should be considered to be material to train the intelligent part of pupils' personality and should be organized as such in its curriculum as well as in its teaching. What is to be learned is

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not only the *technique* but the *way* to develop their personality – it is the fundamental recognition in learning GEI for common (not professional) people and it should also be the primary motivation to learn mathematics for common pupils" (Hirabayashi, 2006, p.58). We may notice such common philosophical thought of humanism in at least Asian countries that school mathematics, especially for the personality and humanity education for all, should have an educational value and should be a material to cultivate the intelligent part of children's personality as a necessary ingredient of humanity.

ISSUES OF SCHOOL MATHEMATICS EDUCATION

We as mathematics educators/teachers really want to realise mathematics education that helps children develop their personality through learning mathematics in school. We never want to make children dislike and hate mathematics. Unfortunately in reality, there is no doubt that there are serious issues surrounding school mathematics education such as children's bullying, violence in school, a long absence from school, enervation of learning and so on. These are not attributed to only school mathematics but we cannot ignore them even in the teaching and learning of mathematics in school. The results of the analysis of international PISA and TIMSS-R studies of mathematics, revealed as a common issue of mathematics education in some East Asian countries, that students' high achievement in mathematics is not positively correlated with a positive attitude towards mathematics (Leung, 2006, p.24).

Most teachers in Japan, as Hirabayashi (2006) recognizes, especially in primary schools, believe that they teach mathematics not merely for the entrance examination. Many teachers seem to teach mathematics in order to give all children not only fundamental knowledge and skills but also habits and attitudes, which are expected as essential to develop children's sound intelligence to think reasonably in their daily work and treat their personal problems logically (p.54). At least in the intended curriculum for school mathematics, two aims of the 'substantial' and the 'formal' are so much blended. For example, in Japan, the overall objectives of the 1998 Course of Study for lower secondary school mathematics is described as follows:

> To help students deepen their understanding of the basic concepts, principles and rules concerning numbers, quantities and figures, and acquire the way of mathematically representing and dealing with them, and to help students to improve their abilities to think and deal with various phenomena mathematically, as well as to help them enjoy mathematical activities and appreciate the mathematical ways of viewing and thinking, and thereby to foster their attitudes of willingly making use of the above mentioned qualities and abilities (Japan Society of Mathematical Education, 2000).

Why do many children have a negative attitude towards mathematics in spite of the above mentioned? As a possible explanation, Hirabayashi (2006) points to a certain inconsistency between the each-grade objectives and the overall objectives of the intended curriculum for school mathematics. He sees in this inclination a clear sign of the decline of GEI-esprit in Japan and demands the reform of mathematics learning as follows:

> If mathematics learning is reformed in a way comparable to GEI-training, the effect of learning would be visible in the learners' way of thinking or activity in many domains of their future lives, and because of this effect mathematics would be able to occupy its paramount place among school subjects for all pupils (p.63).

NEED FOR HUMANISING MATHEMATICS EDUCATION

Now we may recall that more than thirty years ago Wheeler (1975) gave a lecture under the title of Humanising Mathematical Education at an ATM conference in England. He offered the following three ways of humanising mathematics education in order to eliminate children's fear and anxiety generated by mathematics teaching (pp.5-6).

> (a) The substitution of the goal of passing on mathematical knowledge by the goal of facilitating children's mathematical activity is a substantial step in demonstrating that mathematics is a human activity.
>
> (b) The body of mathematics as we know it is an accumulation from the work of many people. It would humanise mathematical education if we could present this accretion of results as it is, warts and all, so that learners might gain a sense of a human activity with all its admirable and foolish qualities.
>
> (c) The third way of humanising mathematical education, and the one which seems to require some special attention, is to utilise lessons on awareness in order to educate children's awareness through the medium of mathematics. Awareness is a characteristically human quality, and awareness of one's awareness is possibly the most human state of all. The principle that "only awareness is educable" was first enunciated by Caleb Gattegno.

Wheeler (1975) suggests that we should pay attention to the third way of humanising mathematics education. Awareness is the act of attention that preserves the significant parts of experience so that they are available for future use. In that conception, for him, the education of awareness is indeed the only answer that is capable of handling the complex challenge of providing an education, and yet respects everyone's right to be educated independently of theories, ideologies, fashions and so on (p.9).

Is humanising mathematics education a peculiar need to that period in one country? No, it is still needed to humanise mathematics education in our present society of high technology, lifelong learning and globalisation. The need for humanising mathematics education is the immutability rather than the fashion. We hold the conception in common that mathematics is a human activity, a creative activity of human mind and that the teaching mathematics is to help children do mathematics as mathematical activities in order to develop their personality and humanity in the process of learning mathematics in school. Even if only awareness is educable, we cannot educate children's awareness without their understanding mathematical knowledge and acquiring mathematical skills. For humanising mathematics education, it is not a wise choice to reduce and lower the mathematical contents to be learned in school. It is rather essential and important for us to help children challenge collaboratively to the difficulties encountered in their process of learning mathematics in a classroom. To do so, we have to realise the dialectic process of children's individual and social construction through discussion among them with their teacher in the classroom. It is an ideal picture of humanised mathematics education that a mathematics teacher enjoys his/her teaching activities in a classroom where children are collaboratively enjoying activities such as mathematising, utilising, communicating mathematically, explaining logically and realising usefulness of mathematics.

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RESEARCH FORUMS

RF01: LEARNING THROUGH TEACHING: DEVELOPMENT OF TEACHERS' KNOWLEDGE IN PRACTICE

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This Research Forum is aimed at explaining how and when teachers' knowledge develops through teaching. We ask: what kinds of knowledge are developed as a result of teaching activities, what are the sources and the pitfalls for this development? Although our primary focus is on teachers' mathematical knowledge, our secondary focus is on the interactions between the development of teachers' mathematical, pedagogical and curricular knowledge in the process of teachers' learning though teaching.

A VIEW ON THE TEACHERS' OPPORTUNITIES TO LEARN MATHEMATICS THROUGH TEACHING

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THEORETICAL BACKGROUND

Mathematics teachers and researchers agree that teachers learn through their teaching experiences (e.g., Cobb and McClain, 2001; Kennedy, 2002; Lampert & Ball, 1999; Lesh & Kelly, 1994; Mason, 1998; Ma, 1999; Shulman, 1986; Wilson, Shulman, & Richert, 1987). Teachers' expertise is usually considered a function of their experience (e.g., Wilson, Shulman, and Richert, 1987; Berliner, 1987; Leinhardt, 1993).

The main source of teachers' learning through teaching (LTT) is their interactions with students and learning materials (Leikin 2005, 2006). This clearly follows from cyclic models of teaching (e.g., Artzt & Armour-Thomas, 2002; Steinbring, 1998; Simon, 1997) which include expectation of development in teacher knowledge from this interactive process (see Figure 1).

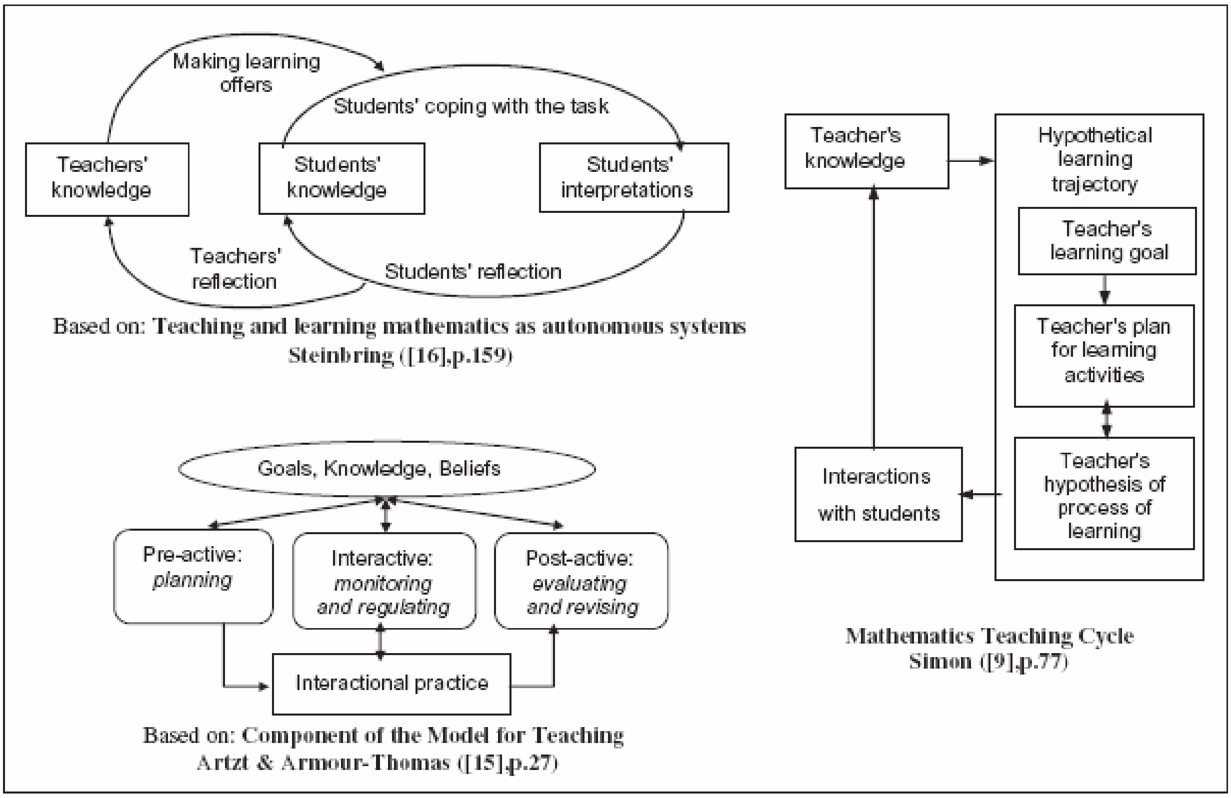


Figure 1: Cyclic models of teaching (from Leikin, 2005a)

Epistemological analysis of teachers' knowledge reveals significant complexities in its structure (e.g., Scheffler, 1965; Shulman, 1986; Wilson, Shulman, & Richert, 1987). Addressing these complexities and combining different approaches to the classification of knowledge, Leikin (2006) identified three dimensions of teachers' knowledge as follows:

Dimension 1 – KINDS OF TEACHERS' KNOWLEDGE – is based on Shulman's (1986) classification: *Teachers' subject-matter knowledge* comprising their own knowledge of mathematics; *Teachers' pedagogical content knowledge* including knowledge of how students approach mathematical tasks, as well as knowledge of learning setting; *Teachers' curricular content knowledge* including knowledge of different types of curricula and understanding different approaches to teaching mathematics.

Dimension 2 – SOURCES OF TEACHERS' KNOWLEDGE – is based on Kennedy's (2002) classification. Teachers' *systematic knowledge* is acquired mainly through studies of mathematics and pedagogy in colleges and universities, *craft knowledge* is largely developed through classroom experiences, whereas *teachers' prescriptive knowledge* is acquired through institutional policies. In the discussion of teachers' learning through teaching craft knowledge is of the main interest.

Dimension 3 – FORMS OF KNOWLEDGE – refers to differentiation between teachers' *intuitive knowledge* as determining teachers' actions that cannot be premeditated and their *formal knowledge,* which is mostly connected to teachers' planned actions (Atkinson & Claxton, 2000, Fischbein, 1984). Additionally this includes distinction between *knowing and believing* (Scheffler, 1965). Knowing has "propositional and procedural nature" whereas believing is "construable as solely propositional" (p. 15, ibid.). In the framework of learning through teaching transformations of intuitive knowledge into formal knowledge is a foci study point.

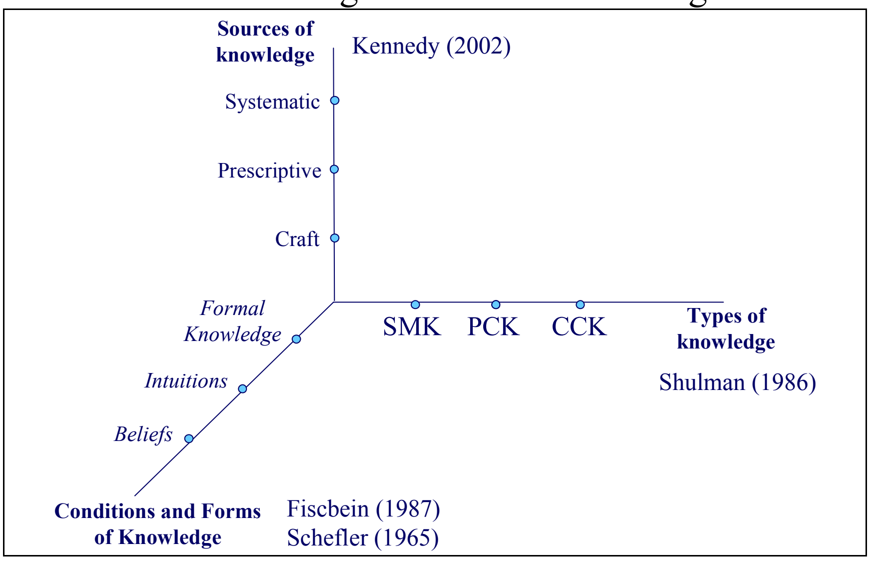


Figure 2: Three dimensions of teachers' knowledge (from Leikin, 2006)

While it is evident that people learn from their practice in general and, in particular, teachers learn from their teaching, what exactly is being learned is often not evident. Leikin (2006, 2005a, 2005b) explored what changes in teachers' knowledge occur through teaching, how development of teachers' knowledge of mathematics and that of their knowledge of pedagogy in the field of mathematics relate to each other, and what mechanisms support those changes.

MECHANISMS OF LTT

Qualities of instructional interactions determine potential of the lesson for students' and teachers' learning (Leikin, 2005). In this context *initiation* of interaction by the teacher or by the students, as well as *motives* for interacting, determine learning processes in the classroom. The motives may be external if they are prescribed by the given educational system, or internal, being mostly psychological, including cognitive conflict, uncertainty, disagreement, or curiosity. Piagetian disequilibration, is the main driving force in intellectual growth or learning. For teachers, unexpected, unforeseen or unplanned situations are the cause of disequilibration and the sources for learning. These sources surface via interaction with students and via reflection on this interaction.

Development of teachers' mathematical knowledge depends on their flexibility (Leikin & Dinur, 2003). By opening opportunities for students to initiate interactions and by managing a lesson according to students' ideas teachers open opportunities for their own learning.

Teachers' noticing and attention (Mason, 1998, 2001) are also of great importance. Of particular interest here is attending to students' responses, both correct and incorrect. When observing students' mistakes during interactions, the teachers search for new explanations or clarifications in order to correct student' understanding, so in the course of the lesson they may construct new mathematical connections (example 1 below). The other interesting source for teacher's LTT is learning from students' unexpected correct ideas (example 2) or from students' surprising questions (example 3) (for elaboration of this phenomenon see Leikin & Dinur, 2003; Leikin & Levav-Waynberg, in press, Leikin, 2005b).

WHAT CHANGES IN TEACHERS' KNOWLEDGE OCCURS THROUGH TEACHING?

Within the space constrains of this paper we mainly focus on teachers learning of mathematics.

From intuitions to formal knowledge and beliefs: Teachers learn mainly in unpredicted (surprising) situations. As Atkinson and Claxton (2000) show in "intuitive practitioner", many of the teachers' actions when teaching are intuitive and unplanned. Teachers' craft knowledge develops as the transformation of their intuitive reactions into formal knowledge or into beliefs. In terms of the relation between knowledge, intuitions, and beliefs suggested in the 3D model of teachers' knowledge (figure 2), the research mainly outlined the transformation of mathematical intuitions into formal mathematical knowledge whereas pedagogical intuitions were transformed into beliefs (Leikin, 2006).

Development of new mathematical knowledge takes place at all the stages of teachers' work: planning, performing and analyzing a lesson. When planning the lesson teachers clearly express their "need to know the material well enough" and their "need to predict students' possible difficulties, answers, and questions". At the planning stage the teachers are involved in designing activities that allow them to reach new insights. Hence *new pieces of information* are sometimes collected and some *familiar ideas are refined* (Leikin, 2006, Leikin, 2005a). The need to "know better than the students" stimulate teachers' thinking about possible students' difficulties. When predicting them teachers reflect on their own uncertainties, thus solve their own questions when planning the lesson. Through interaction with students teachers become aware of new (for them) solutions to known problems, new properties (theorems) of the mathematical objects, new questions that may be asked about mathematical objects and in this way they develop new mathematical connections. In what follows we exemplify this newly acquired awareness.

Example 1: Learning from a student's mistake

Lora, an instructor in a course for pre-service elementary school teachers, taught a lesson on elementary number theory. The following interaction occurred:

Teacher: Is number 7 a divisor of K, where K= $3^4 \times 5^6$?

Student: It will be, once you divide by it

Teacher: What do you mean, once you divide? Do you have to divide?

Student: When you go this [points to K] divided by 7 you have 7 as a divisor, this one the dividend, and what you get also has a name, like a product but not a product…

Lora's intention in choosing this example was to alert students to the unique factorisation of a composite number to its prime factors, as promised by the Fundamental Theorem of Arithmetic, and the resulting fact, that no calculation is needed to determine the answer to her question. This later intention is evident in her probing question.

What Lora learned from the above interaction?

She learned that the term "divisor" is ambiguous and a distinction is essential between *divisor of* a number, as a relationship in a number-theoretic sense and *divisor in* a number sentence, as a role played in a division situation. She learned that the student assigned the meaning based on his prior schooling and not on his recent classroom experience in which the definition for a divisor was given and usage illustrated. Before this teaching incident Lora used the term properly in either case, but was not alert to a possible misinterpretation by learners. The student's confusion helped her make the distinction, increased her awareness of the polysemy (i.e. different but related meanings) of the term *divisor* and definitions that can be conflicting. This resulted in developing a set of instructional activities in which the terminology is practiced (Zazkis, 1998).

Example 2: Learning from a student's solution

Shelly, a teacher with 20 years of experience in secondary school, solved with her Grade 12 students the following a problem:

$$\textit{Prove that: } 1+2t+3t^2+4t^3+...+nt^{n-1}=\frac{1-t^n}{(1-t)^2}-\frac{nt^n}{1-t}$$

She expected her students to prove this equality using mathematical induction but unexpectedly one of the students (Tom) suggested the following solution:

$$S(t)=1+2t+3t^2+4t^3+...+nt^{n-1}=F'(t), \quad \textit{when}$$

$$F(t)=t+t^2+t^3+...+t^{n-1}+t^n=\frac{t^{n+1}-t}{t-1}, \quad \textit{thus}$$

$$S(t)=F'(t)=...=\frac{1-t^n}{(1-t)^2}-\frac{nt^n}{1-t}$$

Shelly's reflective reaction was: "How could I miss this? Oh well, the problem is from the mathematical induction topic and I did not think about derivative at all. The solution is clear, but I did not think about it".

What Shelly learned in this episode?

A connection between the fields of induction and calculus was new for her. She knew about use of mathematical induction in geometry, for example, in proving a theorem about the sum of interior angles in a polygon. Mathematical induction, for her, was also was connected to divisibility principles, since many divisibility rules may be proved using induction. As such, it was naturally connected to the topic of sequences and series, because of the multiple proofs using induction in these topics. She was also aware that many problems in mathematical induction could be solved using different methods.

Shelly: Even in the matriculation exams they say 'prove using induction or in a different way' like 3 is a divisor of n^3-n because n^3-n=(n-1)·n·(n+1)

However, when preparing this lesson Shelly did not think about this solution. Moreover during more than 29-years of experience she never connected induction with calculus.

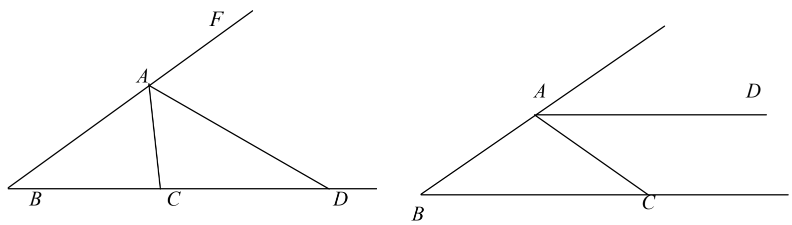
Tom's solution added a new mathematical connection to her subject matter knowledge and this, in turn, became part of her repertoire of problems with multiple solutions drawn from different areas of mathematics.

Example 3: Learning from a student's question

During a geometry lesson Eva, a teacher with 15 years of experience in secondary school, proved with her Grade 10 students the following theorem:

If AD is a hypotenuse of an external angle CAF in a triangle ABC then $\frac{AB}{AC} = \frac{BD}{CD}$ (Figure 3a).

After the theorem was proved one of the students asked: "What happens if AD is parallel to BC (Figure 3b)?" This question led to the classroom discussion in which students drew a conclusion that the theorem was correct for *non-isosceles* triangle.



a *b*

Figure 3

Eva, in her reflective analysis of the situation, reported that she "never had thought about correctness of the theorem for isosceles triangle". Furthermore, when analyzing this situation with the researcher she unexpectedly connected this geometry topic with the topic of limits:

"When AD is parallel to BC $\lim\limits_{AC \to AB} \frac{BD}{CD} = 1$.

Since BD=BC+CD, this situation can demonstrate the rule: $\lim\limits_{x \to \infty} \frac{x + c}{x} = 1$".

What Eva learned through this lesson?

The connection appeared to be surprising both to Eva and our research team. First of all this lesson led her to develop a "novel formulation of a theorem." Eva commented that "this theorem was never mentioned in any familiar textbook or mathematics course". She noted that next time, if students will not consider an isosceles triangle when proving the theorem, she will lead student towards consideration of this special case

As mentioned earlier, a student's question served as a trigger, but it is the teacher's curiosity and deep mathematical knowledge that led her to develop new connections.

IS THIS KNOWLEDGE NEW? IS THIS MATHEMATICS OR PEDAGOGY?

Teachers learn both mathematics and pedagogy when teaching. In many situations teachers' pedagogical content knowledge developed when they become aware of students' unpredicted difficulties. Further, through analysing sources of the difficulties and misconceptions teachers appreciate better the structure of mathematical thought. Example 1 is a case of developing such awareness: in order to help students adopt the meaning of the term implied in a given situation the teacher had to first clarify the disparity in different uses of 'divisor' *for herself.*

In other (less often) situations teachers clearly learn new mathematics. Then this mathematics serves them in the consequent lessons in their pedagogy. In many situations differentiation between mathematical and pedagogical learning is problematic; since teachers' knowledge is situated to the great extent in their teaching practice the distinction is blurred (Leikin & Levav-Waynberg, 2006). Development of teachers' craft knowledge depends strongly on their systematic knowledge. Teachers' mathematical understanding allow them to develop further students' ideas (like in example 3). Teachers' openness and pedagogical knowledge and skills as related to their awareness of the importance of students' autonomy in classroom mathematical discourse allow them to be more open and attentive to students. Finally we found that teacher with more profound mathematical understanding (in terms of Ma, 1999) feel 'safer' and more open to allowing students to present their mathematical ideas and ask questions.

Finally, we note that teachers are not always aware that they learned through their teaching and sometimes they are hesitant to admit their learning. Moreover, when they are aware of learning they are not convinced that they learned mathematics. Very often they report "I knew this but never thought about it". However, we consider this "thinking about it" as an indication of learning when an instructional situation presents such opportunity. In this case LTT occurs not only in acquiring new knowledge but also in transferring existing knowledge from teachers' passive repertoire to an active one. However, clear criteria that indicate teachers' learning of mathematics in LTT need further development and refinement.

INTEGRATING VIRTUAL AND FACE-TO-FACE PRACTICE: A MODEL FOR CONTINUING TEACHER EDUCATION[1]

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A review of the last five PME proceedings reveals that online distance education is a theme that has received little attention; with the exception of a few papers to which I contributed, there are very few presentations addressing the issue (e.g. Brown & Koc, 2003; Nolan, 2006; Rey-Más & Penalva-Martínez, 2006). On the other hand, there is a growing interest in online courses, as a means of creating a bridge between universities and schools. In this paper I will present a report that blurs the categories of knowledge presented by Kennedy (2002) – craft and systematic – as it becomes difficult to determine whether knowledge regarding software use in the classroom

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originates from practice or from courses that teachers have taken. A structure of an online course offered for mathematics teachers will be presented. I will also present excerpts of an assessment made by the teachers enrolled in the courses and raise conjectures regarding why the course was assessed in a surprisingly positive way. Research in the area has pointed out that courses do not necessarily influence practice (Oliveira, 2003)

GPIMEM'S MODEL OF COURSE

In our research group, GPIMEM, we have been researching the possibilities of using the Internet for education since 1997. Despite having no experience with online education at that time, we became convinced that a "download model" was not desirable, as courses based on such a model fail to take advantage of new possibilities offered by the Internet, merely reproducing models of distance education based solely on printed material and the postal service. Downloading functioned as a speedier mail service. Models such as these predominate in courses offered in mathematics education, according to a recent paper on online courses (Engelbrecht & Harding, 2005).

We offered our first online teacher education course in the year 2000. The model we have been building for nearly a decade uses synchronous interaction via interfaces such as "chat rooms" or videoconferences and asynchronous interfaces such as forum and portfolio. For the past two years, we have conducted videoconferences consisting of online interaction using voice and image. This environment includes a feature that enables one participant to begin drawing a geometrical figure which other online participants can then add to or complete. A common screen is used throughout the process (see Borba & Zulatto, 2006, for more details).

We have offered five courses to approximately 25 teachers each time, each consisting of eight or nine synchronous sessions of two hours. Three have focused on geometry and two on functions. The mathematics teachers (fifth grade through high school) who participate in the courses all work in a network of private schools that focuses on improving education in socially-deprived areas. The forty schools - which are located in low-income neighborhoods in big cities, in rural areas, and in the Amazon forest - have an above-average infrastructure and teachers' salaries are above average.

The school foundation approached GPIMEM in 2002 for assistance as they wanted to implement the use of software in their classes. All schools have two labs with 25 computers each which administrators felt were not being used to their full potential. In response to that initial contact, we advised them regarding the purchase of software and taught one face-to-face course for leading mathematics teachers from some of these schools. Although they gave the course a positive evaluation, they also realized that it was only a four hour course, and that more would be needed. Since they had already purchased a platform for video-conferences and knew that we had experience in online courses, they invited us to teach a course about using a geometry software purchased by the school: Geometricks (Sadolin, 2000).

Some findings from research conducted in the process of teaching the courses have been reported previously, including findings related to the type of mathematical learning that takes place in such environments (Zulatto, in progress) and the nature of the collaboration that takes place among researchers and teachers (Borba & Zullato, 2006), as well as how mathematics is transformed by different computer interfaces (Borba, 2005).

However another theme for research that has emerged from the courses is their apparent success. I say "apparent" because more research is needed to verify this, and judge "success" based on the following: the teachers' use of software in their regular face-to-face classes has exceeded our expectations; the school foundation continues to be interested in hiring us for new courses.

Another indication of success, in our opinion, are comments such as the following made by course participants:

> *Teacher 1*: Professor, the activity proposed for today is interesting because it reviews the activity developed during the week. To respond to the activities we researched, we went back, exchanged ideas . . . we went deeper into those things we already knew but are going back to. We would like it, in upcoming classes, if activities were used like this to challenge us, to stimulate our participation more. Thank you.
>
> *Teacher 2*: We have a positive evaluation, as well. Apparently, this thing of passing the pen to colleagues gives the impression of being slow, but it ends up being very interesting; we notice the difficulties each of us has. It is extremely positive.

I do not intend to claim that such excerpts are strong evidence of the positive effect of the course, but rather present them as illustrations of the type of comments that led us to examine the reasons for the apparent success of the courses more closely. Moreover, the results are associated with online courses, which many educators view with ambiguity.

In this research forum, I would like to raise conjectures regarding the reasons for this possible success of the course. Resulting insights can help the GPIMEM team and the mathematics education community design new research to examine whether this should in fact be considered a success, and if so, what are the reasons for it.

The conjectures of our research group include:

a) One possible factor could be that the school was paying attention to the effect of the course. Each semester the school decided whether the course would continue to be offered or not. They needed positive evaluations from the teachers and evidence of increased use of the computer laboratories to convince decision-makers at various levels in the school network that the costs involved (teachers, platform, technicians, etc.) were worth it. This social pressure by the school administrators could be one factor contributing to teachers' use of computers in the classroom;

b) There is evidence in the literature (Nacarato, 2005; Hargreaves, 1998, 2001; Ferreira & Miorin, 2003) that obligating teachers to adopt a given approach is unlikely to be effective. Online systems may engage teachers more, as they facilitate individual and collective work among teachers. Teachers could discuss a topic in the distance course on Saturday and apply it in class during the week, requesting on-line support if necessary. There was more integration between taking a course and acting as a teacher in the classroom;

c) During the course, there was constant interaction among the participants, including sessions during which a given construction in geometry, or a given problem resolution in function, would be developed collectively, using e-mail, which is part of the argument in item b. Throughout the interactions, the teacher-participants were invited to propose activities. One of the classes was dedicated to analysis and discussion of classroom activities developed by the teachers. Following the first course, our team tried to incorporate their activities into subsequent courses. Could the empowerment of teachers as authors be a major factor in itself for such a result?

d) Underlying this course was the view that knowledge is constructed by collectives of humans-with-media (Borba & Villarreal, 2005). In this perspective, non-human actors play an active role in the way knowledge is constructed. We have paid close attention to the way the distance education platform could interact with geometry and function software in order to result in the interactions described in the above paragraph. Developing specific and theoretical pedagogical approaches that include the active role of technology is one of the challenges posed to our community.

As mentioned before, the goal of this paper is to stimulate discussion about issues related to online continuing education courses for teachers. The original goal of our research was not to investigate what makes an online course a good course. This question evolved, in a parallel way, along with other research questions, and is presented to this forum for discussion. The four conjectures related to the social structure of schools, specificities of the online environment, the active model of the course, and a view of technology that supports teacher participation may form a starting point for a deeper understanding of an activity that is gaining importance in our community: online distance education and its integration with face-to-face teaching.

TEACHERS' LEARNING REIFIED: THE PROFESSIONAL GROWTH OF INSERVICE TEACHERS THROUGH NUMERACY TASK DESIGN

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Anyone who knew John when he first became a mathematics teacher eight years ago would say that he has come a long way. He is no longer the idealistic, naïve, teacher he was at the beginning of his career. John is now savvier in the ways of teaching. Although not a specialist in mathematics, over the last eight years John has managed to acquire a large repertoire of ideas and practices that he relies heavily on in his daily teaching of mathematics. These ideas and practices have been gleaned from textbooks, teachers' guides, colleagues, workshops, but mostly from his experiences in the classroom. But there is an incongruity within John. John identifies himself as being a bit of a traditionalist when it comes to teaching mathematics – he espouses the virtues of drills, skills-based assessment, believes in the transmission model of teaching and learning, and bemoans the problems of the 'new new math' movement. Despite these dispositions, however, many of John's favourite lessons and instructional strategies can best be described as being steeped in the traditions of the reform movement.

How is it possible that such an experienced teacher can embody such contradictions between his *knowledge* and his *practice*? What is the *condition of John's knowledge* that allows for such contradictions? How has this contradiction developed, or perhaps more relevant, how has an agreement between *knowledge* and *practice* failed to develop? In this paper I look more deeply at John, and other teachers not too dissimilar from him, whose teacher knowledge has not fully developed through their experience as teachers, but whose knowledge does develop more fully when put in a situation wherein they were required to reify[2] their knowledge, and then act (and enact) that knowledge. My thesis is that through this process teachers' knowledge and practice can develop.

THEORETICAL FRAMEWORK

Leikin (2006) identifies three dimensions of teachers' knowledge: kinds of teachers' knowledge, sources of teachers' knowledge, and conditions of teachers' knowledge. A brief summary of these three dimensions can be found in the introduction of this research forum. Although it is possible to situate the thesis of this paper in each of these three dimensions, for the purposes of brevity I will focus this work on the third

[2] In this paper *reify* and *reification* is used in the tradition of Wenger (1998) rather than in the tradition of Sfard (1994). McClain & Cobb (2004) make a similar notification of distinction.

dimension – the conditions of teachers' knowledge in general, and the transformation in the conditions of teachers' knowledge in particular.

According to Leikin (2006) teachers' knowledge can be situated within their formal knowledge, their intuitive knowledge, and their beliefs. Simply defined, formal knowledge is knowledge that consciously guides practice, intuitive knowledge is knowledge that subconsciously guides practice, and beliefs is *subjective* knowledge that consciously and/or subconsciously guides practice. This simultaneous partitioning of the conditions of knowledge across the knowledge/beliefs divide and the conscious/subconscious divide is somewhat problematic, however.

To begin with, at the level of teachers' action the distinction between knowledge and beliefs is not so clear. In general, knowledge is seen as an "essentially a social construct" (Op 'T Eynde, De Corte, & Verschaffel, 2002). That is, the division between knowledge and belief is the evaluations of these notions against some socially shared criteria. If the truth criterion is satisfied then the conception is deemed to be knowledge. But when teachers operate on their knowledge the distinction between what is true and what they believe to be true is not made. Leatham (2006) articulates this argument nicely:

> Of all the things we believe, there are some things that we "just believe" and other things we "more than believe – we know." Those things we "more than believe" we refer to as knowledge and those things we "just believe" we refer to as beliefs. Thus beliefs and knowledge can profitably be viewed as complementary subsets of the things we believe. (p .92)

Thus, for the purposes of examining teachers' practice I do not make the distinction between beliefs and knowledge. I do, however, make the distinction between what teachers know/believe at the *conscious* level, and what they know/believe at the *subconscious* level. In part, this difference can be summarized by Green's (1971) distinction between evidential and non-evidential beliefs. Evidential beliefs are formed, and held, either on the basis of evidence or logic. Non-evidential beliefs are grounded neither in evidence nor logic but reside at a deeper, tacit level. So, I reinterpret Leikin's (2006) description of the *conditions of teachers' knowledge* as being comprised of the *conscious* knowledge/beliefs that guide their practice, and the *subconscious* knowledge/beliefs that guides their practice. This reinterpretation can be used to describe the discordance between John's practice and his espoused stance on teaching (presented in the introduction). What John espouses is informed by his *conscious* knowledge/beliefs whereas what he does is informed by his *subconscious* knowledge/beliefs. In John's case, what is needed is better articulation between the *subconscious* and the *conscious*.

Wenger (1998) provides us with the language of reification to articulate the movement of knowledge/beliefs from the *subconscious* to the *conscious*. For Wenger (1998) reification is "the process of giving form to our experiences by producing objects that congeal this experience into *thingness*" (p. 58). This congealing of experience can be the movement from tacit knowledge/beliefs to explicit (conscious) knowledge/beliefs, the transformation of abstract thoughts into concrete ideas, or the articulation of fleeting notions into tangible statements.

METHODOLOGY

Participants for the portion of the study presented here are 18 inservice teachers working in two teams (Team A – 10 teachers, Team B – 8 teachers) in two different school districts. Although working in different districts, both teams were formed for the same purpose – to collaboratively design numeracy tasks to be used district wide to assess the level of numeracy of grade 8's. The teachers involved in this project range in age from 25 to 63 (average is 36.4 years), and range in teaching experience from 1 year to 36 years (average is 8.2 years). What does not range, however, is their expertise in mathematics. Like John, none of the 18 teachers in this project has a specialization in mathematics – they are all generalist teachers. And like John, there exists some incongruity within each teacher between their espoused views of teaching and learning and their practice. In fact, John is an amalgamation of these 18 teachers, constructed to exemplify the participants as a whole (Leron & Hazzan, 1997).

The research was conducted over the course of the four, four hour long, planning and implementation meetings allocated for the task design project. The nature of the meetings is summarized below[3]:

Meeting 1 – co-construct a shared understanding of numeracy and begin to design tasks for immediate pilot testing.

Meeting 2 – debrief the pilot testing of tasks, refine tasks, begin to discuss the logistics of scripting/administrating[4] tasks, and prepare to re-pilot test the tasks.

Meeting 3 – refine tasks, finalize scripting/administration of tasks, and prepare to administer the task as an assessment.

Meeting 4 – mark the tasks and do a post-mortem on the process.

Each of these meetings was facilitated by the author. The data for this paper come from the first of theses meetings.

RESULTS AND DISCUSSION

The first meeting with each team began with a prompt for them to think about what numeracy is. In both cases the teams responded with the ideas that numeracy must include the rapid recall of arithmetic facts, fluency with arithmetic algorithms, and

[3] This is really just a summary. There are contextual details about this process, and subtle differences between the two groups that, although relevant, space constraints do not allow for.

[4] This included discussions around time allocation, the role of group work, the provision of graphic organizers, and the use of writing prompts.

good number sense. Although not explicitly stated, there was a sense that this list of characteristics of numeracy was not only *necessary*, but also *sufficient*.

When prompted to think about the qualities of a successful (numerate) student[5], however, the nature of the responses changed. Now, the list of characteristics included notions such as: perseverance of effort, a willingness to engage, an ability to transfer knowledge to new contexts and novel problem solving situations, a broader awareness of mathematics around us, creativity and flexibility in thinking, and an ability to explain their thinking. When asked to synthesize these ideas into a definition the two groups produced the following:

> Numeracy is the willingness and ability to apply and communicate mathematical knowledge and procedures in novel and meaningful problem solving situations.
>
> Numeracy is not only an awareness that mathematical knowledge and understandings can be used to interpret, communicate, analyze, and solve a variety of novel problem solving situations, but also a willingness and ability to do so.

These definitions are a long way from their initial musings about what numeracy is. Initially dominant in the conversations were their *conscious* (traditional) ideas about what it means to 'know' mathematics. As the conversation progressed the nature of the ideas offered changed dramatically. Replacing the traditional views about 'knowing' mathematics were more progressive ideas about the processes of 'doing' mathematics. I argue that the emergent ideas were not new knowledge/beliefs, but rather the explications of previously tacit knowledge/beliefs that had built up from their experiences with teaching. The act of first verbalizing and then synthesizing these tacit ideas reified their good experiences with teaching and with students and moved them into their consciousness. In so doing, "they first projected their knowledge/beliefs into the world [in terms of their definitions] and then began to treat them as having a reality of their own" (paraphrased from Wenger, 1998, p. 58).

This treatment of the definitions as having a reality of their own can be seen in how the teams then acted on them. During the task design phase of the project conversations and questions repeatedly returned to the definitions. In particular, comments regarding the need for creating novel (or non-traditional) tasks that did not rely on recall of arithmetic facts or the regurgitation of arithmetic algorithms dominated the early conversations. Later, questions around the need to avoid or include ambiguity in the tasks arose, as did questions regarding the need for group discussion and written output. Of particular interest were the lengthy conversations around the issue of creating tasks that they (now) felt were vitality central to numeracy (and the teaching and learning of mathematics). Again, I argue that the emergent knowledge/beliefs is the result of the explication of tacit notions and desires about teaching and learning mathematics that have accumulated from their experiences (both positive and negative) in teaching mathematics.

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[5] This prompt is considered a *critical question*. Research into critical questions is part of the larger project within which this paper is situated.

GIVING OUT BONUSES

You are the manager of the Teen-Talk Cell Phone company that employs a number of independent sales people to sell their phones **seven days a week**. These sales people work as much or as little as they want. As a sales manager you don't care how much they work, but you do care how much they sell. So, to motivate them to sell more you give out bonuses based on how productive they have been. There are two bonus plans:

- the top producing individual receives $500.
- the top producing team **shares $500 in a fair manner**.

However, there are also two problems:

- different people have different ways of reporting their productivity.
- the individual sales teams don't have the same number of people on them.

Based on the information provided in the table below, **who should get the bonuses this month, and how much do you think they should get? Justify your answers in writing.**

Sales Person	**Team**	**Sales Reported for the Month of April (30 days)**
John	A	300 cell phones sold this month
Peter	B	An average of 56 cell phones sold every 5 days
Lewis	A	An average of 10 1/3 cell phones sold each day
Amy	A	598 cell phones sold in the last 60 days
La Toya	A	An average of 98¾ cell phones sold every 10 days
Jennifer	B	An average of 11 4/15 cell phones sold each day
Steven	B	An average of 55 cell phones each week
Fantasia	C	4113 cell phone sold in the last year
Diana	C	An average of 10.05 cell phones each day
Matthew	D	An average of 10.87 cell phones each day
Camille	D	An average of 9 1/6 cell phones each day
Jasmine	C	267 cell phones this month

***Figure 1**: The Initial Numeracy Tasks Created by Team B*

In the end, each team created a task that they wished to pilot test (see ***Table 1*** for an example). These tasks, and their implementation, embody the newly acquired conscious knowledge/beliefs of the participants. The tasks are novel, require the combination of mathematical knowledge, and can be attacked with a number of possible different solution strategies. The implementation of the tasks requires the use of group work, an expectation of written output, and does not follow on the heels of lessons designed to 'teach to the task'

CONCLUSIONS

Teachers learn from teaching. What they learn, however, is not always made explicit to them. Teaching experiences may accumulate in disjunctive ways at a very tacit level. The catalyst for unifying and explicating these disjoint and implicit experiences is not always clear. Certainly the culture existing within the school setting has been shown to galvanize some of these experiences into concrete notions (Goos, 2006; Karaagac & Threlfall, 2004). In this paper I offer a different mechanism for galvanizing teachers' experiences. First, teachers' knowledge/beliefs are moved from the subconscious to the conscious through the act of reification (Wenger, 1998). But reification is more than just an explication of tacit knowledge/beliefs. It is also putting that knowledge/beliefs out into the world as if they have a reality of their own. Acting, and then enacting, this knowledge/beliefs through task design and then delivery of these tasks serves to further galvanizes this knowledge/beliefs. Although not comprehensive, and far from conclusive, I have introduced the idea of reification and enactment into the discourse of *Learning through Teaching* in general, and into the area of *conditions of teachers' knowledge* in particular.

CONSTRAINTS ON WHAT TEACHERS CAN LEARN FROM THEIR PRACTICE: TEACHERS' ASSIMILATORY SCHEMES

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INTRODUCTION

Every teacher's greatest opportunity for further learning in mathematics education is her classroom teaching. The number of hours spent, the diversity of situations, and the continual feedback available from students make teaching an opportunity for teacher learning that has no equal. So, of course *all experienced teachers are highly knowledgeable and competent*? Of course, we know that to be untrue. The contrast between the opportunity for learning inherent in teaching and the often-limited knowledge gleaned by teachers suggests a subject of inquiry and discussion. What is it that limits what teachers can learn from their teaching?

An important answer to this question is *their current understandings* (and their current goals, which are based on their current understandings). Just as students' learning of mathematics is afforded and constrained by their extant knowledge (assimilatory schemes), teachers' mathematical and pedagogical learning is similarly afforded and constrained.

In this article, I will focus on two examples of teacher conceptions that we have postulated in the context of recent research projects. These conceptions appear to be widespread and do have significant impact on the perceptions, decisions, and learning of teachers in their classrooms. However, it should be noted that these conceptions are the researchers' descriptions of how teachers' thinking is organized and not necessarily how the teachers would describe their thinking or beliefs (Simon & Tzur, 1999).

PERCEPTION-BASED PERSPECTIVES

The construct *perception-based perspective* developed in the context of a research project in the US in which we worked with and studied a group of practicing and prospective teachers (Simon, et al, 2000). These teachers[6] had come to appreciate the limitations of telling and showing students as their primary teaching medium and were participating in the mathematics education reform that began almost 20 years ago. What is more, these teachers had personally experienced important conceptual learning during recent professional education experiences. As a result, they were committed to providing students with the kinds of conceptual learning experiences that they had had recently, but had not had as primary and secondary school students.

As researchers, we characterized the perspective of these teachers as perception-based[7]. From this perspective, students develop mathematical understanding through personal engagement with particular mathematical tasks and representations that make the concept under study clearly perceivable. The assumptions behind this perspective are that understanding is important, that first-hand experience and active engagement promote understanding, and that particular mathematical tasks and representations give all learners the opportunity to perceive key relationships and gain intended understandings. From this perspective, mathematical relationships exist in an external reality, are perceivable by all learners, and what is perceived is the same for each person.

It is not surprising that teachers develop this perspective. The teachers came to understand particular mathematics while engaged with particular mathematical tasks and representations. There experience was that of coming to *see* the relationships and understand the concepts, because it was so clear in the context of these tasks and representations. Further, people tend to assume that others perceive what they perceive in a given situation. In fact, none of us could communicate, if we did not make that assumption most of the time. Von Glasersfeld (1987) made a similar point in the context of reading:

> When we *understand* what we read, we gain the impression that we have "grasped" the meaning of the printed words, and we believe that this meaning was *in* the words and that we extracted it like kernels out of their shells. . . . This notion . . . is extraordinarily strong and seems so natural that we are reluctant to question it. (p.6)

So when these teachers have powerful learning experiences that they perceive to be a result of active engagement with particular tasks and representations, they assume that such engagement in similar situations will benefit their students in the same way. Thus, they seek a set of mathematical tasks that will allow students to *see* the concepts to be learned.

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6 From this point on, unless otherwise indicated, the word “teachers” refers to both practicing and prospective teachers.

7 This construct is related to earlier work by Cobb, Yackel, and Wood (1992).

As indicated above, teachers, regardless of their perspective, are likely to assume that students perceive what they, the teachers, perceive. However, the difference between a teacher who we would characterize as having a perception-based perspective and a teacher who we would characterize as having a conception-based perspective is that the latter can call that assumption into question when students do not respond in anticipated ways. Further, from a conception-based perspective the teacher focuses on what conceptions the learners bring to the situation and how they are making sense of the situation.

Operating from a perception-based perspective, teachers focus on whether or not the students are perceiving particular relationships and understanding particular concepts. This focus restricts what teachers learn from their practice. Because of this focus, teachers are unlikely to learn about their students' conceptions, obstacles to understanding particular concepts, or the process of making particular conceptual advances. These teachers are not attending to how students are thinking about the situation, but rather whether the students are perceiving what the teachers take to be apparent in the mathematical situation under study. In contrast, working from a conception-based perspective, teachers are likely to learn about students' conceptions and processes by which students can build on those conceptions.

EMPIRICAL LEARNING PROCESSES VERSUS REFLECTIVE ABSTRACTION

I now turn to a second area of teachers' conceptions, one which is related to the one described above. A result of recent efforts to create a more active role for students in mathematics classrooms has been an increase in lessons in which students are asked to look for patterns in outcomes. For example, a teacher asks her students to try some examples and to find out what happens when you add two odd numbers. Students add pairs of numbers, either by hand or with a calculator, and observe that the answers are consistently even. This is an example of an empirical learning process (Simon, 2006a), a process that does not result in conceptual learning. Mathematical concepts are the result of reflective abstraction (Piaget, 2001) not of empirical learning. From the odd-even example, the students learned *that* two odd numbers add to make an even number (a fact), not the logical necessity of that relationship (a concept). In contrast, consider a context in which students thought about a chess club with a particular number of students. In this context, one has an even number when everyone has someone to play with and an odd number is when one person must wait for the next round. One can think about combining two groups with odd numbers as matching the one extra player from each group, thus creating an even number. The abstraction, which can be made from this thought activity, produces an understanding of the logical necessity of two odd numbers adding to make an even.

Teachers who are not aware of the ineffectiveness of empirical learning processes to produce conceptual learning are limited in what they derive from classroom experience. For them, teaching is straightforward. They endeavor to have students generate data and provide them a structure for organizing the data, so the students can see the relationship between the quantities. Such teaching is unlikely to engage them in an inquiry into how a particular concept can be learned (abstracted). In fact, they do not even engage in the often-problematic articulation of the concepts to be learned; they are focused on the perception of relationships and not underlying concepts.

A CLASSROOM EXAMPLE: EXPLORING ONE-HALF

I use the following classroom example to demonstrate the affordances and limitations of the teacher conceptions described above.

> In a fourth-grade class, I asked the students to use a blue rubber band on their geoboards to make a square of a designated size, and then to put a red rubber band around one half of the square. Most of the students divided the square into two congruent rectangles. However, Mary cut the square on the diagonal, making two congruent right triangles. The students were unanimous in asserting that both fit with my request that they show half of the square. Further, they were able to justify that assertion by explaining that each of the parts was 1 of 2 equal parts and that the two parts made up the whole.
>
> I then asked, "Is Joe's (rectangular) half larger; is Mary's half larger, or are they the same size?" Approximately a third of the class chose each option. In the subsequent discussion, students defended their answers. However, few students changed their answers as a result of the arguments presented. (Simon, 2006, p. 361)

When I first encountered this situation, I was quite surprised. It was a situation from which I learned a great deal. Let us examine how the teachers' conceptions described above affect potential learning form this situation.

From a perception-based perspective, teachers are concerned that the students do not see that two halves from identical wholes are the same size. The teachers are unlikely to inquire as to the concept that needs to be developed by these students[8] or to struggle with trying to articulate how the students currently think about one-half. They tend to think about what experience will make the relationship apparent to the students.

On the other hand, the teacher operating from a conception-based perspective learns that students can recognize and define one-half without understanding that one-half is a measure of quantity/amount. She struggles with how to articulate what the students currently understand. My struggle of this type led to the following:

> The students who argued that either the rectangular or the triangular half was larger conceive of halves as an *arrangement* in which a whole is partitioned into two congruent parts. They do not understand that partitioning a whole into two equal parts creates a new unit whose size, relative to the original unit (whole), is determined. That is, they do not understand that "one half" indicates a *quantity* (amount), not just an arrangement.. . .

[8] All claims about the limitations of teachers in response to this situation are based on research data, which is only summarized due to space limitations.

> Educators who understand a fraction as a quantity find it difficult to conceive of this limited understanding of one half (as an arrangement). One can partition a square into two rectangles with any cut parallel to one of the sides. Any such partition will create two parts that can be compared to each other and which sum to the whole. However, in the case where the partition results in equal parts, an important part-whole relationship is determined (from the perspective of those who understand it) – a new, specified unit of quantity is constituted. That is, the whole is twice the size of either of the equal parts. This special relationship between the part and the whole, created by equal partitioning, is neither obvious nor automatic to the young student who is just beginning to explore fractions. (Simon, 2006, p. 361)

Teachers, who do not understand the insufficiency of empirical learning processes, propose to address this issue using an empirical learning process. They propose having students take two identical square pieces of paper, cut one in half horizontally and one in half diagonally, and cut up the diagonal half to see that it can be superimposed on the rectangular half. To reiterate, such activity only demonstrates to the student *that* the two halves are equal in size, not the logical necessity of two halves (or any particular fraction) from identical wholes being the same size. For teachers who understand the learning of mathematical concepts as reflective abstraction, a difficult inquiry ensues as to how to help students understand that equal partitioning produces new units of quantity of a particular size with a particular proportional relationship to the size of the whole. Tzur (1999) undertook this challenge.

CONCLUDING REMARKS

I have tried to demonstrate that teachers' conceptions afford and limit what they can learn form their classroom teaching. Just as students do not see what their mathematics teachers see in a mathematical task or representation, teachers do not necessarily see what researchers and mathematics teacher educators see. However, it is not enough to be aware of this phenomenon. It is incumbent on those concerned with fostering the growth of mathematics teachers to understand teachers' conceptions. This suggests both a program of research and the inquiry of individual teacher educators. Above, I have indicated two of the conceptions derived from research that we have found useful in characterizing teachers thinking.

A final point: Teachers conceptions are in service of the work that they do on a day-to-day basis. As such their conceptions have a certain internal consistency. Teachers construct a network of conceptions that structure how they think about what they do. To understand teachers' thinking, researchers and teacher educators must understand the nature of these networks and the central components around which they are organized. Promoting change in these networks is a complicated process due to the complexity of the networks and the interconnectivity of their components. Significant change (paradigm shift) is unlikely to happen solely as a result of a teacher learning from her own teaching.

WHAT AND HOW MIGHT TEACHERS LEARN VIA TEACHING: CONTRIBUTIONS TO CLOSING AN UNSPOKEN GAP

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INTRODUCTION

Superficially, the four articles of this Research Forum may appear widely divergent. However, in this discussion I articulate deeper undercurrents that can contribute to closing an enduring, disturbing, and mostly unspoken gap—pointed out in Simon's article—between what teachers actually and could potentially learn through teaching. Indeed, these articles stress that teachers' craft (planning, implementing, interacting, reflecting, assessing, etc.) *can* serve as a strategic site for their learning. Concerning this assumed but frequently unrealized potential, I make a twofold argument.

The first part of my argument is that articulating both *what* and *how* teachers can learn through teaching (LTT) is dearly needed for better understanding this gap and bringing the potential to fruition. This is consistent with Borba's concluding comment of the need to articulate what makes certain experiences of mathematics teachers conducive to the substantial learning they must pursue, so that their students acquire the demanding, reform-oriented expectations of "understanding/doing math." The second part of my argument is that theoretical accounts of teacher learning are required to determine what makes particular learning opportunities productive. I point out how the recently elaborated framework of learning a new mathematical conception through reflection on activity-effect relationship (*Ref*AER*) (Simon, Tzur, Heinz, & Kinzel, 2004), provides a good basis for such a theoretical account, though adaptations to the complexities of teacher learning will most likely be needed.

To substantiate this argument, I briefly present key constructs of the *Ref*AER* account. Then, I analyze and synthesize elements of the four articles in keeping with the two leading questions: What might teachers LTT that is worthwhile learning (i.e., likely to benefit student learning)? and How might teachers learn this? Finally, I point to ample key issues the four articles raise that await further discussion.

ACTIVITY-EFFECT RELATIONSHIP: REFLECTION, ANTICIPATION

The theoretical account of how existing mathematical conceptions are transformed into new ones is founded on two core constructs that seem of value for articulating teacher learning: *anticipation* and *reflection*. What a person anticipates and/or reflects on is a relationship the brain creates between an activity and what the mind takes as effects of that activity. The mechanism underlying the conceptual transformation is *reflection on activity-effect relationship* (*Ref*AER*). This mechanism commences with a learner's assimilation of problem situations into her available conceptions. These conceptions set the learner's goal and the activities she calls up and execute to accomplish her goal. Via her activity the learner's goal regulates her noticing of actual effects, including discrepancies between these effects and the anticipated result.

Through reflection on and reasoning about solutions to similar problems, the learner abstracts a new regularity (invariant)—a relationship between an activity and its (newly noticed) anticipated effect(s). *Ref*AER* consists of two types of comparison: between the learner's goal and the actual effects of the activity, which leads to sorting activity-effect records; and among situations in which such activity-effect records are called upon, which leads to abstracting the activity-effect relationship as an anticipated, reasoned regularity. This regularity involves a reorganization of the situation that brought forth the activity in the first place, that is, of the learners' previous assimilatory conceptions.

WHAT MIGHT TEACHERS LEARN THROUGH TEACHING?

Using Leikin's (2006) third dimension–forms of knowledge, a critical goal for teacher learning, pointed out by both Liljedahl and Simon, is progress from intuitive to formal ways of thinking about teaching. Liljedahl illustrates this goal via his focus on changes in teacher reactions to students; Simon illustrates it in his focus on the need for teachers to question and reflect on their hidden epistemological assumptions. All four articles stress further that to accomplish such a difficult transformation, a teacher educator has to promote teachers' mindset of openness to and acceptance of unexpected situations as a key, self-generated source for the teacher's own professional development, hence adoption of active *listening for* the unexpected.

On one hand, it is crucial to point out that a desired transformation from intuitive to formal ways of knowing consists not only of behavioral ("practical") changes but also of a paradigm shift in how teachers think about math knowing and coming to know. Simon (2006) articulates components of such a shift, which was first postulated as advancing from a perception- to a conception-based perspective (Simon, Tzur, Heinz, Kinzel, & Smith, 2000). For most teachers, such a paradigm shift is not likely to happen without substantial, guided, long-term interventions, because teachers' existing conceptions and perspectives seem to serve as an assimilatory trap (Stolzenberg, 1984): What they notice and act upon is structured by the paradigm they have yet to question. For example, it seems that the teacher who concluded that 'there are different ways to solve a problem (Leikin and Zazkis) or those who rethought 'what serves as evidence for numeracy' (Liljedahl) did not transform their epistemological stance toward math knowing.

On the other hand, promoting changes in teacher practices is important because these practices consist of the teachers' goals and activities they employ to accomplish their goals. In terms of *Ref*AER,* reformed practices can become the 'material' for teacher epistemological-oriented reflection. For example, the teacher who learned, through noticing a student's unexpected contribution, that it is important to precisely define a mathematical term ('divisor'), has set a goal that can be capitalized upon for promoting her abstraction of the epistemological role of assimilating an expression. This, in turn, can lead her to examine how using the term 'factor' in the question could have brought forth in her students the activity-effect relationship relevant for figuring out if *any* natural number, presented as multiplication of primes to some power, is divisible by *any* other natural number. Similarly, teachers' learning of how to use computer software, in a course and/or in their own practice (Borba), can become the source for reflection on the role of mathematical activity in a medium, as well as on what role the software user's goal serves in what she or he notices (takes as machine 'feedback'). In this sense, I agree with Borba that the distinction between what teachers learn from their craft and in courses they take is, necessarily, blurred.

Last, but certainly not least, the articles of Leikin & Zazkis, Liljedahl, and Borba manifest the obvious: teachers can learn (and re-learn) mathematics. In the next section, I discuss ways in which teaching affords such learning. Here, I only note two aspects of such learning. First, realizing the potential seems to depend heavily on a teacher's predisposition toward unexpected situations as an opportunity, not as a threat to be eradicated. Welcoming such situations as an opportunity is likely to initiate a constructive cycle because it encourages students to make more contributions, hence more opportunities for the teacher. Second, there seem to be important differences between the mathematics teachers can learn with and without guidance. In particular, teacher educators can assume the key role of prompting teachers' noticing in situations that would otherwise go unnoticed and of orienting teachers' reflection onto relationships the teachers overlook. On the other hand, interaction with teacher educators may also add to the teachers' sense of threat. The impact of these two contexts (with/without guidance) and how to strike a balance between them seems to be an important focus for further research.

HOW MIGHT TEACHERS LEARN THROUGH TEACHING?

All four articles provide ample examples for the role that anticipation and reflection play in teacher LTT. Being aware that these two constructs may potentially be my own conceptual trap, I claim that each of the numerous examples is a specific manifestation of learning as change(s) in anticipation. That is, a teacher essentially learns through noticing unanticipated ways in which others (e.g., one's students or peers) react to plans the teacher executes. Such reactions may become prompts for the teacher's reflection on pedagogical/math activity-effect relationships. That is, the teacher continually considers the extent to which her goal-directed teaching moves foster (or not) certain effects—effects in the sense of inferred students' (or peers') understandings. To substantiate my claim, I briefly discuss three examples.

The teacher in Leikin and Zazkis' second example anticipated student solutions through induction. Her mathematical conceptions afforded her assimilation of the student's calculus solution, her pedagogical conceptions afforded acceptance of different solutions to the same problem, and her goal seemed to be making sense of the student's solution. Consequently, the student's solution served as a prompt that led to extending her mathematical anticipation of proper solutions to such problems and her pedagogical actions—proactively planning for fostering students' understandings of both solutions. The teachers in Liljedahl's study began their work on creating tasks for assessing numeracy while using a narrow and rather procedural understanding of this construct. His follow-up prompt ("the qualities of a successful (numerate) student") fostered teacher-teacher interactions that led to their reflection on what 'numerate students' should be anticipated to do/reason. Following Wenger's (1998) meaning for reification, Liljedahl's examples resonate with Pirie & Kieren's (1994) emphasis on continually expressing one's action-generated ideas as a means for clarifying these ideas to both oneself and others. Borba fostered teachers' learning via (inter)acting on the 'same' virtual geometrical object. Consequently, teachers were exposed to their peers' actions, which sometimes did not match one's own anticipation of actions she/he would take in that situation. As indicated by one of his participants (to cope with math activities for our students we had to revisit our own math), peers' unanticipated actions prompted further reflection, hence learning.

Deeper analyses of these examples must involve further specification of the goals toward which teachers direct their activities, such as correcting student mistakes, predicting student responses, providing students with experiences that differ from one's own school experiences, resolving disagreements and/or one's cognitive conflicts, satisfying school's requirement to use software, improving one's own math, etc., as well as the impact of the medium in which such learning takes place on teacher reflection/anticipation. Such empirically grounded analyses can capitalize on the *Ref*AER* account, as well as other constructs such as Mason's (1998) noticing, for developing powerful explanations of the complex mechanisms, contexts, and stages in teacher change toward productive, reasoned practices. Below, I raise a few issues that deserve further scholarly (theoretical and empirical) attention.

ISSUES FOR FURTHER DISCUSSION

1. What do we mean by and what do we take as evidence for 'the teacher learned' (i.e., what is the meaning/measure of success in math teacher education)?
2. How are guided and non-guided teacher learning different? Similar?
3. How does teachers' continual engagement in expressing their ideas to others contribute to their LTT?
4. How is openness to student unexpected reactions, which is a necessary condition for noticing such reactions and treating them as *contribution* to the teacher's own learning, evolves over time in relation to teachers' confidence (in math, in pedagogy)?
5. How might researchers use/measure changes in teachers' anticipatory schemes of teaching actions (schemes of which teachers are quite often unaware)?
6. Like in quantum mechanics, it seems that the medium through which researchers interact with and observe teachers' behaviors may change what is observed. What are the methodological and educative implications of such changes?
7. How do teachers' goals and implicit assumptions impact their LTT? This question bears both a theoretical elaboration and an articulation of teachers' practical focus (e.g., improve lesson plans, build assessment tasks).

8. Derived from #7, what tasks and prompts can teacher educators employ to foster teacher LTT, including scaffoldings that can serve the teachers in the absence of direct guidance, and why would such tasks work (or how adjusted when not)?
9. As a constructivist, I take for granted that what a teacher educator or a researcher may observe as lack of coherence in teachers' knowledge/beliefs/practice may be un-problematically coherent for the teacher (Liljedahl). Thus, in addressing both #7 and #8 above the onus is on the scholarly community to make explicit teachers' conceptions that afford/constrain hypothetical LTT.

CONCLUDING REMARKS

In this discussion, I argued that the *Ref*AER* account can serve as a good *basis* for articulating what and how teachers might learn through their and others' teaching. Such articulation can greatly contribute to closing the unspoken gap between what teachers actually and could potentially learn. It is my opinion that the four articles portray teacher LTT that is more representative of the potential we, as a field, would like to fulfill than of what most teachers typically do learn. Consequently, following Liljedahl's outlook, this research forum points in the right direction and constitutes a notable step in a long and worthwhile reification process of moving the field's knowledge/beliefs from the tacit/intuitive to the consciously articulated level.

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RF02: RESEARCHING CHANGE IN EARLY CAREER TEACHERS

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INTRODUCTION

Peter Sullivan

The following papers contribute to a research forum on the issue of teacher change. The forum is a development from ongoing interest among teacher educators on the issue of change processes, in particular, researching teacher change.

Essentially, all education is about change. Usually change is through growth in knowledge, understanding, and awareness that leads to change in thought and action. In other words, change is usually developmental and gradual. In most fields of education, educators plan as though changes occur one step at a time, and then only when readiness and awareness allow it. The forum is based on an assumption that, in teacher education, change is considered to be urgent, requiring active facilitation and creating particular ethical dilemmas and methodological challenges.

PERSPECTIVES OF TEACHER EDUCATORS

It is common for teacher educators to believe that there is an urgent imperative to prompt and facilitate changes in knowledge and/or dispositions of prospective and beginning teachers (termed early career teachers). The rationale for seeking change can be due to perceptions by teachers educators that early career teachers:

- Have fixed views of the nature of mathematics and limitations in relevant mathematics discipline knowledge;
- Have anxieties about mathematical knowledge and teaching that can be potentially constraining and even disabling;
- Are unfamiliar with desired pedagogies and curriculum, having not experienced these as school students themselves; and
- See learning to teach as a short-term, once-only event as distinct from a career-long process.

Teacher educators often harbour concerns that, unless these issues can be addressed as part of teacher education, the early career teachers may adopt undesirable and unsustainable practices and orientations that can be restricting, in that these may constrain practice, and may even lead to early attrition from the profession.

CHANGE PROCESSES

It has been argued over some time that change can by affected through modelling of alternate pedagogical approaches (e.g. Bednarz, Gattuso, & Mary, 1996), and that collaboration and teacher collegiality facilitate change (Chazam, ben Chaim, & Gormas, 1996). It is assumed that changes in practice relate to changes in beliefs, and it may be that changes in practice precede changes in orientation (Guskey, 1986).

Teacher educators seek to foster change through growth in mathematical knowledge (both content and pedagogical), through enhancing awareness of the early career teachers' emotional dispositions (attitudes and beliefs about mathematics), by addressing the early career teachers' vocational aspirations (how they see teaching and themselves as teachers of mathematics), and through enhancing sensitivity to themselves as learners, as distinct from disseminators of knowledge.

There are different approaches to change including:

- A professional orientation where even though the initiative for, and the direction of, change comes from teacher educators, the first step is to enlist awareness of the need for change in the beginning teachers, with the eductors facilitating the process of change;
- A therapeutic approach where the intention is to support teachers in addressing mathematics anxiety or other attitudes and beliefs about the nature of mathematics;
- A critical approach where change is fostered as the desirable state and the goal includes challenging structures and approaches that entrench privilege or create barriers to opportunities.

Some of the issues identified in previous PME discussion groups on teacher change include the challenge of ensuring that changes are sustainable, recognising the multifaceted nature of change and constraining factors, the directionality of the impetus for change (e.g., initiated by government policy, teacher educators or the early career teacher themselves), the development of a sense of ownership of the change, identifying relevant motivators for change, and the relationship between knowledge, dispositions, and action.

Recently some important research reports have addressed key issues in teacher change. These have addressed the contribution to teacher learning and change through the development of positive attitudes (Amato, 2004), enhancement of teacher knowledge (Harel & Lim, 2004), and growth through focus on teacher concerns and efficacy (Charalambous, Philippou, & Kyriakides, 2004). Various mechanisms for fostering change have been suggested, including lesson study (Fernandez, 2005), and the development of theoretical models of change drawing on collective understanding and metaphor (Drouijkova, Berenson, Slaten, & Tombes, 2005).

The focus of the following contributions is to examine methodological approaches to the study of change, so that outcomes can be convincingly reported to colleagues and to the profession. This includes processes for evaluating the need for change that can contribute to the change process itself. It is noted that there are significant practical and ethical issues associated with researching change in early career teachers.

CONTRIBUTIONS TO THE RESEARCH FORUM

The goal of the research forum is to allow presentation of different approaches to researching change and change processes, with particular emphases on researching and reporting changes among early career teachers.

The specific questions that are addressed in the following contributions are:

- What are effective and/or innovative methods for researching change in early career teachers?
- What are processes for evaluating effectiveness of tasks and activities seeking to foster changes in early career teachers?
- What are the ethical and practical issues in researching and reporting change in early career teachers?

RESEARCHING RELIEF OF MATHEMATICS ANXIETY AMONG PRE-SERVICE ELEMENTARY SCHOOL TEACHERS

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Mathematics anxiety is a state of discomfort, occurring in response to situations involving mathematical tasks, which are perceived as threatening to one's self-confidence (Cemen, 1987; Hembree, 1990). This affliction is a common phenomenon among pre-service elementary school teachers in many countries and it can seriously interfere with students becoming good mathematics teachers. As such, it is important that elementary teacher education programs help those with mathematics anxiety to overcome it. Fortunately, there exist several examples of how this can be achieved (cf. Kaasila, 2006; Liljedahl, Rolka, & Rösken, in press; Pietilä, 2002; Uusimäki & Nason, 2004). These examples collectively use what we refer to as a therapeutic approach. *In this paper, we will first introduce some examples of these therapeutic approaches and then focus on some of the challenges of doing research within this context.*

THERAPEUTIC APPROACHES PROMOTING RELIEF OF ANXIETY

According to Uusimäki & Nason (2004), teacher educators can reduce math anxiety by identifying a) the origins of teacher trainees' negative beliefs and anxieties about mathematics, b) situations causing anxieties, and c) types of mathematics causing anxieties, and using this information when preparing intervention programs that facilitate change in non-threatening ways. Some of these elements are evident in many of the approaches that have succeeded in relieving mathematics anxiety among pre-service elementary school teachers.

One of these elements is the effort to provide students with positive experiences with mathematics. This can be achieved within the context of elementary mathematics, where hands-on material is used to give student teachers an example of teaching in a constructivist way, whilst at the same time providing an opportunity for many of them to really understand mathematics for the first time in their life (Pietilä, 2002). Another approach is to challenge students with mathematical problem solving that provides them with opportunities for discoveries and AHA! experiences (Liljedahl, 2005). Within this context there is a conscious effort made to keep the learning environment safe. A supportive classroom climate is essential to allow anxious students to express their thoughts and feelings and ask for advice without fear of stigmatisation (Pietilä, 2002).

However, experience alone is not sufficient for a major change in students' mathematical self-concept – it needs to be supported by reflection. Reflection results primarily in new comprehensions, such as an improved ability to carry out the act of reflection, changes to a belief, an attitude, or a value, or an altered emotional state or trait (LaBoskey, 1993). As reflection has a central role in our research methods, we describe our approaches in more detail. We have used four main ideas to reduce mathematics anxiety by handling elementary teacher students' experiences from their years at school or during teacher education – 1) narrative rehabilitation, 2) bibliotherapy, 3) reflective writing, and 4) drawing schematic pictures. In what follows we briefly describe each of these.

Narrative rehabilitation – Teacher trainees are offered opportunities to tell stories about their memories as students and share their experiences with others in small groups (Kaasila, 2002; Pietilä, 2002; Valkonen, 1997).

Bibliotherapy – Prior to their practicum student teachers read mathematical biographies produced by a study on narrative rehabilitation, focusing on the one that most closely resembled their own background (Kaasila, 2002; Kaasila, Hannula, Laine, & Pehkonen, 2006; Lenkowsky, 1987).

Reflective writing – During or after their teaching practicum student teachers produce teaching portfolios, which are comprised of their reflections of their mathematics lessons. They also construct a mathematical autobiography and reflect on their thinking using the mathematical biography that most closely resembled their own background (Pietilä, 2002; Kaasila, 2002; Kaasila et al., 2006). Alternatively, during their mathematics methods course students write reflective journals, which may include their problem-solving strategies (Liljedahl, Rolka, & Rösken, in press a).

Drawing schematic pictures – Students draw mind maps or schematic pictures of their views of mathematics at the beginning and end of the course. These are then reflected upon in groups (Pietilä, 2002; Kaasila et al., 2006).

Within each of these methods it is also typical to encourage collaboration between students. Qualitative results have confirmed that this has an important role in teacher student development (Kaasila et al., 2006). Collaboration can support emotion regulation through the support that peers provide when one of the members of the group is faced with a moment of anxiety that might completely paralyse them if they were alone. Collaboration may also be helpful in encouraging reflection.

EXAMPLES OF RESEARCH ON THE THERAPEUTIC TREATMENT

Liljedahl (2005) began analysing the reflective journals of his elementary education students through case studies that exemplified the effect that experiences in problem solving activities can have on student's affect towards mathematics. In this research, the reflective journaling was explicitly identified as part of the treatment. Liljedahl, Rolka and Rösken have reanalysed the data by categorizing students' beliefs of mathematics and its teaching and learning according to previously established categories of beliefs – toolbox aspect, system aspect, process aspect and utility aspect (Grigutsch, Raatz & Törner, 1997; see also Dionne, 1984, Ernest, 1991 and Törner & Grigutsch, 1994). In the "toolbox aspect", mathematics is seen as a set of rules, formulae, skills and procedures, while mathematical activity means calculating as well as using rules, procedures and formulae. In the "system aspect", mathematics is characterized by logic, rigorous proofs, exact definitions and a precise mathematical language, and doing mathematics consists of accurate proofs as well as of the use of a precise and rigorous language. In the "process aspect", mathematics is considered as a constructive process where relations between different notions and sentences play an important role. Here the mathematical activity involves creative steps, such as generating rules and formulae, thereby inventing or re-inventing the mathematics. In the "utility" aspect, mathematics is seen as useful, considering how it applies to every day experiences.

Liljedahl with colleagues conclude that through their own experiences with mathematics in a non-traditional setting most of the students came to see, and furthermore to believe, in the value of teaching and learning mathematics in the sense of the process aspect (Rolka, Rösken, & Liljedahl, 2006). In order to better understand the process of belief change, they have more recently explored the use of the theory of conceptual change (Liljedahl, Rolka, & Rösken, in press b). The theory of conceptual change is a powerful theory for explaining the phenomena of theory replacement when the rejected theory has been tacitly constructed through lived experiences in the absence of formal instruction (Posner, Strike, Hewson, & Gertzog, 1982). Such organically constructed theories are not too dissimilar from the beliefs that may also be tacitly constructed through lived experiences. The framework was able to discern the difference between instances of belief evolution and belief replacement.

Another example of the research done within the therapeutic approach is reported in Kaasila et al. (2006). They applied narrative analysis (see, e.g., Polkinghorne, 1995) by constructing a narrative of change for four pre-service teachers' mathematical autobiographies and then comparing narratives systematically. These students were selected as representative of a wider spectrum of changes manifested among trainees. The trainees' views of teaching and learning mathematics became more multifaceted during their mathematics methods course. Moreover, ego-defensive orientation changed towards social-dependence orientation. In social-dependence orientation student adaptation is dominated by social motives (e.g., seeking help from the teacher). The student avoids independent effort and easily becomes helpless. Positive emotions are connected with expected satisfaction of the teacher. In an ego-defensive orientation student adaptation is dominated by self-defensive motives. The student is sensitised to task difficulty cues, anticipating a negative response from the teacher, and may try to find compensatory tactics in order not to "lose face". (see Lehtinen, Vauras, Salonen, Olkinuora, & Kinnunen, 1995) The most central facilitators of change seemed to be handling of and reflection on the experiences of learning and teaching mathematics, exploring with concrete materials, and collaboration with a partner or working as a tutor of mathematics. (Kaasila, Hannula, Laine, & Pehkonen, 2006)

CHALLENGES OF RESEARCHING THE TREATMENT OF MATHEMATICS ANXIETY

In this section we look more closely at some of the challenges associated with doing research on the aforementioned therapeutic approaches to teacher education. The first challenge is an ethical one and can emerge in the selection of the research topic. Are we, as teacher educators, changing our students or are the students changing themselves? Who has the agency? Obviously, we cannot change our students. All we can do is to provide opportunities for them to change. Yet, we do have clear goals for how we want them to change. As a solution to this dilemma, we can talk about *empowering* students, or *occasioning change* within those who suffer mathematics anxiety. A stance of either *empowerment* or *occasioning* allows the agency of change to remain with the student teacher while the agency of treatment – through research methodology – to remain with the instructor/researcher.

Another challenge is more of a methodological dilemma – although it can also be construed as being an ethical dilemma. In the previous section, we summarized four *treatments* for mathematics anxiety – 1) narrative rehabilitation, 2) bibliotherapy, 3) reflective writing, and 4) drawing schematic pictures. Each of these treatment methods is also the primary source of data for research of these treatments. That is, there is a confluence between what we are attempting to measure and how we measure it. However, we see this confluence as a strength, rather than as a weakness. By using our treatment method as data we have a direct link between treatment and results, and thus feel that our results speak more directly to the effectiveness of the treatment method.

In researching the therapeutic treatment of mathematics anxiety there is also the challenge of the validity of the data. Although this is not a dilemma unique to this research domain it is a dilemma that is particularly prevalent here. Each of the aforementioned treatments requires a great amount of reliance on student teachers' outputs as data – either in the form of written reflections or mind maps. This elevates the chances that the data can be corrupted by student teachers efforts to present 'correct' responses rather than 'true' responses. However, because the data is first and foremost seen as an integral part of the treatment of mathematics anxiety, both by the student teachers and the researchers, there have been few occurrences of such problems. In fact, in Liljedahl, Rolka, & Rösken (in press a) the authors guarded against occurrences of denial or rhetoric, yet found few occurrences of either.

Finally, there is a challenge pertaining to the deeply personal nature of the data collected. By the very nature of the context, the participants are revealing something about themselves that they are likely view as a weakness. As such, both the collection of the data, and the reporting of the data, need to be treated with great sensitivity. One way to avoid this problem is to report on general categories instead of in-depth case studies. However, in doing this we lose some of the power of qualitative reporting. Another possibility is to create evidence-based fictional stories (Hannula, 2003; Richardson, 1997) that combine experiences of several students.

TEACHERS' LEARNING FROM LEARNING STUDIES: AN EXAMPLE OF TEACHING AND LEARNING FRACTIONS IN PRIMARY FOUR

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In this paper, we suggest an effective model for teachers' professional development – Learning Study. A Learning Study combines students' and teachers' learning. Through a Learning Study, teachers improve their teaching by learning how their students learn. In a Learning Study, a group of teachers explore the relationship between teaching and learning with the aim to improve students' learning in a cyclic process of planning and revising their lessons. An example of a Learning Study about teaching fractions in Primary 4 in a Hong Kong school is given.

INTRODUCTION

Schools have often been blamed for not being able to achieve the educational goals valued by society. Thus, large-scale systemic educational reforms emerged, orchestrated by provincial, state, or national governments (Fullan, 2000). All these reform efforts were found to be futile because they neglected one important factor for the improvement of student learning outcomes - teachers. Recently, more and more researchers have pointed to the fact that teachers are the most important change agent in curriculum reforms. It is now recognized that systematic effort would be of paramount importance to foster teachers' professional development, which will greatly contribute to the success of education reforms. (Darling-Hammond, 2006). Learning Study therefore provides such an opportunity and environment for the professional learning of teachers.

LEARNING STUDIES

A Learning Study is similar to the Japanese lesson study (Yoshida, 1999; Stigler & Hiebert, 1999) in which a group of teachers work collaboratively to explore and develop their teaching practice in a cyclic process of planning, observing and revising lessons. However, the aim of a Learning Study is not to improve the lessons in a general way, (e.g., implementing new methods or new technology) but to enhance students' learning of a specific object of learning. In a Learning Study, recognizing the variation in students' ways of seeing plays a central role in teachers' decision making. Before and after the lessons the students are tested and/or interviewed to obtain information about what is problematic for them to learn a particular concept or skill. This also provides insights into what is critical for students' learning, based on which lessons are subsequently planned. After the lessons, the students are tested and /or interviewed again. Information from the post-test gives immediate feed-back to the teachers and enables them to analyse and reflect on the lesson (already video taped) from the point of view of why the students have been able to or fail to learn. Hence, teachers inquire about and explore the lessons from the perspective of possibilities for learning; whether the necessary conditions for learning are met in the lessons or not. In a Learning Study, the capability we want the students to develop is the very focus. Therefore, how the object of learning is handled in a lesson becomes the object of study for the teachers (as well as for the researchers). After identifying an object of learning they further specify explicitly the critical aspects that need to be handled in the lesson, they plan the lesson and teach it. After the first cycle, they revise the lesson plan according to their post-test findings from the post-test, student interviews and classroom observations. Then, in the second cycle, a new teacher teaches the revised lesson to her students. A post-test is given and the recorded lesson is observed and – in some cases – the lesson plan would be further revised.

The main purpose of LS is to find the relationship between teaching and student learning outcomes. The theory of Variation (Marton, Runesson, & Tsui, 1997) forms the basis of the theoretical framework of Learning Study. According to this theory, in order to learn something, one must be able to discern its critical aspects. And, in order to discern an aspect, one must experience variation in the aspect. The failure to learn something can be interpreted as the inability to discern all the aspects that are necessary to be discerned for a particular way of understanding. For example, it is necessary to discern simultaneously, concepts like equal shares, part-whole comparison, as well as unit and unitizing in order to understand the concept of fractions. The theoretical framework, however, does not prescribe any particular teaching method or arrangements; instead it serves as a guiding principle when investigating students' learning and their learning possibilities in the classroom.

LEARNING STUDIES IN HONG KONG AND IN SWEDEN

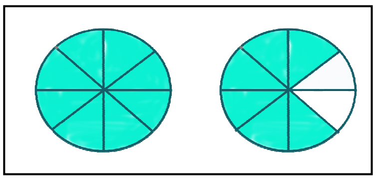
Learning studies have been tried out in a connected project in Hong Kong and Sweden. In Swedish schools, teachers cooperating in working teams is common. Although teams meet regularly to discuss issues related to their teaching, they seldom come together over an extended time to intensely discuss how to teach a particular concept. It is un-common that they get an opportunity to observe a colleague's teaching; and even less common for them to be able to observe teachers' teaching the same topic. In the Swedish project, 18 Learning Studies were carried out in different subjects.

In Hong Kong, the situation is somewhat similar. Peer observation and collaborative lesson planning have become a normal routine in the majority of schools in Hong Kong now. The first attempt at introducing a systemic procedure and learning theory to enhance the effectiveness of such teacher collaboration towards teaching and learning through Learning Study was made in a three-year project in 2000 (Lo, Pong, & Chik, 2005) with only two primary schools. Then in the following six years, over 200 Learning Studies have been developed involving both primary and secondary schools, covering almost every subject in the school curriculum.

ONE EXAMPLE OF TEACHERS' LEARNING IN LEARNING STUDIES

This Learning Study at Primary 4 level was in the area of 'fractions' (Lo, 2001), developed by a Learning Study team of five teachers from a primary school and 3 researchers from the University of Hong Kong. The entire study took nine meetings, which were held in the school at after school hours over a period of six months.

The Learning Study group planned to do a study on fractions because the teachers perceived that 'fraction' was most difficult for school children at this level. As the textbook they used made use of the following diagram to illustrate 8/8 + 6/8 = 14/8:



The teachers complained that some students would just add the numerator and the denominator together and come up with the idea that 8/8 + 6/8 is 14/16. Through the discussions, the teachers began to realize that they had been insensitive to students' ways of seeing which contributed to difficulties in learning. While teachers took for granted that the unit was one circle and expected the answer to be 14/8, some students actually took the whole as two circles and saw it as 14/16. The problem in students' learning arose because the 'unit' under consideration had never been given enough emphasis by the teachers in their teaching. After some discussion, the teachers realized that their previous practice, in which they mainly drilled students on arithmetical operations of fractions in their 'pure form' without any reference to the whole that the fraction relates, may be contributing to the students' habit of ignoring the 'unit' or the whole to which fractions refer. The literature also supports the teachers' views, as Lamon (1999) suggests: there are two aspects of fractions, that are vital but students often fail to grasp, namely, unit and unitising, and part-whole comparison. As a result, the teachers decided that the research lesson should be focused on 'unit' and 'unitizing'.

The teachers then designed a pre-lesson diagnostic test to collect information for planning of the research lesson. The test items were designed to find out students' understanding. Such an exercise further sensitised the teachers to the differences in students' understanding. The teachers then worked collaboratively to plan their lessons to address the critical aspects identified, taking into account students' difficulties as revealed by the results of the pre-test. The Theory of Variation was also used to guide the planning. A number of activities were designed to help students discern the importance of the unit. For example, students were asked to compare the size of the fractions when the units varied and the fraction remained constant. This pattern of variation helped to bring into focus the significance of the unit. There was also an activity to bring about fusion of all the parts to result in better understanding of the whole. In this activity, students were told that TWGH held its yearly fund-raising event. Li Ka Shing (a well known multi-billionaire in Hong Kong) donated five million dollars to help TWGH; Chan Siu Ming took out half of what he had saved, and donated 50 dollars; Wong Tai Yung donated all the 5 dollars that he had. Students were asked to discuss two key questions: 'who has donated the most?' and 'Who was the most generous?' In this case, the unit (the whole), the size of the fraction (the part) and fraction varied simultaneously. This problem-solving exercise motivated students to integrate what they have learned in the first part of the lesson and served the function of fusion.

The research lesson was then taught by the teachers to their own classes in cycles. Permissions were obtained from the school, teachers involved and parents of students for the lessons to be video taped. Both teachers and researchers studied the video recorded lessons and made suggestions for further improvements after each cycle.

After the research lesson, the teachers realized that in the past they had over estimated the students' ability to learn this difficult topic while under estimated the time that the students needed to spend on learning it. In fact, one teacher who appeared to be indifferent at the beginning of the Learning Study said this during one of the meetings: "I really felt ashamed of myself. I'd never realized that 'fraction' could be taught in this way!" This teacher is still leading Learning Studies in his school and has also offered his support to Learning Studies in other schools.

CONCLUSION

A Learning Study aims at enhancing both students' and teachers' learning at the same time. In our interpretation the Learning Study was successful as far as pupils' learning is concerned. Our experience with this and other Learning Studies tell us that it is quite common that some aspects of the object of learning are taken-for-granted or overlooked by the teachers. This happens because such aspects are usually too familiar or well known to the teachers, such that they are considered self-evident and thus failed to take it into consideration when planning the lesson. However, this may contribute to an obstacle for students' learning. Failing to bring out certain aspects of a concept that are critical for learning can result in unsuccessful learning outcomes. It is suggested that through the Learning Study process, teachers dig out what is taken-for-granted or reveal the difficulties that are encountered by students but unknown to them. The teachers can then change their teaching in a way that enhances students' learning. This is, from our point of view, one of the most important learning outcomes for the teachers in a Learning study.

One of the specifics and advantages of a Learning study is the immediate feedback the teachers can get about their teaching from the results of the post-test. Normally, in order to evaluate a single lesson, teachers trust their own intuitions as to how successful the lesson is and what pupils have actually learned. Nevertheless, testing or interviewing the pupils about what has been taught immediately after the lesson, provides a more direct and reliable account of how the teaching enactment had opened up opportunities or made it possible for students to learn what was intended. We also think that the opportunity to observe colleagues teaching *the same topic,* offers specific learning possibilities for the collective discovery of what is critical for pupils' learning.

Pupils' learning progress can more or less be effectively tested. But what about what teachers were learning? In what way is it possible to identify their progress apart from more general reflections and expressed insights reported by the teachers? Since all the meetings were audio-recorded, and the lessons were video-recorded, this made it possible for the changes in individual teachers to be traced. One indication for teachers' learning is the contributions they made when engaged in pre- and post-lesson meetings. As teachers jointly and systematically enquired about their own practice, their professional learning can be manifested in the way they enhanced and contributed to a collective development. The lesson plan that the teachers came up with is another indicator, as it showed whether the teachers managed to identify critical aspects necessary for learning. The enacted lessons reflect the teachers' considerations about pupils' understanding and whether they were able to change their plan during the enactment to take this into account. Of course, the pupils' learning outcomes gave very useful feedback about the effectiveness of the teaching.

From the results of several Learning studies we have noticed that the teachers are much more confident and have a better grasp of what they usually considered difficult to teach and difficult for the students to learn. Finally, it all depends on whether the teachers are able to see a relationship between their teaching enactment and their students' learning outcomes, and be able to act on these findings that contribute to teachers' learning, resulting in better teaching and learning.

TRACKING TEACHERS' LEARNING IN PROFESSIONAL DEVELOPMENT CENTERED ON CLASSROOM ARTIFACTS

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This paper reports on the methodological approach and research findings from the Turning to the Evidence *project, a project that measured teacher learning across two professional development programs focused on using classroom artifacts to study algebraic thinking. The* Turning to the Evidence *project investigates these programs as the context for inquiry into two overarching research questions: (1) what do teachers learn by participating in professional development that uses classroom records and artifacts? And, (2) what aspects of their learning do they apply to their own classroom practice?*

How do mathematics teachers continue to develop the knowledge, skills, and habits of mind that enable them to teach well and to improve their practice over time? This is a fundamental question for researchers and has generated a considerable amount of work in the areas of "teacher learning," much of which is scattered across diverse areas of literature including changes in beliefs, knowledge, decision-making, pedagogical approaches, and even teachers' sense of self-efficacy and identity. Part of the task of getting a handle on what is meant by teacher learning is to be explicit about the aspect of learning that is under investigation, and the reasons for finding it important as a focus of study.

One promising context for promoting and studying teachers' learning is in mathematics professional development (PD) that makes use of classroom artefacts (Ball & Cohen, 1999). Several research projects have suggested that that practice-based PD projects that utilize artifacts of practice, such as classroom video and student work, are effective tools in efforts to increase teachers' opportunity to learn mathematics knowledge for teaching (Smith, 2001). By bringing the everyday work of teaching into the PD setting, these tools enable teachers to unpack the mathematics in classroom activities, examine instructional strategies and student learning, and discuss ideas for improvement (Driscoll et al., 2001; Schifter et al., 1999a, 1999b; Seago et al, 2004).

This paper reports on the *Turning to the Evidence* (TTE) study, which examines the impact on teachers' learning of two PD programs focused on algebraic thinking: *Fostering Algebraic Thinking Toolkit* (Driscoll et al., 2000) and *Learning and Teaching Linear Functions: VideoCases for Mathematics Professional Development*

(Seago, Mumme, & Branca, 2004). The TTE study grew out of the observation that, while there is currently considerable interest in using classroom records and artefacts as a tool for mathematics teachers' PD, as a field we know surprisingly little about what teachers actually learn by working with artifacts or how they integrate their learning into daily classroom practice (Wilson & Berne, 1999). This research effort has two foci: the PD seminars (examining what teachers learn by working with artifacts and records) and the classroom (examining ways that teachers bring learning from their PD seminars into their classroom practice).

THE STUDY CONTEXT

Data have been gathered in four research sites in the U.S.: two groups of teachers on the east coast who participated in *Fostering Algebraic Thinking (AT)* PD and two west coast districts participating in *VideoCases for Mathematics Professional Development (VCM)* PD. All of the groups were facilitated by the lead authors of the respective programs, (Driscoll for the AT groups and Seago for the VCM groups), ensuring high fidelity of implementation. Both seminars involved 12, three-hour sessions. Both PD programs involved 36 hours of PD (12, three-hour sessions). The VCM groups completed all 12 sessions in a single academic year (2003-2004) and the AT groups completed the PD over the course of three semesters (October 2003-January 2004). In all, 49 middle and high school teachers participated in the groups, 20 in the AT groups and 33 in the VCM groups. Sixteen teachers (four from each site) are being followed more closely to create case studies. Seminar participants included both veteran and early career teachers. Slightly more than a quarter of the teachers participating in the PD had been in the classroom for 5 years or fewer; the entire group of seminar participants averaged approximately 10 years of teaching. In addition to the 49 PD participants, 25 teachers served as a comparison group for our pre/post written measures. These comparison teachers came from the same districts as the PD participants.

DATA SOURCES

We have collected both quantitative and qualitative data. The quantitative data includes two paper-and-pencil instruments administered to participants at the beginning and end of the PD described below.

Mathematics survey was designed to assess teachers' algebra knowledge—both their own ability to solve algebra problems and their ability to recognize algebraic thinking that is characteristic of students—that is, aspects of "mathematics knowledge for teaching" (Ball & Bass, 2000; Ball, Hill, & Bass, 2005). In constructing the survey, we drew heavily on items from the University of Michigan "Learning Math for Teaching" database (Hill et al., 2004) and also included items used to assess teachers' learning in California Mathematics Professional Development Institutes (California Professional Development Institute, 2002; Hill & Ball, 2003). Staff added a few additional items. The instrument contains both multiple choice items and open response items. Both administrations of the instrument included confidence scales for each problem answered; in addition, at the end of the post-program administration, we gave teachers their pre-program booklets and asked them to write a few sentences about any differences they noticed between the two.

Artifact Analysis was designed to assess what teachers attended to when analysing classroom artifacts**.** It consists of a mathematical task, a five-minute video clip of a classroom discussion and student solution methods around the mathematical task, a series of four questions about the video, three different written student work to comment upon, a question about the accuracy of each piece of student work and a final question about what lesson would come next.

The qualitative data collected includes video of all of the PD sessions as well as video of the classroom lessons observed by TTE staff. (In addition to the classroom video itself, we audio taped interviews with the teachers before and after each classroom visit).

OVERALL FINDINGS

Because our research goals did not relate specifically to understanding the learning of early career teachers, our data analyses have centered on changes in the participant group as a whole. (However, we expect that the issues our work raises hold equally well for teachers at any point in their careers.) Overall, results indicate that teachers learned to take a more analytic stance to their work with classroom artifacts, attending to the mathematical implications of the thinking embodied in the artifacts and noticing the potential in students' thinking, rather than stopping at an assessment of students' weaknesses.

For example, the post-program artifact analysis indicates that PD participants were more likely than the comparison teachers to comment on specific mathematical ideas in their analysis of both the video clip and the accompanying written work samples. In addition, their work was more grounded in evidence, and they were more attentive to student potential (vs. being evaluative) than the comparison group teachers. Additionally, analyses of the seminar sessions themselves indicated that over the course of the PD, participants' discussions of classroom artifacts not only showed the same kinds of shifts as we observed on the written measures, but that the teachers internalized many of the strategies for attending to artifacts that we had articulated as goals for the project (Nikula, Goldsmith, Blasi, & Seago, 2006).

Analysis of teachers' learning with regard to the development of mathematical knowledge for teaching (MKT) is somewhat more complex. Our analysis of changing discourse over the course of the PD indicates that teachers' discussion of artifacts became deeper and more mathematically coherent, as well more focused on unpacking students' understanding relative to the artifact's underlying mathematical concepts. However, our quantitative data are difficult to interpret. While we do not have comparable data for comparison teachers, and therefore cannot unequivocally ascribe the changes in the mathematical discussions to the PD experience, we are confident that the increasing sophistication of teachers' mathematical analysis was, in fact, a result of their PD experiences and that comparison teachers would not have demonstrated such changes. However, our data from the mathematics survey fails to distinguish between PD and comparison groups in terms of overall score (both made modest improvements from pre- to post-test). A more qualitative analysis of the instrument has suggested that teachers may make subtle shifts in important aspects of their mathematical knowledge for teaching; for example, a number of the seminar participants tended to use the mathematical language (and ideas) of the seminar in their responses to the post instrument, demonstrated more fluent and flexible use of mathematical representations, and identified solutions that involved specific strategies and mathematical content related to the PD experience.

Preliminary analysis of classroom lessons suggests that teachers' transfer of the increased attention to the mathematical ideas behind student thinking observed in the PD to their classroom practice is modest. We did notice that some teachers were making small, subtle changes in their classroom instruction, such as emphasizing the connections between diagrammatic representations of a problem and symbolic representations, or seeking to make students' thinking more public. In general, it seems that the process of reconstructing classroom practice is a slow moving one, subject to relatively undramatic changes as teachers work to integrate their work in PD into their ongoing, daily instruction.

REFLECTIONS ON OUR MEASURES

In conducting our research, we have found a tension between looking for measures that could help us characterize aspects of teachers' learning and honouring the fact that the work of developing one's practice involves deepening both subject area and pedagogical knowledge (and integrating them) and is a difficult, intense, and often emotionally challenging undertaking. Teachers must do this work at the same time they are responsible for the education of their students—a situation that has been likened to redesigning and building an airplane while it is in flight. At times, we have felt that the data from our written measures capture only part of the story.

This has been particularly true for the mathematics survey. The Learning Math for Teaching database of items (Hill et al., 2004) allows researchers to assess the mathematical knowledge for teaching of large numbers of teachers and to compare results across studies. However, our findings of improvement in *both* seminar participants and comparison teachers have left us with questions about how to interpret the data. In addition, it has raised questions for us about the alignment of the survey's "grain size" relative to the kinds of learning we have found through more careful analysis of changing mathematical discourse in the seminars themselves (Seago & Goldsmith, 2006).

This leaves us with a challenge which, we think, is one that is shared by the field as a whole: how to measure dimensions of teacher learning in ways that allow comparisons across studies, seem to capture the essence of the learning, and also respectfully characterize the complex and challenging work that teachers undertake when they engage in PD, and their courage in being willing to share that work with outside researchers.

TEACHER CHANGE IN THE CONTEXT OF ADDRESSING STUDENTS' SPECIAL NEEDS IN MATHEMATICS

Orit Zaslavsky — Technion – Israel Institute of Technology, Haifa

Liora Linchevski — The Hebrew University, Jerusalem

Our presentation will report on ways in which teachers' change was enhanced within the framework of an innovative intervention program aimed at addressing students' special needs for both ends – students that are mathematically talented and promising and students that are mathematically disadvantaged. The program's design included ongoing in-service professional development activities with teachers that required immediate implementation in their classrooms. In a way, the program dictated certain changes in teacher practice. We examine these changes and discuss their implications.

CONTEXT AND BACKGROUND

The context for our study is a national experimental intervention project aimed at implementing a special program in mathematics for low-achieving students hand in hand with a special program for excelling students. The project's main goal was to enhance the mathematics learning opportunities for junior-high school students (7^{th} – 9^{th} grades) at both ends – students that are mathematically capable and promising and students that are mathematically disadvantaged. The goals stem from the standpoint that equity in mathematics education requires addressing students' differential needs by special teaching approaches and appropriate learning material and resources (NCTM, 2000). The unifying goal is for each student in the project to do his or her best to exhaust his/her potential in mathematics. It is important to get the details of this special project in order to better understand the changes that occurred in teachers' knowledge and practices.

We had 80 participating schools, 56 of them were from the Hebrew-speaking sector and 24 were from the Arabic-speaking sector. In each school we identified – out of approximately 140 7^{th} graders about fifteen low-achieving students that were not likely to reach high school with the background and success necessary for continuing their studies even towards the lowest (3-unit) level of the matriculation exam in mathematics. Emphasis was put on identifying low-achievers that were capable of doing better given the proper opportunities. We also identified in each school about twenty five 7^{th} grade potentially excelling students that were either high achievers or with high potential and motivation to study an enhanced mathematics program. We refer to the low-achievers as the LA groups and the high-achievers as the HA groups.

Thus, in each school two special groups were formed. A special program was designed for each group. Each group had specific goals set forth that presented new and ambitious challenges: the LA students were supposed to complete in 3 years the official mathematics curriculum for junior high school students plus some part of the senior high school syllabus. At the end of the 9th grade they were expected to be prepared to take the basic matriculation questionnaire in mathematics, which students normally take only in the 10th grade or later. The program for the groups of excelling students consisted of three main components, concurring with Sheffield (1999): 1. Depth and complexity (within the given curriculum); 2. Breadth and enrichment (extra-curriculum content and activities); and 3. Acceleration (only to the extent of adjusting to their own pace).

The school administrators selected teachers that seemed suitable for teaching the first, the second or both groups. In order to be able to implement this program, we needed to prepare the practising teachers (some early careers and some more experienced ones) to teach their students according to these two special programs. Teaching according to either one of the programs presented great challenges for the teachers. Thus, a critical and integral component of the project was a comprehensive professional development program for the participating teachers who taught these groups of students. The preparation of the teachers addressed the mathematics and pedagogy necessary for teaching mathematics to the students in the project. The meetings with the teachers included activities that aimed at: encountering challenging mathematics; becoming acquainted with innovative/unfamiliar teaching approaches; developing sensitivity to students' special needs, interests and capabilities; understanding student epistemology.

In addition to exposing teachers to the special requirements of our program in terms of the mathematics and pedagogy, regional meetings served as professional support for implementing the program and sharing experiences. We also conducted, three times per year tests for students. These tests reflected the program's goals, and provided the teachers with an 'objective' measure of their students' progress.

Within the context of the project, we identified several manifestations of change in teachers' practices, knowledge and beliefs.

OUR PERSPECTIVE ON CHANGE

Our perspective on teacher change is interconnected with teacher learning. Teacher change can be seen as an outcome or indication of teacher learning. Teacher change is mostly associated with changes in beliefs, knowledge and/or practice (Llinares & Krainer, 2006). These three dimensions are often closely related and inseparable.

We recognize that some changes in teacher practice occur over time naturally with no direct intervention (Richardson and Placier, 2001). In our work, we examine teacher change that can be attributed to an intervention aimed at specific changes.

Thompson (1992) points to ineffective attempts at influencing teacher change:

> We should not take lightly the task of helping teachers change their practices and conceptions. Attempts to increase teachers' knowledge by demonstrating and presenting information about pedagogical techniques have not produced the desired results. (p. 143)

Research on professional development efforts points to interconnections between specific features of professional development program and the outcomes in terms of teacher learning and change. Clearly, in designing a professional development program that aims at certain changes, careful consideration should be given to the design and support provided. In response to Thompson's above concerns, there has been an increasing shift from demonstrating and presenting information to engaging teachers in learning experiences that mirror the goals of the program. We adopt this trend, and view the teacher as an active constructor of his or her knowledge, based on the experience s/he encounters. By engaging in challenging and investigative mathematical tasks, in a cooperative learning setting, and by reflecting on their personal learning – teachers are likely to develop a positive conception of the nature of mathematics, and become aware of ways in which such learning opportunities may be designed and offered to students. This approach concurs with the stand of Zaslavsky, Chapman and Leikin (2003) that stresses the importance of the type of mathematical tasks, which teachers deal with in professional development programmes, in enhancing their "mathematical, pedagogical and educative power" (ibid, p. 899).

Tirosh and Graeber (2003) list three main elements on which several authors agree that professional development programs should include as a necessary condition for change. Accordingly, such programs:

> should reflect the pedagogy that teachers are expected to use with their students, should build teacher collaboration, and should make use of the knowledge and expertise of teachers. (p. 671)

Schwan Smith (2001) considers creating some dissatisfaction with existing teaching practices or outcomes as one of the key features needed for teachers change,. While Schwan Smith considers this dissatisfaction as needed in the first stage, we believe that although such dissatisfaction is important it may occur only in the course of attempts to change teaching practice. Moreover, on the debate whether changing teachers' beliefs should come before attempts to change their practice or vice-versa (Tirosh & Graeber, 2003), in the context we described, changes in teacher beliefs were expected to evolve as they implement the innovative program. We anticipated that hand-in-hand with changing their practices teachers would begin to realize, for example, that even LA students are capable of dealing with mathematical challenges, and that HA students are motivated by 'intellectual rewards' that do not necessarily translate into practical benefits. In fact, we agree with Tirosh and Graeber (2003), that:

> changes in practices occur in a mutually interactive process. Teachers' thoughts influence their classroom practices. Their reflections on these activities and the outcome of changed practice influence teachers' beliefs about mathematics learning and teaching.

> Changes in attitudes and behaviours are iterative. Therefore, well conceived professional learning experiences should consistently address both, knowing that change in one brings about and then reinforces change in the other. (p. 673)

In addition to the features mentioned above, as supporting teacher change, in our view and in the context of our work, there are additional features, which we find critical for enhancing change: institutional support, teacher collaboration, and teachers' commitment to implementation of the program for students.

THE METHODOLOGY EMPLOYED TO STUDY TEACHER CHANGE

Data collection for our study consists of the following: Student achievements in external exams (designed by the project's staff or nationally); teachers' reports and written questionnaires; documented meetings with teachers and staff members; interviews with teachers; classroom observations; conversations with school administers; documented semi-structured conversations between students and teachers from different schools that participate in the program.

Since the context allowed setting rather specific goals for the teachers, we examine the data collected in search of evidence and indicators for the required changes. We identified a number of schools that are particularly successful in implementation of the program and are trying to identify contributing factors and characterize changes in teachers' practices and beliefs. Their development in their mathematical knowledge is assumed indirectly by curriculum they teach, that requires them to deal with more advanced mathematical topics and in a more profound way.

MAIN FINDINGS

Indicators for teacher change were of three types:

1. Students' high performance on tasks that are unique to the program, in terms of content and challenge. The students' performance can be seen as an outcome of a change in their teachers' knowledge and practice: to teach to these goals teachers had to deal with a broader scope of and more challenging mathematics than they had done before.
2. Commonalities across teachers' personal reflections (given independently), in which they indicate similar specific changes in their knowledge, beliefs, and practices.
3. Reports of regional supervisors that are not connected to the project, on their school visits and classroom observations, in which they identified notable differences between 'regular' lessons and lessons within the framework of the project.

We will discuss the strengths and weaknesses of these indicators more thoroughly during our presentation in the conference.

RESEARCHING CHANGE IN PROSPECTIVE AND BEGINNING TEACHERS

Laurinda Brown — Alf Coles

University of Bristol, UK — Kingsfield School, UK

In researching the development of prospective and beginning teachers our focus has been on complex decision-making in the classroom. How is it that new teachers to the profession can act competently and instantaneously, in situations where they literally do not know what to do, without the accumulated wisdom and experience of a practising teacher? As a teacher educator and head of mathematics department in a secondary school our task seems to be to facilitate these teachers to be able to see what is happening in their classrooms in a different way to how they were taught themselves so that they have more choices about how to act. Our take on what 'change' is colours how we collect and analyse data so, firstly, what is change to us?

PERSPECTIVE ON CHANGE AND THEORETICAL FRAME

We change as we open ourselves to, or, in other words, become vulnerable to noticing something different. Most things we do not notice nor remember. We are vulnerable to new distinctions we have done work on and connections might be made. We start to see something we have not seen before and maybe act in a different way in response. We each develop awarenesses over time that deepen our understanding and appreciation of that which we do. Life is not smooth. Personal change is not, as John Mason once commented, like putting on a new suit of clothes. It takes time to integrate the new. We fall over as we try to walk as children. Change is synonomous with learning for us, and, what new teachers of mathematics are learning about is the learning of their students and how to facilitate that. In researching change in (or, 'development of') teachers we can access differences in what they do over time through both observation of lessons and how they speak about what they do during interviews. We look particularly at decision points in lessons, co-observing a video or co-listening to a tape or reflecting together on observation notes. We are interested in teachers reporting, over time, their awarenesses of what they do differently in their classrooms and what seems to engage their students differently in learning mathematics.

Through ten years of joint work in classrooms we have become convinced that groups and individuals become energised by learning through engaging in dialogue *about* the activity in question in relation to a 'purpose' (Brown and Coles, 2000, pp. 168-172). A purpose is a label for a motivation, or idea that can be kept before the mind where there is a link to actions i.e., it can easily be seen in the world. As a teacher, Alf may offer his students the purpose for the whole year of 'becoming a mathematician'. Laurinda may notice an issue arising in a discussion amongst new teachers and give it a label (which may become a 'purpose' for some e.g., 'It seems we are talking about the issue: How do we know what the students know?'). This identifying and naming of an issue is an example of what we term a 'meta-comment' (developed out of Bateson's (1972) writing about metacommunication). It is a comment *about* the conversation. In his teaching, Alf will regularly meta-comment to his students, particularly at the start of the year, in relation to their behaviours that support them becoming mathematicians (e.g., 'Josie has just made a prediction and tested it and found it didn't work, so she's changed her idea – that's a great example of how mathematicians work to spot patterns'). Meta-comments sustain a dialogue about the activity, helping to give meaning and purpose to what individuals do. In our experience individuals in groups where such a level of dialogue becomes commonplace are able develop new relationships. For Alf's students, the new relationship is to the learning mathematics – for Laurinda's student teachers it is to the teaching of mathematics. Laurinda's student teachers work at developing teaching strategies to support the needs that arise as issues from their discussions.

These beliefs and practices place us as enactivist (Reid, 1996; Varela, 1999). As teachers and researchers we believe that cognition is perceptually guided action.

RESEARCHING CHANGE

In 1999/2000 we were involved in a research project funded by the Economic and Social Research Council (ESRC) working with teachers to develop the algebraic activity of students in four year 7 (aged 11) mathematics classrooms in three different secondary schools in the UK. In an earlier project, we worked on ways to characterise students' 'needing to use algebra' and saw this as linked to them being able to ask and answer their own questions. In reflection after lessons, we became aware of a common strand of classification activities which seemed powerful in allowing students to ask questions e.g., when there were two examples to contrast or when students had a disagreement about what they saw or when they wrote an algebraic formula in many seemingly different ways. In activities that had been set up to allow for classification and discrimination, students' mathematical activity seemed natural. The ESRC project was designed to investigate emergent cultures in the classrooms of a group of teachers, some new and some experienced. The project also aimed to help teachers develop teaching strategies that support students using their powers of discrimination, within the context of some 'purpose' given to the students for their work.

Methods

Over one academic year, September 1999 to July 2000 which is split into 3 terms, the project team (3 teachers, 1 teacher-researcher and 3 researchers) investigated the samenesses and differences in the developing algebraic activity in the four classroom cultures through:

- working in a collaborative group, meeting once every half-term for a full day and corresponding through e-mail

- videotaping each teacher for one lesson in every half-term and researchers observing teachers in the classroom at most once a fortnight in teacher/researcher pairs
- every half term interviewing a) each teacher and b) 6 of each teachers' students in pairs, selected to give a range of achievement within the class
- encouraging students to write a) in doing mathematics and b) at the end of an activity, about 'what have I learnt?': photocopies of all the 'what have I learnt?'s are collected from each teacher as well as all the written work of the 6 students interviewed
- each researcher being responsible for viewing the data collected through one or more strands; metacommenting (Laurinda), teacher strategies (Alf), student perspectives (Jan Winter), algebraic activity (Rosamund Sutherland), samenesses and differences in the classroom cultures (Alf and Laurinda).

This structure was to support our looking at what students and teachers did in these classrooms. Schemes of work and organisational structures within the schools were different and it was not our intention to change these. The content of the lessons always has to be decided by the teachers within those structures, but, during the day meetings there was time to plan together, given those constraints, to develop teaching strategies that allow students to use their powers of discrimination.

The teachers gave space within their classrooms for the students to work at making connections and to communicate these to the whole class. The classroom cultures were set up through the teachers sharing with their students the purpose for the year of 'becoming mathematicians'. The teachers made their decisions contingently upon the responses of the students, in relation to the purpose for the year. The teacher cannot be in control of the content nor hear and respond to everything that is happening in such classroom interaction. The teachers set up the possibility of the students making connections through meta-commenting and having 'common boards' used for sharing questions, conjectures and homework.

FINDINGS

So, what do we consider as evidence in the work that we do? Here we will give details of two examples, one from observations and one from a group interview.

In jointly observing extracts from videotapes of the teachers on the project we gain access to teaching strategies noticed by the teachers that they are motivated to put into practice. For instance, the following simple strategy was first noticed through discussing a lesson in this way. It was observed that when a teacher is collecting information from a group, students saying 'I've got another one' almost always meant 'another different' example. Examples from the same class of objects were not offered unless specifically requested. Therefore, when setting up this whole class interactive style of working with a new group, these teachers had choice to ask for 'another, different one' or, 'another, similar one'.

For the project above, the most important data at the end of the project for effective teaching strategies was a group interview of the teachers. Laurinda asked the group to talk about what had changed for them in their practice over the time of the project and to work at finding common aspects that had allowed them to develop their practice effectively. The conversation was taped and analysed. This conversation, offering a final snapshot of the effects of the project, was strong evidence for common development and more generally effective teaching strategies.

One powerful and easily integrated strategy was that of seeing the importance of the act of making distinctions. This was commented on in the conversation leading to an energised discussion amongst the teachers. Starting a lesson by offering an image of more than one example of something and asking students to comment on what they observed could lead to group discussions of concepts without the teacher needing to begin with an explanation.

Another aspect that we have found find useful in collecting data is interviewing visitors to teacher's classrooms. There is a climate of teaching being a less isolated activity in the UK as senior members of staff, teaching assistants, university lecturers and local education authority visitors pass through classrooms. Laurinda would interview these people who, in different schools and years, said similar things about the difference they noticed in the way the children were engaging in mathematics, relative to their own previous experiences of classrooms and, sometimes, in relation to how they had observed the teacher working before the project.

In the end what we were looking at were drivers of change rather than particular tasks or lessons that worked. These teaching strategies seemed to work to develop a culture in the classroom in which children were doing mathematics:

- meta-commenting by the teacher focusing attention of the group on behaviours in the room where students were being mathematical
- students being encouraged to ask and answer their own questions relevant to a group project
- collection of students' thoughts after a homework and discussion of each point arising before a new point was raised
- actively focusing students on making distinctions.

EFFECTIVENESS AND ETHICS

We have worked with each other over a long period of time and like to work with others over extended periods. As we have written above, we gain evidence of change through noticing differences in classroom observations and in interviews with teachers over time. We do not want or need to look at, for example, examination performance of students over time in order to gain a more 'objective' measure of effective change, nor to analyse the tasks or activities that new teachers are using.

This is a question of the ethics of our approach to working with each other and with other beginning teachers. We engage with the issues teachers bring, without judgement. We offer the distinctions we notice - e.g., that one of us would have responded in a different way to how a teacher did at a certain point in a lesson. This can never be to say that an alternative approach would have been better (how could we ever know?). We have learnt to recognise any judgements that do arise internally merely as indications of difference - and aim to track back to what this difference is, so that the awareness can be offered without attachment and lead to a more complex set of awarenesses for all the research group.

As a result of our ethical stance on working with teachers, issues of reporting are less problematic than they can be in projects where teachers are not involved in the analysis. We would never be reporting on, for example a classroom incident, in a way that had not been discussed with the teacher. We are not in the business of judging what happens in classrooms from some 'other' perspective. Analysis of data is not, for us, something that happens at the end of a project. Analysis happens throughout and informs our next actions.

SUMMARY AND CONCLUSIONS

Markku S. Hannula

The research on teacher professional development or teacher change is a large area with different theoretical, methodological and practical approaches. It is clear that the five contributions to this forum can only provide examples of the research done in this field, not cover it. Nevertheless, within this small sample, there are some commonalities that are worth noticing as an indication of the field in general. The studies are exploratory in nature, i.e., there are no clear hypothesises that could be tested. Instead, the aim of the research has been to describe and understand the process of change. This is indicative of the low level of our understanding so far. The research aim is also reflected in research methods that are mainly qualitative in nature, although some quantitative measures have been used.

Although the contributions come from seemingly different frameworks, they acknowledge the Piagetian perspective of change as the learning of an individual. The papers all mainly focus on the individual teacher as the initiator of change. However, in most, the collaboration between participating teachers (teacher students) is an essential aspect of the intervention and at least Zaslavsky and Linchevski explicate institutional support in their study. Furthermore, the mathematics classroom is, in most cases, taken as a unit of analysis when evaluating the effects of the intervention.

All methods except the therapeutic approach focus on students' learning, which is seen in relation to teacher action. Brown & Coles and Hannula, Liljedahl, Kaasila &

Rösken focus explicitly on teacher's reflection while the other three have embedded reflection more or less implicitly in their intervention.

One thing that is in common to the studies reported here is that the duration of the intervention has been at least one semester long. This is in clear contrast with, typically, 1-2 days long in-service training courses. It is perhaps not purely coincidental that all studies also claim to have clearly observable effects due to the intervention. If effective in-service training requires such investment of time, we will face serious problems if we aim to reach all mathematics teachers. Some of the intervention projects were remarkably large, yet there are not enough resources to provide such professional development courses for all teachers. This asks for other solutions than teacher educators providing long-lasting courses for the selected few. One fruitful approach is to engage innovative mathematics teachers as experts or facilitators (teacher-researchers) for new projects, in the cases of Alf Coles and the anonymous teacher in Lo & Runesson.

The issue of agency is dealt with differently in different approaches. Hannula and his colleagues suggest a distinction between agency of treatment and agency of change and, accordingly, talk of empowering students or occasioning change. A similar stance is clearly observable in the approach by Brown and Coles and probably in the Goldsmith and Seago study, although they do not explicate it. In the Learning Study approach reported by Lo and Runesson, the group holds the agency collectively. Each participant in a Learning Study group is required to tryout their teaching action only as a contribution to the group – with no commitment to adopting certain teaching methods. Zaslavsky and Linchevski, on the other hand, report a study where they acknowledge they have taken the agency away from the teachers as "the program dictated certain changes in teacher practice".

The therapeutic approach is seen to be ethically problematic while only Brown and Coles of the other approaches see little reason for considering such issues. True enough, anxieties are a tender spot for anyone, especially if they are in relation to professional competence and possibly under evaluation. In this sense, it is understandable that the ethical problems are more pronounced when studying the mathematics anxiety of future teachers. However, isn't the professional competence of an established teacher in similar risk of public evaluation when we report studies of teachers' professional development? The ethical issues of reporting and the validity of data are relevant questions for reasons similar to those elaborated by Hannula et al. Thus, the stance taken in Brown and Coles, where the teacher's consent for reporting is expected is reasonable.

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DISCUSSION GROUPS

DG01: INDIGENOUS COMMUNITIES AND MATHEMATICS EDUCATION: RESEARCH ISSUES AND FINDINGS

Coordinators
Annette Baturo, QUT, Australia
Hsiu-Fei Lee, National Taitung University, Taiwan

The aim of this Discussion Group is to build a community of PME members who have researched Indigenous mathematics education issues (or who would like to undertake research in the field but are unsure of the protocols involved) in order to support Indigenous mathematics learning students' mathematics outcomes and refine methodologies appropriate for the variety of Indigenous communities.

Research in Indigenous mathematics education has complexities that go beyond that of mainstream mathematics education. Smith (1999) argues that research should focus on improving the capacity and life chances of Indigenous peoples and that such research should be community-driven, collaboratively planned, executed and analysed in order to promote real power-sharing between the researched and the researcher. This second Discussion Group would like to focus on one or more of the following issues:

- the building of Indigenous community capacity through the development of mathematics programs;
- transition to school and early childhood mathematics for Indigenous children and their families;
- culturally-based mathematics programs for Indigenous children - advantages and disadvantages;
- the development of mathematical literacy in Indigenous children;
- respect for Indigenous knowledge / desire for school knowledge

DG02: MASTER TEACHERS IN DIFFERENT SYSTEM CONTEXTS

Yeping Li, Texas A&M University, USA

JeongSuk Pang, Korea National University of Education

Rongjin Huang, University of Macau, Macau SAR

World-wide efforts to improve students' mathematics achievement have led to increased interest in teaching and teacher education practices in some high-achieving education systems, especially those in East Asia. In particular, TIMSS video studies in 1995 have led to findings and further inquiry about specific teaching culture that is formed and nurtured in East Asia. However, there is a lack of research efforts in understanding master mathematics teachers in high-achieving education systems. Such an examination becomes important for understanding what aspects count as an important part of teachers' repertoire in a specific high-achieving education system, how master teachers may function in influencing the formation of a specific teaching culture, and what challenges master teachers may face in an era of education reform in a system context. This discussion group is thus proposed to bring together interested scholars from different education systems to discuss relevant issues on master mathematics teachers.

The discussion group will be organized as a two-part activity. During the first part, three researchers will present brief (about 10 minutes each) overviews and lead discussions of relevant research on three aspects: (a) learning about master mathematics teachers' expertise in past research, (b) master mathematics teachers' adaptation to new challenges in an era of education reform, (c) master mathematics teachers' views and beliefs about effective teaching. The participants will then join small group discussions for the rest of the session as the second part. Based on the three general aspects that have been discussed in the first part, the discussion in small groups will begin with the following questions but will follow the interests of the participants:

- What makes master mathematics teachers in a specific high-achieving education system?
- How may master mathematics teachers function to influence the formation of a specific teaching culture in an education system?

We hope that these questions will be examined across different high-achieving education systems to help us all step outside of our own culture and experience and develop a broader perspective. After the small-group discussions, all participants will come together to generate a collective summary and synthesis of the small-group discussions. A list of potential research questions will be generated/selected and interested participants will be organized to develop further collaborative research activities on this topic after the meeting.

2007. In Woo, J. H., Lew, H. C., Park, K. S. & Seo, D. Y. (Eds.). *Proceedings of the 31st Conference of the International Group for the Psychology of Mathematics Education*, Vol. 1, p. 184. Seoul: PME.

WORKING SESSIONS

WS01: MATHEMATICS AND GENDER: DISCOVERING NEW VOICES IN PME

Proposal for a Working Group

Joanne Rossi Becker, San José State University, USA
Helen Forgasz, Monash University, Australia
KyungHwa Lee, Korea National University of Education, South Korea
Olof Bjorg Steinthorsdottir, University of North Carolina, USA/Iceland

In 2005 and 2006 we had lively discussion group sessions centered on several areas of interest related to gender and mathematics. Noting that this area of research differed greatly by country, we focused on intervention strategies that might be used in countries such as South Korea with large extant gender differences in achievement; how to study linkages among gender, ethnicity and socio-economic status; and, setting a research agenda for future work on gender and mathematics. We discussed the policy issues that influence the collection of data necessary for the study of gender differences/similiarities, and focused on possible new methodological approaches and theoretical frameworks that would enable us to investigate difficult and unresolved issues concerning gender, especially as they relate to ethnicity and socio-economic status.

With the success of several years of discussion groups, and looking ahead to ICME XI in 2008 where there will be Topic Group sessions on gender and meetings of the International Organization of Women and Mathematics Education, we propose a Working Group for PME 31 in which we will invite participants to bring current work or work in progress related to gender and mathematics. We will share papers, solicit feedback and critique participants' developing papers, and develop long-range goals for participation in ICME XI.

Activities

Using the PME newsletter and listserve, we will try to determine who would like to informally present some work, fully or partially developed, to the working group.

Beginning with brief introductions, we will break up into smaller groups on Day 1 around interest areas pre-determined by the organizers. These groups will discuss and critique and offer suggestions to participants who have brought work to share or research ideas with which they want help.

On Day 2, main ideas from the smaller groups will be shared with the whole group, and the Working Group will strategize about how to organize and collaborate to maximize our participation in the sessions related to gender and mathematics at ICME. We will collect participants' email addresses so that all may keep in contact to continue collaboration after the conference.

2007. In Woo, J. H., Lew, H. C., Park, K. S. & Seo, D. Y. (Eds.). *Proceedings of the 31st Conference of the International Group for the Psychology of Mathematics Education*, Vol. 1, p. 187. Seoul: PME.

WS02: TEACHERS RESEARCHING WITH UNIVERSITY ACADEMICS

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Charles University in Prague, Czech Republic,

The University of Queensland, Australia

The PME 30 Research Forum with the same title presented several models of the scientific collaboration of teachers of mathematics and university academics, their forms, advantages and limitations. The discussion not only enriched the questions and topics presented in the description of the Research Forum in the proceedings but certainly opened new perspectives for collaboration of those who are interested in further pursuing this type of research.

The Working Session is the follow-up of this Research Forum. It aims to develop the collaboration of teachers and university academics – with a broader, international dimension. During the first session, examples of successful research collaborations between teachers and university academics will be presented and discussed. Participants will be invited to identify and compare the enabling factors that contributed to the success of each partnership from the perspective of a teacher and a university academic. The benefits of such collaboration will be elaborated, together with possible difficulties and tensions and how these might be overcome.

In the second session, the results of the first session discussions will serve as the basis for choosing topics for this type of research. The coordinators of the working session will engage with participants in small groups, assisting them to sketch out research proposals. One or two proposals will be elaborated to such an extent that it will be possible to realise them after PME 31.

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2007. In Woo, J. H., Lew, H. C., Park, K. S. & Seo, D. Y. (Eds.). *Proceedings of the 31st Conference of the International Group for the Psychology of Mathematics Education*, Vol. 1, p. 188. Seoul: PME.

WS03: GESTURE, MULTIMODALITY, AND EMBODIMENT IN MATHEMATICS

Coordinator: Ferdinando Arzarello (Turin University)
Markku S. Hannula (University of Helsinki and Tallinn University)

The goal of the Working Session will be to continue and deepen the investigation of mathematical thinking, learning, and communication by considering the variety of modalities involved in the production of mathematical ideas. These modalities include gesture, speech, written inscriptions, and physical and electronic artefacts. The central purpose will be to examine how basic communicative modalities such as gesture and speech, in conjunction with the symbol systems and social support provided by culture, are used to construct mathematical meanings. In addition, the role of unconscious conceptual mappings such as metaphors and blends will be investigated in relation to gesture and the genesis of mathematical concepts. Relevant theoretical and empirical work has been carried within cognitive linguistics (Lakoff & Núñez, 2000; Fauconnier & Turner, 2002), semiotics (Radford, 2002; Arzarello, 2006) and psychology (McNeill, 1992, 2000; Goldin-Meadow, 2003). Themes and questions to be addressed include:

How do gestures relate to speech, writing (eg, of formulas), drawing, graphing and other modalities of expression during mathematical learning and problem solving?

What is the role of gesture within different mathematical settings and speech genres, eg, during classroom instruction, small group problem solving, explaining, proving, etc.?

How can gestures be used to condense and manage information during social interaction?

How are conceptual mechanisms such as metaphors and blends involved in students' cognitive processes while learning and doing mathematical activities? How do gestures and unconscious conceptual mechanisms relate to external representations and technologies used in mathematical activity?

The Working Session will consist primarily of small groups working together to: (1) make progress in answering one of the above (or a related) question; and (2) engage in collaborative analysis of videotaped or other data showing the use of various modalities in mathematical activity. The Session will be organized in advance through the discussion group Theory of Embodied Mathematics, and a website to be created at the University of Turin.

Te Working Session will continue the work done in the last years at PME meeting. Unfortunately no one of the last organisers (Laurie Edwards, Janete Bolite Frant, Ornella Robutti) will be in Seoul: last year in Prague it was decided by the group that Arzarello will submit the application this year.

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WS04: TEACHING AND LEARNING MATHEMATICS IN MULTILINGUAL CLASSROOMS

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Mamokgethi Setati, University of the Witwatersrand, South Africa
Susan Staats, University of Minnesota, USA

Multilingualism is a widespread feature of mathematics classrooms around the world. The nature of this multilingualism is, however, highly diverse. In some settings, two or more languages are shared and used by teachers and students, both within the mathematics classroom and in wider society (e.g. in much of Africa or Asia). In some other settings, by contrast, whilst a range of diverse languages may be represented amongst the students, the teacher may know little about any of them (e.g. in North America or Europe). In the first kind of setting, teachers may know a great deal about the different languages used by their students (including having high proficiency in speaking them). In the second, it is possible for teachers to investigate aspects of their students' other languages, such as, for example, obtaining mathematical vocabulary lists (an approach that is not likely to involve any degree of proficiency). The aim of the working group is to explore the question:

> How (if at all) can teachers make use of what they know or can find out about their students' languages in seeking to support mathematics learning?

ACTIVITIES

Over the two working sessions, our exploration will be stimulated by transcripts and video data representing the two settings mentioned above, including: an investigation of Somali mathematical language conducted within an immigrant community in the USA; an example of the planned and deliberate use of multiple languages in a secondary mathematics classroom in multilingual South Africa; and an example of an expatriate mathematics teacher working in Pakistan.

By exploring the above question, we hope to shed light on an underlying issue, of whether or not mathematics teachers need to know or learn about the languages their students bring to the classroom and how this knowing or learning may inform their practice in supporting mathematics learning.

2007. In Woo, J. H., Lew, H. C., Park, K. S. & Seo, D. Y. (Eds.). *Proceedings of the 31st Conference of the International Group for the Psychology of Mathematics Education*, Vol. 1, p. 190. Seoul: PME.

SHORT ORAL COMMUNICATIONS

ON THE NOTION OF COHERENCE IN TECHNOLOGY-ENABLED PROBLEM POSING

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Problem posing has long been recognized as an important tool in the teaching of mathematics. The advent of technology brought about the recognition of the potential of computing to enhance this tool. Most of the existing research on technology-enhanced problem posing concerned dynamic geometry environments within which multiple examples can be explored and, as a result, new hypotheses (or, alternatively, problems) can be formulated. Recent advances in the use of a spreadsheet in mathematics education enable problem-posing activities to be extended to other areas of pre-college mathematics. Already at the elementary level, the appropriate use of the software makes it possible to turn a routine problem into a mathematical investigation (Abramovich & Cho, 2006). Through such an investigation, the numbers involved become parameters that can be altered and tested in a problem-solving situation and then chosen to signify the completion of the problem-posing phase of the activity. This requires teachers' higher order thinking skills that can only be developed through special training.

The main argument of this paper is that the use of technology in problem posing cannot be adequately understood without attending to the notion of didactical coherence of a problem. The proposed notion, for which Bonotto's (2006) discussion on problem posing may serve as an illustration, is structured by three major components: numerical, contextual, and pedagogical. Numerical coherence is related to a problem's formal solvability without regard to context. Contextual coherence comes into play when numerical solution should be interpreted in terms of a context within which problem posing occurs. Pedagogical coherence refers to the problem's appropriateness for a specific grade or developmental level. The paper argues that the notion of didactical coherence has the potential to inform and improve mathematical thinking of elementary pre-teachers engaged in spreadsheet-enabled problem-posing activities.

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THE CONNECTION WITHIN MATHEMATICAL SEQUENCES PERFORMED BY ELEVENTH GRADE STUDENTS THROUGH THE STORY AND DIAGRAM METHOD

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This study focused on the connection of mathematical sequences performed by eleventh grade students through the Story and Diagram Method (SDM). Effectiveness of the selected mathematics teaching approach, SDM, was explored in a secondary school in the northeast of Thailand. Students' mathematical idea and understanding of mathematical sequences and idea of connection, using NTCM (2000) as a basis, revealed through their task and mathematical reasoning presented in the class.

The target group consisted of 41 students. The data collection was performed during mathematics classroom by observation through teaching and learning performance. Field notes, audiotapes, videotapes and interviews were also conducted. The student ability of connection thinking pattern between numbers and things were observed and analyzed. Case studies from the classroom were presented as exemplars.

The results found that, while telling stories of their interested together with diagram drawing, the students had excellent opportunity to expose themselves with critical thinking process. Thus, using SDM and teaching techniques, the students were emerged into critical dialogue and discussion. As a result, they gained more insight and perceived the connection between "adding" and "multiply", and they could understand the interchangeable of "common differences" and "ratio". Furthermore, by using the SDM, the students learned to link mathematical idea from arithmetical sequences to geometrical sequences, and subsequently forwarding the idea of concrete to the more abstract mathematics of which is very useful for daily life (Kongtaln, 2004).

Keyword : Story and Diagram Method, mathematical connection, sequences

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THE RETENTION OF MATHEMATICAL CONCEPTS IN MULTIPLICATION IN THE INQUIRY-BASED PANTOMIME INSTRUCTIONS

Bae, Jong soo	Park, Do-yong	Park, Mangoo
Seoul National University of Education	Illinois State University	Seoul National University of Education

This investigation examined first grade students' retention of concepts and skills in multiplication when using an inquiry-based pantomime approach as intervention in an elementary school. Posttest and retention test after three months were administered to students both in inquiry-oriented pantomime and traditional classes. The results showed that students in the inquiry-oriented Pantomime class longer retained conceptual knowledge of multiplication and equal proficiency in qualitative/pictorial explanations, compared with students in the traditional instructional methods.

The purpose of this study is to investigate first graders' understanding of concepts and skills in multiplication and contribute to the body of research in the understanding of mathematical concepts and skills. This study was conducted to compare first graders' understandings of multiplication concept and graphical representation skills for solving multiplication problems between students in inquiry-oriented pantomime (IP)(Bae & Park, 2004) and traditional class (TC). This research topic reflects the effects of instructional approaches to longer-term retention of mathematical knowledge and skills. Little is known about the retention of learned knowledge in elementary school mathematics.

Students' participation in the IP class produced a positive retention of conceptual knowledge of multiplication as seen in student responses to conceptual knowledge of multiplication and qualitative/pictorial explanation tasks compared to students' retention in its TC counterpart. In particular, as shown in the interviews, students had a common resisting misconception. Their explanations of life examples and drawings also showed unchanged of it.

Long-term retention of school knowledge has been an issue in education across the subject areas in terms of affecting factors. One of the critical factors is an instructional strategy. In mathematics, there are few studies which consider how to promote longer retention of conceptual knowledge in students. This study aimed to investigate the effects of active involving learning environment compared to those of the traditional class. The results of this study will add to the research area of retention of school knowledge in mathematics. In addition, the inquiry-based pantomime approach needs to practice in various levels and areas.

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CHILDREN'S JUSTIFICATION OF MULTIPLICATIVE COMMUTATIVITY IN ASYMMETRIC CONTEXTS

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The commutative property is often taught as a pattern or rule to reduce multiplication facts to learn. In this study, I investigate students' justification of the commutativity in equal grouping and price problem contexts.

THEORETICAL FRAMEWORK

There is conflicting research on the role of contexts in children's understanding of multiplication problems. Some studies (e.g. Baek, 2002) indicate that multiplication contexts are helpful for children to construct sophisticated strategies and to understand distributive and associative properties, and others (e.g. Schielmann et al, 1998) assert that asymmetric context, such as price, hinder children from developing strategies beyond repeated adding, and from using commutative property. In this study, I further examine students' use and justification of the commutativity in asymmetric contexts, including equal grouping and price.

METHODS

Fifty-eight fourth- and fifth-graders were individually interviewed with eight multidigit equal grouping and price problems. Children also asked if a commutative strategy, adding 127 six times, would solve a price problem ,127 children paying $6 each, and why.

RESULTS & CONCLUSION

Among 303 valid strategies (63%) for 8 word problems, I identified 267 strategies (88%) that children treated multiplier and multiplicand differently in, and 36 strategies that children did not make distinction between them. Twenty-two of 58 children agreed that the given commutative strategy would solve problem; 6 of the 22 children did not give any justification, 5 provided other examples, and 11 explained that adding 127 is like all children paying $1 at a time. However, only 1 of the 11 children said that it would work for any numbers. This finding indicates that research and instruction should help children justification of the property, not just recognition.

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EXPLORING STUDENTS' REASONING WITH ALGEBRAIC EXPRESSIONS

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Researchers (e.g. Bell, 1995) have emphasized the role of activities which give meaning and context to understand symbols in algebra and their manipulation/ transformation. Research has also pointed out various means of contextualizing algebra, by generalizing and formalizing patterns and relationships within the domain of mathematics or using practical situations outside the domain of mathematics and requiring which require modeling, representing and problem solving. In a design experiment we have carried out, our effort has been to introduce to students (n=31) studying in grade 6 both the aspects of algebra: syntactic and semantic. In this study we first developed among students understanding of syntactic aspects of expressions in the context of reasoning *about* expressions. This was followed by applying this understanding in contexts which required the use of algebra as a tool for purposes of generalizing, predicting and proving/ justifying which can be termed as reasoning *with* expressions.

Although many tasks were used in the study from time to time to situate the use of algebra, here we would discuss their performance and understanding in two of the tasks: pattern generalizing and think-of-a-number game. Both the tasks required the students to use their syntactic knowledge of transforming expressions as well as appreciate the meaning and purpose of algebra. Students' responses were analyzed for their understanding of the letter, ability to represent the situation using the letter and their appreciation of the manipulation of expressions to arrive at conclusions regarding the situation. It was hard to test this knowledge through a written test. The students showed appreciable understanding of the above criteria in the classroom discussions and during the interview with a subset of the students. Also these tasks proved to be rich ground to discuss issues of representation and syntax which reinforced the ideas encountered in the first part of the study. The students showed understanding of the letter as standing for a number, that situations can be represented using a letter and that expressions could be simplified to draw conclusions about the situations. But their knowledge of syntactic transformations on expressions was not immediately helpful as these tasks required knowledge other than manipulation. It required students to understand ideas about proving and justifying for a general case, make correct representations using conventional symbols and appreciation of the need for manipulation.

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MATHEMATICAL MODELLING AND REFLEXIVE DISCUSSIONS

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There are different emphases in developing Mathematical Modelling activities in the classroom. They could be put in action through of the discussions produced by the teacher and students.

In previous paper (Barbosa, 2006), I've underlined that those discussions that play a role in building the mathematical models may be classified in three types:

- the mathematical discussions: refer to the pure mathematical procedures and concepts;
- the technological discussions: refer to the translation of the elected phenomenon to study in terms of mathematics;
- and the reflexive discussions: refer to the nature of mathematical models and the influence of the criteria used in the results.

From a socio-critical point of view (Barbosa, 2006), there is a strong interest in calling for students to reflect about the nature of the mathematical models, so it's important to motivate the production of reflexive discussions in the classroom.

This study reported here analysed how reflexive discussions are produced and how students develop them when they are engaged in Mathematical Modelling activities. Taking a qualitative approach, two groups of students from a Brazilian public school were filmed. The data have been analysed with inspiration in grounded theory.

Partial conclusions point out that reflexive discussion may take place in Mathematical Modelling activities when students compare their different mathematical models and results. It calls them to review the Modelling process and to notice how different hypothesis create different models and results, giving evidences about a non-neutral nature of the mathematical models.

This study points out the teachers should provoke reflexive discussions in the Mathematical Modelling through of inviting students to compare their results in order to create a window for reflecting about the nature of the mathematical models.

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ADDITION AND SUBTRACTION OF NUMBERS UP TO 100

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The importance of research into children's thinking and learning processes is indisputable. One focus considers prior knowledge as a basis for developing knew knowledge. The data presented were collected at the beginning of year 2. At the time of these interviews the children (7-year-olds) had only been taught numbers and calculation strategies up to 20, they had not been taught anything about numbers and strategies up to 100. Therefore informal strategies predominate.

INTRODUCTION - METHOD

This paper only presents one part of a longitudinal study: the interviews at the beginning of year 2. In this study we examined different aspects of children's problem solving behaviour. Their success rate and methods were analysed. The term *method* here describes whether the children use objects, fingers, notation or if they solved the tasks only in the head. The solution strategies were first categorised into either counting or calculating strategies. When calculating strategies were used, they were then classified into the following major categories: *Split*, *Jump, Split-Jump* and *Holistic*. The subjects were 100 children of varying mathematical abilities. They were presented with 10 addition und 10 subtraction problems. The instrument used was Piagets revised clinical interview technique. The percentages refer only to the number of tasks which were presented. In the first interview not every task was presented since the interviews should not last too long.

FINDINGS

As in many studies of children starting school this study of children in year 2 found high levels of competence and heterogeneity. The children solved correctly 62.1% of the tasks presented. But there is a considerable range of ability amongst the children.

55.1% of the tasks were solved without any help from objects, fingers or notation, while 32.1% of the tasks were solved using objects. The use of fingers was less prevalent at only 9.2%. Pen and Paper were only used for 0.3% of the tasks.

The children solved most of the tasks using counting strategies (44.3%), calculation strategies were used for 38.9% of the tasks. (Other strategies were mixed strategies, guess or known facts). For only 7 of the 20 tasks it was possible to use every calculation strategy. So in analysing the calculation strategies only the 7 two digit tasks are considered. It can be observed that *every* strategy was used. *Split strategy* is the dominant strategy with 14.2% (M=0.82, SD=1.4) followed by *split-jump strategy* with 9.0% (M=0.52, SD=1.0). The *jump strategy* was used for 6.9% (M=0.4, SD=0.8) of the tasks. *Holistic strategies* were hardly employed.

2007. In Woo, J. H., Lew, H. C., Park, K. S. & Seo, D. Y. (Eds.). *Proceedings of the 31st Conference of the International Group for the Psychology of Mathematics Education*, Vol. 1, p. 199. Seoul: PME.

VIRTUAL CENTER FOR MODELING: A PLACE FOR EXCHANGE AMONG RESEARCHERS AND TEACHERS

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The Research Group in Information Technology, other Media, and Mathematics Education (GPIMEM) has been investigating aspects of the Internet in the process of knowledge production since 1997. Continuing online education courses for mathematics teachers have been the scenario for various studies, some of which have been presented at PME (Borba, 2005; Borba & Zulatto, 2006). We have preliminary results from other ongoing projects, including the interdisciplinary TIDIA-Ae project, in which we have been participating with groups from computer science and other areas within the field of education, since 2004, to construct a new virtual environment for implementing learning actions.

In this presentation, we describe our efforts to create the Virtual Center for Modeling (CVM) a virtual community of teachers and researchers who develop modeling in the Brazilian tradition (Borba & Villarreal, 2005). Working with modeling can provoke tension among teachers since, using this pedagogical approach, they are required to deal with the unexpected, beginning, for example, with the students' selection of the themes to be addressed. In this virtual environment, teachers and researchers "meet" to discuss the implementation of such an approach at various instructional levels and to provide mutual support. Researchers study the tools used for the interaction and their role in providing support for the teachers' work. Studies are being designed to investigate the obstacles that present themselves to such a community and changes with respect to research methodology in virtual environment. Thus, the CVM is a locus of research and professional development.

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AN ENACTIVE INQUIRY INTO MATHEMATICS EDUCATION: A CASE STUDY OF NINE PRESERVICE PRIMARY SCHOOL TEACHERS

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It is not uncommon for mathematics learners and teachers to experience a lack of mathematics confidence (Pehkonen, Hannula, Kaasila & Laine, 2004). Research has found that the beliefs teachers have regarding mathematics, including their level of mathematical confidence, have a significant impact on their practice of teaching, and hence on the confidence of their students (Uusimaki & Nason, 2004).

A group of nine pre-service primary school teachers from the University of Cape Town who self-identified themselves as experiencing a lack of mathematics confidence were interviewed. How the participants understand the notion of mathematics confidence and the reasons why they lack confidence was explored. The results of the study indicate that the pre-service teachers understand mathematics confidence to be the ability to do mathematics, as well as understanding the processes involved. In order to understand why they lack confidence, their previous experiences in the mathematics classroom were explored. The experiences leading to a lack of mathematics confidence highlighted in this study include the teacher's pace in the classroom, comparison and competition with other learners, and having a traumatic experience in the classroom. The negative beliefs of family members about mathematics also seemed to impact negatively on their level of confidence. In addition, mathematics anxiety was an important aspect of their prior experiences and it is shown to be intricately intertwined with a lack of mathematics confidence.

The theory of Enactivism, based on Complexity Theory, has been used to explore the concept of mathematics confidence as well as the students' prior experiences in the subject. New perspectives and possibilities for action in teaching and learning offered by Enactivism are also discussed. It is hoped that these new possibilities for action will assist students and teachers cope with this lack of mathematics confidence.

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DIVERSITY OF PROBLEM SOLVING METHODS BY HIGH ACHIEVING ELEMENTARY STUDENTS

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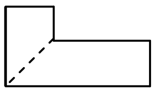
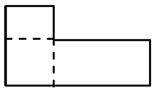
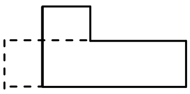
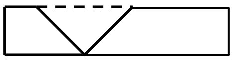
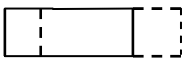
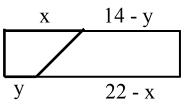
This study investigates how high achievers solve a given mathematical problem. The problem, which comes from ‘SanHakIbMun’, a Korean mathematics book from eighteenth century, is not used in regular courses of study. It requires students to determine the area of a gnomon given four dimensions(4,14,4,22). The subjects are 84 sixth grade elementary school students who, at the recommendation of his/her school principal, participated in the mathematics competition in Jinju, a Korean city of 300,000 inhabitants. The various methods used by these students are as follows.

Methods			Characteristics	#
Numerical Approach	N1		Depending on the only one case relative to the figure	28
	N2		Depending on the only one case irrelative to the figure	10
	N3		Depending on more than one case	1
Decomposing-Reconstructing Approach	R1		$\{(14+22)\times4\}\div2$	2
	R2		$(14\times4)+(4\times4)$	8
	R3		$4\times(22-4)$	9
	R4		$(22\times4)-(8\times4\div2)$	2
	R5		$14\div4=3.5$, in fact 4.5 squares $\therefore\ 4\times4\times4.5$	1
	R6		$x=y+4$ $\Rightarrow 18(\text{length})\times4$	1
Incorrect or No response				22

The methods used can be classified into two approaches: numerical and decomposing-reconstructing, which are subdivided into three and six methods respectively. Of special note are method R6, which assumes algebraic feature, and R1 and R3, which appear in the history of eastern mathematics. Based on the result, we may observe a great variance in methods used, despite the fact that nearly half of the subject group used the numerical approach.

 2007. In Woo, J. H., Lew, H. C., Park, K. S. & Seo, D. Y. (Eds.). *Proceedings of the 31st Conference of the International Group for the Psychology of Mathematics Education*, Vol. 1, p. 202. Seoul: PME.

DOES STUDENT SUCCESS MOTIVATE TEACHERS TO SUSTAIN REFORM-ORIENTED PEDAGOGY?

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This presentation discusses student success as an effective mechanism to motivate and sustain change for teachers implementing reform-oriented teaching approaches. The focus of these teaching approaches is to encourage students to do the mathematical thinking and thereby increase their overall achievement in mathematics. It is this outcome of increased student achievement and student enthusiasm for the new teaching approaches that became a prime motivator for teachers in this reported study to sustain changes to their teaching.

The overall study explored the extent to which eight primary school teachers incorporated the mathematical reforms introduced in the New Zealand Numeracy Development Projects [NDP] in-service professional development. For most of the teachers the reform-oriented, inquiry-based teaching approaches took an extended time of ongoing support to internalise and consolidate. Despite the initial difficulties accommodating the new mathematical knowledge and pedagogy, most of the teachers reported that their students' positive responses to the new teaching approaches was a key motivator to continue the NDP reforms.

For the teachers in this study the firsthand experience of incorporating aspects of the professional development (e.g. eliciting student strategies and encouraging student discussion) in their classroom with their students allowed them to see the immediate success the programme had in engaging their students' learning. Research on teacher change reports that teachers' beliefs and attitudes affect the quality of engagement in professional development and suggests "the most significant changes in teaching attitudes come after they begin using a new practice successfully and see changes in student learning" (Guskey, 1985, p. 1).

The richness of the teachers' voices in recognising the importance of student achievement and enthusiasm as central to successful professional development makes worthwhile academic commentary.

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MATHEMATICAL CLASSIFICATION LEARNING ACTIVITIES FOR SHAPES RECOGNITION - TRIANGLES AS AN EXAMPLE

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A national survey (Chen, 2003) of grade 7 to 9 on shapes recognition presented several important factors affecting students' performances: semantic comprehending, prototype effect, exclusive thinking, and 3-D projection effect. How to facilitate students facing these factors on learning shapes recognition becomes an important issue for teaching.

This report will present a sequence of two mathematical classification learning activities of classifying triangles as an example for enhancing shapes recognition. The designing guide of classification activities consists of five phases: defining a *domain*, explaining the *foundation*, setting *criteria*, *naming/defining* each class, and *cross-relation* between different classifications. The first four are for activity one and the last one is for activity two. *Activity one* is a classification of triangles into acute, obtuse and right triangles. The domain is a collection of triangles. The foundation is that any triangle at least has two acute interior angles. The criterion of the third angle is <, >, or =90° is set to classify. And then came to the next step: naming acute, obtuse, or right triangles. *Activity two* is aimed to construct cross-product table for separating triangles with two different classifications. The learning kits are 20 cards with different triangles on each one, and some strings of two different colours. Grouping by manipulating different set of triangles with two classifications: one is acute, obtuse, and right triangles and the other is equilateral and non-equilateral triangles and then constructing cross-product tables as a goal of this activity. Students' works are exhibited in the following figure and tables. Students are guided to shift their works: Figure 1 into Table 1or 2. In table 1 and table 2, students have to construct examples for cells with ticks and explain the reasons for cells without ticks.

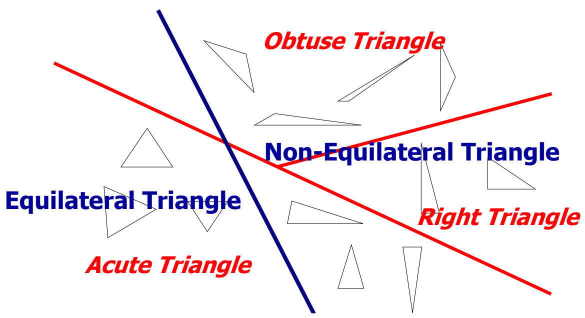

Figure 1

Table 1

	Equilateral Triangle	Non-Equilateral Triangle
Acute Triangle	V	V
Obtuse Triangle		V
Right Triangle		V

Table 2

	Isosceles Triangle	Non-Isosceles Triangle
Acute Triangle	V	V
Obtuse Triangle	V	V
Right Triangle	V	V

In the pre-test in a 7th grade class of 33 students, only 24.4%, 30.3%, and 51.5% of students can recognize acute, obtuse and right triangle correctly, however, after the two activities, 81.8%, 84.8%, and 90.9% of students can correctly recognize three kinds of triangles in the post-test. Such classification learning activities are able to enhance students' performances on shape recognition.

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USING FALSE STATEMENT AS A STARTING POINT TO FOSTER CONJECTURING

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Many issues related to mathematical conjecturing have been concerned, like: conjecturing as a necessary process of problem solving; the strategy to foster conjecturing; the continuity from conjecturing to proving and the role of examples in the process of conjecturing and proving (Polya, 1962; Lakatos, 1976; Mason, 1982). Based on the literatures, one important issue is what we can do for students to develop their conjecturing abilities. In this study, we developed a worksheet according to the PRM (proceduralized refutation model) framework (Lin & Wu, 2005) which was used to cultivate students' abilities of conjecturing and argumentation.

The task on the worksheet was: The square of a given number is even. Following the false statement, there were eight sub-problems to guide students to give examples, use mathematical language to describe the common property of supporting examples and counter examples and make conjectures. 20 students of grade 8 participated in this study. They were asked to finish the worksheet in 40 minutes in the class section. After students wrote down their ideas, the teacher started class discussion and invited them to talk about their ideas. All the data we cited here was collected from the written worksheets.

This report checked three research problems: whether the worksheet could guide students to differentiate supporting examples and counter examples; to use mathematical language to describe the common properties of supporting examples and counter examples; and to make conjectures according to the properties they proposed or not. The results show that: the worksheet could elicit 13 students to give counter examples. 12 students could differentiate supporting and rejected examples. 10 students pointed out the common properties. 8 students made explicit conjectures. The common properties students proposed are: "The square of an even number is even" and "The square of an odd number is odd". The explicit conjectures students proposed are: "The square of an even number is even", "The square of a number is positive", "The square of an odd number is odd". The common properties and the explicit conjectures students proposed were very similar. 10 students' performances were strong connected between pointing out the common properties and making conjectures.

The teaching implication is: using false statement as a starting point could foster students to give supporting and counter examples. Differentiating supporting and counter examples is helpful to find the common property, and finding the common property may be the important key to make a conjecture. This finding has a big contribution to our theoretical construction of mathematical conjecturing.

A SCHOOL-BASED PROFESSIONAL DEVELOPMENT PROGRAM FOR MATHEMATICS TEACHERS

Lu Pien Cheng

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The goal of this study was to investigate a school-based professional development program using laboratory class cycle. This is a qualitative study situated within an interpretive theoretical frame. [1]*Two variations of the laboratory class cycle were experimented with a group of second-grade primary school teachers in the United States over one year. Each teacher was observed and interviewed at the end of each laboratory class cycle. A grounded theory approach and constant comparative analysis were used to analyse the teachers' feedback on each laboratory class cycle. Preliminary findings indicate that the two variations served different purposes, and that the two variations can be merged to better cater to teachers' needs.*

The laboratory class cycle consisted of preparing, observing, and analyzing mathematics lessons. Two variations of the laboratory class cycle were conducted. Both variations began with planning for a demonstration lesson. In the first variation, the teachers observed the professional developer taught in their classrooms in the first demonstration lesson. The second variation required the teachers to observe a mathematics lesson as a team in a second demonstration lesson. In both variations, the teachers completed the cycle by reflecting and critiquing as a group, the two demonstration lessons. This arrangement brought about initial teacher and team change because the teachers needed opportunities to observe and critique a mathematics lesson so that they could come up with ways to improve their mathematics instruction. After the third laboratory class cycle, the second demonstration lessons came to be seen as repetitive. The teachers wanted an opportunity to practice and to test different approaches to teaching mathematics they observed. They also wanted feedback from the professional developer and from their team as they tested those ideas. The two variations were merged in the last three laboratory class cycles. The teachers continued the cycle of planning, observing, and analyzing. They observed the first demonstration lesson before participating in the second demonstration lesson. The second demonstration lesson was modified to engage the teachers in experimentation as they took turns teaching a small group of students at a learning station. The modification afforded the teachers opportunities to observe their peers teach, which fostered greater team growth and individual teachers' growth.

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2007. In Woo, J. H., Lew, H. C., Park, K. S. & Seo, D. Y. (Eds.). *Proceedings of the 31st Conference of the International Group for the Psychology of Mathematics Education*, Vol. 1, p. 206. Seoul: PME.

PROBING REPRESENTATIVENESS: SWITCHES AND RUNS

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This study extends research in probability education by altering a 'classical' problem, referred to as the sequence task, and presenting an unconventional view of the sample space, which helps situate students' ideas within conventional probability.

Students were presented with sequences of heads and tails, derived from flipping a fair coin, and asked to consider their chances of occurrence. Unlike the sequence tasks used in prior research, this iteration maintains the ratio of heads to tails in all of the sequences and, as such, provides additional insight into students' perceptions of randomness.

Which of the sequences is least likely to result from flipping a fair coin five times: *(A) H H H T T* *(B) H H T T H* *(C) T H H H T* *(D) T H H T H* *(E) H T H T H* *(F) All sequences are equally likely to occur* *Provide reasoning for your response…*

Although roughly sixty percent (30/49) of the students incorrectly chose one of the sequences to be the least likely to occur, there was an even split between those who chose *(A) H H H T T* (14 students) and those who chose *(E) H T H T H* (16 students). The results show A and E were deemed least likely to occur because they were not representative of a random process; however, justifications fell into two different categories. Students who chose *(A)* said that the perfect alternation of heads and tails was not reflective of a random process. Alternatively, students who chose *(E)* said that a run of length three was not indicative of a random process either, because it was too long. This study suggests that alternative perspectives can show heuristic reasoning (e.g., representativeness) is not incongruent with probabilistic reasoning. Illustration of this point draws upon the probabilities associated with an alternative view of the sample space, organized according to switches and runs. For sequence *(E)* there is a 2/32 chance that a sequence will have four switches and a longest run of one (i.e., *H T H T H* & *T H T H T*). While for *(A)* there is a 4/32 chance that five flips of a coin will result in one switch with a longest run of three. >From this alternative perspective of the sample space, organized according to switches and runs, all other sequences have higher probabilities of occurring than *(A)* and *(E)* (e.g., three switches and a longest run of two has an 8/32 chance of occurring). As such, it can be inferred that the subjects were unconventionally, but naturally, looking for features in the sequences which are least likely to occur. From this perspective, it can be argued that the subjects were correctly choosing which of the sequences are least likely to occur. In sum: An alternative view of the traditional sample space for five flips of a coin via switches and runs, not the ratio of heads to tails, suggests a probabilistic innateness associated with use of the representativeness when determining, heuristically, the randomness of a sequence.

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PROMOTING NINTH GRADERS' ABILITIES OF INTUITIVE AND ASSOCIATIVE THINKING IN THE PROBLEM-SOLVING PROCESS TOWARD MATHEMATICS LEARNING

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Mathematics could be conceived as problem-solving, reasoning, and communication; in which, problem-solving is the central issue of mathematical inquiry (NCTM, 1989). Based on Krutetskii's elaboration of school children's mathematical abilities (Krutetskii, 1976), it might be concluded that a skilled problem-solver should possess the abilities to clarify primary concepts from irrelative information, to develop good intuitive and associative thinking as well as sharp insight, to think positively from multi- perspectives, and to analyse the structure of problem immediately and precisely. When confronted with a new problem, intuitive inspiration always plays as the pioneer of rigorous deduction. Without intuitive inspiration, it seems to be easy to get lost in finding an explorative direction (Skemp, 1985). Therefore, two main issues are concerned in this research: (1) Whether the empirical instruction design which encourages students' intuitive and associative thinking could promote their problem-solving abilities? (2) Are there any differences within the intuitive and associative thinking abilities of different achievers?

The study was implemented as an action research with research subjects of thirty-four ninth graders. The research period was organised in three times a week, 10 to 15 minutes in each lesson, totally twelve weeks of implementation. Data collection mainly includes student worksheets (each problem was composed of at least two mathematical concepts or solutions), responses to pre-test, mid-test and post-test, and unstructured interviews. These data were encoded and analysed by an intuitive and associative thinking classification table and a mathematical problem-solving ability assessment table. Main research results may concluded as follows: (1) the instruction design encouraging students' intuitive and associative thinking was beneficial to improve their problem-solving abilities, which appeared to be more effective on middle and high achievers; (2) as expected that high achievers showed better intuitive and associative thinking abilities whilst low achievers performed better on the problems with pictures.

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DESIGNING TRACE COMMANDS TO CONNECT EUCLIDEAN AND ANALYTIC GEOMETRY

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In ancient Greek geometry, equations are defined by curves. Coordinates, variables, and equations were subsidiary notions derived from a specific geometry situation (Boyer et al., 1991). But curves are defined by equations in the school mathematics today. As a result of focusing on 'symbols' for curves, most students are not aware of geometrical meaning on curves, and regard them as separated subjects (Cha et al., 2002). We focused on these problems, and designed trace commands in DGS to make a bridge between them. Before introducing the algebraic expressions based on the coordinate system, students can explore geometric relations and algebraic representations using the designed trace commands based on 'angle and length conditions' which is already familiar to them. If students have an opportunity to deal with less formal algebraic form prior to manipulating equations in coordinate system, they may move up to algebraic geometry naturally. In this respect, we introduce a new terminology, 'semi-algebraic condition'. Two primitive concepts of mc(distance) and dc(angle) are used in this condition because they are familiar concepts with students since elementary mathematics. For instance, prior to making an equation like $x^2 + y^2 = 100$, they can express this condition such as $mc(1,2) = 10$.

We focused on two aspects in designing trace command. The first is generative variations. Whenever the students change a factor of the semi-algebraic condition, the traces construct another figure. For a given point 1, we can construct circle as a set of point 2 satisfying the condition like $mc(1,2) = 30$. Then what if 'mc' is changed by 'dc' or the '+' operation is added to it such as $mc(1,2) + mc(2,3) = 50$? The second is local structures. Rather than consider a circle as a regular curve(globally), students may notice local properties of it as a set satisfying given conditions such as 'constant angle'. In analytic geometry, a curve represents the set of all the points satisfying the given algebraic expressions. So students have to notice the pointwise structures. This activity can be a foundation of bridging geometrical sense and formal analytic form. In future work, we can extend this point of view to use logical operation such as 'and,or' as well as algebraic expression, and semi-algebraic expression relating distance and angle.

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AN INVESTIGATION ON THE TIME MISCONCEPTION OF SCHOOL CHILDREN

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Studying the misconception of the learners about the subject is found very useful to the teaching of a subject. Because time concept is more abstract than other quantity measures, the understanding of time misconception become necessary to improve the teaching of time. Recently, Doig, Williams, Wo and Pampaka(2006) used an age-standardized diagnostic assessment to integrate correct and incorrect ideas to describe a developmental map about time. We adopted a two tier multiple choice measurement that catch both the methods of interview and quantified test to avoid the weakness of standardized assessment access the time concept of Taiwanese school children.

A nationwide sample of 1100 school age (9-12 years old) children was randomly drawn from elementary schools in Taiwan. Based on the theories of child's conception of time (Friedman, 1977, 1978 &1986; Piaget, 1969), the TBTC (Test of Basic Time Concepts) was developed to measure the children's understanding of basic time concepts. The collected data was analyzed by several item analyses and statistical methods.

Using 10% as threshold, we pinpoint 24 types of misconception. The heaviest misconception are "the more distance moved, the longer time spent" and "when the clock stops moving, the time stops as well".

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MIND THE GAP: A CASE FOR INCREASING ARITHMETICAL COMPETENCE IN SOUTH AFRICAN PRIMARY SCHOOLS

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This study investigated the achievement of grade 5 to grade 9 children in South Africa in a variety of numeracy tasks. Four primary and two secondary schools participated and involved 360 grade 5 to grade 7 students and 160 grade 8 and grade 9 students. The children's achievement and level of understanding was assessed through the use of four numeracy tasks that included number recognition, computations, measurement, fractions and ratio concepts. An analysis of performance, strategies, misconceptions and errors made by the students in each grade suggested that the majority of students were unable to solve straight calculations, had difficulty with the measurement tasks, and employed the elementary strategy counting all and counting on. Students in the higher grades were unable to solve problems benchmarked at two or more grades below their current status. The results show that there is no progression in terms of conceptual mathematical development across grades 5 to 9. This knowledge gap seems to restrict how these students come to know mathematics and it impacts on their level of understanding any higher order mathematics. Despite, two years of mathematical teaching, the analysis shows that the majority of students tested, especially at the secondary level, could not solve a problem in three attempts. Once they've developed a knowledge gap based on poor procedural and conceptual understanding the mathematical prospects of the children progressing are acutely compromised. Teachers at secondary level, in this context, do not consciously address these deficiencies or knowledge gaps that students bring with them to secondary school.

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HOME SCHOOL MATHEMATICS

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My research aims to identify factors that bridge the gap between what Chapman (2003) describes as 'informal' home mathematics and the 'abstract symbolism' of school mathematics. In this presentation I will compare theoretical positions to examine the mathematical conflicts that occur when young children start school.

Pre school children are members of a community which some refer metaphorically to as a community of practice (Lave & Wenger 1991) whilst others describe explorers navigating their way to an island (Meira & Lins PME 2006). What is common to these models is that the learners undertake a journey, for some this is intentional for others subconscious. Bishop (1988) refers to acculturation and enculturation, Lave and Wenger an apprenticeship and legitimate peripheral practice. What is also well documented is that not all learners are as successful on that journey, nor is success clearly defined. Notions of identity and power apparently interplay to contribute to success or failure on that journey. The journey I refer to, if one subscribes to assumptions about learning as 'part of human nature' and 'inevitable' (Wenger), begins long before children start school. Some young children may start school with no idea what mathematics is (in terms of a school definition) but all are already proficient in using and applying many mathematical skills. This culturally embedded understanding of mathematics conflicts with the beliefs, values and practices of western education systems where 'impersonal learning', a 'technique orientated curriculum' (Bishop) and a view of learning that is hierarchical, teacher directed, with pre determined outcomes positions the child as powerless, the teacher as powerful. As children endeavour to acclimatise themselves to the environment of the classroom the single most important lesson to learn is how to switch discourse away from an understanding of how maths fits into the world and is useful to everyday life to a discourse at odds with this where the emphasis is on the production of context free calculation.

I will examine the problematic notion of identifying opportunities within the classroom that acknowledges the previous experiences of mathematics that young children bring as they start school. I will look at what constitutes evidence of previous learning and examine video footage collected during a pilot study of nursery children to identify how and if teachers cue into these experiences.

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GEOMETRY IN MENO BY PLATO: A CONTRIBUTION FOR HUMANITY EDUCATION THROUGH PROBLEM SOLVING

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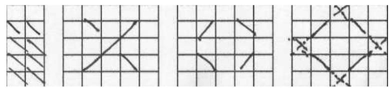
This study aims to examine the possibility of the use of historical problems in problem solving activities. It describes the solutions given by the students of a secondary mathematics classroom of 10th grade, to whom the well known geometrical problem, presented in Meno by Plato, was proposed: “If the side of a square is two feet long, then it’s area is four feet. Doubling the area, we draw another square with an area of eight feet. How long is the side of the new square?” This problem was chosen because, on one hand, the incorporation of the historical aspects of the mathematical development supports a teaching of mathematics that comes closer to a humanity education (Katsap, 2002) and helps the students to realize that the origins of mathematics are social or cultural (Bishop, 1988), on the other hand, the students construct a deeper understanding of mathematical ideas and processes as they are engaged in a challenging problem solving activity (Lester et al., 1994, p.154).



The above problem was proposed to 43 students of the Musical school of Serres in Greece. All the students had a squared paper to draw their solutions (Brock and Price 1980, p.366). Some students chose algebraic solutions but the use of squared paper supported ingenious geometry solutions that will be presented and discussed. The grid helped the students to visualize the area under consideration and by dividing the square units to arrive at interesting solutions. Further research is needed to investigate the heuristics that the students apply while solving this kind of problems, as well as the humanity education they can develop in a problem solving activity that uses historical resources.

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TEACHING HIGH SCHOOL GEOMETRY: PERSPECTIVES FROM TWO TEACHERS

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This paper focuses on how two high school teachers, from two different schools, approached the teaching of geometry and the emphasis they put on algebraic approaches for solving problems in geometry. In this study, one high school geometry class from each of two high schools was selected. Each class was observed for a period of three months and twelve of the lessons were videotaped. In addition, each of the two classroom teachers, Mrs C and Mr D were interviewed twice.

Cobb and Yackels's (1996) emergent perspective provided a framework for analyzing the teachers' pedagogical practice. Regarding classroom social norms, students from each of the two classes were expected to work hard, participate in class discussions, do their homework, show all relevant work in their solutions of problems, and work both collaboratively and individually on the solutions to problems. As for socio-mathematical norms, referred to as meta-discursive rules by Sfard (2001), the students from Mr. D's class were encouraged to find elegant solutions to problems. There was greater rigidity in the format of solutions in Mrs C's class and this implied more emphasis on her part on rules, algorithms and memorization.

Concerning similarities in approaches, both Mrs. C and Mr. D were committed to facilitating the learning of geometry for their respective students. They put significant emphasis on discipline and hard work. Students in both classes were encouraged to do their homework and seek help whenever necessary. Both teachers asked students to use diagrams to illustrate their solutions. They also cautioned the students to show all of their steps in the solution process. The teachers were aware of the importance of algebra in the learning of high school geometry and they both had some reservations about the algebraic skills that the students brought to class. Regarding differences in approaches, Mrs. C and Mr D differed in: their expectations from their students; the review of algebra topics; the format of their solutions; their use of applications of geometry to real-life situations; the use of technology; the flexibility of topic sequencing; and the use of geometrical constructions.

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A DIDACTICAL ANALYSIS TO RECONSIDER THE CURRICULUM AND THE SITUATIONS OF STUDY

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In my thesis (Erdogan, 2006), by dealing with *the autonomous study*, i.e. the autonomous part of work which returns to the pupils' responsibilities in the acquisition of knowledges aimed by the school (learning of the lesson, exercises, homework, revision, preparation of examination, etc.) and with the conditions under which the didactic system places, *de facto,* this work, we arrived at the need for conceiving a didactical analysis which make possible to consider the objects of the curriculum and the situations of study as a field of signification and investigation.

On the basis of a historical and epistemological point of view (Giusti, 1999), we made the hypothesis that in front of such a situation, the mathematical objects must arrange the ones with the others until constituting a relevant field on which will be based the work of the pupil, by giving him the tools of action but also and it is essential, the means of identification and validation of his own steps. We then proposed a didactic concept, that of *mathematical site* (Duchet & Erdogan, 2005) and a model of analysis around this concept.

In this communication, we will introduce the essential principles of this analysis and certain results that it enabled us to obtain. Initially we will show how we built a particular site, *the algebraic-functional site* relating to the 10^{th} grade-class in France. Then, we will present how we used this site as a reference for an analysis of the curriculum and some actual situations of autonomous study.

First of all, the analyses show that the matter of study - i.e. the site of the mathematical objects to study - is strongly split up by the official organization of the curriculum. The decisions of the teacher and the autonomous study of the pupils are strongly influenced by the consequences which result from this.

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LEARNING AS CHANGING PARTICIPATION IN COLLECTIVE MATHEMATICAL DISCUSSIONS

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Situated learning perspectives are characterized by two main epistemological features: 1) learning means changing participation and formation of identities within communities of practice; 2) cognition is seen as a process essentially situated in practices, and so always changing or transforming individuals – including teachers and learners, activities and practices. Some factors that influence participation include the affective domain, the other participants (especially their power relation to the person, e.g. Tatsis & Rowland, 2006), the means of communication (e.g. Frade, Winbourne & Braga, 2006), the artefacts involved and the physical surroundings. Wenger (1998) differentiates participation from 'mere engagement' as the former has the potential of mutual recognition. In doing so, he claims that participation is a process related to the social experience of living in the world 'in terms of membership in social communities and active involvement in social enterprises' (p.55). For him, participation includes: talking, doing, feeling and belonging. On the other hand, Cobb, Stephan, McClain & Gravemeijer (2001) offer an analytical framework for analysing collective (social perspective) and individual (psychological perspective) mathematical activities and learning. The former relates to shared ways of acting, reasoning, and arguing which constitute the classroom norms. Thus, a student's reasoning is understood as an act of participation in these normative activities. Alternatively, the psychological perspective focuses on the nature of a student's reasoning, i.e. on her particular way of participating in collective activities. We propose to discuss Wenger's and Cobb et al.'s ideas, among others, to explore students' learning in terms of participation in collective mathematical discussions. We present some data to suggest possible correspondences between 'signs' of learning and 'local' changes of participation. (*Supported by FAPEMIG and CNPq)

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MATHS ANXIETY IN PRE-SERVICE PRIMARY STUDENT TEACHERS

Gillian Frankcom

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Tell me mathematics and I will forget; show me mathematics and I will remember; involve me…and I will understand mathematics. If I understand mathematics, I will be less likely to have maths anxiety. And if I become a teacher of mathematics, I can thus begin a cycle that will produce less maths-anxious students for generations to come.

W. V. Williams, 1988, p. 101

That quote prompted an inquiry into the maths anxiety of 29 third year pre-service primary student teachers at The University of Auckland, Faculty of Education. These students often complained of not liking mathematics and of having no confidence in their ability to teach the subject in an interesting and engaging manner. Many expressed a fear and loathing of mathematics even when they were completing assignments and tests to a very high standard.

Maths anxiety comprises factors that militate against the learning and understanding of mathematics, such as fear of failure, and physical reactions such as high tension and apprehension. Although maths anxiety has been the subject of many research projects over the years (for example: Betz, 1978; Grootenboer, 2003; Hembree, 1990) there appears to be no diminution of interest, and indeed there seems to have been no remedy for the highly Maths Anxious.

Many people, including students and people in general, believe that mathematics is hard to learn and that only some special people with particular abilities can be successful. In a North American study, Furner and Duffy (2002) investigated this popular view of mathematics and found that about 10% of all school mathematics students are thought to be able to learn mathematics. These beliefs have been labelled “maths myths” by Franks (1990) and she reported that “results showed that these future teachers shared many of the mathematical beliefs held by severely math-anxious people enrolled in math-anxiety clinics” (p. 10). The idea that student teachers hold negative beliefs about the universality of mathematics ability, and that they share mathematics beliefs with severely maths anxious students is cause for concern.

CHILDREN'S ACTIVITIES WITH MATHEMATICAL SUBSTANTIAL LEARNING ENVIRONMENTS

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The aim of our research project is to develop a mathematics curriculum from primary to secondary schools in which students actively engage in rich mathematical activities and acquire a sound knowledge and understanding of mathematics. Our theoretical background is influenced by the 'mathe2000 project' (Wittmann, 2001). In a collaborative research network between schools and university, we have made the designing of 'Substantial Learning Environments (SLEs)' with both flexibility and rich mathematical content the core of the network (Wittmann, 1995; 2000). We also regard 'mathematics' for children and students as a 'science of patterns'. By this we mean patterns not only restricted to mere number-spotting tasks, but including other types of patterns in the relationship amongst numbers, shapes, etc., e.g. patterns found in the multiplication table, the number pyramids (Fig. 1), '321 – 123 = 198, 543 – 345 =198, 765 – 567 = 198', etc.

Our research design is based on 'lesson study' as we first design SLEs and lesson plans collaboratively, then conduct and evaluate lessons. In this paper, we shall report selected episodes from our classroom-based pilot studies which have been undertaken since 2005 in Japan. Data is mainly collected by observation and field notes, focusing on how children and students recognise and investigate mathematical patterns, how they communicate with each other, and how they reason/justify them. For example, in our pilot lesson Year 4 children (aged from 9 to 10) undertook a problem 'Investigate the differences between 3-digit numbers which consist of three consecutive numbers such as 321 – 123'. During the lesson, children not only found the pattern 'the answer is always 198', they also started reasoning by using 'the place value table' with counters (Fig. 2).

Details of the analysis of how children manipulated and interpreted the place value table (321 – 123, 765 – 567 etc. → 200 –2 = 198) will be discussed in our presentation together with other episodes from our pilot studies. We will also discuss how we are going to use these findings in our design of the mathematics curriculum.

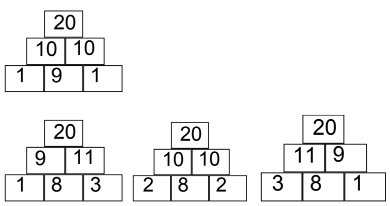

Fig. 1

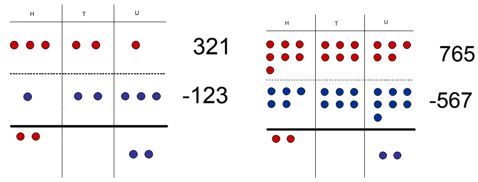

Fig. 2

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STRATEGIES IN LEARNING , TEACHING AND ASSESSMENT IN THE DOMAIN OF "MEASURES"

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Since 2000, Hong Kong has been undergoing a series of educational changes, and in line with the school curriculum reform in particular, the Curriculum Development Institute of the Education and Manpower Bureau has organised collaborative research and development (Seed) projects to gear at promoting the learning capabilities of students. Sections under each Key Learning Area thus work with schools in promoting active engagement in learning and in demonstrating what has been learned (Haertel, 1999, p. 663).

Our team in the Mathematics Education Section collaborated with primary schools in the Seed projects since 2003. Initial findings, as a result of lesson observation and examination scripts analysis, indicated that teachers adopted the transmission mode of instruction, and assessment was the traditional paper-and-pencil type – which could not effectively reveal the real classroom situation. The results also showed that there were still rooms for improvement in the domain of "Measures". [The finding was further confirmed by a government-contracted low-stake survey of the basic mathematics competence of all Primary Three and Primary Six students (Hong Kong Examinations and Assessment Authority, 2006, October)].

To improve the situation, we developed with teachers in the Seed schools learning and teaching strategies in accordance with the new curriculum initiative, and devised diversified modes of assessment to measure what students really learned. The effectiveness of the teaching and the efficacy of the assessment methods were studied via lesson observation, interviews with students and teachers, and practical tests. The topic "volume of a solid" in the "Measures" domain is specifically chosen for illustration and discussion.

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ZERO AND NEGATIVITY ON THE NUMBER LINE

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We are working on a research project currently in process[1]. Our theme addresses the integers in which "simultaneous appearance" of negativity and zero is emphasized in problem and equation solving, Gallardo (2002). It is based on Filloy's theory for empirical observations (1999). In Gallardo and Hernández (2006), we had already reported upon five meanings of zero that were named: **Nil zero**: that which "has no value". **Implicit zero**: that which does not appear in writing. **Total zero**: that which is made up of opposite numbers. **Arithmetic zero:** that which arises as the result of an arithmetic operation. **Algebraic zero**: that which emerges as a solution to an equation. In this study we propose to find other meanings of zero associated with the number line. Our research question is: How does zero contribute to extending the numerical domain of natural numbers to integers? 40 13-15 year-old students were video-taped as they responded to questionnaires. They had to solve addition and subtraction operations using a graduated number line that only bore equally distant marks.

We interpreted the actions of the seven pupils who had achieved the lowest level of academic performance. The results confirmed the first two levels of knowledge reported by Peled (1991), while also demonstrating facts that may contribute to the numerical extension. To wit, the meaning of zero as origin is recognized as 1) a random set point located on a number line, 2) a random movable point whose location changes depending on the numerical values involved in the operations, 3) an unmovable point, that is the half-way point on the number line. The zero avoidance symptom arose when 1) it was not symbolized, 2) it was symbolized, albeit ignored upon undertaking the operations, 3) the students interrupted their one by one counting as they approached the zero from the left, 4) the numbers one and minus one were considered origins on the number line. The link established between zero and negativity was expressed as: 1) the students recognized negatives and the origin zero, 2) they accepted negatives, but avoided the zero, 3) they did not represent negatives nor did they accept the zero. We can conclude that **recognizing negativity does not necessarily entail identifying zero as a number.**

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HOW COULD "CAS" IMPROVE STUDENTS' UNDERSTANDING OF CALCULUS?

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Statistics show that the drop out rate for calculus is quite high and the educational systems are spending a great amount of time and money to compensate for that. This was one of the major reasons that in 1990's, mathematics educators showed a great interest in researching the teaching and learning of advanced mathematical concepts with a specific focus on calculus.

Taking a lead from these findings, a study was conducted to investigate the ways in which computer algebra system (CAS) could be used in calculus classes to improve students' understanding of calculus. The students who volunteered to participate in this study were both high school and university students, and research was conducted in two parts respectively. The data for the study were collected through observations, investigators' field notes, and students' semi-structured interviews.

Part I: High school students helped the researchers to first, investigate the mathematical concepts and skills that are necessary for using CAS and second, to identify the possible obstacles that students might encountering while using CAS. The research findings identified a number of mathematical obstacles that some of them were previously presented in the literature. These obstacles were as follows:

a) the nature of differences between Algebraic representations by CAS and what students expected to see; b) the necessity of having deeper understanding of variables and parameters by students while using CAS; c) students' tendency to accept numeric solutions against algebraic solutions; d) limitations of CAS to the final answer; e) students' weaknesses in deciding when and why a CAS is useful; f) students' insufficient understanding of algebraic solutions; g) students' lack of understanding of an algebraic expression as an object.

Part II: For the second part, the researchers designed a mathematical laboratory in one of the universities in Tehran in which a number of activities were conducted using MAPLE. The researchers were interested in identifying the university students' beliefs about the role of technology – in this case CAS- in learning mathematical concepts. The analysis of interview data and conducted activities at the mathematics laboratory suggested the following four recommendations for integrating teaching of calculus with technology: a) developing positive attitudes in students towards technology; b) starting a mathematical activities using real world context; c) using small group collaborative activities; d) using whole class discussions and reflecting on them.

A STUDY ON DECIMAL COMPARISON WITH TWO-TIER TESTING

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This paper reports the experiment with a two-tier testing instrument. The two-tier test consists of five multiple-choice questions on decimal comparison. The findings revealed some alternative conceptions held by the students in comparing the decimal numbers.

The Study

In this study, I designed a two tier-testing instrument (TTI) with the features of a multiple choice question at the first level (tier-1) and then asked the students to write the reasons for their choice of answers at the second level (tier-2). The TTI was a short test with five items on decimal comparison. The test was administered to 29 (boys 16; girls 13) Year 6 English students of a mixed sex primary school in London on the 1st April 2004. Detailed back reading was done to construct items for TTI on 'decimal comparison'. For example, item 1 in TTI was potential enough to generate 'longer-is-larger' misconception identified by many researchers (Resnick et al, 1989; Stacey et al., 1998; Steinle & Stacey, 1998; Irwin, 2001). Analysis of the responses to TTI revealed that some students showed clear understanding of decimal comparison while some students were not. The TTI items proved to function conversely as well; for example, students who selected the correct answer in the first tier were proved to be wrong by their reasoning in the second tier (Van den Heuvel-Panhuizen, 1996). Analysis of the students' responses revealed some popular misconceptions (*b9: Because decimals are negatives and negatives are the opposite of positive)*. Some students struggled with language to express their reasoning. Difficulty in written explanation is a major issue in TTI as envisaged in the literature (Küchemann & Hoyles; 2003). Some students (b2, b12, b15, g1, g10) supplied procedural explanation as reasoning (Küchemann & Hoyles, 2002). I found the two-tier testing has the potential to stimulate mathematical thinking, to test the quality of mathematical reasoning, and to surface students' misconceptions on a given mathematical concept. I'm now developing a testing instrument using this two-tier structure in my doctoral research program.

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PROBLEM SOLVING WITH A COMPUTER ALGEBRA SYSTEM AND THE PEDAGOGICAL USAGE OF ITS OBSTACLES

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In recent studies, a Computer Algebra System(CAS) is considered as an interesting and new future-oriented tool in teaching mathematics. However a more in-depth study is required to determine whether a CAS actually increases the mathematical thinking ability or not. The present study is a pilot study that attempts to verify the possibility of adopting a CAS in math education. Casio ClassPad 300 which has a built-in CAS was used in this study.

PROCEDURE

Eight Korean Grade 11 students participated in this study, and data were collected through three types of problem solving tasks: modelling in algebra, finding pattern in integration, and optimizing the surface area or volume. After one month of training for the basic skills in using ClassPad, students were asked to solve the three tasks for 2 hours. They were allowed to use ClassPad if necessary.

RESULTS

The findings are as follows: 1. Students solved complicated equations better with the CAS than with paper-and-pencil only; 2. Students showed great interests in solving more complicated modelling problems with the CAS; 3. Students had difficulties in deciding when and how to use the CAS; 4. Several obstacles, as well as those reported by Drijvers (1999), were observed when students adopted the CAS.; 5. The obstacles that students encountered could be served to help and expand students' mathematical understanding in a more meaningful way.

The followings obstacles were observed in the students' problem solving process, which is believed to originate from the CAS: 1) There was lack of recognition of the difference between numerical and algebraic calculations; 2) The representations shown in the CAS are often more complicated than the students expected; 3) The CAS has its own limitations which cannot be overcome without help; 4) Insights into variables and parameters are more needed with a CAS; 5) Students had difficulties in recognizing the equivalence between graphs that look different because of the different settings of the View Window.

The obstacles that students encounter with a CAS are very important in pedagogical view points. The pedagogical usage of these obstacles for the purpose of enhancing mathematical learning will be illustrated and discussed in the present study.

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THE SIGNIFICANCE OF FACILITATING FEEDBACK OF EVOKED CONCEPT IMAGES TO THE CONCEPT DEFINITION: A CASE STUDY REGARDING THE CONCEPT OF FUNCTION

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This research examines one high achieving preservice teacher's conceptions of the function concept and her line of reasoning as she after a calculus course tries to make clear what constitutes a function. A group of twenty-five preservice teachers specializing in mathematics and science participated in the study. One preservice teacher, called Emma, who completed the calculus course with "high pass" is the research subject for the case study in this presentation.

The concepts of function and equation are closely linked for the preservice teacher in question. In the advancement of the preservice teacher's reasoning on functions, feedback from evoked concept images to the concept definition is central, and leads her to a more developed understanding of what constitutes a function. The findings of the research call attention to one component of importance in the development of a concept image, namely, the feedback-process from evoked concept images to the concept definition. Based on the case study results, preservice teachers seemingly need to be exposed to problem formulations concerning the relationships of the function concept to other mathematical concepts (this conclusion is also supported by Hansson, 2006). It was made clear during the interview with Emma that she is not experienced in working with such problems.

During an individual's reasoning, the concept image will almost always be evoked, whereas the concept definition will remain inactive or even be forgotten (Tall & Vinner, 1981). The study indicates that one significant component in the development of preservice teachers' understanding of the function concept could be to stimulate a feedback of evoked concept images to the concept definition in various contexts, as when Emma comes to a better understanding in the interview. The results of the study also support Vinner's (1991, 1992) suggestion that this type of reconnection is primarily possible with problems that are not of the standard variety.

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THE DEVELOPMENT OF MATHEMATICS TEACHING IMAGES FOR EXPERIENCED TEACHERS: AN ACTION RESEARCH

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Mathematics teaching has long been a focus of concern for mathematics education researchers around the world. As educators of teachers, we designed and carried out a one-semester mathematics course to establish positive "teaching image" (Elbaz, 1983) for elementary school teachers. Teaching image is a complex synthesis, comprised of a teacher's knowledge, beliefs and previous learning experiences; it also plays a key role in a teacher's teaching performance and decisions (Calderhead & Robson, 1991; Clandinin, 1986; Johnston, 1992).

This research contains two stages: (I) implementation of our teaching project via action research, and (II) evaluating volunteer teachers by means of qualitative research. For the first stage, 46 in-service teachers participated in action research. In the second stage, one volunteer teacher did trace research for one year. Data were collected through questionnaires, interviews and video taping.

The results from our first stage action research revealed that participated teachers' viewpoints regarding mathematics knowledge emphasised the memorization and calculations, and came mostly from their previous learning experiences. After the completion of our one semester mathematics course, many in-service teachers stated that our teaching method turned their previous learning experiences upside-down. The second stage of our research, the volunteer teacher stated that although she experienced a different mathematics learning process, her original viewpoint of teaching mathematics was only changed a little. However, we found that the volunteer teacher gradually shifted the focus of her teaching to help students understanding mathematics concepts. She further expressed that after participating in our research, she spent more time preparing for and thinking about teaching mathematics. She had formed a clearer picture of the kind of mathematics concepts and abilities her students should learn in her classroom.

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PRE-SERVICE TEACHERS DEVELOPING LOCAL CONCEPTUAL MODEL FROM COMPLEX SITUATIONS

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A mathematical model usually comes from a complex real situation. For developing mathematical model, Mathematicians need to abstract some critical conditions and simplify the complex situation. A conceptual model is a abstraction for understanding the physical situation. The aim of this study is to investigate how pre-service teachers construct a conceptual model. According to Lesh & Doerr (2003), Models are concept systems that are expressed using external notation systems, and that can be used to describe, explain, or predict the complex situation. In general, mathematical modeling is the processes consisting of structuring, mathematizing, working mathematically and interpreting/validating form a real situation to a mathematical model (Blum, 2002).

The subjects of the study were 24 pre-service teachers separated in five groups. The topic is "how the order center of a chain pizza store decided which store to send pizzas to a given location?" The results show that all pre-service teachers' development were two kinds of real models of pre-service teacher, showed as follows:

(1) Object model: a model consisting of physical conditions and focusing on physical objects. They were in the phase of empirical abstraction, so it was not easy for them to construct a mathematical concept. (2) Action model: a model consisting of mathematical objects, but focusing on actions on those objects. They were in the phase of pseudo-empirical abstraction, and they were operating on the relationships of those objects with mathematical tool.

These two kinds of students all idealized the real situation, but their difference is the use of mathematical tools. Those who used mathematical tools could gradually isolate appropriate properties and relationships from real situations, and focused more on actions on mathematical objects. It is not an easy process to make mathematical model form the complex situation. Three of the five groups were constantly entangled in the net of real conditions in the situation. It would seem that a mathematics concept was not so easily established. If even the pre-service teachers encountered these difficulties, the high school students would only have more troubles. Therefore, we have to design proper guiding activities when we use mathematical modeling as teaching approach.

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THE ELEMENTARY SCHOOL GIFTED STUDENTS WITH GSP: CREATIVE GEOMETRY PROBLEM SOLVING AND KNOWLEDGE TRANSFERRING PROCESS

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The purpose of this study is to investigate the effects of applying GSP for elementary school gifted students. Two gifted and 11 regular students were included in this study. The examples of teacher's task designs, students' reasoning nots and the corresponding GSP products of two geometry problems were provide in this paper. The results suggested that the gifted students can benefit from the problem solving process with very limited support. They demonstrate their originality by their knowledge representation designs. Further more, they can successfully transfer their understanding to their community members by modifying their original representations. The multiple representations constructed by the gifted students strongly demonstrate their mathematical fluency and flexibility. Their mathematical giftedness substantially contributes to the other students' learning.

CONCLUSION

GSP could assist the gifted student to solve the geometry problem independently and to construct their geometric knowledge visually.

In the study, the researchers traced the gifted students' problem solving process. At the very beginning, students approach the problem intuitively. At the later stages, they could explain their problem solving process very clearly. They understand the key elements for solving the maximum number of geometry figure divided. They can also verified and expand the abstract functions. Basing upon the notes and the designs of the gifted students' problem solving, we thought GSP could be an applicable mindtool for the elementary school gifted students.

Multiple GSP explanation representations promote knowledge transferring effect

Both gifted students agreed that by using the verbal explanation with GSP illustrations, they can successfully explain their concepts to their learning community members. Multiple explanation designs make the tacit knowledge of gifted students' logical thinking visually explicit. So, the gifted student's problem solving knowledge can easily resolve into the common consensus of learning. With GSP and a collaborative learning climate, we are very glad that the giftedness can also contribute to the equality of mathematics education.

WHAT DO PRIMARY SCHOOL STUDENTS THINK THE EQUAL SIGN MEANS?

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Over the past decade the teaching and learning of algebraic reasoning has been a focus of both national and international research and reform efforts (e.g., Knuth, Stephens, McNeil & Alibabi, 2006; Ministry of Education, 2006; National Council of Teachers of Mathematics, 2000). One approach has advocated increasing student ability to work capably with numbers by developing the students' computational strategies so that their "structural thinking can then be exploited to develop their understanding of algebra" (Hannah, 2006, p. 1). Underpinning this transition from arithmetic to algebraic reasoning requires that students are able to abstract key concepts including those related to equivalence, represented by the equal sign.

The research reported in this paper examines how students from classroom settings where there is a focus on developing efficient computation strategies understand and explain the meaning of equivalence represented by the equal sign. The focus of the study explores the explanations of the equal sign across a band of students aged from eight to thirteen years of age. Student responses were coded into categories using the system devised by Knuth and his colleagues (2006). The main categories included a relational or operational view of the equal sign.

Results indicated that a relatively low number of the students (26%) had a relational view of the equality symbol. In contrast, the operational view held by the majority of the students (61%) showed that it was considered as either an operator sign or as a syntactic indicator. These results illustrated that student understanding of the equality symbol did not improve significantly as the students moved up year levels. Although the students had participated in a mathematics program which focused on developing a flexible range of computational strategies the students demonstrated an inability to generalize the use of equivalence in computation and apply it to the equal sign.

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APPROPRIATE CAS-USE IN CLASS 7 AND 8

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The five-year model test CAliMERO (Computer-Algebra in maths lessons – discovering, calculating, organizing) which is carried out in the German Federal State of Lower Saxony is evaluating the use of computer-algebra-systems from class 7 combined with a teaching concept concentrating on sustainable learning and the development of mathematical competencies. CAliMERO was started in the school year 2005/2006 in six Gymnasiums with 29 classes of level 7 and will be continued up to class level 10.

In order to enhance sustainable maths learning with CAS it is necessary, as described by Stacey (2003), to establish a teaching culture which corresponds to the use of CAS. Therefore a further training course of several days took place at the beginning of the project with representatives of the participating schools and local experts. There were discussions about appropriate teaching methods to support the development of competencies in CAS-supported lessons according to the German education standards. The teaching concept developed with the participating teachers intends to make use of the complex potential of calculators for the discovery of maths and for effective exercises for a better understanding. CAliMERO aims to establish multidiscipline applications of mathematics in lessons, additionally the students' mathematical skills without calculator have to be defined and supported. Regular meetings during the project are organized to improve communication, to develop the next teaching elements and to discuss the state of evaluation. Moreover the TU Darmstadt offers project coaching by means of an internet platform which allows the ideas exchange of the participants and contains all developed materials (www.prolehre.de).

The research centre of the project is: "In which way can CAS combined with an appropriate teaching concept be used from class 7 to support the development of mathematical competencies and which effects can be achieved with it?" To answer these questions the students' perception of maths, their behaviour in dealing with mathematical problems and their mathematical capability are recorded.

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RECONSIDERING THE CONCEPT OF “MATHEMATISATION” IN THE OECD/PISA MATHEMATICS FRAMEWORK

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The definition of “mathematical literacy” in the OECD/PISA mathematics framework refers to Hans Freudenthal’s discription of “the world”. The concept of “Mathematisation” in OECD/PISA traces back to the concept of “mathematizing” in Freudenthal’s didactics. “Mathematisation” in OECD/PISA, however, is not exactly the same as “mathmatizing” in Freudenthal’s didactics. I will compare the concept of “mathematisation” in OECD/PISA with Freudenthal’s “mathmatizing”.

“Mathematizing” in Freudenthal’s didactics means organizing experiences in reality, including mathematics, through mathematical means. Looking at learning processes of mathematics as discontinuous, Freudenthal called mathematical activities which jump the first discontinuous parts, that is, organizing the fields of experiences in real world, “mathematizing reality”. He noted that “mathematizing” should not be restricted to “mathematizing reality” or to mathematical modelling. He argued strongly for attention to be given to mathematical activities that organize the fields of experiences in mathematical world, as “mathematizing mathematics”. The activity of local organising, such as defining a mathematical object, through which students understand the meaning and the aim of formal definitions, exemplify it in school mathematics. Freudenthal included “mathematizing reality” and “mathematizing mathematics” in his definition of “mathematizing”, and emphasized both equally. In Freudenthal’s “mathematizing”, the focus is also placed on long term activities.

In the OECD/PISA mathematics framework “mathematisation” referes to “a fundamental process that students use to solve real-life problems”. The five steps that outline “mathematisation” are almost the same as mathematical modelling.

It is clear that “mathematisation” in OECD/PISA focuses almost exclusively on the “mathematizing reality” component of Freudenthal’s didactics, and consists of relatively short-term activities for the purpose of international comparative study on student assesment. If the concept of “mathematizing” in Freudenthal’s didactics is at the heart of mathematics, then “mathematizing mathematics” and longer term activities that organise fields of mathematical experiences in reality are essential part of a rich definition of mathematical literacy.

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STUDENTS' UNDERSTANDING AND TEACHER'S STYLE

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The purpose of this research project is to investigate the connections between the students' learning progress and the teacher's modes of instruction and interaction in the classroom. For the students' cognitive advance, we based our analysis on the first five levels of growth in mathematical understanding formulated by Pirie and Kieren (1994): a) Primitive knowing, b) Image making, c) Image having, d) Property noticing and e) Formalizing. For the second aspect, we based our analysis on two frameworks. One is Carpenter's et al (2000) four levels of teachers' beliefs that correlate with their mode of instruction. The other is a classification proposed by Jacobs and Ambrose (2003) for the different modes of teachers' interaction: Directive, Observational, Explorative and Responsive. These define a profile of the teacher.

Six teachers of elementary school participated in a didactical experiment, consisting each of eight sessions in a specific topic, assisted by computational software. In addition to the observations of the sessions, an initial and a final evaluation were applied to the students containing notions related to the specific topic treated.

During the didactical experiments each teacher developed differently. Four of them started using a directive type of interaction and three of them moved into a more observational or explorative mode. These four teachers also moved from the first level to the second in Carpenter's classification. The other two teachers however, started at Carpenter's second level, and had a much more noticeable improvement during the teaching sessions on both scales, getting to a responsive mode.

The students' cognitive progress greatly depended on the quality of the interaction and the mode of working of the teacher. For the directive teachers, we observed a low advance on the conceptual interpretations of students. However, for the responsive teachers, we observed a significant advance, getting the students not only to form the concepts but to notice properties and make some generalizations.

This study showed that the teachers' instructional method and their improvement in teaching skills correlated very well with the students' progress in understanding.

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STUDENTS' ATTENTION TO VARIATION IN COMPARING GRAPH

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Variation occurs throughout the diversity of human experience. Moore(1990, p.135) summarizes statistical thinking as the omnipresence of variation in processes, the need for data about processes, the design of data production with variation in mind, the quantification of variation, and the explanation of variation. Recently, research involving reasoning about variation in diverse range of statistical situations has emerged. The research includes investigations into the role of variation in graphical representation, comparison of data sets, and sampling situations. It is the essential for reasoning statistically to reason about variation in different situations.

This study focus on the development of students' reasoning about variation with situations, and presents the results and analysis of three task that investigated students thinking about variation, one on comparing two data sets, the others on comparing sampling distribution. The tasks in this study were used in Shaughnessy et al.(2004) and Canada, D,L.(2004). For example, The Movie Wait Time task, and lolly task.

20 students of grades 9 with high-mathematical ability who has been studied special education for the gifted students solved the tasks individually, and they discussed freely the variation observed on graphs in comparing two data sets, and in sampling distribution, with teacher's intervention without any directions. Only some students are reasonably articulate their thinking. 8 of them were interviewed individually after students' discussion. The response of 20 students provided the source of quantitative data for this study, while the students' discussion and the interview of 8 students were videotaped and transcribed, and then analysed by Reading & Shaughnessy(2004) in their description hierarchy. The characteristic based on the description of the variation was considered the development of hierarchy in this study. The description hierarchy bases around students' description of the variation occurring. Some students concentrated on the middle values, while others were more preoccupied with extreme value. Also, some students considered to both middle value and extreme value, and to what is occurring between them.

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ACTION RESEARCH ON GEOMETRY TEACHING AND LEARNING BY LINKING INTUITIVE GEOMETRY TO FORMAL GEOMETRY IN DYNAMIC GEOMETRY ENVIRONMENT

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The point of departure for this research is that geometry teaching and learning middle school 2nd grade is very difficult, in Korea. My presentation describes action research in which the possibility of integrating intuitive geometry with formal geometry in dynamic geometry environment (DGE) is investigated. The purpose of this research is to provide practical information about the geometry teaching and learning in middle school. For this, research questions have been chosen as follows: 1. How can we integrate intuitive geometry with formal geometry. 2. Does DGE promote students' proof ability by integrating intuitive geometry with formal geometry? 3. How are students changed after participating in this research on geometry teaching and learning?

To solve the research questions, I conducted action research on one 2nd grade class at middle school. Lessons in DGE(Cabri II) were performed during two months. Students learned the property of triangles, the property of parallelograms and quadrangles, and relations between quadrangles. In particular, I focused teaching and learning on the property of parallelograms during three lessons. Each student constructed parallels, quadrangles, and parallelograms and discovered the property of parallels, quadrangles, and parallelograms via a computer during two lessons. In the last lesson, one student in each group presented a proof program constructed with Cabri II and another student described the proof for the property of parallelograms on a blackboard. I played the role as guide. Every activity was video-recorded as well as audio-recorded and computer screen data were saved into files. This study was analysed through recordings, files, worksheets, interviews, and journals.

First, I found the method of this project lesson to be effective on integrating intuitive geometry with formal geometry. Investigative activity on intuitive geometry made students construct parallelograms and discover the property of parallelograms. Presentation activity on formal geometry made students employ the analysis method, design a proof program with Cabri II, and describe the proof process on the property of parallelograms. Second, DGE was a powerful tool that integrated intuitive geometry with formal geometry. DGE helped students investigate intuitive geometry in shorter time, discover the property of parallelograms, and design proof programs using the rotation menu of Cabri II. Third, students became fond of geometry lessons, especially investigative activities with Cabri II. Also, students developed interest and confidence on mathematics. They commented on the difficulty of lessons, but felt them to be very interesting and the memory of the lessons were recorded in journals for future reference.

REFORMING EDUCATIONAL PROGRAMS FOR FUTURE KINDERGARTEN TEACHERS: SOME CONSIDERATIONS

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There is an increasing interest in the mathematics education of early childhood, but the issue of the university programs preparing future kindergarten teachers is less addressed. A significant piece of research conducted on early childhood teachers presents the factors that describes the teaching quality: encouraging problem solving activity related to the children's interests, proposing tasks with cognitive challenge, using high order questions, developing discussions and exchange of ideas, facilitating the students' reflection on activity (Diezmann et al, 2001, Clarke & Clarke, 2002).

Moreover, a number of studies indicate that the mathematical and instructional knowledge of the future teachers considerably affects the quality of their teaching practices and suggest the introduction of these subjects in their educational programs. However, recent studies support that the isolated focus on these subjects does not help the improvement of the pedagogical effectiveness related to mathematics.

The questions addressed in this paper are how and in what way research evidence is integrated and shapes the educational programs preparing future teachers, especially in the courses related to the subject knowledge and to the teaching of mathematics at an early age. To this direction, we comparatively examined the structure and content of educational programs in different European universities, highlighting the distinction between courses in mathematics and mathematics education in early years that many of these programs maintain. In the presentation, we'll discuss the need and examine how to reform educational programs for future kindergarten teachers by introducing the so-called «mathematics-for-teaching» (Davis & Smitt, 2006) as an important component of the studies related to mathematics education in early childhood.

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ANAYSIS ON THE CASE OF UTILIZING MIC IN KOREA

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This research is for applying MiC(Mathematics in Context) to Classroom in Korea, analysing the result based on Korean classroom culture and practice. Considering Mathematics Curriculum, I applied 『Some of the parts』 to 3rd-grade children for 4 lessons, they reached to Formalization and reasoning but I had difficulties during the class, but several difficulties were found.

DIFFICULTIES & ANALYSIS

The following problems were found when proceed to the lesson. First, most students tried not to utilize tools which was provided by teacher, and to calculate with a complex. Second, children had inert attitudes during the whole class discussion. Thus the discussion was decided by the minority, unable to acquire more divergent opinions. Third, children were concentrated on problem solving than creating strategies of solution. They were rather interested in justifying whether their answer was right or wrong while listening to other students' or teacher's utterance.

It had several problems to manage a class implementing MiC as above, the followings were that I could grasp that unique culture of Mathematics classroom in Korea is supposed to be a mixture of many factors, it occur distinctive features.

Sociomathematical norm and culture of mathematical classroom in Korea

The definition of sociomathematical norm is a feature of general learning discussion that is unique in students' mathematical activity by Cobb & Yackel(1996). According to Pang(2004), it showed standardized culture of Korean classroom that children have tendency to focus on what teacher expects of the answer displayed in textbook rather than deliberating on the various strategies. According to this study, the reasons of inactive class discussion were supposed to be closely connected with children's mental pressure to respond to teacher's expectation.

Practices of mathematics learning in Korea

Since our textbooks emphasized on a problem solving itself along with screening the solving strategies and algorithm, for Korean students this process of learning has been habitual. Therefore some of students who thought of completing the instructions turned to be still passive to employ a solving strategy right after getting answers like the case of 'problem solving of exercise', regardless of instructor's assertion of importance of resolving strategies.

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GRADUATE STUDENTS' CONCEPTUAL AND PROCEDURAL UNDERSTANDING OF DERIVATIVE IN KOREA AND THE UNITED STATES

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Results of international studies, such as TIMSS, suggest that students in some Asian countries outperform American students in mathematics. Many factors would contribute to this reported difference in student performance. One factor that may have a significant impact is the teachers' understanding of higher-order mathematics, which may include the concept of derivative, a fundamental concept in calculus. Literature shows that leaning basic calculus ideas is difficult for students and that those observed difficulties are found in teachers', as well as students', performances (e.g., Hitt and Borbón, 2004).

The preliminary results that will be presented and discussed in the short oral are findings associated with an ongoing research study that compares how the U.S. and Korean graduate students understand the concept of derivative and also investigates their algebraic skills to solve traditional derivative problems. 86 graduate students participated from three different masters programs that all have a secondary mathematics teaching focus. The equal number (43) of students participated from each nation; 84% (36) of the U.S. group and 74% (32) of the Korean group were inservice secondary teachers. Each participant was given two tasks: Task 1 that probed how one was making sense of the derivative functions (see below), and Task 2 that sought to determine one's procedural understanding for ascertaining the derivative of a function utilizing the product, quotient and chain rules.

Task 1: Without actually figuring out its derivative, group the functions that have the same derivative function and explain your rationale for your groups. If you can give more than one explanation for your response, please include it (them). Do not use your calculator.

a. $f(x) = (1/2)\sin x$	d. $f(x) = x \cos x$
b. $f(x) = 2x^2 - x + 4$	e. $f(x) = 2(x-1)^2 - (x-1) + 4$
c. $f(x) = 2 \cos x$	f. $f(x) = x \cos x + 4$

Our preliminary analysis suggests that many participants from both nations showed difficulties talking about the derivative without calculating one and that Korean group was significantly better adept at applying rules for determining derivatives in various situations.

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DEVELOPING ETHNOMATHEMATICAL PROBLEM POSING SKILFULNESS IN MATHEMATICS FUTURE TEACHERS

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The discussion of one's customs, tradition and knowledge in the global village, addresses the training of a mathematics teacher capable to cope with multiplex of mathematical knowledge, and bridge between formal mathematical knowledge and the knowledge his pupils derive from the society and culture they belong to. Ethnomathematics, which represents practical mathematical knowledge, has been influenced in its development by various social and identifiable cultural groups, with their codes, symbols, behaviours, and ways of reasoning, is a field where training can assist the teachers in teaching mathematics in pluralistic society.

The research, conducted in 'History of Mathematics' college course, where teachers, Jews and Arabs, form a rich mosaic of backgrounds - examined ways for preparing, through experience in ethnomathematical problem posing, "adaptive" teacher who is crucial for carrying out a multi-cultural society. The data collected through interviews, open questionnaires, report sheets, and lesson units, included ethnomathematical tasks and problems constructing by teachers, and had been analysed using a comparative method according to the Grounded Theory approach.

The theoretical frame of the investigation was a unification of three theoretical frameworks: ethnomathematical approach (D'Ambrosio, 2006), method of problem posing (Brown and Walter, 2005), and conceptualization of the training model, 'Preparing Teachers for a Changing World' (eds., Darling-Hammond and Bransford, 2005). Findings show that the teachers undergo a lot of activity intended for the ethnomathematical problem posing, during a process of investigating, using the "what if not?" strategy, and development of a capability to analyse, "translate", and interpret the mathematics practice from **their own culture**. This presentation will concentrate on revealing what the teachers should be able made in order to succeed in understanding the mathematical practicum, enhance prospective self-knowledge about mathematics concepts, and reinforce learning-teaching skills in posing and producing tasks in ethnomathematical context for teaching school mathematics.

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DEVELOPING MATHEMATICAL LEARNING IN THE TECHNOLOGY-ENHANCED BOUNDARY CROSSING OF MATHEMATICS AND ART

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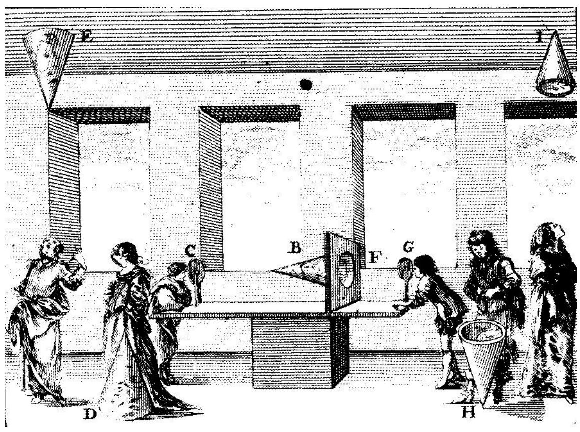
There are numerous opportunities for mathematical learning that present themselves in exploring the connections between mathematics and art. My paper will describe what can happen in the context of an art exhibition, where there are exhibition materials (objects and text), around which learning takes place in both informal and formal ways. "Art of Anamorphosis" is an ongoing project concerned with making mathematics a more accessible subject for "ordinary people". This has taken physical form as an exhibition, but most of the time exists in virtual form as a website [www.anamorphosis.com] and a software application, called "Anamorph Me!"



Anamorphosis is a technique for creating distorted images according to the mathematical rules of perspective and mirror reflection. It offers particularly rich scope for connecting mathematics and art because its 500-year mathematical history, as part of the development of perspective, intertwines with the development of art, science and society since the Renaissance period. The figure above shows a delightful engraving by Jean Du Breuil from 1649, "Cabinet of Anamorphoses", which suggests a playfulness with the idea, and also the use of large objects suspended from walls and ceilings which allow the viewer to have a strong physical engagment with the mathematical objects.

A guiding design principle is that the physical/virtual exhibition should not force attention on the mathematics of anamorphosis, but rather that should be one of the aspects that a visitor might wish to pursue, along various directions offered for investigation. The goal is "boundary crossing" rather than "teaching": trying to provide "boundary objects" usable by the viewer so that personal connections to mathematical ideas might occur. Mathematics must be used and worked on to be meaningful, rather than "consumed" from websites; manipulation of algebra is central to the traditional approach, which is a huge stumbling block for most people's access to mathematical ideas. The software "Anamorph Me!" has been developed as an object to support technology-enhanced boundary crossing with alternative symbolisms, allowing visitors to *make* anamorphic art for themselves, and thus progress from consumption to creation, and hopefully, to engage in a certain amount of mathematical thinking for themselves.

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TURKISH PRE-SERVICE ELEMENTARY SCHOOL TEACHERS' SOLUTION STRATEGIES IN PROPORTIONAL REASONING PROBLEMS

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In this study 86 Turkish Pre-service Elementary school teachers participated from Education faculty. All participants took 5-item Proportional Reasoning Test constructed by the researcher. Analysis of participants' correct responses showed different solution strategies in missing value and numerical comparison problems. In the missing value problems mostly cross multiplication algorithm was used. In the numerical comparison problems unit rate was used mostly. Although a high percentage of pre-service elementary school teachers could solve the missing value and numerical comparison problems using different kind of solution strategies they did not give the example about ratio and proportion from real life and could not say the differences between ratio and proportion. Much lower percentage of participants could solve the problems, give the definitions of ratio&proportion and say the differences between ratio&proportion and give examples from real life.

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TEACHERS' IMPLEMENTATION OF STANDARDS-BASED MATHEMATICS CURRICULUM

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The purpose of this paper is to examine how teachers implement *Standards*-based mathematics curriculum materials in their mathematics classrooms. *Standards*-based mathematics curriculum materials are developed by mathematics educators to support NCTM *Standards*. Successful adoptions of the Standards-based curriculum materials depend on how teachers implement the materials. Teachers select instructional tasks and resources from the curriculum materials and thus, teachers' use of curriculum materials involves teachers' interpretations (Ball & Cohen, 1996; Remillard, 1999, 2005). In this study, I investigate how elementary teachers use *Everyday Mathematics* [*EM*] (UCSMP, 2004) and *Investigations in Number, Data, and Space* [*Investigations*] (TERC, 2004), which are viewed as *Standards*-based curriculum materials, in both planning and enactment of mathematics lesson.

Data were collected in two urban public school districts in 2004, one in the southwest (A) and the other in the northeast (B) in the US that recently mandated the two curriculum programs, *Investigations* and *EM*, respectively. The data include elementary teachers' (N=19) pre- and post-interviews, classroom observation field notes (38 lessons), and those curriculum materials on which the classroom instruction observed is based and are analyzed by using a qualitative research method.

The results suggest that the teachers seem to follow the curriculum materials, follow the materials with alterations, and take suggestions or activities partially from the materials. In particular, all teachers in district A appear to either follow *Investigations* closely (35%) or follow it with alterations (65%). In contrast, 5 out of 9 teachers (56%) in district B seem to partially take suggestions and activities from *EM* in ways that the intention of *EM* is violated and it decreases the opportunities for students learning.

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AN ANALYSIS OF TEACHERS-STUDENTS INTERACTIONS OF MATH CLASSROOMS USING PAPER AND E-TEXTBOOKS

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In Korea, e-textbooks were made by html documents and followed school mathematics curriculum, but didn't based on paper textbooks published by Minister of Education until the early 2000's. Byun el al.(2005a, 2005b) pointed out those type of e-books confused with other e-learning contents, showed problems in developments and applications due to encyclopedic functions, didn't include dynamic aspects of technology, and as a results, and couldn't evoke teacher-students interactions so much as paper books.

Byun el al.(2005a, 2005b) made a new conception of e-textbooks(so called paper-book metaphor e-textbook; PBM e-textbooks) as follow and investigated a methodology of developing mathematics e-textbooks. "E-textbook is a digital learning material that is electronically developed from existing paper textbook, thus possesses the advantages of paper book but has additional functions such as searching and navigation; multimedia learning functions such as animation and 3D to maximize convenience and effectiveness of learning." Mathematics PBM e-textbooks for 5 & 6 graders have been made and modified for these two years and applied 5 schools by way of showing examples. A lot of researcher and educators expect PBM e-textbooks classrooms are even more interactive than paper books classrooms. Also there is less agreement among the others.

The purpose of this study was to investigate how the classroom interactions of PBM e-textbooks different from paper textbooks and find out what make the similarities and differences. Total 24 lessons were videotaped from 4 classrooms using each type of textbooks respectively and the teachers-students interactions of representative 4 lessons were analyzed. A theoretical framework of data analysis was from Hufferd-Ackles et al(2004). Teachers-students interactions are analyzed and compared according to the components of questioning, explaining, the source of mathematical ideas, and responsibility for learning.

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THE EFFECTS OF MATHEMATICS EDUCATION USING TECHNOLOGY IN PRE-SERVICE TEACHERS EDUCATION

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The purpose of this research is to investigate the effects of mathematics education in classroom activities using technology. NCTM(2000) and Korea's mathematics curriculum(1997) proposes the utilization of technology in mathematics education, recommending the use of calculators and computers. Under this background, a design-making activity using graphs on a computer program was conducted with freshmen who entered the mathematics education department(future mathematics teachers), a teacher's college, Jeonju university in Korea.
The technology that was used in this study is Grafeq program. Classroom activity is to construct real-world designs with functions and graphs. From these activities, research subjects experienced re-analysis and re-constructing mathematics. Through the analysis of classroom observation notes and students' individual records, the effects of mathematics education in mathematical design using Grafeq were shown. Especially, the effect of improvement in students' cognitive ability and affective attitude was observed.
Mathematics teachers may also utilize the works of mathematical designs to aid the teaching and learning of functional graphs. They can find, in the design works, the mathematical formulas related to what they are currently teaching and use them to induce the students' interest in the introduction part of the classes. It is also possible to guide the students to mathematically analyze the formulas and conditions used in the works.

This study has a significance as a study which attempted a new approach of learning methodology of mathematics using technology. Considering the effects of mathematics education obtained as a result of the learning process used in the study, mathematical design activities using graphs can be directly applied in mathematics classes at middle and high schools. Moreover, these activities can be utilized for the instruction of gifted students, through diversification of its levels and methods.

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CULTURAL AND INSTITUTIONAL EPISTEMOLOGY OF MATHEMATICS TEACHER KNOWLEDGE

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This study investigates the rationalized aims of education and the curricular contents of secondary mathematics teacher preparation programs in the U.S. and Korea. The findings suggest that there are both "semantically de-contextualized" and "culturally contextualized" dimensions of mathematics teacher knowledge.

SYNOPSIS

Most comparative studies of teacher knowledge tend to assume only national or transnational patterns in it (Alexander, 2000; LeTendre, Baker, Akiba, Goesling, & Wiseman, 2001). We, however, argue that these seemingly different views are not very contrasting. Rather, they describe different dimensions of teacher knowledge: teacher knowledge as a set of "cultural scripts" (Stigler & Hiebert, 1999) and as a set of "institutionalized" (Meyer & Ramirez, 2000) assumptions and beliefs about teaching and learning. The findings of this study suggest that while mathematics teacher knowledge (MTK) in the U.S. is understood as acquired from various field experiences and diverse academic disciplines, MTK in Korea is seen as originating from the deeper understanding of mathematics content and pedagogical content knowledge. These results imply that we need to understand MTK as deeply embedded in the external environment where both *culturally contextualized* and *semantically de-contextualized* components of MTK are constantly constituted. We thus argue that understanding both patterns is useful to capture the whole picture of external influences on MTK. We also argue that a comparison of MTK between different socio-cultural settings is valuable, for it allows a broader perspective on how we may further enrich it by educational borrowing and lending.

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AN ANALYSIS OF ELEMENTARY SCHOOL STUDENTS' CONCEPTION ON THE PURPOSE OF STUDYING MATHEMATICS

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It is significant that students understand why they have to learn mathematics, because such understanding has a powerful impact on not only affective but also cognitive aspects in mathematics education. The recent international studies such as TIMSS and PISA provide us the interesting fact that many Asian students, despite their superior performance on content areas, showed lack of interest and confidence in mathematics and poor understanding on the value of mathematics. Studies related to students' affective aspects have been conducted in various ways but, in particular, studies on in what ways students perceive the purpose of mathematics education are not sufficient. Given this background, this study examined Korean students' conception on the intention of learning mathematics in order to raise subtle but important issues to improve mathematics education.

A comprehensive survey was conducted with 525 sixth grade students. The questionnaire consisted of two parts: (a) goals of mathematics education with 37 items and (b) practical usages of mathematics with 15 items. In particular, the goals were categorized as practicality, preparation for the future, understanding of world culture, development of mathematical ideas, improvement of sociability and communicative skills, aesthetic appreciation, and academic values (Baroody & Coslick, 1998; Heymann, 2003). 7-step response scale was ranged from 'extremely positive' to 'extremely negative'.

With regard to the goals of mathematics education, students showed positive responses to the items related to preparation for the future (68.4%) and practicality (54.5%). The positive percentages decreased in order with regard to development of mathematical ideas (44.7%), understanding of world culture (33.6%), improvement of sociability and communicative skills (32.9%), academic values (27%), and aesthetic appreciation (23.1%). With regard to practical usages of mathematics, students were positive mainly in basic operations of natural numbers, measurement, and time. Understanding the value of mathematics and the various goals of studying mathematics can't be overemphasized. This study urges us to orchestrate mathematics instruction in a way for students to perceive important goals of learning mathematics such as development of mathematical ideas and communication as well as academic values, beyond direct practicality.

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USING THE STORIES AND DIAGRAMS TO CONNECT MATHEMATICAL IDEAS

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An emphasis on **mathematical connections** helps students recognize how ideas in different areas are related. Students should come both to expect and to exploit connections, using insights gained in one context to verify conjectures in another. **"Stories and Diagrams"** is an innovative teaching techniques designed for helping and supporting both students and teachers enhancing their learning and teaching climate and process toward mathematical understanding and ideas. After introduction to the planned topic, students are facilitated to participate in class through stories telling and diagrams drawing based on their own interests.

Stories and Diagrams has been accepted and explored as an innovative teaching technique. Effectiveness of the approach was confirmed by researches through various class examples from preschool to secondary school started in the northeast of Thailand (Kongtaln, 2004). Students' understanding of mathematical idea and recognizing of connection, using NTCM (2000) as a basis, revealed through their mathematical task presentation in classes.

Students should connect mathematical concepts to their daily lives, as well as to situations from science, and the social sciences. Stories and diagrams emerged to be an interesting teaching technique through researcher's long time experiences and vivid observations in teaching mathematics. While telling stories of their interest together with diagram drawing, the students had excellent opportunity to expose themselves with critical thinking process. Thus, using "Stories and Diagrams" teaching approach, both students and teachers were emerged into critical didactic discussion. As a result, the students gained more insight and recognized connections among figures, situations and so on. Furthermore, using connection as a basis applied to various contexts in daily life, the students learned to link mathematical idea from subject to subject, and subsequently forwarding the idea of concrete to the more abstract mathematics of which is very useful for daily life. In addition, students should also recognize the value of mathematics in examining personal and societal issues. (Kongtaln, 2004).

Keyword : connection, Stories and Diagrams, teaching technique.

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DEVELOPMENT OF TEACHER IDENTITY DURING MATHEMATICS TEACHER STUDIES

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In this paper, we will focus on teacher identity and its formation by analysing one case, a student teacher's (Pete's) development process during the pedagogical studies module in mathematics teacher education. The studies in question embody a Master's degree programme (300 ECTS) consisting of approximately four years studies in mathematics, physics or chemistry as a major besides one year of pedagogical studies (60 ECTS). The formation of teacher identity and how a student conceptualises his/her development as a mathematics teacher during the studies are focused on. Here we adopt an understanding that *teacher identity* contains his professional teacher knowledge (in the sense of Shulman 1987) as well as personal processes of teachers as attitudes toward teaching and learning (cf. Bohl & Van Zoest 2002). Especially student teachers' feelings and beliefs come out in discussions on being a teacher.

A case of Pete is represented in order to reveal the student's conceptualisation of mathematics teaching and teachers besides the process of identification as a teacher within the studies. The main question is how Pete has experienced his teacher studies and their influence on his identity. I.e. how does Pete conceptualise his development as a teacher within the teacher education programme? Three semi-structured interviews took place: on September 2005, December 2005 and May 2006 at the Department of Applied Sciences of Education in Helsinki. Some written materials like portfolio folder as well as his essays were used as additional data. The research is engaged in the qualitative research approach following the principles of the analytic induction.

Five themes were essential in the data: a starting point and background of the student; conceptions of good mathematics teaching and being a teacher; identification as a teacher; expectations and aims for the teacher studies; and evaluating the programme. Based on the case, there is a need for addressing the complicated study process of a student teacher that is addressing feelings and beliefs that a student has toward teaching mathematics and being a teacher. The differences between mathematics and education as disciplines emerge. There is a need for educational concepts and models as tools for being able to reflect on their own development and for becoming aware of their teacher identity.

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VISUAL REPRESENTATIONS OF NATIVE AMERICAN HIGH SCHOOL STUDENTS USING CBL-MOTION DETECTOR

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Researchers have identified various types of representation modes including verbal, numerical, graphical, and, analytic, revealing that mathematics cannot be understood in one mode of representation. The mathematical process of converting from one mode of representation to another has been called translation (Janvier, 1987) and demonstrates how mathematical thinking in a learner's mind, or internal representation, relates to understanding of mathematical objects.

Focused on these translation processes, a kinaesthetic learning style, and a graphing calculator in conjunction with a Calculator Based Laboratory (CBL) and a motion detector, this research was designed explore how visual preference and ability mediated Native American high school students' interpretation and sketching ability of graphs of functions. Two cases are reported: one with high spatial ability combined with a preference for visual representation, and one with high spatial ability but with a non-visual preference, to address the following two questions. First, how do the Native American students use CBL to translate among multiple modes of representations of functions? And second, how does visual preference affect the participants' interpretation and sketching ability of graphs of functions?

To answer the first question, the researchers examined the participants' developing ways of talking, acting, and gesturing; visual traits of the graph with qualities of their own motions; and the participants' efforts to make meaning from physical movement for the graphs on the screen of the calculator. To answer the second question the researcher observed the differences of each participant's interpretations of graphs, the differences of each participant's sketchings of verbal descriptions, and the participants' reactions when the graphs on the calculator screen were different from what they expected.

The preliminary results reveal that students developed emotional attachments for the CBL-motion detector in creating meaning for the graphs, indicate the value of multiple representations in the complete understanding of functions. It is also found that the student with visual preference shows better interpretation and graphing ability than the student with non visual preference.

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DIFFICULTIES OF CREATING A PROFESSIONAL LEARNING COMMUNITY

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Wenger (1998) defines learning as having the following components: meaning, practice, community and identity. Building from Wenger's perspective, I tried to understand a professional learning community of teachers and teacher educators from the members' perspective because I consider a professional learning community an important component in teachers' learning. The purposes of this research were to understand teachers and teacher educators in the context of community and to help design research on professional development using communities and a partnership with schools and a university. In particular, I addressed one research question in this paper: what difficulties do the members have in building the learning community?

This study was conducted as part of Partnerships in Reform in Mathematics Education (PRIME), the NSF-funded research, which is a professional development effort related to high school mathematics teachers in the Northeast Georgia region. Seven participants met weekly and discussed pre-service teachers' teaching or their students' work in the meetings for approximately one hour in Norris High School during the 2006 field experience period. Data collected for this research included audiotapes of nine meetings of pre-service and in-service teachers and a university supervisor, two interviews for each participant, and written documents such as observation notes, surveys, and e-mail responses. All interviews were transcribed, and then were considered as the participants' narratives.

Data, the written narratives, were analysed first by being divided into topics – discussion on a learning community, discussion on the value of their activities in the meetings, discussion on the difficulties in building the community. I then analysed the divided pieces and reconstructed three narratives. In this paper, I discuss the difficulties the members had in building the learning community. The preliminary findings showed that the power issue among members came up as one of difficulties; however, the levels of difficulties were different from one group to another. In addition, the members addressed issues about selecting topics to discuss, criticizing others, sharing goals, managing communities such as deciding meeting times and the number of members. This research will help educators understand teachers in a community and further help to design teacher education programs using partnerships with schools and a university.

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VIEWS ON MATHEMATICAL PROOF OF ABLE STUDENTS IN THE 3RD TO 7TH GRADES[1]

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This paper is focused on the views about mathematical proof of able students in the 3rd to 7th grades in Korea. Two contents were studied: One is whether able students can clearly distinguish between empirical evidence and mathematical proof; the other is their views on the role and function of mathematical proof.

The subjects are 457 students who recommended by their school principals and passed the 1st written test in the process of selecting gifted students, who want to enter the institute for the mathematics and science gifted students. They are belonged to the top 1 % of their ages in the mathematical problem solving abilities.

After showing both the deductive proof and justification through measurement of 180°, the sum of interior angles of a triangle, the question was presented that asks whether the fact that the sum of interior angles of a triangle is 180° can be guaranteed. On the other hand, with a view to examine the subjects' views on the role and function of proof, we asked, while presenting the deductive proof, why they thought the textbook (or the teacher who taught proof) proved (or proves) like that, and examined their answers. All the examinations in this research were made in the form of a written test.

Of all the 457students, 86.0% (393 students) had experienced proving; and 96.5% of them thought their proving ability was above average. However, 38.7% of students thought the fact that the sum of interior angles of a triangle is 180° can be guaranteed by the justification through measurement; 31.7% of students thought it might be guaranteed if all triangles are measured; only 6.6% of them answered justification through measurement is not a mathematical proof.

As to the function of proof, able students gave relatively diverse answers including the understanding (18.6%), verification of validity (23.9%), connectivity inside mathematics (5.5%), (external) utilization of proving results (19.0%), etc.

Above results show that despite the fact that able students have confidence in their abilities in mathematical proof and relatively balanced thoughts on the functions of proof, they fail to clearly distinguish between empirical evidence and mathematical proof.

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CONSTRUCTION OF KOREAN TRADITIONAL TESSELLATIONS VIA GSP(GEOMETER'S SKETCHPAD)

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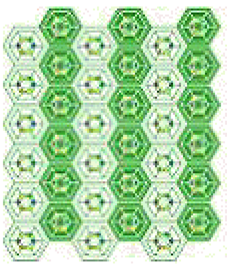
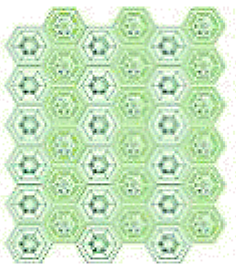
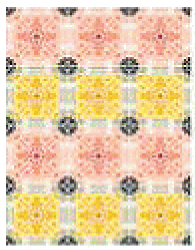
Tessellation mean to completely cover the plane without gaps or overlaps. Mathematicans call such an arrangement of shapes a tessellation or a tiling. When a tessellation uses only one shape as in a honeycomb, it's called a monohedral tiling. In other words, tessellations are the pattern of iterations of geometric symmetry and transformation. We can find them in the works of one of the famous Dutch artist, Escher(1898-1972) and cultural crafts of American Indians. Escher spent many years learning how to use translations, rotations and glide reflections on the grids of equilateral triangles and parallelograms to create tessellations of bird, fish, retile and human. Because Escher found beauty and peace in the categories of regularity, periodic repetition and regeneration. On the otherhand, Indian-Americans of North America investigated the patterns to engrave their own people's identity in baskets, ceramics, carpets, leathers and wooden crafts.

Moreover, we can find the beauty of tessellations in the Korean traditional house doors, Buddhist temple architectures, palace's fences and so on. In this research, we are going to construct Korean traditional pattern, 'Dang-Cho'. Dang-Cho pattern is engraved in ceramics, crafts, stones, palace walls, wall papers etc. Unique Korean temple decoration, 'Dan-Cheng' used to draw Dang-Cho pattern.

In this paper, we'll show various Dang-Cho pattern via GSP(Geometer's sketchPad).

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STUDENT VIEWS OF PROPORTIONAL REASONING ERRORS

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Research has demonstrated that errors are not random, mindless attempts on the part of the student, but that errors " frequently result from systematic strategies that often have sensible origins" (Perso, 1992, p. 12). In accordance with this view, errors are to be used instructionally as opportunities for student learning (Perso, 1992; Siegler, 2003), wherein the teacher uses student mistakes as opportunities to delve into deeper conceptual issues (Borasi, 1996). To inform instructional decision making regarding student errors, further research must be conducted that describes the reasoning that students use when attempting to reconcile their errors. Understanding of how students view the general nature of their errors can provide insight regarding how students use and view the errors they make and how they reconcile them. In our study, we examined how students dealt with errors related to proportional reasoning. Specifically, we posed the following research questions: (a) How do students view the generality of the errors they identify? and (b) How do students' views of generality affect how they reconcile their errors?

During this study two students, Dallas and Lloyd, applied proportional reasoning to a variety of particular instances. Initially it appeared that Dallas and Lloyd had overgeneralized the use of proportional reasoning. As described by Smith, diSessa, and Roschelle (1991), their knowledge had "been extended beyond its productive range of application" (p. 152) causing errors in their application of proportional reasoning. When Dallas and Lloyd were later faced with situations that did not allow for the direct application of proportional reasoning, they grappled with their errors at the local level, and began to extend their reasoning to the problem and across problem levels. However, these two students examined the range of applicability of their errors in different ways, leaving them with distinct differences in their levels of understanding of the use of proportional reasoning. Our study provides a schematised description of the thinking of these two students.

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PRE-SEVICE TEACHERS CONSTRUCTING MENTAL IMAGES BY USING DIFFERENT MEDIA

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It is necessary to take into account the use of instructional media for designing a mathematics learning environment. Vinner proposed a cognitive structure of mathematics concept which consists of concept image and concept definition. Vinner and Dreyfus (1989) discovered further that of most students does not use the concept definition to solve problems. Although they proposed that the construction and manipulation of concept images played an important role in concept understanding and problem solving, they did not elaborate on the forms of image construction.The aim of this research is to investigate how the learners construct and manipulate concept images when different media are used.

Research subjects were 60 pre-service teachers who were into three experimental groups. With a task which involved presenting cardioid by animation, physical objects, and verbal texts, we investigated learners' models of concept images and the influence of media on concept images. Questionnaires were designed for the purpose of collecting types of mental images and concept knowledge.

We have two main findings:(1) subjects formed four kinds of mental images – dynamic images without afterimage: there was a feature of continuity when it presented dynamic information; dynamic images with afterimage: although there was a feature of continuity when it presented dynamic information, there was a phenomenon of overlap; single static images: when it presented dynamic information, there could be no continuity, only a single picture, and separate static images: when it presented dynamic information, the picture consisted of a sequence of several static images. (2) the models of manipulating concept images when stimulated by three different media. The mental images of animation representation group tended to be dynamic image without afterimage, which means the stimuli of animation formed a more complete dynamic simulated image. The mental images of physical objects representation group tended to be dynamic images with or without afterimage, which means the stimuli of animation might not form a complete dynamic simulated image. It is possible to be an influence of the properties of the physical objects. Finally, the mental images of the verbal text group tended to be separated static mental images, which means that thinking under verbal text stimuli is usually fragmented, not continuous or dynamic. Besides, during the referential process, an individual has to draw pictures to help thinking.

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THE CHARACTERISTICS OF STUDENTS' PROOF IN PROBLEM-BASED NUMBER THEORY CLASS

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In this paper, I intend to describe how students developed mathematically logical, critical, creative, and rational thinking and independency by proving theorems on their own in the Number Theory class as taught by the Modified Moore Method (MMM). MMM is a teaching method at the university level, in which students are given a carefully constructed list of problems. In this method, neither collective efforts on the part of the students are allowed nor are they permitted to make use of any source materials for reference. Students have to solve the problems on their own, with little or no direct instruction from the teacher. The other students participate in the class with the task of "refereeing" the argument presented. That is, they evaluate and comment on the presentation with logical and critical views in their own way.

During the class, the students established normative understandings as follows: First, proofs having no logical error and are not advanced relative to the class are mathematically acceptable. Second, mathematically different proofs are those using different approaches, those that use previous theorems or other mathematical knowledge, and those that use different proof techniques. Third, proofs that use notation and previous theorems or lemma are mathematically simple, sophisticated, or efficient proofs.

Since no resources were allowed, students had to use previous definitions and theorems already discussed in previous classes. Proofs using advanced mathematical knowledge over their class level were not accepted in the classroom. Also, since students solved the problems by themselves without reference materials and with little or no direct instruction, each student proved the theorem with his/her unique mathematical knowledge and mathematical disposition. This made the classroom an atmosphere in which the students tried to develop personally meaningful different proofs. This character influenced what counted as mathematically different proofs on the whole. At the same time, while the students evaluated and commented on the presented solutions with their own logical and critical views, they continuously judged if the proof was mathematically simple, sophisticated, and efficient. These criteria were negotiated by the instructor and the students through their interactive discussions. This circumstance established what counted as a mathematically simple, sophisticated, or efficient proof. In addition, because the students acted as referees in the classroom, they revised the presented proof rigorously until they are satisfied that the proof was acceptable.

In the process, students developed their own mathematically logical, critical, creative, and rational thinking. Most especially, they were encouraged to do mathematics independently.

PATTERNS OF JUSTIFICATION IN THE PROCESS OF SOLVING SPACIAL GEOMETRY PROBLEMS [1]

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In this study, preliminary research was made to develop more concrete education program for spatial geometry. Analysis was made on the patterns of spatial reasoning and justification used by 7 mathematically gifted students in the process of solving spatial geometry tasks on cuboctahedron.

The subjects of this research were 7 mathematically gifted students [three 6th graders (E1, E2, E3) and four 8th graders (M1, M2, M3, M4)] who attended spatial geometry lessons in an institute for the gifted elementary school students attached to a university located in a local city, which is supported by the Korean government. The each lesson was held for 3 hours with tasks on cuboctahedron.

The framework of data analysis is as follows: Partial justification and whole justification is made according to whether all component factors that need to be considered in the given task are considered or not. Empirical justification and formal justification is made based on the difference of reasoning type whether one goes further than actual experiences to consider more general and formal methods.

Responses of the 7 mathematically gifted students can be summarized as follows.

Reasoning / Considerations	Empirical Justification	Formal Justification
Partial Justification	E1, E2, M2	M3
Whole Justification	E3	M1, M4

Table 1: Patterns of justification

As seen in Table 1, it would be necessary to pay attention to the value of informal justification, by comparing the response of student E3 (who understood the entire transformation process and provided a reasonable explanation considering all component factors although presenting informal justification) and that of student M3(who showed formalization process based on partial analysis) (Lee, 2005).

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A SURVEY ON THE UNDERSTANDING OF SPATIAL SENSE OF ELEMENTARY SCHOOL STUDENTS

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Although it has been emphasized that spatial sense is needed in everyday life and helps learning mathematics (e.g., Kennedy & Tipps, 2000), it has not been studied in depth how students understand the main contents related to spatial sense in a mathematics curriculum and how teachers may foster students' spatial sense through mathematics instruction. Given this, this study explored in what ways second, fourth, and sixth graders understand the main contents related to spatial sense such as congruence transformation, mirror symmetry, and congruence and symmetry.

A comprehensive survey was conducted. Three kinds of tests (Test I, II, and III) were implemented. Test I contained the main contents taught in second grade, whereas Test Ⅱ and III contained those taught in fourth and sixth grade, respectively. While Test I was employed to second, fourth, and sixth graders, Test II was used to fourth and sixth graders. Test III was used only to sixth graders. Data were collected from the five schools located in Daegu, Korea. The subjects were 161 students in three grades, a total of 483 students. Data were analysed in terms of percentages of correct answers, types of incorrect answers, and significant differences among grades.

Whereas students had good understanding on slide, they had poor understanding on turn. Percentages of correct answers on flip had significant differences among the three grades. Students also showed the lack of understanding on congruence transformation and they experienced difficulties in describing the changes of shapes.

Students understood the fact that the right and the left of an image in a mirror are exchanged, but they had poor understanding on mirror symmetry. The more complicated cubes they had in the problems, the lower percentages of correct answers on cubes they had. Students understood very well congruence per se, but they had difficulties in finding out congruent figures. They also had poor understandings of symmetry and, in particular, symmetric figures for both a line and a point.

In the presentation, it will be addressed in detail how students solved several problems with representative types of incorrect answers and explanations. This study urges us to investigate in depth students' understanding of various contents related to spatial sense and to teach them in connection throughout different grades.

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THE PERFORMANCE OF GEOMETRIC ARGUMENTATION IN ONE STEP REASONING

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Taiwanese students learn geometry mainly focuses on finding the invariant properties of kinds of geometric figures and apply these properties to solve or prove problems (Lin & Cheng, 2003).But students learn geometric concepts always get into trouble (Clements & Battista, 1992). Duval (1998) thinks that using vision to reason isn't a kind of argumentation and can't reduce cognitive loading in reasoning, but its can increase a natural discursive process and make the discourse completely. According to Duval's opinion, we provide "coloring flashcards" to help student to learn geometric argumentation. Coloring flashcards have two kinds of functions :(1)it uses different colors to reveal initial information; and(2) it provides concrete objects to operate to understand the relationship between properties.

The study adopted the one-group pretest-posttest. The subjects were 31 students at school age (CA=12.0 years). There were fours classes with one question designed for each. Each subject was given four kinds of questions in four classes separately. And in the second and forth class, we offered coloring flashcards. The four questions were to distinguish two types of items from each. One kind of items was geometric concepts. Students were asked to explain invariant properties of certain geometric figure to confirm their Geometric concepts. The other one was geometric counting. Geometric counting was confirmed by asking students to calculate to explain invariant properties. The classes were 160 minutes long in total. The results showed that the strategy of coloring flashcards could promote students' performance effectively, and increasing the experience of operating figures was necessary in learning geometric argumentation; the strategy of coloring flashcards had an immediate effect to make students to understand the relationship between different properties. Students took a period of time to show the ability that in finding initial information, and this ability was internal; students needed to help in visual aids and operating to reason in One Step Reasoning.

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AN ACTION RESEARCH ON A MATHEMATICS TEACHER'S ASSESSMENT ACTIVITIES

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The objective of this study as an action research was to analyse and reflect the assessment activities of this researcher who was an elementary mathematics teacher, focused on the open-constructed response problems. And, this study was to attract the practical knowledge that was helpful for elementary mathematics teachers to assess the students' mathematical power by means of open-constructed response problems.

As the 1st stage of action research, this researcher who was an elementary teacher developed an assessment plan, 15 open-constructed response problems, and scoring guides in the area of the congruence and symmetry of geometric figures, according to the PISA 2003 (Program for International Student Assessment 2003) assessment framework. The congruence and symmetry of geometric figures have been dealt with in 5th grade in Korea.

As the 2nd stage of action research, this researcher gave the developed 15 open-constructed response problems to the 40 students on 5th grade and students solved the problems during as much times as they want. This researcher examined the students' responses and misconceptions. Furthermore, this researcher identified the gap between the mathematical knowledge taught by this teacher and the mathematical knowledge possessed by students. In addition to, this researcher confirmed how this teacher-researcher's informal words used in the mathematics class affected on the students' knowledge.

As the 3rd stage of action research, this researcher analysed and reflected the relevance of the developed 15 open-constructed response problems and scoring guides, with considering the students' responses. On the basis of this analysis, this researcher suggested the improved 15 open-constructed response problems and scoring guides in the area of the congruence and symmetry of geometric figures. Furthermore, this researcher reported the practical knowledge obtained by carrying out the whole process of assessment focused on the open-constructed response problems for myself.

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USING CLASSROOM INTERACTION TO DEVELOP STUDENTS' MATHEMATICAL ABILITY AND COGNITIVE SKILLS

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Identifying and adopting diversified teaching strategies to facilitate students' mathematical learning has been teachers' and educators' longstanding concern all over the world. Research (Leung, K.M., 2006) claims teachers should serve as a facilitator in the classroom, help students develop higher-order thinking skills and foster students' learning interest and motivation in mathematics. Increasing classroom interaction is also suggested as helping students towards achieving the above. The rationale behind this study originates in the observation that Hong Kong teachers' classrooms are dominated by traditional teaching practices. This paper deals with successes and difficulties teachers experience in two typical primary schools in Hong Kong in the course of teaching a part of the curriculum (measures). Particularly I focus on their experience as they move towards a student-centred approach through the application of mathematical tasks that aim to increase classroom interaction. The overall aim of the study is to reveal what constitutes effective practice particularly with regard to the extent, nature and usage of mathematical tasks (Stein M K, *et al.*, 2000, Ainley, J. & Pratt, D., 2002). It is doing so in the light of the popularity task-based teaching and group activities in mathematics enjoy in some western countries as reported in numerous studies (e.g. Anghileri J., 2002, Edwards J., 2002).

By using the case-study approach, the study I draw on in this paper is an ongoing doctoral study that started in 2001. Throughout the 3-year longitudinal study in two schools, I have got a closer relationship with the teachers involved, understood the processes taking place and what teachers do and think in relation to the mathematical tasks used in the lessons, and how and why. It is evident from the interviews and observations that the nature and the extent of effectiveness of the mathematical tasks used are very much related to the professional experience of the teachers.

I believe that the study makes a strong case for how increasing and enhancing classroom interaction through task-based teaching can help to foster students' cognitive skills and overall mathematical ability.

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A STUDY OF STUDENTS' STRUCTURING OF TILE ARRAYS AND FIGURE RECONSTRUCTIONS RELATED TO THE AREA OF TRIANGLE

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This paper presents the result of a study of the structural and strategies development of primary school 3-6 grade students' triangular drawing of array, and to investigate students' use on reconstruction strategies as to understand students' conception on the formulation of the formula for finding the area of triangle. Students' array drawing were classified, on basis of numerical properties, into five strategies, that appear to be developmental, and reflected spatial properties of array at four cognitive levels. Students' drawings were sorted in two ways: the numerical properties of arrays, and, the basis of perceived structural similarities that reflected spatial properties of arrays. Types of strategies observed for task of drawing include 1) Use little units to drawing, 2) Use self-order units to drawing, 3) Primitive drawing, 4) Use estimate units to drawing, 5) Array drawing- All lines. The classification of cognitive spatial structure that describes students' increasing level of knowledge of array structure is from Level 1) Unable to draw, Level 2) Incomplete drawing, Level 3) Primitive drawing, to Level 4) Array drawing.

Some particular strategies concerning the process of figure reconstruction producing were also analyzed, and provided us with the notion of teaching for the conception of area formulation of triangle.

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ELEMENTARY STUDENTS' CONCEPTUAL UNDERSTANDING OF MULTIPLICATION IN SOUTH KOREA

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To develop a national textbook with multiplication lessons based on the new 8th National Curriculum, an investigation was carried out on the conceptual understanding of multiplication by elementary students who had learned multiplication according to the current 7th National Curriculum and an analysis compared results with current textbooks. We aimed at obtaining recommendations regarding acceptable parts of current textbooks and new additive contents. We used a questionnaire that contained open-ended questions such as "what is multiplication", creation of a multiplication word problem, and asked students to explain the meaning of '7x6'. 150 elementary students in grade 3 participated in this study.

The second requirement on the questionnaire-creating a problem- and the third requirement-explaining the meaning of '7x6' had rates of correctness of about seventy-five percent and seventy-one percent, respectively. Current textbooks deal with making problems and contain various word problems in each lesson. There is the inclination of instruction which is dependent on mathematics textbooks in elementary school though there are diverse factors which effect learning results. This result implies that current textbooks are effective at teaching students to create a multiplication word problem and explain the meaning of a multiplication expression.

On the other hand, about thirty-seven percent of students used additive ways to answer the first requirement on the questionnaire, in particular, to define multiplication. Also, about twenty-nine percent of students used a metaphorical method based on the relation of addition and multiplication to define multiplication. They wrote that it enables addition to be calculated faster or that multiplication is an upgrade of addition. It is clear that lower grades in elementary school have difficulty defining mathematical concepts in developmental psychology. However, current textbooks do not provide chances to inquire into the meaning or definition of multiplication; instead they offer a numerous variety of multiplication problems. Considering this it may be necessary to consider the communication in mathematics classrooms. Important ideas emerge from group thinking (Yakel, 2001). Moreover, through this thinking students might have an opportunity to establish more clearly the kinds of relationships that connect the formation of intuitive knowledge with consciousness (Vergnaud, 1994) in multiplication conceptual understanding.

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A STUDY OF FIFTH-GRADE STUDENTS' PROBLEM-DEVELOPING AND RELATED FACTORS

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The subjects of this study were 110 randomly selected fifth-grade students from Chang-Hua county in Taiwan. Four instruments [Problem Posing Measure Tools (Leung, 1997), Test of Reading Comprehension (Chen, 2000), Test of Children Mathematical Ability (Ke, 1994)] were used in this study. Eighteen students proportionally stratified to investigate students' concept about problem-developing and the cause of difference and direction of thinking. For data analysis, t test of independent sampling, two-way analysis of variance by dependent sampling, and chi-square test were used.

The results indicated that: (1) there was a positive correlation between students' mathematical abilities and problem-developing abilities. The high mathematical ability group outperformed the middle mathematical ability group in developing problems, and the middle mathematical ability group outperformed the low mathematical ability group in developing problems. (2) There was a positive correlation between students' reading comprehension ability and problem-developing ability. High reading comprehension ability group outperformed the low ability group. (3) Students' problem-developing ability was significantly different among different representation formats. Students' had better problem-developing performance on the drawing format, compared to both word format and answer format. However, there were no differences in developing problem between word format and answer format. There positive correlation between word and graph formats was the highest, followed by the correlation between graph and answer formats, then by the correlation between word and answer formats. (4) The types of mistakes made by students, of the problem-developing, were different among different problem representations. For word and answer formats, mistakes shown were mostly "not a question" and "questions with insufficient information" type of problems. For graph format, the mistakes were mostly "not a questions", "nonmathematical", and "data exceeded". (5) There were significant differences in mistake types of the problem-developing among the different mathematical subjects.

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UNDERSTANDING MATHEMATICAL REPRESENTATIONS AND TRANSFORMATIONS OF FUNCTIONS AND THEIR GRAPHS WITH THE USE OF ICT

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This paper aimed to inquire how a teacher used Information and Communication Technology (ICT) to teach transformations of functions and their graphs could benefit students' learning in a secondary school in the UK. The impact of using ICT in learning was examined in the framework of APOS theory.

The teaching and learning of the concept of function can be problematic due to its various multiple representations. Hennessy et al. (2001) assert that ICT speeds up the graphing process, freeing students to analyse and reflect on the relationships between data. It has been argued that the usage of ICT is highly beneficial in the classroom. Therefore, this paper focused on investigating how students' learning about functions could be illuminated with the use of ICT. A sequence of five lessons was observed and videotaped in a Year 11 class whilst the teacher was teaching transformations of functions and their graphs using both ICT and paper-and-pencil calculation and drawing. Afterwards, interviews with the teacher and six students were carried out and a task was set for the students. Two groups, computer and paper-and-pencil, were asked to create a quadratic function, produce the graphs and transformations and describe how these were achieved. The computer group used Autograph while the paper-and-pencil group worked on duplicated sheets. APOS (Action –Process –Object -Schema) theory (Asiala et al. 1996) was used for data analysis. The data illustrates that the paper-and-pencil group had to repeat procedures including substituting values in functions one by one and drawing the graphs based on the evaluation of independent points to make their transformations (the network of Actions, Processes, and Objects) a multitude of times to approach the task. However, the computer group began with inputting functions (Actions), and then skipped Processes. Their mental Objects of functions were brought out to operate the transformations as they had already learnt about how the graphs could be transformed. It can be shown that, with ICT, students were able to perform the task more flexibly and instantly.

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FROM LEARNERS' PERSPECTIVES: TEACHING FRACTIONS

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Mastering fractions is a major hurdle for students at Key Stage 2 and beyond. Researchers (Aksu, 2001; Smith, 2002; Solange, 2005) have found that the complexity of concepts of fractions would lead to different levels of difficulties in learning and teaching of it. In Asia like Hong Kong (CDC, 2000), the topics of fractions involving mixed operations are designed to implement from Grade 4 and onwards. In this study, we would like to explore and investigate how students can perform and understand the mixed operations in fractions, especially for division of fractions in Grade 5 through their learning outcomes. Students in the lessons were questioned and encouraged to explain and justify their solutions. Thus, while students were invited to work together and conduct thoughtful investigations with appropriate fraction tasks, they would able to build mathematical ideas. This study involved replication for a multiple-case research methodology. Further, the design is considered to be a literal replication since each case was expected to yield similar outcomes as a result of similar conditions being in place (Yin, 1994). Based on a government's mathematics curriculum project (EMB, 2006), we select appropriate teaching strategies and try out different teaching aids to deal with students' problems through analysis on their learning outcomes and scripts collected. All of the research sessions were videotaped and students' original written work and teacher/researchers' notes were also carefully collected. In addition, classroom learning activities and high-level mathematical tasks are designed for students with the collaborative lesson preparation with case-study teachers. Qualitative data including teachers' interviews, classroom observation and students' annotated work are collected from about 200 students in primary schools in Hong Kong.

The study reveals the teaching strategies adopted in this study become a model for classroom instruction of division of fractions and students should give sufficient time and the opportunity to explore mathematical ideas deeply in a supportive environment. Through case-study teachers' experience sharing in effective use of teaching aids such as fraction cards, fraction strips and using IT animations in teaching division of fractions to help students develop the concepts and operations of fractions, teachers agreed that students could easily understand and master the mathematical knowledge.

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DIFFERENCES OF STUDENTS' UNDERSTANDING OF GEOMETRIC FIGURES BASED ON THEIR DEFINITIONS

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Generally we solve various kinds of problems by using properties or relations of geometric figures, so that it is quite important to acquire an understanding of the figures. In spite of this importance, many students don't understand the figures properly. Previously, to improve students' understanding of figures more appropriately, I have identified the ordered states of understanding for relations among figures, between Level 2 and Level 3 of van Hiele (Matsuo, 2000).

The purpose of this study is to clarify differences of students' understanding of geometric figures with respect to their definitions based on the ordered states of understanding. Four states of understanding relations among figures were identified and ordered (Matsuo, 2000); State 1: students are unable to distinguish between two geometric figures, State 2: they are able to identify both figures respectively, State 3: they distinguish between them based on their differences and regard them as the same based on their similarities, State 4: they are able to understand the inclusion relation between the two figures.

By the survey of random classification of quadrilaterals for eighty-eight 6th and sixty-three 8th grade students in Japan, I found that transitions of states of understanding differ depending on the characteristics of figures. Notably, the transition of students' states of understanding in terms of the relations of squares & rectangles was from State 2 or State 3 to State 1 or State 4, while the transition of those of rectangles & parallelograms, rhombi & parallelograms was from State 2 to State 4. These transitional differences might be due to the difficulty of interpreting definitions on which students can identify similarities between two figures. Together, I propose that it is required to devise how to teach geometric figures or these relations on the basis of features of their definitions.

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STAGES OF DEVELOPMENT OF KEY CONCEPTS AND SKILLS IN ELEMENTARY MATHEMATICS

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A major goal of the research was to provide a framework for describing the phases through which students pass as they acquire an understanding of concepts and skills. To validate that there are recognizable, consistent phases that describe student learning in mathematics, a three-year project was undertaken to identify the indicators of each phase and to develop diagnostic tools that will place a student in a phase of development for each of five stands at the elementary level.

Research shows that expertise in teaching mathematics includes developing a deep understanding of mathematical concepts and the relationships among them in order to advance student learning (Borko & Putnam, 1995). If students are able to connect a new concept being taught to previously learned concepts, it is much more likely that the new knowledge will be assimilated. Educational researchers have attempted to describe a few regular patterns, convinced that a shared understanding of these patterns by teachers can lead to optimizing learning for all students.

Method

Data was collected in three provinces in Canada through interviews with Kindergarten to Grade 3 students (N=6000) and paper tests with Grades 4 to 7 students (N=8000). The first step of the research study was to validate a set of developmental maps, specifically designed to show the developmental phases through which students progress in their understanding of mathematics, as well as to provide specific indicators that describe what students know and can do at each developmental phase. The draft developmental maps were validated based on data gathered in two stages.

Findings

Findings arising from this analysis supported the premise that there are developmental stages the five strands of mathematics. The presentation will describe the developmental maps with examples of specific findings for mathematics teaching and learning.

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WORKSHOPS TO IMPROVE TEACHERS' KNOWLEDGE

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This project has as its long range objective to create an effective program to improve teachers' pedagogical content knowledge in Mexico. For this, we implemented pilot workshops with elementary school teachers to reveal and improve their knowledge. Their conducting theme was: "Which are the most important concepts in arithmetic?", and were centered on what Ball and Bass (2000) defined as Mathematical Knowledge for Teaching: a) Figuring out what students understand, b) Analyzing methods different from one's own, c) Unpacking math ideas and d) Choosing representations to effectively convey math ideas. Some sessions of the workshops were dedicated to discussions on Cooper's et al. (2006) three levels of pedagogies, "technical", "domain" and "generic", which, according to these authors are all necessary for successful teaching. Each workshop was organized as a series of tasks, problems and reading materials, given to the teachers or produced by them.

The main instruments to collect data were monthly teachers' classroom observations, homework assigned by them (these two analyzed according to Askew's et al. (2000) four components: tasks, talk, tools and, relationships and norms) and the solutions, discussions and comments of the teachers in the weekly sessions.

Examples of the tasks of the workshops are: "Design a final exam for the arithmetic part of your course with between 6 to 8 questions", "Give some errors or incorrect notions your students have." and "Give some difficulties you have in teaching." In other sessions we proposed a math problem or task, which was solved by the teachers at home and then discussed in the next session. The reading materials distributed to the teachers were summaries of articles related to pedagogical methods and teaching strategies.

The teachers already bring a "primitive", mostly instrumental knowledge based on their experiences ("Do it like this because it works"), which is hard to change. However, during the workshops the teachers showed an increased interest in choosing proper tasks for their students and were more motivated to reflect upon their students' possible thinking, including difficulties they might encounter. Inside their classrooms, we evidenced an effort to use more representations and to interact more.

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HOW ABEL UNDERSTOOD THE SUBSTITUTION METHOD TO SOLVE AN INTEGRAL

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The aim of the study was to observe a first year Life Science classroom at Eduardo Mondlane University, where the “opportunity to speak” (Marr, 2000) is a practice in order to seek how the language used and other ways of communication present in group discussions affected students’ internalization of ideas and ways of reasoning, which emerged from the discussion interaction (Powel, in press). This involves the connection between students understanding of a concept and her (his) ability to use academic language related to the concept, “the means to speak” (Marr, 2000).

The study showed that the classroom interaction helped the students to realize how to choose the function we want to substitute by a new parameter and to understand the method of substitution when they solve an integral. In addition, it was observed that the informal registers were common in the conversation, whereas the students had some difficulties when the formal language was used.

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PHYSICAL EMBODIMENT OF BALANCED EQUATIONS: A MATHEMATICAL PERFORMANCE

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A "digital mathematical performance" is created in the form of a video recorded skit to help both teachers and students learn the gesture of balanced and unbalanced equations. Similar to the use of physical balances, the developed skit introduces new metaphor, albeit in gestured ways, to solving equations – specifically, the metaphors of loading and unloading and of falling out of balance.

A GESTURE, SKIT, AND NEW DISCOURSE

Our overall goal is to use directed role playing to encourage students to learn from physical experiences that have useful interactions with specific mathematical concepts. When solving equations, it is important to keep them "balanced" – a concept that can be visualized with "arm scales" (see Figure 1). To embody the concept of inequality, "imbalanced" equations, a new exaggerated gesture has been created. In addition to the use of physical balances and mnemonic devices, we proposed that students could learn such mathematical concepts and the associated physical gestures through role playing in a skit[1].

Figure 1. The gestures for balanced (left) and unbalanced (right) equations.

A new discourse may be required to understand the role of mathematical skits, or performances in general in learning mathematics. The complexity thinking discourse takes abstract concepts to be emergent entities (Namukasa, 2005). These abstract concepts emerge from human experiences, from recurrent human actions and interactions to form stabilities or patterns (i.e. mathematical concepts). However, to a new learner, these emergent concepts and objects may not necessarily pre-exist. Thus, there are potential benefits in re-enacting, in exaggerated ways some of the actions and interactions which may have led to the stabilized concepts.

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[1] See http://www.atkinson.yorku.ca/~sychen/research/math/DMP.html for the video clips.

THE DEVELOPMENT OF ELECTRONIC PORTFOLIO ON MATHEMATICS TEACHING FOR STUDENT TEACHERS

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An electronic portfolio is a purposeful collection of work that demonstrates effort progress and achievement over time (Barrett, 2006). This paper is a result of a 3-year study of how 40 students learned to create their own electronic portfolios. During this time, I lectured students 18 weeks for 2 hours per week on mathematics methods, followed by 18 weeks of teaching practicum 6 hours each week, and one year field placement course that I supervised.

At the start of the first year, the study focused on learning to teach mathematics in elementary school. The data collected in digitalization included reflective journal of learning mathematics, mathematics paper and teaching video summary, critical incident discussion, lesson plan and teaching practice. The depth of contents was further expanded in the following years. A web-based communication plane was set up by a former student. Most of the documents I assigned were uploaded for sharing and discussion among the students.

By using self-study method, the portfolios included planning and goal setting, teaching philosophy, framework of creativity, learning material and reflective processes at least, others were added separately. The students created their own electronic portfolios to record their learning processes and also self evaluated the works. The growth of their professional development was observed. Benefits of the electronic portfolio included opportunities to reflect, and better access and organization of professional documents, increased technology skills. The results were the same with Wetzel & Strudler (2006), and the cost is really included the amount of time and effort expended. Some students excitedly to show me their resume collected from their electronic portfolios for finding a teacher job.

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TEACHERS' PRACTICES IN TECHOLOGICAL ENVIRONMENT: THE CASE OF USING SPREADSHEET

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This communication reports some resultats of my doctoral thesis (Ozdemir Erdogan, 2006) that focuses on the teachers' practices in a technological environment. It aims at contributing to the dimension of 'teacher' research on the integration of technologies in the teaching of mathematics. The dimension, which has developed recently, aims to take into account the difficulties that this integration meets.

Based upon the results of earlier research, we have defined two axes for our field of study: the complexity of the situations of integration of technological tools for the teacher and the variability of the practices.We choose a model of analysis proposed by Monaghan (2004) borrowed from Saxe (1991) to examine the influences of key factors on the activity of the teachers and to understand 'holistically' the complexities of practices.

In this communication, a comparison of practices, using the Saxe's four parameter model, is proposed. They are the practices of two teachers observed in ordinary conditions teaching the "séquence" in the 11^{th} grade, literary stream in France. In this class, the official texts impose the use of technological tools, in particular, a spreadsheet and this is taken into account in the final secondary school examination – baccalauréat-.We show the positions different from two teachers for the same teaching and the consequences of these positions on their management of class.

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CREATIVE THINKING AND MATHEMATICS LEARNING ABOUT CIRCLE OF NINTH-GRADE STUDENTS THROUGH ART WORK

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"Imagination is more important than knowledge" Albert Einstein, the great mathematician, once said. Imagination is the path leading to creative thinking. Opening an opportunity for students to learn how to create vivid imagination is one of the best methods to develop creative thinking in each student. Creative thinking is also required in students' mathematics classroom.

This study focus on ideas to promote students' creative thinking through art work by using art as a creative tool to enhance learning and understanding in mathematics regarding circle. Art work was selected as creative tool for ninth-grade students' mathematics classroom as it seemed to facilitate the colourful imagination of young students. The target group comprised of six ninth-grade students obtained by simple random sampling from those in the class who interested and volunteered to cooperate. Data analysing based on Guilford framework. Multiple approached and resources for data collecting were obtained including students' art works and mini-diary records, teachers' math- diaries, researchers' field notes, and interviews. Also, video-tape and audio-tape record were used for observations the students' behaviours during their art work presentation.

The research results reflected the students' ability to link between creative thinking and mathematical understanding then transferred and communicated their understandings through art works of which were individual differences. Each student was unique depending on the art work and methods they initiated and presented. Freedom and opportunities to communicate through art work in mathematics classroom enhanced student's higher capability to learn mathematics. The students expressed their clearly understand of mathematical concepts, in this case: about the concept of circle, and successfully presented in various kinds of art works in their own ways. A significant principle of teaching mathematics, the researchers learned from this study was that the teacher should lead and help students to learn mathematics within their own contexts and in daily-live situations. Allowing arts and imagination become parts of mathematical learning activities can enhance and motivate more interests in mathematics. In addition, art work and creative thinking also challenge students' mathematical ideas and facilitate more mathematical activities.

Keywords: Creative Thinking, Mathematical idea about Circle, Art Work

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ON UNDERSTANDING ABSTRACT DEFINITIONS

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We report here on our research into the ab initio understanding of abstract definitions by students in Grade XII. Our findings are based on experiments conducted on these students in which they are given definitions in graph theory which they are not familiar with and asked to answer questions based on these definitions. Our research allows us to propose a model for the formation of a certain primitive concept image of a definition. Our study leads us to conclude that these primitive concept images developed by the students are strongly influenced by the meanings of the keywords and labels used in these definitions.

RESEARCH METHODOLOGY

The questions which initiated our study are the following: (i) What are the factors which influence the formation of what we term "primitive concept images" when a definition is seen for the first time? (ii) what are the initial obstacles in conceptualising a new mathematical definition? (iii) Are the learners able to make use of visual and verbal cues implicit in the given definitions? We attempt to shed light on these questions based on two tests. These tests involved simple questions based on a set of new definitions in graph theory which were not introduced nor discussed in the classroom. The two tests were identical except that some keywords/notations used in the definitions were changed. The participants of the tests had otherwise identical test conditions and background. Detailed analysis of the results of the experiment, including interviews and group discussions, led to the following observations and conclusions.

Observations and conclusions

We observed, based on pictorial representation of `graphs' (respectively `networks'), that some students confounded the combinatorial object with the notion of "graph of a function". The pictorial images were more varied and unusual when the word `network' was used than the word `graph.' Even when these representations were unusual, quite a few students had successfully solved most of the problems. Some students had given rather precise solutions to all the problems. We observed that in the course of the test, the students dynamically modified their images to fit their growing comprehension. A conclusion that we arrived at is that, the labels used in the definition have a strong influence on the formation of primitive concept image when they have multiple meanings or when they have been used in a different context.

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UNDERSTANDING P(X= C)=0 WITHOUT USING INTEGRATION WHERE X IS CONTINUOUS RANDOM VARIABLE

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P(X=c)=0 can be easily derived from integration. But it is difficult to let students accept the justification of the equation, if we do not use the integration. For example, the probability of some baby's weight is 3.5Kg is 0, but we can find the baby whose weight is 3.5kg.

About discrete random variable, the probability 0 means that the incident never happens. If we try to understand the probability 0 in continuous random variable reflecting that of discrete random variable, we should be thrown into confusion.

Gu, Jaheung etc.(1992) described that the value of P(X=c) is not 0 because certain value of continuous quantity always means some interval. That is to say, it is not X=c but it is c-e < X <c+e for some minute e. If we break into continuous quantity, we only see continuous quantity infinitely.

The discussion about P(X=c)=0 may be beyond mathematics. Though we can use integration to show P(X=c)=0, we do not clarify the foundation about relation continuous quantity and infinity. We only provide mathematical explanation for it.

But this discussion about P(X=c)=0 for continuous random variable can be the start line where students think about continuous quantity and infinity.

25 high school students were asked about P(X=c)=0 for continuous random variable. About 50% of them said that they have no idea. One student said that it is 0 because of one among infinity. Some students said that it is 0 because c is not interval. One student answered like 'Let's think that X means the second hand of watch. If we consider the thick of the second hand, we can not decide that the hand points to c correctly'.

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THE PROCESS OF MATHEMATICS TEACHER CHANGING ROLE IN CLASSROOM USING THE STORY AND DIAGRAM METHOD

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This case-study research focused on exploring the Mathematics teaching process using the Story and Diagram Method (SDM) in eleventh grade students. The teaching approach, called SDM, provided an opportunity for students to express themselves by telling stories of their interest together with diagram drawing in order to explain mathematical structure and the connection between things, the students observed, in their Mathematics class.

The results found that the teaching role of the two teachers who were investigated changed and expanded. Teachers reflected that by using SDM, focused more on facilitating the discussion through students' stories telling and diagram drawing. While using SDM, both students and teachers enjoyed critically thinking process and the teacher also challenged their students' ability to solve mathematical problems. Moreover, the teacher expressed that by using S&D approach, teachers and students needed to be more interactive participation in class. The teachers perceived less anxiety than when they taught mathematics by traditional way. Furthermore, the students showed more clearly understanding while they explained or answered the mathematical questions. By using the Story and Diagram Method (SDM), it gave the students an excellent opportunity to discuss independently and think critically. The students gain more insight and perceived the connection between things, from abstract and concrete. In conclusion, the researcher believed that, in an effort to enhance the professional development, teacher needs to explore an innovative teaching technology. This research suggested that SDM was a useful innovative approach for mathematics class. Its implications and usefulness in the other fields of teaching were also suggested.

Keyword : Story and Diagram Method, professional development, changing role

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STUDENTS' REASONING ABOUT CUBE BUILDING RECONSTRUCTION USING ITS TWO-DIMENSIONAL INFORMATION

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Reasoning and proof were essential mathematical skill and process, of which mathematics teacher needs to establish in all students in order to develop their ability for solving mathematics problem. This study demonstrated the use of an innovative "instructional diagram" as a new instrument in the class. The new instructional diagram initiated by the researcher who taught about "spatial sense" aimed at challenging the seventh grade students to express their systematic reasoning process toward cube building reconstruction by using its two-dimensional information. According to the data collection through observations and interviews of teaching and learning performance as well as exploring the students' work, it was found that, the students in this study could learn more about reconstruction and checked their understanding easier than those who were taught by the traditional approach. As a result, the researcher developed a teaching instructional protocol consisted of mathematical language regarding reasoning and proof. The new instructional diagram was suggested to be useful for implication.

Keyword : Reasoning and proof, innovative instructional diagram development, spatial sense, cube reconstruction, mathematical language

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PRE-SERVICE SECONDARY MATHEMATICS TEACHERS: A COMMUNITY OF PRACTICE

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Lave and Wenger (1991) conceptualise learning as a movement from *legitimate peripheral participation* to fuller forms of engagement in a community of shared practice. Learning to teach can be seen as a process of identity formation that brings together one's past experiences, present beliefs and future possibilities. Thus knowledge of teaching which is first established as a high school student and may conflict with the current reform agenda might be recast and re-imagined by pre-service teachers while at university through engagement with theories about teaching and by imagining themselves as teachers. Becoming a teacher is thus an evolutionary process which is deeply connected to ongoing activity in the practice of teaching.

A random sample of sixteen pre-service teachers, eight from each of two universities, was chosen from those applicants who accepted a place in the teacher education program. Each participant was interviewed individually on three separate occasions.

As expected, the participants all had very definite ideas about the characteristics of a good mathematics teacher which they based almost exclusively on memories of their own teachers (see Prescott & Cavanagh, 2006). They discussed why they had decided to embark on a teaching career, generally reflecting affective and idealistic goals such as altruism or a service-orientation. Some participants talked about how the style of teaching depended on the kind of cooperation from the students. A class that was noisy or misbehaving would have one style of teaching (traditional 'chalk and talk') and a cooperative class would have another (activities or group work).

The pre-service teachers adopted a learner's perspective of teaching. They interpreted their teachers' actions from the narrow viewpoint of students who lacked the pedagogical knowledge or expertise to evaluate properly what they saw. The first stage of their identity formation, part of imagining themselves as teachers, was biased in favour of the most obvious aspects of teaching at the expense of developing approaches to teaching that are more closely aligned to reform practices.

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YOUNG CHILDREN'S PERCEPTION OF A RHOMBUS

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The object of my study is to obtain information about what features of geometric objects seem to be observed by 1st grade pupils, and to analyse to what extent the pupils' perception of the objects matches the formal definition as well as the teacher's concept image. In particular I will in this presentation be concerned with what properties of a rhombus seem to be observed by the pupils.

The data are based on video recordings from a class of 18 pupils and their teacher in a whole class situation. They work with naming and classifying 2D geometric objects. The objects are computer generated and depicted in colour on an interactive whiteboard. The objects can be dragged around on the board but their shape can not be altered and neither can they be rotated. The board contains a number of different polygons and a circle. The teacher has also drawn, not very accurately, a rhombus on the board.

A basic framework for discussing development of geometric concepts is the van Hiele levels (van Hiele, 1986). Geometric concepts have a double nature – on the one hand they are conceptual and on the other hand they have a figural nature (Fischbein, 1993). This double nature provides several possibilities as to which properties of the objects are observed. Zaslavski and Shir (2005) have made a distinction between the overt (integral) parts of a geometric object and the more hidden (latent) parts, and they have investigated how the integral and latent parts play different roles in students' view on what to take as acceptable definitions of geometric objects.

My findings seem to provide evidence for the claim that for these pupils the latent parts of the rhombus play a more important role than in the study by Zaslavski and Shir (2005) who worked with older students. My study also reveals that the pupils' focussing on the latent parts of the rhombus leads to a certain tension in the interaction between the teacher and the pupils.

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STUDENTS' BELIEFS AND ATTITUDES IN A MATHEMATICAL LITERACY CLASSROOM

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A Minister of Education once observed that the absence of a clearly-articulated value-system in South African schools lead to fundamental human rights being violated in the classroom (Asmal, 2001). The new curriculum requires educators to instil in learners knowledge, skills and values while engaging them in problems in context. Do the learners believe that this approach enhances learning, and what impact do the values have on their everyday lives?

Success with the integration of values into all aspects of the curriculum has been achieved in Sathya Sai Schools which are spiritual in nature (Taplin, 2005). Is this possible in mainstream schools in South Africa? This study, which forms a part of a larger study, investigates the attitudes and beliefs of learners towards mathematical literacy and the values that are transmitted through the contexts and by the teacher. These five grade 10 learners are from a secondary school in Cape Town, South Africa.

The learners' beliefs were elicited using interviews, questionnaires and journals kept by the learners. Classroom lessons were observed and videotaped to capture these learners' participation and attitudes during the lessons. In this paper, the views of two learners, a male and a female, from the school are interrogated in order to gauge whether their explicit beliefs were constant or contradictory, and whether the views that they expressed were mirrored in their classroom behaviour.

From a single classroom, two opposing views on the values transmitted emerge. For the female student, the values that she identified in her mathematical literacy lessons impact on her social interactions and motivate her into learning the content. Her male counterpart, however, suggests that the context only plays a role in the understanding of the content. Both learners' beliefs were mirrored in their classroom behaviour.

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FOUR FIRST YEAR UNIVERSITY STUDENTS' ALGEBRAIC THINKING AND ITS RELATIONSHIP TO THEIR GEOMETRIC CONCEPTUAL UNDERSTANDING: A CASE STUDY

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This study aims to investigate how first-year university students at Universidade Pedagogica (UP) in Mozambique bring their knowledge and thinking of algebra in understanding and working with geometry. It explores how they connect and use algebraic and geometric concepts and investigates whether this connection promotes students' conceptual understanding and problem solving performance in geometry. This study can best be approached through qualitative research methodology which provides space for explorations. The idea was to collect extensive data with an open mind. As the study progressed, the data were continually examined for emerging patterns and insights. This is a type of qualitative methodology called 'Grounded Theory' (Savenye and Robinson, 2004) through case studies. The study involved eight first-year university students who were enrolled and were participating in Euclidean Geometry (semester 1) and Analytical Geometry (semester 2) courses at UP. The data collected over one academic year involved students' written responses to pre/post-tests, course tests, interviews on their test responses, concept maps, classroom observations on selected geometry topics, and interviews with the course lecturers. This paper only reports on the analysis of pre-test responses (algebraic and geometric knowledge base, connectedness and strategies) of four target students in light of the conceptual model algebraic thinking in geometrical understanding and the framework on learning and transfer drawn from Prawart´s work (Prawart, 1989). From the analysis it can be seen that algebraic thinking aided geometric thinking towards a partial solution of the tasks. For a complete solution of the tasks it was necessary for students to incorporate visualization and construction processes. These results seem to confirm Prawart's assertion that in order for one to develop connectedness, it requires one to possess key concepts and procedures (from different domains) which provide the glue that holds cognitive structures together so as to make representational links.

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GENERALIZATION BY COMPREHENSION AND BY APPREHENSION OF THE KOREAN 5TH GRADE GIFTED STUDENTS

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The aspects of students' understanding may have a kind of varieties depending on the levels of students, the characteristics of the contents, the methods, etc. Freudenthal says every student has a logical ability of an appropriate level (Woo, 2000). Freudenthal(1978) distinguishes generalization by apprehension from comprehension. He says that the former is better than the latter educationally. We usually think comprehension is similar to inductive reasoning, and apprehension a kind of insight from generic examples. The aim of this study is investigating the aspects of Korean 5^{th} grade mathematically gifted students' generalizations and the levels of their reasoning on the basis of Freudenthal's idea.

Subjects of our study are twenty-six fifth grade students involved in an institution for mathematically and scientifically gifted students. We gave lessons them for six hours. The former two hours were assigned to tasks related to Diffy activities (Kang, 2005) presented on questionnaire 1. The rest of times were assigned to tasks related to divisors and multiples presented on questionnaire 2. Subjects' data were collected through their scripts on the papers for their activities. And we analysed on their responses for six tasks showing their characteristics well and we present our main findings by classifying into four categories.

The conclusions for twenty-six subjects are as follows. Firstly, when the tasks are not so complicated, approximately forty percent of subjects showed their understanding by apprehension, but the percent were lower when arbitrary variables were included. Secondly, so many students have an intuitive understanding on the role of counter-example. Thirdly, students' abilities of controlling variables (Inhelder & Piaget, 1958) are so low. Mathematical proof is one of the most difficult subject areas in school mathematics. We think that our study shows a possibility for more gradual transition to mathematical proof by apprehension.

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A PROBABILITY SIMULATION METHOD DESIGNED BY STUDENTS FOR A PROBLEM ARISING FROM REAL LIFE

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Instruction on probability should be incorporated as a modelling process of problems arising from reality(Batanero, 2005). Henry(2001, recited in Batanero, 2005: 32) distinguished three different stages in the process: the pseudoconcrete, formalization and validation model level. Heitele(1975) suggested that as a pseudoconcrete model, simulation can act as an intermediary between reality and the mathematical model. Benko(2006) investigated how the 6th graders identified the fairness of dice games through experiments. The purpose is to describe how students design probability simulation appropriate for a problem arising from real life in the peseudoconcrete modelling process. The participants were four students of the 10th grade and their academic achievements in mathematics were very good. They learned the basic ideas on probability in middle school but haven't taken lessons on probability yet in high school. The task given to them was as follows: 'How many bags of snack with a coupon will you have to buy so that you can obtain a complete set of six kinds of coupons?' To solve it, students designed the simulation as follows: 1)The number of snack bags students first estimated was 10, 15 and 30. 2)They found that the six kinds of coupons correspond to six spots of a dice and the spots obtained when the dice is thrown can determine the kinds of coupons. 3)To identify whether students can collect all the kinds of coupons when buying bags as many as estimated, they were supposed to make an experiment to examine whether they can get all the spots from 1 to 6 when throwing dices as many as estimated. That is, they throw 30 dices when they estimate they have to buy 30 bags. 4)To identify how high probability to get all the spots when throwing as many dices as estimated, they decided to repeat dice-throwing fully. When they repeated 100 times of dice-throwing using computer, they determined that the number of the bags estimated was proper when the frequencies that they got all the spots were about 75. If the frequencies were above 75, they throw less dices about 100 times.

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STUDENTS' CONCEPTUAL DEVELOPMENT OF EIGENVALUE AND EIGENVECTOR BASED ON THE MODEL DEVELOPMENT SEQUENCES

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Evaluating the eigenvalues of a square matrix is a simple process algebraically but understanding the meaning of the value is not easy relatively.

This paper is focused on an instruction design based on situation model process for students' conceptual development of eigenvalue and eigenvector in college linear algebra class. We present a situation model which controls the number of milk cow for a regular production of cheese modifying Leslie population model. The model development sequences-model-eliciting, model-exploration activity, model-adapting activity-described in this research were applied.(Lesh, R., Cramer, K., Doerr, H. M., Post, T. & 856-Zawojewski, J., 2003).

Dorier(2000) pointed formal approach about new concept as the main reason of obstacle of students' learning linear algebra. We already have discussed students' conceptual development of eigenvalue and eigenvector in differential equation course based on reformed differential equation using the mathematical model of mass spring according to historico-generic principle in 2004 with the fund of Ewha Womans University. While former project used visual intuition model by computer in reformed differential equation, this research presents another situation model of eigenvalue and eigenvector in college linear algebra class.

We have collected audio-recording of all the class session, which were transcribed for dicourse analysis. In addition, data such as student interviews, the students' worksheets were collected to supplement the result of the dicourse analysis.

Based on the result of the analysis, students show understandings of conceptual development of eigenvalue and eigenvector through the process of searching a model. And they can learn the method of teaching mathematics as pre-teacher.

The meaningful conceptual development to learners offers the insight and the first step toward the problem solving which includes the concept.

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DIFFERENCES IN TEACHING PRACTICES IN HONG KONG EIGHTH-GRADE MATHEMATICS CLASSROOM

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One Hong Kong classroom from the Learner's Perspective Study (Clarke, Keitel, & Shimizu, 2006) was analysed for the purpose of examining how expository teaching practices across 16 consecutive lessons differ in terms of teacher's PCK and how these differences influence students' understanding and development of mathematical concepts. The results indicate that noticeable differences exist in the expository teaching practices which influence students' understanding and development of mathematical concepts.

The expository teaching style, typically utilized in whole class instruction, is still prevalent in Hong Kong. This type of instruction is described as ineffective for learning mathematics conceptually. Hong Kong students are depicted as passive learners from a Western view (Watkins, & Biggs, 2001). However, Lopez-Real and Mok (2001) argue that mathematical ideas are explained in depth during the expository instruction in Hong Kong classrooms creating opportunities for students to learn conceptually. Research suggests that what teachers know about mathematics is closely related to their instructional decisions and actions. According to Shulman (1987), teachers must have knowledge of how to teach specific facets of a subject, called Pedagogical Content Knowledge (PCK). In this study, one Hong Kong classroom selected from the Learner's Perspective Study (LPS) (Clarke, Keitel, & Shimizu, 2006) was analysed for the purpose of examining how expository teaching practices across 16 consecutive lessons differ in terms of teacher's PCK and how these differences influence students' understanding and development of mathematical concepts. Lesson transcripts and target student post-interviews were analysed. Utilizing a social interaction perspective, the lessons were examined using a coding scheme to identify instances of PCK during expository teaching. Student post-interviews were investigated for the impact of teacher's PCK on students' understanding and development of concepts. The results indicate that differences in the expository teaching practices across class lessons differ noticeably and that this influences students' understanding and development of mathematical concepts in this Hong Kong classroom.

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ABOUT THE CONTEXT-DEPENDENCY OF UNDERSTANDING THE CLASS INCLUSION OF GEOMETRIC FIGURES

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The study is a part of the larger study in which we examined similarities and differences in the concept formation of geometric figures for the Japanese and Finnish students. The theoretical framework is based on the refinements of the van Hiele theory made by Matsuo (2000) and Silfverberg (1999). The data has been collected from the 6th and 8th graders (n=173 in Finland and n=151 in Japan). The concept formation of geometric figures was studied with two different methods: (1) by the test of the free classification (Test1), where a student may group the given figures freely and a student may establish as many groupings as he/she wants and (2) by the test of the forced classification (Test2), where a student is asked what figures apply to the given name of the concepts of geometric figures. Two basic features of the concept formation of the students were studied: (1) the idiosyncratic meanings given to the individual concepts and (2) the idiosyncratic relations between the concepts induced by those meanings. In the test 1 we applied the method developed by Matsuo (2000) for classifying the type of the relations students seemed to have between their concepts. Correspondingly in the analyses of the test 2 we used the method developed by Silfverberg (1999) for recognizing the five possible relations belonging to the so called RCC5 algebra (cf. Sanjiang & Ying 2003)

In our presentation, we examine how consequentially the class inclusion and the disjunctive classification are applied by the students in these different two test situations. As an example of the results, we state that only about half of those pupils who in the test of the free classification considered the relations squre-rect. and rect.-parallelogr. as class inclusions behaved likewise in the test of the forced classification.

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HOW U.S. ELEMENTARY TEACHERS TRANSFORM MATHEMATICS TEXTBOOKS IN TERMS OF COGNITIVE DEMANDS

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Professional Standards for Teaching Mathematics (NCTM, 1991) articulated that students' opportunities for learning are created by, what Stein and Smith (2000) called, "cognitive demands of task", the level and kind of thinking required students to successfully engage with and solve the classroom activities and problems. However, there were few studies looking at how teachers use their textbooks in terms of cognitive aspects and what factors support and constrain teachers' textbook use.

The purpose of this study was to examine elementary teachers' textbook transformation patterns in terms of cognitive demands and its influential factors.

A total of 166 teachers participated in this study from second through sixth grade. Participants were recruited through Master courses at Mid-Western University and professional development programs in the U.S from 2006 to 2007 fall semester.

This study employed a mixed method design that combines both quantitative (survey) and qualitative approaches (interview & observation). However, due to space limit, this paper addressed only one method, quantitative method. The survey was developed based on the previous studies (e.g., Horizon research questionnaires, 2003), which is comprised of five parts: (1) background information, (2) teachers' perceptions of the cognitive demand of their textbooks, (3) individual-level factors, (4) contextual-level factors, (5) teachers' opportunities-to-learn factors. Correlation and regression analysis were used for data analysis.

Three patterns were identified—the L-L, the L-H, and the H-H pattern. Within the L-L pattern, teachers' responses of the cognitive demand of problems both in textbooks and in teaching were categorized into low level cognitive demand. In the L-H pattern, teachers' responses to the cognitive demand of problems in their textbooks were categorized into low level but their responses of those in teaching were counted as high level. In H-H, teachers' responses to the cognitive demand of problems both in textbooks and teaching were categorized into high level.

Interestingly, all teachers who were categorized as having textbook problems of high level cognitive demand reported that they used problems in teaching categorized into high level cognitive demand, indicating that the cognitive demand of textbooks plays an important role in maintaining the cognitive demand of problems in teaching. Three influential factors, textbook types (standard-based vs. conventional curriculum), teachers' teaching objectives, and teachers' view on textbooks were identified to discriminate transformation patterns.

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QUIET STUDENTS ANALYZE CLASSROOM DISCUSSIONS

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Students in a undergraduate introductory algebra class analysed transcriptions of one of their classroom conversations as part of a class exit interview. Although the constructivist, discussion organization of the class consistently involved most students in mathematical discussions, a few students contributed to discussions rarely. This paper presents the transcript analysis produced by two of these "quiet" students. Given the prominence of both discussion-based teaching standards and the importance of discourse analysis for mathematics education research, Wagner (2005) raises the important question of how to understand silence in the classroom. Even videotapes do not give access to the thoughts of quiet students as they follow (or not) classroom conversations. In this study, quiet students were asked to explain their classmates' thinking at conversational moments that exemplify constructivist discussions, e.g., when students offer incorrect answers and explanations, correct answers and explanations, or ask questions.

Quiet students were able to produce explanations of the mathematical thinking of their classmates in each of these situations. Several discourse moves were prominent in their transcript analysis of their classmates' discussions. In the first place, quiet students frequently used reported speech to animate their classmates' thinking and intentions and to characterize their experience of a constructivist classroom (Tannen, 1989). A more subtle discourse strategy was to explain classmates' conversational moves through presupposing indexicality (Silverstein 1976; Wortham, 2003), by linking an explanation to prior phases of the discussion. A final strategy was to identify markers of authority, the phrasings that classmates used that produced a persuasive explanation. Overall, these discourse strategies indicate that quiet students can make precise and cogent interpretations of the classroom construction of mathematical ideas.

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UNDERSTANDING OF NUMERACY IN ICELANDIC PRE-SCHOOL STUDENTS

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Early numeracy has been the emphasis of numerous research projects around the world (Aunio, Niemivirta, Hautamäki et al., 2006; van den Rijt, Godfrey, Aubrey et al, 2003). National organizations like the Australian Association of Mathematics Teachers and Early Childhood Australia have emphasized the need to realize that numeracy experiences in the early years shape later dispositions to mathematical thinking and learning. In this report, we present the findings of a four year study with Icelandic pre-schoolers with implications for early childhood and elementary education.

Participants in the research were 4 and 5 year old students in preschool. The research questions involved student's development of number sense and operation. The children were in 4 groups, each group worked on problem-solving 2 times a week and number-games 2 times. In the problem solving sessions students were encouraged to articulate their thoughts and provide an argument for their reasoning.

To measure growth over the school year, pre- and post-test was given and students' work during the school year was collected. It was analyzed according to their number sense. Three groups of student's could be identified. First, children that successfully finished the pre-test. Their growth was in their complexity of thinking and clarity in articulation. Second, children that could only solve three or less problems on the pre-test. Their growth was in their ability to solve more complex problems, as well as increased confidence in articulation. Thirdly, students that could not solve any problems on the pre-test. Their growth was the ability to solve the simplest problems and beginning to develop a vocabulary to verbalize their thoughts. A comparison group was selected in the beginning of the school year. The selection criteria was a similar size pre-school, located in comparable neighborhood. Pre- and post-test was administrated to the comparison group. Results will be discussed in this report.

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CONCEPTUAL CHANGE AS DIALECTICAL TRANSFORMATION

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This report concerns an inquiry into the nature and character of conceptual transformation through a process of argumentation in a community of mathematical inquiry. Its theoretical approach is based on Vygotsky's theory of social cognition, which in turn is grounded in the broader framework of Hegelian and Marxist dialectical theories, and systems theory. The study explores mechanisms of conceptual transformation in a dialogical group setting in a community of mathematical inquiry format among fifth graders. It concludes that the process follows a dialectical model, which the author illustrates through an analysis of group discussions of the concept of infinity.

The study adopted a dialectical perspective in analyzing the transformation of students' conceptual development. As such, the unit of analysis was defined as the complex system of the whole group. The overarching objective was that each problem presented to the students would be resolved in a context of communal dialogical inquiry, i.e. in a discursive format in which participants were expected to justify their ideas or procedural moves to and with each other. A problem was presented (e.g. Cantor' paradox) at the beginning of each session and the discussion began immediately; thus the agenda for each session was spontaneous, emergent, and guided both by the group and the facilitator. Four consecutive transcripts—taken from a total of 19 audio and videotapes—of conversations about the concept of infinity were chosen for analysis, because they reflected a progressive sequence in which greater depth, complexity and clarity of thinking about the concept emerged over the course of the discussions.

Briefly described, the sequence indicated by the transcripts consists of 1) an **orientation phase**, planned by the facilitator to elicit and question students' spontaneous conceptions of the terms of the problem, which allows for some opposition between those conceptions and reality to emerge, enter into conflict, and be resolved. 2) During the **building phase**, students started to verbalize their statements as possible solutions to the learning task. Here, the clear meaning of students' statements is gradually articulated; through feedback from the community, the arguments and the structure of argument which they form build slowly through collaborative work. During this phase both the thesis and the antithesis are presented, but it might take additional time for them to be recognized as opposites. 3) In the **conflict phase**, the key assertions are sorted out. Here the opposition presents itself in full, and forces the inquirers to focus on this contradiction (inadequacy)—to be consciously aware of it and to search for a resolution. 4) In the last or **synthesis phase**, a resolution is arrived at which closes the cycle figuratively where it began, but with a new conceptual formation which is enriched and more sophisticated.

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BELIEFS AND MATHEMATICAL REASONING

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I present a research project on beliefs as an influence on mathematical reasoning. The results indicate that three major types of beliefs (safety/security, expectations and motivation) dominates the pupils decision-making, primarily their strategy choice and conclusions.

This study looks at what influence beliefs have in problem solving and more specifically how they affect mathematical reasoning used in solving problematic situations. The first indication from this study was that beliefs influence the student's strategy choice and conclusion (Sumpter 2004), but the question remains: How do beliefs influence the central choices students makes in their reasoning while solving problematic situations? To answer this question beliefs are attributed to students' behaviour, and as a test of accuracy, I see if the student behave in ways consistent with them "having" them. The framework about beliefs is heavily influenced by Schoenfeld (1985) and Hannula (2006). Lithner's (2006) framework is used for analysing student's mathematical reasoning. Data were collected by video recording task solving sessions, interviews and a questionnaire. Three major themes of beliefs stands out: safety/ security, expectations and motivation. They interact with students' emotional state and active goal, influencing strategy choices implementations and conclusions. Theses types of beliefs appear to be rather dominant, especially compared to the students' use of mathematical knowledge.

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MATHEMATICAL CHALLENGES ENCOUNTERED IN CLASSROOMS: HOW MATHEMATICALLY GIFTED AND TALENTED SECONDARY STUDENTS DEAL WITH THESE CHALLENGES

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This short oral communication is based on a pilot study which precedes a larger research study aims to develop a substantive theory on how mathematically gifted and talented lower secondary students in Singapore deal with the mathematical challenges posed within the classrooms. Besides reporting on the preliminary understanding of what mathematical challenges are to the talented students and how they deal with it, the pilot study reported here also focused on the way the students responded during the focus group discussion.

The pilot study was conducted during the period from end July 2006 to beginning of November 2006. The purpose of the pilot study was to trial the actual research questions and to refine the interview guide to be used in the research study. For the first part of the pilot study conducted in end July 2006, 7 mathematically gifted lower secondary students were involved in a focus group discussion. The discussion focussed on the central research question: How do mathematically gifted and talented lower secondary students deal with mathematical challenges encountered in classroom. The following sub-questions further guide the discussion: (i) what are the expectations of mathematically gifted and talented students of what will be involved with regards to mathematical challenges encountered in the classroom?, (ii) what are their intentions with regards to these challenges and the reasons they give for having these intentions?, (iii) what are their strategies in dealing with challenging mathematics and the reasons they give for these strategies?, and (iv) what are their actions in light of these intentions and strategies?.

The second part of the pilot study, conducted in end October, was a written response from 4 mathematically gifted upper secondary students. The students responded to the series of questions in the guide and commented on questions that were incomprehensible to them. All data collected from both parts were then analysed. The experience accrued from the pilot study has enabled this researcher to refine the data collection method for the subsequent study conducted in 2007.

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A STUDY OF THE EFFICIENCY OF FRACTION INSTRUCTION THROUGH HANDS-ON ACTIVITIES

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This research is intended as investigation of student's learning effect in mathematical concept about unit fraction and fractional times by making mathematics lessons toward hands-on concrete materials. That is to said by designing teaching activities of two fractional concepts with Chromo paper and playing cards, students are provided the concrete operative opportunity. Hopely, the abstract fractional concepts can be easily conceivable. More over, the designing of experimental course considers through four core viewpoints of compare and analysis: the development of fraction in the history of Mathematics; the kinds of fraction meaning; the designing of teaching material among different curriculum standard; the difficulty of students' fraction learning.

The subjects of this research are the fifth grade students of Guan-Ming elementary school in Hsinchu, Taiwan. We pick two classes, one of the classes is the experimental team which adopts the fractional teaching activity in the course, the other is the contrastive team which takes the original course. Experimental teacher has to do eight weeks sixteen hours preliminary training of experimental course, in order to help teacher familiar with the heart concept and practice strategy of fraction instruction. The test questions contain two parts: idea problems and word problems. Students will take the pre-test before the course, and the post-test after the course.

The research results show: (1)Teacher has found herself improved understanding of fraction concepts and teaching performce; (2) students are interesting in fractional teaching and thought it is easy to learn; (3) Comparing the result of two tests,the scorse of experimental team are higher than contrastive team, and there is a significant difference between two teams in idea problem.

Finally, we believe if teacher uses correct teaching strategy, learning fracion will be no longer a nightmare for students.

AN ACTION RESEARCH ON MENTORING PRE-SERVICE TEACHERS TO IMPLEMENT PROBLEM-CENTERED TEACHING

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The purpose of this study was to mentor traditional-oriented pre-service mathematics teachers to implement problem-centered teaching and to construct an effective model of mentoring.

INTRODUCTION

The traditional mentoring method for practicum teaching courses is unable to assist pre-service teachers to develop competence in solving realistic problems about teaching in classroom. However, in recent years, mathematics education reform has emphasized innovation in mathematics teaching, so mentoring pre-service teachers to implement innovative teaching is very important.

THEORETICAL FRAMEWORK

This study focused on mentoring pre-service teacher to implement Problem Centered Double Cycles [PCDC] instruction (Chang, 1995), so we reviewed the PCDC instruction model. The four important missions of the teacher were to design the task, to guide the students' discussions, to create a safe environment to discuss, and to analyse the achievement of students' learning. Therefore, in the discussion of this study, the authors checked the four missions that pre-service teachers should do.

METHODOLOGY

The action research method and a four-tiered collaborative research paradigm were used. The first author was both researcher and mentor. The subjects as case participants were two pre-service teachers. Both of them took one-year practicum teaching courses in the classroom that the mentor taught. We designed a preliminary mentoring model, and through reflection on the mentoring process we revised it. Data collection included interviews, teaching journals, and mentoring journals.

RESULTS AND DISCUSSIONS

We found that two pre-service teachers moved from being traditional-oriented toward being problem-oriented. Moreover, when they took the classroom situation problems as the center of the practicum teaching and engaged in the problem solving of classroom teaching context, it led them to implement the problem-centered teaching. Finally, a classroom-problem-centered mentoring model was proposed.

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TASKS, TEACHING SEQUENCES, LONGITUDINAL TRAJECTORIES: ABOUT MICRO DIDACTICS AND MACRO DIDACTICS

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Along with generating knowledge about what are powerful learning environments for students and what mechanisms in teaching mathematics can contribute to learning mathematics, a main goal of didactics as a scientific discipline is to design teaching materials to realize these learning environments and make these mechanisms happen. Therefore mathematics education research is identified as a design science. However, what is designed can largely differ in teaching-time scope.

The aim of this theoretical paper is to reflect on characteristics of instructional units which have a different scope in teaching-time. It focuses on tasks, lesson series or teaching sequences, and longitudinal teaching trajectories. By exploring how these instructional units of different sizes differ with respect to their planning nature, their focus and their research methodology, the paper sheds light on differences between micro and macro didactical approaches within mathematics education research.

	Tasks	Teaching sequences	Trajectories
Planning nature	Micro planning	Meso planning	Macro planning
Focus	How can mathematical thinking be prompted?	How can this thinking be strengthened in subsequent lessons?	What mathematical content and competences should be taught?
Research Methodology	Piloting and revising tasks through interviews/observations …	Classroom experiments following a teaching cycle model …	Mathe-didactical analyses …

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TWO TYPES OF CONCEPT CHANGE BY COUNTER-EXAMPLES

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Concepts of geometric shapes provide not only basic elements for pupils to observe and analyse objects but also the basis for reasoning and argumentation. However, literature has shown that pupils have alternative concepts of geometric shapes. Pupils might classify a long, thin parallelogram as a rectangle, or they might believe that a rectangle needed to have not only 4 right angles, but also unequal neighbouring sides. The former (over-extension) includes some non-examples. The latter (under-extension) excludes some positive examples, and that regards the inclusive relationship among quadrilaterals.

This study gives simple feedback by two types of counter-examples in quadrilaterals. For over-extension, the experimenter provides those included non-examples and says, for example, 'This is not a rectangle'. For under-extension, the experimenter oppositely gives those excluded positive examples and says 'this can be categorized as a rectangle'. We concern whether or not pupils change their categorization and how they explain. The participants are 12 fifth graders selected by a paper and pencil categorization task. They all make some mistakes of excluding examples in categorizing the rectangle, rhombus, or parallelogram. But, only 5 participants make mistakes of including non-examples.

The result showed that counter-examples could help correct most categorization performances, but only partially help to change the concept definition. All 5 participants were not hesitant to correct their over-extension. And they could point out the ignored attributes before. Half of all 12 participants could correct their under-extension, but they were more resistant. They could grasp the categorization rules by category-based generalization. For example, after one counter-example a pupil knew all of squares are rectangles because 'that one, you tell me, it is a rectangle'. A few participants could simplify the concept definition by the common attributes of the typical examples and the counter-examples. But most participants they could give acceptable explanations couldn't give up the noncritical attributes. They believe a rectangle has 4 right angles and unequal neighbouring sides, but if it has 4 equal sides that is ok. The pupils can easily pay attention to the ignored attributes by counter examples shows the change involves additive mechanism. But it is difficult to construct a quadrilateral hierarchy and abandon the noncritical attributes. These finding supported the distinction Vosniadou and Bershaggel (2004) drew between knowledge acquisition and conceptual change.

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A STUDY OF THE CONCEPT OF SOLID GEOMETRY OF ELEMENTARY STUDENTS FROM THE 4TH GRADES TO 6TH GRADES IN THE CENTRAL REGION OF TAIWAN

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The purposes of this study were to investigate the thinking levels of van Hiele solid geometry of 1,351 4th, 5th, and 6th graders (from 45 classes and from five counties) in the central region of Taiwan, and the outcome difference among grades, counties and genders. The content presented were partial results of the a project funded by the National Science Council of the Executive Yuan (Project No.: NSC 94 - 2521 - S - 142 - 003).The instrument used in this project was "The Thinking Level Examination of van Hiele Solid Geometry" was used as the measurement instrument in order to investigate the concept of solid geometry shape.

The results were as follows:

1. There were significant differences ($p<.001$) of scores among grades. Sixth graders scored significant higher than 5th graders, and 5th graders did the same to 4th graders on the total scores, at each van Hiele level, and on the different shapes (prism, pyramid and cylinder (column, cone and sphere)).

2. Chi-square results indicated that there were significant differences in concepts as cube, rectangular parallelepiped, triangular prism, quadrihedron, cylinder and cone among grades and among counties.

3. The only significant differences regarding county was: the results from two county (A & C) were significant higher then the result from county E. This finding might due to the gap of pupils between city and countryside.

4. There were no significant differences between genders in the performance scores.

5. The distribution of each grade were: the majority of 4th graders were at below Level 1 (43.88%) and Level 1 (35.75%); 5th graders were at Level 1 (24.83%) and Level 2 (40.81%); and 6th graders were at Level 2 (54.59%) and Level 3 (31.00%).

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THE LIKELIHOOD OF APPLYING LOCAL ORGANIZATION PROGRAM FOR PROMISING STUDENTS[1]

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The purpose of this study is to identify how an activity of the local organization to make a theory about the space would impact the mathematically promising students for the development of their geometric thinking.

Sheffield (1999, p.45) says that it is not only a matter of speed or quantity but also a matter of the depth or complexities that need to be taken into consideration to implement a math program for promising students. We have met two male students (JH and IS) who are in 7th grade and have received special education for the mathematically promising. They say that they have some knowledge about the Euclidean geometry by themselves and have basic knowledge about the polyhedrons. But their perception about geometry does not go beyond "problem-solving activity by drawing auxiliary lines (JH)," or "problem solving activity by drawing figures (IS)". Both have experiences of geometrical proof, but neither of them have experiences of local organization.

Three activities based on Fawcett(1995) were given and lasted 3 or 4 hours for each. These students used to regard geometry as a problem-solving activity. But the program enables these students to grow aware of the need and difficulties in defining terms and deciding on the undefined terms, while it also enabling them to identify factors to determine a conclusion. Table 1 is driven by them.

Assumption necessary	Vertically opposite angles are same. When there are two lines in parallel, and another line that is not in parallel with these two lines, the corresponding angles are same.
Definition	180^{o}, (be) in parallel, half-line, angle, inner angle, corresponding angle, size of angle, the same side, segment of a line
Undefined terms	line, same, rotation, point

Table 1: Factors considered in determining the sum of a triangle's inner angles

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PROBABILISTIC THINKING OF ELEVENTH GRADE STUDENTS USING THE STORY AND DIAGRAM METHOD

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This study focused on encouraging probabilistic thinking of eleventh grade students with different mathematics learning achievements by using the Story and Diagram Method (SDM). The Probabilistic Thinking Framework by Jones et al. (1995) was used as a basis for describing and predicting the students' probabilistic thinking skills. The target group consisted of 36 eleventh-grade students who had learned about probability through SDM. Then, 6 case studies, comprising of 2 students from high, medium, and low mathematics learning achievements, were selected for in-depth exploring regarding their learning performances. Multiple data collection was conducted during the teaching and learning process such as participant observation, interviews with semi-construct, audiotape and videotape. Daily reflects and field notes were also taken. Qualitative analysis examined various sources of data in order to identify students' levels of thinking through reasoning. The invented language was used to describe students' thinking based on the Probabilistic Thinking Framework. The research results revealed that students who had different mathematics learning achievements also had different mathematical idea and thinking process as their tools for understanding the concept of probability. Students' ability to reasoning and initiated invented language to describe their thinking process and ideas to solve the mathematics problems were key patterns in differentiate levels of students' probabilistic thinking and achievement in mathematics learning.

Keywords: Probabilistic thinking, invented language, learning achievement

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POSTER PRESENTATIONS

PERSPECTIVE ON MATHEMATICAL LITERACY IN JAPAN: TOWARDS CURRICULUM CONSTRUCTION

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The purposes of this research are to identify orientation of the mathematical literacy called for in Japan now and to give suggestion to develop curriculum to foster mathematical literacy.

Since 2006, the project to define literacy of science, mathematics and technology was inaugurated in Japan (cf. Kitahara (Ed), 2006). In this project, the references on literacy were investigated (cf. Nagasaki (Ed.), 2006). The papers of the academic journal and the journal in connection with the technology, science education, mathematics education, technical education, and museum education, et al, (since 1970) were taken up, in order to analyse the trend of literacy research. And they are analysed about the present condition of literacy research. Here, the analysis (Abe, et al., 2006) is described, in order to survey the past mathematical literacy research in Japan. As a result, the first appearance of the word of literacy in Japanese mathematics education has been at 1982, and a total is 197. The literacy theory changes from "Matheracy" by Kawaguchi in 1983 to mathematical literacy of OECD/PISA (OECD, 1999) that has most inferences in Japan.

When mathematical literacy of OECD/PISA and past literacy researches are compared, past they consist of “theoretical mathematics” and current one consist of “Functional mathematics”. However, we should call attention to the balance of “functional mathematics” and “theoretical mathematics”. That is to say, “Functional mathematics” should be emphasized, “theoretical mathematics” should not be disregarded on the other hand. Moreover, I think, “mathematization” and “mathematics as the science of patterns” (cf. Devlin, K., 2003) that are key concepts of mathematical literacy of OECD/PISA and AAAS/project2061 (AAAS, 1989) have importance as a bridge.

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PRE-SERVICE TEACHER DEVELOPMENT THROUGH TECHNOLOGY INTEGRATION IN EARLY GRADES MATHEMATICS

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This presentation provides an analysis of data drawn from a mathematics enrichment program taught by the third author, then a student teacher, in the framework of professional development school. The program integrated two major theoretical positions: the effectiveness of learning in context (Schoenfeld, 2006) and the orientation on tomorrow's development in the child (Vygotsky, 1978). The goal of the program was to explore how real-life context and technology (both manipulative and computing) can support mathematical problem-solving activities of second graders that otherwise require a formal use of concepts studied in higher grades. More specifically, the focus of the activities, enabled by specially designed spreadsheet-based environments, has been on exploring weekly weather change in terms of average temperature. Whereas division – an advanced operation for that grade level – was embedded in the software, addition was situated in the advanced context of inverse problems that typically have more than one correct answer. These problems required the grasp of the multiplicity of representations of an integer as a sum of other integers. This made it possible to connect real-life modeling activities and standards-based topics in early grades mathematics, including partition of integers into summands and fair sharing, as prerequisite for division. The presentation shows examples of spreadsheet templates with children's solutions.

Another focus of the program was to support second graders' formulating of mathematically relevant questions similar to those they themselves answered. Encouraging self-generated questions shifts the ownership of mathematical ideas from teachers to pupils enabling a qualitative change in the classroom discourse. In terms of pre-service teacher development, the paper shows how learning to use technology as a mathematical/pedagogical tool could take a path of research intensive practice in the field tailored to scholarly interests of individual teacher candidates and become a good working model for mathematics teacher education programs.

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THE ABILITY TO RECOGNIZE MISTAKES IN TASKS SOLVED BY 3RD GRADE STUDENTS

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Liora Mishna, Maccabi Health Care, Israel
Arie Mishna, Ness Technologies Ltd., Israel

The purpose of this poster is to describe an experiment aimed at testing children's ability to identify mistakes in solved tasks.

Ability to identify mistakes is one of the components of critical thinking, and control skills that are significant components of mathematical reasoning (Pang 2003, De Bono 1994, Zoar 1996). These skills are usually neglected in school mathematics teaching. We were interested to analyze whether elementary school students develop control abilities in regular mathematics classrooms.

Three questionnaires containing series of arithmetical examples related to different arithmetic operations (addition, subtraction, multiplication) were presented to the group of thirty 3rd grade students. Each questionnaire consisted of four examples (related to a particular arithmetic action). Two examples in each questionnaire were correct and two examples included mistaken calculation. The students were asked to identify incorrect examples and to describe mistakes. We compared between the students' success in identifying and explaining mistakes in the examples with different arithmetic operations.

The poster will present the study results and some explanations to our findings.

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TEACHERS' BELIEFS ABOUT THE STRATEGIC INTEGRATION OF MATHEMATICS AND SCIENCE

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The purpose of our research project, **Excellence in Mathematics and Science Teaching**[1] **(eMAST)**, is to study the effects of the strategic integration of science and mathematics on teachers and students. In strategic integration not every math lesson and science lesson are integrated, only those that promote learning of both disciplines in a robust way. Mathematics offers powerful intellectual tools to analyze and study phenomena in the sciences. The relationship between teachers' beliefs and how they teach is strong in mathematics (Brown, 1992; Sztajn, 2003). Research on learning suggests that strategic integration will help teachers and their students better understand content (Huntley, 1998) and develop more positive beliefs and attitudes about science and math (Stipek, Salmon, Givvin, Kazemi, Saxe, & MacGyvers, 1998; Westerback & Primavera, 1992). To test this assertion, we have randomly assigned teachers to two different treatments: professional development focusing on strategic integration and professional development that addresses math and science teaching separately. For this Poster Presentation we will report research questions, design, data collection and analysis and our results on the impact of strategic integration on teachers' knowledge, beliefs and attitudes. We will present the Situational Judgment survey we have developed to assess teachers' pedagogical knowledge of integrating math and science.

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THE VOLUME OF A SPHERE IN SCHOOL MATHEMATICS

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This study analysed 54 mathematics textbooks from 16 countries in order to determine the variance in approaches used to calculate the volume of a sphere–an activity that is dealt with primarily between grades seven and ten. The texts used in this study were from the following countries(standard abbreviations will be provided in parentheses):

Argentine(AG)	China(CH)	Iran(IR)	Singapore(SG)
Australia(AU)	Germany(DE)	Japan(JP)	Thailand(TH)
Brazil(BR)	England(UK)	Korea(KR)	United States(US)
Canada(CA)	France(FR)	Paraguay(PR)	Vietnam(VT)

The various approaches are classified into seven categories: Experiment(E), Application of Cavalieri's Principle(C), Naive Calculus(N), Approximation(A), Limit based on the surface area(L), Statement based on the history of mathematics(H), and Statement without any justification(S). Approaches E, C, and L are subdivided in the following ways: E consists of one of six methods: measuring displaced water when a ball(r) is submerged in a cylinder(r,2r) of water(E1), calculating the empty space in a cylinder containing a ball(E2), measuring the change of water level when a ball is submerged in a container of water(E3), filling a cylinder(r,2r) with a hemisphere(r) of water(E4), filling a hemisphere(r) with a cone(r,r) of water(E5), or filling a cone(r,2r) with a hemisphere(r) of water(E6). C offers two possible methods: comparing the cross sections of a hemisphere(r), a cone(r,r), and a cylinder(r,r)(C1) or comparing the cross sections of a sphere, two cones(r,r), and a cylinder(r,2r)(C2). L uses an infinite number of small pyramids(L1) or adds the volumes of spherical shells(L2). N depends on the lower and upper boundary of the sum of cylinder-partitions. A indicates its range between inner and outer cubes. Illustrations of each approach from the textbooks as shown on the poster demonstrate the different approaches for the purposes of clarification. As shown in the table, E2 and C1 are preferred in relation to experiment and application of the principle. Furthermore, it is revealed that S is used in five textbooks.

App.	Country
E1	CA, KR(4)
E2	AU, DE, FR, JP, KR(9), SG(3), VT
E3	JP, US
E4	KR(2), TH
E5	DE, KR, PR
E6	DE
C1	CH, DE(2), JP(6)
C2	DE(2), PR
N	JP(2)
A	US
L1	JP(4), US
L2	AU
H	BR, FR, JP, UK
S	AG,AU,IR,SG,UK

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EFFECTS OF USING COMPUTER-ASSISTED INSTRUCTION ON THIRD GRADERS' MATHEMATICAL LEARNING IN FRACTION

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THE STUDY

Fraction is introduced to students from grade 2 to grade 6 in the elementary school. The general idea of fraction is important and complex. In a real life situation, however, fraction is novel to the students and they often find them difficult to grasp. If the students cannot comprehend fraction, they will be hindered in their development of math skills. Therefore, the goal of this research was to provide the Computer-Assisted Instruction (CAI) learning environment in the elementary school taking advantage of computer animations to stimulate the interest of the third graders in learning fraction. Further, through evaluating the effectiveness of the use of the computer as a teaching tool in cognitive learning, this tool would provide the learners an opportunity to grasp the concept of fraction and strengthen students' mathematical learning achievement and learning retention. Sixty-four students were randomly selected and assigned into two groups for the quasi-experimental design in order to examine the effectiveness of the CAI learning environment. Learning achievement pre- and post-tests and a CAI usage questionnaire were administered associated with classroom observations and interviews for gathering the data. The results indicated that students who received the Computer-Assisted Instruction had significantly better mathematical leaning achievements than students who received the traditional instruction. Further, the findings from the qualitative data and the CAI usage questionnaire showed that students of the experimental group held and presented positive opinions toward the CAI mathematical learning environment. They thought that the CAI environment provided active and vivid sample questions and operational reviews with interactions, which increased their learning interests, excited the curiosity, and enhancing the degree of concentration. Finally, an analysis in context was applied to provide suggestions for improving elementary school mathematical fraction instruction within the CAI learning environment.

THE DEIGN OF THE CAI LEARNING ENVIRONMENT

The design of the CAI learning environment contains five main sections: design rationale, content of the curriculum, learning objectives, previous knowledge and experience, and the new world of the fraction. In addition to report the research findings, this poster presentation will introduce the design process and the framework of the CAI learning environment, as well as the main features of this CAI learning environment by consecutive pictures and illustrations. The presenter will also show the actual operation of this CAI learning environment by the laptop on-site with free copies of the short papers, free demo CDs, and further contact information online.

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A TEACHING EXPERIMENT TO PROMOTE MATHEMATICS TEACHER'S ABILITY: T-PVDR TEACHING MODEL

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Although Kurt Lewin said that 'theory without practice is sterile; practice without theory is blind' (quoted from Mason & Johnstone-Wilder, 2005), it was usual for mathematics teachers in Taiwan to think they are separated. To help mathematics teachers apply theories to their own teaching, the course related to professional development should be designed with practice-based materials. Steele(2005) promoted the quality of teachers' deeper conversations about practice with practice-based materials. And Nemirovsky, Dimatia, Branca &Lara-Meloy (2005) argued that evaluated discourse is the best way to discuss teaching episodes. However, the participants in their study are not the performers in those materials. Thus they can not reflect themselves immediately. For that, we designed T-PVDR teaching model in the course of professional development to explore the effect of the model and the reason influence teaching change. T-PVDR teaching model which asked participants to apply theories to their practice integrated theories with practice. Teachers in the course talked about their teaching video and shared their experiences with each other.

There are two findings in our research. First, T-PVDR teaching model can promote mathematics teacher's ability, especially in the ability to organize and present teaching materials. Second, Mathematics teacher tries to combine the theory with teaching, their transition was considering the theory and reality from separately to coordinately. They found out the benefit of the theory from the experience of demonstrating, and would like to adopt new ways in their practice. Otherwise, the factors influenced the mathematics teacher to change are : one can understand the shortcoming of his own teaching and improve ; one can learn from others ; one can grasp the content of mathematics; one can understand the mathematics concept that students got.

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THE MONISTIC PROBABILISTIC PERSPECTIVE

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This report investigates theoretical interpretations of probability, institutes the Monistic Probabilistic Perspective (MPP) and discusses possible ramifications of MPP.

> The theory of probability has a mathematical aspect and a foundational or philosophical aspect. There is a remarkable contrast between the two. While an almost complete consensus and agreement exists about the mathematics, there is a wide divergence of opinions about the philosophy. With a few exceptions…all probabilists accept the same set of axioms for the mathematical theory, so that they all agree about what are the theorems (Gillies, 2000, p. 1).

There are (at least) three different philosophical interpretations of probability: Classical (or Theoretical), Frequentist (or Objective) and Bayesian (or Subjective). In order to bring forth differences in these interpretations, a simple problem will be posed and answered from each perspective: What is the probability of obtaining heads when flipping a fair coin? From the Classical interpretation, the answer is one half because there are two equally likely outcomes, of which one is favourable; thus, the ratio of favourable outcomes to total number of (equally likely) outcomes is one to two. From the Frequentist interpretation, the answer is also one half because "the *probability* of a large difference between the empirical probability and the theoretical probability limits to zero as more trials are collected" (Stohl, 2005, p. 348). While from the Bayesian interpretation, all things considered, an individual would bet on heads at odds of 1:1 (even money) because there is no advantage to one side or the other.

If, mathematically, the answer is one half, regardless of the interpretation used, why concern one's self with different interpretations? I contend that probability has developed a concurrent definition, which has manifested itself in, what I call, the *Monistic Probabilistic Perspective* (MPP). Although this monism may be acceptable when engaged in everyday discourse, using the word probability within an academic field requires the definition be more rigorous. I further contend, and will insist on through pictorial representation, that the MPP has led to an ignoring of the Bayesian philosophical interpretation of probability within Mathematics Education.

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DEVELOPING MATHEMATICAL CONCEPTS THROUGH DISCOURSES IN MATHEMATICS INSTRUCTION

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Since the 7th curriculum was introduced to Korean school mathematics, the development in student's competence of communication and problem-solving has been focused. This article considered the question: how does a teacher make students develop their mathematical concepts, through her discourses in her single classroom over a four-month period. Four developmental trajectories suggested by Hufferd-Ackles, Fuson, and Sherin (2004) were the frameworks for explaining levels of discourses. The analysis was drawn by classroom observation on discourses between the students and the teacher to show how the teacher supported students' discussion and guided their development of mathematical concepts.

Hufferd-Ackles, Fuson, and Sherin (2004) suggest developmental trajectories that describe the building of a math-talk learning community where an individual assists one another's learning of mathematics by being engaged in meaningful mathematical discourse. The developmental trajectories are (a) questioning, (b) explaining mathematical thinking, (c) sources of mathematical ideas, and (d) responsibility for learning. Because this article focuses on linguistic interaction, discourses, we choose explaining, questioning, and justifying of ideas as three important components of discourses with students. Data were collected by observing and recording the teacher's mathematics instruction in her classroom. Field notes and protocols from transcripts of recorded videotapes were used to analyse students' behaviours.

The teacher always asked the students to explain, question, and justify their ideas in order to help them develop the competence of communication in discourses with the students. At the beginning of discourses, the students were not willing to participate in discourses and wanted the teacher to explain directly much more about mathematics. The teacher, however, kept encouraging them to get more actively involved with communication in class. Consequently, about a month later, the students started to ask questions voluntarily. Also, they monitored and clarified their mathematical concepts while exchanging their ideas with the teacher and their classmates. They used varied types of communication such tools as diagram, writing, pictorial and mathematical symbols in order to explain and justify themselves. The diversity of communication was related to the differences in levels of students' mathematical concepts.

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THE IMPLEMENTATION OF GAMES IN THE MATHEMATICS TEACHING OF ONE ELEMENTARY SIXTH GRADE CLASS

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Kun Shan University, Taiwan
Shuk-kwan S. Leung,
National Sun Yat-sen University, Taiwan

The purposes of this study are to examine the design of mathematical games for grade 6 pupils and to modify the games according to the opinions of veteran teachers. In order to devise the mathematical games relating 3 units--- Factor, Fraction and Ratio, the researcher considered Alan Bell's game teaching instructions; cooperative learning by Slavin (1985) and action research by McKernan (1991). The students were given a pretest after the design was completed, and then the researcher had modified the design according to pilot results, finally the formal test was given in class by the researcher. The effect of mathematical game teaching is surveyed by pupils' tests and parents' questionnaires.

The results indicated that the pupils have learned the importance of peer-learning and have been capable to deal with mathematics questions, furthermore, the connections made between games and mathematics have promoted pupils' interests toward mathematics. As to the parents, they agreed to the use of games in mathematics curriculum, but they showed more concern of pupils' learning attitude and sufficient teaching materials supplied.

In order to integrate the idea of games into mathematics curriculum, the researcher should devised in response to mathematics curriculum, the ability of pupils and their motivation, thus enabling them to learn from playing, and to establish positive learning attitude.

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INQUIRY CYCLE FOR DEVELOPING MATHEMATICS TEACHING WITH ICT

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In the project ICT and mathematics learning (ICTML) teachers and didacticians work together in a learning community with emphasis on inquiry (Jaworski, 2005). Inquiry means to ask questions, investigate, acquire information and search for knowledge. In the project inquiry has become fundamental idea and guideline for the work on all levels with the aim to develop mathematics teaching with ICT. We inquire into mathematics, into mathematics teaching and how mathematics can be represented and worked on with ICT tools. Furthermore inquiry into how ICT tools can support pupils learning is fundamental in the development work and in research which deals with all levels in the project, including teachers, pupils and didacticians work. An attitude of inquiry is characterised by willingness to wonder, seek to understand by collaborating with others implies being active in dialogic inquiry (Wells, 2001). This implies asking questions, investigating and exploring issues concerned. An aim is to develop the participants approach in line with this attitude into "inquiry as a way of being".

An important part of the work in the project is collaboration in workshops on mathematical tasks, using of ICT as a tool and discussing development of teaching approaches. Another main activity is to work together with teachers in their school teams in development of teaching approaches and plans, observe the implementation and discuss results and further development. This work follow the characteristics of the design cycle: plan, act, observe, reflect and feedback to future planning (Jaworski, 2007). We found this approach fruitful both for development and research.

The poster will display examples of tasks and discussions from the workshops and an example of implementing the design cycle with a teacher team working on some spreadsheet tasks. Result from the research in this connection will be presented.

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VISUALIZATION IN MATHEMATICAL PROBLEM SOLVING: PERFORMANCE OF STUDENTS IN A NATIONAL EXAMINATION-TYPE WORD PROBLEM

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Bishop (1983) suggested that it is "not necessary to have figural stimuli" (p. 184) when considering visualization in the context of mathematics and therefore it can occur not only in geometry, but in arithmetic and algebra as well. Cognizant of the challenge in the "state of the art" of visualization research, he proposed two types of spatial ability constructs, namely interpreting figural information (IFI) and visual processing (VP). The ability for VP consists of two parts, one of which involves translating abstract relationships and nonfigural information into visual terms.

This study investigates Primary Five (aged 10.25 to 11 years old) and Primary Six (aged 11.25 to 12 years old) students' written performance in a national examination-type word problem. This problem is of a nonfigural yet visually provoking nature. The Primary School Leaving Examination (PSLE) in Singapore is a national examination which aims to assess students' attainment in mathematics at the end of their six years of primary school education with respect to the objectives of the Singapore mathematics curriculum. A total of 1167 Primary Five and Primary Six students from five Singapore primary schools were asked to solve the following word problem: *Mr Ramasamy plants trees along a straight path. Trees are planted 15 metres apart. The length of the path is 300 metres. How many trees, at most, can he plant?*

The following research questions were asked:

1. Were Primary Five and Primary Six students able to solve the word problem?

2. What solution method (use or non-use of diagram) did Primary Five and Primary Six students use to solve the word problem?

In the poster the findings to the above two research questions will be presented. In particular, eight categories of solution method were observed. The findings include a surprising find that only 13.6% of the Primary Five students and around one third of the Primary Six students solved the problem correctly. Majority of the students who obtained the wrong answer solved the problem procedurally, i.e., 300÷15=20 trees. The study also found that students who drew a diagram in their solution methods were not always successful in solving the problem correctly. One-to-one interview is required to further investigate students' thinking processes related to use or non-use of imagery and visualization during mathematical problem solving.

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A SURVEY OF CHILDREN'S ACADEMIC ACHIEVEMENT AND LIVING ADAPTATION OF THE FOREIGN BRIDES IN TAIWAN

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Nationa l Chiayi University

Ru-Jer Wang
National Taiwan Normal University

According to relevant statistics, in school year 2003 the total numbers of students with background of foreign brides in primary schools and in junior high schools was 26,623 and 3,395 respectively. The main aim of this research is therefore to understand children's academic achievement and living adaptation of the foreign bridges. A questionnaire was designed with two research null hypotheses formulated to guide the study as the following: H1: There is no significant difference between children's academic achievement of the foreign bridges and their counterparts; H2: There is no significant difference between children's living adaptation of the foreign bridges and their counterparts. In total 1,489 copies of the questionnaire were mailed directly to the respondents of this survey with a response rate of 92%.

With regard to students in primary schools, children' academic achievement and living adaptation of the foreign bridges in the subjects of Chinese (M=47.58) and Mathematics (M=47.38) are lower than the average (T score below 50); In contrast, 75.6 % of the students from the foreign bridges families are good in terms of living adaptation. This percentage is about the same as those students from general students (75.5%).

Regarding students in junior high schools, children' academic achievement and living adaptation of the foreign bridges in the subjects of Chinese (M=48.53), Mathematics (M=48.53), English (M=47.79) and Science (M=48.85) are lower than the average (T score below 50). In contrast, 77.6 % of the students from the foreign bridges families are good in terms of living adaptation. This percentage is about the same as those students from general students (77.5%).

In conclusion, with regard to academic achievement, foreign bridges' children are significantly lower than those from general ones; however, in terms of living adaptation, there is no difference between students from foreign bridges' families and those from general ones.

A SURVEY ON THE COMPREHENSION OF GRAPHS OF SIXTH GRADERS

HyunMi Hwang & JeongSuk Pang

Seoul Myeondong Elementary School   Korea National University of Education

Knowledge of statistics is increasingly important for mathematical literacy and for numerous everyday decisions. However, it has not been studied in-depth how students understand statistics. This study investigated how sixth graders might react to the types of tasks with regard to the comprehension of graphs and what differences were among the kinds of graphs, and raised issues about instructional methods of graphs.

A comprehensive survey was conducted with 187 students from six elementary schools in Seoul, Korea. The test consisted of 48 questions with 4 types of tasks (reading the data, reading between the data, reading beyond the data, and understanding the situations) and 6 kinds of graphs (pictographs, bar graphs, line graphs, stem-and-leaf plots, band graphs, and circle graphs), based on the national mathematics curriculum and the previous studies including Friel, Curcio, and Bright (2001), and Parmar and Signer (2005).

Generally speaking, students had high scores on reading the data and reading between the data, but they had lower scores on reading beyond the data and understanding the situations. Students' difficulties in reading beyond the data resulted from the lack of prior knowledge and the imbalance between the focus on graphs and the use of everyday-experiences and common senses. Many students couldn't answer or mentioned only the items in graph when they were asked to make a story in understanding the situations. While in reading the data and reading between the data students had lower scores mainly on line graphs and stem-and-leaf plots, in reading beyond the data and understanding the situations they had lower scores on different types of graphs.

This study recommends that teachers should know students' difficulties and thinking processes in graphs comprehension and use multiple types of tasks about all kinds of graphs including more challenging tasks. The poster will present the typical examples of 4 types of tasks used in this study, students' representative answers (correct and incorrect), and suggestions to teaching and learning graphs.

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AUTHENTIC INVESTIGATIVE PROPORTIONAL REASONING ACTIVITIES

Bat-Sheva Ilany   Yaffa Keret   David Ben-Chaim

Beit-Berl College   Israeli Science Teaching Center   Oranim-University of Haifa

In this study we created, implemented, and evaluated the impact of proportional reasoning authentic activities on the mathematical content and pedagogical knowledge and attitudes of pre-service elementary and middle school mathematics teachers. For this purpose, a special teaching model was developed (Ilany, Keret & Ben-Chaim, 2004) implemented, and tested as part of the pre-service mathematics teacher training programs conducted in Israeli teacher colleges (see Figure 1). The conclusion of the study is that application of the model, through which the pre-service teachers gain experience and are exposed to authentic proportional reasoning activities with incorporation of theory (reading and analyzing relevant research reports) and practice, leads to a significant positive change in the pre-service teachers' mathematical content and pedagogical knowledge. In addition, improvement occurred in their attitudes and beliefs towards learning and teaching mathematics in general, and ratio and proportion in particular.

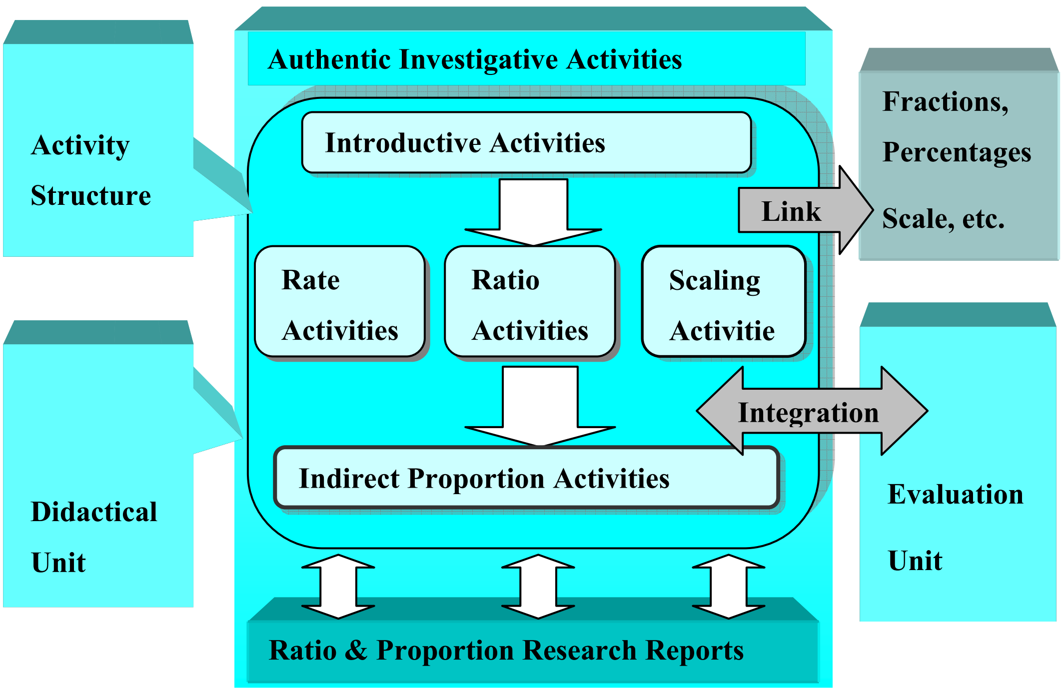

Figure 1: A Model Using Authentic Investigative Activities

The poster will present the main results of our study and will present several detailed examples of the authentic investigative proportional reasoning activities.

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KNOWLEDGE SYSTEM CONSTRUCTED BY STUDENTS IN THE TEACHING/LEARNING OF GEOMETRY

Kazuya KAGEYAMA

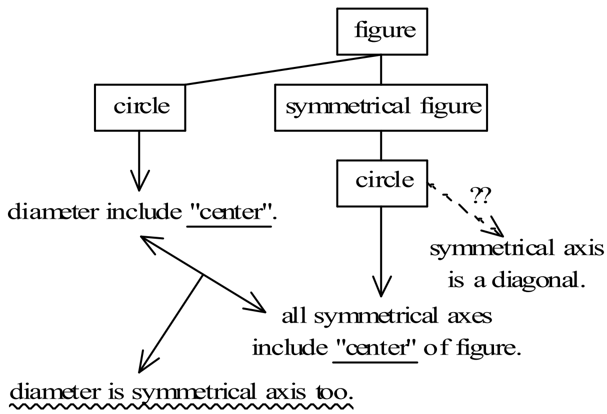
Aichi University of Education

The aim of this presentation is to represent the knowledge system constructed by students in the teaching/learning of geometry as a network. This network including some mathematical and natural words and relations among them is not necessarily same one which teachers expect. For students who encounter many geometrical concepts in the classroom for the first time, mathematical language is a kind of representations which give several meanings to students, if they interpret those by the support of natural language. It may be that polysemy of language, especially of natural language, is not desirable in the teaching, but this "vagueness" of language could be the driving force for the learning by reflecting the meaning of language.

This presentation is concerned about the knowledge system constructed by students in the teaching/learning of geometry as a network. In cognitive science, a network is a kind of diagram which includes some words, properties and relations among them. So, this is frequently referred as a representation of knowledge.

In mathematics education, it has been regarded as one of the problems of the teaching/learning of geometry that knowledge constructed by students is isolated from each other. So, students can't use relevant knowledge in geometrical problem solving and can't "see" a figure from various points of view. As Duval(2006) noticed, the heuristic way of solving and seeing often needs the register of figural transformations of Gestalt order, and the multifunctional CONVERSION which changes register of representations needs interpreting and using mathematical/natural language.

Generally, language is polysemic according to various contexts. Thanks to this nature, students can interpret new words with their knowledge of similar words, and call new concepts with familiar words. If teacher encourage students to reflect the meaning of words, they will find and pay attention to the common word between some isolated knowledge systems. Below, I represent the diagram as an illustration of the network of figure and the common word "center". This word "center" links the property of symmetry with the one of circle.



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TECHNO-MATHEMATICAL LITERACIES IN THE WORKPLACE: MAKING VISIBLE THE MATHEMATICS OF FINANCIAL SERVICES

Phillip Kent, Celia Hoyles, Richard Noss and Arthur Bakker

London Knowledge Lab, Institute of Education, London, UK

This poster presents research which has investigated the needs of intermediate-level employees to gain mathematical knowledge in the form of "Techno-mathematical Literacies" (TmL), that is, knowledge grounded in their workplace situations and the technological artefacts within them. Our aims are: to develop a theoretical framework to describe TmL in workplaces; and to develop principles and methods of training for TmL, and test these out in prototype training interventions.

The workplace context introduces complexity to even the simplest mathematics, since mathematical procedures become part of a set of judgements that have to be made about a complex process. Moreover, this complexity is mostly hidden within models inside IT systems. Our research is therefore concerned with the question of how to support employees in engaging with the invisible mathematical ideas behind workplace artefacts, through the development of appropriate TmL.

We illustrate our research in financial services companies, where we have looked at the work of customer service employees. We will consider the "knowledge gaps" related to mathematical understanding amongst such employees and the prototype learning interventions we co-developed with company trainers to address these knowledge gaps. To engage with customers requires employees to be able to communicate about financial products, and the mathematical models within them. Yet, in looking at the companies' training, we found that mathematical issues are deliberately avoided because they are alienating to many employees. Several companies agreed to work with us to co-develop activities which take mathematical artefacts that learners know as everyday objects in the workplace and make them the focus of a discussion between learners, and between learners and ourselves – that is, to become "symbolic boundary objects" for communication and shared understanding. The activities make use of spreadsheet software as a tool for the modelling of financial-mathematical situations, as well as specially-programmed simulation tools in the form of Flash applications. Activities start with straightforward exercises to develop modelling techniques with spreadsheets, through to open-ended tasks involving the need to make interpretations around calculations, and finally role-play exercises requiring the employees both to interpret and explain in appropriate language to a customer.

Trials have shown the benefits of our approach in a context where successful mathematical training has long been recognised as problematic, and we will present some practical and theoretical implications of the approach.

TEACHING MATHEMATICS BY PICTURE BOOKS FOR CHILDREN

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Jing Chung and Wen-Yun Lin, National Taipei Teachers College

It is common that 5th and 6th graders are lack of interest in learning mathematic. They feel mathematic is useless to them and difficult to learn. As a mathematic teacher, it is a challenge to increase students' interests in learning mathematic and at the same time to make connection between mathematic and students' daily life experiences.

The NCTM (2000) suggests that mathematic is about problem solving, communication and reasoning. Learning mathematic is a constructive process which should be connected to real questions raised from daily life. Moyer (2000) suggests that children learn mathematic through children's books. It is because the content and illustration of children's books support children to discuss math. Whitin and Whitin（2001）believe that by means of stories, illustration and discussion children would learn to use mathematic language for communication. Krech（2003）also believes that except the learning of calculation , stories could help students to learn mathematic through sorting, drawing , planning , and talking.

This is a one year research on a 5th mathematic class. The teacher who is also the researcher taught numbers, quantity and shapes by means of eight pictures books. The discussion of findings are divided into two categories, one is the content of the story and the other is the concept of math. The research points out that picture books with strong story content could promote students' learning interests and also nurture mathematic thought and talk. Besides, these picture books could also lead students to work on problem solving and reasoning and that encouraged students to connect their life experiences to math. On the other hand, picture books with strong mathematic concepts could lead students to learn new mathematic concepts and clarify their thoughts. This research concludes that picture books with strong story content promote student's interest in learning mathematic while picture books with strong mathematic concept promote the effect on learning math.

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DIFFERENT POSSIBILITIES TO LEARN INFINITY OF DECIMAL NUMBERS

Angelika Kullberg

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This poster presents a model called Learning Study (Marton & Tsui, 2004) that has been used to explore the teaching and learning about the infinity of decimal numbers. A Learning Study is a model for teacher collaboration that has the aim to improve students' learning by finding critical aspects for students understanding of a specific content. The model was applied in an investigation in which three classes in grade 6 (12 year old students) and their three teachers took part. The methods of this investigation involved a cyclic process in which a lesson was tested, analysed and revised. The same question was used in all three classes to discuss infinity of decimal numbers, and to observe whether or not there are numbers between the two decimal numbers 0.97 and 0.98. The post-test results showed that;

- When decimal numbers in a class (class A) were treated only as numbers on a number line 21% of the students showed on the post-test that there was infinite number of decimal numbers.
- When decimal numbers was also represented by fractions and percentage (class B and C) 88% and 94% of the students showed on the post-test that there was infinite number of decimal numbers.

Another aspect that seemed to be critical for the students' understanding was also the experience of the part-part-whole relationship of a decimal number, for example that 0.97 could be seen as a part from a whole. The part-part- whole relationship emphasises the relationships of the parts as do fractions. The fractions explicitly points to parts, which could be transformed in to smaller and smaller parts. In conclusion, in class A decimal numbers were treated as countable numbers in an interval juxtaposed to class B and C where decimal numbers were treated as different number of parts in an interval. The different ways of treating the content contributed to different possibilities for learning the infinity of decimal numbers. Previous research (Roche & Clarke, 2006) also indicates that establishing a connection between different forms of rational numbers is a benefit to students' understanding.

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THE VALUE OF LEARNING IN A COMMUNITY

Nayoung Kwon

University of Georgia

Teacher educators have long been interested in research on professional learning communities for teachers' professional development. For example, Hord (1997) examined the characteristics of communities for teachers, and Grossman, Wineburg, & Woolworth (2001) looked at the processes of building the communities. However, little research has been done on the value of learning in a communities. The purpose of my research was to understand the context of community and to help research on professional development using the communities. In particular, I addressed the question of what members value in a learning community in this paper.

Partnerships in Reform in Mathematics Education (PRIME), the NSF-funded research, was conducted for high school mathematics teachers in the Northeast Georgia region. As a case study, part of PRIME research, I investigated a learning community consisting of three mentor teachers, three student teachers, and a university teacher in Norris High School. Seven participants met weekly to discuss the teaching and learning of student teachers, to share their opinions, and to talk about each participant's beliefs about teaching and learning mathematics. Data included nine audio taped meetings, two interviews for each participant, and written documents such as observation notes, surveys, and e-mail responses. All interviews were transcribed and considered as the participants' narratives.

The written data were reconstructed into three narratives according to the research questions. The results showed that the members valued their learning and other member's learning in the community. Their learning came from different perspectives, different environments, and students' work. Their learning such as notice and formation occurred because of sharing different perspectives. Learning from different environment encouraged reflection on their own practices, increased motivation, and helped them recognize partners. Focusing on students' work and using new materials also helped their learning in the community. This research will help educators understand a learning community of teachers and teacher educators and also help to use partnerships for designing teachers' communities.

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VISUAL CALCULATION WITH MENTAL ABACUS

Young-Hee Kwon & Kyung-Yoon Chang

Konkuk University, Korea

IMPORTANCE OF CALCULATION

> "Computational skill is one of the important, primary goals of a school mathematics program. …The nature of learning computational process and skills requires purposeful, systematic, and sensitive instruction. …" - tenets of NCTM(1978)-

The mastery and proficiency in four operations are necessary for school mathematics at all levels. Recent survey, however indicated that a number of elementary and middle school students do not have the minimal calculation abilities to learn school mathematics. This study attempts to investigate the possibility of a instructional method for mental calculation.

ABACUS & MENTAL ABACUS

Abacus, a calculating device based on the decimal number system, can help students to develop computational skills and to understand the decimal number system as well.

A former National Champion of Mental Calculation, Ms. Kwon has been using a mental abacus as a tool for mental four operations. Mental abacus is a dynamic image of an abacus, and the beads of an abacus are visualized and activated as if it is placed in front of one's eyes while calculation is processing.

Mental calculation provides a power in mathematics learning by saving one's working memory for more advanced mathematical thinking. Calculation with mental abacus also has an effect of activating the right hemisphere of a brain.

PROCEDURE & RESULTS

The subjects were two children of age six who enter the elemental school and did not start to learn calculation yet in school. The data were collected with instruction for two months. The objectives of the instruction were addition and subtraction of two or three of 1-digit numbers without- and with- re-grouping, e.g. 1+5+3, 8-7+6, 2+6+9, 7+5-3, etc. those are usually taught in the end of Grade 1.

The teaching sequence was: Solving operation problem by abacus -> Solving operation problem by mental arithmetic -> Listening and solving operation problem by abacus -> Listening and solving operation problem by mental arithmetic -> Flash mental arithmetic. The procedures and the results will be presented and discussed with photos.

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MATHEMATICS EDUCATORS PERSPECTIVE ON MATHEMATICS KNOWLEDGE OF SETMTP

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Researchers developed the Standards for Elementary Teachers' Mathematics Teaching Professional Knowledge and Competence (SETMTP) by literature and modified by 7 mathematics educators of elementary school that they are major at professional development of teachers. We census to 34 mathematics educators of elementary school about the perspective of SETMTP. The SETMTP has six categories: Mathematics Knowledge (MK), Student Cognition (SC), Teaching Method (TM), Teaching Practice (TP), Teaching Assessment (TA) and Professional Accountability(PA) and there are 9, 4, 3, 8, 4, and 18 standards for each category, respectively, with a total of 47 standards. This paper is report mathematics educators perspective on Mathematics Knowledge.

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EXAMINING STUDENTS' OUTCOMES TO IMPROVE THEIR UNDERSTANDING OF MIXED OPERATIONS ON FRACTIONS

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Fractions concepts are one of the complex mathematics ideas which children encounter in the elementary schools (Litwiller, B. & Bright, G., 2002). Nowadays, researchers and educators, especially teachers admitted "Fractions" that have been found in the mathematics curriculum of different countries around the world, to be one of the most difficult topics for students to learn. In Asia like Hong Kong (CDC, 2000) and Japan (1998, amended in 2003), the topics of fractions are designed to implement from Grade 3 in primary schools. On the other hand, researchers (Aksu, 2001; Solange, 2005) have found that the complexity of concepts of fractions would lead to different levels of difficulties in learning and teaching of it. In this study, based on a government's curriculum project (EMB, 2006) that is to investigate mathematics learning and teaching in the primary classroom with a particular focus on assessment for learning (Gardner J., 2006) through two new initiatives, the Learning Outcomes Framework (LOF) and the Basic Competency Assessment (BCA), we select appropriate teaching strategies and try out different teaching aids to deal with students' problems through analysis on their learning outcomes. In addition, this study investigated into teachers' perception on effective teaching of the topic, "Fractions" through students' learning outcomes, especially for students' performance on "developing concept of fractions as a part of one whole and a part of a set of objects" and "mixed operations of fractions with different denominators". More than 200 students' annotated works and scripts are collected from different primary schools in Hong Kong for analysis.

The poster presentation reveals and shares teachers' experience in effective use of teaching aids such as fraction cards, fraction strips and IT animations to help students develop the concepts and operations of fractions. To encourage teachers more non-paper-pencil assessment in the mathematics classroom such as class discussion, oral presentation and etc., these teaching strategies will also be reviewed.

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THE PREDICTIVE EFFECT OF THE FRAMEWORK OF COGNITIVE COMPLEXITY FOR THE ITEM DIFFICULTY VARIANCE OF THE NUMBER PROBLEMS

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Chuan-Ju Lin

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Large scale assessment routinely release part of the sample items to communicate the assessment theme. Proportion correct of each release item is also included in the release documentation. To translate the statistic information into teaching practice adjustment, teachers usually need some professional supports. In this study, an analysis framework on item cognitive complexity is proposed and implemented. The 2006 on-line tests of Taiwan Assessment of Student Achievement on Mathematics (TASA-MAT) for the forth-, sixth-, and eighth-graders were used for the preliminary analysis.

For the number content, a 3 cognitive components coding schema, including the novelty of the item context, the translation of representation, and the logic abstraction, was developed to predict the item difficulty parameters. The data used in this study is derived from 2006 TASA-MAT for the forth-, sixth-, and eighth-grade students. We developed a common metric with concurrent calibration under Three-Parameter-Logistic Item Response Theory. There were three raters to rated the cognitive components for each item. The results suggest that the framework proposed can predict around 43% of the difficulty variance across 20 released items within number content. The correlations of each component between the three raters were .86 to 1.00. The 3 cognitive components coding schema, the contextual novelty, the need of translation, and the abstraction of the number properties, could influence the item difficulty, the more complex of the cognitive components, and the more difficult of the item. The results suggested that 3 cognitive components coding schema for number problems may be feasible. The implications of these results for the math teachers are discussed.

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A FRAMEWORK FOR IMPLEMENTATION OF ELEMENTARY SCHOOL MATHEMATICS

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The purposes of this research were to investigate a curriculum framework for improving mathematics education in elementary schools and to refine the Ten Dimensions of Mathematics Education framework for school and district implementation of elementary school mathematics. The research proposes that teacher improvement in teaching mathematics can be informed by a framework of Ten Dimensions and by working with principals. Findings show that, by identifying two dimensions as personal and school goals, teachers can improve student achievement and their own self-efficacy.

The theoretical framework suggests that teachers make improvement in teaching mathematics when they have goals, have support from other educators, and employ a framework of mathematics dimensions using a continuum of levels of improvement. Research studies report that teachers who have high confidence in their ability to bring about student achievement in standards-based mathematics programs produce higher student achievement.

Method

Principals were introduced to the Ten Dimensions framework at a workshop. The principal and teachers identified two dimensions for further investigation. Phase 1 of the research was an intervention consisting of identifying goals for improvement and mentor-peer coaching. Phase 2 of the research focused on school improvement teams and school improvement plans. Data was collected through interviews with teachers and principals and observations of peer coaching pairs and school improvement team meetings.

Findings

One outcome the study was the improvement in mentoring in mathematics instruction and the implementation of effective and achievable teacher and school improvement plans. This research contributes to our understanding of teacher practice changes when based on a mathematics reform framework. The results describe specific strategies that enable the principal to assist teachers to improve their teaching practice as well as peer teaching strategies that helped teachers improve their practice.

The poster will display the assessment tools such as the teacher beliefs survey and the Ten Dimensions continuum, the peer-coaching results and the changes made to the Ten Dimensions based on the research study.

STARTING FROM TEACHERS MOTIVATION IN DEVELOPING PEDAGOGICAL KNOWLEDGE, TO ENHANCE MATHEMATICAL KNOWLEDGE: A WORK PROJECT APPROACH IN A TEACHING COURSE

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This presentation is focused on a ongoing one year in service teaching course with ten 5th and 6th grade mathematics teachers. This course is integrated in a national program that aims to develop elementary teachers' mathematical knowledge. The course follows a work project methodology in which teachers are invited to identify problems that they have encountered during their teaching of mathematics. This study aims to understand teachers' awareness of their mathematical knowledge for teaching and how the methodology followed during the course can contribute to enhance it.

The mathematical knowledge needed for teaching elementary mathematics has been discussed by some authors such as Ball, D., 2003 and Ma, L., 1999. According to Ma, L. teachers need to have a profound understanding of fundamental mathematics in order to teach mathematics in a comprehensive way. But teachers' knowledge is a complex body of knowledges (e.g. Shulman, 1986) and they should be integrated instead of to be approached in an atomic way. Dewey' ideas about education defend the motivation principle. He also advocates that the development of experience comes through interaction, what means that education is essentially a social process. Following these ideas, the work project methodology followed in this course aims to enhance mathematics understanding, integrating teachers' experience, through cooperative work with their peers and teacher educator and based on teachers' motivation to solve teaching problems. During a school year, each fortnight, teachers attend three hours seminars and they are accompanied in several classroom activities by the teacher educator followed by joint reflections.

The teachers involved had at least four disciplines of mathematics during their pre-service courses considering that the knowledge they have is quite enough to teach mathematical topics of students' curriculum. However, their answers to a questionnaire revealed lack of understanding in mathematical themes such as proportionality, place value, division of fractions and probability. The problems chosen by teachers for the work project *were all in the field of pedagogy and didactics*. Their awareness to deep mathematical knowledge has emerged from the work they are developing with students, the cooperative work with their peers as well as by discussions and reflections in the context of the work developed so far. As the course will finish during June 2007, the poster will display the main results.

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STUDENTS' UNDERSTANDING OF INVARIANCE IN THE CONTEXT OF PROPORTIONAL REASONING

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Proportional reasoning plays a critical role in a student's mathematical development since it is one of the capstones of elementary concepts. According to Piaget's theory proportional reasoning is indicated as the hallmark of the formal operational stage of development (Inhelder and Piaget, 1958). It involves a sense of covariation, multiple comparisons and the ability to mentally store and process several pieces of information (Lesh, Post, & Behr, 1988).

The purpose of this research was to investigate middle school students' understanding of invariance in the context of semantic problems involving proportional reasoning. The students' understanding was analysed using a mixed method. A test given to 180 middle school students was composed of 4 types problems following to Lamon's (1993) semantic types: Associated sets, part-part-whole, well-chunked measures and stretchers and shrinkers. Within each type, there were missing-value problems and numerical comparison problems. After the written test, a semi-structured interview was conducted with some students at various achievement levels in order to inquire students' understanding of invariance.

The data analysis showed that the students had procedural competence rather than conceptual understanding. The students could not understand the structure of problem and recognize the invariance and covariance in the problem although they were able to solve the problem correctly. These results, based on understanding of students' problem solving process, suggest that further research is of essence to develop the concept and strategies of proportional reasoning and effective instructional strategies.

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TRAINING PROGRAMME FOR TEACHERS USING A NETWORK-BASED LEARNING PLATFORM

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The aim of this project funded by the German Research Organisation was to show that subject-specific (problem-solving in mathematics) and interdisciplinary competence (self-regulation) can be improved with the help of special trainings (Collet/Bruder, 2006). Based on the results of this research project, an online programme for further education was developed on the learning platform "moodle" (www.prolehre.de). Contents of the modules are a four step lecture-concept for heuristical education, examples for learning heuristic strategies (Engel, 1998) and information for the support of self-regulation strategies i.e. with the help of homework.

6 modulus are available every 14 days with work instructions, information and tested examples for lectures. Each module takes 2 to 4 hours of work.Expected results of the teachers are a self developed and tested exercise for problem solving, a lecture-concept and a long term assignment. The participants get a feedback for every working result handed in.

For supervision and support are useful a daily availabel teletutor, board and chat for exchange of experiences, a data base for exercises www.madaba.de

and a platform for materials www.problemloesenlernen.de.

The blended learning course was carried out in three semesters with 121 participants in total. It was evaluated throughout questionnaire and the analysis of the working results. For assessment of the teachers' working results the following criteria are used: specialist background, goals and motivation, saving the initial level and internal differentiation, cognitive activation, activities of pupils, elements of self-regulation, meaningful use of calculator and the kind of implementation of the concepts. The poster shows the results of these evaluation.

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BUILDING MATHEMATICAL KNOWLEDGE FOR TEACHING USING TECH-KNOWLEDGY

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This research explored pre-service teachers' experiences with planning and teaching with technology and its impact on their mathematical knowledge for teaching, confidence and attitude about teaching mathematics. The term "tech-knowledgy" was used in the mathematics methods class to focus on specific technologies that promoted knowledge construction, such as concept tutorials and virtual manipulative that explored mathematics concepts, models and representations. Through this project, pre-service teachers selected and evaluated technology tools on effectiveness as teaching and learning tools and planned and taught a mathematics lesson integrating technology. In the process, teachers learned important mathematics and models for mathematical thinking and gained confidence in their ability to effectively use technology to facilitate students' learning.

SUMMARY

Today, our children are growing up in a technology advanced society where working flexibly and thinking critically with technology while solving problems is an increasingly important skill. Jonassen (1996) defined computers as mind tools that should be used for knowledge construction while engaging learners in critical thinking about the content they are studying. This project was interested in these mind tools or as used in this project, "tech-knowledgy" that aimed at building preservice teachers mathematical knowledge for teaching. This project was driven by the following research question: How do preservice teachers' experience with tech-knowledgy impact teachers' mathematical knowledge, their confidence and attitude about teaching mathematics using technology? Results showed that teachers used the technology to represent multiple models and to illustrate abstract mathematics concept while offering differentiation and scaffolding for diverse learners. Most importantly, technology offered opportunities for teachers and students to construct important mathematical knowledge. This project illustrated the importance of preparing teachers to select effective mathematics-technology tools, to design lessons and to create a learning environment that optimizes the potential for construction of mathematical knowledge. Preparing teachers to consider unique design features for planning effective integration of technology in mathematics is critical. Teaching with technology has promising outcomes for our students' learning and can support meaningful knowledge construction.

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THE ANALYSIS OF LEARNING RESOURCES BASED ON THE EFFECTIVENESS OF DGE

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ESSENTIALS OF EFFECTIVENESS FOR LEARNING GEOMETRY OF DGE

The purpose of this study is to reconsider the learning resources in junior high school. In this paper, we especially focus on *the idea of degree of freedom of points*[*1] *in DGE.*

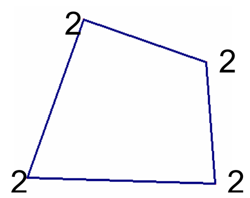
The essentials of effectiveness for learning geometry of DGE require students to not only discover the invariant properties of the geometrical figures, but to investigate why they change while keeping their properties, and construct them in DGE based on their investigation. Through these activities, they make sense of the concept of figures and develop logical thinking for proof. When we do develop the mathematics curriculum in consideration of the effectiveness of DGE in the future, we should reconsider all of the contents of geometry in elementary and junior high schools to focus on that point.

The idea of "degree of freedom of points" is useful in analyzing and explaining the properties of the relations between the figures, and the difference of changes in them based on the procedures of geometric construction in DGE. This idea is the key concept to create the framework for curriculum development in the future.

THE CASE OF INCLUTION RELATIONSHIPS OF QUADRILATERALS

Focusing on the idea of "degree of freedom of points" in geometrical figures in DGE, we can represent the inclusion relationships of quadrilaterals as in the following table:

Ex. Quadrilateral



Degree of freedom	Quad.	Parallel.	Rhom.	Rect.	Squa.
2 (No Constrain)	4 vertex	3	2	2	2
1 (One Constrain)			1	1	
0 (Two or More)		1	1	1	2

It is important that students consider the focus of geometrical figures on the relations between their factors and other figures. It is beneficial for the students to investigate the meaning of the numbers in this table. For example, all of the cases have at least two vertexes of degree "2", because we initially decided an edge when constructing the figures. That is the starting point in the perception of the distinction between the "drawings" and "geometrical figures".

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HELPING TEACHERS COMMUNICATE ABOUT TEACHING MATHEMATICS WITH TECHNOLOGY

Marcia L. Weller Weinhold

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A community of practice (Wenger, 1998) may form when a group of people have a shared enterprise. This ongoing study acknowledges the shared enterprise of teaching secondary mathematics and uses the development of a tool intended to guide the appropriate use of graphing calculators (AUGC) as a focus for teacher inquiry. Design research methodology is used to track the development of an AUGC tool, and results are analysed through the lens of a community of practice. Besides the AUGC tool itself, findings revealed by the process of inquiry relate to teacher beliefs, their own use of calculators, school policies, and the curriculum materials they are using.

Initial interviews and a survey provide baseline information about how teachers think about using graphing calculators for teaching mathematics. In one part of the initial interview teachers are asked to sort tasks according to whether it would never, sometimes or always be appropriate for students to use calculators with the task. Teachers are asked to clarify conditions under which they would allow calculator use with "sometimes" tasks.

This information, along with classroom observations by the researcher, is used to design activities for the first inquiry session. Each of a series of four inquiry sessions has three parts: doing mathematics with graphing calculators; investigating research addressing technology use and mathematical tasks; and working on the AUGC tool. The iterative nature of design research (Zawojewski, in press) and the time between inquiry sessions allow teachers to incorporate their classroom experiences and session activities into a new version of the AUGC tool, which is tested in making classroom decisions. The poster will provide a visualization of this iteration.

Continuous analysis of session and interview transcripts and additional classroom visits guide planning of each inquiry session. Final interviews allow individual comments on the final form of the AUGC tool, and teachers revisit the survey and the task sort to reflect on changes in their own thinking. The resulting tool is considered a work in progress, and it is the major finding of each cohort of teachers studied. Future analysis of such tools will compare them based on previous calculator use of teachers, school policy and curriculum materials used.

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THE DEVELOPMENT OF A COMPUTERIZED NUMBER SENSE DIAGNOSTIC TESTING SYSTEM AND ITS APPLICATION FOR FOURTH GRADERS IN TAIWAN

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To diagnose misconceptions of number sense for 4th graders in Taiwan, a computerized number sense diagnostic testing system was developed. This diagnostic testing system included 40 items based on recent number sense theories (McIntosh et al., 1997; Yang, 2003). 600 fourth graders from remote districts, rural areas and cities in Taiwan were selected to participate in the study through the on-line testing system. The major findings of the study are listed below:

1. Based on the calibration sample (N=600), the computerized number sense testing system demonstrates good reliability and validity (Cronbach's α was .835 and construct reliability was .986).
2. The 4th-grade students in Taiwan perform best on "recognizing the relative number size" and worst on "being able to compose and decompose numbers."
3. The testing system could be also used to diagnose students' misconceptions on number sense related questions. For example:

Q1: Which of the following options is equal to 8/10? 1) The addition of two 4/5 2) The addition of eight 1/10 3) the addition of ten 1/8 4) The addition of 8 and 1/10

Item options	1)	2)	3)	4)
f(%)	183 (30.5%)	198 (33%) *	195 (32.5%)	14 (2.3%)

	reason options (missing value=10 [1.7%])
136 (22.7%)	That's because 4/5+4/5＝8/10.
178 (29.7%)*	That's because the addition of eight 1/10 is equal to 8/10.
165 (27.5%)	That's because the addition of ten 1/8 is equal to 8/10.
4 (.7%)	That's because 8+1/10=8/10.
107 (17.8%)	I guessed.

Item difficulty is 0.5467, Item discrimination is 0.3733 (* indicates the correct answer)

The important point to note is that about 23% of fourth graders had misconceptions on the addition of fractions. They believed that 4/5+4/5＝8/10. Besides, there are about one- fourth of fourth graders could not recognize the relationship between 8/10 and eight 1/10. This suggests that some of them need remedial instructions on related misconceptions.

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READING STRATEGIES FOR COMPREHENDING GEOMETRY PROOF

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This purpose of this study is to explore how students read geometry proof and how their reading strategies are related to their reading comprehension of geometry proof.

INTRODUCTION

(1) Yang and Lin (2007a, 2007b) have investigated a model of reading comprehension of geometry proof and proposed that some other factors should be taken into account in addition to logical reasoning and geometry knowledge. (2) The importance of this purpose is to enhance the understanding of how reading strategies may benefit or damage reading comprehension of geometry proof and then how reading strategies can be involved in the instruction of geometry proof.

METHODOLOGY

Firstly, a questionnaire was modified from a model of reading comprehension of geometry proof (Yang & Lin, 2007a) and designed on the basis of Duval's three modes of reasoning - micro, local and global (Duval, 1998). Next, 243 ninth-tenth graders were tested by the questionnaire, and classified by hierarchical cluster analysis in order to select interviewees. Furthermore, we use semi-structured interview to investigate how the eighteen students read proofs. We interpreted the interview data via comparing and contrasting reading strategies of different students.

CONCLUSIONS

(1) The time to read the applied properties: a. before reading the geometry proof process. b. after answering questions of reading comprehension of geometry proof. (2) The influence of familiarity with proof content on reading strategies: a. skipping the familiar parts. b. skimming the familiar parts. (3) The choice of propositions: a. priority to read familiar propositions even if there are more proof steps. b. priority to read unfamiliar propositions since there are few proof steps.

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